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ON SPECTRAL GAPS OF GROWTH-FRAGMENTATION SEMIGROUPS IN HIGHER MOMENT SPACES

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Abstract. We present a general approach to proving the existence of spectral gaps and asynchronous exponential growth for growth-fragmentation semigroups in moment spaces $L^1(\mathbb{R}_+;\ x^\alpha dx)$ and $L^1(\mathbb{R}_+;\ (1+x)^\alpha dx)$ for unbounded total fragmentation rates and continuous growth rates $r(.)$ such that $\int_0^{+\infty}\frac{1}{r(\tau)}d\tau = +\infty$. The analysis is based on weak compactness tools and Frobenius theory of positive operators and holds provided that $\alpha > \widehat{\alpha}$ for a suitable threshold $\widehat{\alpha} \geq 1$ that depends on the moment space we consider. A systematic functional analytic construction is provided. Various examples of fragmentation kernels illustrating the theory are given and an open problem is mentioned.

Contents

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1. **Introduction.**

1.1. **Notation and general assumptions.**

This paper deals with the existence of *spectral gaps* (see (11) below) for C_0-semigroups $(V(t))_{t\geq 0}$ governing general growth-fragmentation equations

$$\begin{aligned}&\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}\left[r(x)u(x,t)\right]+a(x)u(x,t)\\ =\;&\int_x^{+\infty}a(y)b(x,y)u(y,t)dy,\quad (x,t>0)\end{aligned}\tag{1}$$

in moment spaces

$$X_{0,\alpha}:=L^1(\mathbb{R}_+;\ (1+x)^\alpha\,dx),\quad X_\alpha:=L^1(\mathbb{R}_+;\ x^\alpha dx)\quad (\alpha>0)\tag{2}$$

with nonnegative total fragmentation rate

$$a(\cdot)\in L^1_{loc}(0,+\infty)$$

and a measurable fragmentation kernel $b(\cdot,\cdot)$ such that $b(x,y)=0$ if $x\geq y$,

$$\int_0^y xb(x,y)dx=y\tag{3}$$

and

$$\text{the support of }\ (x,+\infty)\ni y\to a(y)b(x,y)\text{ is } \textit{unbounded}\ \ (x>0)\,.\tag{4}$$

This assumption is required for irreducibility of the growth-fragmentation semigroup, which we also prove under an alternative assumption that $a(y)>0$ for $y\in(0,\infty)$ and

$$\text{there is } p\in[0,1) \text{ such that for any } y>0, \inf\operatorname{supp} b(\cdot,y)\leq py.\tag{5}$$

We note that assumptions (4) and 5 have different physical meaning. If (4) is satisfied, we allow for particles of some sizes not to fragment. This must be offset, however, by the requirement that particles of any size can be obtained by fragmentation of arbitrarily large particles. In this way, non-fragmenting sizes always can be jumped over by daughter particles of parents of a bigger size. Assumption 5 has different interpretation. The fact that $a(y)>0$ means that particles of any size must split and the second part says that the sizes of daughter particles cannot be too close to the parent's size. In physically realistic situations we expect that fragmentation produces at least two daughter particles whose sizes cannot be both close to the parent's size and thus in such a case this assumption is always satisfied. In particular, a common case of the homogeneous fragmentation kernel $b(x,y)=\frac{1}{y}h\left(\frac{x}{y}\right)$ with h of bounded support, is covered by (5).

Our assumptions on the growth rate are

$$r\in C(0,+\infty),\ \ r(x)>0\ \forall x>0\tag{6}$$

and

$$\int_0^1\frac{1}{r(\tau)}d\tau<+\infty,\quad \int_1^\infty\frac{1}{r(\tau)}d\tau=+\infty\tag{7}$$

or

$$\int_0^1\frac{1}{r(\tau)}d\tau=+\infty,\quad \int_1^\infty\frac{1}{r(\tau)}d\tau=+\infty.\tag{8}$$

In the case (7), we complement (1) with the boundary condition

$$\lim_{y\to 0} r(y)u(y,t) = 0.$$

The kinetic equation (1) is the linear part of the growth–coagulation–fragmentation equation where, in the full form, the coagulation part is represented by a quadratic integral term, see [6]. Coagulation and fragmentation processes lay at heart of many fundamental phenomena in ecology, human biology, polymer and aerosol sciences, astrophysics and the powder production industry; see [5] for further details and references, and [11] for the probabilistic context. A common feature of these processes is that each involves a population of inanimate or animate agents that are capable of forming larger or smaller aggregates through, respectively, coalescence or breakup. Coagulation and fragmentation are conservative processes. In many cases, however, they occur alongside other events that result in the growth of the ensemble. For example, in chemical engineering applications we often observe a precipitation of matter from the solute onto the surface of the aggregates. In biological applications, the growth of the aggregates can occur due to births of new individuals with neonates staying inside the parent's aggregate, see e.g. [1] for the application to phytoplankton or [31] in a general context. The interplay of growth and fragmentation plays also an important role in prion proliferation, see e.g. [18].

In (1), the unknown $u(x,t)$ represents the concentration at time t of "agregates" with mass $x > 0$ while $b(x,y)$ $(x < y)$ describes the distribution of mass x aggregates, called daughter aggregates, spawned by the fragmentation of a mass y aggregate. The local mass conservation in the fragmentation process is expressed by (3); we say that the fragmentation kernel $b(\cdot,\cdot)$ is conservative.

In a preliminary step, we provide explicit formulas of the C_0-semigroups $(U(t))_{t\geq 0}$ (with generator T) governing the transport equation

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\left[r(x)u(x,t)\right] + a(x)u(x,t) = 0 \tag{9}$$

in the functional spaces $X = X_\alpha$ or $X_{0,\alpha}$ and discuss the effect of the conditions on the growth rate $r(\cdot)$ on them. This direct approach complements a resolvent Hille-Yosida approach [6]; see also [5] Chapter 5 and [10]. It turns out, at least for *bounded* total fragmentation rate $a(\cdot)$, that under Assumption (7), the problem (9) is *not* well-posed in X_α in the sense of C_0-semigroups; see Remark 3 below. (This does not prevent a generation theory for suitably *singular* functions $a(\cdot)$ but, in this case, the whole construction of the paper would need new technicalities; to keep the coherence of the paper, this special case is treated separately [28].) Hence, in general, under Assumption (7), a C_0-semigroup $(U(t))_{t\geq 0}$ governing (9) is defined *only* in $X_{0,\alpha}$ and we restrict our spectral gap construction to this space. The situation is much more complex under Assumption (8). Indeed, the problem (9) is well-posed in both X_α and $X_{0,\alpha}$ (with suitable assumptions depending on the space) but the full spectral gap theory is completed *only* in X_α. Indeed, in $X_{0,\alpha}$, although all the preliminary results we need can be proved, two of them are based on assumptions which are *not* compatible: indeed, by using the confining role of singular absorptions [26], the resolvent compactness of T (which plays a key role in our construction) follows from the unboundedness of the total fragmentation rate $a(\cdot)$ at infinity *and* at zero, while the existence of a C_0-semigroup $(V(t))_{t\geq 0}$ with generator

$$T + B : D(T) \to X$$

(B is the fragmentation operator (10)) depends on the boundedness of $a(\cdot)$ near zero. Hence, under Assumption (8), we need to restrict our construction to the space X_α.

In summary, *two spectral gap theories* are given in this paper*:* one in X_α under Assumption (8) and another one in $X_{0,\alpha}$ under Assumption (7). A spectral gap theory in X_α under Assumption (7) needs additional technicalities and is given in [28]. Finally, the existence of a spectral gap in $X_{0,\alpha}$ under (8) is an *open problem*; see Remark 12. Our main results are given in Theorem 3.8 and Theorem 4.9 and are consequences of many preliminary results of independent interest. Furthermore, various examples of fragmentation kernels (homogeneous or separable) are given to illustrate the relevance of our assumptions.

The aim of this paper is twofold. The first aim is well-posedness of (1) in the sense of C_0-semigroups. Indeed, because of the unboundedness of the total fragmentation rate $a(\cdot)$, the fragmentation operator (10) is *not* a bounded operator on X (where X is either X_α or $X_{0,\alpha}$). Under suitable assumptions, depending on the space we consider, a generation result is obtained by using a perturbation theorem by W. Desch specific to positive semigroups in L^1-spaces (see below) where the perturbation is given by the fragmentation operator

$$B : D(T) \ni \varphi \mapsto B\varphi \in X, \quad (B\varphi)(x) := \int_x^{+\infty} a(y)b(x,y)\varphi(y)dy, \tag{10}$$

and T is the generator of $(U(t))_{t\geq 0}$ in X. The second aim of this work is proving the existence of a spectral gap of the C_0-semigroup $(V(t))_{t\geq 0}$ governing (1), i.e., showing that

$$r_{ess}(V(t)) < r_\sigma(V(t)) \tag{11}$$

(r_{ess} and r_σ are respectively the essential spectral radius and the spectral radius). If, additionally, $(V(t))_{t\geq 0}$ is irreducible, then, by [29], Corollary 3.16 of Chapter C-III, the spectral bound λ of its generator is its dominant eigenvalue and a simple pole of the resolvent. Moreover, by [17], Proposition 3.4 of Chapter VI, λ is a simple eigenvalue, that is, its eigenspace is one-dimensional. Hence $(V(t))_{t\geq 0}$ has the asynchronous exponential growth property,

$$\left\| e^{-\lambda t}V(t) - P \right\|_{\mathcal{L}(X)} = O(e^{-\varepsilon t}) \tag{12}$$

(for some $\varepsilon > 0$), where P is a one-dimensional spectral projection relative to the isolated algebraically simple dominant eigenvalue λ of the generator, see e.g., [17], Theorem 3.5 of Chapter VI, defined as $P = \langle \mathbf{e}, \cdot \rangle \mathbf{f}$, where $\mathbf{f}$ and $\mathbf{e}$ are strictly positive eigenvectors of, respectively, the generator and its dual, and $\langle \cdot, \cdot \rangle$ is the pairing between X and its dual. A summary of these results can also be found in [34], Appendix C.

The main mathematical ingredients behind the occurrence of the spectral gap are a *local* weak compactness property satisfied by general growth-fragmentation equations (due to the one-dimensionality of the state variable) and the *confining effect* of singular total fragmentation rates ensuring the compactness of the resolvent.

There is an large body of literature dealing with the long term dynamics of solutions to (1) and, in particular, with the asynchronous exponential growth property and the existence of spectral gap. The case when the state space is bounded has been well understood since [14], though the cases with unbounded rates can be tricky, [4]. When the state space is unbounded, a number of results have been obtained by the powerful General Relative Entropy method introduced in [22]. While the method caters for a large class of coefficients in weighted L^p spaces, the exponential rate of convergence has only been established in [32], see also [20], and extensively studied since then. Due to its physical interpretation it is important to study the problem in L^1 spaces. Some results have been established by probabilistic methods, see e.g., [11][12] but we are focused on operator–theoretic results for which we refer to the recent works [23][9][10][13][7]. In particular, quantitative estimates of the gap are obtained by means of Harris's theorem, [13], while [7] contains a comprehensive theory for the discrete case written in the spirit of this paper. A special mention should be given to [16], where the Perron eigenvector and eigenvalue were found and analysed for (1) with fairly general coefficients. That paper has stimulated an active research along these lines, culminating in recent works [9][10].

We note also that most of the known literature on spectral gaps deals with Assumption (7), see however [13]. Our paper is close in spirit to [10] even if our statements are not the same and our constructions are different and more systematic; see below.

We note that

$$\int_0^{+\infty} u(x,t)xdx, \ \int_0^{+\infty} u(x,t)dx,$$

are respectively the total mass and the total number of agregates at time $t \geq 0$. The existence of spectral gaps in the natural functional spaces

$$X_1 = L^1(\mathbb{R}_+;\ xdx), \quad X_0 = L^1(\mathbb{R}_+;\ dx), \ X_{0,1} = L^1(\mathbb{R}_+;\ (1+x)\,dx),$$

has been dealt with systematically in [27] but at the expense of suitable additional *mass loss*

$$\int_0^y xb(x,y)dx = (1-\eta(y))\,y, \quad (0 \leq \eta(y) \leq 1)$$

or *death* assumptions. These assumptions seem to be necessary and play a key role in well-posedness (via W. Desch's perturbation theorem) of growth-fragmentation equations in these natural spaces. Fortunately, W. Desch's theorem can be applied in higher moment spaces without such additional assumptions (see [6], Theorem 2.2). Actually, we adapt the argument of the proof of ([6], Theorem 2.2) in our construction. This allows for a significant extension of the general theory of [27] to higher moment spaces (2) by following a similar construction (without resorting to mass loss or death assumptions) provided that $\alpha > \widehat{\alpha}$ for a suitable *threshold*

$$\widehat{\alpha} \geq 1,$$

depending on the functional space we consider. This is consistent with the existence of thresholds known in the literature [23][10][13]. As in [27], our analysis is based upon few structural assumptions and provides a systematic functional analytic construction relying on weak compactness tools and the Frobenius theory of positive operators.

We recall a fundamental perturbation theorem in L^1 spaces, [15] (see also [35], [24] Chapter 8 or [3] Chapter 5).

Theorem 1.1. *(W. Desch's theorem) Let $(U(t))_{t\geq 0}$ be a positive C_0-semigroup on some $L^1(\mu)$ space with generator T and let $B : D(T) \to L^1(\mu)$ be positive (i.e. $B\varphi \in L^1_+(\mu)$ if $\varphi \in L^1_+(\mu) \cap D(T)$). Then*

$$T + B : D(T) \to L^1(\mu)$$

is a generator of a positive C_0-semigroup on $L^1(\mu)$ if and only if $T + B$ is resolvent positive or, equivalently, if $\lim_{\lambda\to+\infty} r_\sigma\left(B(\lambda - T)^{-1}\right) < 1$.

1.2. Main results.

1.2.1. *Fully singular growth rates* (8).

Let us describe first our main results in the spaces X_α and $X_{0,\alpha}$ under Assumption (8).

Properties of the growth–absorption semigroup in X_α.

The transport C_0-semigroup $(U(t))_{t\geq 0}$ governing (9) exists in X_α (resp. in $X_{0,\alpha}$) and is given by

$$U(t)f = e^{-\int_{X(y,t)}^{y} \frac{a(p)}{r(p)}dp} f(X(y,t)) \frac{\partial X(y,t)}{\partial y}$$

($X(y,t)$ is defined by $\int_{X(y,t)}^{y} \frac{1}{r(\tau)}d\tau = t$) provided that

$$\varpi := \sup_{z>0} \frac{r(z)}{z} < +\infty \ \left(\text{resp. } \sup_{z>1} \frac{r(z)}{z} < +\infty\right). \tag{13}$$

In addition, Assumptions (13) turn out to be also necessary. Note that under (8), the generation theory in $X_{0,\alpha}$ needs no condition on the growth rate at the origin. The resolvent of T is given by

$$\left((\lambda - T)^{-1} f\right)(y) = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda + a(\tau)}{r(\tau)}d\tau} f(x)dx \ (\Re\lambda > s(T))$$

in both spaces X_α and $X_{0,\alpha}$. We show the pointwise estimate in X_α,

$$\left|(\lambda - T)^{-1} f\right|(y) \leq \frac{1}{y^\alpha r(y)} \|f\|_{X_\alpha} \quad (\lambda > \alpha\varpi).$$

If we replace the natural condition $\sup_{z>1} \frac{r(z)}{z} < +\infty$ by the stronger one,

$$\widetilde{C} := \sup_{z>0} \frac{r(z)}{1+z} < +\infty, \tag{14}$$

then we can show the pointwise estimate in $X_{0,\alpha}$,

$$\left|(\lambda - T)^{-1} f\right|(y) \leq \frac{1}{(1+y)^\alpha r(y)} \|f\|_{X_{0,\alpha}} \quad (\lambda > \alpha\widetilde{C}). \tag{15}$$

We show that $(\lambda - T)^{-1}$ has a smoothing effect in X_α in the sense of improving the integrability of the input, that is, for $\lambda > \alpha\varpi$,

$$\int_0^{+\infty} \left|(\lambda - T)^{-1} f\right|(y) a(y) y^\alpha dy \leq \int_0^{+\infty} |f(y)|\, y^\alpha dy.$$

In $X_{0,\alpha}$, if we replace the natural condition $\sup_{z>1} \frac{r(z)}{z} < +\infty$ by (14), we show the smoothing effect in $X_{0,\alpha}$: for $\lambda > \alpha\widetilde{C}$

$$\int_0^{+\infty} \left|(\lambda - T)^{-1} f\right|(y) a(y)(1+y)^\alpha dy \leq \int_0^{+\infty} |f(y)|\,(1+y)^\alpha dy. \tag{16}$$

The above estimates, combined with the general theory, [26], on compactness properties in L^1 spaces induced by the confining effect of *singular absorptions*, show that if the sublevel sets of the total fragmentation rate

$$\Omega_c = \{x > 0; a(x) < c\} \quad (c > 0)$$

are "thin near zero *and* near infinity relatively to r" in the sense

$$\int_0^{+\infty} \frac{1_{\Omega_c}(\tau)}{r(\tau)} d\tau < +\infty \ \ (c > 0), \tag{17}$$

where 1_{Ω_c} is the indicator function of Ω_c (note that $\frac{1}{r(\cdot)} \notin L^1(0, +\infty)$), then T has a compact resolvent in both spaces X_α and $X_{0,\alpha}$. Note that (17) *precludes* $a(\cdot)$ to be bounded near zero or at infinity. Note also that (17) occurs for instance if

$$\lim_{y\to 0^+} a(y) = +\infty, \ \lim_{y\to+\infty} a(y) = +\infty.$$

Properties of the full growth–fragmentation semigroup in X_α.

In X_α, we introduce

$$n_\alpha(y) := \int_0^y x^\alpha b(x, y)dx.$$

We note that n_0, abbreviated as

$$n(y) := \int_0^y b(x, y)dx, \tag{18}$$

is the mean number (which can be infinite) of daughter aggregates spawned by the fragmentation of a mass y aggregate.

We show that if

$$\sup_{y>0} \frac{n_\alpha(y)}{y^\alpha} < +\infty$$

(note that it is automatically satisfied if $\alpha \geq 1$), then the fragmentation operator (10) is T-bounded in X_α and

$$\lim_{\lambda\to+\infty} \left\|B(\lambda - T)^{-1}\right\|_{\mathcal{L}(X_\alpha)} \leq \lim \sup_{a(y)\to+\infty} \frac{n_\alpha(y)}{y^\alpha},$$

where

$$\lim \sup_{a(y)\to+\infty} \frac{n_\alpha(y)}{y^\alpha} := \lim_{c\to+\infty} \sup_{\{y;\ a(y)\geq c\}} \frac{n_\alpha(y)}{y^\alpha}.$$

In particular, by W. Desch's perturbation theorem (Theorem 1.1),

$$T + B : D(T) \subset X_\alpha \to X_\alpha$$

generates a positive C_0-semigroup $(V(t))_{t\geq 0}$ on X_α provided that

$$\lim \sup_{a(y)\to+\infty} \frac{n_\alpha(y)}{y^\alpha} < 1. \tag{19}$$

In addition

$$T + B : D(T) \subset X_\alpha \to X_\alpha$$

is resolvent compact if T is. By exploiting strict comparison results of spectral radii of positive operators in domination contexts [21] and the convex (weak) compactness property of the strong operator topology [33][25] (see below), we deduce that $(V(t))_{t\geq 0}$ has a spectral gap (11) and exhibits the asynchronous exponential growth (12) in X_α provided (4) is satisfied (see Theorem 3.8). We conjecture that no spectral gap can occur in X_α if $a(\cdot)$ is bounded near zero as suggested by [10, Theorem 4.1].

One shows the α-monotony

$$\frac{n_{\alpha'}(y)}{y^{\alpha'}} \leq \frac{n_\alpha(y)}{y^\alpha} \ (y > 0) \ \ (\alpha' > \alpha)$$

as well as the α-convexity while, obviously, $\frac{n_1(y)}{y} = 1$.

It follows that if $a(\cdot)$ is unbounded at infinity then (19) is *never* satisfied for $\alpha \leq 1$. Furthermore, if

$$\lim \sup_{y\to+\infty} \frac{n_\alpha(y)}{y^\alpha} < 1 \tag{20}$$

for some $\alpha > 1$, then it is satisfied for all $\alpha > 1$ so that if (19) is satisfied in some X_α, it is satisfied in any $X_\alpha, \alpha > 1$. Of course, our construction is meaningful only if (20) holds. This can be directly checked for instance in the case of *homogeneous* fragmentation kernels

$$b(x,y) = \frac{1}{y} h\left(\frac{x}{y}\right) \text{ with } \int_0^1 z h(z) dz = 1 \tag{21}$$

for some

$$h \in L^1_+\left((0,1)\,; xdx\right).$$

Indeed, the local conservativeness property

$$\int_0^y x b(x,y) dx = \int_0^y \frac{x}{y} h\left(\frac{x}{y}\right) dx = y \int_0^1 z h(z) dz = y$$

is satisfied and, for all $\alpha > 1$,

$$\frac{\int_0^y x^\alpha b(x,y)dx}{y^\alpha} = y^{-1} \int_0^y \left(\frac{x}{y}\right)^\alpha h\left(\frac{x}{y}\right) dx = \int_0^1 z^\alpha h(z) dz < \int_0^1 z h(z) dz = 1$$

so

$$\lim \sup_{a(y)\to+\infty} \frac{n_\alpha(y)}{y^\alpha} = \int_0^1 z^\alpha h(z) dz < 1 \ (\alpha > 1).$$

We can also check (20) for *separable* (conservative) fragmentation kernels

$$b(x,y) = \beta(x) y \left(\int_0^y s\beta(s) ds\right)^{-1}, \tag{22}$$

introduced in [2], or even by any convex combination of such kernels, see Section A.

Analysis in $X_{0,\alpha}$ — open problems.

The analysis in $X_{0,\alpha}$ follows the same strategy but the construction *fails* under Assumption (8). Indeed, we introduce

$$n_{1,\alpha}(y) := \int_0^y (1+x)^\alpha \, b(x,y) dx \tag{23}$$

and show that if

$$\sup_{y>0} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} < +\infty, \tag{24}$$

then the fragmentation operator (10) is T-bounded in $X_{0,\alpha}$ and

$$\lim_{\lambda\to+\infty} \left\| B(\lambda - T)^{-1} \right\|_{\mathcal{L}(X_{0,\alpha})} \le \lim \sup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} \tag{25}$$

so W. Desch's perturbation theorem shows that if

$$\lim \sup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} < 1, \tag{26}$$

then

$$A := T + B : D(T) \subset X_{0,\alpha} \to X_{0,\alpha} \tag{27}$$

generates a positive C_0-semigroup $(V(t))_{t\ge0}$ on $X_{0,\alpha}$. Unfortunately, if $a(\cdot)$ is *unbounded near zero*,

$$\lim \sup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} \ge \lim \sup_{y\to0} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} = \lim \sup_{y\to0} n_{1,\alpha}(y)$$

and (26) is *not* satisfied since

$$n_{1,\alpha}(y) \ge \int_0^y b(x,y) dx = \frac{1}{y} \int_0^y y b(x,y) dx \ge \frac{1}{y} \int_0^y x b(x,y) dx = 1.$$

On the other hand, under (8), the compactness of the resolvent of T in $X_{0,\alpha}$ (which plays a key role in our construction) depends on the unboundedness of $a(\cdot)$ near zero. *This is why, under Assumption (8), the existence of a spectral gap in $X_{0,\alpha}$ is an open problem and our construction is restricted to the space X_α.*

1.2.2. *Partly singular growth rates* (7).

Let us describe now our main results under Assumption (7).

Negative results in X_α.

We show first, at least for *bounded* total fragmentation kernels, that (9) is not well-posed in X_α in the sense of C_0-semigroups and consequently, we cannot expect a generation theory in X_α for the full problem (1) and therefore we restrict ourselves to the space $X_{0,\alpha}$. The growth C_0-semigroup $(U(t))_{t\geq 0}$ governing (9) with boundary condition

$$\lim_{x\to 0} r(x)u(x,t) = 0 \tag{28}$$

exists in the space $X_{0,\alpha}$ and is given by

$$U(t)f = \chi_{\left\{\int_0^y \frac{1}{r(\tau)}d\tau > t\right\}} e^{-\int_{X(y,t)}^y \frac{a(p)}{r(p)}dp} f(X(y,t))\frac{\partial X(y,t)}{\partial y} \tag{29}$$

($X(y,t)$ is defined by $\int_{X(y,t)}^y \frac{1}{r(\tau)}d\tau = t$ for $\int_0^y \frac{1}{r(\tau)}d\tau > t$) provided that (14) is satisfied. This sufficient condition is also "partly necessary".

Properties of the growth–absorbtion semigroup in $X_{0,\alpha}$.

We show the estimates (15)(16) in $X_{0,\alpha}$. These estimates, combined with the general theory [26] on compactness properties in L^1 spaces induced by the confining effect of singular absorptions, show that if the sublevel sets of the total fragmentation rate

$$\Omega_c = \{x > 0; a(x) < c\} \quad (c > 0)$$

are "thin near *infinity* relatively to r" in the sense

$$\int_1^{+\infty} \frac{1_{\Omega_c}(\tau)}{r(\tau)}d\tau < +\infty \;\; (c > 0),$$

where 1_{Ω_c} is the indicator function of Ω_c (note that $\frac{1}{r(\cdot)} \notin L^1(1,+\infty)$), then T has a compact resolvent in $X_{0,\alpha}$. This occurs for instance if

$$\lim_{y\to+\infty} a(y) = +\infty.$$

We introduce (23) and show that under (24) the fragmentation operator is T-bounded in $X_{0,\alpha}$ and (27) generates a positive C_0-semigroup $(V(t))_{t\geq 0}$ on $X_{0,\alpha}$ provided that (26) is satisfied.

Properties of the full growth–fragmentation semigroup in $X_{0,\alpha}$.

By restricting ourselves to the case where $a(\cdot)$ is unbounded *at infinity only*, (26) amounts to

$$\lim \sup_{y\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} < 1. \tag{30}$$

Under (30), the generator (27) is resolvent compact in $X_{0,\alpha}$ if T is. By arguing as previously, we show that $(V(t))_{t\geq 0}$ has a spectral gap (11) and exhibits the asynchronous exponential growth (12) in $X_{0,\alpha}$ provided (4) is satisfied (see Theorem 4.9).

We show the α-monotony

$$\frac{n_{1,\widehat{\alpha}}(y)}{(1+y)^{\widehat{\alpha}}} \leq \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} \;\; (y > 0) \;\; (0 < \alpha < \widehat{\alpha})$$

as well as the α-convexity, and

$$\lim \sup_{y\to+\infty} \frac{n_{1,1}(y)}{1+y} \geq 1.$$

This implies that (30) is *never* satisfied if $\alpha \leq 1$. It follows that if

$$\lim_{\alpha\to+\infty} \lim \sup_{y\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} < 1, \tag{31}$$

then there exists a unique threshold

$$\widetilde{\alpha} := \inf\left\{\alpha > 1; \lim \sup_{y\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} < 1\right\} \geq 1$$

such that (30) is satisfied if and only if $\alpha > \widetilde{\alpha}$. Similarly, our construction is meaningfull only if (31) holds. To this end, we show that if the growth of (18) at infinity is at most polynomial, i.e., if

$$\eta := \inf\left\{\alpha > 1; \exists c_\alpha > 0, \; \int_0^y b(x,y)dx \leq c_\alpha(1+y)^\alpha\right\} < +\infty, \tag{32}$$

then

$$\lim\sup_{y\to+\infty}\frac{\int_0^y (1+x)^\alpha\, b(x,y)dx}{(1+y)^\alpha}\leq \lim\sup_{y\to+\infty}\frac{\int_0^y x^\alpha b(x,y)dx}{y^\alpha}\quad (\forall\alpha>\eta)$$

and, using α-convexity, we show that in this case $1\leq\widetilde{\alpha}\leq\eta$. Again, for a particular example of homogeneous fragmentation kernels (21)

$$n(y)=\int_0^y \frac{1}{y}h\left(\frac{x}{y}\right)dx=\int_0^1 h(z)dz$$

and consequently the threshold is exactly one

$$\widetilde{\alpha}=\eta=1,$$

provided $\int_0^1 h(z)dz<+\infty$ (note that here $\eta=1$ by definition, as we only consider exponents bigger than 1). We can also check (31) for separable (conservative) fragmentation kernels (22) or by any convex combination of such kernels, see Section A. We can summarize the results in the following table.

Space\ growth	Assumption (7)	Assumption (8)
X_α	No generation in general Under additional assumptions, see [28]	Generation AEG
$X_{0,\alpha}$	Generation AEG	Generation AEG – open problem

2. **The method of characteristics.**

The explicit formulae for solutions to transport equations (9) can be obtained by the method of characteristics and belong to the mathematical folklore, see e.g., [6, 10]; a systematic treatment of them can be found [27]. We recall them for the reader's convenience.

Proposition 1. *([27, Proposition 44]) Let (6) and (7) be satisfied. Then the partial differential equation*

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}\left[r(x)u(x,t)\right]=0,\ \ (x>0,t>0)$$

with initial and boundary conditions

$$u(x,0)=f(x),\quad \lim_{y\to 0}r(y)u(y,t)=0\quad (t>0)$$

has a unique solution given by

$$u(y,t)=\begin{cases}\frac{r(X(y,t))f(X(y,t))}{r(y)}=f(X(y,t))\frac{\partial X(y,t)}{\partial y} & \text{if } \int_0^y\frac{1}{r(\tau)}d\tau>t\\ 0 \text{ if } \int_0^y\frac{1}{r(\tau)}d\tau<t\end{cases}$$

where $X(y,t)$ is defined, for $\int_0^y\frac{1}{r(\tau)}d\tau>t$, by

$$\int_{X(y,t)}^y\frac{1}{r(\tau)}d\tau=t,\quad X(y,t)\in(0,y)\,.$$

Proposition 2. *([27, Proposition 2]) Let (6) and (8) be satisfied. Then the partial differential equation*

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}\left[r(x)u(x,t)\right]=0,\ \ (x>0,t>0)$$

with initial condition $u(x,0)=f(x)$ has a unique solution given by

$$u(y,t)=\frac{r(X(y,t))f(X(y,t))}{r(y)},$$

where $X(y,t)$ $(t>0)$ is defined by

$$\int_{X(y,t)}^y\frac{1}{r(\tau)}d\tau=t,\quad X(y,t)\in(0,y)\,.$$

3. **First construction.**

The first construction is based on Assumption (7). It is devoted to asynchronous exponential growth in the space $X_{0,\alpha}$ only since we cannot expect in general a generation theory in X_α under (7), see Remark 1 below.

3.1. Generation theory.

Our first result on the generation of transport semigroups in $X_{0,\alpha}$ is:

Theorem 3.1. *Let $\alpha > 0$ and let (6) and (7) be satisfied. Let $y(x,t)$ be defined by*

$$\int_x^{y(x,t)} \frac{1}{r(\tau)} d\tau = t. \tag{33}$$

Then $(U_0(t))_{t\geq 0}$ with

$$(U_0(t)f)(y) = \chi_{\left\{\int_0^y \frac{1}{r(\tau)} d\tau > t\right\}} \frac{r(X(y,t)) f(X(y,t))}{r(y)}$$

is a C_0-semigroup on $X_{0,\alpha}$ if and only if

$$\sup_{x>0} \frac{1+y(x,t)}{1+x} < +\infty \quad (t \geq 0)$$

and

$$[0,+\infty) \ni t \mapsto \sup_{x>0} \frac{1+y(x,t)}{1+x}$$

is locally bounded. In this case

$$\|U_0(t)\|_{\mathcal{L}(X_{0,\alpha})} = \sup_{x>0} \frac{(1+y(x,t))^\alpha}{(1+x)^\alpha}.$$

This occurs if there exists $C > 0$ such that

$$r(z) \leq C(z+1) \quad (\forall z > 0). \tag{34}$$

In this case, $\|U_0(t)\|_{\mathcal{L}(X_{0,\alpha})} \leq e^{\alpha C t}$.

Proof. Let us check that $U_0(t)$ is a bounded operator on $X_{0,\alpha}$. Let $y_0(t) > 0$ be defined by

$$\int_0^{y_0(t)} \frac{1}{r(\tau)} d\tau = t. \tag{35}$$

Note that for $\int_0^y \frac{1}{r(\tau)} d\tau > t$ we have

$$\int_{X(y,t)}^y \frac{1}{r(\tau)} d\tau = t, \tag{36}$$

which shows that (for $t > 0$ fixed) $X(y,t)$ is strictly increasing in y and tends to 0 as $y \to y_0(t)$. Note that

$$(y_0(t), +\infty) \ni y \mapsto X(y,t) \in (0,+\infty)$$

is continuous. Note also that (for $t > 0$ fixed)

$$U(y,z) := \int_z^y \frac{1}{r(\tau)} d\tau - t$$

is of class C^1 in (y,z) with

$$\frac{\partial U(y,z)}{\partial z} = -\frac{1}{r(z)} \neq 0,$$

so that the implicit function theorem shows that $X(y,t)$ is a C^1 function in $y \in (y_0(t), +\infty)$. Thus, differentiating (36) with respect to y we obtain

$$\frac{1}{r(y)} - \frac{1}{r(X(y,t))} \frac{\partial X(y,t)}{\partial y} = 0$$

so

$$\frac{1}{r(y)} = \frac{1}{r(X(y,t))} \frac{\partial X(y,t)}{\partial y}$$

and

$$(U_0(t)f)(y) = f(X(y,t)) \frac{\partial X(y,t)}{\partial y}; \quad y \in (y_0(t), +\infty).$$

We have

$$\begin{aligned} \|U_0(t)f\|_{X_{0,\alpha}} &= \int_0^{+\infty} |(U_0(t)f)(y)| (1+y)^\alpha \, dy \\ &= \int_{y_0(t)}^{+\infty} |f(X(y,t))| \frac{\partial X(y,t)}{\partial y} (1+y)^\alpha \, dy. \end{aligned}$$

The change of variable $x = X(y,t)$ gives

$$\|U_0(t)f\|_{X_{0,\alpha}} = \int_0^{+\infty} |f(x)| \, (1 + y(x,t))^\alpha \, dx,$$

where $y(x,t)$ is the unique $y > x$ such that $x = X(y,t)$ i.e. (33). Hence

$$\|U_0(t)f\|_{X_{0,\alpha}} = \int_0^{+\infty} \frac{(1 + y(x,t))^\alpha}{(1+x)^\alpha} |f(x)| \, (1+x)^\alpha \, dx$$

and $U_0(t)$ is a bounded linear operator in $X_{0,\alpha}$ if and only if

$$\sup_{x>0} \frac{(1 + y(x,t))^\alpha}{(1+x)^\alpha} < +\infty.$$

In such a case,

$$\|U_0(t)\|_{\mathcal{L}(X_{0,\alpha})} = \sup_{x>0} \frac{(1 + y(x,t))^\alpha}{(1+x)^\alpha}.$$

Moreover,

$$[0,+\infty) \ni t \mapsto U_0(t) \in \mathcal{L}(X_{0,\alpha})$$

is locally bounded if and only if

$$[0,+\infty) \ni t \mapsto \sup_{x>0} \frac{(1 + y(x,t))^\alpha}{(1+x)^\alpha}$$

is. Using the flow property of $(y,t) \mapsto X(y,t)$, which follows since it is the solution of an autonomous differential equation, we can prove that the semigroup property $U_0(s)U_0(t) = U_0(t+s), t, s \geq 0$, is satisfied. It is also easy to see that $[0,+\infty) \ni t \mapsto U_0(t) \in \mathcal{L}(X_{0,\alpha})$ is locally bounded. Then, by [17, Proposition I.1.3], to prove that it is a strongly continuous semigroup on $X_{0,\alpha}$, it suffices to check that

$$U_0(t)f \to f \ \text{ in } X_{0,\alpha} \text{ as } t \to 0$$

on a dense subspace of $L^1(\mathbb{R}_+;\ (1+x)^\alpha \, dx)$, e.g. for f continuous with compact support in $(0,+\infty)$. Note that for any compact set $\left[c, c^{-1}\right]$

$$\int_0^y \frac{1}{r(\tau)} d\tau > t$$

for t small enough uniformly in $y \in \left[c, c^{-1}\right]$ so

$$\left(U_0(t)f\right)(y) = f(X(y,t)) \frac{\partial X(y,t)}{\partial y} \quad \forall y \in \left[c, c^{-1}\right]$$

for t small enough. In particular

$$\int_{X(y,t)}^y \frac{1}{r(\tau)} d\tau = t \quad \forall y \in \left[c, c^{-1}\right]$$

and

$$\left(U_0(t)f\right)(y) = \frac{r(X(y,t)) f(X(y,t))}{r(y)} \quad \forall y \in \left[c, c^{-1}\right],$$

for t small enough. We note that $X(y,t) \to y$ as $t \to 0$ for any $y > 0$ and uniformly in $y \in \left[\frac{c}{2}, 2c^{-1}\right]$. Hence

$$\left(U_0(t)f\right)(y) = \frac{r(X(y,t)) f(X(y,t))}{r(y)} \to f(y) \quad (t \to 0)$$

and, by the dominated convergence theorem, $U_0(t)f \to f$ in $X_{0,\alpha}$ as $t \to 0$.

Let us continue with the prove of the last statement of the theorem. For fixed $x > 0$, the differentiation in t of $\int_x^{y(x,t)} \frac{1}{r(\tau)} d\tau = t$ gives

$$\frac{1}{r(y(x,t))} \frac{\partial y(x,t)}{\partial t} = 1$$

i.e.

$$\frac{\partial y(x,t)}{\partial t} = r(y(x,t)) \quad \forall t > 0, \ \text{with } y(x,0) = x. \tag{37}$$

Hence,

$$y(x,t) = x + \int_0^t r(y(x,s)) ds \leq x + \int_0^t C\left(y(x,s) + 1\right) ds,$$

where (34) is used in the last step, so

$$y(x,t)+1 \le x+1+\int_0^t C\,(y(x,s)+1)\,ds$$

and Gronwall's lemma gives

$$y(x,t)+1 \le (x+1)\,e^{Ct}.$$

Finally

$$\sup_{x>0}\frac{(1+y(x,t))^\alpha}{(1+x)^\alpha} \le e^{\alpha Ct}$$

and $\|U_0(t)\|_{\mathcal{L}(X_{0,\alpha})} \le e^{\alpha Ct}$. □

Remark 1. Note that we cannot expect a generation theory in X_α. Indeed, $U_0(t)$ is bounded in X_α if and only only if

$$\sup_{x>0}\frac{y(x,t)}{x} < +\infty,$$

while (33) and (35) show that $\lim_{x\to 0} y(x,t) = y_0(t) > 0$.

Remark 2. Assumption (34) is partly necessary for the generation theory in $X_{0,\alpha}$, see Remark 10 below.

To find the formula of the resolvent of the generator, we take the Laplace transform of $(U_0(t))_{t\geqslant 0}$. We point out that the Laplace integral with respect to t of a continuous $L_1(\mathbb{R}_+;(1+x)^\alpha dx)$-valued function $t\mapsto f(\cdot,t)$ in the Bochner sense is a.e. in x equal to the Lebesgue integral with respect to t of f treated as a function of two variables $(x,t)\to f(x,t)$, see [3, Example 2.23]. Thus, with some change of variables, we have

Theorem 3.2. *Let $\alpha>0$, (6), (7) and (34) be satisfied. Let T_0 be the generator of $(U_0(t))_{t\geqslant 0}$. Then*

$$\left((\lambda-T_0)^{-1}f\right)(y) = \frac{1}{r(y)}\int_0^y e^{-\int_x^y \frac{\lambda}{r(s)}ds} f(x)dx, \quad (f\in X_{0,\alpha})\ (\Re\lambda > s(T_0))$$

where $s(T_0)$ is the spectral bound of T_0.

We note that the last statement follows due to the positivity of $(U_0(t))_{t\geqslant 0}$, see [30, Theorem 1.4.1].

3.2. **A pointwise estimate.**

Hereafter we assume that (34) is satisfied.

We give the first key a priori estimate.

Lemma 3.3. *Let $\alpha>0$, (6), (7) and (34) be satisfied and $\lambda \ge \alpha C$. Then*

$$\left|(\lambda-T_0)^{-1}f\right|(y) \le \frac{1}{(1+y)^\alpha\, r(y)}\,\|f\|_{X_{0,\alpha}} \quad (f\in X_{0,\alpha}).$$

Proof. Note that (34), i.e.

$$\frac{1}{r(\tau)} \ge \frac{C^{-1}}{\tau+1},$$

implies

$$e^{-\lambda\int_x^y \frac{1}{r(\tau)}d\tau} \le e^{-\frac{\lambda}{C}\int_x^y \frac{1}{\tau+1}d\tau} = e^{-\frac{\lambda}{C}\ln\left(\frac{y+1}{x+1}\right)} = \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}}, \tag{38}$$

so

$$\begin{aligned}
\left|(\lambda-T_0)^{-1}f(y)\right| &\le \frac{1}{r(y)}\int_0^y e^{-\lambda\int_x^y \frac{1}{r(\tau)}d\tau}\,|f(x)|\,dx\\
&\le \frac{1}{r(y)}\int_0^y \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}} |f(x)|\,dx\\
&= \frac{1}{r(y)}\int_0^y \frac{1}{(1+x)^\alpha}\left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}} |f(x)|\,(1+x)^\alpha\,dx\\
&= \frac{1}{(1+y)^\alpha\, r(y)}\int_0^y \frac{(1+y)^\alpha}{(1+x)^\alpha}\left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}} |f(x)|\,(1+x)^\alpha\,dx\\
&= \frac{1}{(1+y)^\alpha\, r(y)}\int_0^y \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}-\alpha} |f(x)|\,(1+x)^\alpha\,dx.
\end{aligned}$$

Finally

$$\left|(\lambda - T_0)^{-1} f(y)\right| \leq \frac{1}{(1+y)^{\alpha}\, r(y)} \int_0^y |f(x)|\,(1+x)^{\alpha}\, dx \leq \frac{1}{(1+y)^{\alpha}\, r(y)} \|f\|_{X_{0,\alpha}}$$

because $\frac{\lambda}{C} - \alpha \geq 0$ and $\frac{x+1}{y+1} \leq 1$ for $0 \leq x \leq y$. □

3.3. **The first perturbed semigroup.**

We build now a second explicit perturbed C_0-semigroup by solving, using the method of characteristics,

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x}\left[r(x) u(x,t)\right] + a(x) u(x,t) = 0$$

with initial and boundary conditions

$$u(x,0) = f(x), \;\; \lim_{x \to 0} r(x) u(x,t) = 0.$$

The solution is given by

$$\chi_{\left\{\int_0^y \frac{1}{r(\tau)} d\tau > t\right\}} e^{-\int_{X(y,t)}^y \frac{a(p)}{r(p)} dp} \frac{r(X(y,t)) f(X(y,t))}{r(y)}.$$

This defines a perturbed C_0-semigroup $(U(t))_{t \geq 0}$ on $X_{0,\alpha}$, dominated by $(U_0(t))_{t \geq 0}$,

$$(U(t) f)(y) = \chi_{\left\{\int_0^y \frac{1}{r(\tau)} d\tau > t\right\}} e^{-\int_{X(y,t)}^y \frac{a(p)}{r(p)} dp} \frac{r(X(y,t)) f(X(y,t))}{r(y)}. \tag{39}$$

Remark 3. We have seen in Remark 1 that for $a(\cdot) = 0$ we cannot expect a generation theory in X_α. Hence, by the bounded perturbation theory, we cannot expect a generation theory in X_α for bounded $a(\cdot)$. But this does not prevent (39) from defining a C_0-semigroup in X_α for a suitably singular $a(\cdot)$. Actually, this is the case if $a(\cdot)$ is sufficiently singular at zero but then the whole construction of the paper needs additional tehnicalities. For the sake of clarity, this special case is treated separately [28].

As previously, the Laplace transform of $(U(t))_{t \geqslant 0}$ and some change of variables give:

Proposition 3. *Let $\alpha > 0$ and let (6), (7) and (34) be satisfied. Then, the resolvent of its generator T is given by*

$$\left((\lambda - T)^{-1} f\right)(y) = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda + a(\tau)}{r(\tau)} d\tau} f(x) dx, \quad (f \in X_{0,\alpha})\; (\Re\lambda > s(T)).$$

As in Theorem 3.2, the estimate of the abscissa of convergence of the Laplace integral follows from the positivity of $(U(t))_{t \geqslant 0}$.

3.4. **A smoothing effect of the perturbed resolvent.**

The second key a priori estimate is given by:

Lemma 3.4. *Let $\alpha > 0$, (6), (7) and (34) be satisfied and $\lambda \geq \alpha C$. Then*

$$\int_0^{+\infty} \left|\left((\lambda - T)^{-1} f\right)(y)\right| a(y)\,(1+y)^{\alpha}\, dy \leq \int_0^{+\infty} |(f(y)|\,(1+y)^{\alpha}\, dy, \;\; \forall f \in X_{0,\alpha}.$$

Proof. It suffices to consider nonnegative f. Using (38), we have

$$\begin{aligned}
&\int_0^{+\infty} \left((\lambda - T)^{-1} f\right)(y) a(y)\,(1+y)^{\alpha}\, dy \\
= &\int_0^{+\infty} \frac{a(y)\,(1+y)^{\alpha}}{r(y)} \left(\int_0^y e^{-\lambda \int_x^y \frac{1}{r(p)} dp} e^{-\int_x^y \frac{a(p)}{r(p)} dp} f(x) dx\right) dy \\
\leq &\int_0^{+\infty} \frac{a(y)\,(1+y)^{\alpha}}{r(y)} \left(\int_0^y \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}} e^{-\int_x^y \frac{a(p)}{r(p)} dp} f(x) dx\right) dy \\
= &\int_0^{+\infty} \left[\int_x^{+\infty} \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}} \frac{a(y)\,(1+y)^{\alpha}}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)} dp} dy\right] f(x) dx \\
= &\int_0^{+\infty} \left[\int_x^{+\infty} \frac{1}{(1+x)^{\alpha}} \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}} \frac{a(y)\,(1+y)^{\alpha}}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)} dp} dy\right] f(x)\,(1+x)^{\alpha} dx \\
= &\int_0^{+\infty} \left[\int_x^{+\infty} \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C} - \alpha} \frac{a(y)}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)} dp} dy\right] f(x)\,(1+x)^{\alpha}\, dx
\end{aligned}$$

$$\leq \int_0^{+\infty} \left[\int_x^{+\infty} \frac{a(y)}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)}dp} dy\right] f(x)\,(1+x)^\alpha\,dx$$

where $\frac{x+1}{y+1} \leq 1$ and $\frac{\lambda}{C} - \alpha \geq 0$ are used in the last step. Thus

$$\int_0^{+\infty} \left((\lambda - T)^{-1} f\right)(y) a(y)\,(1+y)^\alpha\,dy$$
$$\leq \sup_{x>0} \int_x^{+\infty} \frac{a(y)}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)}dp} dy \left(\int_0^{+\infty} f(x)\,(1+x)^\alpha\,dx\right).$$

Finally, the estimate

$$\int_x^{+\infty} e^{-\int_x^y \frac{a(p)}{r(p)}dp} \frac{a(y)}{r(y)} dy = -\int_x^{+\infty} \frac{d}{dy}\left(e^{-\int_x^y \frac{a(p)}{r(p)}dp}\right) dy$$
$$= -\left[e^{-\int_x^y \frac{a(p)}{r(p)}dp}\right]_{y=x}^{y=+\infty} \leq 1$$

ends the proof. □

3.5. **On the full semigroup.**

We give now the second perturbed semigroup.

Theorem 3.5. *Let $\alpha > 0$ and let (6), (7) and (34) be satisfied. Define*

$$n_{1,\alpha}(y) := \int_0^y (1+x)^\alpha\, b(x,y)dx.$$

If

$$\sup_{y>0} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} < +\infty,$$

then the fragmentation operator B is T-bounded in $X_{0,\alpha}$ and

$$\lim_{\lambda\to+\infty} \left\|B(\lambda - T)^{-1}\right\|_{\mathcal{L}(X_{0,\alpha})} \leq \lim \sup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha}.$$

In particular, if

$$\lim \sup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} < 1, \tag{40}$$

then

$$A := T + B : X_{0,\alpha} \supset D(T) \to X_{0,\alpha}$$

generates a positive C_0-semigroup $(V(t))_{t\geq 0}$ on $X_{0,\alpha}$.

Proof. We note that for nonnegative φ

$$\begin{aligned}
\|B\varphi\|_{X_{0,\alpha}} &= \int_0^{+\infty} \left(\int_x^{+\infty} a(y)b(x,y)\varphi(y)dy\right)(1+x)^\alpha\,dx \\
&= \int_0^{+\infty} a(y)\left(\int_0^y (1+x)^\alpha\, b(x,y)dx\right)\varphi(y)dy \\
&= \int_0^{+\infty} a(y)n_{1,\alpha}(y)\varphi(y)dy.
\end{aligned}$$

Thus, for nonnegative f,

$$\begin{aligned}
&\left\|B(\lambda - T)^{-1} f\right\|_{X_{0,\alpha}} \\
&= \int_0^{+\infty} a(y)n_{1,\alpha}(y)\left((\lambda - T)^{-1} f\right)(y)dy \\
&= \int_0^{+\infty} a(y)\frac{n_{1,\alpha}(y)}{(1+y)^\alpha}\left((\lambda - T)^{-1} f\right)(y)\,(1+y)^\alpha\,dy.
\end{aligned} \tag{41}$$

Let

$$L := \lim \sup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha},$$

that is, for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$a(y) \geq c_\varepsilon \Longrightarrow \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} \leq L + \varepsilon.$$

We decompose (41) into two integrals

$$\begin{aligned}
&\int_0^{+\infty} a(y)\frac{n_{1,\alpha}(y)}{(1+y)^\alpha}\left((\lambda-T)^{-1}f\right)(y)\,(1+y)^\alpha\,dy\\
=\;&\int_{\{a(y)\le c_\varepsilon\}} a(y)\frac{n_{1,\alpha}(y)}{(1+y)^\alpha}\left((\lambda-T)^{-1}f\right)(y)\,(1+y)^\alpha\,dy\\
&+\int_{\{a(y)>c_\varepsilon\}} a(y)\frac{n_{1,\alpha}(y)}{(1+y)^\alpha}\left((\lambda-T)^{-1}f\right)(y)\,(1+y)^\alpha\,dy\\
=\;& I_1+I_2.
\end{aligned}$$

We note that

$$I_1\le c_\varepsilon\left\|\frac{n_{1,\alpha}(\cdot)}{(1+y)^\alpha}\right\|_{L^\infty}\left\|\left((\lambda-T)^{-1}f\right)\right\|_{X_{0,\alpha}},$$

while, using Lemma 3.4,

$$\begin{aligned}
I_2 &\le (L+\varepsilon)\int_0^{+\infty} a(y)\left((\lambda-T)^{-1}f\right)(y)\,(1+y)^\alpha\,dy\\
&\le (L+\varepsilon)\left\|f\right\|_{X_{0,\alpha}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|B(\lambda-T)^{-1}f\right\|_{X_{0,\alpha}} \le\;& c_\varepsilon\left\|\frac{n_{1,\alpha}(\cdot)}{(1+y)^\alpha}\right\|_{L^\infty}\left\|(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_{0,\alpha})}\left\|f\right\|_{X_{0,\alpha}}\\
&+(L+\varepsilon)\left\|f\right\|_{X_{0,\alpha}}
\end{aligned}$$

and

$$\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_{0,\alpha})}\le c_\varepsilon\left\|\frac{n_{1,\alpha}(\cdot)}{(1+y)^\alpha}\right\|_{L^\infty}\left\|(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_{0,\alpha})}+(L+\varepsilon)\quad(\forall\varepsilon>0).$$

Since T is a generator of a semigroup, $\left\|(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_{0,\alpha})}\to 0$ as $\lambda\to+\infty$ and hence

$$\lim_{\lambda\to+\infty}\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_{0,\alpha})}\le L+\varepsilon\quad(\forall\varepsilon>0).$$

Consequently,

$$\lim_{\lambda\to+\infty}\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_{0,\alpha})}\le L.$$

Finally, if (40) is satisfied, $L<1$ and then the generation follows from the Desch theorem, Theorem 1.1. ☐

Remark 4. If $a(\cdot)$ is unbounded near zero, then

$$\lim\sup_{a(y)\to+\infty}\frac{n_{1,\alpha}(y)}{(1+y)^\alpha}\ge\lim\sup_{y\to0}n_{1,\alpha}(y)\ge\lim\sup_{y\to0}\int_0^y b(x,y)dx\ge1$$

because

$$\int_0^y b(x,y)dx=\frac{1}{y}\int_0^y yb(x,y)dx\ge\frac{1}{y}\int_0^y xb(x,y)dx=1,$$

so (40) is not satisfied. Hence Theorem 3.5 is meaningful if $a(\cdot)$ is unbounded only at infinity; see below.

Note the useful observation:

Proposition 4. *Let $(0,\infty)\ni x\to f(x)\in(0,\infty)$ be a non decreasing function and for some there are $y>0,\alpha_0>0$ such that*

$$\int_0^y f^\alpha(x)b(x,y)dx<+\infty$$

for any $\alpha>\alpha_0$. Then

$$(\alpha_0,+\infty)\ni\alpha\to\frac{\int_0^y f^\alpha(x)b(x,y)dx}{f^\alpha(y)}$$

is a non increasing and convex function.

Proof. Since $0 \leq \frac{f(x)}{f(y)} \leq 1$ for $x \in (0, y]$, $(0, \alpha) \ni \alpha \to \left(\frac{f(x)}{f(y)}\right)^{\alpha}$ is non increasing and convex, that is, for $0 < \alpha_1 \leq \alpha \leq \alpha_2$,

$$\left(\frac{f(x)}{f(y)}\right)^{\alpha_1} \geq \left(\frac{f(x)}{f(y)}\right)^{\alpha_2},$$

$$\left(\frac{f(x)}{f(y)}\right)^{\alpha} \leq \left(\frac{f(x)}{f(y)}\right)^{\alpha_1} + \frac{\left(\frac{f(x)}{f(y)}\right)^{\alpha_2} - \left(\frac{f(x)}{f(y)}\right)^{\alpha_1}}{\alpha_2 - \alpha_1}(\alpha - \alpha_1)$$

and the statement follows by multiplying both sides with $b(x, y) \geq 0$ and integrating over $(0, y)$ with respect to x. □

Remark 5. Applying Proposition 4 to $f(x) = (1 + y)^{\alpha}$ we see that if (40) is satisfied for some $\alpha > 0$, then it is satisfied for all $\widehat{\alpha} > \alpha$.

Remark 6. Note first that our assumption (40) precludes the case $\alpha = 1$ if $a(\cdot)$ is unbounded at infinity. Indeed

$$\begin{aligned} n_{1,1}(y) &:= \int_0^y (1 + x)\, b(x, y)dx = \int_0^y b(x, y)dx + \int_0^y x b(x, y)dx \\ &= n_0(y) + y \end{aligned}$$

and then $\limsup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)} \geq 1$. It follows from Proposition 4 that

$$\lim \sup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1 + y)} \geq 1 \quad (0 < \alpha \leq 1). \tag{42}$$

Hence the necessity of higher moments, i.e., $\alpha > 1$. More precisely, if

$$\lim_{\alpha\to+\infty} \lim \sup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1 + y)^{\alpha}} < 1,$$

then the threshold

$$\widetilde{\alpha} := \inf \left\{ \alpha > 1;\ \lim \sup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1 + y)^{\alpha}} < 1 \right\} \tag{43}$$

is such that (40) holds if and only if $\alpha > \widetilde{\alpha}$. See Proposition 5 below for more information about this threshold.

Remark 7. If $a(\cdot)$ is only unbounded at infinity, then (40) amounts to

$$\lim \sup_{y\to+\infty} \frac{n_{1,\alpha}(y)}{(1 + y)^{\alpha}} < 1 \tag{44}$$

and (43) is given by

$$\widetilde{\alpha} := \inf \left\{ \alpha > 1;\ \lim \sup_{y\to+\infty} \frac{n_{1,\alpha}(y)}{(1 + y)^{\alpha}} < 1 \right\}.$$

We end this subsection by an upper estimate of the threshold (43).

Proposition 5. *We assume that*

$$\eta := \inf \left\{ \alpha > 1; \exists c_{\alpha} > 0,\ \int_0^y b(x, y)dx \leq c_{\alpha}(1 + y)^{\alpha}\ \forall y > 0 \right\} < +\infty. \tag{45}$$

Then

$$\lim \sup_{y\to+\infty} \frac{\int_0^y (1 + x)^{\alpha}\, b(x, y)dx}{(1 + y)^{\alpha}} \leq \lim \sup_{y\to+\infty} \frac{\int_0^y x^{\alpha} b(x, y)dx}{y^{\alpha}} \quad (\forall \alpha > \eta).$$

In particular, if (45) is satisfied and $a(\cdot)$ is unbounded only at infinity, then $\widetilde{\alpha} \leq \eta$, provided $\widetilde{\alpha}$ is finite.

Proof. Note first that

$$\lim \sup_{y\to+\infty} \frac{\int_0^y b(x, y)dx}{(1 + y)^{\alpha}} = 0 \quad (\forall \alpha > \eta).$$

Since

$$\begin{aligned} \frac{n_{1,\alpha}(y)}{(1 + y)^{\alpha}} &= \frac{\int_0^y (1 + x)^{\alpha}\, b(x, y)dx}{(1 + y)^{\alpha}} \\ &= \frac{\int_0^y \zeta(x) b(x, y)dx}{(1 + y)^{\alpha}} + \frac{\int_0^y b(x, y)dx}{(1 + y)^{\alpha}} \end{aligned}$$

with $\zeta(x) = (1+x)^\alpha - 1$,

$$\lim\sup_{y\to+\infty} \frac{\int_0^y (1+x)^\alpha\, b(x,y)dx}{(1+y)^\alpha} \le \lim\sup_{y\to+\infty} \frac{\int_0^y \zeta(x) b(x,y)dx}{(1+y)^\alpha}.$$

Let $\varepsilon > 0$ be arbitrary and M_ε be large enough so that

$$\frac{\zeta(x)}{x^\alpha} \le 1+\varepsilon \quad (x \ge M_\varepsilon).$$

Then for $y > M_\varepsilon$

$$\begin{aligned}
\frac{\int_0^y \zeta(x)b(x,y)dx}{(1+y)^\alpha} &= \frac{\int_0^{M_\varepsilon} \zeta(x)b(x,y)dx}{(1+y)^\alpha} + \frac{\int_{M_\varepsilon}^y \zeta(x)b(x,y)dx}{(1+y)^\alpha} \\
&\le \zeta(M_\varepsilon)\frac{\int_0^{M_\varepsilon} b(x,y)dx}{(1+y)^\alpha} + \frac{\int_{M_\varepsilon}^y \zeta(x)b(x,y)dx}{(1+y)^\alpha} \\
&\le \zeta(M_\varepsilon)\frac{\int_0^{y} b(x,y)dx}{(1+y)^\alpha} + (1+\varepsilon)\,\frac{\int_0^y x^\alpha b(x,y)dx}{(1+y)^\alpha}.
\end{aligned}$$

Hence

$$\lim\sup_{y\to+\infty} \frac{\int_0^y \zeta(x)b(x,y)dx}{(1+y)^\alpha} \le (1+\varepsilon) \lim\sup_{y\to+\infty} \frac{\int_0^y x^\alpha b(x,y)dx}{(1+y)^\alpha} \quad (\forall \varepsilon > 0)$$

or, equivalently,

$$\lim\sup_{y\to+\infty} \frac{\int_0^y \zeta(x)b(x,y)dx}{(1+y)^\alpha} \le (1+\varepsilon) \lim\sup_{y\to+\infty} \frac{\int_0^y x^\alpha b(x,y)dx}{y^\alpha} \quad (\forall \varepsilon > 0).$$

Hence

$$\lim\sup_{y\to+\infty} \frac{\int_0^y \zeta(x)b(x,y)dx}{(1+y)^\alpha} \le \lim\sup_{y\to+\infty} \frac{\int_0^y x^\alpha b(x,y)dx}{y^\alpha}.$$

To prove the last statement, we apply Proposition 4. If (44) is not satisfied for any α, then $\widetilde{\alpha} = \infty$. If (44) is satisfied for $\alpha < \eta$, then the statement holds. So, we can assume that (44) is satisfied for some $\alpha_2 > \eta$. Let us take arbitrary $\alpha_1 > \eta$ and $\alpha \in (\alpha_1, \alpha_2)$. By the convexity, for any $y > 0$ we have

$$\begin{aligned}
\frac{n_{1,\alpha}(y)}{(1+y)^\alpha} &\le \frac{n_{1,\alpha_1}(y)}{(1+y)^{\alpha_1}} + \frac{\frac{n_{1,\alpha_2}(y)}{(1+y)^{\alpha_2}} - \frac{n_{1,\alpha_1}(y)}{(1+y)^{\alpha_1}}}{\alpha_2 - \alpha_1}(\alpha - \alpha_1) \\
&= \frac{n_{1,\alpha_1}(y)}{(1+y)^{\alpha_1}} \frac{\alpha_2 - \alpha}{\alpha_2 - \alpha_1} + \frac{n_{1,\alpha_2}(y)}{(1+y)^{\alpha_2}} \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1}.
\end{aligned}$$

For any $\epsilon_1 > 0$ there is y_1 such that for $y > y_1$

$$\frac{n_{1,\alpha_2}(y)}{(1+y)^{\alpha_2}} \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} < (1-\epsilon_1)\frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1}.$$

Next, since $\frac{(1+x)^{\alpha_1}}{1+x^{\alpha_1}} \to 1$ as $x \to \infty$, for any $\epsilon_2 > 0$ we pick $y_2 > y_1$ such that $\frac{(1+x)^{\alpha_1}}{1+x^{\alpha_1}} \le 1 + \epsilon_2$ for $x \ge y_2$ and, since $\alpha_1 > \eta$, for large y and some $1 \le \eta < \alpha' < \alpha_1$, using (45) and

$$\int_{y_2}^y x^{\alpha_1} b(x,y)dx \le y^{\alpha_1 - 1}\int_{y_2}^y x b(x,y)dx \le y^{\alpha_1 - 1}\int_0^y x b(x,y)dx \le y^{\alpha_1},$$

we have

$$\begin{aligned}
&\frac{\int_0^y (1+x)^{\alpha_1} b(x,y)dx}{(1+y)^{\alpha_1}} \\
&\le \frac{\int_0^{y_2} (1+x)^{\alpha_1} b(x,y)dx}{(1+y)^{\alpha_1}} + (1+\epsilon_2)\frac{\int_{y_2}^y b(x,y)dx + \int_{y_2}^y x^{\alpha_1} b(x,y)dx}{(1+y)^{\alpha_1}} \\
&\le (1+y_2)^{\alpha_1}\frac{c_{\alpha'}(1+y)^{\alpha'}}{(1+y)^{\alpha_1}} + (1+\epsilon_2)\frac{c_{\alpha'}(1+y)^{\alpha'} + y^{\alpha_1}}{(1+y)^{\alpha_1}}.
\end{aligned}$$

Thus, for any ϵ_3 we have $y_3 > y_2$ such that for all $y > y_3$ we have

$$\frac{\int_0^y (1+x)^{\alpha_1} b(x,y)dx}{(1+y)^{\alpha_1}} \le 1 + \epsilon_3$$

and, for such y,

$$\frac{n_{1,\alpha}(y)}{(1+y)^{\alpha}} \leq (1+\epsilon_3)\frac{\alpha_2-\alpha}{\alpha_2-\alpha_1} + (1-\epsilon_1)\frac{\alpha-\alpha_1}{\alpha_2-\alpha_1}$$
$$= 1+\epsilon_3\frac{\alpha_2-\alpha}{\alpha_2-\alpha_1} - \epsilon_1\frac{\alpha-\alpha_1}{\alpha_2-\alpha_1}.$$

Since ϵ_3 and ϵ_1 are independent, taking $\epsilon_3 = \frac{\epsilon_1(\alpha-\alpha_1)}{2(\alpha_2-\alpha)}$ we obtain for the corresponding large y,

$$\frac{n_{1,\alpha}(y)}{(1+y)^{\alpha}} \leq 1 - \frac{\epsilon_1}{2}\frac{\alpha-\alpha_1}{\alpha_2-\alpha_1}$$

hence, since $\alpha_1 > \eta$ is arbitrary, we have $\widetilde{\alpha} \leq \eta$. □

Remark 8. Similar estimates appear in [8, Theorem 2.2] and [6, Theorem 2.2], where W. Desch's theorem is used with the weight $1+x^{\alpha}$ instead of $(1+x)^{\alpha}$.

Remark 9. As noted in Introduction, for *homogeneous* fragmentation kernels

$$b(x,y) = \frac{1}{y}h\left(\frac{x}{y}\right) \text{ with } \int_0^1 zh(z)dz = 1,$$

we have

$$\frac{\int_0^y x^{\alpha}b(x,y)dx}{y^{\alpha}} = \int_0^1 z^{\alpha}h(z)dz < \int_0^1 zh(z)dz = 1 \ (\alpha > 1)$$

so,

$$\lim\sup_{y\to+\infty} \frac{\int_0^y x^{\alpha}b(x,y)dx}{y^{\alpha}} = \int_0^1 z^{\alpha}h(z)dz < \int_0^1 zh(z)dz = 1 \ (\alpha > 1)$$

and $\widetilde{\alpha} = 1$. See Appendix A for more examples.

3.6. **Compactness results.**

We start with

Theorem 3.6. *Let $\alpha > 0$, (6), (7) and (34) be satisfied. Let the sublevel sets of $a(\cdot)$ be thin at infinity in the sense that for any $c > 0$*

$$\int_1^{+\infty} 1_{\{a<c\}}\frac{1}{r(y)}dy < +\infty \tag{46}$$

(e.g. let $\lim_{x\to+\infty} a(x) = +\infty$). Then T is resolvent compact.

Proof. Let $\lambda > \alpha C$ and let f be in the unit ball of $X_{0,\alpha}$, i.e.

$$\int_0^{+\infty} |f(x)|\,(1+x)^{\alpha}\,dx \leq 1.$$

According to Lemma 3.4

$$\int_0^{+\infty} \left|\left((\lambda-T)^{-1}f\right)(y)\right| a(y)\,(1+y)^{\alpha}\,dy \leq 1.$$

Let $c > 0$ and $\varepsilon > 0$ be *arbitrary*. We have

$$\begin{aligned}
1 &\geq \int_{\varepsilon^{-1}}^{+\infty} \left|\left((\lambda-T)^{-1}f\right)(y)\right| a(y)\,(1+y)^{\alpha}\,dy \\
&= \int_{\varepsilon^{-1}}^{+\infty} 1_{\{a<c\}} \left|\left((\lambda-T)^{-1}f\right)(y)\right| a(y)\,(1+y)^{\alpha}\,dy \\
&\quad + \int_{\varepsilon^{-1}}^{+\infty} 1_{\{a\geq c\}} \left|\left((\lambda-T)^{-1}f\right)(y)\right| a(y)\,(1+y)^{\alpha}\,dy \\
&\geq \int_{\varepsilon^{-1}}^{+\infty} 1_{\{a<c\}} \left|\left((\lambda-T)^{-1}f\right)(y)\right| a(y)\,(1+y)^{\alpha}\,dy + \\
&\quad c\int_{\varepsilon^{-1}}^{+\infty} 1_{\{a\geq c\}} \left|\left((\lambda-T)^{-1}f\right)(y)\right| (1+y)^{\alpha}\,dy,
\end{aligned}$$

so

$$\sup_{\|f\|_E\leq 1} \int_{\varepsilon^{-1}}^{+\infty} 1_{\{a\geq c\}} \left|\left((\lambda-T)^{-1}f\right)(y)\right| (1+y)^{\alpha}\,dy \leq \frac{1}{c} \quad (\forall\varepsilon > 0).$$

On the other hand, according to Lemma 3.3,

$$\left|(\lambda-T)^{-1}f\right|(y)\leq\left|(\lambda-T_0)^{-1}f\right|(y)\leq\frac{1}{(1+y)^{\alpha}\,r(y)},$$

so

$$\int_{\varepsilon^{-1}}^{+\infty}1_{\{a<c\}}\left|\left((\lambda-T)^{-1}f\right)(y)\right|(1+y)^{\alpha}\,dy\leq\int_{\varepsilon^{-1}}^{+\infty}1_{\{a<c\}}\frac{1}{r(y)}dy$$

and then

$$\begin{aligned}
&\int_{\varepsilon^{-1}}^{+\infty}\left|\left((\lambda-T)^{-1}f\right)(y)\right|(1+y)^{\alpha}\,dy\\
=\;&\int_{\varepsilon^{-1}}^{+\infty}1_{\{a<c\}}\left|\left((\lambda-T)^{-1}f\right)(y)\right|(1+y)^{\alpha}\,dy\\
&+\int_{\varepsilon^{-1}}^{+\infty}1_{\{a\geq c\}}\left|\left((\lambda-T)^{-1}f\right)(y)\right|(1+y)^{\alpha}\,dy\\
\leq\;&\int_{\varepsilon^{-1}}^{+\infty}1_{\{a<c\}}\frac{1}{r(y)}dy+\frac{1}{c}
\end{aligned}$$

can be made arbitrarily small (uniformly in $\|f\|_{X_{0,\alpha}}\leq 1$) by choosing *first* c large enough and then ε small enough.

On the other hand on $\left(0,\varepsilon^{-1}\right)$ we have the uniform domination

$$\left|(\lambda-T)^{-1}f\right|(y)\leq\;\frac{1_{\left(0,\varepsilon^{-1}\right)}(y)}{(1+y)^{\alpha}\,r(y)}\quad(\|f\|_{X_{0,\alpha}}\leq 1),$$

where

$$\frac{1_{\left(0,\varepsilon^{-1}\right)}(y)}{(1+y)^{\alpha}\,r(y)}\in X_{0,\alpha}.$$

Finally $\left\{(\lambda-T)^{-1}f;\ \|f\|_{X_{0,\alpha}}\leq 1\right\}$ is as close to the relatively weakly compact set

$$\left\{1_{\left(0,\varepsilon^{-1}\right)}(y)(\lambda-T)^{-1}f;\ \|f\|_{X_{0,\alpha}}\leq 1\right\}$$

as we want and consequently it is weakly compact. This shows that $(\lambda-T)^{-1}$ is weakly compact operator and consequently (see [26, Lemma 14]) $(\lambda-T)^{-1}$ is compact. □

Corollary 1. *Let (6), (7), (34) and (40) be satisfied and let the sublevel sets of $a(\cdot)$ be thin at infinity in the sense of (46). Then $A:=T+B:D(T)\to X_{0,\alpha}$, where B is defined by* (10), *is resolvent compact.*

Proof. This follows simply from Theorem 3.6 and the fact that for λ large enough, $\sum_{n=0}^{+\infty}\left[B(\lambda-T)^{-1}\right]^n$ is a bounded operator and

$$(\lambda-T-B)^{-1}=(\lambda-T)^{-1}\sum_{n=0}^{+\infty}\left[B(\lambda-T)^{-1}\right]^n.$$

□

3.7. **Spectral gap of $(V(t))_{t\geqslant 0}$ in $X_{0,\alpha}$.**

We start with an irreducibility result which extends [5, Theorem 5.2.21] and is based on ideas from [7, Proposition 2].

Theorem 3.7. *Let $\alpha>0$, (6), (7), (34) and (40) and either*

1. *for any $x>0$, $\operatorname{supp}_{0\leq y\leq+\infty}a(y)b(x,y)$ is unbounded, or*
2. *$a(y)>0$ for a.a. $y>0$ and there is $0\leq p<1$ such that for each $y>0$, $\inf\operatorname{supp}b(\cdot,y)\leq py$*

be satisfied. Then $(\lambda-T-B)^{-1}$ is positivity improving, i.e.,

$$(\lambda-T-B)^{-1}g>0\ \text{a.e.}$$

for any nontrivial nonnegative $g\in X_{0,\alpha}$ or, equivalently, the C_0-semigroup $(V(t))_{t\geq 0}$ is irreducible in $X_{0,\alpha}$.

Proof. We know that

$$\left((\lambda - T)^{-1}g\right)(y) = \frac{1}{r(y)}\int_0^y e^{-\lambda\int_x^y \frac{1}{r(\tau)}d\tau} e^{-\int_x^y \frac{a(s)}{r(s)}ds} g(x)dx$$

and

$$(\lambda - T - B)^{-1} = (\lambda - T)^{-1}\sum_{n=0}^{+\infty}\left[B(\lambda - T)^{-1}\right]^n, \tag{47}$$

Let $g > 0$ and set $z_g = \sup\{z\ : g(z) = 0\ a.a.\text{ on }[0, z]\}$. If $z_g = 0$, then obviously $(\lambda - T)^{-1}g > 0$. and the result is valid. Assume then that $z_g > 0$ and observe that

$$\Psi_0(z) := [(\lambda - T)^{-1}g](z) = \begin{cases} \frac{1}{r(z)}\int_{z_g}^z e^{-\int_x^z \frac{\lambda + a(s)}{r(s)}ds} f(x)dx & \text{for} \quad z \geq z_g, \\ 0 & \text{for} \quad 0 \leq z < z_g \end{cases}$$

and $[(\lambda - T)^{-1}g](z)$ is positive for $x > z_g$. Then, for $x < z_g$,

$$\begin{aligned}&[(\lambda - T)^{-1}B(\lambda - T)^{-1}g](x) = [(\lambda - T)^{-1}B\Psi_0](x)\\ &= \frac{1}{r(x)}\int_0^x e^{-\int_y^x \frac{\lambda + a(s)}{r(s)}ds}\left(\int_{z_g}^{\infty} a(z)b(y, z)\Psi_0(z)dz\right)dy.\end{aligned}$$

We see that if assumption 1. is satisfied, then the inner integrand is positive for any $y > 0$ and hence $\Psi_1(x) := [(\lambda - T)^{-1}B(\lambda - T)^{-1}g](x) > 0$ for any $x > 0$. Otherwise, on using assumption 2, $\Psi_1(x) > 0$ for $x > \inf\operatorname{supp} b(\cdot, z_g)$. Thus, if $p = 0$ (in particular, if $b(x, y) > 0$ for all $y > 0$ and $0 < x < y$), then the result is proved. If $p > 0$, then $\Psi_1(x) > 0$ at least for $x > pz_g$. Next, the third term of (47) is given by

$$\Psi_2(x) := [(\lambda - T)^{-1}(B(\lambda - T)^{-1})^2 f](x) = [(\lambda - T)^{-1}B\Psi_1](x)$$

and thus, by the same argument, $\Psi_2(x) > 0$ for $x > p^2 z_g$. Using induction and $p^n \to 0$, we conclude that $[(\lambda - T - B)^{-1}f](x) > 0$ almost everywhere. ☐

Corollary 2. *Assumption 2. of Theorem 3.7 are satisfied if either*

: *(i) there is $\delta > 0$ such that for any $y > 0$ we have $n_0(y) \geq 1 + \delta$, or*
: *(ii) $b \in L_{\infty,loc}(\mathbb{R}_+ \times \mathbb{R}_+)$.*

Proof. Assume that (i) is satisfied. If $\inf\operatorname{supp} b(\cdot, y) = 0$, then we are done. Otherwise, let for some y, $\inf\operatorname{supp} b(\cdot, y) = p'y$ for some $p' \in (0, 1)$; p' can depend on y. Then

$$1 + \delta \leq n_0(y) = \int_{p'y}^y b(x, y)dx \leq \frac{1}{p'y}\int_{p'y}^y xb(x, y)dx = \frac{1}{p'},$$

which implies $p' \leq (1+\delta)^{-1}$. Hence, if we select p (independent of y) such that $(1+\delta)^{-1} < p < 1$, then for each $y > 0$, $\inf\operatorname{supp} b(\cdot, y) \leq (1 + \delta)^{-1}y < py$. If, instead of (i), assumption (ii) is satisfied, then the constant p of the proof of Theorem 3.7 may be y dependent and though in each step we can prove that the positivity of Ψ_n on $[z_n, \infty)$ implies the positivity of Ψ_{n+1} on $[z_{n+1}, \infty)$, where $z_{n+1} = p(z_n)z_n$, $z_1 = z_g$, this sequence may converge (as a decreasing sequence) to a $z_\infty > 0$. Then, however, we would have

$$\operatorname{supp} b(\cdot, z_n) \subset (z_\infty, z_n], \quad n \in \mathbb{N};$$

that is,

$$z_n = \int_{z_\infty}^{z_n} xb(x, z_n)dx.$$

This, however, leads to a contradiction, since the left-hand side converges to $z_\infty > 0$ and the right-hand side, by (ii), to 0. ☐

We are ready to show the main result of the first construction.

Theorem 3.8. *Under assumptions of Theorem 3.7, let the sublevel sets of $a(\cdot)$ be thin at infinity in the sense of (46). Then $(V(t))_{t\geq 0}$ has a spectral gap, i.e.,*

$$r_{ess}(V(t)) < r_\sigma(V(t)),$$

and it has the asynchronous exponential growth property (12) in $X_{0,\alpha}$.

Proof. Let

$$k(x,y) := 1_{\{x<y\}} a(y) b(x,y)$$

be the kernel of B. Let further

$$\overline{k}(x,y) := k(x,y) \wedge 1$$

and

$$\overline{k}_c(x,y) := \overline{k}(x,y) p(x) p(y),$$

where $p \in C(0,+\infty)$ has a compact support in $(0,+\infty)$ and $0 \le p(x) \le 1$. Note that

$$k(x,y) \geqslant \overline{k}_c(x,y)$$

and

$$\begin{aligned} k(x,y) &= \Big(k(x,y) - \overline{k}_c(x,y)\Big) + \overline{k}_c(x,y) \\ &= \widehat{k}(x,y) + \overline{k}_c(x,y), \end{aligned}$$

where

$$\widehat{k}(x,y) := k(x,y) - \overline{k}_c(x,y).$$

Let $\overline{B}$ be the integral operator with kernel $\overline{k}_c(x,y)$ and let $\widehat{B}$ be the integral operator with kernel $\widehat{k}(x,y)$. Since

$$\widehat{k}(x,y) \le k(x,y),$$

for λ large enough,

$$\left\| \widehat{B}(\lambda - T)^{-1} \right\|_{\mathcal{L}(X_{0,\alpha})} \le \left\| B(\lambda - T)^{-1} \right\|_{\mathcal{L}(X_{0,\alpha})} < 1,$$

so $T+\widehat{B} : D(T) \to X_{0,\alpha}$ generates a positive semigroup $(\widehat{V}(t))_{t\geqslant 0}$. Note that $(V(t))_{t\geqslant 0}$ is generated by

$$\Big(T + \widehat{B}\Big) + \overline{B}$$

where $\overline{B}$ is a *bounded* operator on $X_{0,\alpha}$. Actually the kernel of $\overline{B}$ is compactly supported in $(0,+\infty) \times (0,+\infty)$ and bounded and consequently $\overline{B}$ is a *weakly compact* operator on $X_{0,\alpha}$. On the other hand

$$V(t) = \widehat{V}(t) + \int_0^t \widehat{V}(t-s)\overline{B}\widehat{V}(s)ds$$

and $\int_0^t \widehat{V}(t-s)\overline{B}\widehat{V}(s)ds$ is a weakly compact operator (see [33] or [25]) so that $(\widehat{V}(t))_{t\geqslant 0}$ and $(V(t))_{t\geqslant 0}$ have the same essential spectrum [19] and then the same essential radius

$$r_{ess}(\widehat{V}(t)) = r_{ess}(V(t)). \tag{48}$$

On the other hand, $\widehat{V}(t) \le V(t)$,

$$(\lambda - T - \widehat{B})^{-1} \le (\lambda - T - B)^{-1}$$

and

$$(\lambda - T - \widehat{B})^{-1} \neq (\lambda - T - B)^{-1},$$

because $\overline{B} \neq 0$. Since, by Theorem 3.7, $(\lambda - T - B)^{-1}$ is positivity improving (and thus irreducible) and compact (by Corollary 1),

$$r_\sigma\left[(\lambda - T - \widehat{B})^{-1}\right] < r_\sigma\left[(\lambda - T - B)^{-1}\right],$$

see [21]. Next,

$$r_\sigma\left[(\lambda - T - \widehat{B})^{-1}\right] = \frac{1}{\lambda - s(T+\widehat{B})}, \quad r_\sigma\left[(\lambda - T - B)^{-1}\right] = \frac{1}{\lambda - s(T+B)}$$

(see [29]) implies

$$s(T+\widehat{B}) < s(T+B) \tag{49}$$

and hence, in particular,

$$s(T+B) > -\infty.$$

Note that the type of a positive semigroup on L^1-spaces coincides with the spectral bound of its generator, see e.g. [36]. We combine this with (49) to get

$$r_{ess}(V(t)) = r_{ess}(\widehat{V}(t)) \le r_\sigma(\widehat{V}(t)) = e^{s(T+\widehat{B})t} < e^{s(T+B)t} = r_\sigma(V(t))$$

so $r_{ess}(V(t)) < r_\sigma(V(t))$. Finally, as explained in Introduction, the irreducibility of $(V(t))_{t\geq 0}$ ensures, by [29, Corollary 3.16 of Chapter C-III], that the dominant eigenvalue is a simple pole and, by [17, Proposition 3.4 of Chapter VI], we see that it is simple, that is, its eigenspace is one-dimensional. □

4. Second construction.

This construction is based on Assumption (8).

4.1. Generation results.

We start with the space X_α.

Theorem 4.1. *Let $\alpha > 0$. We assume that (8) is satisfied. Let $X(y,t)$ $(t > 0)$ be defined by $\int_{X(y,t)}^{y} \frac{1}{r(\tau)} d\tau = t$. Then*

$$(U_0(t)f)(y) := \frac{r(X(y,t))f(X(y,t))}{r(y)} = f(X(y,t))\frac{\partial X(y,t)}{\partial y}$$

defines a positive C_0-semigroup $(U_0(t))_{t\geqslant 0}$ on X_α if and only if

$$\sup_{x>0} \frac{y(x,t)}{x} < +\infty \quad (t \geqslant 0)$$

and

$$[0,+\infty) \ni t \to \sup_{x>0} \frac{y(x,t)}{x} \quad \text{is locally bounded}$$

where $y(x,t) > x$ is defined by $\int_x^{y(x,t)} \frac{1}{r(\tau)} d\tau = t$. In this case,

$$\|U_0(t)\|_{\mathcal{L}(X_\alpha)} = \sup_{x>0} \frac{y^\alpha(x,t)}{x^\alpha}.$$

This occurs if

$$C := \sup_{z>0} \frac{r(z)}{z} < +\infty, \tag{50}$$

in which case $\|U_0(t)\|_{\mathcal{L}(X_\alpha)} \leq e^{\alpha C t}$.

Proof. We set

$$U_0(t)f := \frac{r(X(y,t))f(X(y,t))}{r(y)}$$

and argue as in the proof of Theorem 3.1. Let us check that $U_0(t)$ is a bounded operator on X_α. Note that

$$\int_{X(y,t)}^{y} \frac{1}{r(\tau)} d\tau = t \tag{51}$$

and (8) show that (for $t > 0$ fixed) $X(y,t)$ is strictly increasing in y and

$$\lim_{y\to 0} X(y,t) = 0, \ \lim_{y\to+\infty} X(y,t) = +\infty.$$

Since

$$\frac{1}{r(y)} = \frac{1}{r(X(y,t))}\frac{\partial X(y,t)}{\partial y},$$

we have

$$(U_0(t)f)(y) = f(X(y,t))\frac{\partial X(y,t)}{\partial y}, \quad y \in (0,+\infty)$$

and

$$\|U_0(t)f\|_{X_\alpha} = \int_0^{+\infty} |f(X(y,t))| \frac{\partial X(y,t)}{\partial y} y^\alpha dy.$$

The change of variable $x = X(y,t)$ yields

$$\|U_0(t)f\|_{X_\alpha} = \int_0^{+\infty} |f(x)|\, y^\alpha(x,t) dx,$$

where $y(x,t)$ is the unique $y > x$ such that $x = X(y,t)$ i.e., $\int_x^{y(x,t)} \frac{1}{r(\tau)} d\tau = t$. Since

$$\|U_0(t)f\|_{X_\alpha} = \int_0^{+\infty} \frac{y^\alpha(x,t)}{x^\alpha} |f(x)|\, x^\alpha dx,$$

$U_0(t)$ is a bounded operator on X_α if and only if $\sup_{x>0} \frac{y(x,t)}{x} < +\infty$. In this case

$$\|U_0(t)\|_{\mathcal{L}(X_\alpha)} = \sup_{x>0} \frac{y^\alpha(x,t)}{x^\alpha}$$

and

$$[0,+\infty) \ni t \to U_0(t) \in \mathcal{L}(X_\alpha)$$

is locally bounded if and only if

$$[0,+\infty) \ni t \to \sup_{x>0} \frac{y(x,t)}{x}$$

is locally bounded. As in the proof of Theorem 3.1, to show that $(U_0(t))_{t\geqslant 0}$ is strongly continuous on X_α it suffices to check that

$$U_0(t)f \to f \text{ in } L^1(\mathbb{R}_+;\ x^\alpha dx) \text{ as } t \to 0$$

on a *dense* subspace of $L^1(\mathbb{R}_+;\ x^\alpha dx)$, e.g. for f continuous with compact support in $(0,+\infty)$. Note that (51) shows that $X(y,t) \to y$ as $t \to 0$ uniformly in y in compact sets of $(0,+\infty)$. By arguing as in the proof of Theorem 3.1, one sees that

$$U_0(t)f = \frac{r(X(y,t))f(X(y,t))}{r(y)} \to f \quad (t \to 0)$$

in $L^1(\mathbb{R}_+;\ x^\alpha dx)$ by the dominated convergence theorem. Finally (37) implies

$$y(x,t) = x + \int_0^t r(y(x,s))ds \le x + \int_0^t Cy(x,s)ds \tag{52}$$

so, by Gronwall's lemma, $y(x,t) \le xe^{Ct}$ and $\sup_{x>0} \frac{y^\alpha(x,t)}{x^\alpha} \le e^{\alpha Ct}$. □

Remark 10. One can show (see [27, Proposition 6]) that if

$$\lim_{z\to 0} \frac{r(z)}{z} = +\infty \text{ or } \lim_{z\to+\infty} \frac{r(z)}{z} = +\infty,$$

then $\sup_{x>0} \frac{y(x,t)}{x} = +\infty$. In particular we have not a generation theory in X_α. This shows the "optimality" of Assumption (50) in X_α. This shows also that in Theorem 3.1, (34) is partly necessary.

We deal now with $X_{0,\alpha}$.

Theorem 4.2. *Let $\alpha > 0$. We assume that (8) is satisfied. Let $X(y,t)$ $(t > 0)$ be defined by $\int_{X(y,t)}^y \frac{1}{r(\tau)} d\tau = t$. Then*

$$(U_0(t)f)(y) := \frac{r(X(y,t))f(X(y,t))}{r(y)} = f(X(y,t))\frac{\partial X(y,t)}{\partial y}$$

defines a positive C_0-semigroup $(U_0(t))_{t\geqslant 0}$ on $X_{0,\alpha}$ if and only if

$$\sup_{x>0} \frac{1+y(x,t)}{1+x} < +\infty \quad (t \geqslant 0)$$

and

$$[0,+\infty) \ni t \to \sup_{x>0} \frac{1+y(x,t)}{1+x} \quad \text{is locally bounded,}$$

where $y(x,t) \ge x$ is defined by $\int_x^{y(x,t)} \frac{1}{r(\tau)} d\tau = t$. In this case

$$\|U_0(t)\|_{\mathcal{L}(X_{0,\alpha})} = \sup_{x>0} \frac{(1+y(x,t))^\alpha}{(1+x)^\alpha}.$$

This occurs if

$$\widehat{C} := \sup_{z>1} \frac{r(z)}{z} < +\infty. \tag{53}$$

Proof. Arguing as in the previous proof, we obtain

$$\|U_0(t)f\|_{X_{0,\alpha}} = \int_0^{+\infty} |f(X(y,t))| \frac{\partial X(y,t)}{\partial y} (1+y)^\alpha dy,$$

so

$$\begin{aligned}\|U_0(t)f\|_{X_{0,\alpha}} &= \int_0^{+\infty} |f(x)| (1+y(x,t))^\alpha dx \\ &= \int_0^{+\infty} \frac{(1+y(x,t))^\alpha}{(1+x)^\alpha} |f(x)| (1+x)^\alpha dx\end{aligned}$$

and

$$\|U_0(t)\|_{\mathcal{L}(X_{0,\alpha})} = \sup_{x>0} \frac{(1+y(x,t))^\alpha}{(1+x)^\alpha}.$$

Note that $\int_x^{y(x,t)} \frac{1}{r(\tau)} d\tau = t$ implies that $\lim_{x\to 0} y(x,t) = 0$ uniformly for bounded sets of t, so, for any $\overline{t} > 0$,

$$\sup_{t\in[0,\overline{t}]} \sup_{x<1} \frac{1+y(x,t)}{(1+x)} < +\infty.$$

Since

$$y(x,t) = x + \int_0^t r(y(x,s))ds,$$

$y(x,t) \geq x$ and, by (53),

$$y(x,t) \leq x + \int_0^t \widehat{C} y(x,s)ds \quad (x>1).$$

Hence,

$$1 + y(x,t) \leq 1 + x + \int_0^t \widehat{C}\,(1+y(x,s))\,ds \ (x>1)$$

and

$$1 + y(x,t) \leq (1+x)\, e^{\widehat{C}t} \quad (x>1)$$

by Gronwall's inequality. Finally,

$$t \to \sup_{x>0} \frac{(1+y(x,t))^\alpha}{(1+x)^\alpha} < +\infty$$

is locally bounded. The rest of the proof is the same as the previous one. □

Remark 11. As in Remark 10, if

$$\lim_{z\to+\infty} \frac{r(z)}{z} = +\infty,$$

then $\sup_{x>0} \frac{y(x,t)}{x} = +\infty$. This again shows the "optimality" of Assumption (53) in $X_{0,\alpha}$.

4.2. **A pointwise estimate.**
We give now the first a priori estimate in the spaces X_α and $X_{0,\alpha}$.

Lemma 4.3. *Let $\alpha > 0$ and let (8) be satisfied.*
(i) Let (50) be satisfied and $\lambda \geq \alpha C$. Then

$$\left|(\lambda - T_0)^{-1} f\right|(y) \leq \frac{1}{y^\alpha r(y)} \|f\|_{X_\alpha} \quad (f \in X_\alpha).$$

(ii) Let (14) be satisfied and $\lambda \geq \alpha \widetilde{C}$. Then

$$\left|(\lambda - T_0)^{-1} f\right|(y) \leq \frac{1}{(1+y)^\alpha\, r(y)} \|f\|_{X_{0,\alpha}} \quad (f \in X_{0,\alpha}).$$

Proof. (i) Note that $r(\tau) \leq C\tau \ \ (\forall \tau > 0)$, that is,

$$\frac{1}{r(\tau)} \geq \frac{1}{C\tau}$$

implies

$$e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} \leq e^{-\frac{\lambda}{C}\int_x^y \frac{1}{\tau} d\tau} = e^{-\frac{\lambda}{C}\ln\left(\frac{y}{x}\right)} = \left(\frac{x}{y}\right)^{\frac{\lambda}{C}}, \tag{54}$$

so

$$\begin{aligned} \left|(\lambda - T_0)^{-1} f(y)\right| &\leq \frac{1}{r(y)} \int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} |f(x)|\, dx \\ &\leq \frac{1}{r(y)} \int_0^y \left(\frac{x}{y}\right)^{\frac{\lambda}{C}} |f(x)|\, dx. \end{aligned}$$

Since $f \in X_\alpha$,

$$
\begin{aligned}
\left|(\lambda - T_0)^{-1} f(y)\right| &\le \frac{1}{r(y)} \int_0^y x^{-\alpha} \left(\frac{x}{y}\right)^{\frac{\lambda}{C}} |f(x)|\, x^\alpha dx \\
&= \frac{1}{y^\alpha r(y)} \int_0^y y^\alpha x^{-\alpha} \left(\frac{x}{y}\right)^{\frac{\lambda}{C}} |f(x)|\, x^\alpha dx \\
&= \frac{1}{y^\alpha r(y)} \int_0^y \left(\frac{x}{y}\right)^{\frac{\lambda}{C}-\alpha} |f(x)|\, x^\alpha dx \\
&\le \frac{1}{y^\alpha r(y)} \int_0^y |f(x)|\, x^\alpha dx \le \frac{1}{y^\alpha r(y)} \|f\|_{X_\alpha},
\end{aligned}
$$

on account of $\frac{x}{y} \le 1$ and $\frac{\lambda}{C} - \alpha \ge 0$.

(ii) Note that

$$
\frac{1}{r(\tau)} \ge \frac{1}{\widetilde{C}(\tau+1)} \ (\tau > 0)
$$

and

$$
e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} \le e^{-\frac{\lambda}{C} \int_x^y \frac{1}{\tau+1} d\tau} = e^{-\frac{\lambda}{C} \ln\left(\frac{y+1}{x+1}\right)} = \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}} \tag{55}
$$

so that if $f \in X_{0,\alpha}$, then

$$
\begin{aligned}
\left|(\lambda - T_0)^{-1} f(y)\right| &\le \frac{1}{r(y)} \int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} |f(x)|\, dx \\
&\le \frac{1}{r(y)} \int_0^y \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}} |f(x)|\, dx \\
&= \frac{1}{r(y)} \int_0^y \frac{1}{(1+x)^\alpha} \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}} |f(x)|\, (1+x)^\alpha\, dx \\
&= \frac{1}{(1+y)^\alpha\, r(y)} \int_0^y \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}-\alpha} |f(x)|\, (1+x)^\alpha\, dx \\
&\le \frac{1}{(1+y)^\alpha\, r(y)} \int_0^y |f(x)|\, (1+x)^\alpha\, dx \le \frac{1}{(1+y)^\alpha\, r(y)} \|f\|_{X_{0,\alpha}},
\end{aligned}
$$

on account of $\frac{x+1}{y+1} \le 1$ and $\frac{\lambda}{\tilde{C}} - \alpha \ge 0$. □

4.3. **The first perturbed semigroup.**

We solve

$$
\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x}\left[r(x) u(x,t)\right] + a(x) u(x,t) = 0
$$

by the method of characteristics. The solution is given by

$$
e^{-\int_{X(y,t)}^y \frac{a(p)}{r(p)} dp} \frac{r(X(y,t)) f(X(y,t))}{r(y)}.
$$

This defines a perturbed C_0-semigroup $(U(t))_{t \ge 0}$ on both X_α and $X_{0,\alpha}$, dominated by $(U_0(t))_{t\ge 0}$,

$$
U(t) f = e^{-\int_{X(y,t)}^y \frac{a(p)}{r(p)} dp} \frac{r(X(y,t)) f(X(y,t))}{r(y)} = e^{-\int_{X(y,t)}^y \frac{a(p)}{r(p)} dp} U_0(t) f.
$$

As previously, the Laplace transform of $(U(t))_{t\geqslant 0}$ and some change of variables give:

Proposition 6. *Let $\alpha > 0$, (8) and (50) (resp. (53)) be satisfied. The resolvent of the generator T of $(U(t))_{t\ge 0}$ in X_α (resp. in $X_{0,\alpha}$), $\lambda > s(T)$, is given by*

$$
\left((\lambda - T)^{-1} f\right)(y) = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda + \beta(p)}{r(\tau)} d\tau} f(x) dx.
$$

4.4. **A smoothing effect of the perturbed resolvent.**

The second a priori estimate in the spaces X_α and $X_{0,\alpha}$ is given by:

Lemma 4.4. *Let $\alpha > 0$ and (8) be satisfied.*

(i) Let (50) be satisfied and $\lambda \geq \alpha C$. Then, for any $f \in X_\alpha$,

$$\int_0^{+\infty} \left|\left((\lambda - T)^{-1} f\right)(y)\right| a(y) y^\alpha dy \leq \int_0^{+\infty} |(f(y)| \, y^\alpha dy. \tag{56}$$

(ii) Let (14) be satisfied and $\lambda \geq \alpha \widetilde{C}$. Then, for any $f \in X_{0,\alpha}$,

$$\int_0^{+\infty} \left|\left((\lambda - T)^{-1} f\right)(y)\right| a(y) \left(1+y\right)^\alpha dy \leq \int_0^{+\infty} |(f(y)| \left(1+y\right)^\alpha dy. \tag{57}$$

Proof. (i) By using (54) and $f \geq 0$

$$\begin{aligned}
&\int_0^{+\infty} \left((\lambda - T)^{-1} f\right)(y) a(y) y^\alpha dy \\
= &\int_0^{+\infty} \frac{a(y)y^\alpha}{r(y)} \left(\int_0^y e^{-\lambda \int_x^y \frac{1}{r(p)} dp} e^{-\int_x^y \frac{a(p)}{r(p)} dp} f(x) dx\right) dy \\
\leq &\int_0^{+\infty} \frac{a(y)y^\alpha}{r(y)} \left(\int_0^y \left(\frac{x}{y}\right)^{\frac{\lambda}{C}} e^{-\int_x^y \frac{a(p)}{r(p)} dp} f(x) dx\right) dy \\
= &\int_0^{+\infty} \left[\int_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{\lambda}{C}} \frac{a(y)y^\alpha}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)} dp} dy\right] f(x) dx \\
= &\int_0^{+\infty} \left[\int_x^{+\infty} \frac{1}{x^\alpha} \left(\frac{x}{y}\right)^{\frac{\lambda}{C}} \frac{a(y)y^\alpha}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)} dp} dy\right] f(x) x^\alpha dx \\
= &\int_0^{+\infty} \left[\int_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{\lambda}{C} - \alpha} \frac{a(y)}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)} dp} dy\right] f(x) x^\alpha dx \\
\leq &\int_0^{+\infty} \left[\int_x^{+\infty} \frac{a(y)}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)} dp} dy\right] f(x) x^\alpha dx,
\end{aligned}$$

where $\frac{x}{y} \leq 1$ and $\frac{\lambda}{C} - \alpha \geq 0$ are used in the last step. Thus

$$\begin{aligned}
&\int_0^{+\infty} \left((\lambda - T)^{-1} f\right)(y) a(y) y^\alpha dy \\
\leq &\sup_{x>0} \int_x^{+\infty} \frac{a(y)}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)} dp} dy \left(\int_0^{+\infty} f(x) x^\alpha dx\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
\int_x^{+\infty} e^{-\int_x^y \frac{a(p)}{r(p)} dp} \frac{a(y)}{r(y)} dy &= -\int_x^{+\infty} \frac{d}{dy}\left(e^{-\int_x^y \frac{a(p)}{r(p)} dp}\right) dy \\
&= -\left[e^{-\int_x^y \frac{a(p)}{r(p)} dp}\right]_{y=x}^{y=+\infty} \leq 1
\end{aligned}$$

ends the proof.

(ii) The proof of (57) is the same as that of Lemma 3.4 by using (55). ∎

4.5. **On the full semigroup.**

The same proof as that of Theorem 3.5 in $X_{0,\alpha}$ gives the following statement which, unfortunately, is *not* useful for the purpose of spectral gaps, see Remark 12 below.

Theorem 4.5. *Let (8) and (53) be satisfied. Define*

$$n_{1,\alpha}(y) := \int_0^y (1+x)^\alpha \, b(x,y) dx.$$

If $\sup_{y>0} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} < +\infty$, then B is T-bounded in $X_{0,\alpha}$ and

$$\lim_{\lambda \to +\infty} \left\|B(\lambda - T)^{-1}\right\|_{\mathcal{L}(X_{0,\alpha})} \leq \lim \sup_{a(y) \to +\infty} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha}.$$

In particular, if

$$\limsup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^{\alpha}} < 1, \tag{58}$$

then

$$A := T + B : X_{0,\alpha} \supset D(T) \to X_{0,\alpha}$$

generates a positive C_0-semigroup $(V(t))_{t\geq 0}$ on $X_{0,\alpha}$.

Remark 12. If $a(\cdot)$ is *unbounded near zero*, then

$$\limsup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^{\alpha}} \geq \limsup_{y\to 0} n_{1,\alpha}(y) \geq \limsup_{y\to 0} \int_0^y b(x,y)dx \geq 1,$$

because

$$\int_0^y b(x,y)dx = \frac{1}{y}\int_0^y yb(x,y)dx \geq \frac{1}{y}\int_0^y xb(x,y)dx = 1$$

and hence (58) cannot be satisfied. On the other hand, the compactness result we need in the sequel demands the unboundedness of $a(\cdot)$ near zero. Hence, under Assumption (8), it is not possible to finalize our spectral gap construction in the space $X_{0,\alpha}$. We point out that even if $a(\cdot)$ is *unbounded near zero, we can still define a* positive C_0-semigroup $(V(t))_{t\geq 0}$ on $X_{0,\alpha}$ which solve the growth fragmentation equations but in some generalized sense (honesty theory), where the domain of the generator T_B is the closure of $T+B$ only, see [5, Chapter 5]. However, in this case, we cannot infer that T_B is resolvent compact when T is and the key argument behind the existence of the spectral gap fails.

Theorem 4.6. *Let $\alpha > 0$, (8) and (50) be satisfied. Define*

$$n_\alpha(y) := \int_0^y x^\alpha b(x,y)dx.$$

If $\sup_{y>0} \frac{n_\alpha(y)}{y^\alpha} < +\infty$, then B is T-bounded in X_α and

$$\lim_{\lambda\to+\infty} \left\|B(\lambda - T)^{-1}\right\|_{\mathcal{L}(X_\alpha)} \leq \limsup_{a(y)\to+\infty} \frac{n_\alpha(y)}{y^\alpha}.$$

In particular, if

$$\limsup_{a(y)\to+\infty} \frac{n_\alpha(y)}{y^\alpha} < 1, \tag{59}$$

then

$$A := T + B : D(T) \subset X_\alpha \to X_\alpha$$

generates a positive C_0-semigroup $(V(t))_{t\geq 0}$ on X_α.

Proof. We note that for nonnegative φ, standard calculations give

$$\|B\varphi\|_{X_\alpha} = \int_0^{+\infty} a(y)n_\alpha(y)\varphi(y)dy.$$

Thus, for nonnegative f,

$$\begin{aligned}\left\|B(\lambda - T)^{-1}f\right\|_{X_\alpha} &= \int_0^{+\infty} a(y)n_\alpha(y)\left((\lambda - T)^{-1}f\right)(y)dy \\ &= \int_0^{+\infty} a(y)\frac{n_\alpha(y)}{y^\alpha}\left((\lambda - T)^{-1}f\right)(y)y^\alpha dy. \end{aligned} \tag{60}$$

Let

$$L := \limsup_{a(y)\to+\infty} \frac{n_\alpha(y)}{y^\alpha}.$$

For any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$a(y) \geq c_\varepsilon \Longrightarrow \frac{n_\alpha(y)}{y^\alpha} \leq L + \varepsilon.$$

We split (60) into two integrals

$$\begin{aligned} &\int_0^{+\infty} a(y)\frac{n_\alpha(y)}{y^\alpha}\left((\lambda-T)^{-1}f\right)(y)y^\alpha dy \\ = &\int_{\{a(y)\le c_\varepsilon\}} a(y)\frac{n_\alpha(y)}{y^\alpha}\left((\lambda-T)^{-1}f\right)(y)y^\alpha dy \\ &+\int_{\{a(y)> c_\varepsilon\}} a(y)\frac{n_\alpha(y)}{y^\alpha}\left((\lambda-T)^{-1}f\right)(y)y^\alpha dy \\ = &\; I_1+I_2. \end{aligned}$$

We note that

$$I_1 \le c_\varepsilon \left\|\frac{n_\alpha(\cdot)}{y^\alpha}\right\|_{L^\infty}\left\|(\lambda-T)^{-1}f\right\|_{X_\alpha}$$

while, using Lemma 4.4,

$$\begin{aligned} I_2 &\le (L+\varepsilon)\int_0^{+\infty} a(y)\left((\lambda-T)^{-1}f\right)(y)y^\alpha dy \\ &\le (L+\varepsilon)\left\|f\right\|_{X_\alpha}. \end{aligned}$$

Hence,

$$\begin{aligned} \left\|B(\lambda-T)^{-1}f\right\|_{X_\alpha} \le\; & c_\varepsilon\left\|\frac{n_\alpha(\cdot)}{y^\alpha}\right\|_{L^\infty}\left\|(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_\alpha)}\left\|f\right\|_{X_\alpha} \\ &+(L+\varepsilon)\left\|f\right\|_{X_\alpha} \end{aligned}$$

and

$$\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_\alpha)} \le c_\varepsilon\left\|\frac{n_\alpha(\cdot)}{y^\alpha}\right\|_{L^\infty}\left\|(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_\alpha)}+(L+\varepsilon).$$

Since $\left\|(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_\alpha)} \to 0$ as $\lambda\to+\infty$,

$$\lim_{\lambda\to+\infty}\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_\alpha)} \le L+\varepsilon \quad (\forall\varepsilon>0)$$

and consequently, if $L<1$, then

$$\lim_{\lambda\to+\infty}\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_\alpha)} < 1$$

and we end the proof by applying Theorem 1.1. □

Remark 13. Proposition 4 yields that for each $y>0$, $\alpha\to\frac{n_\alpha(y)}{y^\alpha}$ is decreasing and convex. Since

$$\frac{n_1(y)}{y}=\frac{1}{y}\int_0^y xb(x,y)dx=1,$$

$\limsup_{a(y)\to+\infty}\frac{n_1(y)}{y}=1$ and

$$\limsup_{a(y)\to+\infty}\frac{n_\alpha(y)}{y^\alpha}\ge 1 \quad (0<\alpha\le 1),$$

hence the necessity to consider higher moments, i.e., $\alpha>1$. So let $1<\alpha\le\alpha_2$. By the convexity,

$$\frac{n_\alpha(y)}{y^\alpha}\le\frac{n_1(y)}{y}+\frac{\frac{n_{\alpha_2}(y)}{y^{\alpha_2}}-\frac{n_1(y)}{y}}{\alpha_2-1}(\alpha-1)=1+\frac{\frac{n_{\alpha_2}(y)}{y^{\alpha_2}}-1}{\alpha_2-1}(\alpha-1).$$

If (59) holds for α_2, for any $\epsilon>0$, there is c_ϵ such that $\frac{n_{\alpha_2}(y)}{y^{\alpha_2}}\le 1-\epsilon$ on $\{y\in(0,\infty);\ a(y)\ge c_\epsilon\}$ and hence on this set

$$\frac{n_\alpha(y)}{y^\alpha}\le 1-\epsilon\frac{\alpha-1}{\alpha_2-1}.$$

Thus it follows that if (59) holds for some $\alpha_2>1$, then it holds for all $\alpha>1$.

As noted in Introduction, for *homogeneous* fragmentation kernels we have

$$\frac{\int_0^y x^\alpha b(x,y)dx}{y^\alpha}=\int_0^1 z^\alpha h(z)dz<\int_0^1 zh(z)dz=1\ (\alpha>1)$$

and so $\limsup_{a(y)\to+\infty}\frac{n_\alpha(y)}{y^\alpha}=\int_0^1 z^\alpha h(z)dz<1$ for all $\alpha>1$.

See also Section A for more examples.

4.6. **Compactness results.**
By using Lemma 4.3, Lemma 4.4 and arguing as in the proof of Theorem 3.6 we get:

Theorem 4.7. *Let $\alpha > 0$, (8) and (50) (resp. (14)) be satisfied. Let the sublevel sets of $a(\cdot)$ be thin at zero and at infinity in the sense that for any $c > 0$*

$$\int_0^{+\infty} 1_{\{a<c\}} \frac{1}{r(y)} dy < +\infty$$

(e.g., let $\lim_{x\to+\infty} a(x) = +\infty$ and $\lim_{x\to 0} a(x) = +\infty$). Then T is resolvent compact in X_α (resp. in $X_{0,\alpha}$).

Similarly to Corollary 1 we have

Corollary 3. *Let (8), (50) and (59) be satisfied. If the sublevel sets of $a(\cdot)$ are thin at zero and at infinity (e.g. if $\lim_{x\to+\infty} a(x) = +\infty$ and $\lim_{x\to 0} a(x) = +\infty$), then $A := T + B$ is resolvent compact in X_α.*

4.7. **Spectral gap of $(V(t))_{t\geqslant 0}$ in X_α.**
The same proof as for Theorem 3.7 gives:

Lemma 4.8. *Let $\alpha > 0$ and assumptions 1. or 2. of Theorem 3.7, (8), (50) and (59) be satisfied. Then $(\lambda - T - B)^{-1}$ is positivity improving in X_α or, equivalently, the C_0-semigroup $(V(t))_{t\geq 0}$ is irreducible in X_α.*

Finally, the same proof as for Theorem 3.8 gives the main result of the second construction.

Theorem 4.9. *Let $\alpha > 0$ and assumptions 1. or 2. of Theorem 3.7, (8), (50) and (59) be satisfied. If the sublevel sets of $a(\cdot)$ are thin at zero and at infinity, (e.g. if $\lim_{x\to+\infty} a(x) = +\infty$ and $\lim_{x\to 0} a(x) = +\infty$), then $(V(t))_{t\geq 0}$ has an asynchronous exponential growth in X_α.*

Remark 14. We conjecture that the result does not hold without the unboundedness of $a(\cdot)$ at zero as suggested by [10, Theorem 4.1].

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Appendix A. Separable fragmentation kernels. We have seen how homogeneous fragmentation kernels satisfy the key assumptions (40) and (59) of our construction. This last section is devoted to separable fragmentation kernels

$$b(x, y) = \beta(x)\gamma(y),$$

introduced in [2], see also [3][5]. It is easy to see that separable kernels with mass conservation (3) are of the form

$$b(x, y) = \beta(x)y \left(\int_0^y s\beta(s)ds \right)^{-1} \tag{61}$$

where

$$0 < \int_0^y s\beta(s)ds < +\infty \quad \forall y > 0.$$

A particular case of separable kernels are power law kernels

$$b(x, y) = (\nu + 2)\frac{x^\nu}{y^{\nu+1}} \quad (\nu \in (-2, 0]).$$

We can complement Theorem 4.6 by:

Proposition 7. *We assume that the fragmentation kernel is of the form (61) and $a(\cdot)$ is only unbounded at zero and infinity. If*

$$\beta_0^- := \liminf_{x\to 0} x\beta(x) > 0, \; \beta_0^+ := \limsup_{x\to 0} x\beta(x) < +\infty$$

and

$$\int_0^{+\infty} x^\alpha \beta(x) dx < +\infty,$$

then (59) is satisfied provided $\alpha > \frac{\beta_0^+}{\beta_0^-}$.

Proof. Note first that

$$n_\alpha(y) = \left(\int_0^y x^\alpha \beta(x)dx\right) y \left(\int_0^y x\beta(x)dx\right)^{-1}$$

and (59) amounts to

$$\limsup_{y\to+\infty} \frac{1}{y^{\alpha-1}} \frac{\int_0^y x^\alpha \beta(x)dx}{\int_0^y x\beta(x)dx} < 1 \tag{62}$$

and

$$\limsup_{y\to 0} \frac{1}{y^{\alpha-1}} \frac{\int_0^y x^\alpha \beta(x)dx}{\int_0^y x\beta(x)dx} < 1. \tag{63}$$

Let $\varepsilon > 0$ be arbitrary. Then, for y small enough

$$\int_0^y x^\alpha \beta(x)dx \le \left(\beta_0^+ + \varepsilon\right) \int_0^y x^{\alpha-1}dx = \frac{\left(\beta_0^+ + \varepsilon\right) y^\alpha}{\alpha}$$

and

$$\int_0^y x\beta(x)dx \ge \left(\beta_0^- - \varepsilon\right) y,$$

so

$$\frac{1}{y^{\alpha-1}} \frac{\int_0^y x^\alpha \beta(x)dx}{\int_0^y x\beta(x)dx} \le \frac{1}{\alpha} \frac{\beta_0^+ + \varepsilon}{\beta_0^- - \varepsilon}.$$

Therefore,

$$\limsup_{y\to 0} \frac{1}{y^{\alpha-1}} \frac{\int_0^y x^\alpha \beta(x)dx}{\int_0^y x\beta(x)dx} \le \frac{1}{\alpha} \frac{\beta_0^+}{\beta_0^-}.$$

Finally,

$$\frac{1}{y^{\alpha-1}} \frac{\int_0^y x^\alpha \beta(x)dx}{\int_0^y x\beta(x)dx} \le \frac{1}{y^{\alpha-1}} \frac{\int_0^\infty x^\alpha \beta(x)dx}{\int_0^1 x\beta(x)dx} \quad (y \ge 1)$$

and

$$\lim_{y\to+\infty} \frac{1}{y^{\alpha-1}} \frac{\int_0^y x^\alpha \beta(x)dx}{\int_0^y x\beta(x)dx} = 0.$$

This ends the proof. □

Remark 15. If $\beta_0 := \lim_{x\to 0} x\beta(x) > 0$ exists then $\beta_0^+ = \beta_0^-$ and both (62) and (63) are satisfied for any $\alpha > 1$ such that $\int_0^{+\infty} x^\alpha \beta(x)dx < +\infty$.

Similarly, we can complement Theorem 3.5 by:

Proposition 8. *We assume that the fragmentation kernel is of the form (61) and $a(\cdot)$ is only unbounded at infinity. If*

$$\int_0^{+\infty} \beta(x) \left(1 + x^{\overline{\alpha}}\right) dx < +\infty \tag{64}$$

for some $\overline{\alpha} > 1$ then (40) is satisfied for any $\alpha > 1$ and consequently the threshold is equal to one.

Note first that

$$n_{1,\alpha}(y) := y \left(\int_0^y x\beta(x)dx\right)^{-1} \int_0^y \beta(x)\,(1+x)^\alpha\, dx$$

and hence (40) amounts to

$$\limsup_{y\to+\infty} \frac{y}{(1+y)^\alpha} \frac{\int_0^y \beta(x)\,(1+x)^\alpha\, dx}{\int_0^y x\beta(x)dx} < 1.$$

Note that (64) implies that

$$\int_0^{+\infty} \beta(x)\,(1+x^\alpha)\, dx < +\infty \quad (0 < \alpha \le \overline{\alpha}).$$

It is easy to see that $(1+x)^\alpha \le 2^{\alpha-1}(1+x^\alpha)$ so

$$\int_0^y \beta(x)\,(1+x)^\alpha\, dx \le 2^{\alpha-1} \int_0^y \beta(x)\,(1+x^\alpha)\, dx$$

and consequently, for any $1 < \alpha \le \overline{\alpha}$,

$$\frac{y}{(1+y)^\alpha} \frac{\int_0^y \beta(x)\,(1+x)^\alpha\, dx}{\int_0^y x\beta(x)dx} \le \frac{1}{y^{\alpha-1}} \frac{2^{\alpha-1}\int_0^y \beta(x)\,(1+x^\alpha)\, dx}{\int_0^y x\beta(x)dx} \to 0 \quad (y \to +\infty)$$

and this ends the proof.

Remark 16. We note that any convex combination of conservative fragmentation kernels is a conservative fragmentation kernel so

$$b(x,y) = \sum_{j\in J} \lambda_j \beta_j(x) y \left(\int_0^y s\beta_j(s)ds \right)^{-1}, \; \left(\sum_{j\in J} \lambda_j = 1 \right) \tag{65}$$

(J finite or denumerable) is a conservative kernel and we can check the key conditions $\limsup_{a(y)\to+\infty} \frac{n_\alpha(y)}{y^\alpha} < 1$ or $\limsup_{a(y)\to+\infty} \frac{n_{1,\alpha}(y)}{(1+y)^\alpha} < 1$ more generally for (65) by using just the last two propositions above. We could also consider convex combinations of separable kernels and homogeneous ones.

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