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Normalisers in Some Products of Finite Groups

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Abstract

A group $G = AB$ is a weakly totally permutable product of subgroups A and B if for every subgroup, U of A such that $U \leqslant A \cap B$ or $A \cap B \leqslant U$, permutes with every subgroup of B and if for every subgroup V of B such that $V \leqslant A \cap B$ or $A \cap B \leqslant V$, permutes with every subgroup of A. Let the soluble group $G = AB$ be a weakly totally permutable product of subgroups A and B. Suppose that $\mathfrak{F}$ is a saturated formation containing $\mathfrak{U}$. We show that if H is an $\mathfrak{F}$-normaliser of A and K is an $\mathfrak{F}$-normaliser of B, then HK is an $\mathfrak{F}$-normaliser of G. This generalises a corresponding result by Ballester-Bolinches, Pedraza-Aguilera and Pérez-Ramos for totally permutable products.

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1 Introduction

All groups considered in this article are finite.

Recall that a group $G = AB$ is the totally permutable product of subgroups A and B if every subgroup of A permutes with every subgroup of B. This type of factorisation was first considered by Asaad and Shaalan in [1]. They showed that if $G = AB$ is a totally permutable product of supersoluble subgroups A and B, then G is supersoluble (see Theorem 3.1 of [1]). In a seminal paper, Ballester-Bolinches and Pérez-Ramos [6] extended the result above by showing that $G \in \mathfrak{F}$, if $A, B \in \mathfrak{F}$, where $\mathfrak{F}$ is a formation containing the class of finite supersoluble groups. A theory was developed by many authors and compiled in a book by Ballester-Bolinches et al. [2].

A group $G = AB$ is a weakly totally permutable product of subgroups A and B if for every subgroup, U of A such that $U \leqslant A \cap B$ or $A \cap B \leqslant U$, permutes with every subgroup of B and if for every subgroup V of B such that $V \leqslant A \cap B$ or $A \cap B \leqslant V$, permutes with every subgroup of A.

Totally permutable products can be viewed as a generalisation of central products and weakly totally permutable products can be viewed as a generalisation of products $G = AB$ with normal subgroups A and B such that $A \cap B \leqslant Z(G)$, the centre of G.

Since the concept of weakly totally permutable products is strictly more general than that of totally permutable products (see [8], Remark), we ask which results on totally permutable products can be generalised to weakly totally permutable? In [8] and [9] some results on totally permutable products in the framework of formation theory and Fitting classes were extended to weakly totally permutable products. In this note, we shall we focus our attention on $\mathfrak{F}$-normalisers of weakly totally permutable products. We shall recall a characterisation of an $\mathfrak{F}$-normaliser of a group. Note that $G^{\mathfrak{F}}$ denotes the $\mathfrak{F}$-residual subgroup of the group G, $\Phi(G)$ denotes the Frattini subgroup of G and $F(G)$ its Fitting subgroup.

Definition Let $\mathfrak{F}$ be an arbitrary set of groups. A normal subgroup N of a group G is called an *$\mathfrak{F}$-limit normal subgroup* if $N \leqslant G^{\mathfrak{F}}$ and $N/(N \cap \Phi(G))$ is a chief factor of G.

A maximal subgroup M of G is called an *$\mathfrak{F}$-critical subgroup* if $G = MN$ for some $\mathfrak{F}$-limit normal subgroup N of G.

Theorem 1 (see [7], Chapter V, Proposition 3.8) *A subgroup H of a group G is called an $\mathfrak{F}$-normaliser of G if $H \in \mathfrak{F}$ and there is a chain*

$$G = M_0 \geqslant M_1 \geqslant \ldots \geqslant M_s = H, \; s \geqslant 0$$

of subgroups of G such that M_i is a maximal $\mathfrak{F}$-critical subgroup of M_{i-1}, $i \in \{1, 2, \ldots, s\}$.

If $\mathfrak{F}$ is a saturated formation and G is a finite group, then the existence of an $\mathfrak{F}$-normaliser in G is guaranteed and if H is an $\mathfrak{F}$-normaliser of G, then $G = HG^{\mathfrak{F}}$. All $\mathfrak{F}$-normalisers of G are conjugate in G (see [7], Chapter V, Theorem 3.2). For a proper definition an $\mathfrak{F}$-normaliser and more of its properties, see Chapter V of [7].

In general there is no connection between between $\mathfrak{F}$-normalisers of a finite group and those of a proper subgroup. For totally permutable products, Ballester-Bolinches, Pedraza-Aguilera and Pérez-Ramos, in [3], proved the following result.

Theorem 2 (see [3], Theorem C) *Let the soluble group* $G = AB$ *be the product of totally permutable subgroups* A *and* B*. Suppose that* $\mathfrak{F}$ *is a saturated formation containing* $\mathfrak{U}$*, the class of finite supersoluble groups. If* H *is an* $\mathfrak{F}$*-normaliser of* A *and* K *is an* $\mathfrak{F}$*-normaliser of* B*, then* HK *is an* $\mathfrak{F}$*-normaliser of* G*.*

The condition that $\mathfrak{F}$ contains $\mathfrak{U}$ is necessary. For if $\mathfrak{F}$ is the saturated formation of nilpotent groups and $G = S_3$ is the totally permutable product of cyclic subgroups C_2 and C_3, then G is supersoluble but not nilpotent.

The authors in [4] extended the result above to products

$$G = G_1 G_2 \dots G_n$$

of pairwise totally permutable subgroups $G_1, G_2, \dots, G_n$, where n is a positive integer. We generalize Theorem 2 by showing the result holds for products of weakly totally permutable subgroups.

Theorem 3 *Let the soluble group* $G = AB$ *be the product of weakly totally permutable subgroups* A *and* B*. Suppose that* $\mathfrak{F}$ *is a saturated formation containing* $\mathfrak{U}$*. If* H *is an* $\mathfrak{F}$*-normaliser of* A *and* K *is an* $\mathfrak{F}$*-normaliser of* B*, then* HK *is an* $\mathfrak{F}$*-normaliser of* G*.*

2 Preliminaries

The results below will be useful to prove our main theorem.

Lemma 4 (see [8], Lemma 10) *Let a group* $G = AB$ *be the weakly totally permutable product of subgroups* A *and* B*. Suppose that* $\mathfrak{F}$ *is saturated formation containing* $\mathfrak{U}$*. Then* $A^{\mathfrak{F}}$ *and* $B^{\mathfrak{F}}$ *are normal subgroups of* G*.*

Lemma 5 (see [8], Theorem 1) *Let a group* $G = AB$ *be the weakly totally permutable product of subgroups* A *and* B*. Suppose that* $\mathfrak{F}$ *is saturated formation containing* $\mathfrak{U}$*. Then* $A^{\mathfrak{F}}B^{\mathfrak{F}} = G^{\mathfrak{F}}$*.*

Lemma 6 *Let* G *be a soluble group. Let* H, K *and* N *be subgroups of* G *such that* H *is a maximal subgroup of* K *and* N *is a permutable subgroup of* G*. Then either* $NH = NK$ *or* NH *is a maximal subgroup of* NK*.*

Proof — Suppose $NH \neq NK$. We want to show that NH is a maximal subgroup of NK. Suppose that M be a subgroup of NK such that

$$NH \leqslant M \leqslant NK.$$

If $NH \neq M$, then there is a $g \in M \setminus NH$ and $g = nk$ for some $n \in N$ and $k \in K$. Then $k \in M \setminus H$. Now $\langle k, H, N\rangle \leqslant M$. But

$$\langle k, H, N\rangle = \langle k, H\rangle N = KN$$

by the maximality of H in K. Hence the result follows. □

3 $\mathfrak{F}$-normalisers

We begin this section with a lemma on $\mathfrak{F}$-normalisers of weakly totally permutable subgroups.

Lemma 7 *Let the soluble group* $G = AB$ *be the product of weakly totally permutable subgroups* A *and* B. *Suppose that* $\mathfrak{F}$ *is a saturated formation containing* $\mathfrak{U}$. *If* H *is an* $\mathfrak{F}$*-normaliser of* A *and* K *is an* $\mathfrak{F}$*-normaliser of* B, *then* $A \cap B \leqslant H$ *and* $A \cap B \leqslant K$.

Proof — Now $H \in \mathfrak{F}$ and $A \cap B \in \mathfrak{U} \subseteq \mathfrak{F}$ by Lemma 2 (iii) of [8]. Note that $H(A \cap B) \in \mathfrak{F}$ by Lemma 5 since $H(A \cap B)$ is the product of totally permutable subgroups H and $A \cap B$. We show that $H(A \cap B)$ is an $\mathfrak{F}$-normaliser of A. Since H is an $\mathfrak{F}$-normaliser, there is a chain

$$G = M_0 \geqslant M_1 \geqslant \ldots \geqslant M_s = H,\ s \geqslant 0$$

of subgroups of G such that M_i is a maximal $\mathfrak{F}$-critical subgroup of M_{i-1}, $i \in \{1, 2, \ldots, s\}$. Then

$$G = M_0(A \cap B) \geqslant M_1(A \cap B) \geqslant \ldots \geqslant M_s(A \cap B) = H(A \cap B),\ s \geqslant 0.$$

Using Lemma 6, we refine the new chain to

$$G = M_0(A \cap B) \geqslant M_1(A \cap B) \geqslant \ldots \geqslant M_r(A \cap B) = H(A \cap B)$$

with $0 \leqslant r \leqslant s$ by removing $M_j(A \cap B)$ if $M_j(A \cap B) = M_{j-1}(A \cap B)$ for some j. Then M_j is a maximal $\mathfrak{F}$-critical subgroup of M_{j-1}, $i \in \{1, 2, \ldots, r\}$. Hence $H(A \cap B)$ is an $\mathfrak{F}$-normaliser of A. Since H and $H(A \cap B)$ are conjugate (see [7], Chapter V, Theorem 3.2), our result follows.

Arguing in a similar fashion, we also have that $A \cap B \leqslant K$. □

We are ready to prove our theorem which we restate below.

Theorem 8 *Let the soluble group* $G = AB$ *be the product of weakly totally permutable subgroups* A *and* B*. Suppose that* $\mathfrak{F}$ *is a saturated formation containing* $\mathfrak{U}$*. If* H *is an* $\mathfrak{F}$*-normaliser of* A *and* K *is an* $\mathfrak{F}$*-normaliser of* B*, then* HK *is an* $\mathfrak{F}$*-normaliser of* G*.*

Proof — We prove this theorem by induction on $|G| + |A| + |B|$. If $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$, then $G \in \mathfrak{F}$ using Lemma 5. We may assume that A does not belong $\mathfrak{F}$, that is, $A^{\mathfrak{F}} \neq 1$. If $A^{\mathfrak{F}}$ is contained in $\Phi(A)$, then $A \in \mathfrak{F}$ since $\mathfrak{F}$ is a saturated formation, a contradiction. Hence $A^{\mathfrak{F}}$ is not contained in $\Phi(A)$. Since A is soluble, there is a subgroup T of A such that $F(A^{\mathfrak{F}}/(A^{\mathfrak{F}} \cap \Phi(A))) = T/(A^{\mathfrak{F}}/(A^{\mathfrak{F}} \cap \Phi(A)) \neq 1$. Note that $T \cap \Phi(A) = A^{\mathfrak{F}} \cap \Phi(A)$. Using Theorem 3.7 of [5], it follows that T is a nilpotent subnormal subgroup of G. Choose a maximal subgroup M of A such that T is not contained in M. Then

$$A = TM = A^{\mathfrak{F}}M = F(A)M.$$

Therefore M is a maximal $\mathfrak{F}$-critical subgroup of A. By Lemma 3.7 (Chapter V) of [7], an $\mathfrak{F}$-normaliser of M is also an $\mathfrak{F}$-normaliser of A. Since $\mathfrak{F}$-normalisers of A are conjugate by Theorem 3.2 (Chapter V) of [7], we may assume that $H \leqslant M$. Now,

$$G = T(MB) = F(G)(MB) = G^{\mathfrak{F}}(MB).$$

This is because $A^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$ by Lemma 4. Note that $A \cap B \leqslant H \leqslant M$ by Lemma 7. If $G = MB$, then G is the product of weakly totally permutable subgroups M and B and also

$$|G| + |M| + |B| < |G| + |A| + |B|.$$

Using induction, we have that HK is an $\mathfrak{F}$-normaliser of G and the result follows. We may assume that $MB < G$. But MB is a maximal $\mathfrak{F}$-critical subgroup of G. Note that by induction, HK is an $\mathfrak{F}$-normaliser of MB. Using Lemma 3.7 (Chapter V) of [7], we have that HK is also an $\mathfrak{F}$-normaliser of G and the result follows. □

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