AN ERGODIC BSDE RISK REPRESENTATION IN A JUMP-DIFFUSION FRAMEWORK.

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Abstract. We consider the representation of forward entropic risk measures using the theory of ergodic backward stochastic differential equations in a jump-diffusion framework. Our paper can be viewed as an extension of the work considered by Chong et al [6] in the diffusion case. We also study the behaviour of a forward entropic risk measure under jumps when a financial position is held for a longer maturity.

1. Introduction

The purpose of this paper is to study an ergodic risk representation for the forward entropic risk measure in a discontinuous setting, the jump-diffusion framework. In this framework, we investigate the behaviour of the forward entropic risk measure when the underlying stock price process is driven by an independent Brownian motion and the Poisson processes. This risk measure is in a category of maturity-independent risk measures introduced by Zariphopoulou and Žitković in [34]. The weakness of the classical coherent or dynamic risk measures is that of the fixed time horizon. It is set at the beginning of the investment period. If not this presents a challenge to determine whether the risk measure is still the same after the fixed time horizon. This was the focus of the discussion in Chong et al [6] we want to revisit and discuss it in a different framework.

Zariphopoulou and Žitković in [34] proposed maturity-independent risk measures to address how to assess risk positions when the time horizon is not fixed. They formulated the forward entropic risk measures using the forward exponential performance processes. These forward performance processes are introduced and developed by Musiela and Zariphopoulou in ([22], [23], [24], [25]) to measure investment performance across all times

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$t \in [0, \infty)$, which makes the forward entropic risk measures to be defined for all times. Recently, Liang and Zariphopoulou in [16] proposed the use of the ergodic backward stochastic differential equation (ergodic BSDE) to construct the forward performance processes.

Kobylanski introduced BSDEs with the quadratic growth and random terminal time in [15] and her work was developed by Briand and Confortola in [5]. Later, Morlais in [19] proved existence and uniqueness results for the BSDEs with quadratic growth in a jump framework (see also [20], [21] for further contributions).

Fuhrman et. al [11] introduced the notion of ergodic BSDE in and developed further by Debusshe et. al [8]. The ergodic BSDEs are an asymptotic limit of the infinite horizon BSDEs (as shown in [11] and [8]) and are represented as follows

$$dY_t = (-g(V_t, Z_t) + \lambda)dt + Z_t dW_t,$$

where $\lambda \in \mathbb{R}$ is part of the solution. Cohen and Fedyashov [7],[10] extended the ergodic BSDE to a jump-diffusion framework and is represented as follows

$$dY_t = (-g(V_t, Z_t, \Psi_t) + \lambda)dt + Z_t dW_t + \int_{\mathbb{R}\backslash\{0\}} \Psi_t \tilde{N}(dt, d\zeta),$$

where $0 \leq t \leq T < \infty$. We adapt this jump model with a different generator. In our analysis, we extend and study with a quadratic growth in the control variable. The structure of our paper is similar in some respects to that of Zariphopoulou and Žitković [34]. We further study the behaviour of a forward entropic risk measure as the terminal time of the investment period goes to infinity.

The rest of the paper is organized as follows. In Section 2, we introduce the jump-diffusion model and all the notations that will be used in the rest of the paper. Section 3, we provide the representation of the forward entropic risk measure using the classic BSDE and the ergodic BSDE in a jump model setting. Section 4 analyzes the behaviour of a forward entropic risk measure over a long-term horizon. Finally, we conclude the paper.

2. Problem Formulation

Suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is the filtered probability space satisfying the usual conditions (completeness and right-continuous) [33]. The filtration is generated by two independent processes, d-dimensional standard Brownian motion $\{W_t, t \geq 0\}$ defined on $\Omega \times [0, \infty)$ and the compensated Poisson random measure $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$ defined on $\Omega \times [0, \infty) \times \mathbb{R}\backslash\{0\}$. Here, $N(dt, d\zeta)$ counts the number of jumps that occur on or before t, and ν is a positive Lévy measure satisfying the conditions

$$\int_{|\zeta|\leq 1} |\zeta^2| \nu(d\zeta) < \infty$$

and

$$\int_{|\zeta|\geq 1} \nu(d\zeta) < \infty.$$

The last condition implies that the stock process has finite number of jumps with absolute value greater than one [32] (Section 3.4).

Throughout this paper, we consider the following spaces of random and stochastic processes:

(i) $\mathcal{L}^\infty(\mathcal{F}_t)$ is the space of $\mathcal{F}_t$-measurable, essentially bounded random variables. We denote by $\mathcal{L} := \cup_{t\geq 0}\mathcal{L}^\infty(\mathcal{F}_t)$ the space of all risk positions.

(ii) $\mathbb{S}^\infty(\mathbb{R})$ is the space of adapted processes $Y : \mathbb{R}^d \to \mathbb{R}$ with càdlàg path such that

$$ess \sup_t |Y(V_t)| < \infty.$$

(iii) $L^2(W)$ is the space of predictable processes $Z : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\mathbb{E}[\int_0^t |Z(V_u)|^2 du] < \infty.$$

(iv) $L^2_\nu(\tilde{N})$ denotes the space of predictable processes $\Upsilon : \mathbb{R}^d \times \mathbb{R}\backslash\{0\} \to \mathbb{R}^d$, satisfying

$$\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}\backslash\{0\}} |\Upsilon(V_u,\zeta)|^2 \nu(d\zeta)du\Big] < \infty.$$

Furthermore, we present the concept of bounded mean oscillating (BMO) martingale also found in [19] (Page 3) and also in [14]. A process $M := (M(t), 0 \leq t \leq T < \infty)$ belonging to $\mathcal{F}_t$-local martingale is said to be a BMO martingale if it is a square integrable càdlàg $\mathbb{R}$-valued martingales and if there exists a constant $C > 0$ such that

$$\mathbb{E}[\langle M\rangle_T - \langle M\rangle_\tau | \mathcal{F}_\tau] \leq C \quad \text{and} \quad |\Delta M_\tau| \leq C$$

for all $\mathcal{F}$-stopping times τ, with $\langle M\rangle$ denoting the quadratic variation of M. The second condition is to ensure boundedness to the jumps of M.

We recall from [34] (Definition 3.1) the definition of maturity-independent convex risk measure.

Definition 2.1. A functional $\rho : \mathcal{L} \to \mathbb{R}$ is a maturity-independent convex risk measure if it satisfies the following properties for all ξ, $\bar{\xi}$ and $\alpha \in (0, 1)$:

(i) Anti-positivity: $\rho(\xi) \leq 0$, $\forall\, \xi \leq 0$,
(ii) Convexity: $\rho(\alpha\xi + (1-\alpha)\bar{\xi}) \leq \alpha\rho(\xi) + (1-\alpha)\rho(\bar{\xi})$,
(iii) Translation invariance: $\rho(\xi - c) = \rho(\xi) + c$, $\forall\, c \in \mathbb{R}$,
(iv) Replication and maturity independent: $\forall\, t \geq 0$ and admissible investment strategies π,

$$\rho(\xi) = \rho\bigg(\xi + \int_0^t \pi_u \frac{dS_u}{S_u}\bigg).$$

As asserted in [34] (Section 2), the difference between maturity independent and standard risk measure is the choice of the domain $\mathcal{L}$ and the case that in Definition 2.1(iv) is valid for all maturities $t \geq 0$.

We consider a financial market with n risky investments, with price processes, S_t^i for $i = 1, \dots, n$, satisfying the following stochastic differential equation (SDE)

$$\frac{dS_t^i}{S_t^i} = \mu^i(V_t)dt + \sigma^i(V_t)dW_t + \int_{\mathbb{R}\setminus\{0\}} \Upsilon^i(V_t, \zeta)\tilde{N}(dt, d\zeta), \qquad S_0^i > 0. \tag{2.1}$$

The coefficients of the stock price S^i are affected by a stochastic factor, which is modelled by a d-dimensional stochastic process V, satisfying:

$$dV_t = \eta(V_t)dt + \kappa dW_t. \qquad V_0 = v_0 > 0. \tag{2.2}$$

We impose the following assumptions to the coefficients so that Equations (2.1) and (2.2) have solutions.

Assumption 1. The drift $\mu^i(v) \in \mathbb{R}$, volatility $\sigma^i(v) \in \mathbb{R}^{1\times d}$ and jump rate $\Upsilon^i(v, \zeta) > -1$ are $\mathcal{F}_t$-predictable and bounded processes for $v \in \mathbb{R}^d$, satisfying the following condition

$$\int_0^T \big(|\mu(v_t)| + \sigma^2(v_t) + \int_{\mathbb{R}\setminus\{0\}} (\Upsilon^i)^2(v_t, \zeta)\nu(\zeta)\big)dt < \infty, \qquad \text{a.s.}$$

Assumption 2. There exists a large enough constant $C_\eta > 0$, for $v_1, v_2 \in \mathbb{R}^d$ such that the drift coefficient $\eta(v) \in \mathbb{R}^d$ of the factor model satisfies:

$$(\eta(v_1) - \eta(v_2))(v_1 - v_2) \leq -C_\eta |v_1 - v_2|^2.$$

Furthermore, the volatility matrix $\kappa \in \mathbb{R}^{d\times d}$ is positive definite and normalized to $|\kappa| = 1$.

Let π_t^i be a self-financing portfolio representing the amount of wealth invested in stock i. The wealth process X solves

$$dX_t = \sum_{i=1}^n \frac{\pi_t^i dS_t^i}{S_t^i} = \pi_t\big(\mu(V_t)dt + \sigma(V_t)dW_t + \int_{\mathbb{R}\setminus\{0\}} \Upsilon(V_t, \zeta)\tilde{N}(dt, d\zeta)\big), \qquad X_0 > 0, \tag{2.3}$$

where the initial wealth is given by $X_0 = x \in \mathbb{R}$. An investment strategy $\pi_t \in \mathbb{R}^n$ is said to be admissible if it is $\mathbb{R}^n$ valued $\mathcal{F}_t$-progressively measurable satisfying $\mathbb{E}(\int_0^t |\pi_t^2| ds < \infty)$. The process X_t is a unique strong solution of Equation (2.3) using π_t, such that $X_t \geq 0$ for all $t \geq 0$, a.s. The set of all admissible strategies is denoted by $\mathcal{A}$.

We now recall from [6] (Definition 2) the notion of forward performance process.

Definition 2.2. A process $U(t, x)$, $(t, x) \in [0, \infty) \times \mathbb{R}$, is a forward performance process if:

(i) for each $x \in \mathbb{R}$, $U(t, x)$ is $\mathcal{F}_t-$progressively measurable,

(ii) for each $t \geq 0$, the mapping $x \mapsto U(t, x)$ is strictly increasing, strictly concave, continuously differentiable and satisfies the *Inada* conditions, i.e. $\lim_{x\to\infty} U'(x) = 0$ and $\lim_{x\to-\infty} U'(x) = +\infty$.

(iii) for all $\pi \in \mathcal{A}$ and $0 \le t \le s$,

$$U(t, X_t^{\pi}) \ge \mathbb{E}_{\mathbb{P}}[U(s, X_s^{\pi})|\mathcal{F}_t],$$

and there exists an optimal $\tilde{\pi} \in \mathcal{A}$ such that,

$$U(t, X_t^{\tilde{\pi}}) = \mathbb{E}_{\mathbb{P}}[U(s, X_s^{\tilde{\pi}})|\mathcal{F}_t],$$

with X^{π}, $X^{\tilde{\pi}}$ solving Equation (2.3).

We derive the associated stochastic partial differential equation (SPDE) for the performance process by applying the Itô-Ventzell formula to $U(t, x)$ for any strategy $\pi \in \mathcal{A}$ (see [26] on deriving the SPDE and [28] for the Itô-Ventzell formula for a jump process). We first assume that $U(t, x)$ admits the Lêvy decomposition

$$dU(t, x) = b(t, x)dt + a(t, x)dW_t + \int_{\mathbb{R}\setminus\{0\}} \Phi(t, x, \zeta)\tilde{N}(d^{-}t, d\zeta),$$

where the processes $b(t, x)$, $a(t, x)$ and $\Phi(t, x, \zeta)$ are $\mathcal{F}_t-$progressively measurable processes and $\tilde{N}(d^{-}t, d\zeta)$ represents a forward integral. Then we obtain

$$\begin{aligned}
& dU(t, X_t) \\
= \ & b(t, X_t)dt + a(t, X_t)dW_t + U_x(t, X_t)dX_t + \frac{1}{2}U_{xx}(t, X_t)d\langle X\rangle_t + a_x(t, X)d\langle W, X\rangle_t \\
& + \int_{\mathbb{R}\setminus\{0\}} [U(t, X_t + \pi\Upsilon(t, \zeta)) - U(t, X_t) - U_x(t, X_t)\pi\Upsilon(t, \zeta)]\nu(d\zeta)dt \\
& + \int_{\mathbb{R}\setminus\{0\}} [\Phi(t, X_t + \pi\Upsilon(t, \zeta)) - \Phi(t, X_t)]\nu(d\zeta)dt \\
& + \int_{\mathbb{R}\setminus\{0\}} \big[U(t^{-}, X_{t^-} + \pi\Upsilon(t, \zeta)) - U(t^{-}, X_{t^-}) \\
& + \Phi(t^{-}, X_{t^-} + \pi\Upsilon(t, \zeta))\big]\tilde{N}(d^{-}t, d\zeta) \\
= \ & \Bigg[b(t, X_t) + \pi\mu(V_t)U_x(t, X_t) + \pi\sigma(V_t)a_x(t, X_t) + \frac{1}{2}\pi^2\sigma^2(V_t)U_{xx}(t, X_t) \\
& + \int_{\mathbb{R}\setminus\{0\}} \Big([U(t, X_t + \pi\Upsilon(t, \zeta)) - U(t, X_t) - U_x(t, X_t)\pi\Upsilon(t, \zeta)] \\
& + [\Phi(t, X_t + \pi\Upsilon(t, \zeta)) - \Phi(t, X_t)]\nu(d\zeta)\Bigg]dt \\
& + \Big(a(t, X_t) + \pi\sigma(V_t)U_x(t, X_t)\Big)dW_t + \int_{\mathbb{R}\setminus\{0\}} \big[U(t^{-}, X_{t^-} + \pi\Upsilon(t, \zeta)) \\
& - U(t^{-}, X_{t^-}) + \Phi(t^{-}, X_{t^-} + \pi\Upsilon(t, \zeta))\big]\tilde{N}(d^{-}t, d\zeta).
\end{aligned} \tag{2.4}$$

The volatility $a(t, x)$ and the process $\Phi(t, x, \zeta)$ for $t \ge 0$ are model inputs determined by the investor's preference.

From Definition 2.2, we know that the process $U(t,X_t^\pi)$ is a super-martingale for any admissible investment strategy π, that is

$$U(t,X^\pi)\geq \mathbb{E}[U(t,x)].$$

Hence, there exists an optimal strategy $\tilde{\pi}$ when the process $U(t,X_t^\pi)$ is a true martingale. The process $U(t,X_t^\pi)$ is a true martingale when the drift term in Equation (2.4) is zero. Therefore the optimal strategy is given by

$$\begin{aligned}\tilde{\pi} &= \inf_{\pi\in\mathcal{A}}\Bigg[\pi\mu(V_t)U_x(t,X_t)+\pi\sigma(V_t)a_x(t,X_t)+\frac{1}{2}\pi^2\sigma^2(V_t)U_{xx}(t,X_t)\\ &\quad+\int_{\mathbb{R}\setminus\{0\}}\Big([U(t,X_t+\pi\Upsilon(t,\zeta))-U(t,X_t)-U_x(t,X_t)\pi\Upsilon(t,\zeta)]\\ &\quad+[\Phi(t,X_t+\pi\Upsilon(t,\zeta))-\Phi(t,X_t)]\Big)\nu(d\zeta)\Bigg].\end{aligned}$$

We consider an exponential forward performance process given by

$$U(t,x)=-e^{-\gamma x+f(t,V_t)},\quad (t,x)\in[0,\infty)\times\mathbb{R} \tag{2.5}$$

where $\gamma>0$ and a function $f:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}$. By the application of Itô's formula to $U(t,x)$ and setting the resulting drift term to zero, we see that the function f solves a semi-linear partial differential equation of the form

$$0=\frac{\partial}{\partial t}f+\eta(V_t)\nabla f+\frac{1}{2}\kappa^2\nabla^2 f+g(v,\kappa\nabla f,\Upsilon),$$

with g defined as

$$\begin{aligned}g(v,\kappa\nabla f,\Upsilon) &= \frac{1}{2}\gamma^2\sigma^2(v)\left[\pi-\frac{\mu(v)-\frac{1}{2}\sigma(v)\kappa\nabla f}{\gamma\sigma^2(v)}\right]^2+\frac{1}{2}\big(\mu(V_t)-\frac{1}{2}\sigma(v)\kappa\nabla f\big)+\frac{1}{2}\kappa^2(\nabla f)^2\\ &\quad+\int_{\mathbb{R}\setminus\{0\}}\left[e^{-\gamma\pi\Upsilon(t,\zeta)}-1+\gamma\pi\Upsilon(t,\zeta)\right]\nu(d\zeta).\end{aligned} \tag{2.6}$$

We consider the following ergodic backward stochastic differential equation

$$dY_t=(-g(V_t,Z_t,\Psi_t)+\lambda)dt+Z_t dW_t+\int_{\mathbb{R}\setminus\{0\}}\Psi(V_t,\zeta)\tilde{N}(dt,d\zeta), \tag{2.7}$$

for $0\leq t\leq T<\infty$ and a given function $g:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ and $Z_t\in L^2(W)$, $\Psi(V_t,\zeta)\in L^2_\nu(\tilde{N})$. To ensure the solution to (2.7) exists and it is unique we have to impose certain assumptions on g.

Assumption 3. There exist constants $K>0$, $\hat{K}>0$ $C_v>0$ and $C_z>0$ such that the generator g satisfy

$$|g(t,0,0,0)|\leq\hat{K}. \tag{2.8}$$

$$|g(v_1,z,\psi)-g(v_2,z,\psi)|\leq C_v(1+|z|)|v_1-v_2|, \tag{2.9}$$

and

$$|g(v,z_1,\psi)-g(v,z_2,\psi)|\leq C_z(1+|z_1|+|z_2|)|z_1-z_2| \tag{2.10}$$

for any v_1, v_2, z_1, $z_2\in\mathbb{R}$.

Furthermore, there exists $-1<K_1\leq 0$ and $K_2\geq 0$ such that

$$g(v,z,\psi_1)-g(v,z,\psi_2)\leq\int_{\mathbb{R}\backslash\{0\}}(\psi_1-\psi_2)\varphi^{v,z,\psi_1,\psi_2}(\zeta)\nu(d\zeta) \tag{2.11}$$

where $\varphi^{v,z,\psi_1,\psi_2}:\Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\backslash\{0\}\to[-1,\infty)$ is $\mathcal{P}\otimes\mathcal{B}$-measurable and satisfies $K_1(1\wedge|\zeta|)\leq\varphi(\zeta)\leq K_2(1\wedge|\zeta|)$ (see Section 2 of [31]). With $\mathcal{P}$ denoting the predictable σ-field and $\mathcal{B}$ the Borel σ-field on $\mathbb{R}$.

Theorem 2.1. *Suppose Assumption (1), (2) and (3) hold. Then, the ergodic BSDE (2.7) with generator given by*

$$\begin{aligned} g(v,z(v_t),\Psi(v_t,\zeta)) &= \frac{\gamma^2}{2}\left[\pi\sigma(v)-\frac{\mu(v_t)/\sigma(v_t)-z(v_t)}{\gamma}\right]^2+\frac{1}{2}\big(\mu(v_t)/\sigma(v_t)-z(v_t)\big)^2+\frac{1}{2}z^2(v_t) \\ &\quad+\int_{\mathbb{R}\backslash\{0\}}\left[e^{-\gamma\pi\Upsilon(v_t,\zeta)+\Psi(v_t,\zeta)}-1-\gamma\pi\Upsilon(v,\zeta)+\Psi(v,\zeta)\right]\nu(d\zeta), \end{aligned} \tag{2.12}$$

has a unique Markovian solution

$$(Y,Z,\Psi,\lambda)=(y(V_t),z(V_t),\psi(V_t),\lambda),$$

for $0\leq t\leq T<\infty$, *with*

$$|Y_t|\leq\frac{K}{\alpha},\quad |Z_t|\leq C_z:=\frac{C_v}{C_\eta-C_v}\quad\text{and}\quad|\Psi(v_t,\zeta)|\leq\frac{2K}{\alpha}. \tag{2.13}$$

Proof. For the proof, we adapted the method in [6] (see also Appendix section of [16], Page 27) to jump framework. We start by establishing that the driver g satisfies Assumptions (3). We consider truncation functions $\tilde{q}:\mathbb{R}^d\to\mathbb{R}^d$, defined as

$$q(z):=\frac{\min(|z|,C_z)}{|z|}z1_{\{z\neq 0\}},\quad\text{and}\quad\tilde{q}(\psi):=1_{|\psi|\leq 1}.$$

and define a truncated ergodic BSDE

$$dY_t=(-g(V_t,q(Z_t),\tilde{q}(\Psi_t))+\lambda)dt+Z_tdW_t+\int_{\mathbb{R}\backslash\{0\}}\Psi(V_t,\zeta)\tilde{N}(dt,d\zeta), \tag{2.14}$$

for $t\geq 0$. We verify that the generator $g(v,q(z),\tilde{q}(\psi))$ satisfies Assumption (3), i.e.

$$|g(v_1,q(z),\tilde{q}(\psi))-g(v_2,q(z),\tilde{q}(\psi))|\leq C_v(1+C_z)|v_1-v_2|, \tag{2.15}$$

$$|g(v,q(z_1),\tilde{q}(\psi))-g(v,q(z_2),\tilde{q}(\psi))|\leq C_z(1+2C_z)|z_1-z_2| \tag{2.16}$$

and

$$|g(v,q(z),\tilde{q}(\psi_1))-g(v,q(z),\tilde{q}(\psi_2))|\leq\int_{\mathbb{R}\backslash\{0\}}(\psi_1-\psi_2)\varphi^{v,z,\psi_1,\psi_2}(\zeta)\nu(d\zeta). \tag{2.17}$$

We now, have to prove that there exists a Markovian solution $(Y_t, Z_t, \Psi_t, \lambda)$ to the truncated ergodic BSDE (2.14) that satisfies $|Z_t| \leq C_z$ and $|\Psi(v_t, \zeta)| \leq \frac{2K}{\alpha}$ for $t \geq 0$, then $q(Z_t) = Z_t$ and $\tilde{q}(\Psi_t) = \Psi_t$. As a result, this solution $(Y_t, Z_t, \Psi_t, \lambda)$, will also solve the ergodic BSDE (2.7). For this part of the proof, we consider a strictly monotonic BSDE with a constant of monotonicity $\alpha > 0$, on a finite horizon $[0, n]$, i.e.

$$\begin{aligned} Y_t^{v,\alpha,n} &= \int_t^n (g(V_u, q(Z_u^{v,\alpha,n}), \tilde{q}(\Psi_u^{v,\alpha,n})) - \alpha Y_u^{v,\alpha,n})du + \int_t^n Z_u^{v,\alpha,n} dW_u \\ &\quad + \int_t^n \int_{\mathbb{R}\setminus\{0\}} \Psi^{v,\alpha,n}(V_u, \zeta)\tilde{N}(du, d\zeta). \end{aligned} \tag{2.18}$$

We deduce from Cohen and Fedyashov [7], Theorem 8, (see also [4] for the diffusion case), that BSDE (2.18) has a unique solution $(Y_t^{v,\alpha,n}, Z_u^{v,\alpha,n}, \Psi_u^{v,\alpha,n})$ satisfying $|Y_t| \leq \frac{K}{\alpha}$ with $Z_u^{v,\alpha,n} \in L^2(W)$ and $\Psi_u^{v,\alpha,n} \in L^2_\nu(\tilde{N})$. Moreover, we conclude that $(Y_t^{v,\alpha,n}, Z_u^{v,\alpha,n}, \Psi_u^{v,\alpha,n})$, is a unique adapted square integrable solution to the BSDE (2.18) for $t \geq 0$. Hence, there exists an adapted square integrable limiting processes $(Y_t^{v,\alpha}, Z_u^{v,\alpha}, \Psi_u^{v,\alpha})$ such that

$$\lim_{n\to\infty} (Y_t^{v,\alpha,n}, Z_u^{v,\alpha,n}, \Psi_u^{v,\alpha,n}) = (Y_t^{v,\alpha}, Z_u^{v,\alpha}, \Psi_u^{v,\alpha}),$$

with $|Y_t| \leq \frac{K}{\alpha}$. Furthermore, the solution is Markovian, that is, there exist functions $y^\alpha(\cdot), z^\alpha(\cdot)$ and $\psi^\alpha(\cdot)$ such that

$$(Y_t^{v,\alpha}, Z_t^{v,\alpha}, \Psi_t^{v,\alpha}) = (y^\alpha(V_t), z^\alpha(V_t), \psi^\alpha(V_t)),$$

is a solution to the infinite horizon BSDE

$$dY_t^{v,\alpha} = (-g(V_t^v, q(Z_t^{v,\alpha}), \tilde{q}(\Psi_t^{v,\alpha})) + \alpha Y_t^{v,\alpha}) + Z_t^{v,\alpha} dW_t + \int_{\mathbb{R}\setminus\{0\}} \Psi_t^{v,\alpha} \tilde{N}(dt, d\zeta). \tag{2.19}$$

The next part of the proof is to demonstrate that the Lipschitz continuity property

$$|y^\alpha(V_t^{v_1}) - y^\alpha(V_t^{v_2})| \leq C_z |V_t^{v_1} - V_t^{v_2}|,$$

for all $v_1, v_2 \in \mathbb{R}^d$ with the Lipschitz constant C_z. Let $\delta Y_t = Y_t^{\alpha,v_1} - Y_t^{\alpha,v_2}$, $\delta Z_t = Z_t^{\alpha,v_1} - Z_t^{\alpha,v_2}$ and $\delta\Psi_t = \Psi_t^{\alpha,v_1} - \Psi_t^{\alpha,v_2}$, for $t \geq 0$. Subsequently

$$\begin{aligned} d\delta Y_t &= -(g(V_t^{v_1}, q(Z_t^{\alpha,v_1}), \tilde{q}\Psi(V_t^{\alpha,v_1})) - g(V_t^{v_2}, q(Z_t^{\alpha,v_2}), \tilde{q}\Psi(V_t^{\alpha,v_2})))dt + \alpha\delta Y_t dt + \delta Z_t dW_t \\ &\quad + \int_{\mathbb{R}\setminus\{0\}} \delta\Psi_t \tilde{N}(dt, d\zeta) \\ &= -(g(V_t^{v_1}, q(Z_t^{\alpha,v_1}), \tilde{q}\Psi(V_t^{\alpha,v_1})) - g(V_t^{v_2}, q(Z_t^{\alpha,v_2}), \tilde{q}\Psi(V_t^{\alpha,v_2})))dt + \alpha\delta Y_t dt \\ &\quad + \delta Z_t(dW_t - \beta_t dt) + \int_{\mathbb{R}\setminus\{0\}} \delta\Psi_t(N(dt, d\zeta) - \varphi^{v,z,\psi_1,\psi_2}\nu(d\zeta)dt), \end{aligned} \tag{2.20}$$

where

$$\beta_t = \frac{g(V_t^{v_1}, q(Z_t^{\alpha,v_1}), \tilde{q}\Psi(V_t^{\alpha,v_1})) - g(V_t^{v_2}, q(Z_t^{\alpha,v_2}), \tilde{q}\Psi(V_t^{\alpha,v_2}))}{|\delta Z_t|^2} \delta Z_t \mathbf{1}_{\delta \mathbf{Z_t} \neq \mathbf{0}}$$

By Inequality (2.15), β is bounded. From the Girsanov's theorem we define $W_t^\beta = W_t - \int_0^t \beta_u du$ and $\tilde{N}^\varphi(dt,d\zeta) = N(dt,d\zeta) - \int_0^t \varphi^{v,z,\psi_1,\psi_2}\nu(d\zeta)du$, for $0 \le t \le T$, where $\varphi^{v,z,\psi_1,\psi_2}$ is defined in Assumption (2.11). Therefore taking conditional expectation with respect to $\mathbb{Q}$ on $\mathcal{F}_t$ for $0 \le t < T < \infty$, we get

$$\delta Y_t = e^{-\alpha(T-t)}\mathbb{E}_{\mathbb{Q}}[\delta Y_T|\mathcal{F}_t] + \mathbb{E}_{\mathbb{Q}}\big[\int_t^T e^{-\alpha(u-t)}\delta g_u du|\mathcal{F}_u\big].$$

From condition (2.13), we note that the first expectation is bounded by $2K/\alpha$, and therefore will go to zero as $T \to \infty$. We deduce from (2.15) that the second expectation is bounded by

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\big[\int_t^T e^{-\alpha(u-t)}\delta g_u du|\mathcal{F}_u\big] &\le C_v(1+C_z)\mathbb{E}_{\mathbb{Q}}\big[\int_t^T e^{-\alpha(u-t)}|V_u^{v_1} - V_u^{v_2}|du|\mathcal{F}_u\big] \\ &\le C_v(1+C_z)\frac{e^{\alpha t}(e^{-(\alpha+C_\eta)t} - e^{-(\alpha+C_\eta)T})}{\alpha + C_\eta}|v_1 - v_2|. \end{aligned} \tag{2.21}$$

The last inequality is based on the Grownwall Inequality. Hence, as $T \to \infty$ yields

$$|y^\alpha(V_t^{v_1}) - y^\alpha(V_t^{v_2})| \le C_z|V_t^{v_1} - V_t^{v_2}|. \tag{2.22}$$

To obtain the third inequality in Condition (2.15), we consider a stochastic factor with a jump term [1], this yields

$$dV_t = \eta(V_t)dt + \kappa dW_t + \int_{\mathbb{R}\setminus\{0\}} \zeta\tilde{N}(dt,d\zeta)\,,$$

where the coefficients satisfy Assumptions (1) and (2). Suppose that $y^\alpha(\cdot) \in C^2(\mathbb{R}^d)$. By Itô's formula to $y^\alpha(V_t^v)$ we get

$$\begin{aligned} dy^\alpha(V_t^v) &= \nabla y^\alpha(V_t^v)\eta(V)dt + \nabla y^\alpha(V_t^v)\kappa dW_t + \frac{1}{2}\nabla^2 y^\alpha(V_t^v)\kappa^2 dt \\ &\quad \int_{\mathbb{R}\setminus\{0\}}(y^\alpha(V_t^v+\zeta) - y^\alpha(V_t^v) - \nabla y^\alpha(V_t^v)\zeta)\nu(d\zeta)dt \\ &\quad + \int_{\mathbb{R}\setminus\{0\}}(y^\alpha(V_t^v+\zeta) - y^\alpha(V_t^v))\tilde{N}(dt,d\zeta). \end{aligned} \tag{2.23}$$

Comparing terms in the infinite horizon BSDE (2.19) and Equation (2.23), we deduce that

$$Z_t^{\alpha,v} = \nabla y^\alpha(V_t^v)\kappa, \tag{2.24}$$

[1]Note that for this work we consider a stochastic factor in the diffusion case throughout the paper. If we include a jump term in the stochastic factor then our generator will be dependent on the Y variable. Hence, the stochastic factor with jumps will not be ideal for risk representation using BSDE, because the translation invariance property will not hold.

$$\begin{aligned}\alpha Y_t^{v,\alpha} &= \nabla y^\alpha(V_t^v)\eta(V) + \frac{1}{2}\nabla^2 y^\alpha(V_t^v)\kappa^2 + \int_{\mathbb{R}\setminus\{0\}} (y^\alpha(V_t^v+\zeta) - y^\alpha(V_t^v) - \nabla y^\alpha(V_t^v)\zeta)\nu(d\zeta) \\ &\quad + g(V_t^v, q(Z_t^{v,\alpha}), \tilde{q}(\Psi_t^{v,\alpha})) \qquad (2.25)\end{aligned}$$

and

$$\Psi(V_t,\zeta) = y^\alpha(V_t^v+\zeta) - y^\alpha(V_t^v), \qquad (2.26)$$

for $v \in \mathbb{R}^d$. Equation (2.25) is a Partial Integro-Differential Equation (PIDE) with a unique bounded solution, $y^\alpha(\cdot) \in C^2(\mathbb{R}^d)$. We conclude that $|y^\alpha(v)| \le \frac{K}{\alpha}$. Furthermore, using Assumption (2) and Equation (2.24) and from condition (2.22), we conclude that for $t \le 0$, $|Z_t^{\alpha,v}| \le C_z$. From Equation (2.26), we have that $|\Psi(V_t,\zeta)| \le \frac{2K}{\alpha}$.

To show that λ is a constant, the proof follows similarly as in Lian and Zariphopoulou [16] (Appendix Section, Page 30). ☐

In the following theorem, we connect the solution of the ergodic BSDE with jumps (2.7) to the exponential forward performance process (2.5). To do this, we adopt the procedure by Liang and Zariphopoulou [16] (in Theorem 11), where they made the same connection under the diffusion case.

Theorem 2.2. *Suppose that Assumptions 1 and 2 hold, and let $(Y_t, Z_t, \Psi_t, \lambda)$, $t \ge 0$ be a unique Markovian solution to Equation (2.7). Then,*

(i) *the process $U(t,x)$, $(t,x) \in [0,\infty) \times \mathbb{R}$, is an exponential forward performance process defined as*

$$U(t,x) = -e^{-\gamma x + Y_t - \lambda t}, \qquad (2.27)$$

with volatility

$$a(t,x) = -e^{-\gamma x + Y_t - \lambda t} Z_t$$

and jump rate

$$\Phi(t,x,\zeta) = -e^{-\gamma x + Y_t - \lambda t}(e^{-\Psi} - 1).$$

(ii) *The optimal investment strategy is given by*

$$\begin{aligned}\tilde{\pi} &= \inf_{\pi\in\mathcal{A}} \Bigg(\frac{\gamma^2}{2}\left[\pi\sigma(v) - \frac{\mu(v_t)/\sigma(v_t) - z(v_t)}{\gamma}\right]^2 + \frac{1}{2}\big(\mu(v_t)/\sigma(v_t) - z(v_t)\big)^2 + \frac{1}{2}z^2(v_t) \\ &\quad + \int_{\mathbb{R}\setminus\{0\}} \left[e^{-\gamma\pi\Upsilon(v_t,\zeta)+\Psi(v_t,\zeta)} - 1 - \gamma\pi\Upsilon(v,\zeta) + \Psi(v,\zeta)\right]\nu(d\zeta)\Bigg). \qquad (2.28)\end{aligned}$$

Proof. We start by first showing that $U(t,x)$ satisfies the super-martingale property for any admissible investment strategy $\pi \in \mathcal{A}$ for all $0 \le t \le s$, that is

$$\mathbb{E}_{\mathbb{P}}[-e^{-\gamma x + Y_s - \lambda s}|\mathcal{F}_t] \le -e^{-\gamma x + Y_t - \lambda t},$$

and for an optimal investment strategy $\tilde{\pi}$, $U(t,x)$ is a martingale, that is,

$$\mathbb{E}_{\mathbb{P}}[-e^{-\gamma X^{\tilde{\pi}} + Y_s - \lambda s}|\mathcal{F}_t] = -e^{-\gamma X^{\tilde{\pi}} + Y_t - \lambda t}.$$

Based on the wealth process (2.3), $e^{-\gamma X}$ can be written as

$$e^{-\gamma X_s} \;=\; e^{-\gamma X_u}\exp\bigg\{-\int_t^s \gamma\pi\mu(V_u)du - \int_t^s \gamma\pi\sigma(V_u)dW_u - \int_u^s\int_{\mathbb{R}\setminus\{0\}} \gamma\pi\Upsilon(u,\zeta)\tilde{N}(du,d\zeta)\bigg\}. \tag{2.29}$$

On the other hand, the ergodic BSDE (2.7) is given by

$$Y_s - \lambda s = Y_t - \lambda t - \int_t^s g(V_u, Z_u, \Psi_u)du + \int_t^s Z_u dW_u + \int_t^s\int_{\mathbb{R}\setminus\{0\}} \Psi(u,\zeta)\tilde{N}(du,d\zeta).$$

Combining the above expressions yields

$$\begin{aligned} e^{-\gamma X_s + Y_s - \lambda s} \;=\; & e^{-\gamma X_t + Y_t - \lambda t}\exp\bigg\{-\int_t^s \big(\gamma\mu(V_u)\pi + g(V_u, Z_u, \Psi_u)\big)du \\ & - \int_t^s (\gamma\pi\sigma(V_u) - Z_u)dW_u \\ & - \int_t^s\int_{\mathbb{R}\setminus\{0\}} (\gamma\pi\Upsilon(v,\zeta) - \Psi(v,\zeta))\tilde{N}(du,d\zeta)\bigg\}. \end{aligned} \tag{2.30}$$

Then we take expectation under the probability measure $\mathbb{P}$, given $\mathcal{F}_t$, i.e.,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[e^{-\gamma X_s + Y_s - \lambda s} \mid \mathcal{F}_t] \;=\; & e^{-\gamma X_t + Y_t - \lambda t}\mathbb{E}_{\mathbb{P}}\bigg[\exp\bigg\{-\int_t^s \big(\gamma\mu(V_u)\pi + g(V_u, Z_u, \Psi_u)\big)du \\ & - \int_t^s (\gamma\pi\sigma(V_u) - Z_u)dW_u \\ & - \int_t^s\int_{\mathbb{R}\setminus\{0\}} (\gamma\pi\Upsilon(v,\zeta) - \Psi(v,\zeta))\tilde{N}(du,d\zeta)\bigg\}\bigg|\mathcal{F}_t\bigg]. \end{aligned} \tag{2.31}$$

We define a new probability measure $\mathbb{Q}$, for $s \geq 0$ and $\pi \in \mathcal{A}$ using the process $\widetilde{M}_u$, $u \in [0, s]$ defined as the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, therefore

$$\widetilde{M}_u = \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \mathcal{E}(M)_u,$$

where

$$M_u = \exp\bigg\{-\int_t^s (\gamma\pi\sigma(V_u) - Z_u)dW_u - \int_t^s\int_{\mathbb{R}\setminus\{0\}} (\gamma\pi\Upsilon(v,\zeta) - \Psi(v,\zeta))\tilde{N}(du,d\zeta)\bigg\}.$$

Since the processes Z_u, π_u and Ψ_u belong to $BMO(\mathbb{P})$, the process M_u is a BMO-martingale, and consequently the stochastic exponential $\mathcal{E}(M)_u$ is a true martingale (see Lemma 2 in [19]). Hence

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}}\bigg[\exp\bigg(\int_t^s \big(g^\pi(V_u, Z_u, \Psi_u) - g(V_u, Z_u, \Psi_u)\big)du\bigg)\frac{\widetilde{M}_s}{\widetilde{M}_t}\bigg|\mathcal{F}_t\bigg] \\ & = \mathbb{E}_{\mathbb{Q}}\bigg[\exp\bigg(\int_t^s \big(g^\pi(V_u, Z_u, \Psi_u) - g(V_u, Z_u, \Psi_u)\big)du\bigg)\bigg|\mathcal{F}_t\bigg], \end{aligned} \tag{2.32}$$

with

$$g^{\pi}(v, z(v_t), \psi(v_t,\zeta)) := \frac{\gamma^2}{2}\left[\pi\sigma(v) - \frac{\mu(v_t)/\sigma(v_t) - z(v_t)}{\gamma}\right]^2 + \frac{1}{2}\big(\mu(v_t)/\sigma(v_t) - z(v_t)\big)^2 + \frac{1}{2}z^2(v_t) + \int_{\mathbb{R}\setminus\{0\}} \left[e^{-\gamma\pi\Upsilon(v_t,\zeta)+\psi(v_t,\zeta)} - 1 - \gamma\pi\Upsilon(v,\zeta) + \psi(v,\zeta)\right]\nu(d\zeta) \quad (2.33)$$

Since $g^{\pi}(v, z(V_t), \psi(v,\zeta)) \le g(v, z(V_t), \psi(v,\zeta))$, we can conclude that

$$\mathbb{E}_{\mathbb{P}}[-e^{-\gamma X^{\pi}+Y_s-\lambda s}|\mathcal{F}_t] \le -e^{-\gamma X+Y_t-\lambda t}.$$

Further, for $\pi = \tilde{\pi}$ defined in (2.28), we have $g^{\tilde{\pi}}(v, z(V_t), \psi(v_t)) = g(v, z(V_t), \psi(v_t))$ and hence

$$\mathbb{E}_{\mathbb{P}}[-e^{-\gamma X^{\tilde{\pi}}+Y_s-\lambda s}|\mathcal{F}_t] = -e^{-\gamma X^{\tilde{\pi}}+Y_t-\lambda t}.$$

To show the second part of the theorem, we apply Itô's formula to Equation (2.27) that yields,

$$dU(t,x) = (\cdots)dt + U(Z_t - \gamma\pi\sigma(V_t))dW_t + U\int_{\mathbb{R}\setminus\{0\}}(e^{-\gamma\pi\Upsilon(v,\zeta)+\Psi(v,\zeta)} - 1)\tilde{N}(dt,d\zeta).$$

We then, compare the above equation to Equation (2.4) and obtain the following

$$a(t,x) = -e^{-\gamma x+Y_t-\lambda t}Z_t,$$

and

$$\Phi(t,x,\zeta) = -e^{-\gamma x+Y_t-\lambda t}(e^{-\Psi} - 1).$$

It is not difficult to see that the infimum function in Equation (2.28) is convex with respect to π that is the second derivative respect to π of the infimum function is positive. Therefore the minimum in Equation (2.28) exists. □

3. Forward entropic risk measure and ergodic BSDE with jumps

In this section, we recall the definition of forward entropic risk measure. We then provide the representation of a forward entropic risk measure as the solution of a BSDE and ergodic BSDE.

Definition 3.1. Consider the forward exponential performance process
$U(x,t) = -e^{-\gamma x+Y_t-\lambda t}$, with $(t,x) \in [0,\infty)\times\mathbb{R}$. Consider a risk position $\xi_T \in \mathcal{L}_T^{\infty}$, where $T > 0$ is arbitrary and the risk position is entered into at the initial time $t = 0$. Then, the forward entropic risk measure $\rho_t(\xi_T, T), t \in [0,T]$, is the unique $\mathcal{F}_t$-measurable random variable that satisfies the indifference condition

$$ess \sup_{\pi\in\mathcal{A}_{[t,T]}} \mathbb{E}_{\mathbb{P}}\left[U(X_u^{\pi} + \rho_u(\xi_T;T) + \xi_T, T)\middle|\mathcal{F}_t\right] = \sup_{\pi\in\mathcal{A}_{[t,T]}} \mathbb{E}_{\mathbb{P}}\left[U(X_u^{\pi}, T)\middle|\mathcal{F}_t\right] \quad (3.1)$$

for all $(t,x) \in [0,T]\times\mathbb{R}$.

If we let $\xi \in \mathcal{L}$ and consider $T_\xi := \inf\{T \geq 0 : \xi \in \mathcal{F}_T\}$, then the forward entropic risk measure of ξ is defined, for $t \in [0, T_\xi]$, as

$$\rho_t(\xi) := \rho_t(\xi; T_\xi).$$

Therefore, for $\xi_T \in \mathcal{L}^\infty(\mathcal{F}_T)$, we have $\rho_t(\xi) := \rho_t(\xi_T; T)$.

The next theorem gives a representation of the forward entropic risk measure as a solution of an associated BSDE, with a generator that depends on a solution of the ergodic BSDE.

Theorem 3.1. *Let $\xi_T \in \mathcal{L}^\infty(\mathcal{F}_T)$ be a risk position with an arbitrary maturity $T > 0$. Supposes that Assumptions 1, 2 and 3 hold, and the processes Z and Ψ in the ergodic BSDE (2.7) are uniformly bounded. Consider, the BSDE*

$$Y_t^{T,\xi} = -\xi_T + \int_t^T G(V_u, Z_u, Z_u^{T,\xi}, \Psi_u, \Psi_u^{T,\xi})du - \int_t^T Z_u^{T,\xi} dW_u - \int_t^T \int_{\mathbb{R}\setminus\{0\}} \Psi_u^{T,\xi} \tilde{N}(du, d\zeta), \tag{3.2}$$

where the generator $G(v, z, \tilde{z}, \psi, \tilde{\psi}) = \frac{1}{\gamma}\big(g(v, z+\gamma\tilde{z}, \psi+\gamma\tilde{\psi}) - g(v, z, \psi)\big)$, with $g(\cdot,\cdot,\cdot)$ given by (2.33). Then the following statements hold:

(i) *The BSDE (3.2) has a unique solution $(Y_t^{T,\xi}, Z_t^{T,\xi}, \Psi_t^{T,\xi}) \in \mathbb{S}^\infty(\mathbb{R}) \times L^2(W) \times L^2_\nu(\tilde{N})$, for $t \in [0, T]$.*

(ii) *The forward entropic risk measure of a position in ξ_T is given by*

$$\rho_t(\xi_T) = Y_t^{T,\xi},$$

for $t \in [0, T]$.

Proof. Since the associated parameters are bounded and Lipschitz continuous (Assumption (1) and (2)), and the generator g in (2.33) satisfies Assumption (3). These assumptions imply that g is Lipschitz continuous in z and v, a.s.. Therefore, we know from Morlais [19] (Section 3.2, Theorem 1 and 2), (see also Royer [31] and Guambe and Kufakunesu [12]) that there exists a unique solution to the BSDE (3.2) with a generator given by g in (2.33) and the risk position $\xi_T \in \mathcal{L}^\infty(\mathcal{F}_t)$.

(i) For $t \in [0, T]$, the generator $G(v, Z, \tilde{z}, \Psi, \tilde{\psi})$ is Lipschitz continuous in z and ψ, that is,

$$|G(v, Z_t, \tilde{z}_1, \Psi_t, \tilde{\psi}) - G(v, Z_t, \tilde{z}_2, \Psi_t, \tilde{\psi})| \leq C_z(1 + 2Z_t + \gamma|\tilde{z}_1| + \gamma|\tilde{z}_2|)|\tilde{z}_1 - \bar{z}_2|$$

and

$$|G(v, Z_t, \tilde{z}, \Psi_t, \tilde{\psi}_1) - G(v, Z_t, \tilde{z}, \Psi_t, \tilde{\psi}_2)| \leq \int_{\mathbb{R}\setminus\{0\}} |\tilde{\psi}_1 - \tilde{\psi}_2| \varphi^{v,z,\psi_1,\psi_2} \nu(d\zeta)$$

where Z_t and Ψ_t are uniformly bounded in $L^2(W) \times L^2_\nu(\tilde{N})$. Considering that G is a linear combination of g, we deduce that G has the same form as g in (2.33). Therefore, using the fact that $\xi_T \in \mathcal{L}^\infty(\mathcal{F}_t)$, we conclude (following Morlais [19], Royer [31] and Guambe and Kufakunesu [12]) that Equation (3.2) has a unique solution for $t \in [0,T]$.

(ii) We consider the forward performance process in (2.27) and that $\rho_t(\xi_T) \in \mathcal{F}_t$, $t \in [0,T]$. Then we have

$$\begin{aligned}
& ess \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}_{\mathbb{P}}\Big[U(X_u^\pi + \rho_u(\xi_T;T) + \xi_T, T)\Big|\mathcal{F}_t\Big] \\
= \; & e^{-\gamma\rho_t(\xi_T)} ess \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}_{\mathbb{P}}\Big[-\exp\Big\{-\gamma\Big(x + \int_t^T \pi_u\mu(V_u)dt + \int_t^T \pi_u\sigma(V_u)dW_u \\
& + \int_t^T \int_{\mathbb{R}\setminus\{0\}} \pi_u \Upsilon_u \tilde{N}(du,d\zeta)\Big) + Y_T - \lambda T - \gamma\xi_T\Big\}\Big|\mathcal{F}_t\Big].
\end{aligned} \tag{3.3}$$

In order to prove the second part of the theorem, we define for $s \in [t,T]$, the process

$$\begin{aligned}
P_s^\pi \; := \; & -\exp\Big\{-\gamma\Big(x + \int_t^s \pi_s\mu(V_s)dt + \int_t^s \pi_s\sigma(V_s)dW_s + \int_t^s \int_{\mathbb{R}\setminus\{0\}} \pi_s\Upsilon_s\tilde{N}(ds,d\zeta)\Big) \\
& + Y_s - \lambda s + \gamma Y_s^T\Big\}.
\end{aligned} \tag{3.4}$$

As in [6] (Proof of Theorem 6, Page 12), we will show that the process P_s^π is a super-martingale for all $\pi \in \mathcal{A}_{[t,T]}$ and that there exists $\tilde{\pi} \in \mathcal{A}_{[t,T]}$ such that $P_s^{\tilde{\pi}}$ is a martingale.

For $0 \le t \le r \le s \le T$, the exponent of P_s^π satisfies

$$\begin{aligned}
& -\gamma\Big(x + \int_t^s \pi_u\mu(V_u)du + \int_t^s \pi\sigma(V_u)dW_u + \int_t^s \int_{\mathbb{R}\setminus\{0\}} \pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big) + Y_s - \lambda s + \gamma Y_s^T \\
= \; & -\gamma\Big(x + \int_t^r \pi_u\mu(V_u)du + \int_t^r \pi_u\sigma(V_u)dW_u + \int_t^r \int_{\mathbb{R}\setminus\{0\}} \pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big) + Y_r - \lambda r + \gamma Y_r^T \\
& -\gamma\Big(x + \int_r^s \pi_u\mu(V_u)du + \int_r^s \pi_u\sigma(V_u)dW_u + \int_r^s \int_{\mathbb{R}\setminus\{0\}} \pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big) \\
& + (Y_s - Y_r) - (\lambda s - \lambda r) + \gamma(Y_s^T - Y_r^T).
\end{aligned}$$

Furthermore, from the ergodic BSDE (2.7) and BSDE (3.2), we have that

$$(Y_s - Y_r) - \lambda(s-r) = -\int_r^s g(V_u, Z_u, \Psi_u)du + \int_r^s Z_u dW_u + \int_r^s \int_{\mathbb{R}\setminus\{0\}} \Psi_u \tilde{N}(du,d\zeta)$$

and

$$Y_s^T - Y_r^T \; = \; -\frac{1}{\gamma}\int_r^s \big(g(V_u, Z_u + \gamma Z_u^T, \Psi_u + \gamma\Psi_u^T) - g(V_u, Z_u, \Psi_u)\big)du + \int_r^s Z_u^T dW_u$$

$$+\int_r^s\int_{\mathbb{R}\setminus\{0\}}\Psi_u^T\tilde{N}(du,d\zeta).$$

Combining the above three equations and applying the conditional expectation, yields

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[-\exp\Big\{-\gamma\Big(x+\int_t^s\pi_u\mu(V_u)du+\int_t^s\pi_u\sigma(V_u)dW_u+\int_t^s\int_{\mathbb{R}\setminus\{0\}}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big)\\
&+Y_s-\lambda s+\gamma Y_s^T\Big\}\Big|\mathcal{F}_r\Big]\\
=\;&-\exp\Big\{-\gamma\Big(x+\int_t^r\pi_u\mu(V_u)du+\int_t^r\pi_u\sigma(V_u)dW_u+\int_t^r\int_{\mathbb{R}\setminus\{0\}}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big)\\
&+Y_r-\lambda r+\gamma Y_r^T\Big\}\times\mathbb{E}_{\mathbb{P}}\Big[\exp\Big\{\int_r^s\big(-\gamma\pi_u\mu(V_u)-g(V_u,Z_u+\gamma Z_u^T,\Psi_u+\gamma\Psi_u^T)\big)du\\
&+\int_r^s\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)dW_u\\
&+\int_r^s\int_{\mathbb{R}\setminus\{0\}}\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\tilde{N}(du,d\zeta)\Big\}\Big|\mathcal{F}_r\Big].
\end{aligned}\tag{3.5}$$

We consider a process $\mathcal{M}_s:=\exp\big\{\int_r^s\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)dW_u+\int_r^s\int_{\mathbb{R}\setminus\{0\}}\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\tilde{N}(du,d\zeta)\big\}$, with $-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T>-1$ for a.s. (ω,t,ζ). From Assumptions (1)-(2) and the fact that $(Z_t^{T,\xi},\Psi_t^{T,\xi})\in L^2(W)\times L_\nu^2(\tilde{N}_p)$, we conclude that the process $\mathcal{M}_s$ is a BMO-martingale. Define a probability measure $\mathbb{Q}^\pi$ by

$$\frac{d\mathbb{Q}^\pi}{d\mathbb{P}}=\mathcal{E}(\mathcal{M})_T,$$

on $\mathcal{F}_T$, where

$$\begin{aligned}
&\mathcal{E}(\mathcal{M})_T\\
=\;&\exp\Big\{\int_0^T\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)dW_u-\frac{1}{2}\int_0^T\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2du\\
&+\int_0^T\int_{\mathbb{R}\setminus\{0\}}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)du\\
&+\int_0^T\int_{\mathbb{R}\setminus\{0\}}\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\tilde{N}(du,d\zeta)\Big\},
\end{aligned}\tag{3.6}$$

provided that $\int_0^T\int_{\mathbb{R}\setminus\{0\}}(e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1)^2\nu(d\zeta)du<\infty$ {for more on exponential martingale see [2], [27] and [29]}. Therefore, $\frac{d\mathbb{Q}^\pi}{d\mathbb{P}}|_{\mathcal{F}_T}=\mathcal{E}(\mathcal{M})_T$ is uniformly integrable, given that the process $\mathcal{M}_s$ is a BMO-martingale. Now, we have that

$$\begin{aligned}
&\exp\Big\{\int_r^s \big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)dW_u+\int_r^s\int_{\mathbb{R}\setminus\{0\}}\big(e^{-\gamma\pi_u\Upsilon_u}+\Psi_u+\gamma\Psi_u^T\big)\tilde{N}(du,d\zeta)\Big\}\\
=\;&\exp\Big\{\frac{1}{2}\int_r^s\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2du\\
&-\int_r^s\int_{\mathbb{R}\setminus\{0\}}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)du\Big\}\frac{\mathcal{E}(\mathcal{M})_s}{\mathcal{E}(\mathcal{M})_r}.
\end{aligned}\tag{3.7}$$

Hence, from (3.5),

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[-\exp\Big\{-\gamma\Big(x+\int_t^s\pi_u\mu(V_u)du+\int_t^s\pi\sigma(V_u)dW_u+\int_t^s\int_{\mathbb{R}\setminus\{0\}}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big)\\
&+Y_s-\lambda s+\gamma Y_s^T\Big\}\Big|\mathcal{F}_r\Big]\\
=\;&-\exp\Big\{-\gamma\Big(x+\int_t^r\pi_u\mu(V_u)du+\int_t^r\pi_u\sigma(V_u)dW_u+\int_t^r\int_{\mathbb{R}\setminus\{0\}}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big)\\
&+Y_r-\lambda r+\gamma Y_r^T\Big\}\times\mathbb{E}_{\mathbb{P}}\Big[\exp\Big\{\int_r^s\big(-\gamma\pi_u\mu(V_u)-g(V_u,Z_u+\gamma Z_u^T,\Psi_u+\gamma\Psi_u^T)\big)du\\
&+\frac{1}{2}\int_r^s\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2du\\
&-\int_r^s\int_{\mathbb{R}\setminus\{0\}}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)du\Big\}\frac{\mathcal{E}(N)_s}{\mathcal{E}(N)_r}\Big|\mathcal{F}_r\Big].\\
=\;&-\exp\Big\{-\gamma\Big(x+\int_t^r\pi_u\mu(V_u)du+\int_t^r\pi_u\sigma(V_u)dW_u+\int_t^r\int_{\mathbb{R}\setminus\{0\}}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big)\\
&+Y_r-\lambda r+\gamma Y_r^T\Big\}\times\mathbb{E}_{\mathbb{Q}^\pi}\Big[\exp\Big\{\int_r^s\big(-\gamma\pi_u\mu(V_u)-g(V_u,Z_u+\gamma Z_u^T,\Psi_u+\gamma\Psi_u^T)\big)du\\
&+\frac{1}{2}\int_r^s\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2du\\
&-\int_r^s\int_{\mathbb{R}\setminus\{0\}}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)du\Big\}\Big|\mathcal{F}_r\Big].
\end{aligned}\tag{3.8}$$

Following the same procedure as in [6] (Proof of Theorem 6, Page 12), we show that for any $u\in[r,s]$,

$$-\gamma\pi_u\mu(V_u)+\frac{1}{2}\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2-\int_r^s\int_{\mathbb{R}\setminus\{0\}}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1$$

$$+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)\ \geq\ g(V_u,Z_u+\gamma Z_u^T,\Psi_u+\gamma\Psi_u^T), \tag{3.9}$$

then

$$\mathbb{E}_{\mathbb{Q}^\pi}\bigg[\exp\bigg\{\int_r^s\big(-\gamma\pi_u\mu(V_u)-g(V_u,Z_u+\gamma Z_u^T,\Psi_u+\gamma\Psi_u^T)\big)du$$
$$+\frac{1}{2}\int_r^s\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2du$$
$$-\int_r^s\int_{\mathbb{R}\backslash\{0\}}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)du\bigg\}\bigg|\mathcal{F}_r\bigg]\geq 1. \tag{3.10}$$

As a result the super-martingale property

$$\mathbb{E}_{\mathbb{P}}\bigg[-\exp\bigg\{-\gamma\bigg(x+\int_t^s\pi_u\mu(V_u)du+\int_t^s\pi_u\sigma(V_u)dW_u+\int_t^s\int_{\mathbb{R}\backslash\{0\}}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\bigg)$$
$$+Y_s-\lambda s+\gamma Y_s^T\bigg\}\bigg|\mathcal{F}_r\bigg]\leq-\exp\bigg\{-\gamma\bigg(x+\int_t^r\pi_u\mu(V_u)du+\int_t^r\pi_u\sigma(V_u)dW_u$$
$$+\int_t^r\int_{\mathbb{R}\backslash\{0\}}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\bigg)+Y_r-\lambda r+\gamma Y_r^T\bigg\}$$

will hold. Note that the left hand side of the equation (3.9) can be written as

$$-\gamma\pi_u\mu(V_u)+\frac{1}{2}\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2-\int_r^s\int_{\mathbb{R}\backslash\{0\}}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1$$
$$+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)$$
$$=\ \frac{\gamma^2}{2}\big|\pi\sigma-(Z^T+\frac{Z_u+\mu(V_u)/\sigma(V_u)}{\gamma})\big|^2-\frac{1}{2}\big|Z_u+\gamma Z_u^T+\mu(V_u)/\sigma(V_u)\big|^2+\frac{1}{2}|Z_u+\gamma Z_u^T|^2$$
$$-\int_r^s\int_{\mathbb{R}\backslash\{0\}}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta). \tag{3.11}$$

In particular, for any $\pi_u\in\mathcal{A}_{[t,T]}$,

$$\frac{\gamma^2}{2}\big|\pi\sigma-(Z^T+\frac{Z_u+\mu(V_u)/\sigma(V_u)}{\gamma})\big|^2-\frac{1}{2}\big|Z_u+\gamma Z_u^T+\mu(V_u)/\sigma(V_u)\big|^2+\frac{1}{2}|Z_u+\gamma Z_u^T|^2$$
$$-\int_r^s\int_{\mathbb{R}\backslash\{0\}}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)$$
$$\geq\ \inf_{\pi_u\in\mathcal{A}_{[t,T]}}\bigg\{\frac{\gamma^2}{2}\big|\pi\sigma-(Z^T+\frac{Z_u+\mu(V_u)/\sigma(V_u)}{\gamma})\big|^2$$
$$-\frac{1}{2}\big|Z_u+\gamma Z_u^T+\mu(V_u)/\sigma(V_u)\big|^2+\frac{1}{2}|Z_u+\gamma Z_u^T|^2$$

$$-\int_r^s \int_{\mathbb{R}\setminus\{0\}} \big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi^T_u)} - 1 + \big(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi^T_u\big)\big]\nu(d\zeta)\Big\},$$

and using $g(V_u, Z_u + \gamma Z^T_u, \Psi_u + \gamma\Psi^T_u)$ as in (2.6), we conclude that the super-martingale property holds true.

The martingale property of the process $P^{\tilde\pi}$, holds true if $\tilde\pi \in \mathcal{A}_{[t,T]}$ and

$$\begin{aligned}\tilde\pi \;=\; \inf_{\pi_u\in\mathcal{A}_{[t,T]}} &\Big\{\frac{\gamma^2}{2}\big|\pi\sigma - (Z^T + \frac{Z_u + \mu(V_u)/\sigma(V_u)}{\gamma})\big|^2 \\ &-\frac{1}{2}\big|Z_u + \gamma Z^T_u + \mu(V_u)/\sigma(V_u)\big|^2 + \frac{1}{2}|Z_u + \gamma Z^T_u|^2 \\ &-\int_r^s \int_{\mathbb{R}\setminus\{0\}} \big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi^T_u)} - 1 + \big(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi^T_u\big)\big]\nu(d\zeta)\Big\}.\end{aligned}$$

Combining the results from above, we obtain that $\mathbb{E}_{\mathbb{P}}[P^\pi_T|\mathcal{F}_t] \leq P^\pi_t$, and hence, for any $\pi \in \mathcal{A}_{[t,T]}$,

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\Big[-e^{-\gamma\big(x+\int_t^T\pi_u\mu(V_u)du+\int_t^T\pi_u\sigma(V_u)dW_u+\int_t^T\int_{\mathbb{R}\setminus\{0\}}\pi_u\Upsilon_u\tilde N(du,d\zeta)\big)+Y_T-\lambda T-\gamma\xi_T}\Big|\mathcal{F}_t\Big] \\ &\quad\leq\; -e^{-\gamma x+Y_t-\lambda t+\gamma Y^T_t},\end{aligned} \tag{3.12}$$

and for $\pi = \tilde\pi \in \mathcal{A}_{[t,T]}$, we obtain

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\Big[-e^{-\gamma\big(x+\int_t^T\pi_u\mu(V_u)du+\int_t^T\pi_u\sigma(V_u)dW_u+\int_t^T\int_{\mathbb{R}\setminus\{0\}}\pi_u\Upsilon_u\tilde N(du,d\zeta)\big)+Y_T-\lambda T-\gamma\xi_T}\Big|\mathcal{F}_t\Big] \\ &\quad=\; -e^{-\gamma x+Y_t-\lambda t+\gamma Y^T_t}.\end{aligned} \tag{3.13}$$

Subsequently,

$$\begin{aligned}&ess\sup_{\pi\in\mathcal{A}_{[t,T]}} \mathbb{E}_{\mathbb{P}}\Big[-e^{-\gamma\Big(x+\int_t^T\pi_u\mu(V_u)dt+\int_t^T\pi_u\sigma(V_u)dW_u+\int_t^T\int_{\mathbb{R}\setminus\{0\}}\pi_u\Upsilon_u\tilde N(du,d\zeta)\Big)+Y_T-\lambda T-\gamma\xi_T}\Big|\mathcal{F}_t\Big]. \\ &\quad=\; -e^{-\gamma x+Y_t-\lambda t+\gamma Y^T_t},\end{aligned} \tag{3.14}$$

and using condition (3.1), we obtain

$$-e^{-\gamma\rho_t(\xi_T)-\gamma x+Y_t-\lambda t+\gamma Y^T_t} = -e^{-\gamma x+Y_t-\lambda t},$$

and hence,

$$\rho_t(\xi_T) = Y^T_t,$$

which concludes the proof. □

Similar to [6] (Section 3, Page 15), the above representation also satisfies the time-consistent property, which means any risk position defined at time T can be evaluated indifferently at any intermediary time u for any $0 \leq t \leq u \leq T < \infty$. See also Bion [3] for construction of risk measures using BSDE with jumps. Chong [6], highlights the difference between the traditional entropic risk measure and the forward entropic risk measure. The first difference is that the forward entropic risk measure is defined for all time $t \leq 0$, while the traditional entropic risk measure is determined for a finite time $t \in [0, T]$. The second difference is that the generator of the BSDE (3.2), depends on the process Z, which is part of the solution of the ergodic BSDE (2.7) that gives the forward exponential process in (2.27).

4. Long-term maturity behaviour of the forward entropic risk measure

We consider a contingent claim written on the stochastic factor, this position is represented as follow

$$\xi_T = -h(V_T), \tag{4.1}$$

where $h : \mathbb{R} \mapsto \mathbb{R}$ is uniformly bounded and is Lipschitz continuous function with a Lipschitz constant C_h. From Theorem 3.1, we know that the risk of the position is represented as the solution for the BSDE (3.2), that is $\rho(\xi_T) = Y_t^{T,\xi}$, where $Y_t^{T,\xi}$ satisfies

$$Y_t^{T,h} = h(V_T) + \int_t^T G(V_u, Z_u, Z_u^{T,h}, \Psi_u, \Psi_u^{T,h})du - \int_t^T Z_u^{T,h} dW_u - \int_t^T \int_{\mathbb{R}\setminus\{0\}} \Psi_u^{T,h} \tilde{N}(du, d\zeta).$$

To analyse the long term behaviour of the forward risk measure, we associate the above BSDE to the ergodic BSDE given as

$$\hat{Y}_t = \hat{Y}_{T'} + \int_t^{T'} \big(G(V_u, Z_u, \hat{Z}_u, \Psi_u, \hat{\Psi}_u) - \lambda\big)du - \int_t^{T'} \hat{Z}_u dW_u - \int_t^{T'} \int_{\mathbb{R}\setminus\{0\}} \hat{\Psi}_u \tilde{N}(du, d\zeta), \tag{4.2}$$

for $0 \leq t \leq T' < \infty$. We analyze the approximation of $Y_0^{T,h}$ by $\hat{Y}_0 + \hat{\lambda} T$ for large T. In Chong et al. [6] the driver of the ergodic BSDE (4.2) depends only on the solution Z_t of the ergodic BSDE (2.7) of the forward performance process. In our case, the driver of the ergodic BSDE (4.2) will depend on the solution Z and Ψ of the ergodic BSDE (2.7). As was pointed by Chong et al. [6], this creates technical issues, which results in examining the Markovian and non-Markovian forward processes separately. Following a similar route, we analyze the long-term maturity behaviour in the Markovian case. The non-Markovian case follows closely as in Chong et al. [6].

4.1. **Markovian forward performance process.**

Let us consider the case

$$U(t, x) = -e^{-\gamma x + y(V_t) - \lambda t} \tag{4.1.1}$$

where $(Y(V_t), Z(V_T), \Psi(V_t)) = (y(V_t), z(V_t), \psi(V_t), \lambda)$, is the solution of the ergodic BSDE (2.7). The driver $G(V_u, Z_u, \hat{Z}_u, \Psi_u, \hat{\Psi}_u)$ of the ergodic BSDE (4.2) depends on $z(V_t)$ and $\psi(V_t)$. The functions $z(\cdot)$ and $\psi(\cdot)$ are bounded and hence the driver G is Lipschitz continuous in $\hat{z}$ and $\hat{\psi}$ as in (2.10) and (2.11). However, the generator G may not be Lipschitz continuous in v, which affects the existence and uniqueness of the solution to the ergodic BSDE (4.2). To overcome this problem, we consider an auxiliary quadratic BSDE defined by,

$$\begin{aligned}\hat{Y}_t^{T,h} &= h(V_T) + \frac{Y_T - \lambda T}{\gamma} + \int_t^T \frac{1}{\gamma} g(V_u, \gamma \hat{Z}_u^{T,h}, \gamma \hat{\Psi}_u^{T,h}) du - \int_t^T \hat{Z}_u^{T,h} dW_u \qquad (4.1.2)\\ &\quad - \int_t^T \int_{\mathbb{R}\setminus\{0\}} \hat{\Psi}_u^{T,h} \tilde{N}(du, d\zeta),\end{aligned}$$

with $(\hat{Y}_t^{T,h}, \hat{Z}_t^{T,h}, \hat{\Psi}_t^{T,h})$ given as

$$(\hat{Y}_t^{T,h}, \hat{Z}_t^{T,h}, \hat{\Psi}_t^{T,h}) := \left(Y_t^{T,h} + \frac{Y_t - \lambda t}{\gamma}, Z_t^{T,h} + \frac{Z_t}{\gamma}, \Psi_t^{T,h} + \frac{\Psi_t}{\gamma} \right),$$

and g is given in (2.33).

We now recall from [6] (Proposition 7) the following proposition with some results for the stochastic factor model.

Proposition 4.1. [6] *If Assumption 2 holds, then for all $t \geq 0$,*

(i) *the stochastic factor process satisfies $|V_t^{v_1} - V_t^{v_2}|^2 \leq e^{-2C_\eta t}|v_1 - v_2|^2$ where $v_1, v_2 \in \mathbb{R}^d$.*

(ii) *If we assume that the process V^v satisfies the following SDE*

$$dV_t^v = (\eta(V_t^v) + H(V_t^v))dt + \kappa dW_t^H,$$

where $H : \mathbb{R} \mapsto \mathbb{R}$ is a measurable bounded function, $\mathbb{Q}^H$ and $\mathbb{P}$ are equivalent probability measures, and W^H is a $\mathbb{Q}^H$-Brownian motion. Then, for some constant $C > 0$, $\mathbb{E}_{\mathbb{Q}^H}[|V_t^v|^2] \leq C(1 + |v|^2)$.

(iii) *For any measurable function $\phi : \mathbb{R}^d \to \mathbb{R}$ with polynomial growth rate $\vartheta > 0$, and $v_1, v_2 \in \mathbb{R}^d$,*

$$|\mathbb{E}_{\mathbb{Q}^H}[|\phi(V_t^{v_1}) - \phi(V_t^{v_2})|]| \leq C(1 + |v_1|^{1+\vartheta} + |v_2|^{1+\vartheta})e^{-\hat{C}_\eta t},$$

where the constants C and $\hat{C}_\eta$ depend on the function H only through $\sup_{v\in\mathbb{R}^d}|H(v)|$.

The proof of (i) and (ii) follows from the Gronwall's inequality and application of the Lyapunov argument respectively (see [6] and [9] Lemma 3.1). For the proof to the third part of the proposition (*basic coupling estimate*) is given in Lemma 3.4 of [13] and also see Theorem 2.4 of [8] and Theorem 5 of [7].

Theorem 4.2. *Let Assumption 1 and 2 hold, and assume that the forward performance process $U(t, x)$ is given by (2.27). Then*

(i) *there exists a unique solution $(\hat{Y}_t^{T,h}, \hat{Z}_t^{T,h}, \hat{\Psi}_t^{T,h}) = (\hat{y}^{T,h}(V_t), \hat{z}^{T,h}(V_t), \hat{\psi}^{T,h}(V_t))$ of the quadratic BSDE (4.1.2) for $t \in [0, T]$.*

(ii) *For* $(t,v)\in[0,\infty)\times\mathbb{R}^d$, *we have that*

$$|\hat{y}^{T,h}(t,v)|\le C_T(1+|v|)$$

and $\hat{z}^{T,h},\hat{\psi}^{T,g}$ *are uniformly bounded such that,*

$$|\hat{z}^{T,h}(v,t)|\le C_z,\qquad |\psi(v_t,\zeta)|\le\frac{2K}{\alpha}$$

Proof. The existence and uniqueness of the solution to the quadratic BSDE (4.1.2) follows from Morlais [19] (Section 3.2, Theorem 1 and 2). Analogous to Chong et al. [6], the linear growth condition of the function $\hat{y}^{T,g}(t,v)$ follows from the boundedness of $Y_t^{T,h}$ and the linear growth condition of $y(\cdot)$. We consider a truncated BSDE version of (4.1.2)

$$\begin{aligned}\hat{Y}_t^{T,h} &= h(V_T)+\frac{Y_T-\lambda T}{\gamma}+\int_t^T\frac{1}{\gamma}g(V_u,\gamma q(\hat{Z}_u^{T,h}),\gamma\tilde{q}(\hat{\Psi}_u^{T,h}))du-\int_t^T\hat{Z}_u^{T,h}dW_u \\ &\quad -\int_t^T\int_{\mathbb{R}\setminus\{0\}}\hat{\Psi}_u^{T,h}\tilde{N}(du,d\zeta),\end{aligned}\tag{4.1.3}$$

where the truncation functions $q(\cdot):\mathbb{R}^d\to\mathbb{R}^d$ and $\tilde{q}:\mathbb{R}^d\to\mathbb{R}^d$ are defined as

$$q(z):=\frac{\min(|z|,C_z)}{|z|}z1_{\{z\neq0\}},\quad\text{and}\quad\tilde{q}(\psi):=1_{|\psi|\le1}.$$

Now, it then follows that the generator g of the truncated BSDE (4.1.3) is Lipschitz i.e.

$$|g(v_1,\gamma q(z),\gamma\tilde{q}(\psi))-g(v_2,\gamma q(z),\gamma\tilde{q}(\psi))|\le C_v|v_1-v_2|,\tag{4.1.4}$$

$$|g(v,\gamma q(z_1),\gamma\tilde{q}(\psi))-g(v,\gamma q(z_2),\gamma\tilde{q}(\psi))|\le C_z|z_1-z_2|\tag{4.1.5}$$

and

$$|g(v,\gamma q(z),\gamma\tilde{q}(\psi_1))-g(v,\gamma q(z),\gamma\tilde{q}(\psi_2))|\le\int_{\mathbb{R}\setminus\{0\}}|\psi_1-\psi_2|\varphi^{v,z,\psi_1,\psi_2}\nu(d\zeta),\tag{4.1.6}$$

for any $v_1,\ v_2,\ z_1,\ z_2,\ \psi_1,\ \psi_2\in\mathbb{R}^d$. Consequently, we have

$$\begin{aligned}&\hat{Y}_t^{T,t,v_1}-\hat{Y}_t^{T,t,v_2}\\ =\;& h(V_T^{t,v_1})-h(V_T^{t,v_2})+\frac{1}{\gamma}(Y_T^{t,v_1}-Y_T^{t,v_2})\\ &+\int_t^T\frac{1}{\gamma}\big[g\big(V_u^{t,v_1},\gamma q(\hat{Z}_u^{T,t,v_1}),\gamma q(\hat{\Psi}_u^{T,t,v_1})\big)-g\big(V_u^{t,v_2},\gamma q(\hat{Z}_u^{T,t,v_2}),\gamma q(\hat{\Psi}_u^{T,t,v_2})\big)\big]du\\ &-\int_t^T\big(\hat{Z}_u^{T,t,v_1}-\hat{Z}_u^{T,t,v_2}\big)dW_u-\int_t^T\int_{\mathbb{R}\setminus\{0\}}\big(\hat{\Psi}_u^{T,t,v_1}-\hat{\Psi}_u^{T,t,v_2}\big)\tilde{N}(du,d\zeta)\\ =\;& h(V_T^{t,v_1})-h(V_T^{t,v_2})+\frac{1}{\gamma}(y(V_T^{t,v_1})-y(V_T^{t,v_2}))\\ &+\int_t^T\frac{1}{\gamma}\big[g\big(V_u^{t,v_1},\gamma q(\hat{Z}_u^{T,t,v_1}),\gamma q(\hat{\Psi}_u^{T,t,v_1})\big)-g\big(V_u^{t,v_2},\gamma q(\hat{Z}_u^{T,t,v_2}),\gamma q(\hat{\Psi}_u^{T,t,v_2})\big)\big]du\end{aligned}$$

$$
\begin{aligned}
&-\int_t^T \big(\hat{Z}_u^{T,t,v_1} - \hat{Z}_u^{T,t,v_2}\big)(dW_u - \beta du) \\
&-\int_t^T \int_{\mathbb{R}\setminus\{0\}} \big(\hat{\Psi}_u^{T,t,v_1} - \hat{\Psi}_u^{T,t,v_2}\big)(\tilde{N}(du,d\zeta) - \varphi^{v,z,\psi_1,\psi_2}\nu(d\zeta)du)
\end{aligned} \tag{4.1.7}
$$

and denote

$$
\begin{aligned}
\beta_t \;:=\; & \frac{\big[g\big(V_u^{t,v_1}, \gamma q(\hat{Z}_u^{T,t,v_1}), \gamma q(\hat{\Psi}_u^{T,t,v_1})\big) - g\big(V_u^{t,v_2}, \gamma q(\hat{Z}_u^{T,t,v_2}), \gamma q(\hat{\Psi}_u^{T,t,v_2})\big)\big]}{\gamma|\hat{Z}_u^{T,t,v_1} - \hat{Z}_u^{T,t,v_2}|^2} \times \\
& \times|\hat{Z}_u^{T,t,v_1} - \hat{Z}_u^{T,t,v_2}| 1_{\{\hat{Z}_u^{T,t,v_1} \neq \hat{Z}_u^{T,t,v_2}\}}
\end{aligned}
$$

Using the Girsanov's theorem we can define $W_t^\beta := W_t - \int_0^t \beta du$ and $\tilde{N}^\varphi(dt,d\zeta) := \tilde{N}(dt,d\zeta) - \int_0^t \varphi^{v,z,\psi_1,\psi_2}\nu(d\zeta)du$ for $0 \le t \le T$, where $\varphi^{v,z,\psi_1,\psi_2}$ is defined in Assumption (2.11). For all t we define $\delta Z_t := \hat{Z}_t^{T,t,v_1} - \hat{Z}_t^{T,t,v_2}$ and $\delta\Psi_t := \hat{\Psi}_t^{T,t,v_1} - \hat{\Psi}_t^{T,t,v_2}$ and introduce

$$
\mathcal{M}_t = \int_0^t \delta Z_t dW_u^\beta + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \delta\Psi_u \tilde{N}^\varphi(du,d\zeta),
$$

which is a local martingale under the measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, defined on $\mathcal{F}_T$. Thus, taking conditional expectation under the $\mathbb{Q}$ measure on $\mathcal{F}_t$ and using the Lipschitz condition of $h(v)$, in (4.1), $y(v)$ in (2.13) and $g(v,\gamma q(z),\gamma\tilde{q}(\psi))$ in (4.1.4) to (4.1.6), we obtain the following results

$$
\begin{aligned}
|\hat{Y}_t^{T,t,v_1} - \hat{Y}_t^{T,t,v_2}| \;&=\; |\hat{y}_t^{T,t,v_1} - \hat{y}_t^{T,t,v_2}| \\
&\le\; C_h \mathbb{E}_{\mathbb{Q}}[|V_T^{t,v_1} - V_T^{t,v_2}||\mathcal{F}_t] + \frac{K}{\gamma}\mathbb{E}_{\mathbb{Q}}[|V_T^{t,v_1} - V_T^{t,v_2}||\mathcal{F}_t] \\
&\quad + \frac{C_v}{\gamma}\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T |V_u^{t,v_1} - V_u^{t,v_2}||\mathcal{F}_t\Big].
\end{aligned} \tag{4.1.8}
$$

Furthermore, using the results from Proposition 4.1 we conclude that

$$
|\hat{Y}_t^{T,t,v_1} - \hat{Y}_t^{T,t,v_2}| \le \Big(C_h + \frac{K}{\gamma} + \frac{C_v}{\gamma}\Big)|v_1 - v_2|.
$$

□

The proof of the asymptotic behaviour of the forward entropic risk measure is the same as the diffusion can in Theorem 10 of [6], where they show that the forward entropic risk measure converges to a constant as the time horizon increases.

5. Conclusion

In this paper, we have introduced jumps into the ergodic BSDE with quadratic growth in the control variable. We have proved that under certain conditions there exists a unique Markovian solution for a quadratic-exponential ergodic BSDE with bounded jumps.

The solution of the quadratic-exponential ergodic BSDE with bounded jumps was used to derive the representation of a forward entropic risk measure. We have noticed that when the stochastic factor includes jumps, the corresponding generator of the ergodic BSDE contains Y_t and consequently the translation invariance property is not satisfied.
We have also, derived the connection between the ergodic BSDEs with jumps and the PIDE. which allowed us to determine the representation of a forward entropic risk measure using the solution of a quadratic-exponential ergodic BSDE with bounded jumps. This work can be extended to study the differentiability of the ergodic BSDE with jumps in order to determine capital allocation representation of the forward entropic risk measure in the spirit of [17].

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