EOQ models for deteriorating items with substitutable and mutual complementary price, stock, and time-dependent demand

David C Mahlangu
14232350

Submitted in partial fulfilment of the requirements for the degree

Master of Engineering (Industrial Engineering)

in the Faculty of Engineering, Built Environment, and Information Technology

University of Pretoria

February 25, 2022

Department of Industrial and Systems engineering

University of Pretoria

EOQ models for deteriorating items with substitutable and mutual complementary price-, stock-, and time-dependent demand

David Mahlangu[1*] Olufemi Adetunji[1#] Makoena Sebatjane[1#]

davidmahlangu@up.ac.za olufemi.adetunji@up.ac.za makoena.sebatjane@up.ac.za

[1]Department of Industrial and Systems Engineering, University of Pretoria, Pretoria 0002, South Africa

[*]Corresponding author, [#]Supervisor

Abstract

The common assumption made in the classical economic order (EOQ) model is that an item has a finite life and with demand that is constant and independent. In practice, however, this is not always the case. The demand for an item may be dependent on other factors such as time, its stock level and price of other items such as its substitute and complement items. Moreover, some items deteriorate in nature and do not necessarily have a fixed shelf life as the classical EOQ model suggests. These inventory phenomena are apparent in most supply chains in today's market and have added more complexity in management of these type of inventories. The characteristics displayed by these inventories has caught the attention of many researchers of lately.

In this dissertation two inventory models are developed where the second model is the extension of the first model. The first model is developed to find the optimal values of the selling price and cycle length that maximises profit for two mutually complementary items, where these items are subject to deterioration. The demand for these items is given by the exponential function which is dependent on products selling prices, complement product selling price and time. Moreover, the ordering cost of the products is assumed to be made up of fixed and variable components.

The second model as the extension of the first model, considers a three-item EOQ policy, where Product 1 as the main item has a demand that is dependent on the following: its selling price and stock level, the selling price of the mutual complement (Product 2) as well as the selling price of its substitute (Product 3) and time. Product 2 has demand that is dependent on its selling price, the selling price of its complement (Product 1) and time. Product 3 has a demand that is dependent on its selling price, the selling price of its substitute (Product 1) and time. An example of a practical scenario for this model is in a perishable chicken feed supply chain between supplier of medicines used in chicken feed production and a manufacturer of animal feed. Typical in this node of the supply chain the manufacturer of animal feed may source a variety of medications from the same supplier. Due to certain nutritional specification of the feed formula, some of these medicines must be used together (complementarily) as part of the feed Bill of Material (BOM). Moreover, these complementary items can further be used interchangeably or substituted with an alternative item in an instance when there is stock out. Furthermore, stock display or availability at supplier of these medicines has a potential to stimulate demand as purchasing departments of animal feed manufactures are fond with stocking up on these medicines when there is more available.

The models presented in this dissertation can be said to be an extension of two models developed by other researchers namely Karaöz et al. (2011) and Ouyang et al. (2005), their models were developed for complementary and deterioration items, respectively. The two models are combined to address scenario presented in this study since there is currently limited or no study found by the researcher that addresses the combined effect of deteriorating items with substitutable and mutual complementary price-, stock-, and time-dependent demand in a single model. This research gap has given purpose to this dissertation. As such, the inventory models developed will contribute into the body of knowledge

in the observed research gap. The models are aimed at generating an inventory replenishment policy that determines optimal selling prices, cycle length and order quantity that maximizes profit.

The numerical example and sensitivity analysis have been conducted to assess the effectiveness of the results generated by the inventory models. With the test of optimality condition for the models, the results obtained indicate that the profit maximising function is concave, that is, negative (semi)definite.

Findings from the two models developed shows that an increase in either holding or ordering cost results in reduction of profit. Similarly, higher deterioration rate leads to reduction in profit. In practice, these findings indication that these costs need to be constantly reviewed and improved for better profit results. The optimal selling prices have been generated by the models, where the sensitivity tests shows that a change in some parameters of the models may increase or decrease the selling price of an item.

.

CONTENTS

List of Figures 5

List of Tables 6

Chapter 1 7

1 Introduction and background 7

1.1 Motivation and significance 8

1.2 Problem statement, objectives and research questions. 9

1.2.1 Problem statement 9

1.2.2 Objectives 9

1.2.3 Research questions 9

1.3 Scope of the study 9

1.4 Research Methodology 10

1.5 Dissertation outline 10

Chapter 2 11

2 Literature review 11

2.1 The basic work and foundation of EOQ model 11

2.2 Deteriorating items and stock dependent demand 13

2.3 Complementary and substitute items 14

Chapter 3 17

3 EOQ model for two deteriorating items with mutually complementary price dependent demand 17

3.1 Problem definition 17

3.2 Notations and assumptions 17

3.2.1 Notations 17

3.2.2 Assumptions 18

3.3 Model development 18

3.3.1 Mathematical model 18

3.4 Numerical results 24

3.4.1 Numerical example 24

3.4.2 Sensitivity analysis 25

3.5 Conclusion 32

Chapter 4 33

4 EOQ model for three deteriorating items with substitutable and mutual complementary price-, stock -, and time-dependent demand 33

4.1 Problem definition 33

4.2 Notations and assumptions 33

4.2.1 Notations 33
4.2.2 Assumptions 34
4.3 Model development 35
4.3.1 Mathematical model 35
4.4 Numerical results 43
4.4.1 Numerical example 43
4.4.2 Sensitivity analysis 44
4.5 Conclusion 52
Chapter 5 53
5 Conclusions 53
5.1 Summary of findings 53
5.2 Possible practical application of findings 53
5.3 Contributions 54
5.4 Future research 54
6 References 55

List of Figures

Figure 1: Research Methodology. 10
Figure 2: Literature review structure. 11
Figure 3: EOQ inventory system. 12
Figure 4: The EOQ cost model 13
Figure 5: Inventory levels of two deteriorating mutual complementary products at time. 18
Figure 6: Changes in profit due to parameter changes. 30
Figure 7: Change in T due to parameters changes. 31
Figure 8: Products' price dependency. 33
Figure 9: Inventory levels for the 3 products. 35
Figure 10: Changes in profit due to parameter changes. 50

LIST OF TABLES

Table 1: The classical EOQ model assumptions. 11
Table 2: Literature reviewed summary 16
Table 3: Notations used in deriving the inventory model. 17
Table 4 Numerical values 24
Table 5: Model results 25
Table 6: The effect of changing $a1$ while keeping other parameters to original numeric values. 25
Table 7: The effect of changing $a2$ while keeping other parameters to original numeric values. 26
Table 8: The effect of changing $b1$ while keeping other parameters to original numeric values. 26
Table 9: The effect of changing $b2$ while keeping other parameters to original numeric values. 26
Table 10: The effect of changing $\beta1$ while keeping other parameter to original numeric values. 27
Table 11: The effect of changing $\beta2$ while keeping other parameters to original numeric values. 27
Table 12: The effect of changing $\theta1$ while keeping other parameters to original numeric values. 27
Table 13: The effect of changing $\theta2$ while keeping other parameters to original numeric values. 28
Table 14: The effect of changing $h1$ while keeping other parameters to original numeric values. 28
Table 15: The effect of changing $h2$ while keeping other parameters to original numeric values. 28
Table 16: The effect of changing $c1$ while keeping other parameters to original numeric values. 29
Table 17: The effect of changing $c2$ while keeping other parameters to original numeric values. 29
Table 18: Inventory model notations (Parameters and decision Variables). 33
Table 19: Numerical values 43
Table 20: Model results 43
Table 21: Optimality conditions 43
Table 22: The effect of changing $a1$. 44
Table 23: The effect of changing $a2$. 44
Table 24: The effect of changing $a3$. 44
Table 25: The effect of changing $b1$. 45
Table 26: The effect of changing $b2$. 45
Table 27: The effect of changing $\Upsilon1$. 46
Table 28: The effect of changing $\Upsilon3$. 46
Table 29: The effect of changing $\theta1$. 46
Table 30: The effect of changing $\theta2$. 47
Table 31: The effect of changing $\theta3$. 47
Table 32: The effect of changing $h1$. 47
Table 33: The effect of changing $h2$ 48
Table 34: The effect of changing $h3$. 48
Table 35: The effect of changing $c1$. 48
Table 36: The effect of changing $c2$. 49
Table 37: The effect of changing $c3$. 49
Table 38: The effect of changing s. 49
Table 39: The effect of changing ω. 50

CHAPTER 1

1 INTRODUCTION AND BACKGROUND

Every supply chain keeps inventory at various points to ensure smooth and profitable running of the chain. A well-established inventory management system can be a firm's core competency and a differentiator in the market. Inventory management plays a significant role in the value chain of the firm since poorly managed inventory may lead to misalignment in the entire supply chain and result in customer dissatisfaction. The main objectives of inventory management are ensuring that the right level of inventory is kept at the right place and time and sold to customers at the right price. A firm seeks to achieve this objective to find a balance between supply and demand in the chain, such that customer needs are satisfied, and profit is maximized. In practice, the demand for various inventory items react differently to changes in the chain. Hence, inventory managers are compelled to apply varying inventory policies in order to maintain optimal inventory levels. In pursuit of these policies, the general doctrine is that inventory managers need to strike a balance between ordering and holding cost whilst ensuring customer demand is met,

Saeed and Morteza (2000), defines inventory as goods and materials that a business hold for resale or use in production. Inventory management has been the subject of interest in both academia and practice for over a century since the inception of the classical EOQ model, also known as economic lot-size model. It is one of the oldest and widely used inventory model in both practise and academia. The model was developed by Harris (1913), to help inventory managers with a tool to find optimal inventory replenishment policy. Several researchers have extended, and modified assumptions attached to the model to solve various and more realistic inventory scenarios. Similarly, this dissertation challenges some assumptions attached to the traditional EOQ model. The assumptions relaxed play a significant role on the inventory policy of various items in today's market. These are deterioration of items, substitutable and complementary items, stock-, and time-dependent demand items.

The classical EOQ model assumes that inventory depletes through sales demand only. The model ignores other aspects that may contribute to depletion of inventory. In practice, this may not always be the case, certain aspects such as direct spoilage, physical depletion and deterioration contribute to inventory depletion (Ghare & Schrader, 1963). The researchers Ghare and Schrader (1963) developed an inventory model whereby the inventory level of an item is not only consumed by demand only but also through deterioration or decay.

Contrary to the classical EOQ model, which suggests constant demand, an item may have demand that is dependent on the price of other items such as its substitute and complement items. Substitution occurs when the demand for an item is fulfilled by another item with similar features and attributes. Whereas complementarity of items occurs when certain items are sold together or consumed together for full utility by customers, these items are said to experience joint demand (Mokhtari, 2018).

As observed in many retail outlets, physical display of items is practiced in an effort to excite and influence customers to buy displayed stock. This phenomenon has been the subject of interest lately in both academia and practice as physical display of items is likely to stimulate demand.

Seeing that many researchers have relaxed one or more of the assumptions presented in this dissertation at different occasions and in separate papers, a holistic approach has been taken in this dissertation to incorporate all relaxed assumption into a single model. The assumptions are gradually relaxed as established in the two models developed in this dissertation. The first model has fewer assumptions relaxed while the second model is the extension of the initial model with all assumptions incorporated. The mathematical models developed integrate purchasing and marketing decisions which is aimed at assisting inventory managers on taking the right cause of action in the presence of deteriorating, complementary, substitutable items with stock and time-dependent demand.

1.1 MOTIVATION AND SIGNIFICANCE

Inventory management plays a crucial role in many industries and supply chains. In fact, a well-established inventory management policy can give a firm a competitive edge in its industry. In practice, many industries invest a lot of capital on inventory to in trying to improve firm's performance. Therefore, this means that it is becoming more imperative for a firm to understand critical factors that influence the behaviour of the inventory its dealing with, so that an inventory system can be devised to best serve the firm. In some industries this can be a complex and daunting task to perform, especially when the inventory presents complexity such as deterioration, complementary and substitutability of items with demand that is dependent on other factors such as stock and time. These type of inventory problems are normally studied and solved using optimisation models. There seem to be a gap in research on EOQ models covering the combined effect of deteriorating, substitution and complementary products and stock-, and time-dependent demand. This is evident by limited literature found in this area. The classical EOQ model has been used in many instances as an inventory optimisation tool. However, the model has been limited by its attached assumptions. Continued efforts from researchers, indicates that these assumptions have been relaxed and the model has been adapted to more realistic and practical situation. Similarly in this dissertation some assumption made in the classical EOQ model are relaxed.

In practice, it is seldom to find items with finite life especially in essential goods industries such as food and pharmaceutical industries. These industries are known for producing and distributing deteriorating items. This renders deterioration of items as a topic of practical importance in these industries. Moreover, in today's competitive market, demand is not solely dependent on products selling price, demand may be stimulated by factors such as display stock level on the floor, time, the selling price of complement and substitute products. In recently markets, these are typical the practical scenarios that many firms must deal with. An effort is made in this dissertation to develop models with a holistic view of the scenario that considers deteriorating items with substitutable and mutual complementary price, stock, and time-dependent demand.

The first model developed attempts to develop an inventory policy from a retail perspective when two deteriorating complementary products are being offered to the customers. The presences of complement products in the market presents retailers with the opportunity to develop pricing strategies that can optimise profits for the two products while providing optimal replenishment policy for the items. An example of this scenario is that of the bread and butter, these two products are complementary, that is they normally sell together. Bread deteriorates and loses its original quality over time. The same with butter, which loses its shelf life quickly especially when not stored correctly. Therefore, the model formulated aims at developing an inventory management tool that will aid in decision making when a retailer offers these types of products.

The second model developed in this dissertation aims at establishing an inventory optimisation tool which integrates purchasing and marketing decision all together, by using inputs from purchasing to determine optimal pricing strategy for three deteriorating, where two of these items are complements and one of these products is substitutable by the third product. The scenario defined by this model is in the setting of animal feed supply chain between medicines supplier and animal feed producer. Where the medicine supplier seeks to determine the optimal inventory policy to best serve the manufacturer of animal feed in the presence of complementary and substitutability of products (Medicines). Some medicines use in animal feed are complementary, meaning that these medicines should always be formulated together into a specific feed product. There are instances where one of the complement products is unavailable or the product is not in good condition to be used, in such a case, substitution with a product which has similar attribute occurs.

1.2 PROBLEM STATEMENT, OBJECTIVES AND RESEARCH QUESTIONS.

1.2.1 Problem statement

In practice, there exist a relationship between items such as complementary and substitute relationship. Moreover, these items may have deterioration characteristics. Complementarity of items suggest that the demand for these items is not always solely independent on their selling price. The demand for complementary items is sensitive to the prices of each other, as these are items are sold together for full utility. The main scenario considered in this study is in the area of perishable chicken feed supply chain between the animal feed producer and the supplier of complementary medication used in animal feed production for nutritional purposes. In an instant where one of the complement items (medication) is unavailable the customer (animal feed manufacturer) may choose to buy alternative substitute item with similar attributes. This means that in addition to complementary relationship that these items have, there exist a substitute relationship with the next product. Furthermore, inventory availability or levels of medication at supplier is likely to stimulate demand as numerous purchasing departments of animal feed producers aim to buy and secure enough stock for production purposes. As known, these items (medication) deteriorate in nature. Due to complexity brought by this type of inventory it becomes difficult to manage as there is numerous variables to be considered. In response to this need, this study seeks to develop an inventory system which addresses this complexity. Moreover, it has been observed that there is limited research work on EOQ models covering this complexity. As such this dissertation seeks to develop models that will bridge this research gap and make a contribution to the extensions of the classical EOQ model.

1.2.2 Objectives

The compelling needs in the problem statement motivates the following objectives for this study:

- To develop an inventory model that determines optimal values of the selling price and cycle length that maximises profit for two mutual complementary items that are subject to deterioration.
- To develop an inventory model which integrates purchasing and marketing decision all together, by using inputs from purchasing to determine optimal replenishment and pricing strategy for three deteriorating items, where two of these items are complements and one of these complement products is substitutable by the third product and has stock dependent demand.

1.2.3 Research questions

The following are research questions addressed by the two models developed in this study:

What should be the optimal selling price for items such that profit is maximised?

What should be the optimal replenishment cycle length?

What is the optimal order quantity for each item?

1.3 SCOPE OF THE STUDY

The first model considers two mutual complementary items that are subject to deterioration. While the second model considers three deteriorating items, with two of these items being complements and one of these complement products has stock dependent demand and is substitutable by the third product.

The two models developed in this dissertation addresses two scenarios. The first model considers a retail environment scenario, where the retailer wishes to find optimal inventory replenishment and pricing strategy when two mutual complementary deteriorating items are offered. The second model scenario is adapted to a setting of perishable chicken feed supply chain between supplier of medicine used in chicken feed and the manufacturer of chicken feed. The medicine offered by the supplier can be used complementarily or as substitutes by the manufacturer. Moreover, the availability or stock of these medications at supplier has a likelihood of stimulating demand as various manufactures prefer to buy more to mitigate risk. As known medicines do deteriorate over time. The scope of this dissertation is limited to these two models and considers scenarios as described above, however, any interested party may apply the models in similar scenarios.

1.4 RESEARCH METHODOLOGY

Numerous research methodologies have been presented by Bertrand and Fransoo (2002), for carrying out quantitative research under Operations Management (OM) and Operations Research (OR). In conducting this study, the research methodology proposed by Bertrand and Fransoo (2002) has been used as presented in **Figure 1**.

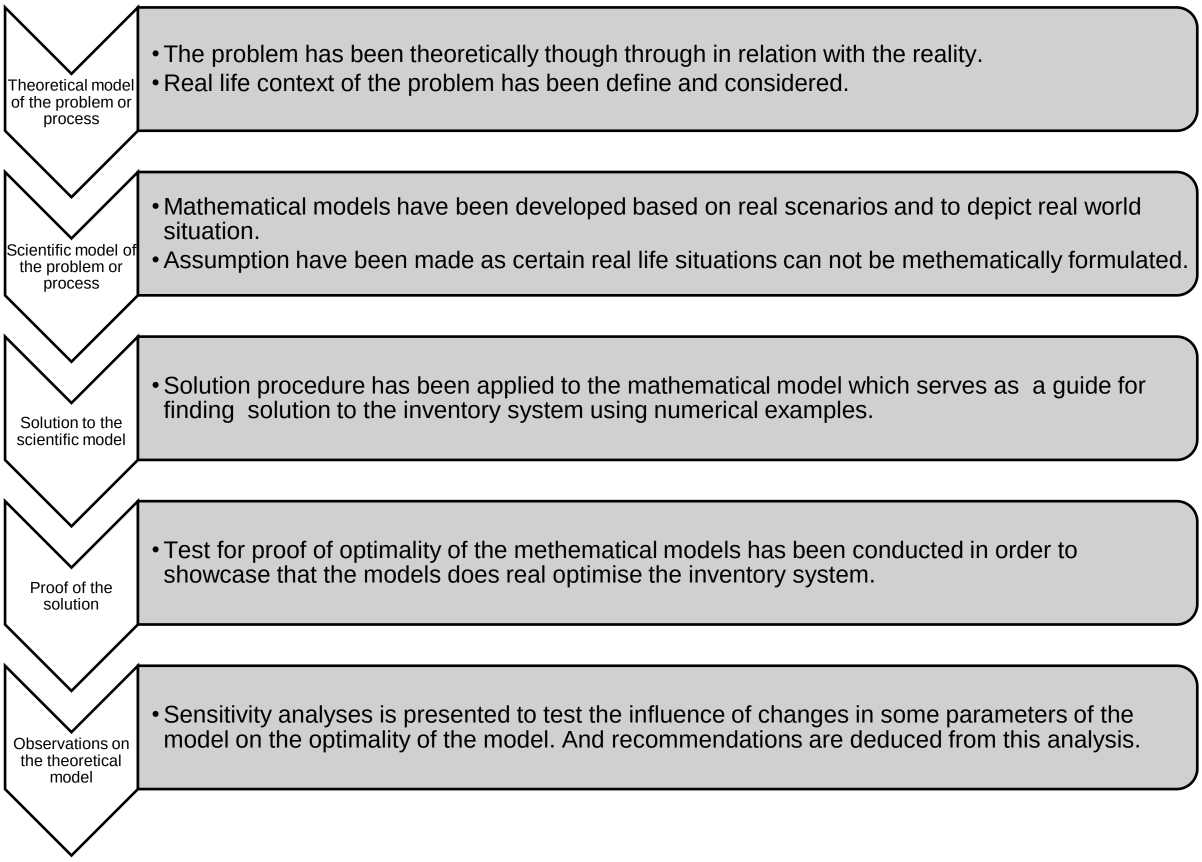

Figure 1: Research Methodology.

1.5 DISSERTATION OUTLINE

The rest of this dissertation is organised as follows:

- Chapter 2 - Covers the literature review, on the topics addressed in this dissertation.
- Chapter 3 - First model development, which considers two mutual complementary items that are subject to deterioration.
- Chapter 4 - Second model development, in which three deteriorating items, with two complements with one of the complement products having stock dependent demand and substitutable by the third product.
- Chapter 5 - Concludes the dissertation by presenting summary of findings, contributions made, limitations and possible future research to extend the models presented.

CHAPTER 2

2 LITERATURE REVIEW

The literature study of this dissertation is structured such that one follows a systematic approach in understanding the problem background and the variants studied. The review expands on the work produced by other researchers in the respective domain. Furthermore, the review covers important concepts addressed in this dissertation and dwells on key contributions and extensions made on variants of interest. These are deterioration, stock dependent-demand, complementary and substitute items. The literature study is organised into three main components as illustrated in **Figure 2**.

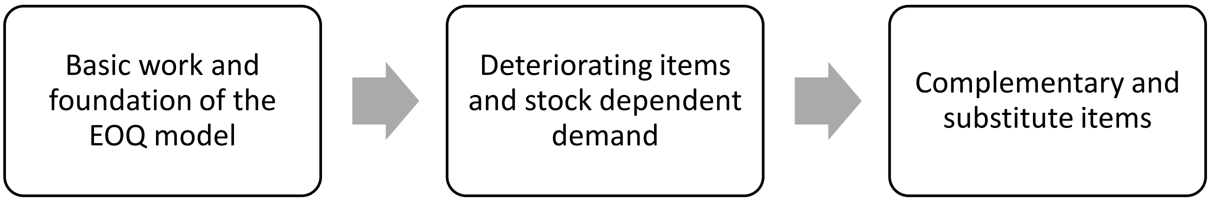

Figure 2: Literature review structure.

2.1 THE BASIC WORK AND FOUNDATION OF EOQ MODEL

The classical EOQ model, also known as economic lot-size model, is one of the oldest and most used inventory models in both practise and academia. The model was developed by Ford W. Harris (1913), to help inventory managers with a tool to manage inventory replenishment. The model is very robust, and it has been used by various industries to manage inventory. These industries include retail, manufacturing, and pharmaceutical industry. The main objective of the model is to determine the optimal economic replenishment policy which ensures that desired customer service level is achieved. The main cost trade-offs that are of importance to the model are ordering (or setup cost) and holding cost. The output of the model seeks to find a balance presented by this trade-off.

Moreover, in view of the fact that there is robustness in the application of the EOQ model, the model is not without any limitations. These limitations are due to the assumptions attached to the model (see **Table 1**). Despite these limitations the model has seen desirable results in its application. To counteract these limitations, the model has received enormous extensions to improve its real-life application. Similarly, in this dissertation, the principles of the EOQ model are incorporated in the model, to help develop inventory models with sound basis.

Table 1: The classical EOQ model assumptions.

Assumptions	**Description**
Demand is constant and continuous.	The model suggest that the demand or consumption rate of the product is known and remains constant over time.
Constant inventory costs	The costs are not variable,
Initial inventory is zero.	Zero initial inventory is assumed.
Instantaneous Leadtime	The model suggests zero Leadtime, meaning that there is no gap between ordering and receiving which is often not the case in practice.
No constraint imposed	No constraint imposed on order quantity, storage capacity, transport capacity and product life.

The inventory system for the traditional EOQ model is illustrated in **Figure 3.** Q represents the quantity of an item ordered at every start of the order cycle whenever Q = 0. The slope of the graph represents the rate of consumption, that is, the demand for the item. As per assumption in **Table 1,** the demand in **Figure 3** is constant and continuous. A straight downward slope line indicates that the demand for the traditional model is known with certainty. Moreover, the annual demand rate (D) slope does not get below Q = 0, this shows that the traditional EOQ model does not allow shortages. When inventory level reaches Q = 0 at the end of cycle T, the classical EOQ model assumes that the order quantity Q is replenished instantaneous, that is, zero lead time.

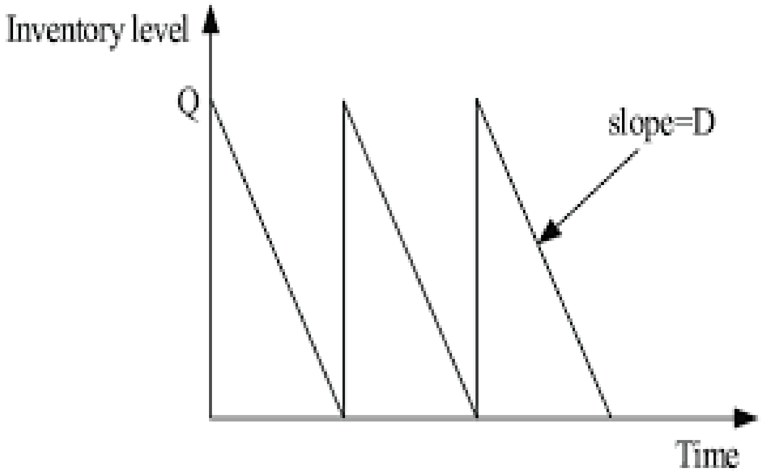

Figure 3: EOQ inventory system.

The basic inventory cost functions of the traditional EOQ model were derived using the **Figure 3**. Ordering costs is incurred every time an order is placed, and the holding cost is incurred for the items that are on hand. The total annual inventory cost per unit is defined by the sum of the ordering and holding costs per unit, as seen in Equation (1).

$$TCU = h\left(\frac{Q}{2}\right) + K\left(\frac{D}{Q}\right) \tag{1}$$

Where

- h is the annual holding cost per unit
- K is the annual ordering cost per unit

The optimal order quantity Q^* is obtained by taking the first derivative of Equation (1) and equate it to zero in order to solve for Q^*, the resulting equation becomes

$$Q^* = \sqrt{\frac{2KD}{h}} \tag{2}$$

The cost functions are illustrated in **Figure 4**, as seen the total cost function is convex, and the optimal cost is reached at EOQ as illustrated by the vertical dash line. The trade-off between holding and ordering cost is clearly inverse, it means that when one decreases the other increases and vice versa. For this reason, the EOQ model, seeks to balance this trade-off.

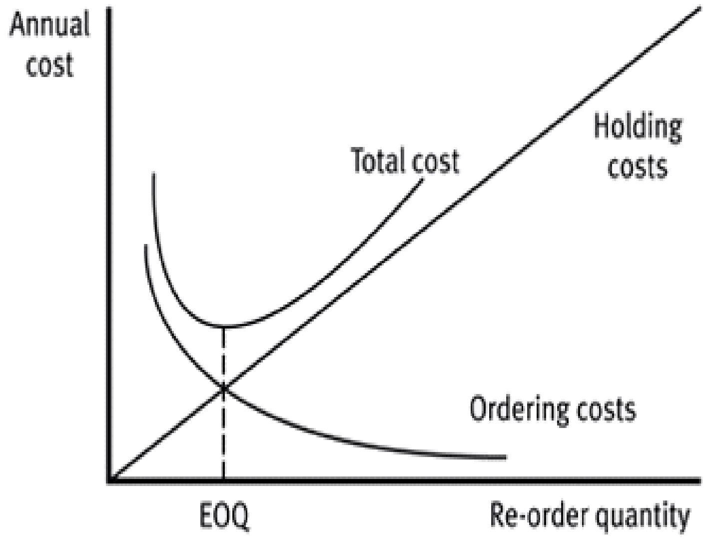

Figure 4: The EOQ cost model

2.2 DETERIORATING ITEMS AND STOCK DEPENDENT DEMAND

Ghare and Schrader (1963) were the first researchers to consider deterioration of items on EOQ models, which then was defined by exponentially decaying function. Later the model was extended by other researchers to explore various types of deterioration rates which includes constant, variable and Weibull distribution rate. Shah and Jaiswal (1977) developed order-level inventory model and considered constant rate of deterioration. On the other hand, Covert and Philip (1973) had a different approach which is taking into account variable deterioration rate for two-parameter Weibull distribution under constant demand and shortages not allowed.

Seeing that constant demand is not practical, researchers started considering alternative ways of defining demand. Dave and Patel (1981) formulated a model for deteriorating items with time proportional demand with shortages not allowed. Hollier and Mak (1983), were first to establish an inventory model with exponentially decreasing demand. Wee (1995) formulated a deterministic inventory model for deteriorating products with demand that declines exponentially over a fixed time limit. Kumar et al. (2013) studied an inventory model for deteriorating item under trade credit. They formulated an inventory model that is subject to the following conditions, the demand is selling price-dependent, and the holding cost is parabolic time varying. Shukla et al., (2013) developed an economic lot-size inventory model for deteriorating items with exponential demand rate and permissible shortages which are partially backlogged. Jaggi and Mittal (2011) developed an economic lot-sizing inventory model for deteriorating items subject to imperfect quality. Maragatham and Palani (2017) formulated an inventory model considering deteriorating items with lead time and price dependent demand under permissible shortages. Aliyu and Sani (2020) investigated a pricing model for deteriorating items under generalised exponential increasing demand with constant holding cost and constant deterioration rate. And lastly, shortages were not permissible in their model. Mashud (2020) considered an EOQ model for deteriorating item with different types of demand and fully backlogged shortages.

Moreover, Ouyang et al. (2005) also developed an EOQ inventory model for deteriorating items under exponentially decreasing demand. Similarly, in this dissertation, the models developed consider deteriorating items with exponential decreasing demand as established in Ouyang et al. (2005) model. This model will be incorporated and adapted in the development of the model addressing the variants of interest of this study. Furthermore, Goyal and Giri (2001) have compiled a comprehensive study on advancement or developments of inventory models for deteriorating items since the early 1990s. In extending this body of work, Perez and Torres (2020) further compiled another comprehensive review on deteriorating items for the period 2001-2018 whereby 317 peer-reviewed articles were thoroughly analysed.

In today's fast paced market, physical display of items has been observed to stimulate demand for the displayed items as this turns to attracts customers. Levin et al. (1972) stipulated that "large piles of consumer goods displayed in a supermarket will lead the customer to buy more". In inventory modelling terms this phenomenon has been referred to as stock-dependent demand. This topic has received attention in recent years as many researchers seem to be keen in exploring the effect of this phenomenon on inventory policies. Datta and Pal (1990), developed an EOQ model which consist of two portions, the first part of the model is made up of demand that is stock dependent and is given by the power function, on the second part is as such that, once the minimum stock level is reached, inventory depletes through constant demand. Their model was further extended by Teng and Ouyang (2005) to incorporate deterioration of items. Ouyang et al (2003), developed an EOQ model for deteriorating items with stock dependent demand under both inflation and time value of money. Kumar et al. (2016) formulated an inventory model for items with non-instantaneous deteriorating rate, as well as stock dependent-demand and linearly increasing holding cost against time. Bansal et al. (2021) established a deterministic inventory model for deteriorating items subject to stock dependent demand rate and inflation. In a single warehouse system, Tripathi and Mishra (2014) presented an EOQ model for deteriorating items with demand that is stock dependent. In their model shortages are allowed and are fully backlogged.

2.3 COMPLEMENTARY AND SUBSTITUTE ITEMS

As much as it is important for a firm to keep certain level of inventory at the right place and time, it is also critical that firms understand the interconnections between these inventories. Understanding the interconnection among items may help managers in making even better inventory decisions. These interconnections among items which are observable in the market includes complementarity and substitutability of items. Two products are said to be complementary when the two are sold together to serve or satisfy a particular need (Karaöz et al., 2011). This creates a market behaviour where, when one of the items is purchased the other complement item is also purchased at the same time and vice versa in the case of mutual complements. On the other hand, substitution occurs in an instance when an intended item to be bought is unavailable and the customer chooses to buy an alternative product which has similar attribute or functionality as the initially intended item. In an instance where these interrelationships (complementarity or substitutability) exist between items, the demand for such items is stimulated by the pricing and availability of these items.

Under complementary items, Cournot (1838), made one of an early study on the interrelationship between the manufacturers of complementary products. In his model, two firms producing complementary goods are considered. The two complementary products were copper and zinc, which are latter mixed to produce a composite product which is brass. The model proved that, irrespective of different cost to produce these products, both firms shared the profits equally. Cournot (1838), further extended the initial model by incorporating quality decisions. The results obtained from this extended model, indicated that the profits among producers was shared unequally due to the asymmetric incentives to invest in quality.

Furthermore, Economides (1999) developed a complementary products model which considers quality decisions from a perspective of complementor firm. The model developed considered instances where complementary products are manufactured at one factory and as well as when the products are manufactured by two distinct factories. The model assumed that the composite product's quality is equal to the quality of the lesser product. The author states that this type of observation is evident in telecom service, where, for instance the overall quality of the call is gauged by the minimum quality of the two terminating points. Taylan (2010), extended Economides (1999) model by defining the quality of composite products as super-modular, meaning that the slight increase in quality of one of the component products results in the absolute increase in quality of the composite product. This observation is common in many complementary items such as smartphone-apps and video game-console. When a quality of one of the components increase the overall experience increases.

The following researchers Chen and Nalebuff (2006), Cheng and Nahm (2007), focused on developing models for one-way complements. One-way complement is an instance where one of the complement products has value and the other product is useless without the other product that has value. Furthermore, the useless product enhances the worth of the product that has value to customer. Cheng and Nahm (2007) look at how the ratio of value item and composite product influences pricing strategy.

These researchers Casadesus-Masanell et al. (2007), Farrell and Katz (2000) focused their research on strictly complement products under consideration of competing industries. They looked at the effect of complementary products, when one of the products monopolised and the product is supplied by a competitor firm.

Under substitute items, Eksler et al. (2019) implemented an inventory model for multiple items which experience varying degrees of substitutability. Zhang and Huang (2011) presented an inventory model with partial backordering, where unmet demand is satisfied through substitute item. Moreover, there are fewer publications that considered the combined effect of complementary and substitute items, this is an indication that there is still more work to be done in this field. Some researchers that have explored this area of research includes Maity and Maiti (2009) who presented an optimisation inventory model for deteriorating items under complementarity and substitutability between items. Karaöz et al. (2011) developed an EOQ model for an item with demand that is dependent on its selling, the selling price of its complement and substitute product and time. Mokhtari (2018) developed joint replenishment model for complementary and substitutable items such that the total cost of inventory policy is optimised. Edalatpour and Mirzapour (2019) developed an inventory model for substitute and complementary items experiencing nonlinear holding cost. Their model was focused on combined effect of optimised pricing and inventory decisions. Taleizadeh et al. (2019) considered a pricing and inventory decision model for complementary and substitute products that are subject to deterioration. Rajesh and Vinod, (2020) presented an inventory model for substitutable and complementary items subject to asymmetrical substitution. In the model they analysed the impact of substitution cost during join replenishment.

The summary of the reviewed papers is illustrated in **Table 2**, where a classification has been made as to whether the specific article addressed the topic raised in the respective columns. As seen in the table several papers consider varying topics in the columns not necessarily all topics at once. Therefore, this dissertation attempts to address all the topics gradually using two models, where the first model concentrates on deteriorating items which are mutual complements and the second model incorporates all the elements into a single model in order to create a more robust model which addresses the current trends on inventory modelling. From the literature reviewed, it can be deduced that the models presented in this dissertation are an extension of the model presented by Karaöz et al (2011) and Ouyang et al. (2005). In which the first model, mainly covered substitute and complementary items with demand that is defined by an exponential function, the later model considered deteriorating items with exponential decreasing demand.

Table 2: Literature reviewed summary

Author(s)	Substitute item	Complementary item	Mutual price dependent	Deterioration	Constant demand (Type)	Joint replenishment	Shared ordering cost	Stock-dependent demand
Datta and Pal (1990)	No	No	No	Yes	No (Power form)	No	No	Yes
Teng and Ouyang (2005	No	No	No	Yes	No (Power form)	Yes	No	Yes
Ouyang et al (2003)	No	No	No	Yes	No (Linear demand)	No	No	Yes
Bansal et al (2021)	No	No	No	Yes	No (Linear demand)	No	No	Yes
Eksle et al (2019)	Yes	No	No	No	Yes	Yes	No	No
Zhang and Huang (2011)	Yes	No	No	No	Yes	Yes	No	No
Edalatpour and Mirzapour (2019)	Yes	Yes	No	Yes	No (Linear)	Yes	No	No
Rajesh and Vinod (2020)	Yes	Yes	No	No	No (Linear)	Yes	No	No
Taleizadeh et al. (2019)	Yes	Yes	Yes	Yes	No (Linear)	Yes	No	No
Mokhtari, (2018)	Yes	Yes	No	No	No (Linear)	Yes	No	No
Maity and Maiti (2009)	Yes	Yes	No	Yes	No (Linear)	No	No	Yes
Karaöz et al. (2011)	Yes	Yes	No	No	No (Exponential)	No	No	No
1st model chapter 3	No	Yes	Yes	Yes	No (Exponential)	Yes	Yes	No
2nd model chapter 4	Yes	Yes	Yes	Yes	No (Exponential)	Yes	Yes	Yes

CHAPTER 3

3 EOQ MODEL FOR TWO DETERIORATING ITEMS WITH MUTUALLY COMPLEMENTARY PRICE DEPENDENT DEMAND

3.1 PROBLEM DEFINITION

Since complementary products are known as goods that are sold together, other researchers for example (Chen and Nalebuf, 2007) and Karaöz et al. (2011) have formulated their model such that one of the products is considered a primary product and the other as its complement. In this case it means that one product is considered more essential to the joint use but not vice versa. In this chapter however, an inventory model or policy is developed for complementary products that are equally essential to the joint use of each other. In which case the consumer values complementarity of both products, that is, if a customer purchases product A they will also purchase product B and vice versa. Typically, there is often no alternative to replace either product in the joint use. An inventory model proposed in this chapter seeks to find optimal values of the selling price and cycle length that maximises profit for mutual complementary products. The complementary products considered in this dissertation are subject to deterioration. The demand for the products is given by the exponential function that is dependent on product's selling prices, its complement product selling price and time. Moreover, the ordering cost of the products is defined by a fixed and a variable component.

3.2 NOTATIONS AND ASSUMPTIONS

3.2.1 Notations

In this section notations used in developing the inventory model are presented together with the assumptions that are attached to the model developed.

Table 3: Notations used in deriving the inventory model.

Notations	***Units***	***Description***
$\boldsymbol{n}$		Product 1,2
$\boldsymbol{D_n}$	R/Unit	Demand for product n.
$\boldsymbol{a_n}$	Constant	Product 1 price coefficient for the demand rate (a > 0) of product n.
$\boldsymbol{b_n}$	Constant	Product 2 price coefficient for the demand rate (b > 0) of product n.
$\boldsymbol{\theta_n}$	Constant	Deterioration rate of product n, where ($0 \leq \theta_n \leq 1$).
$\boldsymbol{k_0}$	R/Unit	Constant portion of order cost per order.
$\boldsymbol{k_n}$	R/Unit	Variable portion of order cost per order for product $\boldsymbol{n}$.
$\boldsymbol{h_n}$	R/Unit	Holding cost per unit of product n..
$\boldsymbol{c_n}$	R/Unit	Purchasing cost per unit of product $\boldsymbol{n}$.
$\boldsymbol{I_n(t)}$	Unit	Inventory level at time t of product $\boldsymbol{n}$.
$\boldsymbol{A_n}$	Unit	Initial demand of product n.
$\boldsymbol{\beta_n}$	Constant	Constant governing decreasing rate of demand of nth product.

Decision variables

Notations	***Units***	***Description***
$\boldsymbol{Q_n}$	Unit	Order quantity for nth Product.
$\boldsymbol{P_n}$	R	Selling price (R/Unit) of product n.
$\boldsymbol{T}$	Month	Length of replenishment cycle.

3.2.2 Assumptions

The following are some assumptions made in formulating the mathematical model:

- The demand for each of the mutually complementary products, that is, product 1 and 2, is dependent on the product's own selling price and the selling price of the complement product.
- Demand for the products is defined by an exponential function that decrease with times
- Each product has its own holding cost.
- The products experience shared ordering cost that consists of constant and variable component.
- The replenishment is instantaneous (Zero lead time).
- Both products have the same replenishment cycle time.
- Shortages are not allowed for the two products.
- The initial inventory is zero.
- Deterioration rate for each product is defined by a constant.

3.3 MODEL DEVELOPMENT

3.3.1 Mathematical model

This study considers, an inventory policy where two deteriorating complement products from a supplier are instantaneously delivered to the retailer as per ordered quantity Q_1 and Q_2, at a unit cost c_n for nth product and the order cost $OC = k_0 + \sum_{i=1}^{n} k_i$ per order. The initial inventory is zero for both products, this is assumed for every cycle. Therefore, for every cycle, products 1 and 2 start with the ordered quantity Q_1 and Q_2 , respectively. The quantity ordered for each product gradually decreases due to demand and deterioration to zero at the end of the cycle at time $t = T$. The inventory level for the items is graphically represented in **Figure 5**. As seen in Figure 1, shortages are not allowed in the model.

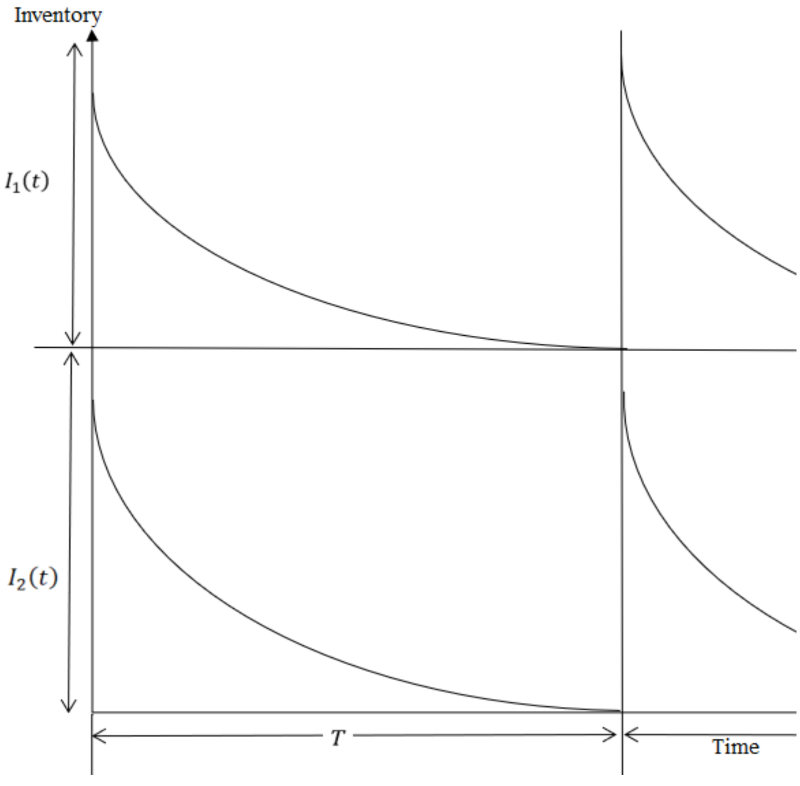

Figure 5: Inventory levels of two deteriorating mutual complementary products at time.

The demand functions for the products are denoted by the following equations.

$$D_1 = A_1 e^{-a_1 P_1 - b_1 P_2 - \beta_1 t} \tag{1}$$

and

$$D_2 = A_2 e^{-a_2 P_1 - b_2 P_2 - \beta_2 t} \tag{2}$$

The change in inventory level for products 1 and 2 at any given time t is governed by the following differential equations

$$\frac{dI_1(t)}{dt} + \theta_1 I_1(t) = -A_1 e^{-a_1 P_1 - b_1 P_2 - \beta_1 t} \qquad 0 \le t \le T \tag{3}$$

$$\frac{dI_2(t)}{dt} + \theta_2 I_2(t) = -A_2 e^{-a_2 P_1 - b_2 P_2 - \beta_2 t} \qquad 0 \le t \le T \tag{4}$$

The boundary conditions are given by the following equations

$$I_1(0) = Q_1\ , I_2(0) = Q_2 \quad and \quad I_1(T) = 0,\ I_2(T) = 0 \tag{5}$$

To derive instantaneous inventory level, $I_1(t)$ and $I_2(t)$, for products 1 and 2, consider (3), for product 1, the equation is of the form

$$\frac{dI_1(t)}{dt} + f(t)y = g(t) \text{ with integrating factor } I.F = e^{\int f(t)} = e^{\theta_1 t} \tag{6}$$

the general solution is

$$I_1(t) \cdot e^{\theta_1 t} = -A_1 e^{-a_1 P_1 - b_1 P_2} \cdot \int e^{t(\theta_1 - \beta_1)} dt \tag{7}$$

Integrating (7) to get the inventory level function for product 1, result in

$$I_1(t) \cdot e^{\theta_1 t} = -\frac{A_1 e^{-a_1 P_1 - b_1 P_2}}{\theta_1 - \beta_1} \cdot e^{t(\theta_1 - \beta_1)} + C \tag{8}$$

Using boundary condition $I_1(T) = 0$ to solve for C in (8) and simplifying the equation results in the inventory level function for product 1 to be

$$I_1(t) = \frac{A_1 e^{-a_1 P_1 - b_1 P_2 - \beta_1 t}}{\theta_1 - \beta_1} \left[e^{(\theta_1 - \beta_1)(T-t)} - 1\right] \tag{9}$$

Similarly, inventory level function $I_2(t)$ can be obtained the same way as Equation (9), therefore

$$I_2(t) = \frac{A_2 e^{-a_2 P_1 - b_2 P_2 - \beta_2 t}}{\theta_2 - \beta_2} \left[e^{(\theta_2 - \beta_2)(T-t)} - 1\right] \tag{10}$$

For simplicity let

$$E = A_1 e^{-a_1 P_1 - b_1 P_2} \quad and \qquad F = A_2 e^{-a_2 P_1 - b_2 P_2} \tag{11}$$

Now by substituting initial conditions from (5) into (9) and (10) respectively results in the following maximum inventory level functions for product 1 and 2

$$I_1(0) = Q_1 = \frac{A_1 e^{-a_1 P_1 - b_1 P_2}}{\theta_1 - \beta_1} \left[e^{(\theta_1 - \beta_1)T} - 1\right] \tag{12}$$

$$I_2(0) = Q_2 = \frac{A_2 e^{-a_2 P_1 - b_2 P_2}}{\theta_2 - \beta_2} \left[e^{(\theta_2 - \beta_2)T} - 1\right] \tag{13}$$

Holding cost ($\boldsymbol{HC}$)

The total $holding\ cost$ per cycle is given by the following function

$$HC = HC_1 + HC_2 = h_1 \cdot \int_0^T I_1(t)dt + h_2 \cdot \int_0^T I_2(t)\ dt$$
$$= h_1 \cdot \frac{Ee^{(\theta_1-\beta_1)T}}{\theta_1-\beta_1}\left[\frac{1-e^{-\theta_1 T}}{\theta_1}\right] - h_1 \cdot \frac{E}{\theta_1-\beta_1}\left[\frac{1-e^{-\beta_1 T}}{\beta_1}\right] + h_2$$
$$\cdot \frac{Fe^{(\theta_2-\beta_2)T}}{\theta_2-\beta_2}\left[\frac{1-e^{-\theta_2 T}}{\theta_2}\right] - h_2 \cdot \frac{F}{\theta_2-\beta_2}\left[\frac{1-e^{-\beta_2 T}}{\beta_2}\right] \tag{14}$$

Order cost ($\boldsymbol{OC}$)

The model assumes one order is placed per cycle. The total ordering cost for the two items constitutes of shared ordering cost given by the following equation.

$OC = k_0 + \sum_{i=1}^{n} k_i$ where $n = 2$ in this case without any loss of generality (15)

Revenue ($\boldsymbol{TR}$) and Purchase cost($\boldsymbol{PC}$)

Revenue for each product is obtained by multiplying order quantity for the specific product with its selling price. While on the other hand, the purchase cost is obtained by multiplying unit purchase price with order quantity.

Profit ($\boldsymbol{TP}$)

Profit per unit is obtained by dividing the total profit by cycle length T. Which is given by the following function

$$TP = \big[TR - PC - [HC + OC\ ]\big]/T$$
$$= \Bigg[(P_1 - c_1) \cdot \frac{E}{\theta_1-\beta_1}\ \left[e^{(\theta_1-\beta_1)T} - 1\right] + (P_2 - c_2) \cdot \frac{F}{\theta_2-\beta_2}\ \left[e^{(\theta_2-\beta_2)T} - 1\right]$$
$$- \Bigg(h_1 \cdot \frac{Ee^{(\theta_1-\beta_1)T}}{\theta_1-\beta_1}\left[\frac{1-e^{-\theta_1 T}}{\theta_1}\right] - h_1 \cdot \frac{E}{\theta_1-\beta_1}\left[\frac{1-e^{-\beta_1 T}}{\beta_1}\right] + h_2$$
$$\cdot \frac{Fe^{(\theta_2-\beta_2)T}}{\theta_2-\beta_2}\left[\frac{1-e^{-\theta_2 T}}{\theta_2}\right] - h_2 \cdot \frac{F}{\theta_2-\beta_2}\left[\frac{1-e^{-\beta_2 T}}{\beta_2}\right] + k_0 + \sum_{i=1}^{n} k_i\Bigg)\Bigg]/T \tag{16}$$

Cycle length

Linearising the exponential terms containing T in (16) by using Maclaurin's expansion for e^x. Now let $x = (\theta_1 - \beta_1)T$ therefore

$$e^{(\theta_1-\beta_1)T} = \sum_{m=0}^{\infty} \frac{(\theta_1-\beta_1)^m T^m}{m!} = 1 + \frac{(\theta_1-\beta_1)^1 T^1}{1!} + \frac{(\theta_1-\beta_1)^2 T^2}{2!} + \frac{(\theta_1-\beta_1)^3 T^3}{3!}$$
$$\approx 1 + (\theta_1 - \beta_1)T \tag{17}$$

All the other exponential terms are approximated the same way as (17). Substituting these approximations into (16), results in

$$TP = \left[(P_1 - c_1) \cdot \frac{E}{\theta_1 - \beta_1} [1 + (\theta_1 - \beta_1)T - 1] + (P_2 - c_2) \cdot \frac{F}{\theta_2 - \beta_2} [1 + (\theta_2 - \beta_2)T - 1] - \left(h_1 \cdot \frac{E(1 + (\theta_1 - \beta_1)T)}{\theta_1 - \beta_1}\left[\frac{1 - (1 - \theta_1 T)}{\theta_1}\right] - h_1 \cdot \frac{E}{\theta_1 - \beta_1}\left[\frac{1 - (1 - \beta_1 T)}{\beta_1}\right] + h_2 \cdot \frac{F(1 + (\theta_2 - \beta_2)T)}{\theta_2 - \beta_2}\left[\frac{1 - (1 - \theta_2 T)}{\theta_2}\right] - h_2 \cdot \frac{F}{\theta_2 - \beta_2}\left[\frac{1 - (1 - \beta_2 T)}{\beta_2}\right] + k_0 + \sum_{i=1}^{n} k_i\right)\right]/T \tag{18}$$

Now, simplifying (18), then

$$TP = \frac{\left[(P_1 - c_1)ET + (P_2 - c_2)FT - \left(h_1 ET^2 + h_2 FT^2 + k_0 + \sum_{i=1}^{n} k_i\right)\right]}{T} \tag{19}$$

Using product differentiation rule to find $TP' = (fg)' = fg' + f'g$ and then equating $TP' = 0$ the following is obtained.

$$TP' = (P_1 - c_1)ET^{-1} + (P_2 - c_2)FT^{-1} - (2h_1 ET + 2h_2 FT)T^{-1} - (P_1 - c_1)ET^{-1} - (P_2 - c_2)FT^{-1} - \left(-h_1 ET^0 - h_2 FT^0 - k_0 T^{-2} - \sum_{i=1}^{n} k_i T^{-2}\right) = 0 \tag{20}$$

Simplify Equation (20) to find the optimum T .Therefore

$$T = \sqrt{\frac{k_0 + \sum_{i=1}^{n} k_i}{h_1 E + h_2 F}} \quad \text{where } E \text{ and } F \text{ have been defined in (11) for simplicity} \tag{21}$$

Proof of optimality

To show that the unit profit function TP is concave, we prove that the Hessian matrix for the profit function Equation (19) is negative (semi)definite.

The Hessian matrix for TP is given by $H(P_1, P_2, T) = \begin{pmatrix} \frac{d^2TP}{dP_1{}^2} & \frac{d^2TP}{dP_1 dP_2} & \frac{d^2TP}{dP_1 dT} \\ \frac{d^2TP}{dP_2 dP_1} & \frac{d^2TP}{dP_2{}^2} & \frac{d^2TP}{dP_2 dT} \\ \frac{d^2TP}{dT dP_1} & \frac{d^2TP}{dT dP_2} & \frac{d^2TP}{dT^2} \end{pmatrix}$ (22)

Second derivatives

Consider (22), the second derivatives of the hessian matrix are obtained using profit function (19), therefore.

$$\frac{d^2TP}{dP_1{}^2} = (P_1 - c_1)a_1{}^2 A_1 e^{-a_1P_1 - b_1P_2} - 2a_1 A_1 e^{-a_1P_1 - b_1P_2} + a_2{}^2(P_2 - c_2)A_2 e^{-a_2P_1 - b_2P_2} - T \times [h_1 a_1{}^2 A_1 e^{-a_1P_1 - b_1P_2} + h_2 a_2{}^2 A_2 e^{-a_2P_1 - b_2P_2}] \tag{23}$$

$$\frac{d^2TP}{dP_2{}^2} = (P_2 - c_2)b_2{}^2 A_2 e^{-a_2P_1 - b_2P_2} - 2b_2 A_2 e^{-a_2P_1 - b_2P_2} + b_1{}^2(P_1 - c_1)A_1 e^{-a_1P_1 - b_1P_2} - T \times \left[h_1 b_1{}^2 A_1 e^{-a_1P_1 - b_1P_2} + h_2 b_2{}^2 A_2 e^{-a_2P_1 - b_2P_2}\right] \tag{24}$$

$$\frac{d^2TP}{dP_2dP_1} = [(P_1 - c_1)(a_1b_1)A_1e^{-a_1P_1-b_1P_2} - b_1A_1e^{-a_1P_1-b_1P_2}] - [-a_2b_2(P_2 - c_2)A_2e^{-a_2P_1-b_2P_2} + a_2A_2e^{-a_2P_1-b_2P_2}] - T[h_1(a_1b_1)A_1e^{-a_1P_1-b_1P_2} + h_2a_2b_2A_2e^{-a_2P_1-b_2P_2}] \quad (25)$$

Equation (23), (24) and (25) are simplified by factorising like terms and substituting with (11), to obtain.

$$\frac{d^2TP}{dP_1^{\,2}} = (P_1 - c_1){a_1}^2\,E - 2a_1E + {a_2}^2(P_2 - c_2)F\ - T[h_1{a_1}^2E + h_2{a_2}^2F\,]$$
$$= a_1E[(P_1 - c_1)a_1 - 2 - Th_1a_1] + {a_2}^2F[(P_2 - c_2) - Th_2] \quad (26)$$

$$\frac{d^2TP}{dP_2^{\,2}} = (P_2 - c_2){b_2}^2\,F - 2b_2F + {b_1}^2(P_1 - c_1)E\ - T\big[h_1{b_1}^2E + h_2{b_2}^2F\,\big]$$
$$= {b_1}^2E[(P_1 - c_1) - Th_1] + b_2F[(P_2 - c_2)b_2 - 2 - Th_2b_2] \quad (27)$$

$$\frac{d^2TP}{dP_2dP_1} = [(P_1 - c_1)(a_1b_1)E - b_1E] - [-a_2b_2(P_2 - c_2)F + a_2F] - T[h_1(a_1b_1)E + h_2a_2b_2F]$$
$$= b_1E[(P_1 - c_1)a_1 - 1 - Th_1a_1] + a_2F[(P_2 - c_2)b_2 - 1 - Th_2b_2] \quad (28)$$

To further simplify (26) to (28). Let,

$$\varepsilon = (P_2 - c_2) - Th_2 \quad (29)$$

$$\omega = (P_1 - c_1) - Th_1 \quad (30)$$

Now

$$\frac{d^2TP}{dP_1^{\,2}} = a_1E(a_1\omega - 2) + {a_2}^2F\varepsilon \quad (31)$$

$$\frac{d^2TP}{dP_2^{\,2}} = {b_1}^2E\omega + b_2F(b_2\varepsilon - 2) \quad (32)$$

$$\frac{d^2TP}{dP_2dP_1} = b_1E(a_1\omega - 1) + a_2F(b_2\varepsilon - 1) \quad (33)$$

For T derivatives, the 1st derivative of TP with respect to T is denoted by

$$\frac{dTP}{dT} = -h_1E - h_2F + T^{-2}\left(k_0 + \sum_{i=1}^{n} k_i\right) \quad (34)$$

With the 2nd derivatives

$$\frac{d^2TP}{dT^2} = -2T^{-3}\left(k_0 + \sum_{i=1}^{n} k_i\right) \quad (35)$$

$$\frac{d^2TP}{dTdP_1} = h_1a_1E + h_2a_2F \quad (36)$$

$$\frac{d^2TP}{dTdP_2} = h_1b_1E + h_2b_2F \quad (37)$$

Now, to prove that TP is negative (semi)definite. The determinants need to satisfy the following condition $|H(P_1)| < 0,\ |\,H(P_2)| > 0$ and $|H(T)| < 0$ where

$$|H(P_1)| = \frac{d^2TP}{dP_1{}^2}, |\,H(P_2)| = \begin{bmatrix} \frac{d^2TP}{dP_1{}^2} & \frac{d^2TP}{dP_1dP_2} \\ \frac{d^2TP}{dP_2dP_1} & \frac{d^2TP}{dP_2{}^2} \end{bmatrix}, |H(T)| = \begin{vmatrix} \frac{d^2TP}{dP_1{}^2} & \frac{d^2TP}{dP_1dP_2} & \frac{d^2TP}{dP_1dT} \\ \frac{d^2TP}{dP_2dP_1} & \frac{d^2TP}{dP_2{}^2} & \frac{d^2TP}{dP_2dT} \\ \frac{d^2TP}{dTdP_1} & \frac{d^2TP}{dTdP_2} & \frac{d^2TP}{dT^2} \end{vmatrix} \tag{38}$$

To derive the conditions for $|\boldsymbol{H}(\boldsymbol{P_1})| < \mathbf{0}$.

Substitute (26) on the relevant determinant in (38)

$$|H(P_1)| = \frac{d^2TP}{dP_1{}^2} = (P_1 - c_1)a_1{}^2\, E + a_2{}^2(P_2 - c_2)F \; - T[h_1a_1{}^2E + h_2a_2{}^2F\,] - 2a_1E \tag{39}$$

It is known that $P_1 > c_1$ and $P_2 > c_2$. Therefore when $(P_1 + P_2) - (c_1 + c_2) \leq T(h_1 + h_2)$ then the condition $|H(P_1)| < 0$ holds.

To derive the conditions for $|\,\boldsymbol{H}(\boldsymbol{P_2})| > \mathbf{0}$.

The determinant is given by $|\,H(P_2)| = \left(\frac{d^2TP}{dP_1{}^2} * \frac{d^2TP}{dP_2{}^2}\right) - \left(\frac{d^2TP}{dP_2dP_1}\right)^2$. Substitute (31) to (33) to into $|\,H(P_2)|$. Now the first term is

$$\begin{aligned}\left(\frac{d^2TP}{dP_1{}^2} * \frac{d^2TP}{dP_2{}^2}\right) &= [a_1E(\omega a_1 - 2) + a_2{}^2F\varepsilon\,] \times \left[b_1{}^2E\omega + b_2F(\varepsilon b_2 - 2)\right] \\ &= a_1b_1{}^2E^2\omega(\omega a_1 - 2) + a_1b_2FE(\omega a_1 - 2)(\varepsilon b_2 - 2) + a_2{}^2b_1{}^2EF\varepsilon\omega \\ &\quad + a_2{}^2b_2F^2\varepsilon(\varepsilon b_2 - 2)\end{aligned} \tag{40}$$

The second term

$$\begin{aligned}\left(\frac{d^2TP}{dP_2dP_1}\right)^2 &= (b_1E(a_1\omega - 1) + a_2F(b_2\varepsilon - 1))^2 \\ &= b_1{}^2E^2(\omega a_1 - 1)^2 + a_2b_1EF(\varepsilon b_2 - 1)(\omega a_1 - 1) + a_2{}^2F^2(\varepsilon b_2 - 1)^2\end{aligned} \tag{41}$$

Therefore, considering (40) and (41) it can be deduced that $|H(P_2)| > 0$ when, $\omega < 0$ and $\varepsilon < 0$.

To derive the conditions for $|\boldsymbol{H}(\boldsymbol{T})| < \mathbf{0}$.

The determinant is given by

$$\begin{aligned}|H(T)| = \frac{d^2TP}{dP_1{}^2}\left(\frac{d^2TP}{dP_2{}^2}\cdot\frac{d^2TP}{dT^2} - \frac{d^2TP}{dTdP_2}\cdot\frac{d^2TP}{dP_2dT}\right) - \frac{d^2TP}{dP_1dP_2}\left(\frac{d^2TP}{dP_2dP_1}\cdot\frac{d^2TP}{dT^2} - \frac{d^2TP}{dP_2dT}\cdot\frac{d^2TP}{dTdP_1}\right) \\ + \frac{d^2TP}{dP_1dT}\left(\frac{d^2TP}{dP_2dP_1}\cdot\frac{d^2TP}{dTdP_2} - \frac{d^2TP}{dP_2{}^2}\cdot\frac{d^2TP}{dTdP_1}\right)\end{aligned} \tag{42}$$

The three terms are thus,

$$\begin{aligned}&\frac{d^2TP}{dP_1{}^2}\left(\frac{d^2TP}{dP_2{}^2}\cdot\frac{d^2TP}{dT^2} - \frac{d^2TP}{dTdP_2}\cdot\frac{d^2TP}{dP_2dT}\right) \\ &\quad = [a_1E(a_1\omega - 2) + a_2{}^2F\varepsilon] \\ &\quad \cdot\left[\left[b_1{}^2E\omega + b_2F(b_2\varepsilon - 2)\right] \times -2T^{-3}\left(k_0 + \sum_{i=1}^{n} k_i\right) - \; (h_1b_1E + h_2b_2F)^2\right]\end{aligned} \tag{43}$$

$$-\frac{d^2TP}{dP_1dP_2}\left(\frac{d^2TP}{dP_2dP_1}\cdot\frac{d^2TP}{dT^2}-\frac{d^2TP}{dP_2dT}\cdot\frac{d^2TP}{dTdP_1}\right)$$
$$= -[b_1E(a_1\omega-1)+a_2F(b_2\varepsilon-1)]$$
$$\cdot\left[[b_1E(a_1\omega-1)+a_2F(b_2\varepsilon-1)]\times -2T^{-3}\left(k_0+\sum_{i=1}^{n}k_i\right)\right.$$
$$\left.-\ (h_1b_1E+h_2b_2F)(h_1a_1E+h_2a_2F\ )\right] \quad (44)$$

$$\frac{d^2TP}{dP_1dT}\left(\frac{d^2TP}{dP_2dP_1}\cdot\frac{d^2TP}{dTdP_2}-\frac{d^2TP}{d{P_2}^2}\cdot\frac{d^2TP}{dTdP_1}\right)$$
$$=[h_1a_1E+h_2a_2F]$$
$$\cdot\ \left[\big(b_1E(a_1\omega-1)+a_2F(b_2\varepsilon-1)\big)\ \ (h_1b_1E+h_2b_2F)\right.$$
$$\left.-\left({b_1}^2E\omega+b_2F(b_2\varepsilon-2)\right)(h_1a_1E+h_2a_2F\ \ )\ \ \right] \quad (45)$$

Combining all terms, finally

$$|H(T)|=[a_1E(\omega a_1-2)+{a_2}^2F\varepsilon]$$
$$\cdot\left[-\left[{b_1}^2E\omega+b_2F(\varepsilon b_2-2)\right]\ \cdot 2T^{-3}\left(k_0+\sum_{i=1}^{n}k_i\right)-\ (h_1b_1E+h_2b_2F)^2\right]$$
$$+\ (b_1E(\omega a_1-1)+a_2F(\varepsilon b_2-1)\ )^2\times 2T^{-3}\left(k_0+\sum_{i=1}^{n}k_i\right)$$
$$+\ 2\times[h_1a_1E+h_2a_2F]\cdot[(b_1E(\omega a_1-1)+a_2F(\varepsilon b_2-1))\ (h_1b_1E+h_2b_2F)]$$
$$-\left({b_1}^2E\omega+b_2F(\varepsilon b_2-2)\right)(h_1a_1E+h_2a_2F\ \ )^2<0 \quad (46)$$

Therefore $|H(T)|<0$ when, $\varepsilon<0$ and $,\omega<0$.

3.4 NUMERICAL RESULTS

3.4.1 Numerical example

The numerical example considers a retail environment, where a retailer desires to find an optimal inventory policy for managing two deteriorating items that are mutual complements. Numerical values have been selected to formulate the scenario described. The numerical values are shown in **Table 4**. Sensitivity analysis on these values is performed next.

Table 4 Numerical values

Parameters	Product 1	Product 2
a_n	0.30	0.10
b_n	0.22	0.25
θ_n	0.40	0.50
h_n	0.60	0.50
c_n	6.00	5.00

A_n	200	300
β_n	0.16	0.18
e	2.718	
k_0	100	
k_n	10	20

The chosen numerical values in **Table 4** are substituted into the optimisation model developed on excel. The model yields the following results:

Table 5: Model results

P_1	P_2	T	TP	Q_1	Q_2
8.84	7.58	5.33	49.68	63.95	118.22

Note from the results above that profit function is negative (semi)definite since the following conditions are satisfied $\varepsilon = (P_2 - c_2) - Th_2 = (7.58 - 5) - 5.33 \times 0.5 = -0.085 < 0$ and $\omega = (P_1 - c_1) - Th_1 = (8.84 - 6) - 5.33 \times 0.6 = -0.358 < 0$. Next sensitivity analysis is performed to test the robustness of the model.

3.4.2 Sensitivity analysis

Sensitivity analysis is performed by changing parameters a_n, b_n , β_n , θ_n , h_n and c_n for each product while keeping other parameters at chosen numeric value as given in Table 3. The sensitivity of parameters on to the model is tested by taking 25% reduction and increment around the chosen value. The test has been performed asymmetric for deterioration rate of the products such that optimality conditions are adhered to. The effect of changes in parameters is exhibited in Table 5 to 16. Moreover, the summary graphs for the percentage change in parameters vs profit and percentage changes in parameter change vs cycle length are presented in Figure 2 and 3 respectively.

Table 6: The effect of changing a_1 while keeping other parameters to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	a_1	Change in parameter
22.27	5.00	3.66	791.16	190.68	55.29	0.075	-75%
14.24	5.00	3.68	212.99	120.46	124.20	0.15	-50%
10.55	6.73	4.62	82.48	86.10	118.78	0.225	-25%
8.84	7.58	5.33	49.68	63.95	118.22	0.30	0%
7.79	7.97	5.73	39.04	48.87	122.29	0.375	25%
7.03	8.15	5.93	35.73	38.36	127.67	0.45	50%
6.43	8.24	6.02	35.33	30.89	133.02	0.525	75%

Table 7: The effect of changing a_2 while keeping other parameters to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	a_2	Change in parameter
9.32	7.87	4.51	88.87	42.59	166.14	0.025	-75%
9.01	7.79	4.78	72.67	51.13	147.98	0.05	-50%
8.87	7.70	5.05	59.87	58.10	132.20	0.075	-25%
8.84	7.58	5.33	49.68	63.95	118.22	0.10	0%
8.89	7.44	5.60	41.63	68.95	105.78	0.125	25%
9.01	7.25	5.87	35.38	73.29	94.72	0.15	50%
9.18	7.00	6.13	30.72	77.12	84.97	0.175	75%

Table 8: The effect of changing b_1 while keeping other parameters to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	b_1	Change in parameter
9.72	7.66	4.34	116.29	133.57	82.29	0.055	-75%
9.51	7.47	4.67	84.76	104.29	99.15	0.11	-50%
9.23	7.47	5.00	63.60	81.63	110.47	0.165	-25%
8.84	7.58	5.33	49.68	63.95	118.22	0.22	0%
8.29	7.76	5.62	41.10	50.35	123.70	0.275	25%
7.54	7.98	5.85	36.58	40.12	127.91	0.33	50%
6.55	8.23	6.01	35.27	32.59	131.62	0.385	75%

Table 9: The effect of changing b_2 while keeping other parameters to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	b_2	Change in parameter
6.00	17.44	2.96	1092.86	8.21	276.89	0.1	-75%
6.00	11.27	3.55	311.63	40.57	210.16	0.2	-50%
7.29	8.94	4.36	108.11	59.45	154.94	0.3	-25%
8.84	7.58	5.33	49.68	63.95	118.22	0.40	0%
9.56	6.74	6.07	31.01	72.21	91.11	0.5	25%
9.89	6.11	6.54	25.61	81.80	71.52	0.6	50%
10.06	5.58	6.78	25.43	91.13	57.52	0.7	75%

Table 10: The effect of changing β_1 while keeping other parameter to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	β_1	Change in parameter
8.71	7.67	5.35	38.69	50.22	116.04	0.04	-75%
8.74	7.67	5.36	41.82	53.65	116.17	0.08	-50%
8.78	7.64	5.35	45.37	58.07	116.86	0.12	-25%
8.84	7.58	5.33	49.68	63.95	118.22	0.16	0%
8.92	7.49	5.28	55.40	72.15	120.47	0.2	25%
9.03	7.34	5.20	63.80	84.36	124.06	0.24	50%
9.17	7.11	5.05	77.95	104.51	130.01	0.28	75%

Table 11: The effect of changing β_2 while keeping other parameters to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	β_2	Change in parameter
8.79	7.45	5.19	31.39	64.78	99.70	0.045	-75%
8.84	7.51	5.26	37.12	64.02	103.95	0.09	-50%
8.86	7.55	5.31	43.09	63.72	109.93	0.135	-25%
8.84	7.58	5.33	49.68	63.95	118.22	0.18	0%
8.77	7.61	5.32	57.51	64.79	129.75	0.225	25%
8.65	7.64	5.29	67.62	66.38	146.23	0.27	50%
8.44	7.68	5.23	82.03	69.06	171.03	0.315	75%

Table 12: The effect of changing θ_1 while keeping other parameters to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	θ_1	Change in parameter
9.63	5.75	4.13	207.52	258.66	168.01	0.2	-50%
9.11	7.23	5.14	69.30	92.76	126.78	0.3	-25%
8.84	7.58	5.33	49.68	63.95	118.22	0.40	0%
8.69	7.70	5.36	41.24	52.15	115.45	0.5	25%
8.60	7.70	5.33	35.64	45.96	115.31	0.6	50%
8.55	7.63	5.24	30.74	42.43	117.01	0.7	75%

Table 13: The effect of changing θ_2 while keeping other parameters to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	θ_2	Change in parameter
7.06	7.86	4.75	203.49	89.73	374.06	0.25	-50%
8.48	7.68	5.25	77.00	68.58	164.11	0.375	-25%
8.84	7.58	5.33	49.68	63.95	118.22	0.50	0%
8.88	7.47	5.24	35.43	63.62	100.45	0.625	25%
8.74	7.31	5.04	24.02	65.67	93.49	0.75	50%
8.49	7.10	4.77	12.43	69.23	92.23	0.875	75%

Table 14: The effect of changing h_1 while keeping other parameters to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	h_1	Change in parameter
8.48	7.53	6.00	55.95	82.86	143.45	0.15	-75%
8.60	7.55	5.73	53.76	75.34	133.33	0.3	-50%
8.72	7.57	5.51	51.67	69.15	125.09	0.45	-25%
8.84	7.58	5.33	49.68	63.95	118.22	0.6	0%
8.96	7.60	5.17	47.80	59.53	112.37	0.75	25%
9.07	7.61	5.04	46.00	55.70	107.31	0.9	50%
9.18	7.62	4.93	44.29	52.35	102.88	1.05	75%

Table 15: The effect of changing h_2 while keeping other parameters to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	h_2	Change in parameter
8.85	7.18	7.05	64.96	96.20	190.90	0.125	-75%
8.83	7.33	6.23	59.00	81.34	156.88	0.25	-50%
8.83	7.46	5.70	54.02	71.30	134.45	0.375	-25%
8.84	7.58	5.33	49.68	63.95	118.22	0.5	0%
8.85	7.70	5.05	45.82	58.28	105.79	0.625	25%
8.86	7.81	4.83	42.32	53.73	95.89	0.75	50%
8.88	7.92	4.65	39.12	49.98	87.76	0.875	75%

Table 16: The effect of changing c_1 while keeping other parameters to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	c_2	Change in parameter
5.25	6.73	3.34	150.94	127.35	135.26	1.5	-75%
6.48	7.04	3.95	103.55	101.87	130.05	3	-50%
7.68	7.33	4.62	71.51	80.77	124.15	4.5	-25%
8.84	7.58	5.33	49.68	63.95	118.22	6	0%
9.94	7.79	6.04	34.56	50.89	112.69	7.5	25%
10.97	7.95	6.74	23.82	40.89	107.82	9	50%
11.91	8.07	7.40	15.97	33.27	103.71	10.5	75%

Table 17: The effect of changing c_2 while keeping other parameters to original numeric values.

P_1	P_2	T	TP	Q_1	Q_2	c_2	Change in parameter
7.99	4.34	2.84	216.29	76.26	216.48	1.25	-75%
8.28	5.40	3.50	137.22	73.14	180.55	2.5	-50%
8.57	6.49	4.32	84.34	68.79	147.22	3.75	-25%
8.84	7.58	5.33	49.68	63.95	118.22	5	0%
9.07	8.67	6.52	27.14	59.26	94.19	6.25	25%
9.23	9.71	7.84	12.36	55.33	75.13	7.5	50%
9.28	10.66	9.18	2.39	52.61	60.59	8.75	75%

Profit Graph

The graph in **Figure 6** shows combined summary from previous tables, the graph shows the effect of parameter changes on profit.

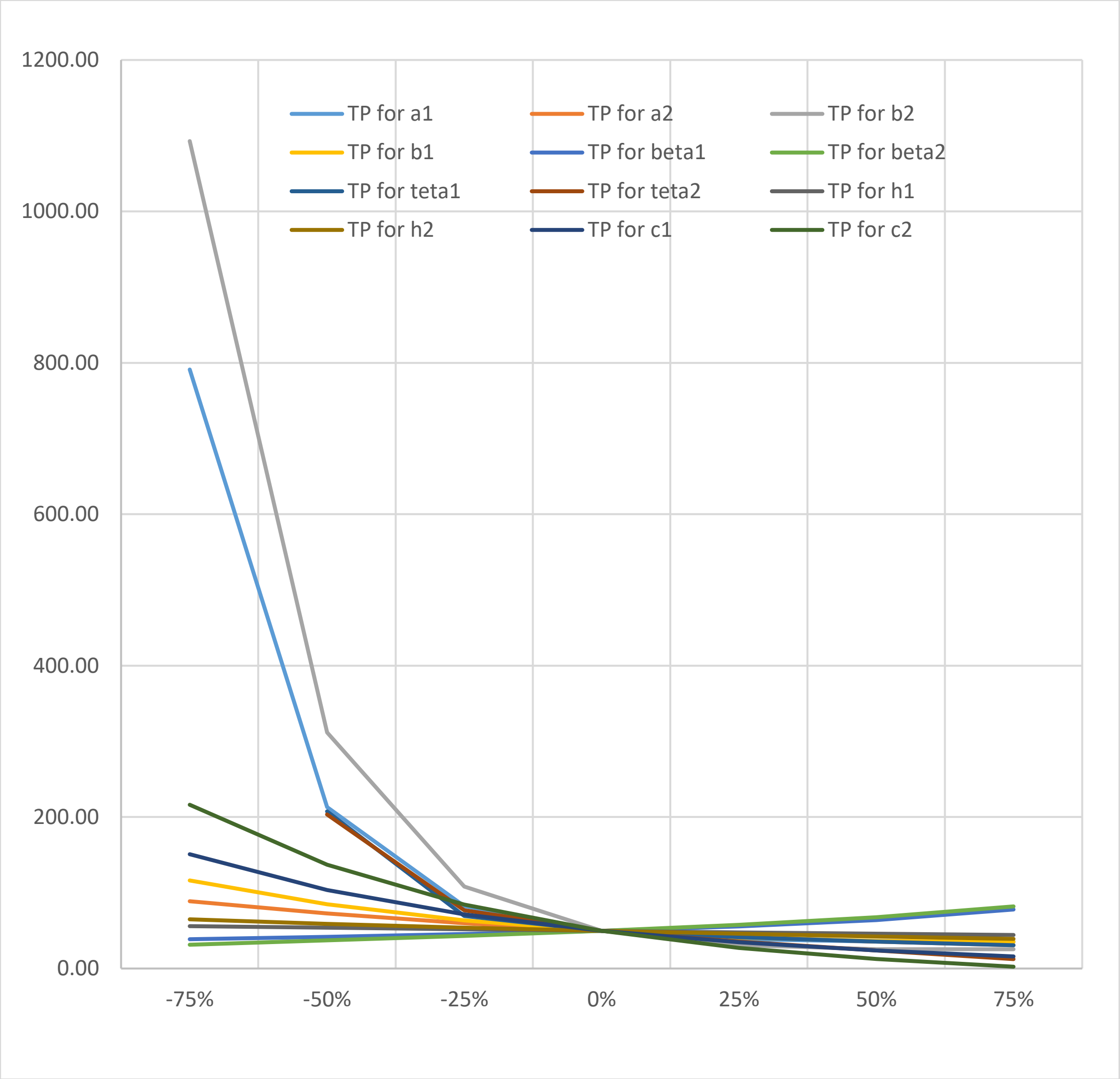

Figure 6: Changes in profit due to parameter changes

Cycle Time graph

The graph in **Figure 7** shows the effect of percentage change in parameter on cycle time.

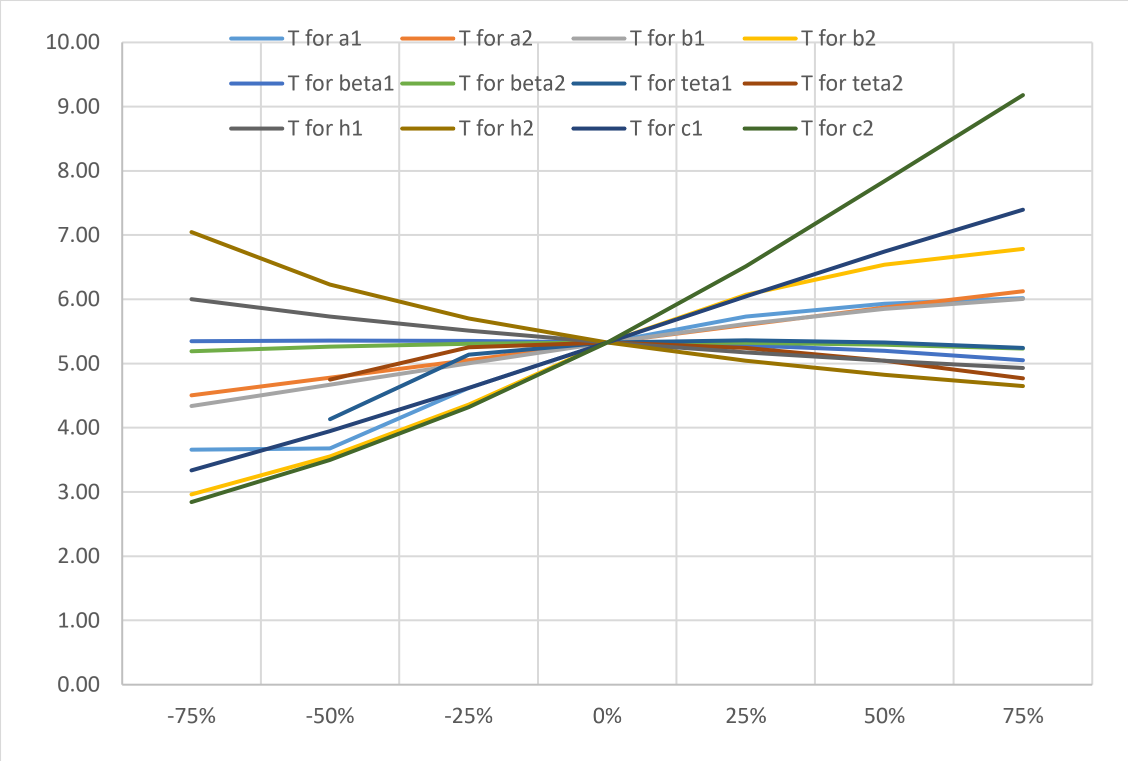

Figure 7: Change in T due to parameters changes.

Observations from the above tables and graphs is summarised below.

- Table 6

As the price coefficient for product 1 a_1 increases the selling price P_1 decreases while product 2 selling price P_2 increases. The cycle length T increases slightly. The unit profit decreases as seen in **Figure 6**.

- Table 7

Similarly, when a_2 increases, the selling price P_1 for product 1 increases slightly while the selling price P_2 for product 2 decreases marginally. The cycle length increases slightly, and the unit profit decreases as seen **Figures 6** and **7.**

- Table 8

When b_1 increases the selling price for product 1 P_1 decreases while the selling price for product 2 P_2 increases marginally. The cycle length increases slightly while the unit profit decreases as seen in **Figures 4** and **5.**

- Table 9

An increase in b_2 results in an increase in the selling price P_1 and a decrease in P_2. The cycle length increases while the unit profit decreases drastically as apparent from **Figures 6** and **7**.

- Table 10 and 11

When either time sensitivity for product 1 or 2 increases, the unit profit increases as observed from **Figure 6** while on the other hand the cycle length decreases as seen in **Figure 7**. P_1 decrease with an increase either β_1 or β_2. Whereas P_2 decrease with an increase in β_1 and on the other hand P_2 increase with an increase in β_2.

- Table 12 and 13

When the deterioration rate of either product 1 or 2 increases the cycle length increases while the unit profit decreases drastically as seen from **Figures 6** and **7**. The price for product 1 decreases with an increase in θ_1 and increases with an increase in θ_2. Product 2 selling price follows a similar trend as product 1 selling price. changes as well which indicates that it is sensitive to changes in either θ_1 or θ_2.

- Table 14 and 15

As the unit holding cost increase for either product the unit profit decreases, as expected. And the prices for products 1 and 2 increase due to increase in either unit holding cost of the products. The cycle length increases.

- Table 16 and 17

Similarly, as the unit purchasing cost increase for either product the unit profit decreases. As anticipated, the selling prices for products 1 and 2 increases with an increase in either unit purchasing cost of the products. On the other hand, the cycle length decreases with an increase in unit purchase cost.

In summary, it is observed that the variables P_1,P_2, T and TP are all sensitive to changes in the model parameters $a_n b_n$, β_n, θ_n, h_n and c_n.When the price sensitivity parameter $a_n, b_n,$ increase results in a decrease in TP as seen from the graphs in Figure 2. On the other hand, an increase in deterioration rate θ_n for products results in a decrease in TP. Conversely, an increase in β_n result in an increase in TP. Finally, an increase in either unit holding or purchasing cost result in a decrease in profit. This is an indication that the retailer should always try to find ways to keep costs low in order to maximises profit.

3.5 CONCLUSION

In this chapter, an EOQ model for two deteriorating complementary items has been developed. The model has been thematically formulated in order to find optimal values of the selling price and cycle length that maximises profit. Excel solver function has been used in this regard to perform numerical analysis in order to find optimised values. Sensitivity analysis has been performed to draw understanding on how the changes in input parameters influences the outputs of the model. As seen from the analysis, changes in the parameters of the model have influence on the cycle length, pricing, and the profitability of the inventory policy. Complementarity of items has often been implicitly implied by many researchers. In this chapter complementarity of items has been explicitly defined and incorporated in the development of the model. Furthermore, to this, the contribution made by the model developed in this chapter is in the domain of deteriorating items specifically considering mutual complements.

Since the model presented in this chapter only considers complementarity of items, further extension can be made to incorporate substitutable items. This will allow the model to address current trend in the market as more products are becoming substitutable by other products. The model can also be further expanded to allow for shortages as this is often encountered in inventory management. As observed in some markets the demand for some items may be stimulated by physical display or presence, therefore one may be interested in extended the model to items which have stock dependent demand. Some of these possible extensions are explored in the next chapter of this dissertation.

CHAPTER 4

4 EOQ MODEL FOR THREE DETERIORATING ITEMS WITH SUBSTITUTABLE AND MUTUAL COMPLEMENTARY PRICE-, STOCK -, AND TIME-DEPENDENT DEMAND

4.1 PROBLEM DEFINITION

In today's market, numerous customers tend to buy their day-to-day goods all at the same time and preferably from the same retail shop which is a phenomenon referred by Edalatpour and Mirzapour (2019) as "holus-bolus buying" meaning buying all at once. Among the items bought at once some items are deteriorating in nature with demand that is price dependent on each other such as complementary and substitute items. Moreover, some of these item's demand is dependent on the displayed stock level of these items. Consequently, this observation in the market gives rise for the need for joint replenishment of these items.

In addressing this scenario, an inventory model is developed in this chapter which considers three deteriorating items namely, Product 1, 2 and 3. Where Product 1 as the main item has demand that is dependent on the following: its selling price, stock level, the selling price of the mutual complement (Product 2), and the selling price of its substitute (Product 3) as well as time. Product 2 has demand that is dependent on its selling price, the selling price of its complement (Product 1) and time. Product 3 has a demand that is dependent on its selling price and the selling price of its substitute item (Product 1), and time. This scenario is visually illustrated in **Figure 8**.

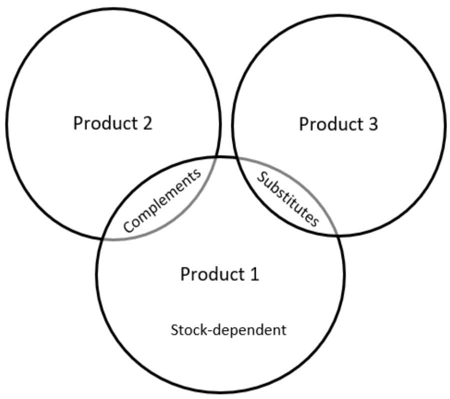

Figure 8: Products' price dependency.

The practical scenario defined by the model developed is in the context animal feed supply chain between medicines supplier and animal feed producer. Where the medicine supplier seeks to determine the optimal inventory policy to best serve the manufacturer of animal feed in the presence of complementary and substitutability of products (Medicines). Where these medicines are used complementarily and substitutable by the animal feed producers for various nutritional needs.

4.2 NOTATIONS AND ASSUMPTIONS

4.2.1 Notations

The notations presented in **Table 18** are use throughout model development together with the assumption made in the next subsection.

Table 18: Inventory model notations (Parameters and decision Variables).

Parameters

Notations	***Units***	***Description***
$\boldsymbol{n}$		Product 1,2,3
$\boldsymbol{D_n}$	R/Unit	Demand for product n.
$\boldsymbol{a_n}$	Constant	Product 1 price coefficient for the demand rate (a > 0) of product n.
$\boldsymbol{b_n}$	Constant	Product 2 price coefficient for the demand rate (b > 0) of product n.
$\boldsymbol{\gamma_n}$	Constant	Product 3 price coefficient for the demand rate γ_n > 0) of product n.
$\boldsymbol{\theta_n}$	Constant	Deterioration rate of product n, where ($0 \leq \theta_n \leq 1$).
$\boldsymbol{s}$	Constant	Stock level dependent coefficient for Product 1 demand
$\boldsymbol{k_0}$	R/Unit	Constant portion of order cost per order.
$\boldsymbol{k_n}$	R/Unit	Variable portion of order cost per order for product $\boldsymbol{n}$.
$\boldsymbol{h_n}$	R/Unit	Holding cost per unit of product $\boldsymbol{n}$..
$\boldsymbol{c_n}$	R/Unit	Purchasing cost per unit of product $\boldsymbol{n}$.
$\boldsymbol{I_n(t)}$	Unit	Inventory level at time t of product $\boldsymbol{n}$.
$\boldsymbol{A_n}$	Unit	Initial demand of product n.
$\boldsymbol{\beta_n}$	Constant	Constant governing decreasing rate of demand of n th product.

Decision variables

Notations	***Units***	***Description***
$\boldsymbol{Q_n}$	Unit	Order quantity for nth Product.
$\boldsymbol{P_n}$	R	Selling price (R/Unit) of product n.
$\boldsymbol{T}$	Month	Length of replenishment cycle.

4.2.2 Assumptions

The following are assumptions made in the model:

- The demand for items is given by exponential decreasing function (with some products having stock, price, and time dependent).
- Each item has constant holding cost.
- Products experience joint ordering cost with constant and variable portion.
- Instantaneous replenishment (Zero Leadtime).
- All items experience the same replenishment cycle length.
- All items have constant deteriorating rate.

- Substitution occurs
- Shortages are not allowed

4.3 MODEL DEVELOPMENT

4.3.1 Mathematical model

The study considers three deteriorating items, where the first product has a demand that is dependent on its stock, selling price, complementary (Product 2) and substitute (Product 3) prices. The second product has demand that is dependent on its selling price and the selling price of its complement (Product 1). The third product has demand that is dependent on its selling price and the selling price of its substitute product (Product 1). All items experience unique constant deterioration rates. The items arrive at the supplier instantaneously as per order quantities Q_1, Q_2 and Q_3. The stock level of product 1 is reduced to zero at t_1 due to deterioration and its demand. At $t > t_1$ the demand for product 1 is satisfied through substitution with product 3 until the end of cycle T. Product 2's inventory level is reduced to zero at T through deterioration as well and its demand. Due to deterioration and its demand product 3 inventory level is reduced to $I(t_1) = q_3(t_1)$ at t_1, during substitution $t > t_1$ inventory is reduced drastically to zero at the end of cycle. The inventory level for the described policy is illustrated in **Figure 9**.

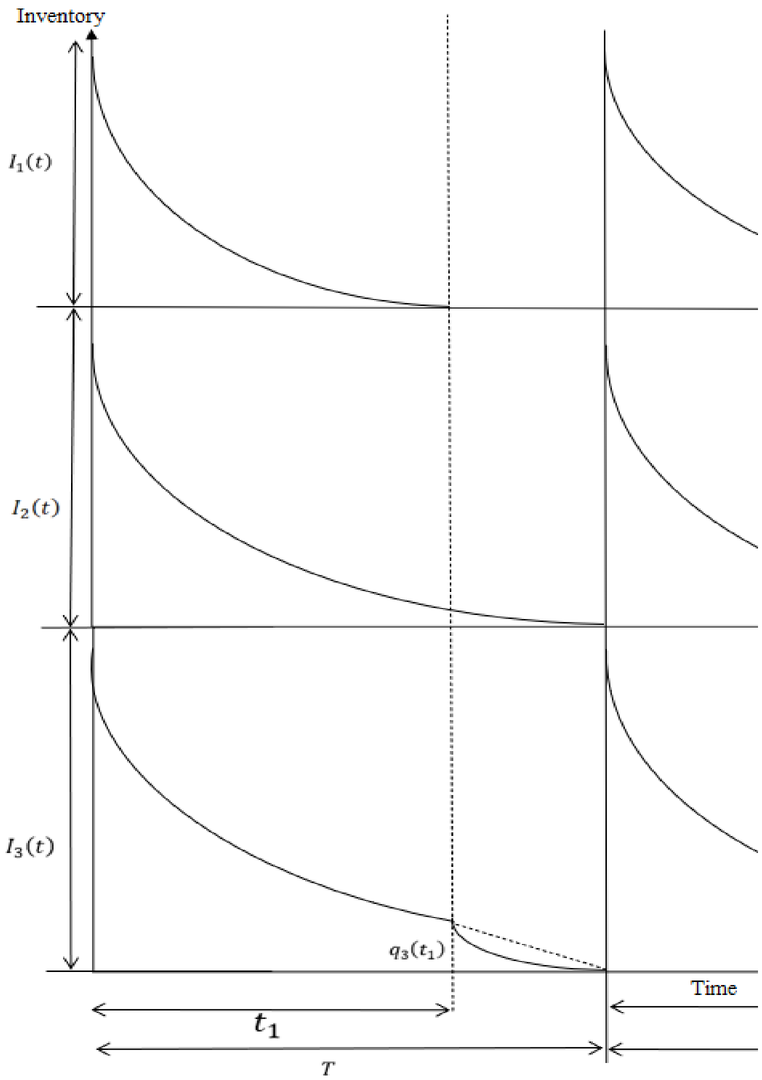

Figure 9: Inventory levels for the 3 products.

The three products' demand functions are denoted by

$$D_1 = A_1 e^{-a_1P_1 - b_1P_2 + \gamma_1P_3 - \beta_1 t} + sI_1(t) \qquad 0 \le t \le t_1 \qquad (1)$$

where s is the stock-dependent coefficient for the demand

$$D_2 = A_2 e^{-a_2P_1 - b_2P_2 - \beta_2 t} \qquad 0 \le t \le T \qquad (2)$$

and

$$D_3 = A_3 e^{a_3P_1 - \gamma_3P_3 - \beta_3 t} \qquad 0 \le t \le t_1 \qquad (3)$$

At $t \ge t_1$ during substitution, product 3 experiences its demand and the demand for product 1 which is without the component of stock-dependency as a result of substitution. The demand is given by

$$d_3 = A_3 e^{a_3P_1 - \gamma_3P_3 - \beta_3 t} + A_1 e^{-a_1P_1 - b_1P_2 + \gamma_1P_3 - \beta_1 t} \qquad t_1 \le t \le T \qquad (4)$$

The inventory level functions for the items are denoted by

$$\frac{dI_1(t)}{dt} + I_1(t)[\theta_1 + s] = -A_1 e^{-a_1P_1 - b_1P_2 + \gamma_1P_3 - \beta_1 t} \qquad 0 \le t \le t_1 \qquad (5)$$

$$\frac{dI_2(t)}{dt} + \theta_2 I_2(t) = -A_2 e^{-a_2P_1 - b_2P_2 - \beta_2 t} \qquad 0 \le t \le T \qquad (6)$$

$$\frac{dI_3(t)}{dt} + \theta_3 I_3(t) = -A_3 e^{a_3P_1 - \gamma_3P_3 - \beta_3 t} \qquad 0 \le t \le t_1 \qquad (7)$$

$$\frac{dq_3(t)}{dt} + \theta_3 q_3(t) = -A_3 e^{a_3P_1 - \gamma_3P_3 - \beta_3 t} - A_1 e^{-a_1P_1 - b_1P_2 + \gamma_1P_3 - \beta_1 t} \qquad t_1 \le t \le T \qquad (8)$$

With the following boundary conditions

$$I_1(0) = Q_1, I_2(0) = Q_2, I_3(0) = Q_3 \text{ and } I_1(t_1) = 0,\ I_2(T) = 0,\ I_3(t_1) = q_3(t_1),\quad q_3(T) = 0 \quad (9)$$

Now consider product 1, equation (5) is of the form

$\frac{dI_1(t)}{dt} + f(t)y = g(t)$ with the integrating factor $I.F = e^{\int f(t)} = e^{(\theta_1+s)t}$ (10)

The general solution is given by

$$I_1(t) \cdot e^{(\theta_1+s)t} = -A_1 e^{-a_1P_1 - b_1P_2 + \gamma_1P_3} \cdot \int e^{t(\theta_1+s-\beta_1)} dt \qquad (11)$$

Integrating (11) results in inventory level function

$$I_1(t) \cdot e^{(\theta_1+s)t} = -\frac{A_1 e^{-a_1P_1 - b_1P_2 + \gamma_1P_3}}{\theta_1 + s - \beta_1} \cdot e^{t(\theta_1+s-\beta_1)} + C \qquad (12)$$

Considering boundary condition in (8), which is $I_1(t_1) = 0$ to solve for C in (12) and simplifying the equation results in

$$I_1(t) = \frac{A_1 e^{-a_1P_1 - b_1P_2 + \gamma_1P_3 - \beta_1 t}}{\theta_1 + s - \beta_1} \left[e^{(\theta_1+s-\beta_1)(t_1-t)} - 1\right] \qquad (13)$$

Now consider product 2, the instantaneous inventory level for product 2 is derive similarly as product 1, however over its boundary conditions in (8).

$$I_2(t) = \frac{A_2 e^{-a_2P_1 - b_2P_2 - \beta_2 t}}{\theta_2 - \beta_2} \left[e^{(\theta_2-\beta_2)(T-t)} - 1\right] \qquad (14)$$

Similarly, consider the following for product 3,

$$I_3(t) \cdot e^{(\theta_3)t} = -\frac{A_3 e^{a_3 P_1 - \gamma_3 P_3}}{\theta_3 - \beta_3} \cdot e^{t(\theta_3 - \beta_3)} + C \tag{15}$$

Substitute with boundary condition $I_3(t_1) = q_3$ to solve for C in (15) and simplifying the equation results in

$$I_3(t) = \frac{A_3 e^{a_3 P_1 - \gamma_3 P_3 - \beta_3 t}}{\theta_3 - \beta_3} \left[e^{(\theta_3 - \beta_3)(t_1 - t)} - 1\right] + q_3(t_1) \cdot e^{\theta_3 (t_1 - t)} \tag{16}$$

Consider $t > t_1$ during substitution, inventory level for product 3 is defined by $q_3(t)$ given by

$$q_3(t) \cdot e^{(\theta_3)t} = -\frac{A_3 e^{a_3 P_1 - \gamma_3 P_3}}{\theta_3 - \beta_3} \cdot e^{t(\theta_3 - \beta_3)} - \frac{A_1 e^{-a_1 P_1 - b_1 P_2 + \gamma_1 P_3}}{\theta_3 - \beta_1} \cdot e^{t(\theta_3 - \beta_1)} + C \tag{17}$$

Using boundary condition $q_3(T) = 0$ now

$$q_3(t) = \frac{A_3 e^{a_3 P_1 - \gamma_3 P_3 - \beta_3 t}}{\theta_3 - \beta_3} \left[e^{(\theta_3 - \beta_3)(T - t)} - 1\right] + \frac{A_1 e^{-a_1 P_1 - b_1 P_2 + \gamma_1 P_3 - \beta_1 t}}{\theta_3 - \beta_1} \left[e^{(\theta_3 - \beta_1)(T - t)} - 1\right] \tag{18}$$

with

$$\begin{aligned} q_3(t_1) = &\frac{A_3 e^{a_3 P_1 - \gamma_3 P_3 - \beta_3 t_1}}{\theta_3 - \beta_3} \left[e^{(\theta_3 - \beta_3)(T - t_1)} - 1\right] \\ &+ \frac{A_1 e^{-a_1 P_1 - b_1 P_2 + \gamma_1 P_3 - \beta_1 t_1}}{\theta_3 - \beta_1} \left[e^{(\theta_3 - \beta_1)(T - t_1)} - 1\right] \end{aligned} \tag{19}$$

Substitute initial conditions from (8) into inventory level functions, then

$$I_1(0) = Q_1 = \frac{A_1 e^{-a_1 P_1 - b_1 P_2 + \gamma_1 P_3}}{\theta_1 + s - \beta_1} \left[e^{(\theta_1 + s - \beta_1)t_1} - 1\right] \tag{20}$$

$$I_2(0) = Q_2 = \frac{A_2 e^{-a_2 P_1 - b_2 P_2}}{\theta_2 - \beta_2} \left[e^{(\theta_2 - \beta_2)T} - 1\right] \tag{21}$$

$$\begin{aligned} I_3(0) = Q_3 &= \frac{A_3 e^{a_3 P_3 - \gamma_3 P_3}}{\theta_3 - \beta_3} \left[e^{(\theta_3 - \beta_3)t_1} - 1\right] + q_3(t_1) \cdot e^{\theta_3 (t_1)} \\ &= \frac{A_3 e^{a_3 P_1 - \gamma_3 P_3}}{\theta_3 - \beta_3} \left[e^{(\theta_3 - \beta_3)t_1} - 1\right] \\ &\quad + \left[\frac{A_3 e^{a_3 P_1 - \gamma_3 P_3 - \beta_3 t_1}}{\theta_3 - \beta_3} \left[e^{(\theta_3 - \beta_3)(T - t_1)} - 1\right] \right. \\ &\quad \left. + \frac{A_1 e^{-a_1 P_1 - b_1 P_2 + \gamma_1 P_3 - \beta_1 t_1}}{\theta_3 - \beta_1} \left[e^{(\theta_3 - \beta_1)(T - t_1)} - 1\right]\right] \cdot e^{\theta_3 t_1} \end{aligned} \tag{22}$$

Note the following for simplicity

$$E = A_1 e^{-a_1 P_1 - b_1 P_2 + \gamma_1 P_3} \quad \text{and} \quad F = A_2 e^{-a_2 P_1 - b_2 P_2} \quad G = A_3 e^{a_3 P_1 - \gamma_3 P_3} \tag{23}$$

Holding cost (HC)

The total holding cost per cycle is given by

$$HC = HC_1 + HC_2 + HC_3 = h_1 \cdot \int_0^{t_1} I_1(t)dt + h_2 \cdot \int_0^{T} I_2(t)\ dt + h_3 \cdot \int_0^{t_1} I_3(t)\ dt + h_3 \cdot \int_{t_1}^{T} q_3(t)\ dt$$
$$= h_1 \cdot \frac{Ee^{(\theta_1+s-\beta_1)t_1}}{\theta_1+s-\beta_1}\left[\frac{1-e^{-(\theta_1+s)t_1}}{\theta_1+s}\right] - h_1 \cdot \frac{E}{\theta_1+s-\beta_1}\left[\frac{1-e^{-\beta_1 t_1}}{\beta_1}\right] + h_2 \cdot \frac{Fe^{(\theta_2-\beta_2)T}}{\theta_2-\beta_2}\left[\frac{1-e^{-\theta_2 T}}{\theta_2}\right] - h_2 \cdot \frac{F}{\theta_2-\beta_2}\left[\frac{1-e^{-\beta_2 T}}{\beta_2}\right] + h_3 \cdot \frac{Ge^{(\theta_3-\beta_3)t_1}}{\theta_3-\beta_3}\left[\frac{1-e^{-\theta_3 t_1}}{\theta_3}\right] - h_3 \cdot \frac{G}{\theta_3-\beta_3}\left[\frac{1-e^{-\beta_3 t_1}}{\beta_3}\right] + h_3 \cdot [q_3(t_1)] \cdot \left[\frac{e^{\theta_3 t_1}-1}{\theta_3}\right] + h_3 \cdot \frac{Ge^{(\theta_3-\beta_3)t_1}}{\theta_3-\beta_3}\left[\frac{e^{-\theta_3 t_1}-e^{-\theta_3 T}}{\theta_3}\right] - h_3 \cdot \frac{G}{\theta_3-\beta_3}\left[\frac{e^{-\beta_3 t_1}-e^{-\beta_3 T}}{\beta_3}\right] + h_3 \cdot \frac{Ee^{(\theta_3-\beta_1)T}}{\theta_3-\beta_1}\left[\frac{e^{-\theta_3 t_1}-e^{-\theta_3 T}}{\theta_3}\right] - h_3 \cdot \frac{E}{\theta_3-\beta_1}\left[\frac{e^{-\beta_1 t_1}-e^{-\beta_1 T}}{\beta_1}\right] \tag{24}$$

Order cost ($\boldsymbol{OC}$)

The total ordering cost for the two items constitutes of shared ordering cost given by.

$$OC = k_0 + \sum_{i=1}^{n} k_i\ \textit{where}\ n = 3\ \textit{in this case without any loss of generality} \tag{25}$$

Revenue ($\boldsymbol{TR}$) and Purchase cost($\boldsymbol{PC}$)

Revenue for each product is obtained by multiplying order quantity for the specific product with its selling price. While on the other hand, the purchase cost is obtained by multiplying unit purchase price with order quantity. These is formulated into equation (26) represented by first and second terms.

Profit ($\boldsymbol{TP}$)

Profit per unit time is obtained by dividing the total profit by cycle length T. Which is given by the following function

$$TP = \big[TR - PC - [HC + OC\ ]\big]/T$$
$$= \Bigg[(P_1 - c_1) \cdot \frac{E}{\theta_1+s-\beta_1}\ \left[e^{(\theta_1+s-\beta_1)t_1} - 1\right] + (P_2 - c_2) \cdot \frac{F}{\theta_2-\beta_2}\ \left[e^{(\theta_2-\beta_2)T} - 1\right] + (P_3 - c_3) \cdot \left[\frac{G}{\theta_3-\beta_3}\ \left[e^{(\theta_3-\beta_3)t_1} - 1\right] + [q_3(t_1)] \cdot e^{\theta_3 t_1}\right] - \Bigg(h_1 \cdot \frac{Ee^{(\theta_1+s-\beta_1)t_1}}{\theta_1+s-\beta_1}\left[\frac{1-e^{-(\theta_1+s)t_1}}{\theta_1+s}\right] - h_1 \cdot \frac{E}{\theta_1+s-\beta_1}\left[\frac{1-e^{-\beta_1 t_1}}{\beta_1}\right] + h_2 \cdot \frac{Fe^{(\theta_2-\beta_2)T}}{\theta_2-\beta_2}\left[\frac{1-e^{-\theta_2 T}}{\theta_2}\right] - h_2 \cdot \frac{F}{\theta_2-\beta_2}\left[\frac{1-e^{-\beta_2 T}}{\beta_2}\right] + h_3 \cdot \frac{Ge^{(\theta_3-\beta_3)t_1}}{\theta_3-\beta_3}\left[\frac{1-e^{-\theta_3 t_1}}{\theta_3}\right] - h_3 \cdot \frac{G}{\theta_3-\beta_3}\left[\frac{1-e^{-\beta_3 t_1}}{\beta_3}\right] + h_3 \cdot [q_3(t_1)] \cdot \left[\frac{e^{\theta_3 t_1}-1}{\theta_3}\right] + h_3 \cdot \frac{Ge^{(\theta_3-\beta_3)t_1}}{\theta_3-\beta_3}\left[\frac{e^{-\theta_3 t_1}-e^{-\theta_3 T}}{\theta_3}\right] - h_3 \cdot \frac{G}{\theta_3-\beta_3}\left[\frac{e^{-\beta_3 t_1}-e^{-\beta_3 T}}{\beta_3}\right] + h_3 \cdot \frac{Ee^{(\theta_3-\beta_1)T}}{\theta_3-\beta_1}\left[\frac{e^{-\theta_3 t_1}-e^{-\theta_3 T}}{\theta_3}\right] - h_3 \cdot \frac{E}{\theta_3-\beta_1}\left[\frac{e^{-\beta_1 t_1}-e^{-\beta_1 T}}{\beta_1}\right] + k_0 + \sum_{i=1}^{n} k_i\Bigg)\Bigg] /T \tag{26}$$

Cycle length

Linearising the exponential terms containing T in (26) by using Maclaurin's expansion for e^x. Therefore, the following is obtained,

$$e^{(\theta_1+s-\beta_1)t_1} = \sum_{m=0}^{\infty} \frac{(\theta_1+s-\beta_1)^m {t_1}^m}{m!} \approx 1 + (\theta_1+s-\beta_1)t_1 \tag{27}$$

$$e^{(\theta_2-\beta_2)T} \approx 1 + (\theta_2-\beta_2)T \tag{28}$$

$$e^{(\theta_3-\beta_3)t_1} \approx 1 + (\theta_3-\beta_3)t_1 \tag{29}$$

$$e^{-(\theta_3-s)t_1} \approx 1 - (\theta_1-s)t_1 \tag{30}$$

$$e^{(\theta_3-\beta_1)T} \approx 1 + (\theta_3-\beta_1)T \tag{31}$$

$$e^{(\theta_3-\beta_3)(T-t_1)} \approx 1 + (\theta_3-\beta_3)(T-t_1) \tag{32}$$

$$e^{(\theta_3-\beta_1)(T-t_1)} \approx 1 + (\theta_3-\beta_1)(T-t_1) \tag{33}$$

$$e^{-\beta_1 t_1} \approx 1 - \beta_1 t_1 \tag{34}$$

$$e^{-\beta_2 T} \approx 1 - \beta_2 T \tag{35}$$

$$e^{-\beta_3 t_1} \approx 1 - \beta_3 t_1 \tag{36}$$

$$e^{\theta_1 T} \approx 1 + \theta_1 T \tag{37}$$

$$e^{-\theta_2 T} \approx 1 - \theta_2 T \tag{38}$$

$$e^{\theta_3 t_1} \approx 1 + \theta_3 t_1 \tag{39}$$

$$e^{-\theta_3 t_1} \approx 1 - \theta_3 t_1 \tag{40}$$

$$e^{-\theta_3 T} \approx 1 - \theta_3 T \tag{41}$$

$$e^{-\beta_1 T} \approx 1 - \beta_1 T \tag{42}$$

Substituting these approximations into (26), results in

$$TP = \big[TR - PC - [HC + OC\,]\big]/T$$

$$= \Bigg[(P_1 - c_1) \cdot \frac{E}{\theta_1 + s - \beta_1}\ [1 + (\theta_1 + s - \beta_1)t_1 - 1] + (P_2 - c_2)$$
$$\cdot \frac{F}{\theta_2 - \beta_2}\ [1 + (\theta_2 - \beta_2)T - 1] + (P_3 - c_3)$$
$$\cdot \left[\frac{G}{\theta_3 - \beta_3}\ [1 + (\theta_3 - \beta_3)t_1 - 1] + [q_3(t_1)] \cdot [1 + \theta_3 t_1]\right]$$
$$- \Bigg(h_1 \cdot \frac{E[1 + (\theta_1 + s - \beta_1)t_1]}{\theta_1 + s - \beta_1}\left[\frac{1 - [1 - (\theta_1 + s)t_1]}{\theta_1 + s}\right] - h_1$$
$$\cdot \frac{E}{\theta_1 + s - \beta_1}\left[\frac{1 - [1 - \beta_1 t_1]}{\beta_1}\right] + h_2 \cdot \frac{F[1 + (\theta_2 - \beta_2)T]}{\theta_2 - \beta_2}\left[\frac{1 - [1 - \theta_2 T]}{\theta_2}\right] - h_2$$
$$\cdot \frac{F}{\theta_2 - \beta_2}\left[\frac{1 - [1 - \beta_2 T]}{\beta_2}\right] + h_3 \cdot \frac{G[1 + (\theta_3 - \beta_3)t_1\,]}{\theta_3 - \beta_3}\left[\frac{1 - [1 - \theta_3 t_1]}{\theta_3}\right] - h_3$$
$$\cdot \frac{G}{\theta_3 - \beta_3}\left[\frac{1 - [1 - \beta_3 t_1]}{\beta_3}\right] + h_3 \cdot [q_3(t_1)]\left[\frac{1 + \theta_3 t_1 - 1}{\theta_3}\right] + h_3$$
$$\cdot \frac{G[1 + (\theta_3 - \beta_3)t_1\,]}{\theta_3 - \beta_3}\left[\frac{[1 - \theta_3 t_1] - [1 - \theta_3 T\,]}{\theta_3}\right] - h_3$$
$$\cdot \frac{G}{\theta_3 - \beta_3}\left[\frac{(1 - \beta_3 t_1) - (1 - \beta_3 T)}{\beta_3}\right] + h_3$$
$$\cdot \frac{E[1 + (\theta_3 - \beta_1)T]}{\theta_3 - \beta_1}\left[\frac{(1 - \theta_3 t_1\,) - (1 - \theta_3 T)}{\theta_3}\right] - h_3$$
$$\cdot \frac{E}{\theta_3 - \beta_1}\left[\frac{(1 - \beta_1 t_1) - (1 - \beta_1 T)}{\beta_1}\right] + k_0 + \sum_{i=1}^{n} k_i\Bigg)\Bigg]/T \qquad (43)$$

Now, simplifying (43), then

$$TP = \big[TR - PC - [HC + OC\,]\big]/T$$
$$= \Bigg[(P_1 - c_1)E\omega T + (P_2 - c_2)FT + (P_3 - c_3) \cdot \big[G\omega T + [q_3(t_1)] \cdot [1 + \theta_3 \omega T]\big]$$
$$- \Bigg(h_1 E \cdot (\omega T)^2 + h_2 \cdot FT^2 + h_3 \cdot E(\omega T)^2 + h_3 \cdot [q_3(t_1)]\omega T + h_3 G\omega T(T - \omega T)$$
$$+ h_3 E\omega T(T - \omega T) + k_0 + \sum_{i=1}^{n} k_i\Bigg)\Bigg]/T \qquad (44)$$

Substitution occurs at $t > t_1$ whereby

$$t_1 = \omega T \qquad (45)$$

where $0 < \omega < 1$ is the fraction of the cycle length during which product one is still available.

Now differentiate (44) to find the 1st order differential equation to obtain optimal T using $TP' = 0$,

Let

$$TP'(T) = -(P_3 - c_3) \cdot q_3(t_1)T^{-2}$$
$$- \left[h_1 E\omega^2 + h_2 \cdot F + h_3 G(\omega - \omega^2) + h_3 E\omega - \left(k_0 + \sum_{i=1}^{n} k_i\right) T^{-2}\right] \qquad (46)$$

Simplify (46)

$$T = \sqrt{\frac{k_0 + \sum_{i=1}^{n} k_i - (P_3 - c_3) \cdot q_3(t_1)}{h_1 E\omega^2 + h_2 \cdot F + h_3 G(\omega - \omega^2) + h_3 E\omega}} \tag{47}$$

Therefore, the optimum cycle length T function is defined by (47).

Proof of optimality

To show that the unit profit function TP is concave, we prove that the Hessian matrix of the profit function (44) is negative (semi)definite.

With Hessian matrix for TP given by $H(P_1, P_2, P_2, T) = \begin{pmatrix} \frac{d^2TP}{dP_1{}^2} & \frac{d^2TP}{dP_1dP_2} & \frac{d^2TP}{dP_1dP_3} & \frac{d^2TP}{dP_1dT} \\ \frac{d^2TP}{dP_2dP_1} & \frac{d^2TP}{dP_2{}^2} & \frac{d^2TP}{dP_2dP_3} & \frac{d^2TP}{dP_2dT} \\ \frac{d^2TP}{dP_3dP_1} & \frac{d^2TP}{dP_3dP_2} & \frac{d^2TP}{dP_3{}^2} & \frac{d^2TP}{dP_3dT} \\ \frac{d^2TP}{dTdP_1} & \frac{d^2TP}{dTdP_2} & \frac{d^2TP}{dTdP_3} & \frac{d^2TP}{dT^2} \end{pmatrix}$ (48)

Second derivatives

Starting with T, the 1st derivative $TP'(T)$ is obtained in (47) now the second derivative is given by

$$TP''(T) = 2(P_3 - c_3) \cdot q_3(t_1)T^{-3} - \left[2\left(k_0 + \sum_{i=1}^{n} k_i\right) T^{-3}\right] \tag{49}$$

From (44) the 1st and 2nd derivatives of P_1 are given by the following

$$\begin{aligned} TP'(P_1) = {} & [(P_1 - c_1)(-a_1)\, E + E]\omega - a_2(P_2 - c_2)F + (P_3 - c_3) \cdot [a_3 G\omega] \\ & - \big(-a_1 h_1 E \cdot \omega^2 T - a_2 h_2 \cdot FT - a_1 h_3 \cdot E\omega^2 T + a_3 h_3 G\omega(T - \omega T) \\ & - a_1 h_3 E\omega(T - \omega T)\big) \end{aligned} \tag{50}$$

$$\begin{aligned} TP''(P_1) = {} & [(P_1 - c_1)a_1{}^2\, E - 2a_1 E]\omega + a_2{}^2(P_2 - c_2)F + (P_3 - c_3) \cdot [a_3{}^2 G\omega] \\ & - \big(a_1{}^2 h_1 E \cdot \omega^2 T + a_2{}^2 h_2 \cdot FT + a_1{}^2 h_3 \cdot E\omega T + a_3{}^2 h_3 G\omega(T - \omega T)\big) \end{aligned} \tag{51}$$

The 1st and 2nd derivatives of P_2 are given by the following

$$\begin{aligned} TP'(P_2) = {} & [(P_2 - c_2)(-b_2)F + F] - b_1(P_1 - c_1)E\omega \\ & - \big(-b_1 h_1 E \cdot \omega^2 T - b_2 h_2 \cdot FT - b_1 h_3 \cdot E\omega^2 T - b_1 h_3 E\omega(T - \omega T)\big) \end{aligned} \tag{52}$$

$$\begin{aligned} TP''(P_2) = {} & \left[(P_2 - c_2)b_2{}^2 F - 2b_2 F\right] + b_1{}^2(P_1 - c_1)E\omega \\ & - \left(b_1{}^2 h_1 E \cdot \omega^2 T + b_2{}^2 h_2 \cdot FT + b_1{}^2 h_3 \cdot E\omega T\right) \end{aligned} \tag{53}$$

The 1st and 2nd derivatives of P_3 are given by the following

$$\begin{aligned} TP'(P_3) = {} & \gamma_1(P_1 - c_1)E\omega + (P_3 - c_3) \cdot [-\gamma_3 G\omega] + G\omega \\ & - \big(\gamma_1 h_1 E \cdot \omega^2 T + \gamma_1 h_3 \cdot E\omega T - \gamma_3\, h_3 G\omega(T - \omega T)\big) \end{aligned} \tag{54}$$

$$\begin{aligned} TP''(P_3) = {} & \gamma_1{}^2(P_1 - c_1)E\omega + (P_3 - c_3) \cdot (\gamma_3{}^2 G\omega) - 2\gamma_3 G\omega \\ & - \big(\gamma_1{}^2 h_1 E \cdot \omega^2 T + \gamma_1{}^2 h_3 \cdot E\omega T + \gamma_3{}^2\, h_3 G\omega(T - \omega T)\big) \end{aligned} \tag{55}$$

Other second derivatives are

$$TP''(P_2P_1) = [(P_2 - c_2)a_2b_2\,F - a_2F] - b_1\omega[(P_1 - c_1)(-a_2E) + E] - (a_1b_1h_1E \cdot \omega^2T + a_2b_2h_2 \cdot FT + a_1b_1h_3 \cdot E\omega T) \tag{56}$$

$$TP''(P_2P_3) = -\gamma_1(P_1 - c_1)E\omega - (-\gamma_1b_1h_1E \cdot \omega^2T - \gamma_1b_1h_3 \cdot E\omega T) \tag{57}$$

$$TP''(P_2T) = -(-b_1h_1E \cdot \omega^2 - b_2h_2 \cdot F - b_1h_3 \cdot E\omega) \tag{58}$$

$$TP''(P_1P_3) = [(P_1 - c_1)(-a_1\gamma_1E) + \gamma_1E] + (P_3 - c_3)(-\gamma_3a_3G\omega) + a_3G\omega - (-a_1\gamma_1h_1E \cdot \omega^2T - a_1\gamma_1h_3 \cdot ET - a_3\gamma_3h_3 \cdot G\omega(T - \omega T)) \tag{59}$$

$$TP''(P_1T) = -(-a_1h_1E \cdot \omega^2 - a_2h_2 \cdot F + a_3h_3G\omega(1 - \omega) - a_1h_3E\omega) \tag{60}$$

$$TP''(P_3T) = -(1) \cdot q_3(t_1)T^{-2} - [\gamma_1h_1E\omega^2 - \gamma_3h_3G(\omega - \omega^2) + \gamma_1h_3E\omega] \tag{61}$$

Now, to prove that TP is negative (semi)definite. The determinants must satisfy the following conditions $|H(P_1)| < 0,\ |\,H(P_2)| > 0, |H(P_3)| < 0$ and $|H(T)| > 0$ where

$$|H(P_1)| = \frac{d^2TP}{dP_1{}^2}, |\,H(P_2)| = \begin{bmatrix} \frac{d^2TP}{dP_1{}^2} & \frac{d^2TP}{dP_1dP_2} \\ \frac{d^2TP}{dP_2dP_1} & \frac{d^2TP}{dP_2{}^2} \end{bmatrix}, |H(P_3)| = \begin{vmatrix} \frac{d^2TP}{dP_1{}^2} & \frac{d^2TP}{dP_1dP_2} & \frac{d^2TP}{dP_1dP_3} \\ \frac{d^2TP}{dP_2dP_1} & \frac{d^2TP}{dP_2{}^2} & \frac{d^2TP}{dP_2dP_3} \\ \frac{d^2TP}{dP_3dP_1} & \frac{d^2TP}{dP_3dP_2} & \frac{d^2TP}{dP_3{}^2} \end{vmatrix},$$

$$|H(T)| = \begin{vmatrix} \frac{d^2TP}{dP_1{}^2} & \frac{d^2TP}{dP_1dP_2} & \frac{d^2TP}{dP_1dP_3} & \frac{d^2TP}{dP_1dT} \\ \frac{d^2TP}{dP_2dP_1} & \frac{d^2TP}{dP_2{}^2} & \frac{d^2TP}{dP_2dP_3} & \frac{d^2TP}{dP_2dT} \\ \frac{d^2TP}{dP_3dP_1} & \frac{d^2TP}{dP_3dP_2} & \frac{d^2TP}{dP_3{}^2} & \frac{d^2TP}{dP_3dT} \\ \frac{d^2TP}{dTdP_1} & \frac{d^2TP}{dTdP_2} & \frac{d^2TP}{dTdP_3} & \frac{d^2TP}{dT^2} \end{vmatrix} \tag{62}$$

Given by the following determinants.

$$|H(P_1)| = \frac{d^2TP}{dP_1{}^2} = TP''(P_1) \tag{63}$$

$$|\,H(P_2)| = \left(\frac{d^2TP}{dP_1{}^2} \times \frac{d^2TP}{dP_2{}^2}\right) - \left(\frac{d^2TP}{dP_2dP_1}\right)^2 \tag{64}$$

$$|H(P_3)| = \frac{d^2TP}{dP_1{}^2}\left(\frac{d^2TP}{dP_2{}^2} \cdot \frac{d^2TP}{dP_3{}^2} - \frac{d^2TP}{dP_3dP_2} \cdot \frac{d^2TP}{dP_2dP_3}\right) - \frac{d^2TP}{dP_1dP_2}\left(\frac{d^2TP}{dP_2dP_1} \cdot \frac{d^2TP}{dP_3{}^2} - \frac{d^2TP}{dP_2dP_3} \cdot \frac{d^2TP}{dP_3dP_1}\right) + \frac{d^2TP}{dP_1dP_3}\left(\frac{d^2TP}{dP_2dP_1} \cdot \frac{d^2TP}{dP_3dP_2} - \frac{d^2TP}{dP_2{}^2} \cdot \frac{d^2TP}{dP_3dP_1}\right) \tag{65}$$

The determinant for $|H(T)|$ is derived in a similar manner for a 4 by 4 matrix. Due to the complexity of the hessian second derivatives, it becomes difficult to obtain close form solutions for the determinants. Therefore, the optimality of the model is proved by numerical analysis by carefully investigating optimal values which satisfy optimality conditions as established by determinants of the hessian matrix.

4.4 NUMERICAL RESULTS

4.4.1 Numerical example

The numerical example presented considers a supplier of medication to producer of animal feed, the supplier wishes to find optimal values of the selling prices for medication with substitutable and complementary characteristics. The supplier also wishes to know the optimum replenishment as well the quantity required to satisfy the demand. The model developed in this dissertation is used to help in decision making of this nature. The numerical values have been chosen as shown in **Table 19**. Sensitivity analysis is performed next to assess the significance of these values.

Table 19: Numerical values

Parameters	Product 1	Product 2	Product 2
$\boldsymbol{a_n}$	0.12	0.16	0.24
$\boldsymbol{b_n}$	0.1	0.20	
$\boldsymbol{\theta_n}$	0.2	0.25	0.3
$\boldsymbol{h_n}$	0.3	0.35	0.4
$\boldsymbol{c_n}$	5	6	11
$\boldsymbol{A_n}$	150	160	200
$\boldsymbol{\beta_n}$	0.2	0.16	0.18
$\boldsymbol{e}$	2.718		
$\boldsymbol{k_0}$	100		
$\boldsymbol{k_n}$	20	40	30
$\boldsymbol{\Upsilon_n}$	0.1		0.35
$\boldsymbol{s}$	0.2		
$\boldsymbol{\omega}$	0.7		

The selected numerical values illustrated in **Table 19** are used in the optimisation model developed in this dissertation and the model has been programmed on excel. The model yields the optimal results as shown in **Table 20** as this satisfies optimality conditions established by determinants of the hessian matrix. The results of proof of optimality are depicted in **Table 21**, this result indicate that the profit function is negative (semi)definite since optimality conditions are satisfied

Table 20: Model results

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3
8.2	8.0	20.0	7.00	4.9	1876	1553.8	85.2	1100.2

Table 21: Optimality conditions

$\lvert H(P_1)\rvert < 0$	$\lvert H(P_2)\rvert > 0$	$\lvert H(P_3)\rvert < 0$	$\lvert H(T)\rvert > 0$
-33.1	144.0	-1352.1	7167.1

4.4.2 Sensitivity analysis

Sensitivity analysis has been conducted whereby the parameters a_n, b_n , Υ_n , θ_n , h_n, c_n , s and ω were changed for each item whilst maintaining other parameters at selected numeric value as established in **Table 19**. A 25% reduction and increment around the chosen value is used to test the sensitivity of parameter on the model outputs. The results of the analysis are summarised in the following tables. **Figure 10** gives graphical representation of the profit as the consequence of parameters changes.

Table 22: The effect of changing a_1.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	a_1	Change in parameter
20.00	8.00	20.00	7.00	4.90	6885.26	2274.63	12.84	1878.21	0.03	-75%
20.00	8.00	20.00	7.00	4.90	3917.14	1248.34	12.84	1162.85	0.06	-50%
11.31	8.00	20.00	7.00	4.90	2495.11	1498.26	51.61	1080.66	0.09	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.12	0%
6.42	8.00	20.00	7.00	4.90	1511.41	1582.04	112.76	1113.97	0.15	25%
5.28	8.00	20.00	7.00	4.90	1270.40	1603.16	135.40	1126.00	0.18	50%
5.00	8.00	20.00	6.91	4.83	1093.07	1419.87	138.98	1003.35	0.21	75%

Table 23: The effect of changing a_2.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	a_2	Change in parameter
8.18	8.00	20.00	7.00	4.90	1900.50	1553.85	227.13	1100.22	0.04	-75%
8.18	8.00	20.00	7.00	4.90	1889.60	1553.85	163.78	1100.22	0.08	-50%
8.18	8.00	20.00	7.00	4.90	1881.73	1553.85	118.09	1100.22	0.12	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.16	0%
8.18	8.00	20.00	7.00	4.90	1871.98	1553.85	61.40	1100.22	0.20	25%
8.18	8.00	20.00	7.00	4.90	1869.03	1553.85	44.27	1100.22	0.24	50%
8.18	8.00	20.00	7.00	4.90	1866.91	1553.85	31.92	1100.22	0.28	75%

Table 24: The effect of changing a_3.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	a_3	Change in parameter
8.01	8.00	20.00	7.00	4.90	1861.47	1585.08	87.44	1108.75	0.06	-75%

8.01	8.00	20.00	7.00	4.90	1864.19	1585.08	87.44	1111.15	0.12	-50%
8.08	8.00	20.00	7.00	4.90	1868.66	1572.10	86.49	1106.11	0.18	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.24	0%
8.39	8.00	20.00	7.00	4.90	1888.89	1513.88	82.25	1085.09	0.30	25%
9.09	8.00	20.00	7.00	4.90	1914.30	1392.42	73.57	1034.09	0.36	50%
20.0	8.00	14.95	7.00	4.90	26360.44	226.87	12.84	62934.16	0.42	75%

Table 25: The effect of changing b_1.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	b_1	Change in parameter
8.13	8.00	20.00	7.00	4.90	3412.75	2845.17	85.71	2000.14	0.03	-75%
8.13	8.00	20.00	7.00	4.90	2795.37	2329.43	85.71	1640.65	0.05	-50%
8.13	8.00	20.00	7.00	4.90	2289.89	1907.17	85.71	1346.33	0.08	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.10	0%
8.20	8.00	20.00	7.00	4.90	1537.25	1268.98	84.87	901.75	0.13	25%
8.20	8.00	20.00	7.00	4.90	1259.86	1038.96	84.87	741.41	0.15	50%
8.26	8.00	20.00	7.00	4.90	1032.78	844.46	84.05	606.10	0.18	75%

Table 26: The effect of changing b_2.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	b_2	Change in parameter
7.96	8.00	20.00	7.00	4.90	1910.67	1595.44	292.85	1128.33	0.05	-75%
8.06	8.00	20.00	7.00	4.90	1894.19	1575.61	193.06	1114.92	0.10	-50%
8.13	8.00	20.00	7.00	4.90	1883.30	1562.53	127.98	1106.08	0.15	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.20	0%
8.21	8.00	20.00	7.00	4.90	1871.25	1548.07	56.80	1096.32	0.25	25%
8.21	8.00	20.00	7.00	4.90	1868.02	1548.07	38.07	1096.32	0.30	50%
8.24	8.00	20.00	7.00	4.90	1865.88	1541.63	25.38	1091.98	0.35	75%

Table 27: The effect of changing Υ_1.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	Υ_1	Change in parameter
8.62	8.00	20.00	7.00	4.90	424.39	328.61	79.28	248.13	0.03	-75%
8.36	8.00	20.00	7.00	4.90	694.68	559.16	82.69	407.66	0.05	-50%
8.24	8.00	20.00	7.00	4.90	1140.65	935.30	84.29	669.34	0.08	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.10	0%
8.14	8.00	20.00	7.00	4.90	3088.64	2572.85	85.64	1810.35	0.13	25%
8.14	8.00	20.00	7.00	4.90	5087.86	4241.91	85.64	2973.73	0.15	50%
8.14	8.00	20.00	7.00	4.90	8384.01	6993.73	85.64	4891.83	0.18	75%

Table 28: The effect of changing Υ_3.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	Υ_3	Change in parameter
20.00	8.00	20.00	7.00	4.90	64284.08	375.99	12.84	56044.94	0.09	-50%
20.00	8.00	18.12	7.00	4.90	12466.40	311.66	12.84	13678.85	0.18	-25%
20.00	8.00	16.46	7.00	4.90	3322.88	263.87	12.84	4451.20	0.26	0%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.35	25%
8.03	8.00	20.00	7.00	4.90	1860.30	1581.44	87.18	1105.19	0.44	50%
8.03	8.00	20.00	7.00	4.90	1857.61	1581.44	87.18	1102.81	0.53	75%

Table 29: The effect of changing θ_1.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	θ_1	Change in parameter
5.24	8.00	20.00	7.00	4.90	1749.64	1474.88	136.25	1549.35	0.05	-75%
6.31	8.00	20.00	7.00	4.90	1761.06	1476.31	114.72	1365.33	0.10	-50%
7.29	8.00	20.00	7.00	4.90	1802.85	1502.33	98.09	1218.14	0.15	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.20	0%
8.97	8.00	20.00	7.00	4.90	1982.94	1632.89	75.03	1005.70	0.25	25%
9.67	8.00	20.00	7.00	4.90	2126.99	1742.67	67.07	930.01	0.30	50%
10.28	8.00	20.00	7.00	4.90	2313.20	1887.64	60.78	869.50	0.35	75%

Table 30: The effect of changing θ_2.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	θ_2	Change in parameter
8.23	8.00	20.00	7.00	4.90	1868.08	1544.27	43.94	1093.75	0.06	-75%
8.23	8.00	20.00	7.00	4.90	1869.94	1544.27	53.77	1093.75	0.13	-50%
8.23	8.00	20.00	7.00	4.90	1872.50	1544.27	66.85	1093.75	0.19	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.25	0%
8.14	8.00	20.00	7.00	4.90	1881.04	1559.82	109.82	1104.25	0.31	25%
8.10	8.00	20.00	7.00	4.90	1888.08	1568.27	144.09	1109.96	0.38	50%
8.03	8.00	20.00	7.00	4.90	1898.10	1580.32	192.35	1118.10	0.44	75%

Table 31: The effect of changing θ_3.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	θ_3	Change in parameter
10.18	8.00	20.00	7.00	4.90	1481.68	1221.90	61.81	554.23	0.08	-75%
9.52	8.00	20.00	7.00	4.90	1600.56	1322.06	68.65	693.40	0.15	-50%
9.13	8.00	20.00	7.00	4.90	1680.23	1385.71	73.10	844.88	0.23	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.30	0%
7.23	8.00	20.00	7.00	4.90	2094.11	1740.20	99.04	1436.34	0.38	25%
6.17	8.00	20.00	7.00	4.90	2369.53	1975.82	117.31	1906.54	0.45	50%
5.00	8.00	20.00	7.00	4.90	2722.51	2274.63	141.54	2573.21	0.53	75%

Table 32: The effect of changing h_1.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	h_1	Change in parameter
7.80	8.00	20.00	7.00	4.90	1955.85	1625.33	90.42	1148.57	0.08	-75%
7.93	8.00	20.00	7.00	4.90	1928.86	1601.21	88.63	1132.23	0.15	-50%
8.05	8.00	20.00	7.00	4.90	1902.26	1577.39	86.88	1116.12	0.23	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.30	0%
8.30	8.00	20.00	7.00	4.90	1850.26	1530.59	83.46	1084.53	0.38	25%
8.43	8.00	20.00	7.00	4.90	1824.84	1507.61	81.79	1069.06	0.45	50%
8.55	8.00	20.00	7.00	4.90	1799.81	1484.90	80.15	1053.79	0.53	75%

Table 33: The effect of changing h_2 .

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	h_2	Change in parameter
8.13	8.00	20.00	7.00	4.90	1883.35	1562.59	85.79	1106.12	0.09	-75%
8.13	8.00	20.00	7.00	4.90	1880.91	1562.59	85.79	1106.12	0.18	-50%
8.13	8.00	20.00	7.00	4.90	1878.48	1562.59	85.79	1106.12	0.26	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.35	0%
8.19	8.00	20.00	7.00	4.90	1873.65	1550.95	84.94	1098.26	0.44	25%
8.19	8.00	20.00	7.00	4.90	1871.24	1550.95	84.94	1098.26	0.53	50%
8.19	8.00	20.00	7.00	4.90	1868.82	1550.95	84.94	1098.26	0.61	75%

Table 34: The effect of changing h_3.

P_1	P_2	P_3	T	t_1	TP	Q_1	Q_2	Q_3	h_3	Change in parameter
7.74	8.00	20.00	7.00	4.90	1973.63	1636.46	91.24	1156.12	0.10	-75%
7.89	8.00	20.00	7.00	4.90	1940.58	1608.43	89.16	1137.12	0.20	-50%
8.03	8.00	20.00	7.00	4.90	1908.06	1580.89	87.14	1118.48	0.30	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.40	0%
8.32	8.00	20.00	7.00	4.90	1844.59	1527.30	83.22	1082.31	0.50	25%
8.46	8.00	20.00	7.00	4.90	1813.61	1501.23	81.33	1064.77	0.60	50%
8.61	8.00	20.00	7.00	4.90	1783.12	1475.65	79.49	1047.58	0.70	75%

Table 35: The effect of changing c_1.

P_1	P_2				T	TP	Q_1	Q_2	c_1	Change in parameter
4.28	8.00	20.00	7.00	4.90	2939.28	2480.23	158.85	1735.52	1.25	-75%
5.56	8.00	20.00	7.00	4.90	2528.68	2127.00	129.42	1491.73	2.50	-50%
6.85	8.00	20.00	7.00	4.90	2176.84	1820.76	105.20	1281.61	3.75	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	5.00	0%
9.54	8.00	20.00	7.00	4.90	1619.99	1318.95	68.44	943.13	6.25	25%

10.99	8.00	20.00	7.00	4.90	1403.57	1108.17	54.26	806.13	7.50	50%
12.64	8.00	20.00	7.00	4.90	1223.45	909.68	41.70	684.08	8.75	75%

Table 36: The effect of changing c_2.

P_1	P_2				T	TP	Q_1	Q_2	c_2	Change in parameter
7.82	8.00	20.00	7.00	4.90	1932.39	1621.63	90.14	1146.06	1.50	-75%
7.94	8.00	20.00	7.00	4.90	1913.26	1598.56	88.44	1130.44	3.00	-50%
8.06	8.00	20.00	7.00	4.90	1894.49	1575.97	86.77	1115.16	4.50	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	6.00	0%
8.29	8.00	20.00	7.00	4.90	1857.99	1532.18	83.57	1085.60	7.50	25%
8.35	9.00	20.00	7.00	4.90	1675.94	1377.43	67.84	977.97	9.00	50%
8.35	10.50	20.00	7.00	4.90	1442.46	1185.57	50.25	844.23	10.50	75%

Table 37: The effect of changing c_3.

P_1	P_2				T	TP	Q_1	Q_2	c_3	Change in parameter
5.00	8.00	20.00	7.00	4.90	3588.54	2274.63	141.54	1593.48	2.75	-75%
5.00	8.00	20.00	7.00	4.90	2962.53	2274.63	141.54	1593.48	5.50	-50%
6.19	8.00	20.00	7.00	4.90	2361.89	1971.47	116.96	1384.82	8.25	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	11.00	0%
10.16	8.00	20.00	7.00	4.90	1488.84	1224.95	62.01	881.41	13.75	25%
12.06	8.00	20.00	7.00	4.90	1175.67	974.38	45.71	722.76	16.50	50%
13.72	8.00	20.00	7.00	4.90	913.37	799.10	35.09	621.80	19.25	75%

Table 38: The effect of changing s.

P_1	P_2				T	TP	Q_1	Q_2	s	Change in parameter
5.24	8.00	20.00	7.00	4.90	1749.64	1474.88	136.25	1549.35	0.05	-75%
6.31	8.00	20.00	7.00	4.90	1761.06	1476.31	114.72	1365.33	0.10	-50%
7.29	8.00	20.00	7.00	4.90	1802.85	1502.33	98.09	1218.14	0.15	-25%

8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.20	0%
8.97	8.00	20.00	7.00	4.90	1982.94	1632.89	75.03	1005.70	0.25	25%
9.67	8.00	20.00	7.00	4.90	2126.99	1742.67	67.07	930.01	0.30	50%
10.28	8.00	20.00	7.00	4.90	2313.20	1887.64	60.78	869.50	0.35	75%

Table 39: The effect of changing ω.

P_1	P_2				T	TP	Q_1	Q_2	ω	Change in parameter
5.00	8.00	20.00	7.00	1.23	3140.99	379.39	141.54	2772.29	0.18	-75%
5.00	8.00	20.00	7.00	2.45	3730.06	864.12	141.54	3200.12	0.35	-50%
5.00	8.00	20.00	7.00	3.68	2953.08	1483.41	141.54	2573.61	0.53	-25%
8.18	8.00	20.00	7.00	4.90	1876.07	1553.84	85.15	1100.22	0.70	0%
12.91	8.00	20.00	7.00	6.13	1644.58	1271.22	39.91	281.69	0.88	25%

Note from **Table 39** that the values of w cannot be greater than 1 hence it limits the test.

Profit Graph

Figure 10 shows a summary of results from previous tables, which depicts the influence of the parameter change on the profit.

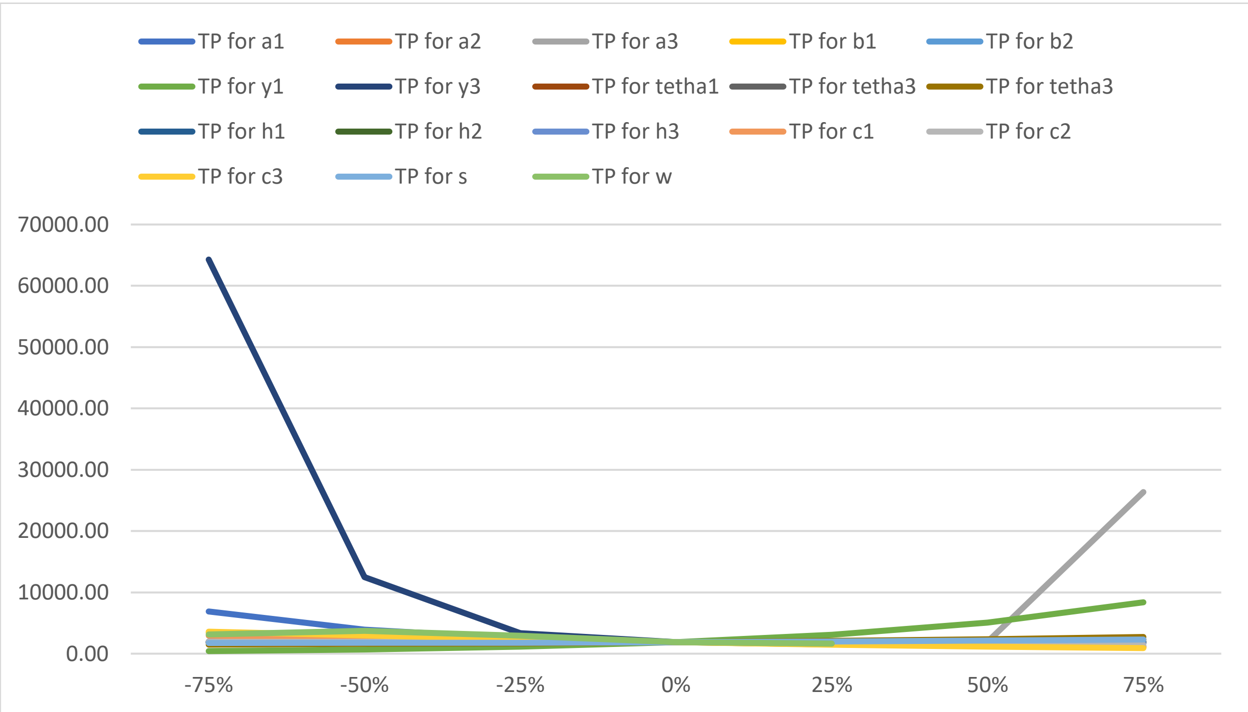

Figure 10: Changes in profit due to parameter changes.

The following can be deduced from the summary tables and graph presented:

- Table 22,23 and 24

On **Table 22**, an increase in price coefficient for product 1 a_1 results in selling price P_1 decreases while the selling prices of products 2 and 3 remain less sensitive to this change. The cycle length T remains less sensitive to the change in a_1 how decrease slightly at higher value of a_1. The unit profit decreases as seen in **Figure 10**. In **Table 23**, an increase in a_2 results in a slightly decrease of the unit profit decreases as seen **Figures 10**. In **Table 24**, when a_3 increase selling price of product 1 increase consequently the unit profit increase. At higher values of a_3 the selling for product 3 drops.

- Table 25 and 26

An increase in b_1 and b_2 result in a slight increase in the selling price for product 1. On the other hand, this results in a decrease in unit profit.

- Table 27 and 28

Conversely an increase in Υ_1 and Υ_2 result in a slightly decrease in the selling price for product 1. An increase in Υ_1 results in an increase in unit profit while and increase in Υ_2 decrease in profit.

- Table 29,30 and 31

The unit profit is sensitive to changes in deterioration rates as seen in the tables When the deterioration rate θ_1 for product 1 increase the price for product 1 increases. Conversely, an increase in θ_1 and θ_2.results in price decrease for product 1.

- Table 32, 33 and 34

As the unit holding cost for each item increases the unit profit decreases, as anticipated. Product 1 is more sensitive to this change resulting in a slight increase in its selling price.

- Table 35, 36 and 37

As expected, when the unit purchasing cost increase for each product the unit profit decreases. Again product 1 price increases in trying to counteract the increase in unit purchase cost

- Table 38

As product 1 stock dependency ratio s increases the selling price for product 1 which results in an increase in unit profit.

- Table 39

When the stock out ratio of product 1 increases the price, the price for product 1 increases. An increase in this ration result in an increase in profit at lower value and at higher values

In summary, product 1 as the main item has a demand that is dependent on numerous factors or parameters which were tested. As expected, unlike other products, product 1 is more sensitive to changes of these parameters which consequently influence the price change of product 1. Product 2 and 3 prices in most cases are less sensitive while the unit profit on the other hand is sensitive to any change in parameters as anticipated. An increase in deterioration rate θ_n has an influence on unit profit TP. Lastly, an increase in unit holding and purchasing cost leads to a decrease in profit. This means it remains critical for inventory managers to seek ways to better manage this cost as they have direct impact on profitability.

4.5 CONCLUSION

In this chapter, the model developed is an extension of the model presented in **Chapter 3** of this dissertation. In addition to deterioration and mutual complementary items, as presented in the previous model, the second model incorporates substitutability of items and stock dependent demand. The main contribution of this model on EOQ models is the consideration of all variants presented in this dissertation into a single optimisation model. The inventory model developed integrates purchasing and marketing decision making into a single model. The model uses inputs from the purchasing side to find optimal replenishment policy while on the other hand finds the pricing strategy that optimises profit. The model has been developed from the perspective of the supplier who sources and supplies medication to animal feed manufacturer. Numerical analysis has been performed to find optimal values of the selling price and cycle length that maximises profit. Sensitivity analysis has been conducted to test the effectiveness of the model. As observed the outputs of the model are stimulated by the change in parameter values.

CHAPTER 5

5 CONCLUSIONS

5.1 SUMMARY OF FINDINGS

This dissertation was aimed at developing two inventory models, which integrates purchasing and marketing decision making into a single model. The models seek to find the optimal values of the selling price and cycle length that maximises profit. The models are adapted from the models presented by Karaöz et al (2011) and Ouyang et al. (2005). As apparent in the study, the demand for the products does not necessarily deplete through sales demand only, factors such as deterioration do contribute to inventory depletion for an item. Moreover, the reviewed studies show that in today's market, demand for items is seldomly constant. Aspects such as display stock level, time, selling price, substitutability and complementarity of items have a huge influence on the overall demand of an item. This study's focus has been on developing inventory optimisation models for deteriorating items with stock that does not only deplete through demand only but also through deterioration nature of items and stock display. The demand for these items has been defined by an exponential function that is dependent on the products selling prices, complement and substitute product selling price and time.

The models have been mathematically formulated in this dissertation to find optimal values of the selling price and cycle length that maximises profit. The models have been programmed on a excel sheet using Excel solver function which is an optimisation tool on Excel. The numerical values have been chosen to define the scenario, and the results were obtained which show a positive profit. Sensitivity analysis indicated that changes in certain parameters of the models have influence on the cycle length, pricing, and the profitability of the inventory policy. Furthermore, it has been observed that an increase in either unit holding or purchasing cost result in a decrease in profit. This indicates that a firm dealing with this inventory needs to further look for ways and means to minimize holding and purchasing costs.

5.2 POSSIBLE PRACTICAL APPLICATION OF FINDINGS

The first model proposed an inventory policy from a retail perspective when two deteriorating complementary products are being offered to the customers. When items exhibit complementary relationship, retailers are likely to take advantage of the opportunity. As such, this model may be used to device a pricing strategy that will optimise profits and provide optimal replenishment policy for the items. Findings from the outputs of the model, indicate that the model can yield optimal results when optimality conditions are observed. The result of the model indicates that an increase in deterioration rate results in a decrease in profit, therefore, firms dealing with this type of inventory should take note of the deterioration characteristics of the items and plan accordingly to avoid loss in profits.

The second model developed in this dissertation establishes inventory optimisation strategy which integrates purchasing and marketing decision altogether, by using inputs from purchasing to determine optimal replenishment and pricing strategy for three deteriorating items, where two of these items are complements and one of these products is substitutable by the third product. The scenario defined by the model is in the context animal feed supply chain between medicines supplier and animal feed producer, where the medicine supplier seeks to determine the optimal inventory policy to best serve the manufacturer of animal feed in the presence of complementary and substitutability of products (Medicines). Some medicines used in animal feed are complementary, meaning that these medicines must be formulated together into a specific feed product. There are instances where one of the complement products is unavailable or the product is not in good condition to be used, in such a case, substitution with a product which has similar attribute occurs. The findings from model outputs indicates that the model can yield optimal values under the scenario described. As seen from the sensitivity test, an increase in either purchasing or holding cost results in reduced profit. This emphasises that these costs still need to be managed in order to obtain better results. The model can be applied to any industry that deals with these type of inventories as described through this dissertation.

5.3 CONTRIBUTIONS

The contribution made by this study is summarised below.

- The EOQ models developed considers mutual complementary items. Instead of one-sided complementary.
- The EOQ model developed incorporates multiple items which deteriorate and exhibit substitution and mutual complementary relationship with stock and time dependent demand all in a single model.

5.4 FUTURE RESEARCH

The two models developed in this dissertation are adapted from the models presented by Karaöz et al (2011) and Ouyang et al. (2005). The first model by these authors covered two of the main elements of this study which are complementary and substitute products. However, this model did not cater for deterioration of items. The second model by Ouyang et al. (2005) considers deteriorating items under exponentially decreasing demand. The basis of these models is used in this dissertation to formulate more robust models. The first model presented in this study only considered two deteriorating items and specifically adapts to mutually complementarity relationship. The model was further extended to into the second model by considering three items and incorporating substitution and stock dependent demand.

In both models it is assumed that shortages are not allowed. In reality, this is not always true, in numerous industries and markets there is always a likelihood that items may run short due to various reason such unpredictable demand and poor planning. In such situation where items have run out, there is a possibility that a firm may lose profit and in an extreme case the firm might even loose reliable and loyal customers, who might not satisfy with the service. In order to counteract the limitation brought by the models presented in this dissertation, the models can be further expanded to allow for shortages.

The second model considers stock-out based substitution which is frequently observed in the markets, which occurs when a customer buys an alternative item in a situation where the intended item to be bought is unavailable. This happens frequently due to the presence of common features on items. Stock-out based substitution is not the only manner or reason that the customer buys a substitute item. There are other means of substitution that this dissertation does not consider these are assortment-based substitution and price-based substitution.

Given that this dissertation only focuses on deterioration defined by a constant, future research can focus on extending the model to considering none-constant deterioration rate, such variable deterioration rate and Stochastic deterioration rate. Furthermore, it may also be interesting to extend the model to multi-echelon inventory system. As this will allow for system wide optimisation. The models may also be adapted to incorporate random demand, as this is frequently observed in various markets. Finally, the models can be improved to include none-zero lead-time as this is the better representation of the reality.

6 REFERENCES

Aliyu, I., & Sani, B. (2020). An Inventory Model for Deteriorating Items with a Generalised Exponential Increasing Demand Constant Holding Cost and Constant Deterioration Rate. *Applied Mathematical Sciences*, 14, 725 - 736.

Arcelus, F. J., & Rowcroft, J. E. (1991). Inventory Policies with Freight Discounts and Disposals. *International Journal of Operations & Production Management, 11*(4), 89-91. doi:10.1016/j.mcm.2011.04.028.

Bansal, K. K., Malik, A., & Pandey, M. (2021). Study Of Deterministic Inventory Model For Deteriorating Items With Stock- Dependent Demand Rate Under Inflation. *European Journal of Molecular & Clinical Medicine*, 8, 1263-1267.

Benton, W. C. (1991). Quantity Discount Decisions under Conditions of Multiple Items, Multiple Suppliers and Resource Limitations. *International Journal of Production Research*, 10, 1953-1961.

Benton, W. C., & Park, S. (1996). A classification of Literature on Determining the Lot Size Under Quantity Discounts. *European Journal of Operationalal research,* 92, 219-238.

Bertrand, J., & Fransoo, J. (2002). Operations management research methodologies using quantitative modeling. *International Journal of Operations & Production Management,* 22 ,241-264.

Carbaugh, Robert (2006). Contemporary Economics: An Applications Approach. Cengage Learning. p. 35. ISBN 978-0-324-31461-8.

Casadesus-Masanell, R., B. Nalebuff and D. B. Yoffie. (2007). Competing Complements. Working paper 07-44.

Chang, C.-T., Cheng, Y.-J., Tsai, T.-R., & WU, S.-J. (2010). Inventory models with stock and price dependent demand for deteriorating items based on limited shelf space. *Yugoslav Journal of Operations Research*, 20, 55-69.

Chang, H.J., and Dye, C.Y. (1999), "An EOQ model for deteriorating items with time varying demand and partial backlogging", *Journal of the Operational Research* Society, 50, 1176-1182.

Chen, M. K., & Nalebuf, B. (2007). One-Way Essential Complements.

Cheng, L. K. and J. Nahm. (2007). Product Boundary, Vertical Competition, and the Double Mark-up Problem. *RAND Journal of Economics* 38(2), 447-466.

Cournot, A. (1838). Researches into the Mathematical Principles of the Theory of Wealth. (Originally published in French, translated by Nathaniel Bacon, Macmillan, New York, 1927).

Covert, R.B., and Philip, G.S. (1973). "An EOQ model with Weibull distribution deterioration", *AIIE Transactions*, 5 323-326.

Dave, U., and Patel, L.K. (1981). "(T, Si) Policy inventory model for deteriorating items with time proportional demand", *Journal of the Operational Research Society*, 32 137-142.

Datta, T., & Pal, A. (1990). A note on an inventory model with inventory level dependent demand. *Journal of the Operational Research Society*, Vol. 41, pp.971-975.

Diaby, M., & Martel, A. (1993). Dynamic Lot Sizing for Multiechelon Distribution Systems with Purchasing and Transportation Price Discounts. *Operations Research, 41*(1), 48-59. doi:10.1007/s00170-011-3422-3.

Eastman, W. (1959). A Note on the Multi-commodity Warehouse Problem. *The Computation Laboratory of Harvard University*.

Economides, N. (1999). Quality, choice and vertical integration. *International Journal of Industrial Organization*, 17(6), 903-914.

Edalatpour, M. A., & Mirzapour Al-e-Hashem, S. M. (2019). Simultaneous pricing and inventory decisions for substitute. *Production Engineering*, 13:305–315.

Eksler, L., Aviram, R., Elalouf, A., & Kamble, A. (2019). An EOQ Model for multiple products with varying degrees of substitutability. *Economics E-Journal* , 13(1).

Farrell, J. and M. Katz. (2000). Innovation, Rent Extraction, and Integration in Systems Markets. *Journal of Industrial Economics*, 48(4), 413-432.

Ghare, P.M., and Schrader, G.H.(1963). "A model for exponentially decaying inventory system", *International Journal of Production Research*, 21, 449-460.

Goyal, S.K., and Giri, B.C. (2001). “Recent trends in modeling of deteriorating inventory“, *European Journal of Operational Research*, 134 1-16.
Hariga, M., and Benkherouf, L. (1994), “Optimal and heuristic replenishment models for deteriorating items with exponential time varying demand“, *European Journal of Operational Research*, 79 123-137.
Harris, F. W. (1913). How many parts to make at once. *Factory: The Magazine of Management*(10), 135-136.
Hemmati, M., Ghomi, S., & Sajadieh, M. (2017). Inventory of complementary products with stock dependent demand under vendor managed inventory with consignment policy. *Scientia Iranica*, 25(4), 2347-2360.
Hollier, R.H., and Mak, K.L. (1983). “Inventory replenishment policies for deteriorating items in a declining market“, *International Journal of Production Research*, 21 813-826.
Jacobs, F., Berry, W. L., Whybark, D., & Vollmann, T. E. (2011). *Manufacturing Planning and Control for Supply Chain Management*: McGraw-Hill Professional.
Jaggi, C. K., & Mittal, M. (2011). Economic order quantity model for deteriorating items with imperfect quality. *International Journal of Industrial Engineering Computations*, 2, 237-247.
John, M., & Cengiz, H. (2013). Models for Ordering Multiple Products Subject to Multiple Constraints, Quantity and Freight Discounts. *American Journal of Operations Research*(3), 521-535.
Karaöz, M., Eroğlu, A., Sütçü, A., & Sulak, H. (2011). An EOQ model with price and time dependent demand under the influence of complement and substitute product’s selling-prices. *J. Alanya Fac. Bus.—Alanya Isletme Fakültesi Dergisi*, *3*, 21-32.
Kumar, S., Malik, D. A., Sharma, A., Yadav, S., & Singh, Y. (2016). An Inventory Model with linear holding cost and Stock- Dependent Demand for Non-Instantaneous Deteriorating Items. *AIP conference proceedings* (p. 1715(1)). AIP Publishing.
Levin, R. I., McLaughlin, C. P., Lamone, R. P., & Kottas, J. F. (1972). Contemporary Policy for Managing Operating Systems. *Productions / Operations Management,McGraw-Hill*, 373.
Maity, K., & Maiti, M. (2009). Optimal inventory policies for deteriorating complementary and substitute items. *International Journal of Systems Science*, 267-276.
Maragatham, M., & Palani, R. (2017). An Inventory Model for Deteriorating Items with Lead Time price Dependent Demand and Shortages. *Advances in Computational Sciences and Technology* , 10, 1839-1847.
Mashud, A. H. (2020). An EOQ deteriorating inventory model with different types of demand and fully backlogged shortages. *International Journal of Logistics Systems and Management*, 36, 16-45.
Mokhtari, H. (2018). Economic order quantity for joint complementary and substitutable items. *Mathematics and Computers in Simulation*, 154,34-47. Retrieved from https://doi.org/10.1016/j.matcom.2018.06.004.
Ouyang, L. Y., Hsieh, T. P., & Chang, H. C. (2003). An inventory model for deteriorating items with stock-dependent demand under the conditions of inflation and time-value of money. *The enineering economist*, 48(1), 52-68.
Ouyang, L. Y., Wu, K. S., & Cheng, M. C. (2005). An inventory model for deteriorating items with exponential declining demand and partial backlogging. *Yugoslav Journal of Operations Research*, 15(2), 277-288.
Perez, F., & Torres, F. (2020). Inventory models for managing deteriorating products: a literature review. *Authorea.*
Poormoaied, S. (2021). Inventory decision in a periodic review inventory model with two complementary products. *Annals of Operations Research* . Retrieved from https://doi.org/10.1007/s10479-021-03949-w
Rajesh, K. M., & Vinod, K. M. (2020). Impact of cost of substitution and joint replenishment on inventory decisions for joint substitutable and complementary items under asymmetrical substitution. *WPOM-Working Papers on Operations Management*, 1-26. Retrieved from https://doi.org/10.4995/wpom.v11i2.1373
Rubin, P. A., & Benton, W. C. (1993). Evaluating Jointly Constrained Order Quantities with All-Units Discounts. *Naval Research logistics*, *40*(2), 255-278.

Saeed, G., & Morteza, K. B. (2018). A multi-product inventory management model in a three-level supply chain with multiple members at each level. *Uncertain Supply Chain Management*(7), 109-120.

Sarbjit S. (2015). Replenishment Policy for Items Having a Fixed Shelf Life under Permissible Delay and Variable Lead Time. *Universal Journal of Accounting and Finance* , 3(5), 171-174.

Shah, Y.K., & Jaiswal, M.C., (1977) .An order-level inventory model for a system with constant rate of deterioration, *Operations research*, 14, 174-184.

Shukla, H., Shukla, V., & Yadav, S. K. (2013). EOQ model for deteriorating items with exponential demand rate and shortages. *Uncertain Supply Chain Management*, 1, 67-76.

Taleizadeh, A. A., Masoumeh, S. B., Shib, S. S., & Biswajit, S. (2019). Pricing Decision within an Inventory Model for Complementary and Substitutable Products. *Mathematics 7*, 7- 568. Retrieved from https://doi.org/10.3390/math7070568

Taylan Y, E. O. (2010). Complementary Goods: Creating and Sharing Value.

Teng, J.-T., & Ouyang, L.-Y. (2005). An EOQ Model for Deteriorating Items with Power-Form Stock-Dependent Demand. *International Journal of Information and Management Sciences* , 16(1):1-16.

Tersine, R. J., & Barman, S. (1991). Lot Size Optimization with Quantity and Freight Rate Discounts. *Logistics and Transportation Review*, *27*(4), 319-332. doi:10.1016/j.apm.2010.02.004

Tersine, R. J., Barman, S., & Toelle, R. A. (1995). Composite Lot Sizing with Quantity and Freight Discounts. *Computers and Industrial Engineering, 28*(1), 107-122. doi:10.1016/j.cor.2006.01.016

Tripathi, R., & Mishra, S. (2014). Inventory model with inventory-dependent demand for deteriorating items in a single warehouse system. *Uncertain Supply Chain Management*, 2, 209–218.

Valdecy, P., & Helder, G. C. (2015). A literature review on lot size with quantity discounts: 1995-2013. *Journal of Modelling in Management*(3), 341-359. doi:10.1108/JM2-07-2013-0029

Wee, H.M., (1995) .“Joint pricing and replenishment policy for deteriorating inventory with declining market“, *International Journal of Production Economics*, 40 163-171.

Yanhai, L., & Jinwen, O. (2020). Optimal ordering policy for complementary components with partial backordering and emergency replenishment under spectral risk measure. *European Journal of Operational Research,Volume 284, 284*(Issue 2), 538-549. Retrieved from https://doi.org/10.1016/j.ejor.2020.01.006.

Zhang, R., & Huang, Y. (2011). A two-item deterministic EOQ model with partial backordering and substitution. *2010 International Conference on Economics, Business and Management* . Manila, Philippines: IAC S IT Press.