First-order theories of bounded trees

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Abstract

A maximal chain in a tree is called a *path*, and a tree is called *bounded* when all its paths contain leaves. This paper concerns itself with first-order theories of bounded trees. We establish some sufficient conditions for the existence of bounded end-extensions that are also partial elementary extensions of a given tree. As an application of tree boundedness, we obtain a conditional axiomatisation of the first-order theory of the class of trees whose paths are all isomorphic to some ordinal $\alpha < \omega^\omega$, given the first-order theories of certain classes of bounded trees.

Keywords: bounded tree, ordinal tree, elementary substructure, end-extension, axiomatisation

Mathematics Subject Classification (2000): 03C07, 03C64, 06A06

1 Introduction

1.1 Motivation

Given a tree $\mathfrak{T} = (T; <)$, the substructure $(A; <\restriction_A)$ is called a *path* in $\mathfrak{T}$ when A is a maximal linearly ordered subset of T. A path is called *bounded* when it contains a greatest element (otherwise it is called *unbounded*) and a tree is called *bounded* when each of its paths is bounded. Bounded trees are of natural interest because every path in a bounded tree can be defined by a first-order formula using its leaf as parameter, in essence therefore permitting for quantification over paths.

A tree $\mathfrak{T}' = (T'; <_{\mathfrak{T}'})$ is called an *end-extension* of $\mathfrak{T} = (T; <_{\mathfrak{T}})$ when $\mathfrak{T}$ is a substructure of $\mathfrak{T}'$ and for each $a \in T$ and $b \in T'$, if $b <_{\mathfrak{T}'} a$ then $b \in T$. This paper is motivated by the problem of axiomatising the first-order theory of the class of bounded trees (although such an axiomatisation is not actually obtained here). Related results in the literature include (i) an explicit description of the class of trees that are k-equivalent (satisfy the same sentences of quantifier rank up to k) to the full binary tree of which each path has n nodes, given in [Doe89], and (ii) an axiomatisation of the first-order theory of the class of finite trees, given in [BRVS95]. In both of these cases, one encounters trees that are not bounded but are shown to be k-equivalent to bounded trees.

Our notion of an axiomatisation differs from the usual model theoretical notion, where a set of sentences Σ axiomatises a class of structures $\mathcal{K}$ if and only if each model of Σ is a member of $\mathcal{K}$. We have instead looked at the problem of axiomatising the *first-order theories* of those classes, not the classes themselves. The idea is this: given a class of structures $\mathcal{K}$ and a set of sentences Σ, Σ is said to *axiomatise* the first-order theory of $\mathcal{K}$ when (i) $\Sigma \subseteq \mathrm{Th}(\mathcal{K})$, (ii) Σ is a recursive set, and (iii) $\Sigma \models \mathrm{Th}(\mathcal{K})$. In turn, $\Sigma \models \mathrm{Th}(\mathcal{K})$ if and only if for each natural number n and each model $\mathfrak{A}$ of Σ, there exists a structure $\mathfrak{B}$ in $\mathcal{K}$ such that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of quantifier rank at most n. This notion of an axiomatisation is also used in [BRVS95], [Doe89], [Gor99], [GK] and [Sch77].

The following more general problem is also considered here: given a natural number k and a tree $\mathfrak{T}$ that is not bounded, does there exist a bounded tree $\mathfrak{T}'$ that is an end-extension of $\mathfrak{T}$ with $\mathfrak{T} \preceq_k \mathfrak{T}'$ (where $\preceq_k$ denotes the elementary substructure relation restricted to sentences of rank k)? Requiring of the tree $\mathfrak{T}'$ to be not just k-equivalent to $\mathfrak{T}$, but to be an end-extension of $\mathfrak{T}$ with $\mathfrak{T} \preceq_k \mathfrak{T}'$, is useful if one wants to preserve the structure of $\mathfrak{T}$ within $\mathfrak{T}'$. The problem further becomes relevant in the context of axiomatising the first-order theory of the class of trees whose paths are isomorphic to some fixed infinite successor ordinal. Such axiomatisations admit non-standard models that are not bounded and a natural approach to showing completeness of the axiomatisation is to augment the unbounded paths of such a non-standard model with suitable bounded forests. To this end, a conditional axiomatisation (which assumes axiomatisations of the first-order theories of certain classes of bounded trees) of the first-order theory of the class of trees whose paths are all isomorphic to a fixed ordinal $\alpha < \omega^\omega$ (which may be a successor ordinal or a limit ordinal), is obtained. The more fundamental problem of axiomatising the first-order theory of the class of bounded trees remains open however and is not settled in this paper.

For the sake of illustration, consider the following two examples, each of a tree $\mathfrak{T}$ that is not bounded which, in the first example, *can* be extended to a bounded tree $\mathfrak{T}'$ that is an end-extension of $\mathfrak{T}$ such that $\mathfrak{T} \preceq \mathfrak{T}'$, and in the second example, *cannot* be extended in this way.

Example 1 *Let $\mathfrak{T}' = (T'; <)$ be the tree that is minimal with respect to the property that each of its paths is isomorphic to the ordinal $\omega+1$ and each of its non-leaf nodes has exactly two immediate successors (so $\mathfrak{T}'$ is simply $2^{\leqslant\omega}$ with the usual tree structure). Since $\mathfrak{T}'$ has $2^{\aleph_0}$ many paths, and since each path has exactly one corresponding leaf node, the cardinality of $\mathfrak{T}'$ is at least $2^{\aleph_0}$. By the downwards Lőwenheim-Skolem theorem, $\mathfrak{T}'$ has a countably infinite elementary substructure $\mathfrak{T}$. Each non-leaf node in $\mathfrak{T}$ must have exactly two immediate successors so $\mathfrak{T}$ will differ from $\mathfrak{T}'$ only in the distribution of its leaf nodes. In particular, $\mathfrak{T}$ must contain paths that are isomorphic to ω, and $\mathfrak{T}'$ can be obtained from $\mathfrak{T}$ by extending each of these paths by a single node.*

Example 2 *Any unbounded linear order, viewed as a tree, clearly cannot have a bounded end-extension that is also an elementary extension of that linear order. More generally, any tree that has an unbounded definable path cannot have a bounded end-extension that is also an elementary extension of the given tree. As an example of a "proper" – one that is not a linear order – tree with this behaviour, let $\mathfrak{T} = (T; <)$ be the tree that consists of a copy of the ordinal ω with a copy of the ordinal $\omega + 1$ attached to each of its points, and let A be the path in $\mathfrak{T}$ that consists of all nodes that have two distinct immediate successors (see Fig. 1).*

Let $\varphi(x)$ be a first-order formula that states that x has at least two distinct immediate successors, let path_φ be a sentence which expresses that the formula $\varphi(x)$ defines a path (the sentence path_φ will be formally defined in Section 2.1), and let

$$\mu_\varphi = \mathsf{path}_\varphi \wedge \forall x\,(\varphi(x) \rightarrow \exists y\,(x < y \wedge \varphi(y)))$$

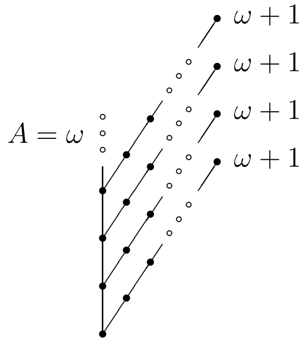

Fig. 1 The tree $\mathfrak{T}$ of Example 2.

which states that $\varphi(x)$ defines a path that has no greatest node. Since $\mathfrak{T} \models \mu_\varphi$ then if $\mathfrak{T}'$ is any tree that is elementarily equivalent to $\mathfrak{T}$, $\mathfrak{T}' \models \mu_\varphi$ so $\mathfrak{T}'$ will also contain an unbounded path. Therefore there is no bounded end-extension $\mathfrak{T}'$ of $\mathfrak{T}$ that is also an elementary extension of $\mathfrak{T}$.

1.2 Outline of the paper

The remainder of this paper is structured as follows. Subsection 1.3 (Notation and terminology) fixes logic related notation and terminology. Section 2 (Trees) recalls some definitions about trees and introduces various tree operations for which composition results are proved. Section 3 (Axiomatising the first-order theory of the class of α-trees) shows, for any ordinal $\alpha < \omega^\omega$, how axiomatisations of the first-order theories of certain classes of bounded trees can be used to axiomatise the first-order theory of the class of trees whose paths are all isomorphic to α. Section 4 (Definable sets of leaves) derives a general formula for defining certain sets of leaves and shows that at most one parameter is necessary for defining such sets. Section 5 (Approximations of boundedness) suggests several axiom schemes for formalising the property of tree boundedness, and examines how these schemes are related to each other. Section 6 (Bounded extensions of trees) obtains various sufficient conditions for the existence of bounded end-extensions $\mathfrak{T}'$ of trees $\mathfrak{T}$ that are not bounded but for which $\mathfrak{T} \preceq_k \mathfrak{T}'$, and Theorem 29 – the main result of the paper – gives a general construction for producing such $\mathfrak{T}'$. Section 7 (A counterexample) shows by way of a counterexample that a tree that is not bounded cannot in general be made into a k-equivalent bounded tree by the mere addition of missing leaves, even when that tree is a model of the first-order theory of the class of bounded trees; a more elaborate construction of the kind that is described in Theorem 29 is needed for this. Finally, directions for future research are suggested in Section 8 (Concluding remarks).

1.3 Notation and terminology

1.3.1 Tuples

For α any ordinal, $\bar{0}_\alpha$ and $\bar{1}_\alpha$ will denote the sequences $(0)_{i\in\alpha}$ and $(1)_{i\in\alpha}$ respectively, and $\bar{x}\bar{y}$ will denote the concatenation of the sequences $\bar{x}$ and $\bar{y}$. The sequence $(x_i)_{i\in\alpha}$ will also be written as $x_0x_1x_2\cdots$ and ϵ will denote the empty sequence which will be treated as a finite sequence of length 0. When dealing with a first-order formula $\varphi(x,\bar{y})$, it will be assumed that the tuple $\bar{y}$ may be empty, i.e. that φ may have the form $\varphi(x)$, unless otherwise stated or clear from the context.

1.3.2 Elementary equivalence and characteristic formulas

The quantifier rank of a formula φ is denoted $\mathrm{qr}(\varphi)$. Given structures $\mathfrak{A}$ and $\mathfrak{B}$ in the same signature, the notation $\mathfrak{A} \equiv_n \mathfrak{B}$ denotes that $\mathfrak{A}$ and $\mathfrak{B}$ are *n-equivalent*, i.e. that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same first-order sentences of quantifier rank at most n. The *rank* (see also [Sch77]) of a first-order formula φ is the sum of its quantifier rank and the number of free variables in φ. Given structures $\mathfrak{A}$ and $\mathfrak{B}$ over the same signature, $\mathfrak{B}$ is called a *k-extension* of $\mathfrak{A}$, and $\mathfrak{A}$ is called a *k-substructure* of $\mathfrak{B}$, denoted $\mathfrak{A} \preceq_k \mathfrak{B}$, when $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and any one of the following four equivalent[1] conditions hold:

(1) for each $\bar{a} \in A^m$ (with $m \geqslant 0$) and every formula $\varphi(\bar{x})$ (where $\bar{x}$ is an m-tuple of variables) of rank at most k,
$$\mathfrak{A} \models \varphi(\bar{a}) \iff \mathfrak{B} \models \varphi(\bar{a});$$

(2) for each $\bar{a} \in A^m$ (with $m \geqslant 0$) and every formula $\varphi(y, \bar{x})$ (where $\bar{x}$ is an m-tuple of variables) of rank at most k,
$$\mathfrak{A} \models \exists y\big(\varphi(y, \bar{a})\big) \iff \mathfrak{B} \models \exists y\big(\varphi(y, \bar{a})\big);$$

(3) for each $\bar{a} \in A^m$ (with $m \geqslant 0$) and every formula $\varphi(y, \bar{x})$ (where $\bar{x}$ is an m-tuple of variables) of rank at most k,
$$\mathfrak{B} \models \exists y\big(\varphi(y, \bar{a})\big) \implies \mathfrak{A} \models \exists y\big(\varphi(y, \bar{a})\big);$$

(4) for each $\bar{a} \in A^m$ (with $m \geqslant 0$) and every formula $\varphi(y, \bar{x})$ (where $\bar{x}$ is an m-tuple of variables) of rank at most k,
$$\mathfrak{B} \models \exists y\big(\varphi(y, \bar{a})\big) \implies \mathfrak{B} \models \varphi(d, \bar{a})$$
for some $d \in A$.

Observe that the relation $\preceq_k$ is transitive. Moreover, the following fact can be proved similarly to the Tarski-Vaught theorem on the union of elementary chains (see e.g. [Rot00, Theorem 10.1.1] for the elementary chain version of the Tarski-Vaught theorem):

Fact 3 *If $(\mathfrak{A}_i)_{i\in\gamma}$ (where γ is an ordinal) is a chain of structures over the same signature such that $\mathfrak{A}_i \preceq_k \mathfrak{A}_{i+1}$ for each i and $\mathfrak{A}_\delta = \bigcup_{i\in\delta} \mathfrak{A}_i$ for each limit ordinal $\delta < \gamma$, then $\mathfrak{A}_n \preceq_k \bigcup_{i\in\gamma} \mathfrak{A}_i$ for each $n \in \gamma$.*

The terminology and notation of [Doe89, Section 1.6] will be used when working with characteristic formulas. We briefly state the definition and main result about characteristic formulas that will be needed in the paper. Given a structure $\mathfrak{A}$, a natural number n, a tuple $\bar{a} = (a_0, a_1, \ldots, a_{k-1}) \in A^k$ and a tuple of variables $\bar{x} = (x_0, x_1, \ldots, x_{k-1})$, the *$n$-characteristic formula* of the structure $\mathfrak{A}$ over the tuple $\bar{a}$ is denoted as $[\![(\mathfrak{A}; \bar{a})]\!]^n(\bar{x})$ and is defined as follows:
$$[\![(\mathfrak{A}; \bar{a})]\!]^0(\bar{x}) = \bigwedge \big\{\varphi(\bar{x}) : \varphi \text{ an atomic or negated atomic formula with } \mathfrak{A} \models \varphi(\bar{a})\big\};$$
$$[\![(\mathfrak{A}; \bar{a})]\!]^{m+1}(\bar{x}) = \bigwedge\nolimits_{a_k\in A} \Big(\exists x_k [\![(\mathfrak{A}; \bar{a}a_k)]\!]^m(\bar{x}x_k)\Big) \wedge \forall x_k \Big(\bigvee\nolimits_{a_k\in A} [\![(\mathfrak{A}; \bar{a}a_k)]\!]^m(\bar{x}x_k)\Big).$$

[1] The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate and the implication (1) $\Rightarrow$ (4) is straightforward. The other implications can be proved similarly to the Tarski-Vaught criterion for elementary substructures: refer to [Hed04, Proposition 4.31] for the proof of the elementary substructure analogue of (3) $\Rightarrow$ (1); this proof can be modified to prove the implication (4) $\Rightarrow$ (1) as well.

For languages with finite relational signatures it can be shown that, for all natural numbers n and k, there are, up to logical equivalence, only finitely many n-characteristic formulas, taken over the class of all structures in that signature and all k-tuples in those structures. If $\bar{a}$ is the empty tuple then $[\![(\mathfrak{A};\bar{a})]\!]^n(\bar{x})$ is written as $[\![\mathfrak{A}]\!]^n$ and is called the *n-characteristic sentence* of $\mathfrak{A}$. The formula $[\![(\mathfrak{A};\bar{a})]\!]^n(\bar{x})$ has quantifier rank n and $\mathfrak{A} \models [\![(\mathfrak{A};\bar{a})]\!]^n(\bar{a})$. If $\mathfrak{B}$ is a structure in the signature of $\mathfrak{A}$ and $\bar{b}$ is a k-tuple of elements from $\mathfrak{B}$ then the following statements are equivalent for every natural number n:

(i) $(\mathfrak{A};\bar{a}) \equiv_n (\mathfrak{B};\bar{b})$;

(ii) $\mathfrak{B} \models [\![(\mathfrak{A};\bar{a})]\!]^n(\bar{b})$;

(iii) the formulas $[\![(\mathfrak{A};\bar{a})]\!]^n(\bar{x})$ and $[\![(\mathfrak{B};\bar{b})]\!]^n(\bar{x})$ are logically equivalent.

If $\chi_0, \chi_1, \ldots, \chi_p$ are, up to logical equivalence, all the n-characteristic sentences in some finite relational language, then the tuple $\bar{\chi} = (\chi_0, \chi_1, \ldots, \chi_p)$ will be called an *n-spectrum* in the language.

1.3.3 Ehrenfeucht-Fraïssé games

The reader is referred to [Doe96] for more information on Ehrenfeucht-Fraïssé games. The n-round Ehrenfeucht-Fraïssé game on structures $\mathfrak{A}$ and $\mathfrak{B}$ will be denoted $\mathrm{EF}(\mathfrak{A},\mathfrak{B},n)$ and will be treated as starting on round 1 (not round 0). If Σ is a winning strategy for Player II for the game $\mathrm{EF}(\mathfrak{A},\mathfrak{B},n)$ then $\mathrm{EF}(\mathfrak{A},\mathfrak{B},n,\Sigma)$ will denote the game $\mathrm{EF}(\mathfrak{A},\mathfrak{B},n)$ with Player II using the strategy Σ, and if the elements chosen during the first k rounds of $\mathrm{EF}(\mathfrak{A},\mathfrak{B},n,\Sigma)$ are $\bar{a} \in A^k$ and $\bar{b} \in B^k$, then $\mathrm{II}_{\mathfrak{A},\mathfrak{B},n,\Sigma}(\bar{a}a_k,\bar{b})$ (respectively $\mathrm{II}_{\mathfrak{A},\mathfrak{B},n,\Sigma}(\bar{a},\bar{b}b_k)$) will denote the response of Player II to Player I having chosen the element $a_k \in A$ (respectively $b_k \in B$) in round $k+1$ of the game $\mathrm{EF}(\mathfrak{A},\mathfrak{B},n,\Sigma)$.

1.3.4 Relativisations

Given formulas $\sigma(x)$ (σ may also be a sentence) and $\varphi(y,\bar{z})$ where $\bar{z}$ is an n-tuple of variables for some $n \geqslant 0$, the relativisation of σ to φ will be denoted by σ^φ. A detailed treatment of relativisations can be found in e.g. [Ros82, p. 259]; the relevant facts are briefly stated here. The formula $\sigma^\varphi = \sigma^\varphi(x,\bar{z})$ (or simply $\sigma^\varphi = \sigma^\varphi(\bar{z})$ when σ is a sentence) is defined recursively as follows:

$\sigma^\varphi = \sigma$ when σ is atomic;

$(\neg\sigma)^\varphi = \neg(\sigma^\varphi)$;

$(\sigma_1 \star \sigma_2)^\varphi = (\sigma_1^\varphi) \star (\sigma_2^\varphi)$ for $\star$ any of the connectives $\vee$, $\wedge$, $\rightarrow$, and $\leftrightarrow$;

$(\exists x(\sigma(x)))^\varphi = \exists x\,(\varphi(x,\bar{z}) \wedge \sigma(x))$;

$(\forall x(\sigma(x)))^\varphi = \forall x\,(\varphi(x,\bar{z}) \rightarrow \sigma(x))$.

Given a structure $\mathfrak{A}$ with underlying set A and an n-tuple $\bar{a}$ from A, define

$$A^{\varphi(y,\bar{a})} = \{u \in A : \mathfrak{A} \models \varphi(u,\bar{a})\}$$

to be the subset of A that is defined by φ using the parameters $\bar{a}$, and let $\mathfrak{A}^{\varphi(y,\bar{a})}$ be the substructure of $\mathfrak{A}$ that has underlying set $A^{\varphi(y,\bar{a})}$. It then holds for any $b \in A^{\varphi(y,\bar{a})}$ that

$$\mathfrak{A} \models \sigma^\varphi(b,\bar{a}) \iff \mathfrak{A}^{\varphi(y,\bar{a})} \models \sigma(b).$$

If φ is the formula $\varphi(y,z) = z \leqslant y$, with z here fulfilling the role of a parameter, then σ^φ will be written simply as $\sigma^{\geqslant z}$; the formulas $\varphi^{>z}$, $\varphi^{\leqslant z}$ and $\varphi^{<z}$ are to be similarly interpreted. If $\varphi(y,z_1,z_2) = z_1 \leqslant y < z_2$ (with z_1 and z_2 now fulfilling the roles of parameters) then σ^φ will be written as $\sigma^{[z_1,z_2)}$, and similarly for other bounded intervals.

2 Trees

2.1 Basic definitions

The simplest first-order language for trees has no constant symbols, two relation symbols (the usual equality symbol = and an order symbol <), and no function symbols; this language will be called the *language of trees*. Define the following formula:

$$x \smile y \;=\; (x < y \vee x = y \vee y < x)\,.$$

The expressions $x \leqslant y$ and $x < y < z$ will be shorthand for $x < y \vee x = y$ and $x < y \wedge y < z$ respectively. A *tree* is a non-empty structure $\mathfrak{T} = (T; <)$ that satisfies the following first-order sentences:

A1: $\forall x\, (\neg\, (x < x))$ (irreflexivity);

A2: $\forall x \forall y \forall z\, ((x < y \wedge y < z) \rightarrow x < z)$ (transitivity);

A3: $\forall x \forall y \forall z\, ((y < x \wedge z < x) \rightarrow y \smile z)$ (downwards linear);

A4: $\forall x \forall y \exists z\, (z \leqslant x \wedge z \leqslant y)$ (downwards connected).

Denote $\mathsf{Tree} = \{\mathsf{A1}, \mathsf{A2}, \mathsf{A3}, \mathsf{A4}\}$. If $\mathfrak{T}$ satisfies only $\mathsf{A1} - \mathsf{A3}$ then it is called a *forest*. The signature of $\mathfrak{T}$ may sometimes be enriched by adding to it a tuple $\bar{a}$ of elements from T that are treated as constants, a subset A of T that is treated as a unary relation, or a tuple $\bar{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n)$ where each $\mathbf{c}_i$ (called a *colour*) is a subset of T that is treated as a unary relation, to obtain structures of the form $(\mathfrak{T}; \bar{a})$, $(\mathfrak{T}; A)$ and $(\mathfrak{T}; \bar{\mathbf{c}})$ respectively. If additional symbols are included in the signature of $\mathfrak{T}$, the resulting structure will be called an *enriched tree*. The underlying set of $\mathfrak{T}$ will sometimes be denoted as $|\mathfrak{T}|$. When different structures – say $\mathfrak{T}$, $\mathfrak{S}$ and $\mathfrak{F}_i$ – occur together, it should be understood that $|\mathfrak{T}| = T$, $|\mathfrak{S}| = S$, $|\mathfrak{F}_i| = F_i$, and that $<_{\mathfrak{T}}$, $<_{\mathfrak{S}}$ and $<_{\mathfrak{F}_i}$ are the order relations of $\mathfrak{T}$, $\mathfrak{S}$ and $\mathfrak{F}_i$ respectively, etc. Given $G \subseteq T$, the substructure $(G; <\restriction_G)$ of $\mathfrak{T}$ will often be denoted simply as $(G; <)$.

The elements of T are called *nodes*. The least node of $\mathfrak{T}$, if it exists, is called the *root* of $\mathfrak{T}$, and will be denoted as $e_{\mathfrak{T}}$. A node a is called a *leaf* when a is maximal with respect to $<$. The set of all leaves in $\mathfrak{T}$ will be denoted as $L\,(\mathfrak{T})$. A node is a leaf if and only if it satisfies the formula

$$\mathsf{leaf}(x) \;=\; \forall y\, (\neg\, (x < y))\,.$$

A subset B of T is called a *barrier* in $\mathfrak{T}$ when B is an antichain in $\mathfrak{T}$ and, for each $a \in T$, there exists $b \in B$ such that $a \smile b$. The tree $\mathfrak{T}$ is called *downwards discrete* when, for all $a, b \in T$, if $a < b$ then there exists c with $a \leqslant c < b$ such that there is no $d \in T$ for which $c < d < b$. $\mathfrak{T}$ is called *upwards discrete* when, for all $a, b \in T$, if $a < b$ then there exists c with $a < c \leqslant b$ such that there is no $d \in T$ for which $a < d < c$.

Recall that $\mathfrak{L} = (L; <)$ is called a *path* in $\mathfrak{T}$ when L is a maximal linearly ordered subset of T. The path $\mathfrak{L}$ will be called *singular* when there exists $a \in L$ such that $\{x \in T : a \leqslant x\}$ forms a linear order. Observe that $\mathfrak{L}$ is a path in $\mathfrak{T}$ if and only if the following three conditions hold: (i) L is linearly ordered by $<$, and (ii) L is downwards closed in $\mathfrak{T}$ (i.e. if $x \in L$ and $y < x$ then $y \in L$), and (iii) L is not bounded above in $\mathfrak{T}$ (i.e. there is no $x \in T$ such that $y < x$ for each $y \in L$). For $\varphi(x, \bar{z})$ any formula, define the formula

$$\begin{aligned} \mathsf{path}_{\varphi}(\bar{z}) \;=\;& \exists x \big(\varphi\,(x, \bar{z})\,\big) \wedge \forall x \forall y \Big(\big(\varphi\,(x, \bar{z}) \wedge \varphi\,(y, \bar{z})\,\big) \rightarrow x \smile y\Big) \wedge \\ & \forall x \forall y \Big(\big(x < y \wedge \varphi\,(y, \bar{z})\,\big) \rightarrow \varphi\,(x, \bar{z})\,\Big) \wedge \neg \exists x \forall y \big(\varphi\,(y, \bar{z}) \rightarrow y < x\big). \end{aligned}$$

Observe that if $\bar{a}$ is a tuple consisting of nodes from T then $\varphi\,(x, \bar{a})$ defines a path in $(\mathfrak{T}; \bar{a})$ if and only if $(\mathfrak{T}; \bar{a}) \models \mathsf{path}_{\varphi}\,(\bar{a})$.

Given a path $\mathfrak{L} = (L; <)$ in $\mathfrak{T}$ and $A \subseteq L$, the substructure $(A; <)$ of $\mathfrak{T}$ is called a *stem* when A is downwards closed in $\mathfrak{T}$. As a notational convenience, the stem $(A; <)$ (which may itself be a path) will often be identified with the set A.

If $S \subseteq T$ then the structure $\mathfrak{S} = (S; <)$ is called a *subforest* (respectively, *subtree*) of $\mathfrak{T}$ when $\mathfrak{S}$ is a forest (respectively, tree) and every path in $\mathfrak{S}$ is an upwards closed subset of a path in $\mathfrak{T}$ (i.e. if A is a path in $\mathfrak{S}$, then there exists a path B in $\mathfrak{T}$ such that $A \subseteq B$ and if $x \in A$, $y \in B$ and $x < y$ then $y \in A$). The sentences $\mathsf{A1}$ - $\mathsf{A3}$ persist in any substructure of $\mathfrak{T}$ so it follows that, for any $S \subseteq T$, $\mathfrak{S} = (S; <)$ will be a subforest of $\mathfrak{T}$ if and only if every path in $\mathfrak{S}$ is an upwards closed subset of a path in $\mathfrak{T}$, and $\mathfrak{S}$ will be a subtree of $\mathfrak{T}$ if and only if $\mathfrak{S} \models \mathsf{A4}$ and every path in $\mathfrak{S}$ is an upwards closed subset of a path in $\mathfrak{T}$. For $\varphi(x, \bar{z})$ any formula, define the formula

$$\begin{aligned}\mathsf{sub}_{\varphi}(\bar{z}) \quad = \quad & \forall x \forall y \forall u \Big[\Big(\varphi(x, \bar{z}) \wedge \varphi(y, \bar{z}) \wedge x < u < y\Big) \rightarrow \varphi(u, \bar{z})\Big] \wedge \\ & \forall x \Big[\Big(\neg\varphi(x, \bar{z}) \wedge \exists y\big(y < x \wedge \varphi(y, \bar{z})\big)\Big) \rightarrow \\ & \quad \exists u\Big(\varphi(u, \bar{z}) \wedge \forall y\big((y < x \wedge \varphi(y, \bar{z})) \rightarrow y < u\big)\Big)\Big].\end{aligned}$$

It is straightforward to check that $(\mathfrak{T}; \bar{a}) \models \mathsf{sub}_{\varphi}(\bar{a})$ if and only if $\varphi(x, \bar{a})$ defines a subforest of $\mathfrak{T}$ in $(\mathfrak{T}; \bar{a})$.

Given nodes $a, b \in T$ with $a \leqslant b$, define the sets $a^{<} = \{x \in T : a < x\}$, $T_a = a^{\leqslant} = a^{<} \cup \{a\}$, $b^{>} = \{x \in T : x < b\}$, $b^{\geqslant} = b^{>} \cup \{b\}$, $T^b = T \setminus T_b$ and $T_a^b = T_a \cap T^b$. Bounded intervals will be denoted in the usual manner, e.g. $[a, b) = \{x \in T : a \leqslant x < b\}$. Define the trees $\mathfrak{T}_{>a} = (a^{<}; <)$, $\mathfrak{T}_a = (T_a; <)$, $\mathfrak{T}^b = \big(T^b; <\big)$, $\mathfrak{T}_a^b = \big(T_a^b; <\big)$, ${}_{\star}\mathfrak{T}^b = \big(\mathfrak{T}^b; b^{>}\big)$ and ${}_{\star}\mathfrak{T}_a^b = \big(\mathfrak{T}_a^b; [a, b)\big)$.

Given a linearly ordered subset L of T, $L < a$ will denote that $x < a$ for each $x \in L$, $L^{<}$ will denote the set $\{x \in T : L < x\}$ and $\mathfrak{T}_L$ will denote the tree $(L^{<}; <)$. For $u \in L$, define $\mathfrak{T}_u^L = (T_u \backslash L^{<}; <)$,

$$T_{u,L} = T_u \backslash \left(\bigcup_{x \in L \cap u^{<}} T_x \right)$$

and $\mathfrak{T}_{u,L} = (T_{u,L}; <, u)$. Similarly, for A any subset of T, $a < A$ will denote that $a < x$ for each $x \in A$ and $A^{>}$ will denote the set $\{x \in T : x < A\}$.

The *height* of a node x is taken as the order type of the linear order $(x^{>}; <)$.

2.2 Operations on trees

2.2.1 Sums of trees

Let $\mathfrak{T}$ be a (possibly enriched) tree, let $\mathcal{L} = \{L_i\}_{i \in I}$ be a non-empty set of stems in $\mathfrak{T}$, and let $\mathcal{F} = \{\mathfrak{F}_i\}_{i \in I}$ be a set of (possibly enriched) forests, where I is an index set for which $0 \notin I$. $\mathfrak{T} +_{\mathcal{L}} \mathcal{F}$ will denote the tree obtained from $\mathfrak{T}$ by adding the forest $\mathfrak{F}_i$ to the end of the stem L_i for each i, in such a way that the only nodes in $\mathfrak{T}$ that are comparable to nodes in $\mathfrak{F}_i$, are those in L_i. Formally, $\mathfrak{T} +_{\mathcal{L}} \mathcal{F}$ is defined as follows:

> $|\mathfrak{T} +_{\mathcal{L}} \mathcal{F}| = (T \times \{0\}) \cup \big(\bigcup_{i \in I} (F_i \times \{i\})\big)$ (i.e. the underlying set of $\mathfrak{T} +_{\mathcal{L}} \mathcal{F}$ is the disjoint union of the underlying set of $\mathfrak{T}$ and the underlying sets of all of the forests $\mathfrak{F}_i$);
>
> the order relation $<_{\mathfrak{T} +_{\mathcal{L}} \mathcal{F}}$ of $\mathfrak{T} +_{\mathcal{L}} \mathcal{F}$ is given by
>
> $$\begin{aligned}<_{\mathfrak{T} +_{\mathcal{L}} \mathcal{F}} \quad = \quad & \Big\{\big((x, 0), (y, 0)\big) : x <_{\mathfrak{T}} y\Big\} \cup \\ & \quad \Big(\bigcup\nolimits_{i \in I} \Big\{\big((x, i), (y, i)\big) : x <_{\mathfrak{F}_i} y\Big\}\Big) \cup \\ & \quad\quad \Big(\bigcup\nolimits_{i \in I} \Big\{\big((x, 0), (y, i)\big) : x \in L_i \text{ and } y \in F_i\Big\}\Big);\end{aligned}$$

for every constant a in $\mathfrak{T}$, the element $(a,0)$ occurs as a constant in $\mathfrak{T} +_{\mathcal{L}} \mathcal{F}$, and for every constant b in any forest $\mathfrak{F}_i$, the element (b,i) occurs as a constant in $\mathfrak{T} +_{\mathcal{L}} \mathcal{F}$;

for every relation A (including when A is a colour $\mathbf{c}_j$) in $\mathfrak{T}$, the set $A \times \{0\}$ occurs as a relation in $\mathfrak{T} +_{\mathcal{L}} \mathcal{F}$, and for every relation B (including when B is a colour $\mathbf{c}_j$) in any forest $\mathfrak{F}_i$, the set $B \times \{i\}$ occurs as a relation in $\mathfrak{T} +_{\mathcal{L}} \mathcal{F}$.

If, for some forest $\mathfrak{F}$, $\mathfrak{F}_i = \mathfrak{F}$ for each i, then $\mathfrak{T} +_{\mathcal{L}} \mathcal{F}$ will be denoted as $\mathfrak{T} +_{\mathcal{L}} \mathfrak{F}$, and if, in addition, $\mathcal{L} = \{L\}$, then $\mathfrak{T} +_{\mathcal{L}} \mathfrak{F}$ will be denoted simply as $\mathfrak{T} +_L \mathfrak{F}$. Given a stem M in $\mathfrak{F}$, and assuming without loss of generality that $I = \{1\}$, the set

$$N = (L \times \{0\}) \cup (M \times \{1\})$$

(i.e. the disjoint union of L and M in $\mathfrak{T} +_L \mathfrak{F}$) will sometimes be added as a unary relation to $\mathfrak{T} +_L \mathfrak{F}$, to obtain the tree $(\mathfrak{T} +_L \mathfrak{F}; N)$, which will be denoted as $\mathfrak{T} +_{L,M} \mathfrak{F}$. To keep the notation simple, the node $(a,0) \in |\mathfrak{T} +_L \mathfrak{F}|$ will often be identified with the node $a \in T$, and $(b,1) \in |\mathfrak{T} +_L \mathfrak{F}|$ will often be identified with $b \in F$.

The following composition result can be proved by a straightforward application of an Ehrenfeucht-Fraïssé game.

Lemma 4 *Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be trees that are possibly enriched, and L_1 and L_2 be stems in $\mathfrak{T}_1$ and $\mathfrak{T}_2$ respectively such that $(\mathfrak{T}_1; L_1) \equiv_n (\mathfrak{T}_2; L_2)$. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be forests that are possibly enriched and M_1 and M_2 be stems in $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively.*

1. *If $\mathfrak{F}_1 \equiv_n \mathfrak{F}_2$ then $\mathfrak{T}_1 +_{L_1} \mathfrak{F}_1 \equiv_n \mathfrak{T}_2 +_{L_2} \mathfrak{F}_2$.*
2. *If $(\mathfrak{F}_1; M_1) \equiv_n (\mathfrak{F}_2; M_2)$ then $\mathfrak{T}_1 +_{L_1,M_1} \mathfrak{F}_1 \equiv_n \mathfrak{T}_2 +_{L_2,M_2} \mathfrak{F}_2$.*

2.2.2 Multiples of trees

Now suppose that $\mathfrak{T} = (T; <_{\mathfrak{T}})$ is a non-enriched tree. We will describe two types of tree multiples, namely $\mathfrak{T} \times_L \mathfrak{A}$, where L is a stem in $\mathfrak{T}$ and $\mathfrak{A} = (A; <_{\mathfrak{A}})$ is a linear order, and $\mathfrak{T} \times_{\mathcal{L}} \alpha$, where $\mathcal{L} = \{L_i\}_{i \in I}$ is a set of stems in $\mathfrak{T}$ and α is an ordinal.

The tree $\mathfrak{T} \times_L \mathfrak{A}$ consists of a copy of $\mathfrak{T}$ for each element of A, attached to each other at the end of the stem L. Formally, $\mathfrak{T} \times_L \mathfrak{A} = (|\mathfrak{T} \times_L \mathfrak{A}|; <_{\mathfrak{T} \times_L \mathfrak{A}})$ has underlying set $|\mathfrak{T} \times_L \mathfrak{A}| = T \times A$ and order relation

$$<_{\mathfrak{T} \times_L \mathfrak{A}} = \bigcup_{x \in A} \Big\{ \big((u,x), (v,x) \big) : u <_{\mathfrak{T}} v \Big\} \cup \Big\{ \big((u,x), (v,y) \big) : u \in L \text{ and } x <_{\mathfrak{A}} y \Big\}.$$

The tree $(\mathfrak{T} \times_L \mathfrak{A}; L \times A)$ that is obtained by enriching $\mathfrak{T} \times_L \mathfrak{A}$ with the unary predicate $L \times A$ will be denoted as $\mathfrak{T} \times'_L \mathfrak{A}$.

The following composition result can again be proved using a straightforward application of an Ehrenfeucht-Fraïssé game.

Lemma 5 *Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be non-enriched trees, L_1 and L_2 be stems in $\mathfrak{T}_1$ and $\mathfrak{T}_2$ respectively, and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be linear orders, such that $(\mathfrak{T}_1; L_1) \equiv_n (\mathfrak{T}_2; L_2)$ and $\mathfrak{A}_1 \equiv_n \mathfrak{A}_2$. Then $\mathfrak{T}_1 \times'_{L_1} \mathfrak{A}_1 \equiv_n \mathfrak{T}_2 \times'_{L_2} \mathfrak{A}_2$ (hence also $\mathfrak{T}_1 \times_{L_1} \mathfrak{A}_1 \equiv_n \mathfrak{T}_2 \times_{L_2} \mathfrak{A}_2$).*

Next, the tree $\mathfrak{T} \times_{\mathcal{L}} \alpha$ is a generalisation of $\mathfrak{T} \times_L \mathfrak{A}$ in the case where $\mathfrak{A}$ is an ordinal. Let $X = I^{<\alpha}$ denote the set of sequences of elements of I of length less than α (including the empty tuple ϵ) and for $\bar{x}, \bar{y} \in X$, let $\bar{x} < \bar{y}$ when $\bar{x}$ is a proper initial subsequence of $\bar{y}$ (with $\epsilon < \bar{y}$ for each $\bar{y} \in X$). Let $\ell(\bar{x})$ denote the

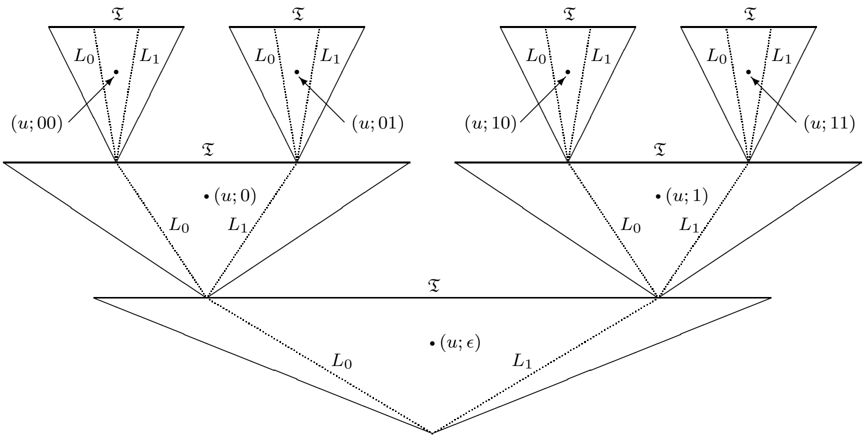

Fig. 2 A depiction of the tree $\mathfrak{T}\times_{\mathcal{L}}\alpha$ with $\mathcal{L}=\{L_0,L_1\}$ and $\alpha=3$. The forms $(u,\bar{x})$ that nodes in different parts of the tree take, are shown.

ordinal length of the sequence $\bar{x}$ (with $\ell(\epsilon)=0$), and let $\pi_j(\bar{x})$ denote the usual projection onto the coordinate of $\bar{x}$ that has index j (where the first component of a non-empty tuple has index 0).

Then $\mathfrak{T}\times_{\mathcal{L}}\alpha=(|\mathfrak{T}\times_{\mathcal{L}}\alpha|\,;<_{\mathfrak{T}\times_{\mathcal{L}}\alpha})$ has underlying set $|\mathfrak{T}\times_{\mathcal{L}}\alpha|=T\times X$ and order relation

$$
\begin{aligned}
<_{\mathfrak{T}\times_{\mathcal{L}}\alpha} \quad = \quad & \bigcup_{\bar{x}\in X}\left\{\Big((u,\bar{x}),(v,\bar{x})\Big) : u<_{\mathfrak{T}} v\right\} \;\cup \\
& \left\{\Big((u,\bar{x}),(v,\bar{y})\Big) : \bar{x}<\bar{y} \text{ and } u\in L_k \text{ where } k=\pi_{\ell(\bar{x})}(\bar{y})\right\}.
\end{aligned}
$$

For an example, a depiction of the tree $\mathfrak{T}\times_{\mathcal{L}}\alpha$ when $\mathcal{L}=\{L_0,L_1\}$ and $\alpha=3$ is shown in Fig. 2.

The following lemma will be used to prove Lemma 7, which in turn will be needed in the proof of Proposition 35.

Lemma 6 *Let $\mathfrak{T}=(T;<_{\mathfrak{T}})$ and $\mathfrak{S}=(S;<_{\mathfrak{S}})$ be trees with roots $e_{\mathfrak{T}}$ and $e_{\mathfrak{S}}$ respectively and such that $\mathfrak{T}\equiv_n\mathfrak{S}$. Let I be the set of leaves in $\mathfrak{T}$, J be the set of leaves in $\mathfrak{S}$, $\mathcal{A}=\left\{i^{\geqslant_{\mathfrak{T}}}\right\}_{i\in I}$, $\mathcal{B}=\left\{j^{\geqslant_{\mathfrak{S}}}\right\}_{j\in J}$, $\mathfrak{T}'=\mathfrak{T}\times_{\mathcal{A}}\omega$ and $\mathfrak{S}'=\mathfrak{S}\times_{\mathcal{B}}\omega$. Then $\mathfrak{T}'\equiv_n\mathfrak{S}'$.*

Proof For $\bar{x}$ any finite tuple over I and $p'=(p,\bar{y})$ any node in T', define the node $(p')^{\bar{x}}\in T$ as follows:

$$
\left(p'\right)^{\bar{x}}=\begin{cases} p & \text{when } \bar{x}=\bar{y},\\ i & \text{when } \bar{x}<\bar{y}, \text{ where } i \text{ is the leaf in } \mathfrak{T} \text{ for which } (i,\bar{x})<_{\mathfrak{T}'}p',\\ e_{\mathfrak{T}} & \text{otherwise}\end{cases}
$$

(informally, $(p')^{\bar{x}}$ is the node a in $\mathfrak{T}$ for which $(a,\bar{x})$ is the node in $\mathfrak{T}'$ that is closest to p').

If $\bar{p}=\left(p_0',p_1',\ldots,p_{r-1}'\right)\in(T')^r$ then define the r-tuple $(\bar{p})^{\bar{x}}\in T^r$ as $(\bar{p})^{\bar{x}}=\left((p_0')^{\bar{x}},(p_1')^{\bar{x}},\ldots,\left(p_{r-1}'\right)^{\bar{x}}\right)$. For $q'\in S'$, $\bar{q}\in(S')^r$ and $\bar{x}$ a finite tuple over J, the node $(q')^{\bar{x}}\in S$ and tuple $(\bar{q})^{\bar{x}}\in S^r$ are defined similarly.

Let Σ be a winning strategy for Player II for the game $\mathrm{EF}(\mathfrak{T},\mathfrak{S},n)$. We describe a winning strategy Θ for Player II for the game $\mathrm{EF}(\mathfrak{T}',\mathfrak{S}',n)$ recursively. The idea behind it is that whenever Player I chooses a node $(u,\bar{x})$ from $\mathfrak{T}'$, Player

II chooses a node $(v, \bar{y})$ from $\mathfrak{S}'$ where first the components of $\bar{y}$ are obtained from those of $\bar{x}$ using the strategy Σ, and then v is chosen also using the strategy Σ, and similarly when Player I instead chooses a node from $\mathfrak{S}'$.

Hence let $0 \leqslant m \leqslant n-1$ and suppose that the nodes that have been chosen by the two players after m rounds of $\mathrm{EF}(\mathfrak{T}, \mathfrak{S}, n, \Theta)$ are given by the tuples $\bar{t} = (t_0, t_1, \ldots, t_{m-1}) \in (T')^m$ and $\bar{s} = (s_0, s_1, \ldots, s_{m-1}) \in (S')^m$, and assume that $\bar{t}$ and $\bar{s}$ form a local isomorphism between $\mathfrak{T}'$ and $\mathfrak{S}'$. Suppose that, for his $(m+1)$-th move, Player I chooses the node $t_m = (u, \bar{x})$ from $\mathfrak{T}'$, where $u \in T$ and $\bar{x} = (x_0, x_1, \ldots, x_{\beta-1})$ is some sequence over I (the argument where he instead chooses a node from $\mathfrak{S}'$ is similar). The response of Player II will be the node $s_m = (v, \bar{y})$ from $\mathfrak{S}'$, with v and $\bar{y}$ described below.

If $\bar{x} = \epsilon$ then take $\bar{y} = \epsilon$ and

$$v = \mathrm{II}_{\mathfrak{T},\mathfrak{S},n,\Sigma}\left((\bar{t})^{\epsilon}\, u, (\bar{s})^{\epsilon}\right).$$

Otherwise let $\bar{x}_0 = \epsilon$ and $\bar{x}_r = (x_0, x_1, \ldots, x_{r-1})$ for $1 \leqslant r \leqslant \beta$. Let $\bar{y}_0 = \epsilon$ and define, for $1 \leqslant r \leqslant \beta$, the r-tuple $\bar{y}_r = (y_0, y_1, \ldots, y_{r-1})$ over J recursively as follows: given $\bar{y}_r$, where $0 \leqslant r < \beta$, let

$$y_r = \mathrm{II}_{\mathfrak{T},\mathfrak{S},n,\Sigma}\left((\bar{t}t_m)^{\bar{x}_r}, (\bar{s})^{\bar{y}_r}\right).$$

Observe that y_r is a leaf in $\mathfrak{S}$ and take $\bar{y}_{r+1} = \bar{y}_r y_r$. Finally, take $\bar{y} = \bar{y}_\beta$ and

$$v = \mathrm{II}_{\mathfrak{T},\mathfrak{S},n,\Sigma}\left((\bar{t})^{\bar{x}}\, u, (\bar{s})^{\bar{y}}\right).$$

It remains to show that the tuples $\bar{t}t_m$ and $\bar{s}s_m$ form a local isomorphism between $\mathfrak{T}'$ and $\mathfrak{S}'$. To this end, it is readily verified that for each r with $0 \leqslant r \leqslant m-1$, $t_m = t_r \Leftrightarrow s_m = s_r$, and $t_m <_{\mathfrak{T}'} t_r \Leftrightarrow s_m <_{\mathfrak{S}'} s_r$, and $t_r <_{\mathfrak{T}'} t_m \Leftrightarrow s_r <_{\mathfrak{S}'} s_m$, as required. ⊣

Lemma 7 *Let $\mathfrak{T} = (T; <_{\mathfrak{T}})$ and $\mathfrak{S} = (S; <_{\mathfrak{S}})$ be trees with roots $e_{\mathfrak{T}}$ and $e_{\mathfrak{S}}$ respectively such that $\mathfrak{T} \equiv_n \mathfrak{S}$, and let $\mathcal{A}$, $\mathcal{B}$, $\mathfrak{T}'$ and $\mathfrak{S}'$ be defined as in Lemma 6. Let $\mathcal{A}' = \{X_k\}_{k \in K}$ be the set of paths X_k in $\mathfrak{T}'$ for which X_k contains an infinite number of nodes of the form $(e_{\mathfrak{T}}, \bar{x})$, and let $\mathcal{B}' = \{Y_l\}_{l \in L}$ be the set of paths Y_l in $\mathfrak{S}'$ for which Y_l contains an infinite number of nodes of the form $(e_{\mathfrak{S}}, \bar{x})$ (informally each X_k, respectively Y_l, is an infinite disjoint union of paths from $\mathcal{A}$, respectively $\mathcal{B}$). Let $\mathfrak{D}$ be the tree with a single node d, $\mathfrak{T}'' = \mathfrak{T}' +_{\mathcal{A}'} \mathfrak{D}$ and $\mathfrak{S}'' = \mathfrak{S}' +_{\mathcal{B}'} \mathfrak{D}$. Then $\mathfrak{T}'' \equiv_n \mathfrak{S}''$.*

Proof For $\bar{x}$ any finite tuple over I and $w'' \in T''$, define the node $(w'')^{\bar{x}} \in T$ as follows:

If w'' has the form $w'' = (w', 0)$ (where $w' \in T'$) then $(w'')^{\bar{x}} = (w')^{\bar{x}}$ where $(w')^{\bar{x}}$ is defined as in the proof of Lemma 6.

If w'' has the form $w'' = (d, k)$ for some $k \in K$ then

$$(w'')^{\bar{x}} = \begin{cases} i & \text{when } ((e_{\mathfrak{T}}, \bar{x}), 0) <_{\mathfrak{T}''} w'', \text{ where } i \text{ is the leaf in } \mathfrak{T} \\ & \qquad\qquad \text{for which } ((i, \bar{x}), 0) <_{\mathfrak{T}''} w'', \\ e_{\mathfrak{T}} & \text{when } ((e_{\mathfrak{T}}, \bar{x}), 0) \not<_{\mathfrak{T}''} w''. \end{cases}$$

If $\bar{p} = \left(p''_0, p''_1, \ldots, p''_{r-1}\right) \in (T'')^r$ then define the r-tuple $(\bar{p})^{\bar{x}} \in T^r$ as $(\bar{p})^{\bar{x}} = \left((p''_0)^{\bar{x}}, (p''_1)^{\bar{x}}, \ldots, (p''_{r-1})^{\bar{x}}\right)$. For $q'' \in S''$, $\bar{q} \in (S'')^r$ and $\bar{x}$ a finite tuple over J, the node $(q'')^{\bar{x}} \in S$ and tuple $(\bar{q})^{\bar{x}} \in S^r$ are defined similarly.

Let Σ be a winning strategy for the game $\mathrm{EF}(\mathfrak{T}, \mathfrak{S}, n)$. We will now describe a winning strategy Θ for Player II for the game $\mathrm{EF}(\mathfrak{T}'', \mathfrak{S}'', n)$.

Let $0 \leqslant m \leqslant n-1$ and suppose that the nodes that were chosen by the two players after m rounds of the game $\mathrm{EF}(\mathfrak{T}, \mathfrak{S}, n, \Theta)$ are given by the tuples $\bar{t} = (t_0, t_1, \ldots, t_{m-1}) \in (T'')^m$ and $\bar{s} = (s_0, s_1, \ldots, s_{m-1}) \in (S'')^m$, and assume that $\bar{t}$ and $\bar{s}$ form a local isomorphism between $\mathfrak{T}''$ and $\mathfrak{S}''$. Suppose that, for his $(m+1)$-th move, Player I chooses the node t_m from $\mathfrak{T}''$ (the argument where he instead chooses a node from $\mathfrak{S}''$ is similar). Two cases are distinguished:

Case 1: t_m is not a leaf in $\mathfrak{T}''$, i.e. t_m has the form $t_m = ((u, \bar{x}), 0)$ for some $u \in T$ and where $\bar{x}$ is a finite sequence over I. The response of Player II will be the node $s_m = ((v, \bar{y}), 0) \in S''$ where v and $\bar{y}$ are obtained as in the proof of Lemma 6 but using the operation $(\cdot)^{\bar{x}}$ as defined in the larger structures $\mathfrak{T}''$ and $\mathfrak{S}''$ in this proof.

Case 2: t_m is a leaf in $\mathfrak{T}''$, i.e. t_m has the form $t_m = (d, k)$ for some $k \in K$. Let $((e_{\mathfrak{T}}, \bar{x}_r))_{r \in \omega}$ be a sequence of nodes in the path X_k in $\mathfrak{T}'$ such that $\ell(\bar{x}_r) = r$ for each r. For each $r \in \omega$, define $\bar{y}_r$ as in the proof of Lemma 6, i.e. $\bar{y}_0 = \epsilon$ and given $\bar{y}_r$ with $r \geqslant 0$, obtain $\bar{y}_{r+1}$ by taking

$$y_r = \mathrm{II}_{\mathfrak{T}, \mathfrak{S}, n, \Sigma}\left((\bar{t}t_m)^{\bar{x}_r}, (\bar{s})^{\bar{y}_r}\right)$$

(with the operation $(\cdot)^{\bar{x}}$ as defined in the larger structures $\mathfrak{T}''$ and $\mathfrak{S}''$) and $\bar{y}_{r+1} = \bar{y}_r y_r$. Let Y_l be the path in $\mathfrak{S}'$ that contains the sequence of nodes $((e_{\mathfrak{S}}, \bar{y}_r))_{r \in \omega}$. Finally, take $s_m = (d, l) \in S''$.

As in the proof of Lemma 6, it is readily verified that the tuples $\bar{t}t_m$ and $\bar{s}s_m$ form a local isomorphism between $\mathfrak{T}''$ and $\mathfrak{S}''$. $\dashv$

2.3 Leaf coloured trees

A tree $\mathfrak{T}$ that is enriched with colours $\bar{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p)$ will be called *leaf coloured* when, for each node u in T that is not a leaf, $\mathfrak{T} \not\models \mathbf{c}_i(u)$ for each colour $\mathbf{c}_i$, while for each leaf t in $\mathfrak{T}$, there is exactly one colour $\mathbf{c}_i$ such that $\mathfrak{T} \models \mathbf{c}_i(t)$.

Lemma 8 *Let $(\mathfrak{T}; \bar{\mathbf{c}})$ and $(\mathfrak{S}; \bar{\mathbf{c}})$ be leaf coloured trees (where $\mathfrak{T}$ and $\mathfrak{S}$ may themselves be enriched) such that $(\mathfrak{T}; \bar{\mathbf{c}}) \equiv_n (\mathfrak{S}; \bar{\mathbf{c}})$. Let $\mathcal{A} = \{t^{\geqslant}\}_{t \in L(\mathfrak{T})}$ and $\mathcal{B} = \{s^{\geqslant}\}_{s \in L(\mathfrak{S})}$ and let $\mathcal{F} = \{\mathfrak{F}_t\}_{t \in L(\mathfrak{T})}$ and $\mathcal{G} = \{\mathfrak{G}_s\}_{s \in L(\mathfrak{S})}$ be sets of (possibly enriched) forests such that, for each colour $\mathbf{c}_k$ in $\bar{\mathbf{c}}$, $\mathfrak{F}_t \equiv_n \mathfrak{G}_s$ if and only if $\mathfrak{T} \models \mathbf{c}_k(t)$ and $\mathfrak{S} \models \mathbf{c}_k(s)$. Then $\mathfrak{T} +_{\mathcal{A}} \mathcal{F} \equiv_n \mathfrak{S} +_{\mathcal{B}} \mathcal{G}$.*

Proof Let Σ be a winning strategy for Player II for the game $\mathrm{EF}(\mathfrak{T}, \mathfrak{S}, n)$, and given $\mathfrak{F}_t$ and $\mathfrak{G}_s$ for which $\mathfrak{F}_t \equiv_n \mathfrak{G}_s$, let $\Theta_{t,s}$ be a winning strategy for Player II for the game $\mathrm{EF}(\mathfrak{F}_t, \mathfrak{G}_s, n)$. We will describe a winning strategy for Player II for the game $\mathrm{EF}(\mathfrak{T} +_{\mathcal{A}} \mathcal{F}, \mathfrak{S} +_{\mathcal{B}} \mathcal{G}, n)$.

Let the nodes chosen by the players from the two trees after k rounds (with $0 \leqslant k < n$) be given by the tuples $\bar{a} = (a_0, a_1, \ldots, a_{k-1})$ and $\bar{b} = (b_0, b_1, \ldots, b_{k-1})$ respectively (where $\bar{a} = \epsilon = \bar{b}$ when $k = 0$). For each a_m and b_m, define

$$a_m^{\mathfrak{T}} = \begin{cases} a'_m & \text{when } a_m \text{ has the form } a_m = (a'_m, 0) \\ t & \text{when } a_m \text{ has the form } a_m = (a'_m, t) \end{cases}$$

and

$$b_m^{\mathfrak{S}} = \begin{cases} b'_m & \text{when } b_m \text{ has the form } b_m = (b'_m, 0) \\ s & \text{when } b_m \text{ has the form } b_m = (b'_m, s) \end{cases}$$

and for t a leaf in $\mathfrak{T}$ and s a leaf in $\mathfrak{S}$, define

$$a_m^{\mathfrak{F}_t} = \begin{cases} a'_m & \text{when } a_m \text{ has the form } a_m = (a'_m, t) \\ e_{\mathfrak{F}_t} & \text{otherwise} \end{cases}$$

and

$$b_m^{\mathfrak{G}_s} = \begin{cases} b'_m & \text{when } b_m \text{ has the form } b_m = (b'_m, s) \\ e_{\mathfrak{G}_s} & \text{otherwise} \end{cases}$$

and define the tuples $\bar{a}^{\mathfrak{T}} = \left(a_0^{\mathfrak{T}}, a_1^{\mathfrak{T}}, \ldots, a_{k-1}^{\mathfrak{T}}\right)$, $\bar{b}^{\mathfrak{S}} = \left(b_0^{\mathfrak{S}}, b_1^{\mathfrak{S}}, \ldots, b_{k-1}^{\mathfrak{S}}\right)$, $\bar{a}^{\mathfrak{F}_t} = \left(a_0^{\mathfrak{F}_t}, a_1^{\mathfrak{F}_t}, \ldots, a_{k-1}^{\mathfrak{F}_t}\right)$ and $\bar{b}^{\mathfrak{G}_s} = \left(b_0^{\mathfrak{G}_s}, b_1^{\mathfrak{G}_s}, \ldots, b_{k-1}^{\mathfrak{G}_s}\right)$.

Now suppose that, for his $(k+1)$-th move, Player I chooses the node a_k from $\mathfrak{T} +_{\mathcal{A}} \mathcal{F}$ (the argument where he instead chooses a node from $\mathfrak{S} +_{\mathcal{B}} \mathcal{G}$ is similar).

First consider the case where a_k has the form $a_k = (a'_k, 0)$. Player II then chooses the node $b_k = (b'_k, 0)$ as her response, where $b'_k = \mathrm{II}_{\mathfrak{T},\mathfrak{S},n,\Sigma}\left(\bar{a}^{\mathfrak{T}} a'_k, \bar{b}^{\mathfrak{S}}\right)$.

Next consider the case where a_k has the form $a_k = (a'_k, t)$ for some leaf t in $\mathfrak{T}$. Player II then responds with the node $b_k = (b'_k, s)$, where $s = \mathrm{II}_{\mathfrak{T},\mathfrak{S},n,\Sigma}\left(\bar{a}^{\mathfrak{T}} t, \bar{b}^{\mathfrak{S}}\right)$ and $b'_k = \mathrm{II}_{\mathfrak{F}_t,\mathfrak{G}_s,n,\Theta_{t,s}}\left(\bar{a}^{\mathfrak{F}_t} a'_k, \bar{b}^{\mathfrak{G}_s}\right)$.

It is straightforward to check that the pair of tuples $(a_0, a_1, \ldots, a_{n-1})$ and $(b_0, b_1, \ldots, b_{n-1})$ that are eventually obtained in this manner form a local isomorphism between the trees $\mathfrak{T} +_{\mathcal{A}} \mathcal{F}$ and $\mathfrak{S} +_{\mathcal{B}} \mathcal{G}$ hence $\mathfrak{T} +_{\mathcal{A}} \mathcal{F} \equiv_n \mathfrak{S} +_{\mathcal{B}} \mathcal{G}$. $\dashv$

Corollary 9 *Let $\mathfrak{T}$ and $\mathfrak{S}$ be trees, possibly enriched, such that $\mathfrak{T} \equiv_n \mathfrak{S}$. Let $\mathcal{A} = \left\{t^{\geqslant}\right\}_{t \in L(\mathfrak{T})}$ and $\mathcal{B} = \left\{s^{\geqslant}\right\}_{s \in L(\mathfrak{S})}$ and let $\mathfrak{F}$ and $\mathfrak{G}$ forests, possibly enriched, such that $\mathfrak{F} \equiv_n \mathfrak{G}$. Then $\mathfrak{T} +_{\mathcal{A}} \mathfrak{F} \equiv_n \mathfrak{S} +_{\mathcal{B}} \mathfrak{G}$.*

Proof Expand $\mathfrak{T}$ and $\mathfrak{S}$ into leaf coloured trees $(\mathfrak{T}; \mathbf{c})$ and $(\mathfrak{S}; \mathbf{c})$ by setting $\mathfrak{T} \models \mathbf{c}(t)$ and $\mathfrak{S} \models \mathbf{c}(s)$ for all leaves t in $\mathfrak{T}$ and all leaves s in $\mathfrak{S}$. Let $\mathfrak{F}_t = \mathfrak{F}$ and $\mathfrak{G}_s = \mathfrak{G}$ for all leaves t in $\mathfrak{T}$ and all leaves s in $\mathfrak{S}$, and apply Lemma 8. $\dashv$

Suppose that $\bar{\chi} = (\chi_1, \chi_2, \ldots, \chi_p)$ is an n-spectrum (defined at the end of Section 1.3.2) in the language of trees. Given a non-enriched tree $\mathfrak{T}$ and a formula $\varphi(x)$ for which T^{φ} is a barrier in $\mathfrak{T}$, let $\varphi{\downarrow}(x)$ be the formula

$$\varphi{\downarrow}(x) = \exists y\,(x \leqslant y \wedge \varphi(y))$$

and define the leaf coloured tree

$$\mathfrak{T}_{\varphi,\bar{\chi}} = \left(\mathfrak{T}^{\varphi\downarrow}; \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p\right)$$

by specifying, for each colour $\mathbf{c}_i$ and each $t \in T^{\varphi}$, that $\mathfrak{T}_{\varphi,\bar{\chi}} \models \mathbf{c}_i(t)$ if and only if $\mathfrak{T}_{>t} \models \chi_i$. Note that the leaves in $\mathfrak{T}_{\varphi,\bar{\chi}}$ are precisely the nodes in T^{φ}.

Lemma 10 *Let $\bar{\chi} = (\chi_1, \chi_2, \ldots, \chi_p)$ be an n-spectrum in the language of trees. Let $\mathfrak{T}$ and $\mathfrak{S}$ be non-enriched trees and $\varphi(x)$ a formula of quantifier rank k such that T^{φ} and S^{φ} are barriers in $\mathfrak{T}$ and $\mathfrak{S}$ respectively, and let $m = n + \max\{n, k\} + 1$. If $\mathfrak{T} \equiv_m \mathfrak{S}$ then $\mathfrak{T}_{\varphi,\bar{\chi}} \equiv_n \mathfrak{S}_{\varphi,\bar{\chi}}$.*

Proof Let Σ be a winning strategy for Player II for the game $\mathrm{EF}(\mathfrak{T}, \mathfrak{S}, m)$. Observe that, for each i with $0 \leqslant i < n$, if $\bar{t} = (t_0, t_1, \ldots, t_{i-1})$ and $\bar{s} = (s_0, s_1, \ldots, s_{i-1})$ (with $\bar{t} = \epsilon = \bar{s}$ when $i = 0$) are tuples of nodes from $\mathfrak{T}$ and $\mathfrak{S}$ respectively that have been played during the first i rounds of the game $\mathrm{EF}(\mathfrak{T}, \mathfrak{S}, m, \Sigma)$, and if Player I chooses the node t_i in $\mathfrak{T}$ (respectively s_i in $\mathfrak{S}$) in round $i+1$ of the game $\mathrm{EF}(\mathfrak{T}, \mathfrak{S}, m, \Sigma)$ and Player II responds with the node s_i in $\mathfrak{S}$ (respectively t_i in $\mathfrak{T}$), then the following must hold:

(i) $(\mathfrak{T}; \bar{t}t_i) \equiv_{m-i-1} (\mathfrak{S}; \bar{s}s_i)$ (by properties of Ehrenfeucht-Fraïssé games) hence

(ii) $t_i \in T^{\varphi\downarrow}$ if and only if $s_i \in S^{\varphi\downarrow}$ (this is because the sets $T^{\varphi\downarrow}$ and $S^{\varphi\downarrow}$ are definable in $\mathfrak{T}$ and $\mathfrak{S}$ respectively using the formula $\varphi{\downarrow}(x)$, with $\mathrm{qr}(\varphi{\downarrow}) = k + 1 \leqslant m - i - 1$), and

(iii) $\left(t_i \in T^{\varphi} \text{ and } \mathfrak{T}_{\varphi,\bar{\chi}} \models \mathbf{c}_j\left(t_i\right)\right)$ if and only if $\left(s_i \in S^{\varphi} \text{ and } \mathfrak{S}_{\varphi,\bar{\chi}} \models \mathbf{c}_j\left(s_i\right)\right)$ (this is because the set of leaves in $\mathfrak{T}_{\varphi,\bar{\chi}}$, respectively $\mathfrak{S}_{\varphi,\bar{\chi}}$, that have the colour $\mathbf{c}_j$, can be defined in $\mathfrak{T}$, respectively $\mathfrak{S}$, using the formula $\psi(x) = \varphi(x) \wedge \chi_j^{>x}$, with $\operatorname{qr}(\psi) = \max\{k, n\} \leqslant m - i - 1$).

We describe a winning strategy for Player II in the game $\mathrm{EF}\left(\mathfrak{T}_{\varphi,\bar{\chi}}, \mathfrak{S}_{\varphi,\bar{\chi}}, n\right)$. Let $0 \leqslant i < n$ and suppose that $\bar{a} = (a_0, a_1, \ldots, a_{i-1})$ and $\bar{b} = (b_0, b_1, \ldots, b_{i-1})$ are the tuples of nodes from $\mathfrak{T}_{\varphi,\bar{\chi}}$ and $\mathfrak{S}_{\varphi,\bar{\chi}}$ respectively that have been played during the first i rounds of the game, and such that

(†) $\bar{a}$ and $\bar{b}$ are also tuples that are generated in some instance of the game $\mathrm{EF}(\mathfrak{T}, \mathfrak{S}, m, \Sigma)$ after i rounds

(this latter condition is needed to ensure that the strategy Σ can be applied to tuples that are generated in the game $\mathrm{EF}\left(\mathfrak{T}_{\varphi,\bar{\chi}}, \mathfrak{S}_{\varphi,\bar{\chi}}, n\right)$, and that properties (i) – (iii) above can be applied to these tuples).

Suppose that Player I chooses a_i from $\mathfrak{T}_{\varphi,\bar{\chi}}$ (respectively b_i from $\mathfrak{S}_{\varphi,\bar{\chi}}$) in round $i+1$ of $\mathrm{EF}\left(\mathfrak{T}_{\varphi,\bar{\chi}}, \mathfrak{S}_{\varphi,\bar{\chi}}, n\right)$. Then Player II responds with $b_i = \mathrm{II}_{\mathfrak{T},\mathfrak{S},m,\Sigma}\left(\bar{a}a_i, \bar{b}\right)$ from $\mathfrak{S}_{\varphi,\bar{\chi}}$ (respectively $a_i = \mathrm{II}_{\mathfrak{T},\mathfrak{S},m,\Sigma}\left(\bar{a}, \bar{b}b_i\right)$ from $\mathfrak{T}_{\varphi,\bar{\chi}}$) as her move in round $i+1$ of $\mathrm{EF}\left(\mathfrak{T}_{\varphi,\bar{\chi}}, \mathfrak{S}_{\varphi,\bar{\chi}}, n\right)$. That this response of Player II is indeed a node in $\mathfrak{S}_{\varphi,\bar{\chi}}$ (respectively $\mathfrak{T}_{\varphi,\bar{\chi}}$) follows from property (ii) above, while properties (i) and (iii) ensure that the tuples $\bar{a}a_i$ and $\bar{b}b_i$ form a local isomorphism between $\mathfrak{T}_{\varphi,\bar{\chi}}$ and $\mathfrak{S}_{\varphi,\bar{\chi}}$, from which it follows that $\mathfrak{T}_{\varphi,\bar{\chi}} \equiv_{i+1} \mathfrak{S}_{\varphi,\bar{\chi}}$ (in particular, $\mathfrak{T}_{\varphi,\bar{\chi}} \equiv_n \mathfrak{S}_{\varphi,\bar{\chi}}$ when $i = n-1$). Finally, condition (†) clearly holds for the tuples $\bar{a}a_i$ and $\bar{b}b_i$ in place of $\bar{a}$ and $\bar{b}$. $\dashv$

2.4 Other composition results

If a and b are any nodes in a tree $\mathfrak{T}$ then the *ramification point* of a and b is the greatest lower bound (if it exists) of the set $\{a, b\}$, and $\mathfrak{T}$ is called *ramification complete* when the ramification point of any two nodes from $\mathfrak{T}$ exists.

Lemmas 11 – 13 below can all be proved by the straightforward use of Ehrenfeucht-Fraïssé games.

Lemma 11 *Let $\mathfrak{T} = (T; <_{\mathfrak{T}})$ and $\mathfrak{S} = (S; <_{\mathfrak{S}})$ be ramification complete trees and let L and M be paths in $\mathfrak{T}$ and $\mathfrak{S}$ respectively. For each $u \in L$ and $v \in M$, define the sets*

$$T'_u = T_u \backslash \left(\bigcup_{x \in L \cap u^{<_{\mathfrak{T}}}} T_x \right) \text{ and } S'_v = S_v \backslash \left(\bigcup_{x \in M \cap v^{<_{\mathfrak{S}}}} S_x \right)$$

and the trees $\mathfrak{T}'_u = (T'_u; <_{\mathfrak{T}}, u)$ and $\mathfrak{S}'_v = (S'_v; <_{\mathfrak{S}}, v)$. If $(L; <_{\mathfrak{T}}) \equiv_n (M; <_{\mathfrak{S}})$ and, for each $u \in L$ and $v \in M$, $\mathfrak{T}'_u \equiv_n \mathfrak{S}'_v$, then $(\mathfrak{T}; L) \equiv_n (\mathfrak{S}; M)$.

Observe that, in Lemma 11, the ramification completeness of $\mathfrak{T}$ and $\mathfrak{S}$ ensures that the sets $\{T'_u : u \in L\}$ and $\{S'_v : v \in M\}$ are partitions of T and S respectively.

Lemma 12 *Let $\mathfrak{T} = (T; <_{\mathfrak{T}})$ and $\mathfrak{S} = (S; <_{\mathfrak{S}})$ be trees, α an ordinal with $\alpha \leqslant \omega$ and $(a_i)_{i \in \alpha}$ and $(b_i)_{i \in \alpha}$ be increasing (but not necessarily cofinal) sequences in T and S respectively. Let $A = \bigcup_{i \in \alpha} [a_0, a_i)$ and $B = \bigcup_{i \in \alpha} [b_0, b_i)$ and define the trees*

$$\mathfrak{T}' = \left(T_{a_0} \backslash A^{<}; <_{\mathfrak{T}}, A\right) \text{ and } \mathfrak{S}' = \left(S_{b_0} \backslash B^{<}; <_{\mathfrak{S}}, B\right).$$

If ${}_\star\mathfrak{T}_{a_i}^{a_{i+1}} \equiv_n {}_\star\mathfrak{S}_{b_i}^{b_{i+1}}$ for each i for which $i, i+1 \in \alpha$ then $\mathfrak{T}' \equiv_n \mathfrak{S}'$ (hence also $(T_{a_0} \backslash A^{<}; <_{\mathfrak{T}}) \equiv_n (S_{b_0} \backslash B^{<}; <_{\mathfrak{S}})$).

Lemma 13 *Let $\mathfrak{T} = (T;<)$ be a tree, M be a stem in $\mathfrak{T}$, and $A \subseteq T \backslash M$ be an antichain. For each $x \in A$, let $\mathfrak{F}_x$ be a tree such that $\mathfrak{F}_x \equiv_n \mathfrak{T}_x$ and let $L_x = x^{>}$. Let $T' = T \backslash \left(\bigcup_{x \in A} T_x\right)$, $\mathfrak{T}' = (T';<)$, $\mathcal{L} = \{L_x\}_{x \in A}$, $\mathcal{F} = \{\mathfrak{F}_x\}_{x \in A}$, and $\mathfrak{S} = \mathfrak{T}' +_{\mathcal{L}} \mathcal{F}$ (informally $\mathfrak{S}$ is the tree that is obtained from $\mathfrak{T}$ by replacing each of the subtrees $\mathfrak{T}_x$ of $\mathfrak{T}$ with the tree $\mathfrak{F}_x$). Then $(\mathfrak{T}; M) \equiv_n (\mathfrak{S}; M)$.*

3 Axiomatising the first-order theory of the class of α-trees

In this section it will be seen how the property of boundedness can be used when axiomatising the first-order theory of the class of trees of which each path is isomorphic to some ordinal α. A more general notion of boundedness will be needed for this. First, given a formula $\varphi(x, \bar{z})$ where $\bar{z}$ is a k-tuple of variables for some $k \geqslant 0$, again define the formula $\varphi{\downarrow}(x, \bar{z})$ by

$$\varphi{\downarrow}(x, \bar{z}) = \exists y \left(x \leqslant y \wedge \varphi\left(y, \bar{z}\right)\right).$$

Then a tree $\mathfrak{T} = (T;<)$ is *φ-bounded* when for each tuple $\bar{a} \in T^k$ for which $T^{\varphi(x,\bar{a})} \neq \emptyset$, the tree $\mathfrak{T}^{\varphi\downarrow(x,\bar{a})}$ is bounded.

Using the terminology from [GK10], call a forest (respectively tree) $\mathfrak{F}$ an *α-forest* (respectively *α-tree*), where α is some ordinal, when each path in $\mathfrak{F}$ is isomorphic to α. It will be shown below how the first-order theory of the class of α-trees, where α is any ordinal with $\alpha < \omega^\omega$, can be axiomatised (using a set of sentences that will be denoted as Ψ_α) when given a suitable axiomatisation of the first-order theory of the class of φ-bounded trees for certain formulas $\varphi\left(x, \bar{z}\right)$.

To this end, it will be assumed that an axiomatisation $\mathsf{Tree} \cup \mathsf{B}_\varphi$ of the first-order theory of the class of φ-bounded trees (with B_φ having the role of approximating φ-boundedness) is known, i.e.

> $\mathsf{BD1}(\varphi)$: For each first-order formula $\varphi\left(x, \bar{z}\right)$ for which any φ-bounded tree exists, there exists a set of sentences B_φ such that if $\mathfrak{T} = (T; <_{\mathfrak{T}})$ is a model of $\mathsf{Tree} \cup \mathsf{B}_\varphi$ then for each natural number n, there exists a φ-bounded tree $\mathfrak{S} = (S; <_{\mathfrak{S}})$ such that $\mathfrak{T} \equiv_n \mathfrak{S}$.

Moreover, since we will be axiomatising classes of α-trees for ordinal α, the tree $\mathfrak{S}$ will need to be well-founded, so it will be assumed that B_φ respects well-foundedness in the following sense:

> $\mathsf{BD2}(\varphi)$: With reference to the property $\mathsf{BD1}(\varphi)$, if the tree $\mathfrak{T}$ is well-founded then a well-founded such tree $\mathfrak{S}$ exists.

Taking $\iota\left(x\right)$ to be the formula $x = x$, ι-boundedness coincides with boundedness. The scheme B_ι will be denoted simply as B.

We do not know whether, for arbitrary φ, such an axiom scheme B_φ *that respects well-foundedness*, necessarily exists. However, for our purposes, less is actually needed. Firstly, we will make use of the scheme B_φ only for φ equal to the formulas ι, θ_n (to be defined below), and η_β (also to be defined below). Secondly, any axiomatisation of the first-order theory of the class of well-founded trees that are φ-bounded, will suffice in place of the scheme $\mathsf{Tree} \cup \mathsf{B}_\varphi \cup \mathsf{WF}$ that will be used in Theorems 15, 17 and 18 below.

However, we opt to use the scheme B_φ, with the assumptions $\mathsf{BD1}(\varphi)$ and $\mathsf{BD2}(\varphi)$ as above, since it may be easier to find an axiomatisation of the first-order theory of the class of φ-bounded trees that happens to also respect well-foundedness, than to find an axiomatisation of the first-order theory of the class of trees that are both well-founded and φ-bounded. In proving the completeness of an axiomatisation of the first-order theory of the class of φ-bounded trees, it may well happen that the construction for producing a standard model from a non-standard one, copies and pastes parts of the original tree in such a way that well-foundedness is maintained.

In general, B_φ may be infinite and contain sentences of arbitrarily large quantifier rank but observe that, for given n, there exists a finite subset B'_φ of B_φ with the property that every model of $\mathsf{Tree} \cup \mathsf{B}'_\varphi$ is n-equivalent to some φ-bounded tree. For let $\tau_1, \tau_2, \ldots, \tau_k$ be (up to logical equivalence) all n-characteristic sentences that hold in φ-bounded trees and let $\tau = \bigvee_{i=1}^k \tau_i$. Then $\mathsf{Tree} \cup \mathsf{B}_\varphi \models \tau$ hence by the compactness theorem for first-order logic, $\mathsf{Tree} \cup \mathsf{B}'_\varphi \models \tau$ for some finite $\mathsf{B}'_\varphi \subseteq \mathsf{B}_\varphi$, and the claim follows.

In the remainder of this section, $f_\varphi(n)$, or simply $f(n)$ when φ is the formula ι above, will denote the maximum of the quantifier ranks of all sentences in such B'_φ. The set B'_φ can therefore simply be taken to consist of all sentences in B_φ of quantifier rank at most $f_\varphi(n)$. This set B'_φ will be denoted as $\mathsf{B}_{\varphi, f_\varphi(n)}$, or as $\mathsf{B}_{f(n)}$ in the case where φ is the formula ι above. Hence assuming $\mathsf{BD1}(\varphi)$ and $\mathsf{BD2}(\varphi)$ then for each natural number n and each tree $\mathfrak{T}$, $\mathfrak{T} \models \mathsf{B}_{\varphi, f_\varphi(n)}$ if and only if there exists a well-founded and φ-bounded tree $\mathfrak{S}$ such that $\mathfrak{T} \equiv_n \mathfrak{S}$. Without loss of generality, it may be assumed that $f_\varphi(n) \geqslant n$ for each formula φ.

For α some ordinal, a linear order $\mathfrak{L}$ will be called *α-like* when $\mathfrak{L} \equiv \alpha$. It is known (see e.g. [Ros82]) that for each ordinal α with $\alpha < \omega^\omega$, there exists a first-order sentence Φ_α that admits as its class of models precisely the α-like linear orders, and α is the only well-ordered model of Φ_α. Moreover, the ω-like linear orders are precisely all linear orders of the form $\omega + \zeta \cdot \gamma$, where ζ is the order type of the integers and γ is any linear order, and the ω^{n+1}-like linear orders (with $n \geqslant 1$) are precisely all linear orders that are ω^n-like sums of ω-like linear orders, equivalently, ω-like sums of ω^n-like linear orders. Finally, if α has Cantor normal form

$$\alpha = \omega^{n_1} \cdot a_1 + \omega^{n_2} \cdot a_2 + \ldots + \omega^{n_k} \cdot a_k$$

where $n_1 > n_2 > \ldots > n_k$ and $a_i \neq 0$ for all i, then the α-like linear orders are precisely all linear orders of the form

$$\left(W_1^{n_1} + \ldots + W_{a_1}^{n_1}\right) + \left(W_1^{n_2} + \ldots + W_{a_2}^{n_2}\right) + \ldots + \left(W_1^{n_k} + \ldots + W_{a_k}^{n_k}\right)$$

where each $W_j^{n_i}$ is an ω^{n_i}-like linear order.

A tree $\mathfrak{T}$ is called *definably well-founded* when every parametrically definable non-empty set of nodes in T contains a minimal element. The property of being definably well-founded can be formalised using the scheme WF that consists of the sentences

$$\forall \bar{z}\Big(\exists x\big(\varphi(x, \bar{z})\big) \rightarrow \exists x\Big(\varphi(x, \bar{z}) \wedge \forall y\big(\left(\varphi(y, \bar{z}) \wedge y \leqslant x\right) \rightarrow y = x\big)\Big)\Big)$$

for each formula $\varphi(x, \bar{z})$. The following result is proved in [Doe89].

Fact 14 *Let $\mathfrak{T}$ be a definably well-founded tree. For each natural number n there exists a well-founded tree $\mathfrak{S}$ such that $\mathfrak{S} \equiv_n \mathfrak{T}$.*

Theorem 15 *Assume $\mathsf{BD1}(\iota)$ and $\mathsf{BD2}(\iota)$. Let α be a successor ordinal with $\alpha < \omega^\omega$. The first-order theory of the class of α-trees can be axiomatised by the theory*

$$\Psi_\alpha = \mathsf{Tree} \cup \mathsf{B} \cup \Big\{\forall x\Big(\mathsf{leaf}(x) \rightarrow \Phi_\alpha^{\leqslant x}\Big)\Big\} \cup \mathsf{WF}.$$

Proof Pick $n \geqslant \mathrm{qr}(\Phi_\alpha) + 1$ and let $k = f(n)$. Given a model $\mathfrak{T}$ of Ψ_α, there exists a well-founded tree $\mathfrak{S}'$ such that $\mathfrak{S}' \equiv_k \mathfrak{T}$, and since $\mathfrak{S}' \models \mathsf{B}_k$ then there exists a bounded and well-founded tree $\mathfrak{S}$ such that $\mathfrak{S} \equiv_n \mathfrak{S}'$ hence also $\mathfrak{S} \equiv_n \mathfrak{T}$.

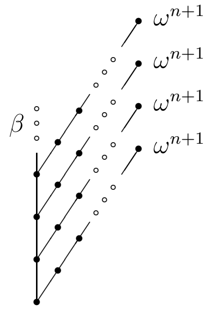

Fig. 3 The example referred to in the proof of Theorem 17.

Then $\mathfrak{S} \models \forall x \left(\mathsf{leaf}\,(x) \to \Phi_{\alpha}^{\leqslant x}\right)$ so each path in $\mathfrak{S}$ is elementarily equivalent to α. Since $\mathfrak{S}$ is well-founded then each of its paths is in fact isomorphic to α, as required. ⊣

The properties of a tree having a root, being downwards discrete, and being upwards discrete, can all be expressed by first-order sentences of quantifier rank at most 4. Let root, ddiscr and udiscr be such first-order sentences that express respectively the existence of a root, downwards discreteness and upwards discreteness. The following result is taken from [GK]:

Fact 16 *The first-order theory of the class of ω-trees can be axiomatised by the theory*

$$\Psi_{\omega} = \mathsf{Tree} \cup \{\mathsf{root}, \neg\exists x\,(\mathsf{leaf}\,(x))\,, \mathsf{ddiscr}, \mathsf{udiscr}\} \cup \mathsf{WF}.$$

Proof Let $n \geqslant 4$. Given a model $\mathfrak{T}$ of Ψ_{ω}, there exists a well-founded tree $\mathfrak{S}$ such that $\mathfrak{S} \equiv_n \mathfrak{T}$. $\mathfrak{S}$ will satisfy the sentences root, $\neg\exists x\,(\mathsf{leaf}\,(x))$, ddiscr and udiscr hence each of its paths will contain a least element but no greatest element and will be downwards and upwards discrete, from which it follows that each of its paths will be isomorphic to ω, as required. ⊣

We next derive an axiomatisation $\Psi_{\omega^{n+1}}$ of the first-order theory of the class of ω^{n+1}-trees. A few formulas that will be needed for this, must first be introduced. First,

$$\Psi'_{\omega} = \mathsf{root} \wedge \neg\exists x\,(\mathsf{leaf}\,(x)) \wedge \mathsf{ddiscr} \wedge \mathsf{udiscr}.$$

Next, the formula $\mathsf{lim}_n\,(x)$, which will be denoted simply as $\mathsf{lim}\,(x)$ when $n = 1$, expresses that x is an n-limit node, and is defined recursively as follows:

$$\begin{aligned} \mathsf{lim}_1\,(x) \;=\; \mathsf{lim}\,(x) \;&=\; \forall y \big(y < x \to \exists z\,(y < z < x)\big) \quad \text{and} \\ \mathsf{lim}_{n+1}\,(x) \;&=\; \mathsf{lim}^{\,\mathsf{lim}_n}\,(x) \end{aligned}$$

($\mathsf{lim}^{\,\mathsf{lim}_n}\,(x)$ is the relativisation of lim to lim_n). The sentence dom_n below expresses that each node is dominated by some n-limit node:

$$\mathsf{dom}_n = \forall x \exists y\,(x < y \wedge \mathsf{lim}_n\,(y))\,.$$

The formula $\theta_n\,(x, z)$ expresses that x is a minimal n-limit node greater than z in the subtree $\mathfrak{T}_z$:

$$\theta_n\,(x, z) = z < x \wedge \mathsf{lim}_n\,(x) \wedge \neg\exists y\,(z < y < x \wedge \mathsf{lim}_n\,(y))\,.$$

Theorem 17 *Assume* $\mathsf{BD1}(\theta_n)$ *and* $\mathsf{BD2}(\theta_n)$ *for each* $n \geqslant 1$. *The first-order theory of the class of* ω^{n+1}*-trees (where* $n \geqslant 1$*) can be axiomatised by the theory*

$$\Psi_{\omega^{n+1}} = \mathsf{Tree} \cup \left\{ \left(\Psi'_{\omega}\right)^{\mathsf{lim}_n}, \, \mathsf{dom}_n \right\} \cup \mathsf{B}_{\theta_n} \cup \mathsf{WF}.$$

Proof Let $\mathfrak{T} = (T; <_{\mathfrak{T}})$ be a model of $\Psi_{\omega^{n+1}}$ and let k be a positive integer. It will be shown that there exists an ω^{n+1}-tree $\mathfrak{S}$ such that $\mathfrak{S} \equiv_k \mathfrak{T}$. Without loss of generality, assume that

$$k \geqslant \max \left\{ \mathrm{qr}\left(\left(\Psi'_\omega \right)^{\mathsf{lim}_n} \right), \mathrm{qr}\left(\mathsf{dom}_n \right) \right\}.$$

By Fact 14 there exists a well-founded tree $\mathfrak{W}$ such that $\mathfrak{W} \equiv_{f_{\theta_n}(k)} \mathfrak{T}$. Since $\mathfrak{W} \models \mathsf{B}_{\theta_n, f_{\theta_n}(k)}$ then there exists a well-founded θ_n-bounded tree $\mathfrak{S}$ such that $\mathfrak{W} \equiv_k \mathfrak{S}$.

In particular, $\mathfrak{S} \models (\Psi'_\omega)^{\mathsf{lim}_n}$ hence $\mathfrak{S}^{\mathsf{lim}_n} \models \Psi'_\omega$ from which it follows that $\mathfrak{S}^{\mathsf{lim}_n}$ is an ω-tree. Since $\mathfrak{S}$ is well-founded then for $u, v \in S^{\mathsf{lim}_n}$ such that v is an immediate successor to u in $\mathfrak{S}^{\mathsf{lim}_n}$, $([u, v); <_{\mathfrak{S}}) \cong \omega^n$. Since $\mathfrak{S} \models \mathsf{dom}_n$ then every node in S is actually contained in some such interval $[u, v)$. It cannot yet be concluded that $\mathfrak{S}$ is an ω^{n+1}-tree however: $\mathfrak{S}$ may contain paths of the form β where $\beta < \omega^{n+1}$ is any limit ordinal, an example of which is shown in Fig. 3.

Hence let $\mathfrak{L} = (L; <_{\mathfrak{S}})$ be any path in $\mathfrak{S}$ and suppose (for a contradiction) that $\mathfrak{L} \not\cong \omega^{n+1}$. Since $\mathfrak{S} \models \mathsf{dom}_n$, $\mathfrak{L}$ must be unbounded hence $\mathfrak{L}$ must have the form $\omega^n \cdot i + \alpha$ for some natural number i and some limit ordinal $\alpha \leqslant \omega^n$. Let $a \in L$ be the node of which the height in $\mathfrak{S}$ is $\omega^n \cdot i$. Observe that $\mathfrak{L}_a \cong \alpha$ hence $\mathfrak{L}_a$ is an unbounded path in $\mathfrak{S}^{\theta \downarrow (x,a)}$. Since $\mathfrak{S}$ is θ_n-bounded however, the tree $\mathfrak{S}^{\theta \downarrow (x,a)}$ is bounded, a contradiction. This completes the proof. $\dashv$

Finally, we consider the case where α is a limit ordinal that is not a power of ω. Such α can be written in the form $\alpha = \beta + \omega^n$ for some positive integer n. Let

$$\begin{aligned} \eta_\beta(x) &= \Phi_\beta^{<x} \quad \text{and} \\ \mathsf{bar}_\beta &= \forall x \exists y \left(x \smile y \wedge \eta_\beta(y) \right) \end{aligned}$$

which respectively express in a well-founded tree that the height of x is β, and that the set of nodes of which the height is β, forms a barrier.

Theorem 18 *Assume* $\mathsf{BD1}(\eta_\beta)$ *and* $\mathsf{BD2}(\eta_\beta)$. *Let* $\alpha = \beta + \omega^n < \omega^\omega$ *be an ordinal that is not a power of* ω. *The first-order theory of the class of* α*-trees can be axiomatised by the theory*

$$\Psi_\alpha = \mathsf{Tree} \cup \{\mathsf{bar}_\beta\} \cup \mathsf{WF} \cup \mathsf{B}_{\eta_\beta} \cup \left\{ \forall x \left(\eta_\beta(x) \to \sigma^{\geqslant x} \right) : \sigma \in \Psi_{\omega^n} \right\}.$$

Proof Let $\mathfrak{T} = (T; <_{\mathfrak{T}})$ be a model of Ψ_α. Fix a natural number k. It will be shown that $\mathfrak{T} \equiv_k \mathfrak{S}$ for some α-tree $\mathfrak{S}$.

From $\mathfrak{T} \models \left\{ \forall x \left(\eta_\beta(x) \to \sigma^{\geqslant x} \right) : \sigma \in \Psi_{\omega^n} \right\}$ follows that, for each node $u \in T$ of height β, there exists an ω^n-tree $\mathfrak{C}(u) = \left(C(u); <_{\mathfrak{C}(u)} \right)$ with $\mathfrak{C}(u) \equiv_{k+1} \mathfrak{T}_u$. Observe that the structure $\mathfrak{C}(u)' = \left(C(u) \setminus \{u\}; <_{\mathfrak{C}(u)} \right)$ will then be an ω^n-forest and $\mathfrak{C}(u)' \equiv_k \mathfrak{T}_{>u}$. Let $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_p$ be representatives of all the k-equivalence classes of such forests $\mathfrak{C}(u)'$ and let $\chi_1, \chi_2, \ldots, \chi_p$ be k-characteristic sentences of $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_p$ respectively. Let $\bar{\chi}$ be a k-spectrum in the language of trees, say $\bar{\chi} = (\chi_1, \chi_2, \ldots, \chi_p, \ldots, \chi_q)$ with $q \geqslant p$. Let

$$m = k + \max\{k, \mathrm{qr}(\eta_\beta)\} + 1.$$

By Fact 14 there exists a well-founded tree $\mathfrak{W}$ such that $\mathfrak{T} \equiv_{f_{\eta_\beta}(m)} \mathfrak{W}$. Then $\mathfrak{W} \models \mathsf{B}_{\eta_\beta, f_{\eta_\beta}(m)}$ hence there exists a well-founded η_β-bounded tree $\mathfrak{V}$ such that $\mathfrak{W} \equiv_m \mathfrak{V}$. Since $\mathrm{qr}(\mathsf{bar}_\beta) = \mathrm{qr}(\eta_\beta) + 2 \leqslant m$ and $\mathfrak{T} \models \mathsf{bar}_\beta$ and $\mathfrak{T} \equiv_m \mathfrak{W} \equiv_m \mathfrak{V}$, it follows that W^{η_β} and V^{η_β} are barriers in $\mathfrak{W}$ and $\mathfrak{V}$ respectively. Let $\mathfrak{W}' = \mathfrak{W}_{\eta_\beta, \bar{\chi}}$ and $\mathfrak{V}' = \mathfrak{V}_{\eta_\beta, \bar{\chi}}$. By Lemma 10, $\mathfrak{W}' \equiv_k \mathfrak{V}'$.

The tree $\mathfrak{S}$ is constructed as follows. Let $\mathcal{A} = \left\{ u^{\geqslant \mathfrak{V}'} \right\}_{u \in L(\mathfrak{V}')}$ and for each leaf u in $\mathfrak{V}'$, take $\mathfrak{F}_u$ to be the forest $\mathfrak{C}_i$, where i is that integer for which $\mathfrak{V}' \models \mathbf{c}_i(u)$.

Let $\mathcal{F} = \{\mathfrak{F}_u\}_{u \in L(\mathfrak{V}')}$. Take $\mathfrak{S}$ to be the tree $\mathfrak{S} = \mathfrak{V}' +_{\mathcal{A}} \mathcal{F}$ and observe that $\mathfrak{S}$ is an α-tree and that for each leaf u in $\mathfrak{V}'$ and each colour $\mathbf{c}_i$, $\mathfrak{V}' \models \mathbf{c}_i(u)$ if and only if $\mathfrak{F}_u \models \chi_i$.

Now observe that $\mathfrak{W}$ can be expressed in the form $\mathfrak{W} \cong \mathfrak{W}' +_{\mathcal{B}} \mathcal{G}$ where $\mathcal{B} = \left\{v^{\geqslant \mathfrak{W}'}\right\}_{v \in L(\mathfrak{W}')}$ and $\mathcal{G} = \{\mathfrak{G}_v\}_{v \in L(\mathfrak{W}')}$ with $\mathfrak{G}_v = \mathfrak{W}_{>v}$ for each leaf v in $\mathfrak{W}'$. Moreover, for each leaf v in $\mathfrak{W}'$ and each colour $\mathbf{c}_i$, $\mathfrak{W}' \models \mathbf{c}_i(v)$ if and only if $\mathfrak{G}_v \models \chi_i$.

Using Lemma 8, it follows that $\mathfrak{S} \equiv_k \mathfrak{W} \equiv_k \mathfrak{T}$, as required. $\dashv$

4 Definable sets of leaves

In this section we will briefly investigate formulas that define leaves. The following fact is proved in [Kel15, Lemma 4.1]:

Fact 19 *Let $\mathfrak{T} = (T; <)$ be a tree. Let $\bar{a} = (a_0, a_1, \ldots, a_k)$ be a tuple of nodes from T such that $a_k \not< a_i$ for $i = 0, 1, \ldots, k-1$ and let $b, c \in T$ be any nodes such that $a_k \leqslant b, c$. The following statements are equivalent for each natural number n:*

(i) $(\mathfrak{T}; b, \bar{a}) \equiv_n (\mathfrak{T}; c, \bar{a})$;

(ii) $(\mathfrak{T}; b, a_k) \equiv_n (\mathfrak{T}; c, a_k)$;

(iii) $(\mathfrak{T}_{a_k}; b) \equiv_n (\mathfrak{T}_{a_k}; c)$.

Fact 19 will be used to show that certain definable sets can be defined using at most one parameter.

Proposition 20 *Let $\mathfrak{T} = (T; <)$ be a tree and let $A \subseteq T$ be defined in $(\mathfrak{T}; \bar{c})$ by a formula $\varphi(x, \bar{c})$ of quantifier rank n for some tuple $\bar{c} = (c_0, c_1, \ldots, c_{j-1})$. Suppose that the set $B = A^{>} \setminus \left(\bigcup_{i=0}^{j-1} c_i^{>}\right)$ is non-empty. Let $\{a_i\}_{i \in I}$ be a finite subset of A with the property that, for each $a \in A$ and $b \in B$, there exists a_i such that $(\mathfrak{T}; b, a) \equiv_n (\mathfrak{T}; b, a_i)$.*[2]

Then for each $b \in B$, A is defined in $(\mathfrak{T}; b)$ by the formula

$$\psi(x, b) = \bigvee_{i \in I} \left([\![(\mathfrak{T}; b, a_i)]\!]^n (b, x) \right).$$

Proof Let $b \in B$. It follows from the choice of the set $\{a_i\}_{i \in I}$ that $(\mathfrak{T}; b) \models \psi(u, b)$ for each $u \in A$. Hence it remains only to show that $(\mathfrak{T}; b) \not\models \psi(v, b)$ for all $v \notin A$. Suppose to the contrary that there exists a node $d \notin A$ such that $(\mathfrak{T}; b) \models \psi(d, b)$; then $(\mathfrak{T}; b) \models [\![(\mathfrak{T}; b, a_k)]\!]^n (b, d)$ for some k. Then $(\mathfrak{T}; b, a_k) \equiv_n (\mathfrak{T}; b, d)$. Since $b < a_k$ then $b < d$. It follows from Fact 19 that $(\mathfrak{T}; \bar{c}, b, a_k) \equiv_n (\mathfrak{T}; \bar{c}, b, d)$ hence also $(\mathfrak{T}; \bar{c}, a_k) \equiv_n (\mathfrak{T}; \bar{c}, d)$ and since $(\mathfrak{T}; \bar{c}, a_k) \models \varphi(a_k, \bar{c})$ then $(\mathfrak{T}; \bar{c}, d) \models \varphi(d, \bar{c})$ which contradicts the assumption that $\varphi(x, \bar{c})$ defines A in $(\mathfrak{T}; \bar{c})$. This completes the proof. $\dashv$

In Lemma 21 and Proposition 22, we hence consider sets of nodes A that are definable using one parameter $b \in A^{>}$.

Lemma 21 *Let $\mathfrak{T} = (T; <)$ be a tree and let $A \subseteq T$ such that A is defined in $(\mathfrak{T}; b)$ by a formula $\varphi(x, b)$ of quantifier rank n for some $b \in A^{>}$. Let $A_b = \bigcup_{a \in A} (b, a]$. For each $u \in A_b$ and each $v \notin A_b$, $(\mathfrak{T}; b, u) \not\equiv_{n+1} (\mathfrak{T}; b, v)$.*

[2] By the properties of characteristic formulas, a finite such set $\{a_i\}_{i \in I}$ exists.

Proof Let $\psi(x,b) = \exists y\,(b < x \leqslant y \wedge \varphi(y,b))$. For $u \in (b,A]$ and $v \notin (b,A]$, $(\mathfrak{T}; b, u) \models \psi(u,b)$ while $(\mathfrak{T}; b, v) \not\models \psi(v,b)$ hence $(\mathfrak{T}; b, u) \not\equiv_{n+1} (\mathfrak{T}; b, v)$. $\dashv$

The next proposition gives a general defining formula (albeit not of lowest quantifier rank) for certain definable sets of leaves.

Proposition 22 *Let A be a set of leaves in a tree $\mathfrak{T} = (T; <)$ such that each $a \in A$ has no immediate predecessor. Let $b \in A^{>}$ be such that, for every leaf $d \in b^{<} \backslash A$, the set $(b,d) \backslash A_b$ (again with $A_b = \bigcup_{a \in A} (b,a]$) is non-empty, and suppose that A is defined in $(\mathfrak{T}; b)$ by a formula $\varphi(x,b)$ of quantifier rank n. Then there exists a finite set of nodes $\{c_i\}_{i \in I}$ in A_b such that A is defined in $(\mathfrak{T}; b)$ by the formula*

$$\begin{aligned}\psi(x,b) \quad = \quad & \mathsf{leaf}(x) \wedge b < x \wedge \\ & \forall y\Big(b < y < x \rightarrow \exists z\Big(y < z < x \wedge \Big(\bigvee\nolimits_{i \in I} [\![(\mathfrak{T}; b, c_i)]\!]^{n+1}(b,z)\Big)\Big)\Big)\end{aligned}$$

of quantifier rank $n+3$.

Proof We make use of the fact that there are only finitely many pairwise non-equivalent $(n+1)$-characteristic formulas with two free variables in the language of trees. It follows that there must exist a finite set of nodes $\{c_i\}_{i \in I}$ in A_b such that, for each $u \in A$, there exists $i \in I$ and an increasing sequence $(u_j)_{j \in \alpha_u}$ that is cofinal in $[b,u)$ (where α_u is the cofinality of $[b,u)$) such that $(\mathfrak{T}; b) \models [\![(\mathfrak{T}; b, c_i)]\!]^{n+1}(b, u_j)$ for all $j \in \alpha_u$. It is then clear that $(\mathfrak{T}; b) \models \psi(u,b)$ for each $u \in A$.

On the other hand, for any leaf $d \in b^{<} \backslash A$ and for each node $v \in (b,d) \backslash A_b \neq \emptyset$, Lemma 21 guarantees that $(\mathfrak{T}; b) \not\models [\![(\mathfrak{T}; b, c_i)]\!]^{n+1}(b,v)$ for each $i \in I$, hence $(\mathfrak{T}; b) \not\models \psi(d,b)$. It follows that $\psi(x,b)$ defines A in $(\mathfrak{T}; b)$, as required. $\dashv$

In particular, every leaf a for which the set $A = \{a\}$ satisfies the conditions of Proposition 22, can be defined by a formula of the form

$$\begin{aligned}\psi(x,b) \; = \; & \mathsf{leaf}(x) \wedge b < x \wedge \\ & \forall y\,\big(b < y < x \rightarrow \exists z\,\big(y < z < x \wedge [\![(\mathfrak{T}; b, c)]\!]^{n+1}(b,z)\big)\big)\end{aligned}$$

for some b and c with $b < c < a$.

5 Approximations of boundedness

Here we look at three first-order approximations of boundedness, namely definable boundedness, almost boundedness, and sequential boundedness. A tree $\mathfrak{T}$ will be called *definably bounded* when each of its definable paths is bounded, and *almost bounded* when each of its definable subforests contains a leaf. Given a formula $\varphi(x, \bar{z})$, consider the sentences

$$\begin{aligned}\delta_\varphi \quad &= \quad \forall \bar{z}\Big(\mathsf{path}_\varphi(\bar{z}) \rightarrow \exists x\big(\mathsf{leaf}(x) \wedge \varphi(x,\bar{z})\big)\Big) \quad \text{and} \\ \alpha_\varphi \quad &= \quad \forall \bar{z}\Big(\mathsf{sub}_\varphi(\bar{z}) \rightarrow \exists x\big(\mathsf{leaf}(x) \wedge \varphi(x,\bar{z})\big)\Big).\end{aligned}$$

Clearly a tree is definably bounded if and only if it satisfies the theory

$$\mathsf{DB} \;=\; \{\delta_\varphi : \ \bar{z} \text{ is any (possibly empty) tuple and } \varphi(x,\bar{z}) \text{ is any formula}\},$$

and a tree is almost bounded if and only if it satisfies the theory

$$\mathsf{AB} \;=\; \{\alpha_\varphi : \ \bar{z} \text{ is any (possibly empty) tuple and } \varphi(x,\bar{z}) \text{ is any formula}\}.$$

Moreover, every almost bounded tree is definably bounded, but the converse does not hold: the tree $\mathfrak{B}_{2,\omega}$ of which each path is isomorphic to ω and each node has exactly two immediate successors, is vacuously definably bounded by virtue of not having any definable paths, but $\mathfrak{B}_{2,\omega}$ is not almost bounded because $\mathfrak{B}_{2,\omega}$ itself does not contain any leaf.

Let $\mathfrak{T}$ be a tree, $\varphi(x,\bar{z})$ a formula with $\bar{z}$ a k-tuple of variables for some $k \geqslant 0$, and $\bar{c}$ a k-tuple of nodes from T. A sequence $(a_i)_{i\in\omega}$ of nodes in T will be called a $\varphi(x,\bar{c})$-*sequence* when the sequence is strictly increasing in $\mathfrak{T}$ and $a_i \in T^{\varphi(x,\bar{c})}$ for each i. A node $b \in T$ will be said to *dominate* the sequence $(a_i)_{i\in\omega}$ when $a_i < b$ for each i. A tree $\mathfrak{T}$ will be called *sequentially bounded* when, for any tuple $\bar{c}$ and formula $\varphi(x,\bar{c})$ for which the set $T^{\varphi(x,\bar{c})}$ is non-empty and has no maximal element, there is a leaf that dominates a $\varphi(x,\bar{c})$-sequence $(a_i)_{i\in\omega}$. Consider the sentence

$$\begin{aligned}\sigma_\varphi \;=\; & \forall\bar{z}\Big[\Big(\exists x\big(\varphi(x,\bar{z})\big)\wedge\forall x\big(\varphi(x,\bar{z})\to\exists y\,(x<y\wedge\varphi(y,\bar{z}))\big)\Big)\;\to\\ & \quad\exists x\Big(\mathsf{leaf}(x)\wedge\exists y\,(y<x\wedge\varphi(y,\bar{z}))\;\wedge\\ & \qquad\forall y\big((y<x\wedge\varphi(y,\bar{z}))\to\exists u\,(y<u<x\wedge\varphi(u,\bar{z}))\big)\Big)\Big].\end{aligned}$$

Then a tree is sequentially bounded if and only if it satisfies the theory

$$\mathsf{SB} \;=\; \{\sigma_\varphi :\ \bar{z} \text{ is any (possibly empty) tuple and } \varphi(x,\bar{z}) \text{ is any formula}\}.$$

A tree that is definably bounded need not be sequentially bounded (the tree $\mathfrak{B}_{2,\omega}$ again serves as an example). Moreover, a sequentially bounded tree need not be definably bounded, as the next example shows. We first state a result that is used in the example.

Fact 23 *(See e.g. [Ros82, Theorem 6.21].) For $\alpha_2,\beta_2 < \omega^\omega$ it is the case that $\omega^\omega\cdot\alpha_1+\alpha_2 \equiv \omega^\omega\cdot\beta_1+\beta_2$ if and only if $\alpha_2=\beta_2$ and either $\alpha_1=\beta_1=0$ or $\alpha_1,\beta_1>0$.*

Example 24 *Let $\eta=\omega^\omega\cdot\omega$. Observe that, since η is additively indecomposable then for each tail χ of η, $\chi\cong\eta$. Consider the tree $\mathfrak{T}=(T;<)$ that is defined as follows:*

> *T consists of all sequences of the form $\bar{0}_\alpha$ for $0\leqslant\alpha<\eta$ any ordinal, and all sequences of the form $\bar{0}_\alpha\bar{1}_\beta$ for $\alpha,\beta\geqslant 0$ any ordinals with $\alpha+\beta\leqslant\eta$;*
>
> *for all $\bar{x},\bar{y}\in T$, $\bar{x}<\bar{y}$ when $\bar{x}$ is a proper initial subsequence of $\bar{y}$.*

The height of any node $\bar{x}$ in T is simply the length $\ell(\bar{x})$ of $\bar{x}$. Informally, $\mathfrak{T}$ consists of a copy of the limit ordinal η that has attached to each of its points a copy of the successor ordinal $\eta+1$. It will be shown that $\mathfrak{T}$ is sequentially bounded but not definably bounded.

Note that the path $\mathfrak{L}=(L;<)=(\{\bar{0}_\alpha : 0\leqslant\alpha<\eta\};<)$ consists of all nodes x for which $x^<$ is not totally ordered hence L can be defined in $\mathfrak{T}$ by the formula $\psi(x)=\exists y\exists z\,(x<y\wedge x<z\wedge y\not\sim z)$. However, L is not bounded so $\mathfrak{T}$ is not definably bounded.

Next it will be shown that $\mathfrak{T}$ is sequentially bounded. Suppose that $\bar{c}$ is a tuple of nodes in T, and $\varphi(x,\bar{c})$ is a formula of quantifier rank n, such that $T^{\varphi(x,\bar{c})}$ is non-empty and contains no maximal element. If there exists a node of the form $\bar{0}_\alpha\bar{1}_\beta$ with $\beta\geqslant 1$ for which $\mathfrak{T}\models\varphi(\bar{0}_\alpha\bar{1}_\beta,\bar{c})$, then clearly there must be a leaf that dominates a $\varphi(x,\bar{c})$-sequence, while if $T^{\varphi(x,\bar{c})}\subseteq L$ and $T^{\varphi(x,\bar{c})}$ is not cofinal in L then it is also clear that there must be a leaf that dominates a $\varphi(x,\bar{c})$-sequence. Hence consider the case where $T^{\varphi(x,\bar{c})}\subseteq L$ and $T^{\varphi(x,\bar{c})}$ is cofinal in L.

Let $d=\bar{0}_\gamma$ be such that each node in the tuple $\bar{c}$ belongs to T^d. There must exist a $\varphi(x,\bar{c})$-sequence $(a_i)_{i\in\omega}$ such that each a_i has the form $\bar{0}_\gamma\bar{0}_{\alpha_i}$ for some α_i

with $\alpha_i \geqslant \omega^\omega \cdot 2$. This α_i can be written in the form $\alpha_i = \omega^\omega \cdot m_i + \beta_i$ for some m_i with $2 \leqslant m_i < \omega$ and some β_i with $0 \leqslant \beta_i < \omega^\omega$, i.e. $a_i = \bar{0}_\gamma \bar{0}_{\omega^\omega \cdot m_i + \beta_i}$. For each $i < \omega$, let $b_i = \bar{0}_\gamma \bar{0}_{\omega^\omega + \beta_i}$.

For $u \in L$, define the tree $\mathfrak{T}'_u = (T'_u; <, u)$ as in Lemma 11. Note that, for each $u \in L$, $\mathfrak{T}'_u \cong (\eta + 1; 0)$ hence $\mathfrak{T}'_u \equiv \mathfrak{T}'_v$ for all $u, v \in L$. Moreover, using Fact 23, $\mathfrak{L}_d^{a_i} \cong \omega^\omega \cdot m_i + \beta_i \equiv \omega^\omega + \beta_i \cong \mathfrak{L}_d^{b_i}$ hence, by Lemma 11, ${}_\star\mathfrak{T}_d^{a_i} \equiv_n {}_\star\mathfrak{T}_d^{b_i}$ for each i. Also, $\mathfrak{L}_{a_i} \cong \eta \cong \mathfrak{L}_{b_i}$ so, again by Lemma 11, $(\mathfrak{T}_{a_i}; a_i) \equiv_n (\mathfrak{T}_{b_i}; b_i)$. Hence, using Lemma 4,

$$\begin{aligned}
(\mathfrak{T}; \bar{c}a_i) &\cong \left(\mathfrak{T}^d; \bar{c}\right) +_{d>} \left(\mathfrak{T}_d^{a_i} +_{[d,a_i)} (\mathfrak{T}_{a_i}; a_i)\right) \\
&\equiv_n \left(\mathfrak{T}^d; \bar{c}\right) +_{d>} \left(\mathfrak{T}_d^{b_i} +_{[d,b_i)} (\mathfrak{T}_{b_i}; b_i)\right) \cong (\mathfrak{T}; \bar{c}b_i)
\end{aligned}$$

from which it follows that $(\mathfrak{T}; \bar{c}) \models \varphi(b_i, \bar{c})$ for each i. Hence $(b_i)_{i\in\omega}$ is a $\varphi(x, \bar{c})$-sequence in L that is not cofinal in L so there is a leaf that dominates $(b_i)_{i\in\omega}$, which completes the proof.

Hence, since every tree that is almost bounded is also definably bounded, it follows that a tree that is sequentially bounded need not be almost bounded.

We do not know whether every tree that is almost bounded is also sequentially bounded. To see how the property of almost boundedness may not be strong enough to prove the property of sequential boundedness, consider a tree $\mathfrak{T}$ that is almost bounded, a tuple of nodes $\bar{c}$ in T and a formula $\varphi(x, \bar{c})$ for which $T^{\varphi(x,\bar{c})}$ is non-empty and contains no maximal element. To show that $\mathfrak{T}$ contains a leaf that dominates a $\varphi(x, \bar{c})$-sequence, a natural strategy would be to consider some suitable definable subtree $\mathfrak{T}'$ of $\mathfrak{T}$ for which $T^{\varphi(x,\bar{c})} \subseteq T'$ and to then deduce, using the almost boundedness of $\mathfrak{T}$, that $\mathfrak{T}'$ contains a leaf that dominates a $\varphi(x, \bar{c})$-sequence. Suppose that the underlying set T' of this subtree $\mathfrak{T}'$ is defined in $(\mathfrak{T}; \bar{d})$ by the formula $\psi(x, \bar{d})$ for some tuple $\bar{d}$ of nodes in T. Suppose however that $\mathfrak{T}'$ contains a path X with leaf a and with the property that X contains only finitely many nodes that satisfy the formula $\varphi(x, \bar{c})$ in $\mathfrak{T}$ (see Fig. 4). Then when invoking the almost boundedness of $\mathfrak{T}$ to deduce that $\mathfrak{T}'$ contains a leaf, it may happen that a is this leaf and a does not dominate a $\varphi(x, \bar{c})$-sequence.

Finally, it will be shown that the theories DB, AB and SB need contain only those sentences δ_φ, α_φ and σ_φ for which the formula φ uses at most one parameter, i.e. has one of the forms $\varphi = \varphi(x)$ or $\varphi = \varphi(x, z)$. To this end, define

$$\begin{aligned}
\mathsf{DB}_1 &= \{\delta_\varphi : \ \varphi \text{ is any formula of the form } \varphi = \varphi(x) \text{ or } \varphi = \varphi(x, z)\}, \\
\mathsf{AB}_1 &= \{\alpha_\varphi : \ \varphi \text{ is any formula of the form } \varphi = \varphi(x) \text{ or } \varphi = \varphi(x, z)\}, \\
\mathsf{SB}_1 &= \{\sigma_\varphi : \ \varphi \text{ is any formula of the form } \varphi = \varphi(x) \text{ or } \varphi = \varphi(x, z)\}.
\end{aligned}$$

Proposition 25 *Let $\mathfrak{T} = (T; <)$ be any tree. Then (i) $\mathfrak{T} \models \mathsf{DB} \iff \mathfrak{T} \models \mathsf{DB}_1$, (ii) $\mathfrak{T} \models \mathsf{AB} \iff \mathfrak{T} \models \mathsf{AB}_1$, and (iii) $\mathfrak{T} \models \mathsf{SB} \iff \mathfrak{T} \models \mathsf{SB}_1$.*

Proof It is immediate that $\mathfrak{T} \models \mathsf{DB} \implies \mathfrak{T} \models \mathsf{DB}_1$, $\mathfrak{T} \models \mathsf{AB} \implies \mathfrak{T} \models \mathsf{AB}_1$ and $\mathfrak{T} \models \mathsf{SB} \implies \mathfrak{T} \models \mathsf{SB}_1$. The reverse implications will be proved for each of (i) – (iii).

(i) Any path in $\mathfrak{T}$ that contains a leaf a can be defined in $(\mathfrak{T}; a)$ by the formula $\varphi(x, a) = x \leqslant a$, while in [Kel15, Theorem 4.3] it is shown that if L is a leafless path in $\mathfrak{T}$ that can be defined in $(\mathfrak{T}; \bar{c})$ by some formula $\varphi(x, \bar{c})$, then there exists a node $b \in L$ such that L can be defined in $(\mathfrak{T}; b)$ using a formula $\psi(x, b)$. Hence every path that is at all definable, can be defined using at most one parameter. It follows that $\mathfrak{T} \models \mathsf{DB}_1 \implies \mathfrak{T} \models \mathsf{DB}$.

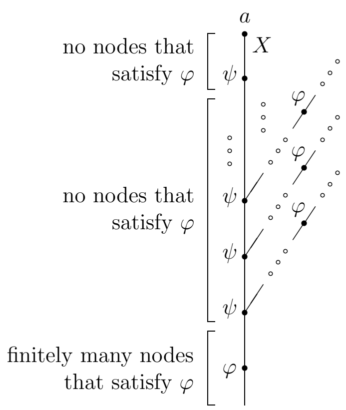

Fig. 4 The tree $\mathfrak{T}$ used in discussing whether almost boundedness implies sequential boundedness.

(ii) Assume that $\mathfrak{T} \models \mathsf{AB}_1$. Let $\bar{c}$ be a k-tuple of nodes from T with $k \geqslant 2$ and let $\varphi(x, \bar{c})$ be a formula for which $T^{\varphi(x,\bar{c})}$ is the underlying set of a subtree of $\mathfrak{T}$. If some c_i in $\bar{c}$ happens to be a leaf that belongs to $T^{\varphi(x,\bar{c})}$ then the proof is complete, hence consider the case where no c_i in $\bar{c}$ is both a leaf and an element of $T^{\varphi(x,\bar{c})}$. It follows that there must exist a node $b \in T^{\varphi(x,\bar{c})}$ for which $b \not< c_i$ for each c_i in $\bar{c}$. If b is a leaf then $T^{\varphi(x,\bar{c})}$ contains a leaf, as required, so consider the case where b is not a leaf. Let T' be the set that is defined in $(\mathfrak{T}; \bar{c}b)$ by the formula $\varphi(x, \bar{c}) \wedge b < x$. By Proposition 20, there exists a formula $\psi(x, b)$ such that $\psi(x, b)$ defines the set T' in $(\mathfrak{T}; b)$. The formula $\psi'(x, b) = \psi(x, b) \vee x = b$ defines the underlying set $T^{\varphi(x,\bar{c})} \cap T_b$ of a subtree of $\mathfrak{T}$ in $(\mathfrak{T}; b)$. Since $\mathfrak{T} \models \alpha_{\psi'}$ then $T^{\psi'(x,b)}$ contains a leaf d and $d \in T^{\varphi(x,\bar{c})}$ too, as required.

(iii) Assume that $\mathfrak{T} \models \mathsf{SB}_1$. First observe that for each node $u \in T$, the set T_u contains a leaf of $\mathfrak{T}$: taking $\varphi(x, z) = z \leqslant x$, if $T_u = T^{\varphi(x,u)}$ did not contain a leaf then T_u would be non-empty and without any maximal element hence, since $\mathfrak{T} \models \sigma_\varphi$ there would exist a leaf v that dominates a $\varphi(x, u)$-sequence from which it would follow that $v \in T_u$, which is a contradiction.

Now let $\bar{c}$ be a k-tuple with $k \geqslant 2$ and $\varphi(x, \bar{c})$ be a formula such that $T^{\varphi(x,\bar{c})}$ is non-empty and does not contain any maximal element. It will be shown that $\mathfrak{T} \models \sigma_\varphi$. Two cases are distinguished:

Case 1: There is a node c_i in $\bar{c}$ that dominates a $\varphi(x, \bar{c})$-sequence $(a_i)_{i\in\omega}$. Then T_{c_i} contains a leaf b that also dominates the sequence $(a_i)_{i\in\omega}$, hence $\mathfrak{T} \models \sigma_\varphi$.

Case 2: There is no node c_i in $\bar{c}$ that dominates a $\varphi(x, \bar{c})$-sequence. Then there must exist a node $a \in T^{\varphi(x,\bar{c})}$ such that $a \not< c_i$ for each c_i in $\bar{c}$. By Proposition 20 the set $a^< \cap T^{\varphi(x,\bar{c})}$ can be defined in $(\mathfrak{T}; a)$ by some formula $\psi(x, a)$; moreover, $a^< \cap T^{\varphi(x,\bar{c})} = T^{\psi(x,a)}$ is non-empty and does not contain a maximal element. From the fact that $\mathfrak{T} \models \sigma_\psi$ it then follows that there exists a leaf b and a $\psi(x, a)$-sequence $(a_i)_{i\in\omega}$ in $\mathfrak{T}$ such that $a_i < b$ for each $i \in \omega$. The sequence $(a_i)_{i\in\omega}$ is a $\varphi(x, \bar{c})$-sequence too. Hence $\mathfrak{T} \models \sigma_\varphi$ as required. $\dashv$

6 Bounded extensions of trees

We now look at conditions for the existence of a bounded end-extension $\mathfrak{T}'$ of a tree $\mathfrak{T}$ that is not bounded, and such that $\mathfrak{T} \preceq_k \mathfrak{T}'$. Let $\mathfrak{T}$ be a tree and let L be a path in $\mathfrak{T}$. If $\mathfrak{F}$ is a bounded forest for which $\mathfrak{T} \preceq_k \mathfrak{T} +_L \mathfrak{F}$ then the tree $\mathfrak{T} +_L \mathfrak{F}$, which is an end-extension of $\mathfrak{T}$, will be called an (L, k)-*completion* of $\mathfrak{T}$. The tree $\mathfrak{T}$ will be called (L, k)-*complete* when no (L, k)-completion of $\mathfrak{T}$ exists.

Note that if L is a path with leaf a and $k \geqslant 2$ then $\mathfrak{T}$ is (L,k)-complete: setting $\varphi(x) = \exists y\,(x < y)$ one obtains $(\mathfrak{T}; a) \not\models \varphi(a)$ while $(\mathfrak{T} +_L \mathfrak{F}; a) \models \varphi(a)$ for any forest $\mathfrak{F}$, so $\mathfrak{T} \not\preceq_k \mathfrak{T} +_L \mathfrak{F}$ for every forest $\mathfrak{F}$, i.e. there is no (L,k)-completion of $\mathfrak{T}$.[3] Hence a tree $\mathfrak{T}$ is (L,k)-complete when either L is bounded, or when L is unbounded but there is no bounded forest $\mathfrak{F}$ for which $\mathfrak{T} \preceq_k \mathfrak{T} +_L \mathfrak{F}$. Clearly for every tree $\mathfrak{T}$ and every path L in $\mathfrak{T}$, $\mathfrak{T}$ is k-equivalent to an (L,k)-complete tree. A tree $\mathfrak{T}$ will be called *k-complete* when $\mathfrak{T}$ is (L,k)-complete for each of its paths L.

Proposition 26 *Let k be a positive integer, $\mathfrak{T}$ be a tree, L be an unbounded path in $\mathfrak{T}$, $\mathfrak{F}$ be a forest and let $\mathfrak{S} = \mathfrak{T} +_L \mathfrak{F}$.*

1. *If $\mathfrak{T} \preceq_k \mathfrak{S}$ then $\mathfrak{T}_a \equiv_{k-1} \mathfrak{S}_a$ for each $a \in L$.*
2. *If there exists a cofinal increasing sequence $(a_i)_{i\in\alpha}$ in L such that $\mathfrak{T}_{a_i} \equiv_k \mathfrak{S}_{a_i}$ for every $i \in \alpha$ (in particular, if $\mathfrak{T}_a \equiv_k \mathfrak{S}_a$ for every $a \in L$) then $\left(\mathfrak{T}; \bar{b}\right) \equiv_k \left(\mathfrak{S}; \bar{b}\right)$ for each finite (possibly empty) tuple $\bar{b}$ of nodes from T (in particular, $\mathfrak{T} \preceq_k \mathfrak{S}$).*

Proof 1. Assume that $\mathfrak{T} \preceq_k \mathfrak{S}$ and let $a \in L$. Since $\mathfrak{S}$ is a k-extension of $\mathfrak{T}$ then $(\mathfrak{T}; a) \equiv_{k-1} (\mathfrak{S}; a)$, hence $\mathfrak{T}_a \equiv_{k-1} \mathfrak{S}_a$.

2. Assume that $(a_i)_{i\in\alpha}$ is a cofinal increasing sequence in L such that $\mathfrak{T}_{a_i} \equiv_k \mathfrak{S}_{a_i}$ for every $i \in \alpha$. Let $\bar{b}$ be a finite tuple of nodes in T and let $j \in \alpha$ be such that $b_i \in T^{a_j}$ for each b_i in $\bar{b}$ (if $\bar{b}$ is the empty tuple then any $j \in \alpha$ will suffice). Since $\left({}_\star\mathfrak{T}^{a_j}; \bar{b}\right) \cong \left({}_\star\mathfrak{S}^{a_j}; \bar{b}\right)$ (hence also $\left({}_\star\mathfrak{T}^{a_j}; \bar{b}\right) \equiv_k \left({}_\star\mathfrak{S}^{a_j}; \bar{b}\right)$) and $\mathfrak{T}_{a_j} \equiv_k \mathfrak{S}_{a_j}$ then by Lemma 4,

$$\left(\mathfrak{T}; (a_j)^{>}, \bar{b}\right) \cong \left({}_\star\mathfrak{T}^{a_j}; \bar{b}\right) +_{(a_j)^{>}} \mathfrak{T}_{a_j} \equiv_k \left({}_\star\mathfrak{S}^{a_j}; \bar{b}\right) +_{(a_j)^{>}} \mathfrak{S}_{a_j} \cong \left(\mathfrak{S}; (a_j)^{>}, \bar{b}\right).$$

In particular, $\left(\mathfrak{T}; \bar{b}\right) \equiv_k \left(\mathfrak{S}; \bar{b}\right)$. $\dashv$

Proposition 27 *Let $\mathfrak{T}$ be a tree that is not bounded with at most countably many unbounded paths. Suppose, for each unbounded path L in $\mathfrak{T}$, that there exists a bounded forest $\mathfrak{F}_L$ for which $\mathfrak{T} \preceq_{k+1} \mathfrak{T} +_L \mathfrak{F}_L$. Then there exists a bounded end-extension $\mathfrak{T}'$ of $\mathfrak{T}$ such that $\mathfrak{T} \preceq_k \mathfrak{T}'$.*

Proof Let $\{L_m\}_{m<\gamma}$ (where γ is an ordinal with $\gamma \leqslant \omega$) be a well-ordering of the set of unbounded paths in $\mathfrak{T}$. For each m, let $\mathfrak{F}_m$ be a bounded forest such that $\mathfrak{T} \preceq_{k+1} \mathfrak{T} +_{L_m} \mathfrak{F}_m$. Define a chain of trees $(\mathfrak{T}_m)_{m\leqslant\gamma}$ as follows:

$\mathfrak{T}_0 = \mathfrak{T}$;

for each $m < \gamma$, $\mathfrak{T}_{m+1} = \mathfrak{T}_m +_{L_m} \mathfrak{F}_m$;

if $\gamma = \omega$ then $\mathfrak{T}_\omega = \bigcup_{m\in\omega} \mathfrak{T}_m$.

It will first be shown that $\mathfrak{T}_m \preceq_k \mathfrak{T}_{m+1}$ for each m with $0 \leqslant m < \gamma$. Let $(a_i)_{i\in\alpha}$ (where α is a limit ordinal) be a cofinal increasing sequence in L_m such that $a_i \notin L_j$ for each a_i and each j with $0 \leqslant j < m$. Since $\mathfrak{T} \preceq_{k+1} \mathfrak{T} +_{L_m} \mathfrak{F}_m$ then by Part 1 of Proposition 26, $\mathfrak{T}_{a_i} \equiv_k (\mathfrak{T} +_{L_m} \mathfrak{F}_m)_{a_i}$, hence

$$(\mathfrak{T}_m)_{a_i} \cong \mathfrak{T}_{a_i} \equiv_k (\mathfrak{T} +_{L_m} \mathfrak{F}_m)_{a_i} \cong (\mathfrak{T}_m +_{L_m} \mathfrak{F}_m)_{a_i} = (\mathfrak{T}_{m+1})_{a_i}$$

for each a_i hence, by Part 2 of Proposition 26, $\mathfrak{T}_m \preceq_k \mathfrak{T}_{m+1}$.

Now let $\mathfrak{T}' = \mathfrak{T}_\gamma$. If γ is finite then, by the transitivity of $\preceq_k$, it follows that $\mathfrak{T} = \mathfrak{T}_0 \preceq_k \mathfrak{T}_\gamma = \mathfrak{T}'$, while if $\gamma = \omega$ then it follows from Fact 3 that $\mathfrak{T} = \mathfrak{T}_0 \preceq_k \bigcup_{m\in\omega} \mathfrak{T}_m = \mathfrak{T}_\omega = \mathfrak{T}_\gamma = \mathfrak{T}'$.

[3] Here the node $(a,0) \in |\mathfrak{T} +_L \mathfrak{F}|$ is identified with the node $a \in T$, and $(b,1) \in |\mathfrak{T} +_L \mathfrak{F}|$ is identified with $b \in F$, in order to keep the notation simple.

To see that $\mathfrak{T}'$ is bounded, consider any path L in $\mathfrak{T}'$. We consider two cases:

Case 1: $L \subseteq T$. Then L is a path in $\mathfrak{T}$ as well. Were L one of the unbounded paths L_m in $\mathfrak{T}$ then L would not be a path in $\mathfrak{T}'$ (since there would then exist $t \in F_m$ such that $L < t$), a contradiction, hence L must be one of the bounded paths in $\mathfrak{T}$.

Case 2: There exists a node $t \in L \backslash T$. Then $t \in F_m$ for some $m < \gamma$. Since L is a path in $\mathfrak{T}'$ then it follows that $L \backslash T$ must be a path in $\mathfrak{F}_m$ hence $L \backslash T$ must contain a leaf b, and b will also be a leaf of L in $\mathfrak{T}$.

Finally, it is clear from the construction of the tree $\mathfrak{T}'$ that $\mathfrak{T}'$ is an end-extension of $\mathfrak{T}$. $\dashv$

Let k be a natural number and let L be an unbounded path in a tree $\mathfrak{T}$ of which the signature is finite. Owing to the fact that there are, up to logical equivalence, only finitely many characteristic formulas of the form $[\![(\mathfrak{T}; a)]\!]^k$, there must exist a cofinal increasing sequence $(a_i)_{i\in\alpha}$ in L such that $(\mathfrak{T}; a_i) \equiv_k (\mathfrak{T}; a_j)$ for all i and j. This observation can be combined with an additive version of Ramsey's Theorem (see e.g. [She75]) to give Fact 28 below (refer also to the motivation of a certain partition property that is discussed in [Kel15], for a more detailed justification of Fact 28).

Fact 28 *Let $\mathfrak{T}$ be any tree and let L be a path in $\mathfrak{T}$ that is not bounded. For every cofinal increasing sequence $(b_i)_{i\in\beta}$ in L and every natural number k, there exists a subsequence $(a_i)_{i\in\alpha}$ of $(b_i)_{i\in\beta}$ that satisfies the following properties:*

1. *$(a_i)_{i\in\alpha}$ is a cofinal increasing sequence in L, and*
2. *$(\mathfrak{T}; a_p) \equiv_k (\mathfrak{T}; a_q)$ for all $p < q$, and*
3. *${}_\star\mathfrak{T}_{a_p}^{a_q} \equiv_k {}_\star\mathfrak{T}_{a_s}^{a_t}$ for all $p < q$ and $s < t$.*

A sequence $(a_i)_{i\in\alpha}$ that satisfies the above three properties will be called a *k-homogeneous* sequence in L.

The next result describes a general construction that can be used for producing a forest $\mathfrak{F}$ for which $\mathfrak{T} \preceq_k \mathfrak{T} +_L \mathfrak{F}$ when given an unbounded path L in a tree $\mathfrak{T}$ that is not $(L; k)$-complete. Given a tree $\mathfrak{T}$, $\mathcal{U}(\mathfrak{T})$ will denote the set of unbounded paths in $\mathfrak{T}$.

Theorem 29 *Let $\mathfrak{T}$ be a tree, L be an unbounded path of countable cofinality in $\mathfrak{T}$, $k \geqslant 2$ be any natural number and $(a_i)_{i\in\omega}$ be a k-homogeneous sequence in L. Suppose that there exists a tree $\mathfrak{S}$ and a stem M in $\mathfrak{S}$ for which $\mathcal{U}(\mathfrak{S}) \setminus \{M\} = \emptyset$ (i.e. each path in $\mathfrak{S}$, except possibly M if M is itself a path, is bounded), and a bounded tree $\mathfrak{S}'$, such that ${}_\star\mathfrak{T}_{a_0}^{a_1} \equiv_k (\mathfrak{S}; M)$ and $\mathfrak{T}_{a_0} \equiv_k \mathfrak{S}'$. Then there exists a bounded tree $\mathfrak{F}$, namely*

$$\mathfrak{F} = (\mathfrak{S} \times_M \omega^\star) +_{M\times\omega^\star} \mathfrak{S}',$$

such that $\mathfrak{T} \preceq_k \mathfrak{T} +_L \mathfrak{F}$.

Proof To show that $\mathfrak{T} \preceq_k \mathfrak{T} +_L \mathfrak{F}$ it suffices, by Proposition 26, to show that $\mathfrak{T}_{a_n} \equiv_k (\mathfrak{T} +_L \mathfrak{F})_{a_n}$ for each a_n. Hence fix n and let $l = n + 2^k - 1$. Observe that the tree $\mathfrak{S}'$ contains a root $e_{\mathfrak{S}'}$ since $\mathfrak{S}' \equiv_k \mathfrak{T}_{a_0}$ and $\mathfrak{T}_{a_0}$ has a root. Let $c = (e_{\mathfrak{S}'}, 1) \in F$. Since[4]

$$\mathfrak{T}_{a_n} \cong \mathfrak{T}_{a_n}^{a_l} +_{[a_n,a_l)} \mathfrak{T}_{a_l} \quad \text{and} \quad (\mathfrak{T} +_L \mathfrak{F})_{a_n} \cong (\mathfrak{T} +_L \mathfrak{F})_{a_n}^c +_{[a_n,c)} (\mathfrak{T} +_L \mathfrak{F})_c$$

[4]Again the nodes $(a_n, 0), (c, 1) \in |\mathfrak{T} +_L \mathfrak{F}|$ are identified with the nodes $a_n \in T$ and $c \in F$ to keep the notation simple, with similar conventions elsewhere in the proof.

and $\mathfrak{T}_{a_l} \equiv_k \mathfrak{T}_{a_0} \equiv_k \mathfrak{S}' \cong (\mathfrak{T} +_L \mathfrak{F})_c$ then, to show that $\mathfrak{T}_{a_n} \equiv_k (\mathfrak{T} +_L \mathfrak{F})_{a_n}$ it suffices, by Lemma 4, to show that $\left(\mathfrak{T}^{a_l}_{a_n}; [a_n, a_l)\right) \equiv_k \left((\mathfrak{T} +_L \mathfrak{F})^c_{a_n}; [a_n, c)\right)$, which can be re-written simply as ${}_\star\mathfrak{T}^{a_l}_{a_n} \equiv_k {}_\star(\mathfrak{T} +_L \mathfrak{F})^c_{a_n}$.

Observe that, since ${}_\star\mathfrak{T}^{a_1}_{a_0} \equiv_k {}_\star\mathfrak{T}^{a_{i+1}}_{a_i}$ for each i then it follows from Lemma 12 that

$$\mathfrak{T}^{a_1}_{a_0} \times'_{[a_0,a_1)} \omega \equiv_k (\mathfrak{T}_{a_n}; T_{a_n} \cap L), \tag{1}$$

and since ${}_\star\mathfrak{T}^{a_1}_{a_0} \equiv_k (\mathfrak{S}; M)$ then by Lemma 5,

$$\mathfrak{T}^{a_1}_{a_0} \times'_{[a_0,a_1)} \omega^\star \equiv_k \mathfrak{S} \times'_M \omega^\star. \tag{2}$$

Now

$$\begin{aligned}
{}_\star\mathfrak{T}^{a_l}_{a_n} &\equiv_k \mathfrak{T}^{a_1}_{a_0} \times'_{[a_0,a_1)} \left(2^k - 1\right) && \text{(since } {}_\star\mathfrak{T}^{a_{i+1}}_{a_i} \equiv_k {}_\star\mathfrak{T}^{a_1}_{a_0} \text{ for all } i \\
&&& \text{and by Lemma 12)} \\
&\equiv_k \mathfrak{T}^{a_1}_{a_0} \times'_{[a_0,a_1)} (\omega + \omega^\star) && \text{(since } 2^k - 1 \equiv_k \omega + \omega^\star \text{ and by Lemma 5)} \\
&\cong \left(\mathfrak{T}^{a_1}_{a_0} \times_{[a_0,a_1)} \omega\right) +_{[a_0,a_1)\times\omega,\ [a_0,a_1)\times\omega^\star} \left(\mathfrak{T}^{a_1}_{a_0} \times_{[a_0,a_1)} \omega^\star\right) \\
&\equiv_k \mathfrak{T}_{a_n} +_{T_{a_n}\cap L,\ [a_0,a_1)\times\omega^\star} \left(\mathfrak{T}^{a_1}_{a_0} \times_{[a_0,a_1)} \omega^\star\right) && \text{(by (1) and Lemma 4)} \\
&\equiv_k \mathfrak{T}_{a_n} +_{T_{a_n}\cap L,\, M\times\omega^\star} (\mathfrak{S} \times_M \omega^\star) && \text{(by (2) and Lemma 4)} \\
&\cong {}_\star(\mathfrak{T} +_L \mathfrak{F})^c_{a_n}
\end{aligned}$$

as required. $\dashv$

Definition 30 *[Doe89, Definition 2.2.1] Let $\mathfrak{T} = (T; <)$ be any tree. For $k \geqslant 1$ define the following four statements about $\mathfrak{T}$:*

> Q.1 (k) *: If $k = 1$ then there exists a node, if $k = 2$ then there exists a leaf and if $k \geqslant 3$ then for every node x there exists a leaf y with $x \leqslant y$.*
>
> Q.2 (k) *: Every path in $\mathfrak{T}$ contains at least $2^k - 2$ nodes.*
>
> Q.3 (k) *: Some path in $\mathfrak{T}$ contains at least $2^k - 1$ nodes.*
>
> Q.4 (k) *: For every node $x \in T$ and for $n < 2^{k-1} - 1$, if some path in $\mathfrak{T}_{>x}$ contains exactly n nodes then every path in $\mathfrak{T}_{>x}$ contains exactly n nodes.*

Let Q (k) *be the conjunction of* Q.1 (k) – Q.4 (k).

For $m \geqslant 2$, $\mathfrak{B}_{m,n}$ will denote the non-enriched tree of which each non-leaf node has exactly m immediate successors, and each path has exactly n nodes. When $m = 2$, the binary tree $\mathfrak{B}_{m,n}$ will also be denoted simply as $\mathfrak{B}_n$.

Fact 31 *[Doe89, Theorem 2.3.8] Let $k \geqslant 1$ and let $\mathfrak{T} = (T; <)$ be a well-founded binary tree of height at most ω. Then $\mathfrak{T}$ is k-equivalent to the binary tree $\mathfrak{B}_n$ if and only if either $n < 2^k - 1$ and $\mathfrak{T} \cong \mathfrak{B}_n$, or $n \geqslant 2^k - 1$ and $\mathfrak{T}$ satisfies* Q (k).

Working through the proofs of the sequence of propositions and theorems in [Doe89] that are used to establish Fact 31, it is evident that the result also holds, for each $m \geqslant 2$, when $\mathfrak{T}$ is an m-ary tree and the binary tree $\mathfrak{B}_n$ is replaced with the m-ary tree $\mathfrak{B}_{m,n}$.

Given a tree $\mathfrak{T}$, recall that $\mathcal{U}(\mathfrak{T})$ denotes the set of unbounded paths in $\mathfrak{T}$. Define $U(\mathfrak{T}) = \bigcup_{X\in\mathcal{U}(\mathfrak{T})} X$ and $\mathfrak{U}(\mathfrak{T}) = (U(\mathfrak{T}); <)$.

Proposition 32 *Let $k \geqslant 2$ and let $\mathfrak{T}$ be a tree and L be an unbounded path of countable cofinality in $\mathfrak{T}$. Each of the following conditions is sufficient for the existence of a bounded tree $\mathfrak{F}$ for which $\mathfrak{T} \preceq_k \mathfrak{T} +_L \mathfrak{F}$:*

1. *$\mathfrak{T}$ is sequentially bounded and L is a singular path in $\mathfrak{U}(\mathfrak{T})$.*

2. $\mathfrak{T}$ *is upwards discrete and there exists a bounded tree* $\mathfrak{T}'$ *for which* $\mathfrak{T} \equiv_{k+1} \mathfrak{T}'$.

3. *For some* $m \geqslant 2$, $\mathfrak{T}$ *is a well-founded* m*-ary tree of height* ω *that satisfies* $\mathrm{Q}(k+1)$.

4. *There exists a bounded tree* $\mathfrak{T}'$ *for which* $\mathfrak{T} \equiv_{k+2} \mathfrak{T}'$.

Proof 1. Let $a \in L$ be such that $(\mathfrak{U}(\mathfrak{T}))_a$ is a linear order. Then $\mathfrak{T}_x$ must be bounded for each $x \in T_a \backslash L$. Let $(a_i)_{i\in\omega}$ be a k-homogeneous sequence in $T_a \cap L$. Suppose, for a contradiction, that $\mathfrak{T} \not\models [\![(\mathfrak{T}; a_0)]\!]^k(u)$ for each $u \in T_a \backslash L$. Letting $\varphi(x,a) = (x > a) \wedge [\![(\mathfrak{T}; a_0)]\!]^k(x)$, it follows that the set $T^{\varphi(x,a)}$ is non-empty and has no maximal element (since $(\mathfrak{T}; a) \models \varphi(a_i, a)$ for each a_i while $(\mathfrak{T}; a) \not\models \varphi(u,a)$ for each $u \in T_a \backslash L$) hence, by the sequential boundedness of $\mathfrak{T}$, there is a leaf that dominates a $\varphi(x,a)$-sequence. But this leaf will have to be an element of L, which contradicts the fact that L is unbounded. Hence $\mathfrak{T} \models [\![(\mathfrak{T}; a_0)]\!]^k(b)$ for some $b \in T_a \backslash L$ so $\mathfrak{T}_{a_0} \equiv_k \mathfrak{T}_b$ and the tree $\mathfrak{T}_b$ will be bounded. Now apply Theorem 29 with $(\mathfrak{S}; M) = {}_\star\mathfrak{T}_{a_0}^{a_1}$ and $\mathfrak{S}' = \mathfrak{T}_b$ to obtain the result.

2. Let $(a_i)_{i\in\omega}$ be any k-homogeneous sequence in L. For each node $u \in [a_0, a_1)$, let $N(u)$ be the set that consists of all the immediate successors of u in $\mathfrak{T}$, let $N = \left(\bigcup_{u\in[a_0,a_1)} N(u)\right) \backslash L$ and let $N' = \{x \in N : \mathfrak{T}_x \text{ is not bounded}\}$. For each node $x \in T$, let $t_x \in T'$ be any node for which $\mathfrak{T}_x \equiv_k (\mathfrak{T}')_{t_x}$ (such t_x exists from the fact that $\mathfrak{T} \equiv_{k+1} \mathfrak{T}'$). Let $\mathfrak{R}$ be the tree that is obtained from $\mathfrak{T}_{a_0}^{a_1}$ by replacing, for each $x \in N'$, the subtree $\left(\mathfrak{T}_{a_0}^{a_1}\right)_x = \mathfrak{T}_x$ of $\mathfrak{T}_{a_0}^{a_1}$ with the tree $(\mathfrak{T}')_{t_x}$. By Lemma 13, ${}_\star\mathfrak{T}_{a_0}^{a_1} \equiv_k (\mathfrak{R}; [a_0, a_1))$. Now apply Theorem 29 with $(\mathfrak{S}; M) = (\mathfrak{R}; [a_0, a_1))$ and $\mathfrak{S}' = (\mathfrak{T}')_{t_{a_0}}$ to obtain the result.

3. Since $\mathfrak{T}$ is well-founded, it must be upwards discrete. By Fact 31 and the remark that follows it, $\mathfrak{T} \equiv_{k+1} \mathfrak{B}_{m,2^{k+1}-1}$. The result now follows by Part 2 above.

4. Let $(a_i)_{i\in\omega}$ be any k-homogeneous sequence in L. Let

$$\sigma = \exists x_0 \exists x_1 \left([\![(\mathfrak{T}; a_0, a_1)]\!]^k(x_0, x_1)\right).$$

Since $\mathfrak{T} \models \sigma$ and $\mathfrak{T} \equiv_{k+2} \mathfrak{T}'$ then $\mathfrak{T}' \models \sigma$ hence there exist $b_0, b_1 \in T'$ such that $(\mathfrak{T}'; b_0, b_1) \equiv_k (\mathfrak{T}; a_0, a_1)$ hence ${}_\star(\mathfrak{T}')_{b_0}^{b_1} \equiv_k {}_\star\mathfrak{T}_{a_0}^{a_1}$ and $(\mathfrak{T}')_{b_0} \equiv_k \mathfrak{T}_{a_0}$. Now apply Theorem 29 with $(\mathfrak{S}; M) = {}_\star(\mathfrak{T}')_{b_0}^{b_1}$ and $\mathfrak{S}' = (\mathfrak{T}')_{b_0}$ to obtain the result. ⊣

Corollary 33 *Let* $k \geqslant 2$ *and let* $\mathfrak{T}$ *be a tree that is not bounded with at most countably many unbounded paths such that each unbounded path has countable cofinality. Each of the following conditions is sufficient for the existence of a bounded end-extension* $\mathfrak{T}'$ *of* $\mathfrak{T}$ *for which* $\mathfrak{T} \preceq_k \mathfrak{T}'$:

1. $\mathfrak{T}$ *is sequentially bounded and each of its unbounded paths is a singular path in* $\mathfrak{U}(\mathfrak{T})$.

2. $\mathfrak{T}$ *is upwards discrete and there exists a bounded tree* $\mathfrak{T}'$ *for which* $\mathfrak{T} \equiv_{k+2} \mathfrak{T}'$.

3. *For some* $m \geqslant 2$, $\mathfrak{T}$ *is a well-founded* m*-ary tree of height* ω *that satisfies* $\mathrm{Q}(k+2)$.

4. *There exists a bounded tree* $\mathfrak{T}'$ *for which* $\mathfrak{T} \equiv_{k+3} \mathfrak{T}'$.

Proof By Proposition 32, for each unbounded path L in $\mathfrak{T}$, each of the four conditions is sufficient for the existence of a bounded tree $\mathfrak{F}_L$ for which $\mathfrak{T} \preceq_{k+1} \mathfrak{T} +_L \mathfrak{F}_L$. The result then follows by Proposition 27. ⊣

7 A counterexample

Given a tree $\mathfrak{T}$ and a path L in $\mathfrak{T}$ for which $\mathfrak{T}$ is not (L,k)-complete, it might be expected that $\mathfrak{T}$ contains a subforest $\mathfrak{F}$ for which $\mathfrak{T} \preceq_k \mathfrak{T} +_L \mathfrak{F}$, as was the case in Example 1 where each unbounded path in the tree $\mathfrak{T}$ could be augmented by a single leaf to obtain a bounded tree $\mathfrak{T}'$ for which $\mathfrak{T} \preceq \mathfrak{T}'$. Example 34 below shows that this need not generally be the case, even when $\mathfrak{T}$ is a model of the first-order theory of the class of bounded trees.

Example 34 *Let*

$$U = \bigcup\nolimits_{n\geqslant 1, n\in\mathbb{N}} \left\{ \bar{0}_n 1\bar{x} : \bar{x} \in \{0,1\}^{n-1} \right\},$$

let $U^0 = \{\epsilon\}$, and for $m \geqslant 1$ let U^m denote the set that consists of all sequences of the form $\bar{y}_0\bar{y}_1\bar{y}_2\cdots\bar{y}_{m-1}$, and U^ω denote the set that consists of all sequences of the form $\bar{y}_0\bar{y}_1\bar{y}_2\cdots$, where $\bar{y}_i \in U$ for each i. Define the relation $\leqslant'$ on pairs of sequences by specifying that $\bar{x} \leqslant' \bar{y}$ when $\bar{x}$ is a non-empty initial subsequence (not necessarily a proper subsequence) of $\bar{y}$. Finally, let $\mathfrak{T} = (T; <_\mathfrak{T})$, where

$$T = \bigcup\nolimits_{\bar{y}\in U^\omega} \{\bar{x} : \bar{x} \leqslant' \bar{y}\}$$

and for all $\bar{x},\bar{y} \in T$, $\bar{x} <_\mathfrak{T} \bar{y}$ when $\bar{x}$ is a proper initial subsequence of $\bar{y}$. The tree $\mathfrak{T}$ is depicted in Fig. 5.
Observe the following:

$\mathfrak{T}_{\bar{y}0} \cong \mathfrak{T}$ for each $\bar{y} \in U$.

For each $\bar{y} \in U^\omega$, the set $L_{\bar{y}} = \{\bar{x} \in T : \bar{x} \leqslant' \bar{y}\}$ is a bounded path in $\mathfrak{T}$ with leaf $\bar{y}$.

For each $\bar{z} \in \bigcup_{m\geqslant 0} U^m$, the set $L_{\bar{z}\bar{0}_\omega} = \{\bar{x} \in T : \bar{x} \leqslant' \bar{z}\bar{0}_\omega\}$ is an unbounded path in $\mathfrak{T}$ since $\bar{z}\bar{0}_\omega \notin T$. Hence $\mathfrak{T}$ is not bounded.

Let L be the unbounded path $L_{\epsilon\bar{0}_\omega} = \{\bar{0}_n : n \geqslant 1\}$ in $\mathfrak{T}$. It will be shown in Propositions 35 and 36 below that $\mathfrak{T}$ is a model of the first-order theory of the class of bounded trees and that, for $k \geqslant 6$, $\mathfrak{T}$ is not (L,k)-complete, and that $\mathfrak{T} \not\preceq_k \mathfrak{T} +_L \mathfrak{F}$ for each subforest $\mathfrak{F}$ of $\mathfrak{T}$.

Proposition 35 *The tree $\mathfrak{T}$ of Example 34 is a model of the first-order theory of the class of bounded trees.*

Proof We will show, for each natural number k, that there exists a bounded tree to which $\mathfrak{T}$ is k-equivalent; to ensure that the node c that is used below is defined, assume without loss of generality that $k \geqslant 2$. Let U be as in Example 34 and let $\mathfrak{U} = (U; <_\mathfrak{T})$, $c = \bar{0}_{2^k-3}$, $C = c^{>_\mathfrak{T}}$ and $\mathfrak{U}' = \mathfrak{U}^c +_C \mathfrak{B}_{2^k-1}$. Observe that the tree $\mathfrak{U}$ is not bounded while $\mathfrak{U}'$ is bounded. The tree $\mathfrak{U}_c$ satisfies $\mathrm{Q}(k)$ hence, by Fact 31, $\mathfrak{U}_c \equiv_k \mathfrak{B}_{2^k-1}$. From Lemma 4,

$$\mathfrak{U} \cong \mathfrak{U}^c +_C \mathfrak{U}_c \equiv_k \mathfrak{U}^c +_C \mathfrak{B}_{2^k-1} = \mathfrak{U}'.$$

Let $\mathcal{A}$ be the set that consists of all bounded paths in $\mathfrak{U}$, let $\mathcal{B}$ be the set that consists of all paths in $\mathfrak{U}'$, and let $\mathcal{A}'$ be the set that consists of all paths in $\mathfrak{U} \times_\mathcal{A} \omega$ that contain infinitely many nodes of the form $(e_\mathfrak{U}, \bar{x})$. Observe that every path in $\mathfrak{U}' \times_\mathcal{B} \omega$ contains infinitely many nodes of the form $(e_{\mathfrak{U}'}, \bar{y})$ and let $\mathcal{B}'$ be the set that consists of all paths in $\mathfrak{U}' \times_\mathcal{B} \omega$. Let $\mathfrak{D}$ be a singleton tree. By Lemma 7,

$$\mathfrak{T} \cong (\mathfrak{U} \times_\mathcal{A} \omega) +_{\mathcal{A}'} \mathfrak{D} \equiv_k \left(\mathfrak{U}' \times_\mathcal{B} \omega\right) +_{\mathcal{B}'} \mathfrak{D}$$

and $(\mathfrak{U}' \times_\mathcal{B} \omega) +_{\mathcal{B}'} \mathfrak{D}$ is bounded, as required. $\dashv$

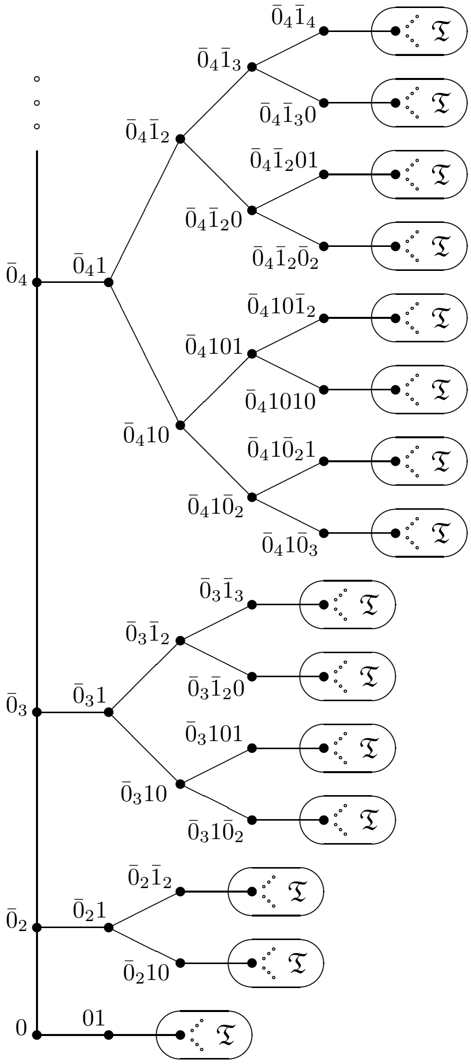

Fig. 5 The tree $\mathfrak{T}$ of Example 34.

Proposition 36 *Let $k \geqslant 6$ and let $\mathfrak{T}$ and L be as in Example 34. $\mathfrak{T}$ is not (L,k)-complete, and for each subforest $\mathfrak{F}$ of $\mathfrak{T}$, $\mathfrak{T} \not\preceq_k \mathfrak{T} +_L \mathfrak{F}$.*

Proof To see that $\mathfrak{T}$ is not (L,k)-complete, observe that $\mathfrak{T}$ is clearly upwards discrete and by Proposition 35, $\mathfrak{T}$ is $(k+1)$-equivalent to a bounded tree hence, by Proposition 32, there exists an (L,k)-completion of $\mathfrak{T}$.

Next it will be shown that $\mathfrak{T} \not\preceq_k \mathfrak{T} +_L \mathfrak{F}$ for every subforest $\mathfrak{F}$ of $\mathfrak{T}$. Let $\rho(x,y) = (x < y) \wedge \neg\exists z\,(x < z < y)$ (which expresses that x is an immediate predecessor of y) and $\tau(x) = \exists y_1 \exists y_2\,(y_1 \neq y_2 \wedge \rho(x,y_1) \wedge \rho(x,y_2))$ (which expresses that x has more than one immediate successor). Consider the formula

$$\varphi(x) = \neg\exists y\,(\rho(y,x)) \wedge \exists y\,(y < x \wedge \forall z\,((y < z < x) \rightarrow \tau(z)))$$

of quantifier rank 5 (which expresses that x does not have an immediate predecessor but that it does have a predecessor y such that each node between y and x has more than one immediate successor). Let v be a minimal node in $\mathfrak{F}$ and let $v' = (v,1)$. Then $\mathfrak{T} +_L \mathfrak{F} \models \varphi(v')$ hence $\mathfrak{T} +_L \mathfrak{F} \models \exists x\,(\varphi(x))$ while $\mathfrak{T} \not\models \exists x\,(\varphi(x))$ so $\mathfrak{T} \not\equiv_6 \mathfrak{T} +_L \mathfrak{F}$ hence $\mathfrak{T} \not\preceq_k \mathfrak{T} +_L \mathfrak{F}$, as required. $\dashv$

8 Concluding remarks

It was shown in Section 3 how axiomatisations of the first-order theories of certain classes of bounded trees can be adapted for use in axiomatising the first-order theory of the class of α-trees for α an ordinal in the range $\alpha < \omega^\omega$. The construction that was used in the proof of the completeness of this axiomatisation required of the first-order theories of these classes of bounded trees to respect well-foundedness in the sense of the property $\mathsf{BD2}(\varphi)$. The reason for this is that the only known completeness proof of an axiomatisation of the first-order theory of the class of well-founded trees, given in [Doe89], uses a construction, for producing a k-equivalent well-founded tree from one that is not well-founded, that does not respect boundedness. Hence, rather than requiring of the first-order theory of the class of well-founded trees to respect boundedness, the approach of requiring of the first-order theories of the abovementioned classes of bounded trees to respect well-foundedness, was taken.

Axiomatisations of the first-order theories of the classes of bounded trees and φ-bounded trees are not presently known, nor is it known whether these first-order theories necessarily respect well-foundedness in the sense of $\mathsf{BD2}(\varphi)$. Hence the axiomatisation of the first-order theory of the class of α-trees that is given in Section 3 is only a conditional axiomatisation. The subsequent remainder of the paper therefore concerned itself with results and observations that could be used towards determining an actual axiomatisation of the first-order theory of the class of bounded trees.

One natural method for constructing a k-equivalent bounded tree from a tree $\mathfrak{T}$ that is not bounded, is by augmenting the unbounded paths in $\mathfrak{T}$ with suitable bounded forests. In order that one may employ a chain construction as was used in Proposition 27, the tree $\mathfrak{T} +_L \mathfrak{F}$ that results from adding a bounded forest $\mathfrak{F}$ to the end of an unbounded path L must, in addition to being k-equivalent to $\mathfrak{T}$, also be such that $\mathfrak{T} \preceq_k \mathfrak{T} +_L \mathfrak{F}$. One may expect that the raw material for constructing such a forest $\mathfrak{F}$ should be found within the tree $\mathfrak{T}$ itself. The tree of Example 34 shows however that one cannot, in general, simply add missing leaves, or even entire subforests of $\mathfrak{T}$, to the unbounded paths of $\mathfrak{T}$ to turn it into a bounded tree; a more intricate construction is needed.

One such construction for producing the required bounded forest $\mathfrak{F}$ is given in Theorem 29; it should be noted however that this construction does not respect well-foundedness – the forest $\mathfrak{F}$ is not well-founded, even when the tree $\mathfrak{T}$ is. In fact, a construction for producing such a bounded forest $\mathfrak{F}$ that also respects well-foundedness does not exist: the tree of Example 34 is a model of the first-order theory of the class of bounded trees, yet for $k \geqslant 6$, this tree has no well-founded bounded end-extension that is k-equivalent to it.

The main problem of axiomatising the first-order theories of the classes of bounded and φ-bounded trees, and of devising a construction for producing, from models of these axiomatic theories, k-equivalent trees that are bounded and φ-bounded respectively, remains. From the perspective of the axiomatisations given in Section 3, it is furthermore needed that these constructions respect well-foundedness. This, apart from being mathematically interesting in its own right, will then also give a full axiomatisation of the first-order theory of the class of α-trees.

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