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Brief paper

Semi-global finite-time observers for nonlinear systems☆

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ABSTRACT

It is well known that high gain observers exist for single output nonlinear systems that are uniformly observable and globally Lipschitzian. Under the same conditions, we show that these systems admit semi-global and finite-time converging observers. This is achieved with a derivation of a new sufficient condition for local finite-time stability, in conjunction with applications of geometric homogeneity and Lyapunov theories.

1. Introduction

Research on nonlinear observers has achieved remarkable progress since the formal introduction of the concept and the Lyapunov approach based results of existence and design in Thau (1973). With the advance of the nonlinear observability theory (Hermann & Krener, 1997) in the differential geometric framework (Isidori, 1995), quite a number of early works have been devoted to establishing the link between nonlinear observer and nonlinear observability. The existence of exponential observers is closely related to the observability of the linearized system (Kou, Elliott, & Tarn, 1975; Xia & Gao, 1988). Uniform observability of a single output nonlinear system results in a triangular structure useful for observer design (see Gauthier, Hammouri, and Othman (1992); Gauthier and Kupka (1994); Hammouri, Targui, and Armanet (2002) and their other works). These findings are employed in all three major classes of nonlinear observer design methods that abound in the literature. Linearized observability is a standing assumption for both the Lyapunov based approach (Raghavan & Hedrick, 1994; Thau, 1973) and the observer canonical form approach (Bestle & Zeitz, 1983; Krener & Isidori, 1983). High-gain observers are very much associated with the triangular structure derived from the uniform observability of nonlinear systems (Gauthier et al., 1992; Gauthier & Kupka, 1994). New developments of all three design methods have been carried out in various directions (Kazantzis & Kravaris, 1998; Krener & Respondek, 1985; Rajamani & Cho, 1998; Shim, Son, & Seo, 2001; Xia & Gao, 1989).

Observers with finite-time convergence have certain advantages and are therefore desirable in some situations of control and supervision (Menold, Findeisen, & Allgöwer, 2003a). There is a series of methods that achieve finite-time convergence (Engel & Kreisselmeier, 2002; Haskara, Ozguner, & Utkin, 1998; Hong, Huang, & Xu, 2001; Michalska & Mayne, 1995). Some of these observers, such as the sliding mode observers, are not continuous. The continuity property and its importance in finite-time stability are realized in Bhat and Bernstein (2000, 2005). It is also interesting to point out that continuous observers are realized to be different and unique in the nonlinear context (Krener, 1986; Xia & Zeitz, 1997). For instance, linearized observability is no longer necessary for the existence of a continuous observer (Xia & Zeitz, 1997). A first approach to design such an observer is a dedicated introduction of time-delay in the observers (Engel & Kreisselmeier, 2002). This approach was extended to linear time-varying systems in Menold et al. (2003a) and to nonlinear systems that can be transformed into the observer canonical form Menold, Findeisen, and Allgöwer (2003b). Sauvage, Guay, and Dochain (2007) also proposed nonlinear finite-time observers for a class of nonlinear systems, with a time-delay in the observers. A finite-time observer for a class of observer error linearizable systems has recently been constructed in Perruquetti, Floquet, and Moulay (2008). The major technique used is homogeneity (Qian & Lin, 2001).

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The aim of this paper is to prove a general result: a uniformly observable and globally Lipschitzian single output nonlinear system admits semi-global finite-time observers. This paper is organized as follows. The definition of finite-time stability and its criteria are reviewed in Section 2. In Section 3, we present the semi-globally finite-time stable observers for single output nonlinear systems. Finally, the paper is concluded in Section 4.

2. Preliminaries

Consider the following system

$$\dot{x} = f(x(t)), \quad f(0) = 0, \quad x \in \mathcal{R}^n, \quad x(0) = x_0, \tag{1}$$

where $f : \mathcal{D} \to \mathcal{R}^n$ is continuous on an open neighborhood $\mathcal{D}$ of the origin $x = 0$.

Definition 1 (*Bhat & Bernstein, 2000*). The zero solution of (1) is *finite-time convergent* if there is an open neighborhood $\mathcal{U} \subset \mathcal{D}$ of the origin and a function $T : \mathcal{U} \setminus \{0\} \to (0, \infty)$, such that $\forall x_0 \in \mathcal{U}$, the solution $\psi(t, x_0)$ of system (1) is defined and $\psi(t, x_0) \in \mathcal{U} \setminus \{0\}$ for $t \in [0, T(x_0))$, and $\lim_{t \to T(x_0)} \psi(t, x_0) = 0$. Then, $T(x_0)$ is called *the settling time*. If the zero solution of (1) is finite-time convergent, the set of point x_0 such that $\psi(t, x_0) \to 0$ is called the *domain of attraction of the solution*. The zero solution of (1) is *finite-time stable* if it is Lyapunov stable and finite-time convergent. When, $\mathcal{U} = \mathcal{D} = \mathcal{R}^n$, the zero solution is said to be *globally finite-time stable*.

For example

$$\dot{y}(t) = -l\lceil y(t) \rfloor^{\alpha} + ky(t), \quad y(0) = x, \tag{2}$$

where $\lceil y \rfloor^{\alpha} = |y|^{\alpha}\mathrm{sign}(y)$, $l, k > 0$, $\alpha \in (0, 1)$, is continuous everywhere and locally Lipschitzian everywhere except at the origin. Hence every initial solution in $\mathcal{R} \setminus \{0\}$ has a unique solution. If $|x|^{1-\alpha} < \frac{l}{k}$, multiplying (2) by e^{-kt}, we have

$$\frac{\mathrm{d}(\mathrm{e}^{-kt}y(t))}{\mathrm{d}t} = -l|y(t)\mathrm{e}^{-kt}|^{\alpha}\mathrm{e}^{(\alpha-1)kt}\mathrm{sign}(y(t)).$$

The solution trajectories are unique and described by

$$\mu(t, x) = \begin{cases} \mathrm{sign}(x)\mathrm{e}^{kt}\left[|x|^{1-\alpha} - \dfrac{l}{k} + \dfrac{l}{k}\mathrm{e}^{k(\alpha-1)t}\right]^{\frac{1}{1-\alpha}}, \\ \quad t < \dfrac{\ln(1 - \frac{k}{l}|x|^{1-\alpha})}{k(\alpha - 1)}, \quad 0 < |x|^{1-\alpha} < \dfrac{l}{k}, \\ 0, \quad t \geq \dfrac{\ln(1 - \frac{k}{l}|x|^{1-\alpha})}{k(\alpha - 1)}, \\ 0, \quad t \geq 0, x = 0. \end{cases} \tag{3}$$

Clearly, the solutions initiated at $x : |x|^{1-\alpha} < \frac{l}{k}$, converge to $y = 0$ in finite time.

Lemma 1. *Suppose there is a Lyapunov function $V(x)$ defined on a neighborhood $\mathcal{U} \subset \mathcal{R}^n$ of the origin, and*

$$\dot{V}(x) \leq -lV(x)^{\alpha} + kV(x), \quad \forall x \in \mathcal{U} \setminus \{0\}. \tag{4}$$

Then, the origin of (1) is finite-time stable. The set

$$\Omega = \left\{x | V(x)^{1-\alpha} < \frac{l}{k}\right\} \cap \mathcal{U} \tag{5}$$

is contained in the domain of attraction of the origin. The settling time satisfies $T(x) \leq \frac{\ln(1-\frac{k}{l}V(x)^{1-\alpha})}{k(\alpha-1)}$, $x \in \Omega$.

Proof. Note that the following inequality holds:

$$\dot{V}(x) \leq -lV(x)^{\alpha}\left(1 - \frac{k}{l}V(x)^{1-\alpha}\right) < 0, \quad \forall x \in \Omega \setminus \{0\}.$$

Since V is positive definite and $\dot{V}$ takes negative values on $\Omega \setminus \{0\}$, Ω is forward invariant. Moreover, $x = 0$ is the unique solution of (1) satisfying $x(0) = 0$ (Yoshizawa, 1966). Thus every initial condition $x \in \Omega$ has a unique solution $\psi(t, x) \in \Omega$. Consider $x \in \Omega \setminus \{0\}$, which results in

$$\dot{V}(\psi(t, x)) \leq -lV(\psi(t, x))^{\alpha} + kV(\psi(t, x)). \tag{6}$$

Next, applying the comparison lemma to differential inequality (6) and the differential equation (2) yields

$$V(\psi(t, x)) \leq \mu(t, V(x)), \tag{7}$$

where μ is given by (3). It follows from (3) and (7) that

$$\psi(t, x) = 0, \quad t \geq \frac{\ln(1 - \frac{k}{l}V(x)^{1-\alpha})}{k(\alpha - 1)}, \forall x \in \Omega. \tag{8}$$

Obviously, the set Ω is contained in the domain of attraction of the origin.

Now, consider the following system:

$$\dot{x} = f(x, u), \tag{9}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^p$ are the states and inputs of the system, respectively. $f : \mathcal{R}^n \times \mathcal{R}^p \to \mathcal{R}^n$ is assumed to be smooth enough, and $f(0, 0) = 0$. The state variables x are not available for direct measurement, only outputs $y \in \mathcal{R}^m$ are available:

$$y = h(x), \tag{10}$$

where $h : \mathcal{R}^n \to \mathcal{R}^m$ and is smooth enough. We give the following definition: □

Definition 2. Let a dynamic system be described by

$$\dot{z} = g(z, y, u), \tag{11}$$

in which $z \in \mathcal{R}^n$, and $g : \mathcal{R}^n \times \mathcal{R}^m \times \mathcal{R}^p \to \mathcal{R}^n$ is continuous. Denote the solution of (9) and (11) with respect to the corresponding input functions and passing through x_0 and z_0 respectively as $x(t, x_0, u)$ and $z(t, z_0, y, u)$, respectively. We denote $x(t, x_0, u)$ simply by $x(t)$, and $z(t, z_0, h(t, x_0, u), u)$ by $z(t)$. If

(i) $z_0 = x_0$ implies $z(t) = x(t)$, for $t \geq 0$ and u;

(ii) there exists an open neighbourhood $\mathcal{U} \subset \mathcal{R}^n$ of the origin such that $e_0 = z_0 - x_0 \in \mathcal{U}$ implies $z(t) - x(t) \in \mathcal{U}$ and a function $T : \mathcal{U} \setminus \{0\} \to (0, \infty)$, such that

$$\|z(t) - x(t)\| \to 0, \quad \text{as } t \to T(e_0), \tag{12}$$

then, the system (11) is called a *finite-time observer* of the system (9) and (10). All points $e_0 = z_0 - x_0$ such that (12) holds constitute *a domain of observer attraction*. If the open set $\mathcal{U}$ can be chosen as $\mathcal{R}^n$, then (11) is called a *global finite-time observer*. If for any given compact $\mathcal{W} \subset \mathcal{R}^n$ containing the origin, there exists a finite-time observer of the form (11), such that $\mathcal{W}$ is contained in the domain of observer attraction, then (9) and (10) are said to admit *semi-global finite-time observers*.

3. Finite-time observers

Consider a single output nonlinear system

$$\Gamma : \begin{cases} \dot{z} = F(z) + \displaystyle\sum_{i=1}^{p} G_i(z)u_i, \\ y = h(z), \end{cases} \tag{13}$$

where $z \in \mathcal{R}^n$, $u = [u_1, \dots, u_p]^{\mathrm{T}} \in \mathcal{R}^p$ and $y \in \mathcal{R}$. If (Γ) is uniformly observable for any uniformly bounded input (Gauthier et al., 1992). Then, a coordinate change can be found to transform the system (13) into the form (Hammouri et al., 2002)

$$\begin{cases} \dot{x}_1 = x_2 + \sum_{i=1}^{p} g_{1i}(x_1)u_i, \\ \dot{x}_2 = x_3 + \sum_{i=1}^{p} g_{2i}(x_1, x_2)u_i, \\ \vdots \\ \dot{x}_n = f(x_1, \dots, x_n) + \sum_{i=1}^{p} g_{ni}(x_1, \dots, x_n)u_i, \\ y = x_1 = C_0 x, \quad C_0 = [1, \dots, 0], \end{cases} \tag{14}$$

where f and g_{ij} $(i = 1, \dots, n, j = 1, \dots, p)$ are continuous functions with $f(0) = 0$, $g_{ij}(0, \dots, 0) = 0$. In addition, g_{ij} and f satisfy the global Lipschitzian condition with Lipschitzian constant l. For $p = 1$, by introducing $S(\theta) = S^{\mathrm{T}}(\theta) > 0$ which satisfies $-\theta S(\theta) - A_0^{\mathrm{T}} S(\theta) - S(\theta)A_0 + C_0^{\mathrm{T}} C_0 = 0$ and $S(\theta) \geq \delta_0 I$, an exponential observer has been built in Gauthier et al. (1992), where A_0 is the anti-shift operator $A_0 : \mathcal{R}^n \to \mathcal{R}^n$, $A_{0i,j} = \delta_{i,j-1}$, and $\delta_0 > 0$ is a scalar. In this paper, the observer of the system (14) can be designed as follows:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + s_1 \lceil e_1 \rfloor^{\alpha_1} + \sum_{j=1}^{p} g_{1j}(\hat{x}_1)u_j, \\ \dot{\hat{x}}_2 = \hat{x}_3 + s_2 \lceil e_1 \rfloor^{\alpha_2} + \sum_{j=1}^{p} g_{2j}(\hat{x}_1, \hat{x}_2)u_j, \\ \vdots \\ \dot{\hat{x}}_n = s_n \lceil e_1 \rfloor^{\alpha_n} + f(\hat{x}_1, \dots, \hat{x}_n) + \sum_{j=1}^{p} g_{n,j}(\hat{x}_1, \dots, \hat{x}_n)u_j, \end{cases} \tag{15}$$

where $[s_1\ s_2\ \cdots\ s_n]^{\mathrm{T}} = S^{-1}(\theta)C_0^{\mathrm{T}}$ and $\alpha_i = i\alpha - (i-1)(i = 1, \dots, n)$, $\alpha \in (0, 1]$. The dynamics of the observation error $e = x - \hat{x}$ is given by

$$\begin{cases} \dot{e}_1 = e_2 - s_1 \lceil e_1 \rfloor^{\alpha_1} + \tilde{f}_1, \\ \dot{e}_2 = e_3 - s_2 \lceil e_1 \rfloor^{\alpha_2} + \tilde{f}_2, \\ \vdots \\ \dot{e}_n = -s_n \lceil e_1 \rfloor^{\alpha_n} + \tilde{f}_n, \end{cases} \tag{16}$$

where $\tilde{f}_1 = \sum_{j=1}^{p}(g_{1j}(x_1) - g_{1j}(\hat{x}_1))u_j$, $\tilde{f}_2 = \sum_{j=1}^{p}(g_{2j}(x_1, x_2) - g_{2j}(\hat{x}_1, \hat{x}_2))u_j, \dots, \tilde{f}_n = f(x_1, \dots, x_n) - f(\hat{x}_1, \dots, \hat{x}_n) + \sum_{j=1}^{p}(g_{nj}(x_1, \dots, x_n) - g_{nj}(\hat{x}_1, \dots, \hat{x}_n))u_j$, $S(\theta)$ is the same as in Gauthier et al. (1992).

Now, we are ready to state our main result.

Theorem 1. *Assume that the input $u \in \mathcal{R}^p$ uniformly bounded by some $u_0 \geq 0$, and the nonlinear system (13) is uniformly observable and globally Lipschitzian. Then, it admits semi-global finite-time high gain observers.*

The proof of Theorem 1 is divided into the following several parts.

First, we focus on (16) without $\tilde{f}_i$, i.e.,

$$\begin{cases} \dot{e}_1 = e_2 - s_1 \lceil e_1 \rfloor^{\alpha_1}, \\ \dot{e}_2 = e_3 - s_2 \lceil e_1 \rfloor^{\alpha_2}, \\ \vdots \\ \dot{e}_n = -s_n \lceil e_1 \rfloor^{\alpha_n}. \end{cases} \tag{17}$$

Lemma 2 (Perruquetti et al., 2008). *For $\alpha > 1 - \frac{1}{n-1}$, the system (17) is homogeneous of degree $\alpha - 1$ with respect to the weights $\{(i-1)\alpha - (i-2)\}_{1 \leq i \leq n}$.*

Lemma 3 (Perruquetti et al., 2008). *There exists $\varepsilon_1 \in (1 - \frac{1}{n-1}, 1]$ such that for all $\alpha \in (1 - \varepsilon_1, 1)$, (17) is globally finite-time stable.*

A proof of this can be found in Perruquetti et al. (2008), with the following Lyapunov function

$$V_\alpha(e) = \tilde{e}^{\mathrm{T}} S(\theta) \tilde{e}, \tag{18}$$

where $\tilde{e} = \left(\lceil e_1 \rfloor^{\frac{1}{r}} \dots \lceil e_n \rfloor^{\frac{1}{\alpha_{n-1} r}}\right)^{\mathrm{T}}$, $e = (e_1 \cdots e_n)^{\mathrm{T}}$, $r = \prod_{i=1}^{n-1}[(i-1)\alpha - (i-2)]$. Moreover, by Lemma 4.2 (Bhat & Bernstein, 2005), we have

$$\begin{aligned} &-c_1(\alpha, \theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2} + \alpha - 1}{\frac{1}{r^2}}} \leq L_{f_\alpha} V_\alpha(e) \\ &\leq -c_2(\alpha, \theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2} + \alpha - 1}{\frac{1}{r^2}}}, \end{aligned} \tag{19}$$

where $c_1(\alpha, \theta) = -\min_{\{z: V_\alpha(z)=1\}} L_{f_\alpha} V_\alpha(z)$ and $c_2(\alpha, \theta) = -\max_{\{z: V_\alpha(z)=1\}} L_{f_\alpha} V_\alpha(z)$.

The above construction of homogeneity and proof are also similar to those in Perruquetti et al. (2008), which are actually rooted in Bhat and Bernstein (2000). The above proof is independent of θ. However, $c_2(\alpha, \theta)$ in (19) has the following property.

Lemma 4. *$c_2(\alpha, \theta)$ satisfies $\lim_{\alpha \to 1} c_2(\alpha, \theta) = \theta$.*

Proof. It can be easily verified that $\max_{\{e: V_1(e)=1\}} L_{f_1} V_1(e) = \max_{\{e: V_1(e)=1\}} \left[-\theta e^{\mathrm{T}} S(\theta) e - e_1^2\right] = -\theta$. It is obvious that $L_{f_1} V_1(e^*) = -\theta$, where $e^* = [0\ 0\ \cdots\ 0\ \frac{1}{\sqrt{s_{nn}}}]^{\mathrm{T}}$ and $s_{nn} = [S(\theta)]_{n,n}$. Because there is a one-to-one correspondence between the set $\{z : V_a(z) = 1\}$ and $\{z : V_1(z) = 1\}$, that is for any $z = [z_1, \dots, z_n]^{\mathrm{T}} \in \{z : V_a(z) = 1\}$, there is a $\bar{z} = \left[\lceil z_1 \rfloor^{\frac{1}{r}}, \dots, \lceil z_n \rfloor^{\frac{1}{\alpha_{n-1} r}}\right] \in \{z : V_1(z) = 1\}$ and $\lim_{\alpha \to 1} \|\bar{z} - z\|_2 = 0$. Since $L_{f_\alpha} V_\alpha(z)$ is continuous, then, for any $\epsilon, \epsilon_1 > 0$, there exists $\eta > 0$, when $|\alpha - 1| < \eta$, $\|z - \bar{z}\|_2 < \epsilon_1$, resulting in $L_{f_1} V_1(\bar{z}) - \epsilon < L_{f_\alpha} V_\alpha(z) < L_{f_1} V_1(\bar{z}) + \epsilon$. Therefore, $\max_{\{z: V_\alpha(z)=1\}} L_{f_\alpha} V_\alpha(z) < \max_{\{\bar{z}: V_1(\bar{z})=1\}} L_{f_1} V_1(\bar{z}) + \epsilon = -\theta + \epsilon$. Then, $\overline{\lim}_{\alpha \to 1} \max_{\{e: V_\alpha(e)=1\}} L_{f_\alpha} V_\alpha(e) \leq -\theta$.

On the other hand, let $e^{**} = \left[0\ 0\ \cdots\ 0\ s_{nn}^{\frac{-\alpha_{n-1} r}{2}}\right]^{\mathrm{T}}$, then $e^{**} \in \{e : V_\alpha(e) = 1\}$, and $\lim_{\alpha \to 1} L_{f_\alpha} V_\alpha(e^{**}) = L_{f_1} V_1(e^*) = -\theta$. Then, $\max_{\{e: V_\alpha(e)=1\}} L_{f_\alpha} V_\alpha(e) \geq L_{f_\alpha} V_\alpha(e^{**})$. Therefore, $\underline{\lim}_{\alpha \to 1} \max_{\{e: V_\alpha(e)=1\}} L_{f_\alpha} V_\alpha(e) \geq \lim_{\alpha \to 1} L_{f_\alpha} V_\alpha(e^{**}) = -\theta$. Then, $\lim_{\alpha \to 1} \max_{\{e: V_\alpha(e)=1\}} L_{f_\alpha} V_\alpha(e) = -\theta$. Thus, the proof is completed. □

Lemma 5. *When $\alpha = 1$, for $u \in \mathcal{R}^p$ uniformly bounded by some $u_0 \geq 0$, there exists a large enough $\theta_1 \geq 1$, such that if $\theta \geq \theta_1$, then (16) is exponentially stable.*

Proof. Using the techniques in Gauthier et al. (1992), we can obtain the result easily. □

For the system (14) with $x_0 \in \mathcal{R}^n$, and the system (15) initiated at $\hat{x}_0 \in \mathcal{R}^n$, we have the following proposition.

Lemma 6. *For the system (16), there exists $\varepsilon_2 \in [1 - \frac{1}{n-1}, 1)$ such that for all $\alpha \in (1 - \varepsilon_2, 1]$, the following inequalities hold:*

$$V_\alpha(e) \leq S\|e_0\|_2, \quad \forall t > 0, \tag{20}$$

$$\|\tilde{e}\|_2 \leq \frac{S}{\delta_0}\|e_0\|_2, \quad \forall t > 0, \tag{21}$$

where $V_\alpha(e)$ and $\tilde{e}$ are given by (18), $e_0 = x_0 - \hat{x}_0$, $S = \max_{i,j} |S(1)|_{i,j}$ and $\delta_0 > 0$ is a scalar. Moreover, for $i = 2, \ldots, n$, $k = 1, \ldots, i$, there exists $\theta_2 \geq 1$ such that if $\theta \geq \theta_2$, the following inequalities hold

$$|e_k(t)|^{\frac{1}{\alpha_{i-1}r}}/\theta^i \leq |e_k(t)|^{\frac{1}{\alpha_{k-1}r}}/\theta^k. \tag{22}$$

Proof. Let $d = e_0^{\mathrm{T}} S(\theta) e_0$, $\mathcal{A}' = V_\alpha^{-1}([0,d])$, $\mathcal{S}' = V_1^{-1}(\{d\})$. Let f'_α denote the vector field of system (16). Then, $\mathcal{A}'$ and $\mathcal{S}'$ are compact. Define $\varphi' : (0,d] \times \mathcal{S}' \to \mathcal{R}$ by $\varphi'(\alpha, e) = L_{f'_\alpha} V_\alpha(e)$. Then φ' is continuous and by Lemma 5 satisfies $\varphi'(1,e) < 0$, therefore, there exists $\varepsilon_2 > 0$ such that $\varphi'((1-\varepsilon_2, 1] \times \mathcal{S}') \subset (-\infty, 0)$. Thus, for $\alpha \in (1-\varepsilon_2, 1]$, $L_{f'_\alpha} V_\alpha$ takes negative values on $\mathcal{S}'$. Therefore, $\mathcal{A}'$ is strictly positive invariant under f'_α for every $\alpha \in (1-\varepsilon_2, 1]$, then $\tilde{e}^{\mathrm{T}} S(\theta)\tilde{e} \leq e_0^{\mathrm{T}} S(\theta) e_0$. Since $S(\theta) \geq \delta_0 I$ (Gauthier et al., 1992), we have $\delta_0 \|\tilde{e}\|_2 \leq \tilde{e}^{\mathrm{T}} S(\theta)\tilde{e} \leq e_0^{\mathrm{T}} S(\theta) e_0 \leq S\|e_0\|_2$. If $\|e_0\|_2 \leq 1$, since $1 \leq \frac{1}{r} \leq \frac{1}{\alpha_1 r} \leq \cdots \leq \frac{1}{\alpha_{n-1} r}$ and $\theta \geq 1$, it is obvious that inequalities (22) hold. If $|e_k(t)| > 1$, it follows from (21) that $e_k(t)$ is bounded. Then, there exists θ_2 such that if $\theta \geq \theta_2$, the inequalities (22) hold. $\square$

Now, calculating the derivative of $V_\alpha(e)$ as defined in (18) along the solution of system (16) by noting that $\frac{\mathrm{d}}{\mathrm{d}t}\lceil e_i \rfloor^{\alpha_i} = \alpha_i |e_i|^{\alpha_i - 1}$ (Hong, 2002), we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\alpha(e)_{(16)} = \frac{\mathrm{d}}{\mathrm{d}t} V_\alpha(e)_{(17)} + 2\tilde{e}^{\mathrm{T}} S(\theta) \begin{bmatrix} \frac{1}{r}|e_1|^{\frac{1}{r}-1}\tilde{f}_1 \\ \frac{1}{\alpha_1 r}|e_2|^{\frac{1}{\alpha_1 r}-1}\tilde{f}_2 \\ \vdots \\ \frac{1}{\alpha_{n-1} r}|e_n|^{\frac{1}{\alpha_{n-1} r}-1}\tilde{f}_n \end{bmatrix}$$

$$\leq -c_2(\alpha,\theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}} + 2l(u_0+1)p\left[\tilde{e}^{\mathrm{T}} S(\theta)\tilde{e}\right]^{\frac{1}{2}}$$

$$\times \left[\sum_{i,j} \frac{|S(1)_{i,j}|}{\theta^{i+j-1}} \frac{|e_i|^{\frac{1}{\alpha_{i-1}r}-1} \sum_{k=1}^{i} |e_k|}{\alpha_{i-1} r} \times \frac{|e_j|^{\frac{1}{\alpha_{j-1}r}-1} \sum_{k=1}^{j} |e_k|}{\alpha_{j-1} r}\right]^{\frac{1}{2}}.$$

By Lemma 2.4 (Qian & Lin, 2001), there exist positive constants $\bar{c}_i$ $(1 \leq i \leq n)$ such that the following inequalities hold.

$$\sum_{k=1}^{i} |e_i|^{\frac{1}{\alpha_{i-1}r}-1}|e_k|$$

$$\leq \sum_{k=1}^{i}\left[\bar{c}_i |e_i|^{\frac{1}{\alpha_{i-1}r}} + \alpha_{i-1} r \left(\frac{1-\alpha_{i-1}r}{\bar{c}_i}\right)^{\frac{1}{\alpha_{i-1}r}-1} |e_k|^{\frac{1}{\alpha_{i-1}r}}\right]$$

$$\triangleq \sum_{k=1}^{i} b_{i,k} |e_k|^{\frac{1}{\alpha_{i-1}r}},$$

where $b_{i,k} > 0$. Let $b = \max_{i,k} b_{i,k}$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\alpha(e)_{(16)} \leq -c_2(\alpha,\theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}} + \frac{2bl(u_0+1)pS^{\frac{1}{2}}}{\alpha_{n-1} r}$$

$$\times \left[\tilde{e}^{\mathrm{T}} S(\theta)\tilde{e}\right]^{\frac{1}{2}} \left[\sum_{i,j}\left(\sum_{k=1}^{i} \frac{e_k^{\frac{2}{\alpha_{k-1}r}}}{\theta^{2k}}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{j} \frac{e_k^{\frac{2}{\alpha_{k-1}r}}}{\theta^{2k}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{23}$$

Let $\xi_k = \frac{\lceil e_k \rfloor^{\frac{1}{\alpha_{k-1}r}}}{\theta^k}$, for $\theta \geq \max\{\theta_1, \theta_2\} \geq 1$, which results in

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\alpha(e)_{(16)} \leq -c_2(\alpha,\theta)[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}}$$

$$+ \frac{2n^2 l(u_0+1)pbS^{\frac{1}{2}}\theta^{\frac{1}{2}}}{\alpha_{n-1} r}\left[\tilde{e}^{\mathrm{T}} S(\theta)\tilde{e}\right]^{\frac{1}{2}}\left(\sum_{k=1}^{n}\xi_k^2\right)^{\frac{1}{2}}. \tag{24}$$

On the other hand, let $\xi = [\xi_1, \xi_2, \ldots, \xi_n]^{\mathrm{T}}$, note that $S(\theta) \geq \delta_0 I$, then,

$$\sum_{k=1}^{n}\xi_k^2 \leq \frac{1}{\delta_0}\xi^{\mathrm{T}} S(1)\xi = \frac{1}{\theta\delta_0}\left[\lceil e_i \rfloor^{\frac{1}{\alpha_{i-1}r}} \frac{S(1)_{i,j}}{\theta^{i+j-1}} \lceil e_j \rfloor^{\frac{1}{\alpha_{j-1}r}}\right]_{i,j}$$

$$= \frac{1}{\theta\delta_0}\left[\lceil e_i \rfloor^{\frac{1}{\alpha_{i-1}r}} S(\theta)_{i,j} \lceil e_j \rfloor^{\frac{1}{\alpha_{j-1}r}}\right]_{i,j} = \frac{1}{\theta\delta_0} V_\alpha(e). \tag{25}$$

It follows from (24) and (25) that

$$\dot{V}_\alpha(e) \leq -c_2[V_\alpha(e)]^{\frac{\frac{1}{r^2}+\alpha-1}{\frac{1}{r^2}}} + c_3 V_\alpha(e), \tag{26}$$

where

$$c_3 = \frac{2n^2 l(u_0+1)pbS^{\frac{1}{2}}}{\alpha_{n-1} r \delta_0^{\frac{1}{2}}}. \tag{27}$$

Now, we can summarize the proof for our main theorem.

Proof of Theorem 1. For any given compact set $\mathcal{U} \subset \mathcal{R}^n$ containing the origin, for system (14) on $\mathcal{R}^n \times \mathcal{R}^p$, define a system (15) on $\mathcal{R}^n \times \mathcal{R}^p$, we can choose an $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ such that for all $\alpha \in [1-\varepsilon, 1)$ and $\theta \geq \max\{\theta_1, \theta_2\}$, $c_2(\alpha,\theta)$ satisfies $c_2(\alpha,\theta) \geq \frac{\theta}{2}$. By (26) and Lemma 1, "the domain of observer attraction", by an abuse of terminology (since observer convergence has not yet been obtained), is given by

$$\Omega = \left\{e : V_\alpha(e) < (c_2/c_3)^{\frac{1}{r^2(1-\alpha)}}\right\}. \tag{28}$$

Due to the properties of c_2 (Lemma 4) and the specific form of c_3 in (27), we can choose sufficiently large $\theta \geq \max\{\theta_1, \theta_2\}$ such that $\mathcal{U} \subset \left\{e : S\|e\|_2 < (c_2/c_3)^{\frac{1}{r^2(1-\alpha)}}\right\}$. Then, by (20) and (28), $\mathcal{U} \subset \Omega$. Thus, the system (13) admits semi-global finite-time observers. $\square$

By incorporating an update law for gain and higher order output error terms, an extension of the well-known high gain observer was recently presented by Andrieu, Praly, and Astolfi (2007). However, our technique in this paper allows us to obtain semi-global results. It might be possible to obtain a global result instead of the semi-global ones expressed here by adding a linear term to the homogeneous gain. We will discuss this issue elsewhere.

4. Conclusion

There are high gain observers for single output nonlinear systems, that are uniformly observable and globally Lipschitzian. Under the same conditions, we showed that for these systems the uniform observability and the global Lipschitzian properties imply the existence of semi-global and finite-time converging observers. This was achieved with a derivation of a new sufficient condition for local finite-time stability, together with applications of geometric homogeneity and Lyapunov theories. It could however be noted that non-locally Lipschitzian functions are employed in the observer dynamics. At a digital implementation level, discretizing such dynamics and disturbances may introduce chattering before achieving convergence.

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