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Operational quantities in C^*-algebras

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CONTENTS

NOTATIONS

CHAPTERS

0. INTRODUCTION 1

1. PREREQUISITES 6

2. NORM CLOSED IDEALS IN AW^*-ALGEBRAS 22

3. THE REDUCED MINIMUM MODULUS IN OPERATOR ALGEBRAS 34

4. THE ESSENTIAL LOWER BOUND 55

REFERENCES 74

SUMMARY 81

OPSOMMING 83

Notations

Symbols and their meanings

Symbol	Meaning
$\mathcal{A}$	an AW^*-algebra
$P(\mathcal{A})$	complete lattice of projections in $\mathcal{A}$
a, b, c	usually denotes elements in a C^*-algebra
a^+	the generalised inverse of a
$\mathcal{B}(H)$	all bounded linear operators on H
bdy $(\mathcal{G})$	boundary of the set $\mathcal{G}$
$\mathbf{C}$	set of complex numbers
$\mathcal{C}$	C^*-algebra
dim $(\cdot)$	dimension function
dist $(T, \mathcal{S})$	distance from the operator T to the set $\mathcal{S}$
E, F, G, P, Q	projections
$E_{(\cdot)}$, $F_{(\cdot)}$	spectral measures
E_λ , F_λ	spectral projections
$\mathcal{G}$	group of invertible elements in $\mathcal{M}$
$\bar{\mathcal{G}}$	closure of $\mathcal{G}$
int $(\mathcal{G})$	interior of $\mathcal{G}$
$\gamma(T)$	reduced minimum modulus of T
$\gamma_{\mathcal{I}}(T)$	essential minimum modulus of T

H	a Hilbert space
$\mathcal{I}$	closed two-sided ideal
$\text{index}(T)$	index of an element T
K	stonian space
$\mathcal{K}(\mathcal{M})$	closed ideal generated by finite projections in $\mathcal{M}$
$\mathcal{K}(H)$	ideal of compact operators in $\mathcal{B}(H)$
L_a	left regular representation of a
$\mathcal{M}$	a von Neumann algebra
$m(T)$	$\inf \sigma(\|T\|)$
$m_{\mathcal{I}}(T)$	lower bound of T relative to $\mathcal{I}$
$\mathbf{N}$	set of natural numbers
N_T	null projection of T
$P(\mathcal{M})$	complete lattice of projections in $\mathcal{M}$
$\rho(T)$	resolvent set of T
$R(\mathcal{S})$	right annihilator of the set $\mathcal{S}$
$\pi_{\mathcal{I}}(\cdot)$	quotient map
Φ_0	Fredholm elements of index zero
$\Phi_\ell(\mathcal{M},\mathcal{I})$	left Fredholm elements in $\mathcal{M}$ relative to $\mathcal{I}$
$\Phi_r(\mathcal{M},\mathcal{I})$	right Fredholm elements in $\mathcal{M}$ relative to $\mathcal{I}$
$\Phi_s(\mathcal{M},\mathcal{I})$	semi-Fredholm elements in $\mathcal{M}$ relative to $\mathcal{I}$
$\Phi_u(\mathcal{M},\mathcal{I})$	union of $\Phi_\ell(\mathcal{M},\mathcal{I})$ and $\Phi_r(\mathcal{M},\mathcal{I})$

$\mathbf{R}$	set of real numbers
R_T	range projection of T
$r(a)$	spectral radius of a
$\sigma(T)$	spectrum of T
$\sigma_e(T)$	essential spectrum of T
$s(T)$	semi-Fredholm radius of T
T, S, U, V	operators
T^*	adjoint of T
$\|T\|$	absolute value of T
$\{T\}'$	commutant of T
$\mathcal{Z}$	center of $\mathcal{M}$
$Z_\ell(\mathcal{M}, \mathcal{I})$	set of T in $\mathcal{M}$ for which $\pi_\mathcal{I}(T)$ is a left topological zero divisor in $\mathcal{M}/\mathcal{I}$
$Z_r(\mathcal{M}, \mathcal{I})$	set of T in $\mathcal{M}$ for which $\pi_\mathcal{I}(T)$ is a right topological zero divisor in $\mathcal{M}/\mathcal{I}$
$Z(\mathcal{M}, \mathcal{I})$	set of T in $\mathcal{M}$ for which $\pi_\mathcal{I}(T)$ is a topological zero divisor in $\mathcal{M}/\mathcal{I}$

INTRODUCTION

Operational quantities characteristic of the semi-Fredholm operators have been introduced in the theory of bounded operators between Banach spaces and applied successfully to for example perturbation theory [Gol66], [Kat66], [Zem81], [Zem84a], [Zem84b].

In recent years, there have been various attempts at generalising various classical results relating to compact operators (in particular results that consider the compact operators as an ideal in the full algebra of bounded operators on a Hilbert space) to the setting of a von Neumann algebra containing a closed ideal.

In all these developments the existence of projections plays a crucial role. In fact it is well known (via the spectral theorem) that a von Neumann algebra is generated by its projections and that the class of projections is a complete lattice in the sense of Boolean algebra. For a closed ideal in the algebra it was shown by Wright [Wri54] that the ideal is exactly the closure of the ideal generated by its projections. This result generalises an important characterisation of the ideal of compact operators, namely that it is the closure of the ideal of finite rank operators, and hence the closed ideal generated by the finite dimensional projections.

In [Kaf77,Kaf78], Kaftal has considered the ideal of so called algebraically compact operators which is defined to be the closed ideal generated by the "Murray and von Neumann" finite projections in the von Neumann algebra, and has shown that the ideal consists of those operators which map the unit ball to sets which have compact-like properties. This characterisation was generalised to arbitrary norm-closed ideals by Ströh [Str89].

M. Breuer [Bre68,Bre69] has developed a complete Fredholm theory and index theory relative to the closed ideal generated by the finite projections in the algebra and applied his results successfully in the study of vector bundles relative to a von Neumann algebra. By using Breuer's ideas, left and right Fredholm theory relative to the compact ideal was introduced by V. Kaftal [Kaf77,Kaf78] and results like the Weyl-von Neumann theorem was extended to this setting. With the Fredholm theory available, the notion of Riesz elements was introduced by A. Ströh and J. Swart [StS91] and a complete Riesz decomposition theorem was obtained.

It was first noted by Olsen [Ols84] that the study of a Fredholm theory in a von

Neumann algebra setting has successful generalisations towards any closed two-sided ideal in the algebra. In particular, by making use of the dimension function of Tomiyama [Tom58] and the work of Wils [Wil70], Olsen has developed a comprehensive Fredholm theory and index theory for closed two-sided ideals in von Neumann algebras.

In this thesis we will focus mainly on the role that the minimum modulus and the reduced minimum modulus play with regards to perturbation theory in a von Neumann algebra setting and some possible extensions to AW^*-algebras.

Chapter 1 mainly deals with basic facts that will be used throughout the thesis. The main tool in all of this work is the existence of a spectral representation for self-adjoint elements in a von Neumann algebra. In this chapter we show that any self-adjoint element in an AW^*-algebra possesses a spectral decomposition within the algebra. Closely related is the notion of a polar decomposition of an element in the algebra. In applications one usually apply the spectral theorem to the absolute value of an element. Since this will be a key idea in obtaining our results we include a proof of the polar decomposition of an element in an AW^*-algebra. In order to make the thesis as self-contained as possible we give a short introduction of the index theory of von Neumann algebras developed by Olsen [Ols84].

In Chapter 2 we focus on the norm-closed two-sided ideals in an AW^*-algebra. We show that closed ideals have properties very similar to the ideal of compact operators on a Hilbert space. The spectral type of characterisations obtained in this chapter of elements from a given closed ideal generalises results of Kaftal [Kaf77] and Ströh [Str89].

A fairly easy application of the spectral theorem shows that any AW^*-algebra has real rank zero. We finally use this fact and some work of Hadwin [Had95] to show that the lifting of algebraic elements is possible in an AW^*-algebra setting relative to any closed two-sided ideal.

In Chapter 3 we study some properties of the reduced minimum modulus and the reduced essential minimum modulus in a von Neumann algebra setting. We answer a question raised in [Str94] in the affirmative, namely that the reduced essential minimum modulus of an element in a von Neumann algebra relative to any norm closed two sided ideal is equal to the reduced minimum modulus of the element perturbed by an element from the ideal. As a corollary, we extend some basic perturbation results on semi-Fredholm elements to a von Neumann algebra setting. In particular, we show that if an operator is semi-Fredholm relative to any closed ideal, then it is a point of continuity of the reduced essential minimum modulus map. More generally, we find a complete characterisation of the points of continuity of the reduced essential minimum modulus in terms of Fredholm properties. These results generalise the work of Mbekhta and Paul [MbP96]. We conclude the chapter by a study of the asymptotic behaviour of the reduced essential minimum modulus.

In [Ols84], Olsen used another quantity, the essential lower bound, to characterise the classes of left and right Fredholm elements. In Chapter 4 we show that by using similar techniques, as for the case of the reduced quantities, the essential lower bound of an element can be perturbed by an element from the ideal to the lower bound. An important consequence is the lifting of invertible elements from the quotient algebra to the algebra, namely that any Fredholm element of index zero can be perturbed by an element from the ideal to give an invertible element in the algebra. In [Ver95], the essential lower bounds are used to characterise the topological zero divisors in the quotient algebra. We continue further our study on these bounds, by results, connecting the topological divisors of zero with the boundary of the group of invertible elements.

We conclude this chapter by finding some necessary and sufficient conditions for regular elements in a von Neumann algebra to be in the closure of the group of invertible elements. For similar results in the classical theory of operators on a Hilbert space we refer the reader to [Rog77], [IzK85], [Wu89], and [Gal94].

Chapter 1

PREREQUISITES

An important part of our study is to have a complete understanding of the ideal structure in von Neumann algebras. Results on this subject are studied in detail by Wils [Wil70], Ströh [Str89] and West [Wes93]. A careful investigation of the proofs of these results indicates that one could easily generalise most of the result to a more general class of C^*-algebras, namely the AW^*-algebras.

Essential tools in the whole study are the use of the spectral theorem and the existence of a polar decomposition in the algebra. Kaplansky [Kap51], [Kap52], [Kap68], showed AW^*-algebras as an appropriate setting for certain parts of the algebraic theory of von Neumann algebras.

An AW^*-algebra $\mathcal{A}$ is a C^*-algebra which is also a Baer *-ring. i.e. for every nonempty subset $\mathcal{S}$ of $\mathcal{A}$, the right annihilator $R(\mathcal{S}) = P\mathcal{A}$ for a suitable projection P in $\mathcal{A}$. It is clear from the definition that any AW^*-algebra has an identity I.

Hence the definition implies already the existence of projections in the algebra. A W^*-algebra (von Neumann algebra) is a weakly closed *-sub algebra of operators on a Hilbert space. It is not hard to show that any W^*-algebra is an AW^*-algebra. In the commutative case the difference is this: A commutative AW^*-algebra is characterised by a $C(K)$ space, where K is compact and stonian, while in the W^*-case K need to be hyperstonian.

Recall that a compact space K is stonian if the closure of every open set is open. A useful characterisation of this notion is that if $\{f_\alpha\}$ is any decreasing set in $C_{\mathbf{R}}(K)$ bounded from below, then infimum $\{f_\alpha\}$ exists and is in $C_{\mathbf{R}}(K)$.

Now suppose that T is a self-adjoint element of an AW^*-algebra $\mathcal{A}$, and let $\mathcal{A}(T, I)$ be the commutative AW^*-algebra generated by T and I. Then $\mathcal{A}(T, I)$ is isomorphic to a $C(K)$ space where K is stonian. Since T is self-adjoint it corresponds to a real valued function $T(t)$ on K. Let $K_\lambda = K \backslash \overline{T^{-1}(\lambda, \infty)}$. Clearly K_λ is an open and closed set on which T takes values not exceeding λ. The characteristic function of K_λ is a projection in $C(K)$. Let $E_\lambda = E_{(-\infty, \lambda]}$ be the corresponding projection in $\mathcal{A}(T, I)$.

Theorem 1.1 *If T is a self-adjoint element in $\mathcal{A}$, then the family $\{E_\lambda\}$ of projections defined above satisfies the following properties*

(a) $E_\lambda = 0$ *if* $\lambda < -||T||$ *and* $E_\lambda = I$ *if* $\lambda \geq ||T||$

(b) $E_\lambda \leq E_{\lambda'}$ *if* $\lambda \leq \lambda'$

(c) $E_\lambda = \bigwedge_{\lambda' > \lambda} E_{\lambda'}$

(d) $TE_\lambda \leq \lambda E_\lambda$ *and* $\lambda(I - E_\lambda) \leq T(I - E_\lambda)$ *for each* λ

(e) $T = \int_{-||T||}^{||T||} \lambda dE_\lambda$, *meaning T is the norm limit of finite linear combinations with coefficients in the spectrum of T of orthogonal projections $E_{\lambda'} - E_\lambda$.*

Proof

(a) Let E_λ be as above. If $\lambda < -||T||$, then $K_\lambda = K \backslash \overline{T^{-1}(\lambda, \infty)}$ is the empty set since $T(K) = \sigma(T) \subset [-||T||\ ,\ ||T||]$. Similarly if $\lambda > ||T||$, $K_\lambda = K$ which implies that $E_\lambda = I$ (the identity of the AW^*-algebra $\mathcal{A}$).

(b) Let $\lambda \leq \lambda'$, then $K_\lambda \subseteq K_{\lambda'}$ and since a $*$-isomorphism preserves order $E_\lambda \leq E_{\lambda'}$.

(c) Note that K_λ is a clopen set on which T takes values not exceeding λ. If K' is another clopen subset of K on which T takes values not exceeding λ, then $K' \subseteq K \backslash T^{-1}(\lambda, \infty)$, hence $T^{-1}(\lambda, \infty) \subset K \backslash K'$. Since $K \backslash K'$ is closed, $\overline{T^{-1}(\lambda, \infty)} \subset K \backslash K'$, hence $K' \subseteq K_\lambda$. Thus K_λ is the largest such set. To show

(c) note first that E_λ is a lower bound for $\{E'_\lambda \ : \ \lambda' > \lambda\}$. Let P be any projection in $\mathcal{A}$ such that $P \leq E_{\lambda'}$ for all $\lambda' < \lambda$ (i.e. $PE_{\lambda'} = E_{\lambda'}P$ for all $\lambda' < \lambda$). Then P corresponds to a characteristic function of a clopen set K_0 which is contained in $K_{\lambda'}$ except for a nowhere dense set for each $\lambda' < \lambda$. Since K_0 and $K_{\lambda'}$ are both open and closed, K_0 is actually a subset of $K_{\lambda'}$. Thus K_0 is a clopen subset of K on which P takes values not exceeding λ' for each $\lambda' > \lambda$. Hence P takes values not exceeding λ on K_0. Thus $P \leq E_\lambda$.

(d) Note that T takes values not exceeding λ on $K \backslash T^{-1}(\lambda, \infty)$. Hence $TE_\lambda \leq \lambda E_\lambda$ for each $\lambda > 0$. Also as T is continuous and takes values greater than or equal to λ on $\overline{T^{-1}(\lambda, \infty)}$, we have

$$\lambda(I - E_\lambda) \leq T(I - E_\lambda) \text{ for each } \lambda > 0.$$

(e) Let $\{\lambda_0, \lambda_1, \ldots \lambda_n\}$ be a partition of $[-||T|| \ , \ ||T||]$ such that $\lambda_0 = -||T||$ and $\lambda_n = ||T||$. Let λ'_j be a point of $[\lambda_{j-1}, \lambda_j] \cap \sigma(T) \neq \phi$. If $[\lambda_{j-1}, \lambda_j] \cap \sigma(T) = \phi$, then $T^{-1}[\lambda_{j-1}, \lambda_j] = \phi$. Hence

$$T^{-1}(\lambda_{j-1} \ , \ \infty) = T^{-1}(\lambda_j \ , \ \infty) \text{ which implies } E_{\lambda_{j-1}} = E_{\lambda_j}.$$

Now, $\sum_{j=1}^{n} \lambda'_j (E_{\lambda_j} - E_{\lambda_{j-1}})$ is a linear combination of mutually orthogonal characteristic functions $E_{\lambda'} - E_\lambda$ with coefficients in $\sigma(T)$. Since

$$||T - S|| \leq \ \max \ \{|\lambda_j - \lambda_{j-1}|\} \text{ it follows that } T = \int_{-||T||}^{||T||} \lambda dE_\lambda. \qquad \square$$

Since K is stonian i.e. $C_{\mathbf{R}}(K)$ is order complete, projections like $E_{(-\infty,\lambda)}$; $E_{(\lambda,\mu)}$; $E_{[\lambda,\mu]}$; $E_{(\lambda,\infty)}$ etc will all make sense as projections in $\mathcal{A}(T,I)$.

We will see that the existence of spectral projections of a self-adjoint element in $\mathcal{A}$ plays an important role in the structure theory of ideals in AW^*-algebras.

Proposition 1.2 *Let T be a positive element in $\mathcal{A}$. Then for any $\varepsilon > 0$ there exists a positive $S \in \mathcal{A}$ such that $TS = P$, P a non-zero projection and $||T - TP|| < \varepsilon$.*

Proof Let $\mathcal{A}(T,I)$ be the commutative $\mathcal{A}W^*$-algebra generated by T and I. Then $\mathcal{A}(T,I)$ is isomorphic to $C(K)$, where K is stonian. Since T is positive, it corresponds to a positive real valued function on K. We suppose $0 < \varepsilon < ||T||$. Define

$$U = \{t \in K : T(t) > \varepsilon\}.$$

Then $\bar{U}$ is a non-empty clopen set. Consider the spectral projection $P = I - E_\varepsilon$ i.e. P is the projection onto $\bar{U}$. Since

$$T(t) \leq \varepsilon \text{ on } K\backslash\bar{U}, \text{ we have}$$

$$0 \leq T(I-P) \leq \varepsilon(I-P) \leq \varepsilon\, I.$$

Hence

$$||T - TP|| \leq \varepsilon.$$

Let

$$S(t) = \frac{1}{T(t)} \text{ for } t \in \bar{U} \text{ and } S(t) = 0 \text{ for } t \in K\backslash\bar{U}.$$

Since $\bar{U}$ is open and closed, S is a continuous function and ST corresponds to the characteristic function of $\bar{U}$. Hence $ST = TS = P$. □

Let $P(\mathcal{A})$ denote the set of all projections in $\mathcal{A}$. Projections in $\mathcal{A}$ are partially ordered by $P \leq Q$ if $PQ = P$. It was shown by Kaplansky [Kap51] that with this order relation $P(\mathcal{A})$ is a complete lattice. We say that projections P and Q are equivalent, written $P \sim Q$, if there exists an element $U \in \mathcal{A}$ such that $P = UU^*$ and $Q = U^*U$. We write $P \preceq Q$ if there exists a projection G such that $P \sim G \leq Q$. We call a projection $P \in P(\mathcal{A})$ finite if $P \sim G \leq P$ implies $G = P$. It was shown by Kaplansky [Kap51] that basically all the results that Murry and von Neumann [MvN36] proved on these relations in W^*-algebras hold in general AW^*-algebras. We list the important facts:

PL: For any two projections P and Q, we have $P \vee Q - Q \sim P - P \wedge Q$.

C: For any two projections P and Q there exists a central projection G such that $PG \preceq QG$ and $P(I-G) \succeq Q(I-G)$.

For $T \in \mathcal{A}$, we define the null projection of T by $N_T := \sup\{P \in P(\mathcal{A}) : TP = 0\}$ and the range projection of T by $R_T : = \inf \{Q \in P(\mathcal{A}) : (I-Q)T = 0\}$

By making use of the structure of commutative AW^*-subalgebras, it was shown in Kaplansky [Kap51] that $R_T \sim R_{T^*}$.

The following relations is known in an AW^*-algebra setting and will prove to be useful in our study.

Proposition 1.3 *Let P and Q be any two projections in $\mathcal{A}$, then*

$$R_{PQ} = P - P \wedge (I - Q) \text{ and } R_{QP} = P \vee (I - Q) - (I - Q).$$

Proof Let $V = PQ$. Since $V(I - Q) = 0$, one has $R_{QP}(I - Q) = 0$. Thus

$(I - Q) + R_{QP}$ is a projection. Let $U = (I - Q) + R_{QP}$. We will show that

$U = \sup\ \{P, (I - Q)\}$. Obviously $(I - Q) \leq U$. Since

$$\begin{aligned} 0 = VR_{QP} - V &= PQR_{QP} - PQ \\ &= [P - P(I - Q)]R_{QP} - [P - P(I - Q)] \\ &= PR_{QP} - P[(I - Q)R_{QP}] - P + P(I - Q) \\ &= PR_{QP} - P + P(I - Q) \end{aligned}$$

we have

$$\begin{aligned} P &= P(I - Q) + PR_{QP} \\ &= P[(I - Q) + R_{QP}] \\ &= PU \end{aligned}$$

Hence

$$P \leq U$$

On the other hand, suppose $P \leq W$ and $(I - Q) \leq W$ for some projection W, it is to be shown that $U \leq W$. Then

$$\begin{aligned} VW &= PQW \\ &= P(W-(I-Q)) \\ &= PW - P(I-Q) \\ &= P - P(I-Q) \\ &= PQ = V \end{aligned}$$

Thus

$$R_{QP} \leq W.$$

Combined with $(I-Q) \leq W$, this yields $(I-Q) + R_{QP} \leq W$. Thus $U = P \vee (I-Q)$ which proves the second identity. It follows at once that $P \wedge (I-Q)$ exists and is equal to $I-(I-Q) \vee (I-P)$. The first identity will hence follow from the second one. □

We are now ready to state and prove the polar decomposition for elements in an AW^*-algebra setting. For $a \in \mathcal{C}$,$\mathcal{C}$ a C^*-algebra it is clear by the continuous functional calculus that $|a| = (a^*a)^{\frac{1}{2}}$ is an element of $\mathcal{C}$. However in a C^*-algebra setting one does not have in general a decomposition of the form $a = u|a|$.

Theorem 1.4 *Let $\mathcal{A}$ be an arbitrary AW^*-algebra. Then for any $T \in \mathcal{A}$ there exists a partial isometry U in $\mathcal{A}$ such that $T = U|T|$, $UU^* = R_T$ and $U^*U = R_{T^*}$.*

Proof Let $\mathcal{A}\{T^*T\ ,I\}$ be a commutative AW^*-algebra containing T^*T. By Proposition 1.2 we have for every $n = 1, 2, 3, \ldots$ there exists a positive element

$S_n \in \mathcal{A}\{T^*T, I\}$ such that

(a) $T^*T\ S_n^2$ is a projection $P_n \leq R_{T^*}$

(b) $T^*T \geq \frac{1}{n}P_n$ and $T^*T \leq (\frac{1}{n})(R_{T^*} - P_n)$ on $R_{T^*} - P_n$

(c) (S_n) is increasing and $S_{n-1}(S_n - S_{n-1}) = 0 \quad n = 2, 3, \ldots$

It is clear that (P_n) is increasing and $\sup P_n = R_{T^*}$. Let $U_n = TS_n$, then $U_n^*U_n = S_nT^*TS_n = T^*TS_n^2 = P_n$ and if we let

$$Q_n = U_nU_n^* = TS_n^2T^* \text{ then}$$

$$Q_n \leq R_T.$$

Moreover (Q_n) is increasing, since

$$\begin{aligned} Q_{n-1}Q_n &= (TS_{n-1}^2T^*)(TS_n^2T^*) \\ &= TT^*TS_{n-1}^2S_n^2T^* \\ &= TT^*TS_{n-1}^4T^* \\ &= (TS_{n-1}^2T^*)(TS_{n-1}^2T^*) \\ &= Q_{n-1}. \end{aligned}$$

By [Kap52, Lemma 20] there exists a partial isometry U in $\mathcal{A}$ such that $U^*U = R_{T^*}$,

$$UU^* = R_T \text{ and } UP_n = U_n. \text{ We show that } T = U|T|.$$

Note that $U_n|T| = TS_n|T| = T(S_n^2T^*T)^{\frac{1}{2}} = TP_n$

Hence by (b),

$$\begin{aligned} ||T - U_n|T||| &= ||T(R_{T^*} - P_n)|| \\ &= ||T^*T(R_{T^*} - P_n)||^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{n}} \end{aligned}$$

Thus

$$\begin{aligned} ||T - U|T||| &\leq ||T - U_n|T||| + ||(U - U_n)|T||| \\ &\leq \frac{1}{\sqrt{n}} + ||(U - U_n)(R_{T^*} - P_n)|T||| \\ &\leq \frac{2}{\sqrt{n}} \end{aligned}$$ □

It is not hard to show the uniqueness of this polar decomposition under the given conditions.

It is clear from the work of Kaplansky [Kap51], and the results illustrated here, that AW^*-algebras has a rich projection structure. This in fact carries over to the ideal structure in AW^*-algebras. It was shown by F. Wright [Wri54] that there exists a 1-1 correspondence between the norm closed two-sided ideals of $\mathcal{A}$ and the equivalence closed lattice ideals of $P(\mathcal{A})$. In Chapter 2 we will give more attention towards this structure theory of closed ideals and its projections. In order to obtain any generalisation of a Fredholm theory, one needs the notion of a dimension function on the lattice of projections.

In the case where $\mathcal{A}$ is a W^*-algebra, J Tomiyama [Tom58], provided such a notion. Olsen [Ols84] used this function to develop a complete index theory in W^*-algebras.

We give a short introduction to this theory. To avoid misunderstanding we will denote a W^*-algebra (von Neumann algebra) by $\mathcal{M}$.

If $\mathcal{Z}$ is the center of $\mathcal{M}$, with spectrum Ω, the abelian von Neumann algebra $\mathcal{Z}$ is identified with $C(\Omega)$ where the compact space Ω is hyperstonian, i.e. the closure of every open set in Ω is open and Ω admits the structure of a perfect Borel measure [Dix51]. Hence for every projection P in $\mathcal{Z}$, there corresponds a characteristic function of a unique clopen subset K in Ω. Since any von Neumann algebra can be decomposed into three different types, we can partition Ω into three clopen subsets Ω_i, $i = I, II, III$, such that $\mathcal{M}_{\chi\Omega_i}$ is of type i. Let

$$V_I := \{0\} \cup \mathbf{N} \cup \{\mathcal{N}/\mathcal{N} \text{ an infinite cardinal, } \mathcal{N} \leq \text{ dimension of } H\}\ ;$$

$$V_{II} := [0,\infty) \cup [\mathcal{N}/\mathcal{N} \text{ an infinite cardinal, } \mathcal{N} \leq \text{ dimension of } H\}\ ;$$

$$V_{III} := \{0\} \cup \{\mathcal{N}/\mathcal{N} \text{ an infinite cardinal, } \mathcal{N} \leq \text{ dimension } H\}.$$

Each V_i is compact when considered with the order topology. Let

$$F := \{f\ :\ \Omega \to V\ :\ f \text{ is continuous and } f(\Omega_i) \subset V_i \times \{i\}\},$$

where V denotes the disjoint union $\bigcup_{i\in\{I,II,III\}} V_i \times \{i\}$

The following theorem is due to Tomiyama [Tom58]

Theorem 1.5 *For any von Neumann algebra $\mathcal{M}$ on a Hilbert space H, there exists a function dim :$P(\mathcal{M}) \to F$ with the following properties:*

(a) $0 \leq$ *dim* $P \leq$ *dimension of* H, *for each* $P \in P(\mathcal{M})$ *and dim* $P = 0$ *iff* $P = 0$;

(b) $\dim P \leq \dim Q$ *if* $P \preceq Q$;

(c) for mutually orthogonal projections P *and* Q, $\dim (P+Q) = \dim P + \dim Q$

(d) for a central projection Z, $\dim (ZP) = Z \dim P$ *for each* $P \in P(\mathcal{M})$. □

In the classical index theory every Fredholm operator has index some positive or negative integer. In the general theory, in order to include 'negative' values we need to extend F to a larger class of continuous functions.

For V_i, let $-V_i = \{-a : a \in V_i\}$ with the natural ordering $-a \leq -b$ if $a \geq b$. Identify -0 with 0 and with the order topology on $-V_i \cup V_i$ we let $C_c(\Omega)$ be the set of all continuous functions f such that $f(\Omega_i) \subset -V_i \cup V_i$.

For any two functions f and g in $C_c(\Omega)$, we define addition as follows:

If $X = \{t \in \Omega : f(t) \neq -g(t)\}$ then we define

$$(f+g)(t) = \begin{cases} f(t) + g(t) & \text{on } \bar{X} \\ 0 & \text{on } \Omega \setminus \bar{X} \end{cases}$$

Then since $\bar{X}$ is clopen, addition is well defined and continuous.

For each $T \in \mathcal{M}$, define the map index: $\mathcal{M} \to C_c(\Omega)$ by

$$\text{index}(T) := \dim N_T - \dim N_{T^*}.$$

Notice that index$(T^*) = -$ index(T) and from Theorem 1.5 index$(T) \geq 0$ if and only if $N_{T^*} \preceq N_T$.

Let $\mathcal{I}$ be any norm closed two sided ideal in $\mathcal{M}$ and let $\pi_{\mathcal{I}} : \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{I}$ denote the canonical quotient map. An element $T \in \mathcal{M}$ is called left Fredholm relative to $\mathcal{I}$ if $\pi_{\mathcal{I}}(T)$ is left invertible in $\mathcal{M}/\mathcal{I}$. We denote $\Phi_\ell(\mathcal{M},\mathcal{I})$ by the class of left Fredholm elements. The class $\Phi_r(\mathcal{M},\mathcal{I})$ of right Fredholm elements is defined in the obvious similar way. Then by $\Phi(\mathcal{M},\mathcal{I}) := \Phi_\ell(\mathcal{M},\mathcal{I}) \cap \Phi_r(\mathcal{M},\mathcal{I})$ we will denote the class of Fredholm operators relative to $\mathcal{I}$.

Call an element $T \in \mathcal{M}$ semi-Fredholm relative to $\mathcal{I}$ if there exists a central projection $P \in \mathcal{M}$ such that

$$PT \in \Phi_\ell(P\mathcal{M}\ ,\ P\mathcal{I}) \text{ and } (I-P)T \in \Phi_r((I-P)\mathcal{M},(I-P)\mathcal{I}).$$

We denote this class by $\Phi_s(\mathcal{M},\mathcal{I})$. Note that $\Phi_s(\mathcal{M},\mathcal{I}) = \Phi_\ell(\mathcal{M},\mathcal{I}) \cup \Phi_r(\mathcal{M},\mathcal{I})$ when $\mathcal{M}$ is a factor i.e. the center $\mathcal{Z} = \mathbf{C}I$.

We give two examples to illustrate some of the notions above.

Examples 1.6

1. Let $\mathcal{N}_0$ be the first infinite cordinal number (i.e. the number corresponding to countable sets) and let $\mathcal{N}_1$ be the ordinal number of the continuum. Let $\mathcal{M} = \mathcal{B}(H)$ where H is of dimension $\mathcal{N}_1$. Then the center $\mathcal{Z}$ of $\mathcal{M}$ is $\mathbf{C}I$.

   Since $\mathcal{M}$ is of type I,

$$V = -V_1 \cup V_1 = \{-\mathcal{N}_1, -\mathcal{N}_0\} \cup \mathbf{Z} \cup \{\mathcal{N}_0, \mathcal{N}_1\}$$

and $C_c(\Omega) = V$. In this case index$(\Phi(\mathcal{M}, \mathcal{K}(H))) = \mathbf{Z}$ which implies the classical index theory.

2. Let $\mathcal{M} = \ell^\infty$ and $\mathcal{I} = c_0$. It is well known that $\mathcal{M}$ can be embedded as a W^*-subalgebra of $\mathcal{B}(\ell^2)$. Then $\mathcal{Z} = \ell^\infty = C(\beta\mathbf{N})$. For this type I algebra

$$V = \{-\mathcal{N}_0\} \cup \mathbf{Z} \cup \{\mathcal{N}_0\}$$

and $C_c(\beta\mathbf{N}) = C(\beta\mathbf{N}, V) = V^{\mathbf{N}}$.

□

We refer the interested reader for more detailed exposition of Fredholm theory relative to any closed ideal in $\mathcal{M}$ to [Ols84].

Remark 1.7 In the development of [Ols84], Olsen used the fact that the stonian space Ω has the structure of a perfect Borel measure. For more general AW^*-algebras we do not necessarily have that Ω is hyperstonian. However, it was shown by [Sas55] that for a general AW^*-algebra there exists a notion of a dimension function which has properties very similar to the result of Tomiyama. It is hence an interesting problem to investigate to which extend an index theory can be developed in general AW^*-algebras. This however will not be dealt with in this thesis.

Now if $\mathcal{I}$ is a compact ideal in $\mathcal{M}$ (i.e. $\mathcal{I}$ is contained in the norm closed ideal generated by the finite projections) the index map is invariant under perturbations by elements of $\mathcal{I}$, and constant on components of the open sets of Fredholm elements relative to $\mathcal{I}$.

We state without proofs the results from [Ols84] which will be used throughout this work. For $T \in \mathcal{M}$ and $\mathcal{I}$ any norm closed ideal in $\mathcal{M}$ let

$$m_{\mathcal{I}}(T) \;:=\; \inf \sigma(\pi_{\mathcal{I}}(|T|))$$

Clearly $m_{\mathcal{I}}(T) > 0$ if and only if $\pi_{\mathcal{I}}(|T|)$ is invertible in $\mathcal{M}/\mathcal{I}$. Hence by using the polar decomposition $m_{\mathcal{I}}(T) > 0$ if and only if $\pi_{\mathcal{I}}(T)$ is left invertible in $\mathcal{M}/\mathcal{I}$ if and only if $T \in \Phi_\ell(\mathcal{M}, \mathcal{I})$.

Proposition 1.8 [Ols84] *Let $\mathcal{I}$ be any closed ideal in $\mathcal{M}$. Then*

1) $T \in \Phi_\ell(\mathcal{M}, \mathcal{I})$ *if and only if* $m_{\mathcal{I}}(T) > 0$

2) $T \in \Phi_r(\mathcal{M}, \mathcal{I})$ *if and only if* $m_{\mathcal{I}}(T^*) > 0$

3) $T \in \Phi(\mathcal{M}, \mathcal{I})$ *if and only if* $m_{\mathcal{I}}(T) > 0$ *and* $m_{\mathcal{I}}(T^*) > 0$ *and in this case* $m_{\mathcal{I}}(T) = m_{\mathcal{I}}(T^*)$. □

Theorem 1.9 [Ols84] *Let $\mathcal{I}$ be a compact ideal in $\mathcal{M}$. Let T and S in $\mathcal{M}$ be such that $||T - S|| < m_{\mathcal{I}}(T)$ then $T, S \in \Phi_\ell(\mathcal{M}, \mathcal{I})$ with index $(T) =$ index (S).* □

Theorem 1.10 [Ols84] *Let $\mathcal{I}$ be a compact ideal in $\mathcal{M}$ and $K \in \mathcal{I}$. If $T \in \Phi_\ell(\mathcal{M}, \mathcal{I})$ then $T + K \in \Phi_\ell(\mathcal{M}, \mathcal{I})$ and index $(T + K) =$ index (T).* □

Remark 1.11 If the ideal $\mathcal{I}$ is not contained in the relative compact ideal, the index map does not have these desired properties. Olsen [Ols84] showed that for any norm closed ideal $\mathcal{I}$ there exists a unique central projection P such that $P\mathcal{I}$ is compact in $P\mathcal{M}$ and $(I-P)\,\mathcal{I}$ is completely non compact in $(I-P)\,\mathcal{M}$.

For the compact summand, the result above applies and for the completely non-compact part Olsen modified the index map to obtain an index with all the desired properties. Hence if $\mathcal{I}$ is any norm closed two-sided ideal in $\mathcal{M}$ and P the central projection, we define

$$\text{index}(T) := \ \text{index}_P(TP) + \ \text{index}_{(I-P)}((I-P)T)$$

where index_P (resp. $\text{index}_{(I-P)}$) is the index map on the compact (resp. completely non-compact) part.

Theorem 1.12 [Ols84, 7.1, 10.8] *Let $\mathcal{I}$ be any norm closed ideal in $\mathcal{M}$, and let $T, S \in \mathcal{M}$ be such that $||T-S|| < m_{\mathcal{I}}(T)$ then $T, S \in \Phi_\ell(\mathcal{M},\mathcal{I})$ and index $(T) =$ index (S).* □

Theorem 1.13 [Ols84] *Let $\mathcal{I}$ be any norm closed ideal in $\mathcal{M}$. The class $\Phi_s(\mathcal{M},\mathcal{I})$ is an open partial semigroup in $\mathcal{M}$ on which the index map is continuous and $\Phi_s(\mathcal{M},\mathcal{I})$ is norm dense in $\mathcal{M}$. Moreover if $T \in \mathcal{M}$, then T is the limit of elements from $\Phi_s(\mathcal{M},\mathcal{I})$ having the same index as T.* □

Chapter 2

NORM CLOSED IDEALS IN AW^*-ALGEBRAS

In the study of W^*-algebras and more generally an AW^*-algebra $\mathcal{A}$, the interplay between the algebra structure and the set of projections of $\mathcal{A}$ is very important. For such an algebra the ideals of $\mathcal{A}$ are subsets of central importance.

In this chapter we want to focus on the role on how projections can be used to characterise the elements in the ideals. Particularly we are interested in the study of norm-closed two-sided ideals in $\mathcal{A}$. As mentioned earlier it was shown by F. Wright [Wri54] that there exists a one to one correspondence between the closed ideals of $\mathcal{A}$ and the equivalence closed lattice ideals of $P(\mathcal{A})$, the so called p-ideals.

Here we explore the extend to which norm closed ideals in AW^*-algebras resemble the ideal of compact operators on a Hilbert space. This is a generalisation of the theory developed for W^*-algebras by Kaftal [Kaf77,Kaf78] and Ströh [Str89].

The following spectral type of characterisation for AW^*-algebras proved to be important in the case of W^*-algebras.

Theorem 2.1 *Let $\mathcal{I}$ be a norm-closed two-sided ideal in $\mathcal{A}$ and let $T \in \mathcal{A}$. Then the following statements are equivalent.*

1) $T \in \mathcal{I}$

2) For every $\varepsilon > 0$ there exists a spectral projection P_ε of $|T|$ such that $P_\varepsilon \in \mathcal{I}$ and $||T - TP_\varepsilon|| < \varepsilon$.

Proof From the polar decomposition in AW^*-algebras, Theorem 1.4, $T \in \mathcal{I}$ if and only if $|T| \in \mathcal{I}$. Clearly (2) implies (1), since $\mathcal{I}$ is a norm closed ideal. If (1) holds, then $|T| \in \mathcal{I}$ and by Proposition 1.2 for any $\varepsilon > 0$, there exists a positive $S \in \mathcal{A}$ such that $|T|S$ is a projection, say $|T|S = P_\varepsilon$, and $||T - TP_\varepsilon|| \leq |||T| - |T|P_\varepsilon|| < \varepsilon$. Hence $P_\varepsilon \in \mathcal{I}$ and therefore the result follows. □

For a closed ideal $\mathcal{I}$, let $\mathcal{I}_{finite} = \{T \in \mathcal{A} : R_T \in \mathcal{I}\}$. In the case of $\mathcal{A} = \mathcal{B}(H)$ and $\mathcal{I} = \mathcal{K}(H)$, it is clear that $\mathcal{I}_{finite}$ will correspond to the finite rank operators. Since the TP_ε in Theorem 2.1 is contained in $\mathcal{I}_{finite}$ this characterisation corresponds to the classical result which says that any compact operator on a Hilbert space can be approximated by finite rank operators.

Corollary 2.2 *Let $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$. Then $\mathcal{I} = \bar{\mathcal{I}}_{finite}$.* □

Recall from Chapter 1 that if $T \in \mathcal{A}$ is self-adjoint we may identify the AW^*-algebra generated by T and I with a $C(K)$ space where K is stonian. We denoted the spectral projections by $E_\lambda = E_{(-\infty,\lambda]}$ i.e. the projection corresponding the characteristic function of the clopen set $K_\lambda = K \backslash \overline{T^{-1}(\lambda,\infty)}$. Let $E_{(\lambda,\infty)} = I - E_\lambda$ and if $\mu < \lambda$, let $E_{(\mu,\lambda]}$ be the projection corresponding to the characteristic function of the clopen set $K_\lambda \cap \overline{T^{-1}(\mu,\infty)}$.

Theorem 2.3 *Let $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$ and let $T \in \mathcal{I}$ be self-adjoint, then*

$$E_\lambda \in \mathcal{I} \quad for\ every\ \lambda < 0$$

and

$$E_{(\lambda,\infty)} \in \mathcal{I} \quad for\ every\ \lambda > 0$$

Proof For $\lambda > 0$ let $U = \{t \in K : T(t) > \lambda\}$. Then $\bar{U}$ is a clopen set and $E_{(\lambda,\infty)}$ corresponds to the characteristic function of $\bar{U}$. Define $S(t) = \dfrac{1}{T(t)}$ for $t \in \bar{U}$ and $S(t) = 0$ for $t \in K \backslash \bar{U}$. Then as in the proof of Proposition 1.2 $ST = E_{(\lambda,\infty)}$. Since $T \in \mathcal{I}$ it follows that $E_{(\lambda,\infty)} \in \mathcal{I}$.

For $\lambda < 0$, let $K_\lambda = K \backslash \overline{T^{-1}(\lambda,\infty)}$. Then E_λ corresponds to the characteristic function of K_λ. Similarly, define

$$S(t) = \frac{1}{T(t)} \quad \text{on} \quad K_\lambda \quad \text{and} \quad S(t) = 0 \quad \text{for} \quad t \in \overline{T^{-1}(\lambda, \infty)}.$$

Hence $TS = E_\lambda$ and since $T \in \mathcal{I}$, it follows that $E_\lambda \in \mathcal{I}$. □

We give a characterisation of the essential spectrum of a positive element in an AW^*-algebra which will be used in further chapters. Let $\pi_\mathcal{I} : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the canonical quotient map and for any $T \in \mathcal{A}$, let

$$\sigma_\mathcal{I}(T) = \sigma(\pi_\mathcal{I}(T)).$$

A similar result for the case of W^*-algebras appears in [LSS95].

Theorem 2.4 *Let $\mathcal{I}$ be a closed ideal in an AW^*-algebra $\mathcal{A}$ and $T \in \mathcal{A}$ be self-adjoint. Then*

$$\sigma_\mathcal{I}(T) = \{\lambda \in \mathbf{R} \ : \ E_{(\lambda-\varepsilon, \lambda+\varepsilon]} \notin \mathcal{I} \quad \textit{for all } \varepsilon > 0\}.$$

Proof Suppose first $\lambda \notin \sigma_\mathcal{I}(T)$. Then $T_\lambda = T - \lambda I$ has an essential inverse $S \in \mathcal{A}$ i.e.

$$\pi_\mathcal{I}(T) \cdot \pi_\mathcal{I}(S) = \pi_\mathcal{I}(I). \quad \text{Let} \quad \varepsilon < \frac{1}{||\pi_\mathcal{I}(S)||}.$$

Since $T_\lambda E_{(\lambda-\varepsilon,\lambda+\varepsilon]} \leq \varepsilon E_{(\lambda-\varepsilon,\lambda+\varepsilon]}$, by Theorem 1.1, it follows that

$$\begin{aligned} ||\pi_{\mathcal{I}}(E_{(\lambda-\varepsilon,\lambda+\varepsilon]})|| &= ||\pi_{\mathcal{I}}(E_{(\lambda-\varepsilon,\lambda+\varepsilon]}T_\lambda S)|| \\ &\leq \varepsilon \; ||\pi_{\mathcal{I}}(S)|| \\ &< 1 \end{aligned}$$

But since $\pi_{\mathcal{I}}(E_{(\lambda-\varepsilon,\lambda+\varepsilon]})$ is a projection in $\mathcal{A}/\mathcal{I}$ of norm less than one, it follows that

$$\pi_{\mathcal{I}}(E_{(\lambda-\varepsilon,\lambda+\varepsilon]}) = \mathcal{I} \; .$$

Hence

$$E_{(\lambda-\varepsilon,\lambda+\varepsilon]} \in \mathcal{I}.$$

Conversely, suppose there exists an $\varepsilon > 0$ such that $E_{(\lambda-\varepsilon,\lambda+\varepsilon]} \in \mathcal{I}$. Again, view T as a continuous function on a stonian space K. Define

$$S(t) = \begin{cases} (T(t)-\lambda)^{-1} & \text{if} \quad t \in K_{\lambda-\varepsilon} \\ 0 & \text{if} \quad t \in K_{\lambda+\varepsilon} \cap \overline{T^{-1}(\lambda-\varepsilon,\infty)} \\ (T(t)-\lambda)^{-1} & \text{if} \quad t \in \overline{T^{-1}(\lambda+\varepsilon,\infty)} \end{cases}$$

Since all the sets in consideration are clopen, $S \in C(K)$ and

$$T_\lambda S = ST_\lambda = I - E_{(\lambda-\varepsilon,\lambda+\varepsilon]}.$$

Hence $\pi_{\mathcal{I}}(T_\lambda)\pi_{\mathcal{I}}(S) = \pi_{\mathcal{I}}(S)\pi_{\mathcal{I}}(T_\lambda) = \pi_{\mathcal{I}}(I)$, which implies that $\lambda \notin \sigma_{\mathcal{I}}(T)$. □

We now generalise part of [Ols84] Theorems 4.5 and 4.7 towards an AW^*-algebra setting.

Lemma 2.5 *Suppose T, S are two elements in any C^*-algebra $\mathcal{C}$. If T is positive and S is a left inverse for T, then S is a positive two-sided inverse for T.*

Proof By the Gelfand-Naimark theorem we may assume that T and S acts on some Hilbert space H. Since $ST = I, TS^* = I$ and it suffices to show that S is positive. This follows from the fact that

$$< Sx, x > \quad = \quad < STS^*x, x > \quad = \quad < TS^*x, S^*x >$$

for every $x \in H$. □

Theorem 2.6 *Let $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$. Then the following are equivalent:*

a) *T is left invertible mod $\mathcal{I}$.*

b) *$|T|$ is invertible mod $\mathcal{I}$.*

c) $0 \notin \sigma_{\mathcal{I}}(|T|)$.

Proof The implications b) $\Rightarrow$ a) and b) $\Leftrightarrow$ (c) are obvious. That a) $\Rightarrow$ b) follows from Lemma 2.5, for if $T = U|T|$ is the polar decomposition of T (Theorem 1.4) and S is a left inverse for T mod $\mathcal{I}$ then SU will be a left inverse for the positive element $|T|$ modulo $\mathcal{I}$. □

Corollary 2.7 *The following conditions are equivalent:*

a) *T is invertible mod $\mathcal{I}$.*

b) *Both $\sigma_{\mathcal{I}}(|T|)$ and $\sigma_{\mathcal{I}}(|T^*|)$ do not contain zero and inf $\sigma_{\mathcal{I}}(|T|) =$ inf $\sigma_{\mathcal{I}}(|T^*|)$.*

Proof Suppose T is invertible mod $\mathcal{I}$ and let $S \in \mathcal{A}$ be the two-sided inverse mod $\mathcal{I}$, i.e. there exists $K_1, K_2 \in \mathcal{I}$ such that $ST = I + K_1$ and $TS = I + K_2$. From Theorem 2.6 $0 \notin \sigma_{\mathcal{I}}(|T|)$ and if we let $T^* = V|T^*|$ be the polar decomposition of T^*, it follows from $S^*T^* = I + K_2^*$ in a similar way that $0 \notin \sigma_{\mathcal{I}}(|T^*|)$. a) $\Rightarrow$ b) then follows from the equality

$$\sigma_{\mathcal{I}}(|T|) \cup \{0\} = \sigma_{\mathcal{I}}(|T^*|) \cup \{0\}$$

The implication b) $\Rightarrow$ a) follows from the fact that if T has a left inverse mod $\mathcal{I}$ and a right inverse mod $\mathcal{I}$, the two inverses should be the same. □

By definition we have T is left invertible mod $\mathcal{I}$ if there exists an $S \in \mathcal{A}$ such that $ST - I \in \mathcal{I}$. In the classical characterisation theorem for operators that are Fredholm with respect to the compact operators, one can arrange these members of $\mathcal{K}(H)$ to be projections. We generalise this result in the following Theorem.

Theorem 2.8 *If T is left invertible mod $\mathcal{I}$ then there exists an $S \in \mathcal{A}$, $I - E \in \mathcal{I}$ such that $ST = E$.*

Proof It follows from Theorem 2.6 that $|T|$ is invertible mod $\mathcal{I}$ and from Theorem 2.4 we can find an $\varepsilon > 0$ such that $E_{[0,\varepsilon]} \in \mathcal{I}$. View $|T|$ as a continuous function on a stonian space K and define

$$S_0(t) = \begin{cases} (|T|(t))^{-1} & \text{if} \quad t \in \overline{|T|^{-1}(\varepsilon,\infty)} \\ 0 & \text{if} \quad t \in K \backslash \overline{|T|^{-1}(\varepsilon,\infty)} \end{cases}$$

Then

$$S_0|T| = E_{(\varepsilon,\infty)} = I - E_{[0,\varepsilon]}. \text{ Hence}$$

if we let $S = S_0 U^*$, where $T = U|T|$ is the polar decomposition of T, the result follows. □

From these results it seems natural to ask whether one can develop a complete Fredholm theory in an AW^*-algebra setting. However, the main task will be to introduce an index map which possesses all the desired properties. We have mentioned in Chapter 1, Remark 1.7 that it seems possible to define an index map by making use of the existence of a dimension function on $P(\mathcal{A})$ due to Sasaki [Sas55]. The theory of Olsen [Ols84] made extensive use of the fact that in case of a von Neumann algebra, the center has the structure of a perfect Borel measure. Hence in order to develop an index theory for general AW^*-algebra one should overcome this difficulty and explore the topological properties of our stonian space Ω.

We conclude this chapter by another application of our spectral theory in AW^*-algebras by showing that the lifting of algebraic elements in any AW^*-algebra is possible. Lifting of an element with a certain property from the quotient algebra to an element in the algebra with the same property became important during the mid seventees when Brown, Douglas and Fillmore [BDF73] started their famous programme of extension theory of C^*-algebras.

To explain the basic idea around lifting we consider the following exact sequence:

$$0 \to \mathcal{I} \xrightarrow{i} \mathcal{C} \xrightarrow{\pi_{\mathcal{I}}} \mathcal{C}/\mathcal{I} \to 0$$

where $\mathcal{I}$ is a closed ideal in a C^*-algebra $\mathcal{C}$, i the inclusion map and $\pi_{\mathcal{I}}$ the canonical quotient map. It was shown [BDF73] that in the case where $\mathcal{C} = \mathcal{B}(H)$, $\mathcal{I} = \mathcal{K}(H)$ and $T \in \mathcal{C}$ is essentially normal (i.e., $\pi_{\mathcal{I}}(T)$ is normal in $\mathcal{C}/\mathcal{I}$) one cannot in general find a compact operator K such that $T + K$ is normal in $\mathcal{C}$. Brown, Douglas and Fillmore provided precise conditions, relating to Fredholm theory, under which normality can be lifted. Today various references exist of results relating to liftings and we refer the reader to [AkP77], [BrP91], [Had95], [Ols77] and [Str94] etc.

We are interested in the recent results obtained by Hadwin [Had95] who studied the lifting of algebraic elements in a C^*-algebra setting. An element $a \in \mathcal{C}$ is called algebraic if there exists a complex polynomial f such that $f(a) = 0$. It was shown by Olsen and Pedersen [OlP89] that in the case $f(t) = t^n, n \geq 1$ and $b \in \mathcal{C}/\mathcal{I}$ is such that $f(b) = 0$ then there exists an $a \in \mathcal{C}$ with $f(a) = 0$ and $\pi_{\mathcal{I}}(a) = b$.

In [Had95] it was shown that the lifting of algebraic elements in general is not possible. For example if $f(t) = t^2 - t$, $\mathcal{C} = C[0,1]$ and $\mathcal{I} = \{g : g(0) = g(1) = 0\}$

then for the function $h(t) = t$ it follows that $f(\pi_{\mathcal{I}}(h)) = 0$. But, since $\mathcal{C}$ has no non-trivial idempotents it will not be possible to find a $q \in \mathcal{C}$ such that $\pi_{\mathcal{I}}(q) = \pi_{\mathcal{I}}(h)$ and $f(q) = 0$.

Hadwin showed that this lifting holds for a large class of C^*-algebras. For example if $\mathcal{C}$ has real rank zero then the lifting of algebraic elements is possible. Recall that a C^*-algebra has real rank zero if every self-adjoint element in $\mathcal{C}$ is a limit of self-adjoint elements with finite spectrum.

The following characterisation of real rank zero due to Brown and Pedersen [BrP91] will be the essential key to our result.

Theorem 2.9 *A C^*-algebra $\mathcal{C}$ has real rank zero if and only if for each pair of positive elements $a, b \in \mathcal{C}$ with $ab = 0$ and $\varepsilon > 0$ there exists a projection $p \in \mathcal{C}$ such that $||(1-p)a|| \leq \varepsilon$ and $||pb|| \leq \varepsilon$.* □

Theorem 2.10 *Any AW^*-algebra has real rank zero.*

Proof Let T and S be two positive elements of $\mathcal{A}$ such that $TS = 0$. Then by [Ber72], § 3 Proposition 9 and § 4 Proposition 7, $\{T, S\}''$ is a commutative AW^*-algebra and hence isomorphic to $C(K)$ where K is stonian. View S and T as functions on K and let $0 < \varepsilon < ||T||$ be given. Let P be the projection corresponding to the characteristic function of $\bar{U}$ where $U = \{t \in K \ : \ T(t) > \varepsilon\}$. It clearly follows that

$$0 \leq T(I-P) \leq \varepsilon(I-P) < \varepsilon I$$

and hence $$||T(I-P)|| \leq \varepsilon$$

Since $TS = 0$ it follows for any $t \in \bar{U}$ that $S(t) = 0$ which implies that $PS = 0$. □

From [Had95], Corollary 3 we obtain the following:

Theorem 2.11 *Let $\mathcal{A}$ be an AW^*-algebra, and $\mathcal{I}$ a closed two-sided ideal in $\mathcal{A}$. For $\pi_{\mathcal{I}}(T) \in \mathcal{A}/\mathcal{I}$, and f a polynomial such that $f(\pi_{\mathcal{I}}(T)) = 0$, there exists a $K \in \mathcal{I}$ such that $f(T+K) = 0$.* □

This directly solves an interesting problem of lifting projections in an AW^*-algebra setting. We have seen that this is not the case for a general C^*-algebra.

Corollary 2.12 *Any projection in $\mathcal{A}/\mathcal{I}$ has a projection lifting in $\mathcal{A}$.*

Proof Let $f(t) = t^2 - t$ and apply Theorem 2.11. □

Open problems 2.13

1. Is it possible to develop a meaningful index theory in the framework of AW^*-algebras?

2. We call $a \in \mathcal{A}/\mathcal{I}$ quasinilpotent if $\sigma(a) = \{0\}$, i.e. $r(a) = 0$, where r denotes the spectral radius. The question whether a quasinilpotent element of $\mathcal{A}/\mathcal{I}$

has a quasinilpotent lifting is still unresolved in general. For the case where $\mathcal{A} = \mathcal{B}(H)$ and $\mathcal{I} = \mathcal{K}(H)$ we have the well-known West decomposition theorem [Wes66], which states that any quasinilpotent element in $\mathcal{B}(H)/\mathcal{K}(H)$ has a quasinilpotent lifting. The method of proof of this result is analogous to the process of super-diagonalizing a matrix and then splitting it into the sum of diagonal and nilpotent matrices. Super-diagonalization of compact operators depends essentially on the existence of proper closed invariant subspaces.

The best result in this direction can be found in Rogers [Rog90]. In this paper Rogers proved that if $\mathcal{I}$ is any closed two-sided ideal in a C^*-algebra $\mathcal{C}$, then a quasinilpotent element $\pi_{\mathcal{I}}(T)$ of $\mathcal{C}/\mathcal{I}$ has a quasinilpotent lifting if $\sigma(T)$ is totally disconnected.

The open question mentioned would be answered in the affirmative if one could 'lift' the spectral radius for arbitrary elements of $\mathcal{A}/\mathcal{I}$. In other words we need to prove that for any $T \in \mathcal{A}$ there exists a $K \in \mathcal{I}$ such that $r(T+K) = r(\pi_{\mathcal{I}}(T))$, where $r(T)$ denotes the spectral radius of T. In [AkP77] Akemann and Pedersen proved this result for any T with $r(\pi_{\mathcal{I}}(T)) > 0$. Further partial answers to the question under consideration can be found in [StS91]. Even in the case of type II_∞ W^*-factors the question is to the best of our knowledge still open.

Chapter 3

THE REDUCED MINIMUM MODULUS IN OPERATOR ALGEBRAS

A number of operational quantities characteristic of the semi-Fredholm operators have been introduced in the theory of bounded operators between Banach spaces and applied successfully to for example perturbation theory.

Recall that for an operator T on a Banach space X, the reduced minimum modulus of T is defined by

$$\gamma(T) = \inf \{||Tx|| \ : \ d(x, N(T)) = 1\}$$

where $N(T)$ is the kernel of the operator T. It is well known that $\gamma(T) > 0$ if and only if T has closed range. In the study of operators and the quantities associated with them, the concept of stability under small perturbation is central. The word small may refer to an operator with small norm, or an operator belonging to some special class, such as the ideal of compact operators. One of the classical results in the perturbation theory of semi-Fredholm operators is the following stability theorem due to Kato [Kat66], that if T is semi-Fredholm and S is arbitrary with $||S|| < \gamma(T)$, then $T + S$ is also semi-Fredholm and the Fredholm index is preserved.

Zemánek [Zem84a], obtained asymptotic formulas for the semi-Fredholm radius of an operator in terms of various essential minimum moduli of the operator. Here the semi-Fredholm radius $s(T)$ is defined to be the supremum of all $\varepsilon > 0$ such that $T - \lambda I$ are semi-Fredholm for $|\lambda| < \varepsilon$. Now if we let

$$\gamma_\infty(T) = \sup \{\gamma(T + F) \ : \ \dim(\text{range}(F)) < \infty\},$$

it was shown by Zamánek [Zem84a] that

$$\lim_{n\to\infty} \gamma_\infty(T^n)^{\frac{1}{n}} = s(T),$$

hence showing that the perturbation theorem of Kato is asymptotically sharp.

In the Hilbert space setting these quantities have important spectral interpretations. In fact, C. Apostal [Apo85] obtained a useful spectral characterisation of γ, namely

$$\gamma(T) = \inf \{\sigma(|T|)\backslash\{0\}\}.$$

This characterisation provides a natural way of extending the concept of a reduced minimum modulus to the C^*-algebra setting, as was shown by Harte and

Mbekhta [HaM92,HaM93], in their study of generalised inverses in C^*-algebras. Now if $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ is the quotient map it is natural to consider the reduced minimum modulus of $\pi(T)$ in the Calkin algebra.

$$\text{i.e.} \quad \gamma_e(T) = \gamma(\pi(T)) = \inf\{\sigma(\pi(|T|))\backslash\{0\}\}.$$

This quantity is called the reduced essential minimum modulus. Mbekhta and Paul [MbP96], proved that $\gamma_e(T) = \sup\{\gamma(T+K) : K \in \mathcal{K}(H)\}$ and gave a beautiful argument to show that the supremum is attained. These results were independently obtained by Ströh [Str94], in his study of the lifting of regular elements in C^*-algebras.

In this chapter we study the role of the reduced minimum modulus in a von Neumann algebra setting. One of the main results gives a similar relation as in [Str94] between the reduced essential minimum modulus and the minimum modulus of an element in the algebra. To be more precise, if $\mathcal{I}$ is any norm closed ideal in $\mathcal{M}$, then

$$\gamma(\pi_{\mathcal{I}}(T)) = \sup_{K \in \mathcal{I}} \gamma(T+K).$$

In [Str94], it was not known whether the supremum in the above formula is attained. The main obstacle with the argument presented in [Str94] is the absence of a Stampfli type of decomposition in general operator algebras. i.e. For $T \in \mathcal{M}$, can one find a $K \in \mathcal{I}$ such that

$$\sigma(T+K) = \sigma(\pi(T)).$$

A positive solution of this result will solve a long standing open problem in W^*-algebras, namely that lifting of quasinilpotent elements from $\mathcal{M}/\mathcal{I}$ to $\mathcal{M}$ is possible [Rog90].

However, we show by using different spectral techniques that in any von Neumann algebra relative to any closed two-sided ideal the supremum in our formula is attained. We show that basically all the perturbation results in Mbekhta and Paul [MbP96] extend to a von Neumann algebra setting and give a complete characterisation of semi-Fredholm elements in terms of the points of continuity of the map: $T \to \gamma(\pi_{\mathcal{I}}(T))$.

Lemma 3.1 [Str94, Theorem 7] *Let $\mathcal{I}$ be any closed two-sided ideal in $\mathcal{M}$. Then for any $T \in \mathcal{M}$,*

$$\gamma_{\mathcal{I}}(T) = \sup_{K \in \mathcal{I}} \gamma(T+K).$$

Proof Since $|\pi(T)| = \pi(|T|) = \pi(|T+K|)$ for all $K \in \mathcal{I}$, it follows that $\sigma(|\pi(T)|) \subset \sigma(|T+K|)$ for all $K \in \mathcal{I}$. Hence

$$\sup_{K \in \mathcal{I}} \gamma(T+K) \leq \gamma(\pi(T)).$$

Conversely, we may assume without loss of generality that $\gamma(\pi(T)) > 0$. Hence for any interval $[\alpha, \beta) \subset (0, \gamma(\pi(T)))$, using Theorem 2.4 we have $E_{[\alpha,\beta)} \in \mathcal{I}$. From the inequality $||TE_{[0,\beta)} - TE_{[\alpha,\beta)}|| \leq \alpha$ it is clear that $TE_{[0,\beta)} \in \mathcal{I}$ for any

$\beta \in (0, \gamma(\pi(T)))$, and hence $\gamma(T - TE_{[0,\beta)}) \geq \beta$. From this it follows that

$$\sup_{K \in \mathcal{I}} \gamma(T + K) \geq \gamma(\pi(T)).$$

□

Theorem 3.2 *Let $T \in \mathcal{M}$. Then there exists a $K \in \mathcal{I}$ such that $\gamma_{\mathcal{I}}(T) = \gamma(T + K)$.*

Proof If $\gamma_{\mathcal{I}}(T) = 0$ we are done, since then $\gamma(T) = \gamma_{\mathcal{I}}(T) = 0$ and therefore the theorem will hold for the choice $K = 0$. Suppose that $\gamma_{\mathcal{I}}(T) > 0$. Then, by Lemma 3.1 there is a $K_1 \in \mathcal{I}$ such that $\gamma(T + K_1) > 0$. Hence we may assume without loss of generality that $\gamma_{\mathcal{I}}(T) > 0$ and $\gamma(T) > 0$. Let $P = E_{(0,\gamma_{\mathcal{I}}(T))} = E_{[\gamma(T),\gamma_{\mathcal{I}}(T))}$ and let $K_0 = \gamma_{\mathcal{I}}(T)P - |T|P$, where E denotes the spectral measure for $|T|$. We want to show that $K_0 \in \mathcal{I}$.

By an application of Theorem 2.4 it follows that $t \in \sigma_{\mathcal{I}}(|T|)$ if and only if $E_{[t-\varepsilon,t+\varepsilon]} \notin \mathcal{I}$ for every $\varepsilon > 0$. Hence for any $0 < \alpha < \beta < \gamma_{\mathcal{I}}(T)$ it follows by a compactness argument that $E_{[\alpha,\beta]} \in \mathcal{I}$. Now for any choice of $\beta > 0$ such that $\gamma(T) < \beta < \gamma_{\mathcal{I}}(T)$, it follows from

$$||K_0 - \int_{\gamma(T)}^{\beta} (\gamma_{\mathcal{I}}(T) - \lambda) dE_\lambda|| = ||\int_{\beta}^{\gamma_{\mathcal{I}}(T)} (\gamma_{\mathcal{I}}(T) - \lambda) dE_\lambda|| \leq \gamma_{\mathcal{I}}(T) - \beta$$

and

$$\int_{\gamma(T)}^{\beta} (\gamma_{\mathcal{I}}(T) - \lambda) dE_\lambda = \gamma_{\mathcal{I}}(T) E_{[\gamma(T),\beta]} - |T| E_{[\gamma(T),\beta]} \in \mathcal{I}$$

that $K_0 \in \mathcal{I}$. Furthermore, since $|T| = (I - P)|T| + \gamma_{\mathcal{I}}(T)P - K_0$, we have

$$\begin{aligned}
\gamma_{\mathcal{I}}(T) &= \gamma_{\mathcal{I}}(|T|) \\
&= \gamma_{\mathcal{I}}((I-P)|T| + \gamma_{\mathcal{I}}(T)P) \\
&= \gamma((I-P)|T| + \gamma_{\mathcal{I}}(T)P) \\
&= \gamma(|T| + K_0).
\end{aligned}$$

Let $T = U|T|$ be the polar decomposition of T and let $K = UK_0$. Then, since

$$\begin{aligned}
|T+K|^2 &= (T+K)^*(T+K) \\
&= T^*T + T^*K + K^*T + K^*K \\
&= |T|^2 + |T|U^*UK_0 + K_0U^*U|T| + K_0U^*UK_0 \\
&= |T|^2 + |T|R_{|T|}K_0 + K_0R_{|T|}|T| + K_0R_{|T|}K_0 \\
&= |T|^2 + |T|K_0 + K_0|T| + K_0^2 \\
&= (|T| + K_0)^2,
\end{aligned}$$

it follows from the uniqueness of the square root of a positive element in a C^*-algebra that $|T+K| = |T| + K_0$. Hence $\gamma_{\mathcal{I}}(T) = \gamma(T+K)$. □

Corollary 3.3 *If T is normal then, for every $n \geq 1$, there exists a $K \in \mathcal{I} \cap \{T\}'$ such that*

$$\gamma_{\mathcal{I}}(T^n) = \gamma((T+K)^n).$$

Proof By a consequence of the functional calculus for a normal element T, it follows that if we choose K as in Theorem 3.2, then T and K commute. Direct computation shows that $T+K$ is normal, and by the uniqueness of the square root it again follows that $|(T+K)^n| = |T+K|^n$ and $|\pi(T)^n| = |\pi(T)|^n$. Applying the spectral mapping theorem, we obtain

$$\gamma_{\mathcal{I}}(T^n) = \gamma_{\mathcal{I}}(T)^n = \gamma(T+K)^n = \gamma((T+K)^n). \qquad \square$$

Theorem 3.4 *Let $T \in \mathcal{M}$. Then T is left Fredholm relative to $\mathcal{I}$ if and only if $N_T \in \mathcal{I}$ and there exists a $B \in \mathcal{M}$ such that $R_B \in \mathcal{I}$ and $\gamma(T+B) > 0$.*

Moreover, if $||T-S|| < \gamma_{\mathcal{I}}(T)$ then S is left Fredholm and index$(S) =$ index(T).

Proof If T is left Fredholm, then clearly $|T|$ is Fredholm, which implies that $0 \notin \sigma_{\mathcal{I}}(|T|)$. Hence $\gamma_{\mathcal{I}}(T) = \gamma_{\mathcal{I}}(|T|) > 0$. Let E be the spectral measure with respect to $|T|$, then $N_T = E_{[0]} \in \mathcal{I}$. In fact, an argument similar to that in the proof of Theorem 3.2 shows that for any $0 < \beta < \gamma_{\mathcal{I}}(T), E_{[0,\beta]} \in \mathcal{I}$. Let $B = -TE_{[0,\beta]}$, then $R_B = E_{(0,\beta]} \leq E_{[0,\beta]} \in \mathcal{I}$ and

$$\gamma(T+B) = \gamma(\int_{\beta}^{\infty} \lambda dE_\lambda) \geq \beta > 0.$$

Conversely, suppose that $N_T = E_{[0]} \in \mathcal{I}$ and that there exists a $B \in \mathcal{M}$ such that $R_B \in \mathcal{I}$ and $\gamma(T+B) > 0$. Then $\gamma_{\mathcal{I}}(T) = \gamma_{\mathcal{I}}(T+B) \geq \gamma(T+B) > 0$ and $E_{[0]} \in \mathcal{I}$ implies $0 \notin \sigma_{\mathcal{I}}(|T|)$, which implies that $|T|$ is Fredholm. Hence T is left Fredholm.

To obtain the last statement of our theorem, let $||T-S|| < \gamma_{\mathcal{I}}(T)$. By Remark 1.11 it was shown that one can reduce the Fredholm and index perturbation theory to two special cases by making use of a central decomposition. Now a direct application of Theorem 1.12 it follows that S is left Fredholm relative to $\mathcal{I}$ and that $\text{index}(S) = \text{index}(T)$. $\square$

Harte and Mbekhta [HaM92; HaM93] introduced the notion of regular elements in a general C^*-algebra setting. If $\mathcal{C}$ denotes a C^*-algebra, an element $a \in \mathcal{C}$ is called regular if there exists a $b \in \mathcal{C}$ such that $a = aba$. For operator T on a Hilbert space, this relation is equivalent to the closedness of the range of T. Note that ab and ba are idempotents. It is shown [HaM92; HaM93] that if $a \in \mathcal{C}$ is regular then there exists a unique (Moore-Penrose) generalised inverse a^+ such that aa^+ and a^+a are orthogonal projections in $\mathcal{C}$. In fact, it is shown [Str94] that if T is an element of a von Neumann algebra then $TT^+ = R_T$ and $T^+T = R_{T^*}$.

For a general C^*-algebra $\mathcal{C}$, Harte and Mbekhta [HaM92; HaM93] defined the reduced minimum modulus by using the left regular representation i.e. if $a \in \mathcal{C}$ and $L_a : \mathcal{C} \to \mathcal{C}$ is defined by $L_a(b) = ab$ then $\gamma(a)$ is the reduced minimum modulus of the operator L_a acting as a Banach space operator on $\mathcal{C}$. It was shown [HaM93], [Str94] that $\gamma(a) = \inf \sigma(|a|)\backslash\{0\}$ where $|a| = (a^*a)^{\frac{1}{2}}$. As in the classical theory one has the following proposition which can be found in [HaM92].

Proposition 3.5 *Let $\mathcal{C}$ be a C^*-algebra and $a \in \mathcal{C}$. Then a is regular if and only if $\gamma(a) > 0$. Moreover if a is regular then $||a^+||\gamma(a) = 1$.* $\square$

From this proposition and Theorem 3.2 the following result holds.

Theorem 3.6 *If $T \in \mathcal{M}$ is regular modulo $\mathcal{I}$ (i.e. $\pi_{\mathcal{I}}(T)$ is regular) then there exists a $K \in \mathcal{I}$ such that $T + K$ is regular.* □

Remark 3.7 We illustrate with an example that the lifting of regular elements in a C^*-algebra setting does not hold in general. Let

$$\mathcal{C} = C[0,1] \text{ and } \mathcal{I} = \{f \in \mathcal{C} \ : \ f(0) = f(1) = 0\}.$$

If we let $f(x) = x$, then $f + \mathcal{I}$ is a non-trivial idempotent in $\mathcal{C}/\mathcal{I}$ and hence regular. Moreover, the nonzero regular elements in $\mathcal{C}$ are exactly the invertible functions. Any perturbation of f by an element in $\mathcal{I}$ will have the value zero at the point $x = 0$. This implies that there exists no $g \in \mathcal{I}$ such that $f + g$ is regular in $\mathcal{C}$. □

Theorem 3.8 [HaM93, Theorem 5] *Let $T, S \in \mathcal{M}$ be regular, then*

$$S^+ - T^+ = -S^+(S-T)T^+ + S^+S^{*+}(S^* - T^*)(I - TT^+) + (I - S^+S)\ (S^* - T^*)T^{*+}T^+$$

and if both T and S are non-zero, then

$$||S^+S - T^+T|| < 1 \Rightarrow |\ \gamma(S) - \gamma(T)| \leq ||S - T||.$$

Proof View $\mathcal{M}$ as a concrete von Neumann algebra on some Hilbert space H. It was shown in [LSS95], Proposition 1.2 that for any $T \in \mathcal{M}$

$$\gamma(T) = \sup\{\lambda : ||Tx|| \geq \lambda \ \text{dist}(x, N(T)) \text{ for all } x \in H\}.$$

Since $T = TT^+T$; $T^+ = T^+TT^+$, $(T^+T)^* = T^+T$, $(TT^+)^* = TT^+$ and $(T^*)^+ = (T^+)^*$, we have

$$S^+S^{*+}(S^* - T^*)(I - TT^+) = S^+(I - TT^+)$$

and

$$(I - S^+S)(S^* - T^*)T^{*+}T^+ = -(I - S^+S)T^+.$$

Hence

$$\begin{aligned} S^+ - T^+ + S^+(S-T)T^+ &= S^+(I - TT^+) - (I - S^+S)T^+ \\ &= S^+S^{*+}(S^* - T^*)(I - TT^+) - (I - S^+S)T^+. \end{aligned}$$

Thus $S^+ - T^+ = -S^+(S-T)T^+ + S^+S^{*+}(S^* - T^*)(I - TT^+)$

$$+(I - S^+S)(S^* - T^*)T^{*+}T^+$$

Let $C = (I + T^+T - S^+S)^{-1}$ and $x \in H$. Then since for any element $B \in \mathcal{M}$,

$$\begin{aligned} ||BB^+|| &= ||B^+B|| = 1 \text{ and} \\ \text{dist}(x, N(B)) &= \text{ dist } (R_{B^*}x, N(B)) \leq ||R_{B^*}x|| \\ &\leq ||R_{B^*}|| \text{ dist } (x, N(B)) \end{aligned}$$

we have

$$\begin{aligned} \text{dist } (T^+TCx, N(T)) = ||T^+TCx|| &= ||S^+Sx|| \\ &= \text{dist } (x, N(S)) \end{aligned}$$

which implies

$$||Sx|| \geq ||TT^+TCx|| - ||S - T|| \; ||T^+TCx||$$

$$\geq (\gamma(T) - ||S - T||)\ \mathrm{dist}(x, N(S)).$$

This gives

$$\gamma(S) \geq \gamma(T) - ||S - T|| \qquad \square$$

Lemma 3.9 *If T is left Fredholm and $||T - S|| < \gamma_{\mathcal{I}}(T)$, then*

$$|\gamma_{\mathcal{I}}(T) - \gamma_{\mathcal{I}}(S)| \leq ||T - S||.$$

Proof Since T is left Fredholm, $\pi(T)$ is left invertible. Let $\pi(T)^+$ be the left inverse of $\pi(T)$ in $\mathcal{M}/\mathcal{I}$. From Proposition 3.5 it follows that $\gamma_{\mathcal{I}}(T) = ||\pi(T)^+||^{-1}$. Now let $S \in \mathcal{M}$ be such that $||T - S|| < \gamma_{\mathcal{I}}(T) = ||\pi(T)^+||^{-1}$. Then

$$||\pi(T) - \pi(S)|| \leq ||T - S|| < ||\pi(T)^+||^{-1},$$

which implies that

$$||\pi(T)^+(\pi(T) - \pi(S))\ || < 1.$$

Hence

$\pi(I) - \pi(T)^+(\pi(T) - \pi(S)) = \pi(T)^+\pi(S)$ is invertible in $\mathcal{M}/\mathcal{I}$. As

$[\pi(T)^+\pi(S)]^{-1}\pi(T)^+\pi(S) = \pi(I)$,

it follows that $\pi(S)$ is left invertible. Let $\pi(S)^+$ denote its left inverse. Since

$$||\pi(S)^+\pi(S) - \pi(T)^+\pi(T)|| = 0 < 1$$

it follows from Theorem 3.8 that

$$|\gamma_{\mathcal{I}}(T) - \gamma_{\mathcal{I}}(S)| \leq ||\pi(T) - \pi(S)|| \leq ||T - S||. \qquad \square$$

Lemma 3.10 *Let $Q \in P(\mathcal{M})$ be a central projection and $P = I - Q$. Then*

$$\gamma_{\mathcal{I}}(T) = \ min\{\gamma_{\mathcal{I}Q}(TQ), \gamma_{\mathcal{I}P}(TP)\}.$$

Proof Let $T \in \mathcal{M}$ be such that $\pi(T)$ is invertible in $\mathcal{M}/\mathcal{I}$, i.e. there exists an $S \in \mathcal{M}$ such that

$$\pi(T)\pi(S) = \pi(S)\pi(T) = \pi(I).$$

Hence

$$\pi(PTP)\pi(PSP) = \pi(P)\pi(T)\pi(S)\pi(P) = \pi(P)$$

and similarly

$\pi(QTQ)\pi(QSQ) = \pi(Q)$, which imply that $\pi(PTP)$ is invertible in $\mathcal{M}P/\mathcal{I}P$ and

$\pi(QTQ)$ is invertible in $\mathcal{M}Q/\mathcal{I}Q$.

Thus

$$\rho_{\mathcal{I}}(T) \subset \rho_{\mathcal{I}P}(TP) \cap \rho_{\mathcal{I}Q}(TQ).$$

Now suppose that

$$\pi(TP)\pi(SP) = \pi(SP)\pi(TP) = \pi(P)$$

and

$$\pi(TQ)\pi(RQ) = \pi(RQ)\pi(TQ) = \pi(Q).$$

Then

$$\pi(T)(\pi(PSP) + \pi(QRQ)) = \pi(PT)\pi(SP) + \pi(QT)\pi(QR) = \pi(P) + \pi(Q) = \pi(\mathcal{I}).$$

Hence

$$\rho_{\mathcal{I}P}(TP) \cap \rho_{\mathcal{I}Q}(TQ) \subset \rho_{\mathcal{I}}(T).$$

In particular,

$$\sigma_{\mathcal{I}}(|T|) = \sigma_{\mathcal{I}P}(|T|P) \cup \sigma_{\mathcal{I}Q}(|T|Q) = \sigma_{\mathcal{I}P}(|TP|) \cup \sigma_{\mathcal{I}Q}(|TQ|).$$

Hence

$$\begin{aligned} \gamma_{\mathcal{I}}(T) &= \inf(\sigma_{\mathcal{I}}(|T|)\backslash\{0\}) \\ &= \inf(\sigma_{\mathcal{I}P}(|TP|)\backslash\{0\} \cup \sigma_{\mathcal{I}Q}(|TQ|)\backslash\{0\}) \\ &= \min\{\gamma_{\mathcal{I}P}(TP), \gamma_{\mathcal{I}Q}(TQ)\}. \end{aligned}$$

□

Theorem 3.11 *If T is semi-Fredholm, then T is a point of continuity of $\gamma_{\mathcal{I}}(\cdot)$.*

Proof Let T be semi-Fredholm and let $T_n \to T$. Since T is semi-Fredholm there exists a central projection P such that TP is left Fredholm in $\mathcal{M}P/\mathcal{I}P$ and $T(I-P)$ is right Fredholm in $\mathcal{M}(I-P)/\mathcal{I}(I-P)$. Clearly $T_nP \to TP$. Thus given any $\varepsilon > 0$, there exists an N such that $||T_nP - TP|| < \min(\varepsilon, \gamma_{\mathcal{I}P}(TP))$ for all $n \geq N$. By Lemma 3.9, $|\gamma_{\mathcal{I}P}(TP) - \gamma_{\mathcal{I}P}(T_nP)| < ||TP - T_nP|| < \varepsilon$ for all $n \geq N$, i.e. $\gamma_{\mathcal{I}P}(T_nP)$ converges to $\gamma_{\mathcal{I}P}(TP)$. By taking adjoints we see that $T^*(I - P)$ is left Fredholm, and by using the fact that the minimum modulus preserves adjoints we deduce that

$$\gamma_{\mathcal{I}(I-P)}(T_n(I - P)) \to \gamma_{\mathcal{I}(I-P)}(T(I - P)).$$

Since

$$\gamma_{\mathcal{I}}(T) = \min\{\gamma_{\mathcal{I}P}(TP), \gamma_{\mathcal{I}(I-P)}(T(I-P))\}$$

it follows that

$\gamma_{\mathcal{I}}(T_n)$ converges to $\gamma_{\mathcal{I}}(T)$. Hence T is a point of continuity of $\gamma_{\mathcal{I}}(\cdot)$. □

Theorem 3.12 *If $T \in \mathcal{M}$ is a point of continuity of $\gamma_{\mathcal{I}}(\cdot)$ and $\gamma_{\mathcal{I}}(T) > 0$, then T is semi-Fredholm.*

Proof If $\gamma_{\mathcal{I}}$ is continuous at T and $\gamma_{\mathcal{I}}(T) > 0$ then there exists an $\alpha > 0$ such that if $||T - S|| < \alpha$ then $\gamma_{\mathcal{I}}(S) \geq \gamma_{\mathcal{I}}(T)/2 := \delta$. Since the set of semi-Fredholm elements is uniformly dense in $\mathcal{M}$, Theorem 1.13, there exists a sequence (T_n) of semi-Fredholm elements such that T_n converges to T. Let $\varepsilon = \min\{\alpha, \delta\} > 0$. Then there exists an N such that $||T_N - T|| < \varepsilon$, which implies that $\gamma_{\mathcal{I}}(T_N) \geq \delta$ and $||T_N - T|| < \delta$. Since T_N is semi-Fredholm there exists a central projection P such that $T_N P$ is left Fredholm and $T(I - P)$ is right Fredholm. Note that

$$\gamma_{\mathcal{I}P}(T_N P) \geq \gamma_{\mathcal{I}}(T_N) \geq \delta \text{ and}$$

$$||T_N P - TP|| \leq ||T_N - T|| < \delta \leq \gamma_{\mathcal{I}}(T_N) \leq \gamma_{\mathcal{I}P}(T_N P).$$

Hence Theorem 1.12 implies that TP is left Fredholm. On the other hand, from

$$||T_N^*(I-P) - T^*(I-P)|| \leq ||T_N^* - T^*|| < \delta \leq \gamma_{\mathcal{I}}(T_N^*) < \gamma_{\mathcal{I}(I-P)}(T_N^*(I-P))$$

it follows that $T(I - P)$ is right Fredholm. Thus T is semi-Fredholm. □

Theorem 3.13 *An element $T \in \mathcal{M}$ is a point of continuity of $\gamma_{\mathcal{I}} : \mathcal{M} \to [0, \infty]$ if and only if either $\gamma_{\mathcal{I}}(T) = 0$ or T is semi-Fredholm.*

Proof Suppose that $T \in \mathcal{M}$ is a point of continuity of $\gamma_{\mathcal{I}}$. Then either $\gamma_{\mathcal{I}}(T) = 0$ or $\gamma_{\mathcal{I}}(T) > 0$. If $\gamma_{\mathcal{I}}(T) > 0$ then Theorem 3.12 implies that T is semi-Fredholm.

Conversely, if $\gamma_{\mathcal{I}}(T) = 0$ and (T_n) is any sequence in $\mathcal{M}$ that converges to T, then $\limsup_n \gamma_{\mathcal{I}}(T_n) \leq \gamma_{\mathcal{I}}(T)$ by Theorem 7 of [HaM93]. Hence $\gamma_{\mathcal{I}}(T_n) = \gamma_{\mathcal{I}}(T) = 0$ for every n, which implies that T is a point of continuity of $\gamma_{\mathcal{I}}$. Alternatively, if T is semi-Fredholm, then it follows from Theorem 3.11 that T is a point of continuity. □

It is worth noting that Theorem 3.4 holds for semi-Fredholm elements in $\mathcal{M}$.

Theorem 3.14 *Let $T \in \Phi_s(\mathcal{M}, \mathcal{I})$ and let $S \in \mathcal{M}$ be such that $||T - S|| < \gamma_{\mathcal{I}}(T)$. Then $S \in \Phi_s(\mathcal{M}, \mathcal{I})$ and index$(S) =$ index(T).*

Proof Let P be a central projection such that $TP \in \Phi_\ell(\mathcal{M}P, \mathcal{I}P)$ and $T(I-P) \in \Phi_r(\mathcal{M}(I-P), \mathcal{I}(I-P))$. Then from Lemma 3.9 $||TP - SP|| < \gamma_{\mathcal{I}P}(TP)$ and from Theorem 3.4 it follows that $SP \in \Phi_\ell(\mathcal{M}P, \mathcal{I}P)$ with $\text{index}_P(SP) = \text{index}_P(TP)$. Similarly $S(I-P) \in \Phi_r(\mathcal{M}(I-P), \mathcal{I}(I-P))$, with $\text{index}_{(I-P)}(S(I-P)) = \text{index}_{(I-P)}(T(I-P))$. Hence $S \in \Phi_s(\mathcal{M}, \mathcal{I})$ and $\text{index}(S) = \text{index}(T)$. □

Corollary 3.15 *Let $T \in \mathcal{M}$ be such that $\gamma_{\mathcal{I}}(T) > 0$. Then the following conditions are equivalent:*

(a) T is a point of continuity of $\gamma_{\mathcal{I}}(\cdot)$.

(b) T is semi-Fredholm.

(c) There exist $\alpha, \beta > 0$ such that for any $S \in \mathcal{M}$,

$||T - S|| < \alpha$ implies $\gamma_{\mathcal{I}}(S) \geq \beta$.

(d) There exists $\alpha, \beta > 0$ such that for any $S \in \mathcal{M}$,

$||T - S|| < \alpha$ implies $(0, \beta) \subset \rho_{\mathcal{I}}(|S|)$

(e) There exists $\alpha, \beta > 0$ such that for any $S \in \mathcal{M}$,

$||T - S|| < \alpha$ implies $(0, \beta) \subset \rho_{\mathcal{I}}(|S^*|)$.

Proof (a) $\Leftrightarrow$ (b) follows from Theorem 3.13.

(b) $\Rightarrow$ (c) Let $0 < \alpha < \gamma_{\mathcal{I}}(T)$. Then, if $||T - S|| < \alpha$, S is semi-Fredholm by Theorem 3.14. Hence $\gamma_{\mathcal{I}}(S) = m_{\mathcal{I}}(S) > 0$ and we may choose $0 < \beta \leq \gamma_{\mathcal{I}}(S)$.

(c) $\Rightarrow$ (d) If $\gamma_{\mathcal{I}}(S) = \inf\,(\sigma(\pi(|S|))\backslash\{0\}) \geq \beta$ then $(0, \beta) \subset \rho_{\mathcal{I}}(|S|)$

(d) $\Leftrightarrow$ (e) follows from $\gamma_{\mathcal{I}}(S) = \gamma_{\mathcal{I}}(S^*)$

(d) $\Rightarrow$ (b): From Theorem 1.13 $\Phi_s(\mathcal{M}, \mathcal{I})$ is norm dense in $\mathcal{M}$, hence we can find a semi-Fredholm element $S \in \mathcal{M}$ such that $||T - S|| < \min\{\alpha, \beta\}$. Thus by our hypothesis we have $\beta \leq \gamma_{\mathcal{I}}(S)$ which implies that $||T - S|| < \gamma_{\mathcal{I}}(S)$. Since S is semi-Fredholm there exists a central projection $P \in P(\mathcal{M})$ such that

$$SP \in \Phi_\ell(\mathcal{M}P, \mathcal{I}P) \text{ and } S(I-P) \in \Phi_r(\mathcal{M}(I-P), \mathcal{I}(I-P))$$

Also from Lemma 3.10 it follows that

$$||TP - SP|| \leq ||T - S|| < \gamma_{\mathcal{I}}(S) \leq \gamma_{\mathcal{I}P}(SP)$$

Hence by Theorem 3.4, $TP \in \Phi_\ell(\mathcal{M}P, \mathcal{I}P)$

Similarly we can show that $T(I-P) \in \Phi_r(\mathcal{M}(I-P), \mathcal{I}(I-P))$

Thus $T \in \Phi_s(\mathcal{M}, \mathcal{I})$. □

Another interesting fact to note is that in Lemma 3.1 $\gamma_{\mathcal{I}}(T)$ can not be obtained by taking the supremum only over all perturbation from the ideal $\mathcal{I}_{finite}$. This was observed for the case where $\mathcal{M} = \mathcal{B}(H)$ and $\mathcal{I} = \mathcal{K}(H)$ in [MbP96], Example 1.

We give a condition for which such a result holds in a general von Neumann algebra setting. This generalises Theorem 7 of Mbekhta and Paul [MbP96].

Theorem 3.16 *Let $T \in \mathcal{M}$ such that $\gamma(T) > 0$, then*

$$\gamma_{\mathcal{I}}(T) = \ sup\{\gamma(T+K) : \ K \in \mathcal{I}_{finite}\}.$$

Proof This result follows by the same arguments used in proving Lemma 3.1. For, if $\gamma(T) > 0, \ \gamma_{\mathcal{I}}(T) > 0$ from which we have shown that for any $\beta \in (0, \gamma_{\mathcal{I}}(T))$ we have $\gamma(T - TE_{[0,\beta)}) \geq \beta$ and $TE_{[0,\beta)} \in \mathcal{I}_{finite}$ Hence sup $\{\gamma(T+K) : \ K \in \mathcal{I}_{finite}\} \geq \gamma_{\mathcal{I}}(T)$ and since the converse inequality is clear the result follows. □

For general $T \in \mathcal{M}$, let $\gamma_\infty(T) = \sup\{\gamma(T+K) : K \in \mathcal{I}_{finite}\}$.

With regards to the asymptotic behaviour of $\gamma_{\mathcal{I}}(T)$ we have the following generalisation of Zemánek [Zem84a]. Let

$$\sigma_{\Phi_s}(T) = \{\lambda \in \mathbf{C} \ : \ T - \lambda I \notin \Phi_s(\mathcal{M}, \mathcal{I})\}$$

and we define the semi-Fredholm radius

$$s_{\mathcal{I}}(T) \ := \ \sup\{\varepsilon > 0 : \ T - \lambda I \in \Phi_s(\mathcal{M}, \mathcal{I}) \text{ for } |\lambda| < \varepsilon\}$$

Theorem 3.17 *If $T \in \Phi_s(\mathcal{M}, \mathcal{I})$, then*

$$\lim_{n\to\infty} sup \ \gamma_\infty(T^n)^{\frac{1}{n}} = \lim_{n\to\infty} sup \ \gamma_{\mathcal{I}}(T^n)^{\frac{1}{n}} \leq s_{\mathcal{I}}(T).$$

Proof Note that from Theorem 3.16 $\gamma_\infty(T) = \gamma_{\mathcal{I}}(T)$. We first show that

$$s_{\mathcal{I}}(T^n)^{\frac{1}{n}} \leq s_{\mathcal{I}}(T) \text{ for every } n \in \mathbf{N}.$$

Let $\lambda \in \mathbf{C}$ be such that

$$\lambda^n \notin \sigma_{\Phi_s}(T^n).$$

i.e. $\lambda^n I - T^n \in \Phi_s(\mathcal{M}, \mathcal{I})$. Then there exists a central projection $P \in \mathcal{M}$ such that $(\lambda^n I - T^n)P$ is left Fredholm in $\mathcal{M}P$ and $(\lambda^n I - T^n)(I - P)$ is right Fredholm in $\mathcal{M}(I - P)$.

Note that

$$\lambda^n I - T^n = (\lambda^{n-1} I + \lambda^{n-1} T + \ldots + T^{n-1})(\lambda I - T)$$

Hence $\pi_P(\lambda P - TP)$ is left invertible in $\mathcal{M}P/\mathcal{I}P$ and $\pi_{(I-P)}(\lambda(I-P) - T(I-P))$ is right invertible in $\mathcal{M}(I-P)/\mathcal{I}(I-P)$ which implies that $\lambda I - T \in \Phi_s(\mathcal{M},\mathcal{I})$ and hence $\lambda \notin \sigma_{\Phi_s}(T)$ which proves our assertion. Hence

$$\lim_{n\to\infty} \sup s_{\mathcal{I}}(T^n)^{\frac{1}{n}} \leq s_{\mathcal{I}}(T).$$

Now, for $T \in \Phi_s(\mathcal{M},\mathcal{I})$ and for every $\lambda \in \mathbf{C}$ with $|\lambda| < \gamma_{\mathcal{I}}(T)$, we have

$$||(\lambda I - T) - T|| = |\lambda| < \gamma_{\mathcal{I}}(T)$$

Thus from Theorem 3.14, it follows that $\lambda I - T \in \Phi_s(\mathcal{M},\mathcal{I})$. This implies that

$$\gamma_{\mathcal{I}}(T) \leq s_{\mathcal{I}}(T).$$

Since T^n is semi-Fredholm for every $n \in \mathbf{N}$, we have

$$\lim_{n\to\infty} \sup \gamma_{\mathcal{I}}(T^n)^{\frac{1}{n}} \leq \lim_{n\to\infty} \sup s_{\mathcal{I}}(T^n)^{\frac{1}{n}} \leq s_{\mathcal{I}}(T). \qquad \square$$

We show that the limit in the above theorem exists and equality holds.

Theorem 3.18 *Let* $T \in \Phi_s(\mathcal{M},\mathcal{I})$, *then*

$$\lim_{n\to\infty} \gamma_\infty(T^n)^{\frac{1}{n}} = \lim_{n\to\infty} \gamma_{\mathcal{I}}(T^n)^{\frac{1}{n}} = s_{\mathcal{I}}(T) = dist(0, \sigma_{\Phi_s}(T)).$$

Proof For the converse inequality: Let $L_{\pi(T)} : \mathcal{M}/\mathcal{I} \to \mathcal{M}/\mathcal{I}$ be the left regular representation. Since T is in $\Phi_s(\mathcal{M},\mathcal{I})$, there exists a central projection $P \in \mathcal{M}$ such that $TP \in \Phi_\ell(\mathcal{M}P, \mathcal{I}P)$ and $T(I-P) \in \Phi_r(\mathcal{M}(I-P), \mathcal{I}(I-P))$.

If $\pi_P : \mathcal{M}P \to \mathcal{M}P/\mathcal{I}P$ denotes the canonical quotient map, it follows that $L_{\pi_P(TP)}$ is injective. Clearly

$$\gamma(L_{\pi_P(TP)}) = m_{\mathcal{I}P}(TP).$$

Now, if we view $L_{\pi_P(TP)}$ as an operator on the Banach space $\mathcal{M}P/\mathcal{I}P$, then it follows from [MaZ83] Theorem 3 that

$$\lim_{n\to\infty} \gamma(L^n_{\pi_P(TP)})^{\frac{1}{n}} = b(L_{\pi_P(TP)})$$

where $b(L_{\pi_P(TP)}) := \sup\{\varepsilon > 0 \ : \ \lambda P - TP \in \Phi_\ell(\mathcal{M}P, \mathcal{I}P)\}$ for all $|\lambda| < \varepsilon.\}$

Note that [HaM93], Theorem 4 implies that

$$\gamma(L_{\pi_P(TP)}) = \gamma_{\mathcal{I}P}(TP) \quad , \quad \text{hence}$$

$$\lim_{n\to\infty} \gamma_{\mathcal{I}P}(T^nP)^{\frac{1}{n}} = b(L_{\pi_P(TP)}).$$

By using the facts that $T(I-P) \in \Phi_r(\mathcal{M}(I-P), \mathcal{I}(I-P))$ if and only if

$$T^*(I-P) \in \Phi_\ell(\mathcal{M}(I-P), \mathcal{I}(I-P)) \text{ and}$$

$$\gamma_{\mathcal{I}(I-P)}(T^*(I-P)) = \gamma_{\mathcal{I}(I-P)}(T(I-P)), \text{ we have}$$

$$\lim_{n\to\infty} \gamma_{\mathcal{I}(I-P)}(T^n(I-P))^{\frac{1}{n}} = b\left(L_{\pi_{I-P}}(T^*(I-P))\right).$$

Since $\gamma_{\mathcal{I}}(T) = \min \left\{\gamma_{\mathcal{I}P}(TP), \gamma_{\mathcal{I}(I-P)}(T(I-P))\right\}$, by Lemma 3.10, we have

$$\lim_{n\to\infty} \gamma_{\mathcal{I}}(T^n)^{\frac{1}{n}} = \min \left\{ \lim_{n\to\infty} \gamma_{\mathcal{I}P}(T^n P)^{\frac{1}{n}}, \lim_{n\to\infty} \gamma_{\mathcal{I}(I-P)}(T^n(I-P))^{\frac{1}{n}} \right\}$$

Moreover it follows directly that

$$s_{\mathcal{I}}(T) = \min\left\{b(L_{\pi_P(TP)}) \ \ , b(L_{\pi_{I-P}(T^*(I-P))})\right\}. \text{ Hence}$$

$$\lim_{n\to\infty} \gamma_{\infty}(T^n)^{\frac{1}{n}} = \lim_{n\to\infty} \gamma_{\mathcal{I}}(T^n)^{\frac{1}{n}} = s_{\mathcal{I}}(T).$$

□

Open problems 3.19

1. Regarding Theorem 3.2, if p is a polynomial with complex coefficients, can one find a $K \in \mathcal{I}$ such that $\gamma_{\mathcal{I}}(p(T)) = \gamma(p(T+K))$. This conjecture is open even in the case where $\mathcal{M} = \mathcal{B}(H)$ and $\mathcal{I} = \mathcal{K}(H)$ [MbP96].

2. Does Theorem 3.2 hold for any C^*-algebra and any closed ideal in the algebra?

Chapter 4

THE ESSENTIAL LOWER BOUND

As in the case of the reduced minimum modulus, the lower bound (minimum modulus) of an operator on a Banach space has been studied extensively [MaZ83], [Bou81], [Gal94], [LeS71], [Rog77] and [Wu89] etc. For an operator T on a Banach space X, the lower bound of T is defined by

$$m(T) \ := \ \inf \{||Tx|| \ : \ x \in X \ , \ \ ||x|| = 1\}.$$

It is clear from this definition that $m(T) > 0$ if and only if T is bounded from below, hence injective. In [MaZ83], a useful spectral radius type result on the asymptotic behaviour of $m(T)$ was proved, namely if

$$b(T) \ := \ \sup\{\varepsilon \geq 0 \ : \ T - \lambda I \ \text{ is bounded below } \ \text{for } |\lambda| < \varepsilon\}$$

then

$$\lim_{n\to\infty} m(T^n)^{\frac{1}{n}} = b(T).$$

It was shown [MaZ83] that this formula can be used to estimate the size of the spectrum of a given operator. If H is a Hilbert space it can be shown easily that

$$m(T) = \inf \sigma(|T|).$$

Since the main focus of this thesis is on the connection of quantities with Fredholm theory, we need to consider the notion of an essential lower bound of an operator. With the above spectral characterisation of $m(T)$ (T a Hilbert space operator) in hand, it is clear how one should define the essential lower bound of T, namely

$$m_e(T) = \inf \sigma_e(|T|)$$

where $\sigma_e(|T|)$ denotes the spectum of $\pi(T)$ in $\mathcal{B}(H)/\mathcal{K}(H)$. This quantity has been important in understanding the structure of the Calkin algebra [Bou81], [Gal94].

It is well known that T is invertible modulo the compact operators if and only if $m_e(T)$ and $m_e(T^*)$ are both positive. These quantities appear naturally in the study of Fredholm theory. More importantly, it was shown [Bou81], [Rog77] and [Wu89] that these quantities can also be used to determine the distance from an arbitrary operator to the sets of unitary and more generally invertible operators. For example if the Fredholm index of an operator T is not zero then max $\{m_e(T), m_e(T^*)\}$ is precisely the distance from T to the group of invertible operators.

In this chapter we are interested in the study of these quantities in a von Neumann algebra setting. To explain the setting, let $\mathcal{M}$ denote a von Neumann algebra and $\mathcal{I}$ a closed two-sided ideal in $\mathcal{M}$. For a $T \in \mathcal{M}, m(T)$ will denote the lower bound and $m_{\mathcal{I}}(T)$ the essential lower bound relative to the quotient algebra $\mathcal{M}/\mathcal{I}$. It was shown [Ols84], [OlP89] and [Olp86] that these generalised quantities play, as in the classical theory, a very important role in the study of Fredholm elements relative to $\mathcal{I}$. The main result of this chapter will be the following lifting.

"*If $T \in \mathcal{M}$ is Fredholm with index zero then there exists a $K \in \mathcal{I}$ such that*

$$m_{\mathcal{I}}(T) > 0 \text{ and } m_{\mathcal{I}}(T) = m(T+K) = m(T^* + K^*) = m_{\mathcal{I}}(T^*).\text{''}$$

An important consequence of this result is the lifting of invertible elements of $\mathcal{M}/\mathcal{I}$ to $\mathcal{M}$, namely if T is invertible modulo $\mathcal{I}$ and index $(T) = 0$, then there exists a $K \in \mathcal{I}$ such that $T + K$ is invertible in $\mathcal{M}$.

In the last part of this chapter we apply the lower bounds to study elements in the algebra which are essentially topological divisors of zero. We show that the semi-Fredholm elements in $\mathcal{M}$ can be characterised in terms of these elements, and that any element in the boundary of $\mathcal{G}$ (the group of invertible elements) with non-zero index is essentially a topological divisor of zero. In a type II_∞ factor we obtain more, namely that if index $(T) \neq 0$, then $T \in \bar{\mathcal{G}}$ if and only if T is essentially a topological divisor of zero. Finally we characterise regular elements in $\bar{\mathcal{G}}$, by showing that an element T is regular if and only if index $(T) = 0$.

Lifting of the essential lower bound

In [Ver95], Vermaak obtained the following perturbation result on the essential lower bound $m_{\mathcal{I}}(T)$, where $T \in \mathcal{M}$.

Theorem 4.1 [Ver95, Theorem 3.7] *Let $T \in \mathcal{M}$, then*

$$m_{\mathcal{I}}(T) = \sup_{K \in \mathcal{I}} m(T + K).$$

□

In his thesis, Vermaak posed the question whether the supremum in the above is attained in any von Neumann algebra relative to any closed ideal. By using a Stampfli type of decomposition (which do not hold for a general von Neumann algebra setting), Vermaak proved that the supremum is attained for the special case where $\mathcal{M} = \mathcal{B}(H)$ and $\mathcal{I} = \mathcal{K}(H)$.

We show that this is the case in general. The proof of our theorem will be similar to Theorem 3.2.

Theorem 4.2 *Let $T \in \mathcal{M}$ and $\mathcal{I}$ be any closed ideal in $\mathcal{M}$. Then there exists a $K \in \mathcal{I}$ such that*

$$m_{\mathcal{I}}(T) = m(T + K).$$

Proof For any $T \in \mathcal{M}$, let $E(\lambda)$ denote the spectral measure of $|T|$ and $F(\lambda)$ the corresponding spectral measure of $|T^*|$. If $m_{\mathcal{I}}(T) = 0$, the claim holds for the choice $K = 0$. Hence we may assume without loss of generality that $m_{\mathcal{I}}(T) > 0$. Then by Proposition 1.8 T is left Fredholm and in particular $E_{[0]} \in \mathcal{I}$. Let $P = E_{[0, m_{\mathcal{I}}(T)]}$ and $K_0 = m_{\mathcal{I}}(T)P - |T|P$. By Theorem 2.4 it follows for any $0 \leq \beta < m_{\mathcal{I}}(T)$, that $E_{[0,\beta]} \in \mathcal{I}$. Hence $\int_0^\beta (m_{\mathcal{I}}(T) - \lambda) dE_\lambda \in \mathcal{I}$. Since $\mathcal{I}$ is a norm closed ideal and

$$\begin{aligned} ||K_0 - \int_0^\beta (m_{\mathcal{I}}(T) - \lambda) dE_\lambda|| &= ||\int_\beta^{m_{\mathcal{I}}(T)} (m_{\mathcal{I}}(T) - \lambda) dE_\lambda|| \\ &\leq m_{\mathcal{I}}(T) - \beta \end{aligned}$$

it follows that $K_0 \in \mathcal{I}$. Also, since $|T| = (I - P)|T| + m_{\mathcal{I}}(T)P - K_0$ we have

$$\begin{aligned} m_{\mathcal{I}}(T) &= m_{\mathcal{I}}(|T|) \\ &= m_{\mathcal{I}}((I - P)|T| + m_{\mathcal{I}}(T)P) \\ &= m((I - P)|T| + m_{\mathcal{I}}(T)P) \\ &= m(|T| + K_0) \end{aligned}$$

Let $K = UK_0$ where U is the partial isometry in the polar decomposition of T.

Since

$$\begin{aligned} |T + K|^2 &= (T + K)^*(T + K) \\ &= T^*T + T^*K + K^*T + K^*K \\ &= |T|^2 + |T|U^*UK_0 + K_0U^*U|T| + K_0U^*UK_0 \\ &= |T|^2 + |T|R_{|T|}K_0 + K_0R_{|T|}|T| + K_0R_{|T|}K_0 \end{aligned}$$

$$= |T|^2 + |T|K_0 + K_0|T| + K_0^2$$

$$= (|T| + K_0)^2$$

it follows from the uniqueness of the square root of a positive element in a C^*-algebra that $|T+K| = |T| + K_0$ and hence $m_{\mathcal{I}}(T) = m(T+K)$. □

From this theorem it immediately follows that if $T \in \mathcal{M}$ is left invertible mod $\mathcal{I}$, then there exists a $K_\ell \in \mathcal{I}$ such that $T + K_\ell$ is left invertible in $\mathcal{M}$. By replacing T with T^* it follows that if T is right invertible mod $\mathcal{I}$, then there exists a $K_r \in \mathcal{I}$ such that $T + K_r$ is right invertible mod $\mathcal{I}$. However to ensure that we obtain a single element $K \in \mathcal{I}$ such that $T + K$ is invertible in $\mathcal{M}$ we have to assume that T is invertible mod $\mathcal{I}$ with index zero. This decomposition of Fredholm elements of index zero proved to be very important in the classical theory [Ber70]. For a general von Neumann algebra a proof can be found in [Ver95]. It will be clear that this will also follow from the following perturbation result.

Theorem 4.3 *If $T \in \mathcal{M}$ is Fredholm with index zero then there exists a $K \in \mathcal{I}$ such that $m_{\mathcal{I}}(T) > 0$ and $m_{\mathcal{I}}(T) = m(T+K) = m(T^* + K^*) = m_{\mathcal{I}}(T^*)$.*

Proof Since T is Fredholm of index zero it follows from Proposition 1.8 that $m_{\mathcal{I}}(T) = m_{\mathcal{I}}(T^*) > 0, E_{[0]}, F_{[0]} \in \mathcal{I}$ and $E_{[0]} \sim F_{[0]}$. The last equivalence means there exists a partial isometry V in $\mathcal{M}$ such that $V^*V = E_{[0]}$ and $VV^* = F_{[0]}$. If we let $W = U + V$, where $T = U|T|$ it follows that W is unitary and $WE_{[0,\lambda]}W^* = F_{[0,\lambda]}$ for every $\lambda \geq 0$. To see this one needs the following relations

a) $E_{[0]} = N_T = I - R_{T^*}$, $F_{[0]} = N_{T^*}$

b) $U^*U = R_{T^*}$, $UU^* = R_T$

c) $U = UU^*U = R_T U = U R_{T^*}$

d) $V = VV^*V = VN_T = N_{T^*}V$

Then $W^*W = (U+V)^*(U+V) = R_{T^*} + N_T = I$.

Similarly $WW^* = I$.

By similar arguments we see that

$$WE_{[0]}W^* = F_{[0]}$$

and since from the spectral theorem $UE_{(0,\lambda]}U^* = F_{(0,\lambda]}$, for every $\lambda > 0$, one easily obtains

$$WE_{(0,\lambda]}W^* = F_{(0,\lambda]}$$

If we let $K = WK_0$ where K_0 is choosen as in the proof of Theorem 4.2, it follows that $K \in \mathcal{I}$. Since

$$|T| + K_0 = m_{\mathcal{I}}(T)E_{[0,m_{\mathcal{I}}(T)]} + (I - E_{[0,m_{\mathcal{I}}(T)]})|T|$$

and

$$W(|T| + K_0)W^* = m_{\mathcal{I}}(T^*)F_{[0,m_{\mathcal{I}}(T^*)]} + (I - F_{[0,m_{\mathcal{I}}(T^*)]})|T^*|,$$

by a similar computation as in the proof of Theorem 4.2 we have $|T+K| = |T|+K_0$ and $m_{\mathcal{I}}(T) = m(T+K)$. Since $|T^*+K^*| = W|T|W^* + WK_0W^*$ it follows that $m_{\mathcal{I}}(T^*) = m(T^*+K^*)$ and hence the result follows. □

Corollary 4.4 *Let $T \in \mathcal{M}$ be Fredholm of index zero. Then there exists $K \in \mathcal{I}$ such that $T+K$ is invertible in $\mathcal{M}$.*

Proof Note that T is Fredholm if and only if $m_{\mathcal{I}}(T) = m_{\mathcal{I}}(T^*) > 0$. It follows from Theorem 4.3 that there exists a $K \in \mathcal{I}$ such that $m(T+K) = m(T^*+K^*) > 0$. This implies that $T+K$ is invertible in $\mathcal{M}$ (see also [Bou81]). □

Topological divisors of zero

Let $\mathcal{B}$ be a Banach algebra and $L_a : \mathcal{B} \to \mathcal{B}$ be the left regular representation of $\mathcal{B}$. We define

$$m(a) := m(L_a) = \inf\{||ab|| : b \in \mathcal{B} \text{ and } ||b|| = 1\}$$

Up to now we were mostly interested in the case where the lower bound of an element is positive. It follows directly that $m(a) = 0$ if and only if there exists a sequence (b_n) in $\mathcal{B}$ with $||b_n|| = 1$ such that $(||ab_n||)$ converges to zero. Hence $m(a) = 0$ if and only if a is a left topological divisor of zero.

In the case of C^*-algebras we have the following spectral characterisation which motivates our definition of the lower bounds in the first section of this chapter.

Theorem 4.5 *Let $\mathcal{C}$ be a C^*-algebra and $a \in \mathcal{C}$, then $m(a) = inf\, \sigma(|a|)$*

Proof If $m(a) = 0$ then $0 \in \sigma(|a|)$. For, if not $|a|$ is invertible in $\mathcal{C}$ and hence $a^*a = |a|^2$ is invertible in $\mathcal{C}$. Let $c = (a^*a)^{-1}$. Since $m(a) = 0$ there exists a sequence (b_n) in $\mathcal{C}$ such that $||b_n|| = 1$ and (ab_n) converges to zero. From this it follows that $b_n = ca^*ab_n \to 0$, which contradicts the fact that $||b_n|| = 1$ for all n. Hence we have

$$m(a) = \inf \sigma(|a|) = 0.$$

Suppose $m(a) > 0$, then from

$$m(a) = m(L_a) = \inf\{||L_a b|| \ : \ ||b|| = 1\}$$

it follows that $L_a : \mathcal{C} \to \mathcal{C}$ is bounded from below i.e. there exists a constant $k > 0$ such that $||ab|| \geq k\ ||b||$ for all $b \in \mathcal{C}$.

If $N(L_a)$ denotes the null space of L_a then $N(L_a) = \{0\}$, hence

$$m(a) = m(L_a) = \gamma(L_a) = \gamma(a)$$

Recall that $\gamma(a) = \inf \sigma(|a|)\backslash\{0\}$. In order to complete the proof we only need to show that $0 \notin \sigma(|a|)$. Since $\gamma(a) > 0$, $|a|$ is regular. Hence there exists $c \in \mathcal{C}$ such that $|a| = |a|c|a|$.

Since $|||a|b|| = ||ab||$ for every $b \in \mathcal{C}$ it follows that $|||a|b|| \geq k||b||$ for all $b \in \mathcal{C}$. In particular

$$|||a|(1 - c|a|)|| \geq k||1 - c|a|||.$$

Since $|a|(1 - c|a|) = |a| - |a|c|a| = 0$ it follows that $|a|c = c|a| = 1$.

Hence $|a|$ is invertible in $\mathcal{C}$ which implies that $0 \notin \sigma(|a|)$. □

We will be mainly interested in the case where $\mathcal{C} = \mathcal{M}/\mathcal{I}$ where $\mathcal{M}$ is a von Neumann algebra and $\mathcal{I}$ a closed ideal in $\mathcal{M}$. Recall we defined $m_{\mathcal{I}}(T) = \inf \sigma(\pi_{\mathcal{I}}(|T|))$.

From Theorem 4.5 we see that $m_{\mathcal{I}}(T) = m(\pi_{\mathcal{I}}(T))$ where $m(\cdot)$ is defined using the left regular representation.

Theorem 4.6 *Let $T \in \mathcal{M}$. Then*

$$m_{\mathcal{I}}(T) = m(\pi_{\mathcal{I}}(T)) = \textit{ inf } \{||\pi_{\mathcal{I}}(TP)|| \ : \ P \in P(\mathcal{M}) \ \ \textit{and } P \notin \mathcal{I}\}$$

Proof The first equality follows from Theorem 4.5. Let $E_{(\cdot)}$ denote the spectral measure for $|T|$. Let

$$F = \{||\pi_{\mathcal{I}}(TP)|| \ : \ P \in P(\mathcal{M}) \text{ and } P \notin \mathcal{I}\}$$

and

$$G = \{||\pi_{\mathcal{I}}(TS)|| \ : \ S \in \mathcal{M} \text{ and } ||\pi_{\mathcal{I}}(S)|| = 1\}.$$

Let $\varepsilon > 0$ be given and let $Q_\varepsilon = E_{[0, m_{\mathcal{I}}(T)+\varepsilon]}$. Then by Theorem 2.4 we have for any $\varepsilon > 0$ that $Q_\varepsilon \notin \mathcal{I}$. Moreover, $||\pi_{\mathcal{I}}(TQ_\varepsilon)|| \leq ||TQ_\varepsilon|| \leq m_{\mathcal{I}}(T) + \varepsilon$ and therefore $\inf F \leq m_{\mathcal{I}}(T)$. Now consider any $\varepsilon > 0$ and let $S_\varepsilon = |T|E_{[m_{\mathcal{I}}(T)-\varepsilon, \infty)}$ and $R \in \mathcal{M}$

arbitrary. Then, since $E_{[0,m_{\mathcal{I}}(T)-\varepsilon)} \in \mathcal{I}$ it follows directly that

$$\pi_{\mathcal{I}}(R^*T^*TR) = \pi_{\mathcal{I}}(R^*S_\varepsilon^2 R).$$

Then

$$\begin{aligned} ||\pi_{\mathcal{I}}(TR)||^2 &= ||\pi_{\mathcal{I}}(R^*T^*TR)|| \\ &= ||\pi_{\mathcal{I}}(R^*S_\varepsilon^2 R)|| \\ &\geq (m_{\mathcal{I}}(T)-\varepsilon)^2 ||\pi_{\mathcal{I}}(R^*E_{[m_{\mathcal{I}}(T)-\varepsilon,\infty)}R)|| \\ &= (m_{\mathcal{I}}(T)-\varepsilon)^2 ||\pi_{\mathcal{I}}(R^*R)|| \\ &= (m_{\mathcal{I}}(T)-\varepsilon)^2 ||\pi_{\mathcal{I}}(R)||^2, \end{aligned}$$

so that $m_{\mathcal{I}}(T) \leq \inf G$. All that is still needed to be shown in order to obtain the second equality is that $\inf G \leq \inf F$ and this is clear since $F \subset G$. □

We call $T \in \mathcal{M}$ an essential left topological divisor of zero if $\pi_{\mathcal{I}}(T)$ is a left topological divisor of zero in $\mathcal{M}/\mathcal{I}$. We denote this class of elements by $Z_\ell(\mathcal{M},\mathcal{I})$. The class $Z_r(\mathcal{M},\mathcal{I})$ of essentially right topological divisors of zero is defined in a similar way. Let $Z(\mathcal{M},\mathcal{I}) = Z_\ell(\mathcal{M},\mathcal{I}) \cap Z_r(\mathcal{M},\mathcal{I})$.

Theorem 4.7 *$T \in Z_\ell(\mathcal{M},\mathcal{I})$ if and only if $m_{\mathcal{I}}(T) = 0$.*

Proof This follows directly from Theorem 4.5 and our comment that $m(\pi_{\mathcal{I}}(T)) = 0$ if and only if $\pi_{\mathcal{I}}(T)$ is a left topological divisor of zero in $\mathcal{M}/\mathcal{I}$. □

By a similar argument we can show that $T \in Z_r(\mathcal{M},\mathcal{I})$ if and only if $m_{\mathcal{I}}(T^*) = 0$. Hence we obtain the following.

Corollary 4.8 *Let $T \in \mathcal{M}$. Then $T \in Z(\mathcal{M},\mathcal{I})$ if and only if $m_{\mathcal{I}}(T) = m_{\mathcal{I}}(T^*) = 0$.*

□

The following characterisation of the left and right Fredholm elements follows from Theorem 4.7 and Proposition 1.8

Corollary 4.9 *Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{I}$ any norm closed two-sided ideal in $\mathcal{M}$. Then $\Phi_\ell(\mathcal{M},\mathcal{I}) = (Z_\ell(\mathcal{M},\mathcal{I}))^c$ and $\Phi_r(\mathcal{M},\mathcal{I}) = (Z_r(\mathcal{M},\mathcal{I}))^c$.* □

We would like to mention that a part of Theorem 4.6, Theorem 4.7 and the corollaries also appear in [Ver95]. However we made use of the left regular representation on a C^*-algebra to at first obtain a useful characterisation of $m_{\mathcal{I}}(T)$ (see Theorem 4.5 and the first equality in Theorem 4.6). From this the important characterisations, Theorem 4.7 and its corollaries, follow immediately.

It was shown in [IzK85], Theorem 4.2 that in the case where $\mathcal{M} = \mathcal{B}(H)$ and index $(T) \neq 0$, then *dist* $(T,\mathcal{G}) = \max\{m_e(T), m_e(T^*)\}$. Hence for $T \in \mathcal{B}(H)$ with nonzero index, $T \in \bar{\mathcal{G}}$ if and only if $T \in Z(\mathcal{B}(H),\mathcal{K}(H))$. In this section we show to what extend this result holds in a von Neumann algebra setting.

We recall that two projections E, F in $\mathcal{M}$ are Murry and von Neumann equivalent, and write $E \sim F$, if there exists a partial isometry $U \in \mathcal{M}$ such that $U^*U = E$ and $UU^* = F$. A projection E is called finite relative to $\mathcal{M}$ if the relation $E \sim F \leq E$ implies $E = F$. The set of finite projections play an important role in the structure theory of von Neumann algebras. Let $\mathcal{K}$ denote the norm closed two-sided ideal generated by the finite projections. In the case where $\mathcal{M}$ is the set of bounded operators on a Hilbert space, $\mathcal{K}$ corresponds to the ideal of compact operators.

In [Ols89], Olsen considered the quantity $\alpha(T) = \inf\{\lambda : E_{[0,\lambda]} \sim F_{[0,\lambda]}\}$ and showed that $\alpha(T) = dist\ (T, \mathcal{G})$. Clearly if index$(T) = 0$ then $\alpha(T) = 0$. It is also shown that in the case where $\mathcal{M}$ is a type II_∞ factor and $\mathcal{K}$ the norm closed ideal generated by the finite projections that if index $(T) \neq 0$ then $\alpha(T) = \max\{m_{\mathcal{K}}(T), m_{\mathcal{K}}(T^*)\}$.

Proposition 4.10 *Let $\mathcal{M}$ be a type II_∞ factor and $\mathcal{K}$ the ideal generated by finite projections in $\mathcal{M}$. Then if index $(T) \neq 0$, we have $\alpha(T) = 0$ if and only if $T \in Z(\mathcal{M}, \mathcal{K})$.*

Proof This follows directly from the above observations and Theorem 4.7. □

Corollary 4.11 *Let Φ_0 be the set of index zero Fredholm elements in $\mathcal{M}$, then Φ_0 is an open subset of $\bar{\mathcal{G}}$, i.e. $\Phi_0 \subset int(\bar{\mathcal{G}})$.*

Proof We know that $\Phi_0 \subset \bar{\mathcal{G}}$. Let $T \in \Phi_0$, then $m_{\mathcal{K}}(T) > 0$. Hence for every $S \in \mathcal{M}$ such that $||S - T|| < m_{\mathcal{K}}(T)$ has index zero by Theorem 1.9 and hence $\alpha(S) = dist(S, \mathcal{G}) = 0$. Thus $S \in \bar{\mathcal{G}}$, which implies that T is an interior point of $\bar{\mathcal{G}}$.

□

Theorem 4.12 *Let $\mathcal{M}$ be a type II_∞ factor then*

(1) $Z(\mathcal{M},\mathcal{K}) \subset \bar{\mathcal{G}}\backslash\mathcal{G}$

(2) $bdy(\bar{\mathcal{G}}) \subset Z(\mathcal{M},\mathcal{K})$

Proof (1) If $T \in Z(\mathcal{M},\mathcal{K})$ then $m_{\mathcal{K}}(T) = m_{\mathcal{K}}(T^*) = 0$. If index$(T) = 0$ then $\alpha(T) = 0$. Hence $T \in \bar{\mathcal{G}}$. If index$(T) \neq 0$ then it follows directly from Proposition 4.10 that $T \in \bar{\mathcal{G}}$.

(2) Suppose $T \in bdy(\bar{\mathcal{G}})$ and say $m_{\mathcal{K}}(T) > 0$ (i.e. T is left Fredholm). Then there exists an invertible $S \in \mathcal{M}$ such that $||T - S|| < m_{\mathcal{K}}(T)$. By Theorem 1.9 index$(T)$ = index$(S) = 0$ which implies that T is Fredholm of index zero. Hence $m_{\mathcal{K}}(T) = m_{\mathcal{K}}(T^*) > 0$. By Corollary 4.11, Φ_0 is an open subset of $\bar{\mathcal{G}}$ and hence T is an interior point of $\bar{\mathcal{G}}$, which is a contradiction. Thus $m_{\mathcal{K}}(T) = 0$. Similarly we can show that $m_{\mathcal{K}}(T^*) = 0$. Hence $T \in Z(\mathcal{M},\mathcal{K})$. □

Remark 4.13 From Theorem 4.12 it seems interesting to ask in the case of type II_∞ factor whether $bdy(\bar{\mathcal{G}})$ coincides with $Z(\mathcal{M},\mathcal{K})$. In the case of non-factors we could manage to prove only one direction of Proposition 4.10.

The following Lemma is well known but since it plays such an important role we include a proof.

Lemma 4.14 *Let $E_0 \leq E$, $F_0 \leq F$ be projections in $\mathcal{K}$ such that $E \sim F$ and $E_0 \sim F_0$, then $E - E_0 \sim F - E_0$.*

Proof By comparability of projections [KaR86], there exists a projection Q in the center of $\mathcal{M}$ such that

$$(E - E_0) \preceq Q(F - F_0)Q \text{ and } (F - F_0)(I - Q) \preceq (E - E_0)(I - Q).$$

Suppose $(E - E_0)Q \not\sim (F - F_0)Q$. Then there exists a projection P such that $(E - E_0)Q \sim P < (F - F_0)Q$. Since $E_0 \sim F_0$ we have $QE_0 \sim QF_0$ and thus

$$EQ = (E - E_0)Q + E_0Q \sim P + F_0Q < FQ.$$

Hence $FQ \sim EQ < FQ$ which contradicts the finiteness of FQ. Using the relation

$$(F - F_0)(I - Q) \preceq (E - E_0)(I - Q),$$

we similarly obtain

$$(F - F_0)(I - Q) \sim (E - E_0)(I - Q).$$

Hence $E - E_0 \sim F - F_0$. □

Theorem 4.15 *Let $\mathcal{K}$ be the closed ideal generated by the finite projections. For $T \in \mathcal{M}$ such that index $(T) \neq 0$ and $\alpha(T) = 0$ we have $T \in Z(\mathcal{M}, \mathcal{K})$.*

Proof Suppose $T \notin Z(\mathcal{M}, \mathcal{K})$. Then either $m_{\mathcal{K}}(T)$ or $m_{\mathcal{K}}(T^*)$ is positive. Since $m_{\mathcal{K}}(T) = \inf \{\beta : E_{[0,\beta+\varepsilon]} \notin \mathcal{K} \text{ for every } \varepsilon > 0\}$ and $\alpha(T) = 0$ it follows that both $m_{\mathcal{K}}(T)$ and $m_{\mathcal{K}}(T^*)$ are positive and hence equal. Hence there exists $\lambda > 0$ such that $E_{[0,\lambda]} \sim F_{[0,\lambda]} \in \mathcal{K}$. Let $E = E_{[0,\lambda]}$, $F = F_{[0,\lambda]}$, $E_0 = E_{(0,\lambda]}$ and $F_0 = F_{(0,\lambda]}$.

Then $E_0 \sim F_0$ since $F_{(0,\lambda]} = UE_{(0,\lambda]}U^*$ where U is the partial isometry in the polar decomposition of T. Applying Lemma 4.14 we have $E_{[0]} \sim F_{[0]}$. Hence index$(T) = 0$ which contradicts our assumption. □

We conclude this section by finding some necessary and sufficient conditions for regular elements in any von Neumann algebra to be in $\bar{\mathcal{G}}$.

Lemma 4.16 *If P and Q are projections in $\mathcal{M}$ such that $||P - Q|| < 1$, then $P \sim Q$.*

Proof Let $P \wedge (I - G)$ denote the projection onto the closed subspace $P(H) \cap (I - Q)(H)$. It follows from Proposition 1.3 that the range projection $R_{PQ} = P - P \wedge (I - Q)$. Since $||P - Q|| < 1$, it follows that $P \wedge (I - Q) = 0$. For, otherwise we could find a norm one vector $x \in P(H) \cap (I - Q)(H)$ which would mean that

$$1 = ||x|| = ||P(x)|| = ||P(x) - Q(x)|| \leq ||P - Q|| < 1.$$

Hence $R_{PQ} = P$. Similarly by replacing P by Q in the above argument, we have $R_{QP} = Q$. But since $R_{PQ} \sim R_{(PQ)^*} = R_{QP}$, it follows that $P \sim Q$. □

Theorem 4.17 [HaM93, Theorem 6]. *If T, T_n are nonzero regular elements in a von Neumann algebra $\mathcal{M}$, with $||T_n - T|| \to 0$, then the following are equivalent:*

a) $||T_n^+ - T^+|| \to 0$

b) $\gamma(T_n) \to \gamma(T)$

c) $\sup\limits_n \|T_n^+\| < \infty$.

Proof

(a) $\Rightarrow$ (b): If $(\|T_n^+ - T^+\|)$ converges to zero, the sequences of projections $(T_n^+T_n)$ and $(T_nT_n^+)$ converge to T^+T and TT^+ respectively. Hence by Theorem 3.8, we have $\gamma(T_n)$ converges to $\gamma(T)$.

(b) $\Rightarrow$ (c): Since T is regular, we have $\gamma(T) > 0$. $\gamma(T_n)$ converges to $\gamma(T)$ implies that $\gamma(T_n)$ is bounded below and since $\|T_n^+\|\gamma(T_n) = 1$ we have $\sup\limits_n \|T_n^+\| < \infty$.

(c) $\Rightarrow$ (a): Since by Theorem 3.8

$$T_n^+ - T^+ = -\, T_n^+(T_n - T)T^+ + \; T_n^+T_n^{*+}(T_n^* - T^*)(I - TT^*)$$

$$+ \, (I - T_n^+T_n)(T_n^* - T^*)T^{*+}T^+$$

and $\sup\limits_n \|T_n^+\| < \infty$ we have

$$\begin{aligned}
\|T_n^+ - T^+\| &\leq \|-T_n^+(T_n - T)T\| + \|T_n^+T_n^{*+}(T_n^* - T^*)(I - TT^*)\| \\
&\quad + \|(I - T_n^+T_n)(T_n^* - T^*)T^{*+}T^+\| \\
&\leq \|T_n^+\| \, \|T^+\| \, \|T_n - T\| + \|T_n^+\| \, \|T_n^+\| \, \|T_n - T\| \\
&\quad + \|T^+\| \, \|T^+\| \, \|T_n - T\| \\
&\leq c \, \|T_n - T\| \;\; \text{for some constant } c.
\end{aligned}$$

Hence, as $(||T_n - T||)$ converges to zero, it follows that $(||T_n^+ - T^+||)$ converges to zero. □

Theorem 4.18 *Let $T \in \mathcal{M}$ be regular. Then the following conditions are equivalent:*

(1) $T \in \bar{\mathcal{G}}$

(2) *T is of index zero*

(3) *$T = SP$ for some $S \in \bar{\mathcal{G}}$ and P an orthogonal projection.*

Proof (1) $\Rightarrow$ (2): Let $T \in \bar{\mathcal{G}}$. Then there exists a sequence $(T_n) \subset \mathcal{G}$ such that (T_n) converges to T. Let $S_n = T_nT^+$ and $S = TT^+$. Clearly (S_n) converges to S and since $R_{S_n^*} = R_T$ and $R_T(H) = T(H)$, we have $S_n^+S_n = TT^+ = S = S^+S$. Hence by Theorem 4.17, it follows that $S_nS_n^+$ converges to $SS^+ = TT^+$. This implies that for sufficiently large n, we can find two projections $S_nS_n^+$ and $S_n^+S_n$ of distance strictly less than one. Since $R_{S_n} = I - N_{S_n^*}$ and $R_{S_n^*} = I - N_{S_n}$, we have $||N_{S_n} - N_{S_n^*}|| < 1$. Hence by Lemma 4.16, $N_{S_n} \sim N_{S_n^*}$ i.e. index $(S_n) = 0$. Since $S_n = T_nT^+$ and index $(T_n) = 0$, it follows that index $(T^+) = 0$. Hence index $(T) = 0$.

(2) $\Rightarrow$ (3): If index $(T) = 0$, then as was indicated in the proof of Theorem 4.3 there exists a unitary $U \in \mathcal{M}$ such that $T = U|T|$. If we let $S = U(|T| + N_T)$ and $P = R_{T^*}$ then $S \in \mathcal{G}$ and $T = SP$.

(3) $\Rightarrow$ (1): It follows immediately from the fact that

$$\begin{aligned}\text{index }(T) &= \text{index }(SP)\\ &= \text{index }(S)+\text{ index }(P)\\ &= 0\end{aligned}$$

which implies that $\alpha(T) = 0$. Hence $T \in \bar{\mathcal{G}}$. □

Open problems 4.19

1. See Remark 4.13.

2. For $\mathcal{M} = \mathcal{B}(H)$ and $\mathcal{I} = \mathcal{K}(H)$ the following interesting inequality holds [Izu79]: For any $T, S \in \mathcal{B}(H)$ with index$(T) \neq$ index(S), it follows that

$$||T - S||_e \geq m_e(T) + m_e(S)$$

Does this inequality hold in general? A positive solution for a type II_∞ factor with respect to the compact ideal might be a first step in solving this question.

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Operational quantities in C^*-algebras

by

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SUMMARY

In this thesis we study operational quantities characteristic of semi-Fredholm elements relative to a norm-closed ideal in a von Neumann algebra. In recent years, there have been various attempts at generalising various classical results relating to compact operators on a Banach space to the setting of a von Neumann algebra containing a closed ideal. In the study of operators and the quantities associated with them, the concept of stability under small perturbations is central.

In this work we mainly concentrate on results relating quantities like the reduced minimum modulus and the lower bound of an element to the study of Fredholm theory. We answer an open question in the affirmative, namely that the reduced essential

minimum modulus of an element in a von Neumann algebra relative to a closed ideal is equal to the reduced minimum modulus of the element perturbed by an element from the ideal. As a corollary, we extend some basic perturbation results on semi-Fredholm elements. We also find a complete characterisation of the points of continuity of the reduced essential minimum modulus in terms of Fredholm properties and study the asymptotic behaviour of this quantity.

On the other hand it is known in the classical theory of operators on a Hilbert space that the lower bound and the essential lower bound of an operator measures the distance from the operator to the sets of unitary and more generally invertible operators. We study these bounds, by results, connecting the topological divisors of zero with the boundary of the group of invertible elements. We also find necessary and sufficient conditions for regular elements in a von Neumann algebra to be in the closure of the group of invertible elements.

Operatormate in C^*-algebras

deur

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OPSOMMING

Die proefskrif handel oor die studie en toepassings van operatormate in Fredholm teorie relatief tot enige geslote ideaal in 'n von Neumann algebra raamwerk. Daar bestaan reeds beduidende literatuur waarin verskeie belangrike klassieke resultate wat handel oor kompakte operatore op 'n Banach ruimte en die rol van hierdie operatore in Fredholm teorie, veralgemeen is na 'n von Neumann algebra raamwerk. Stabiliteit van operatore onder "klein" steurings vorm 'n sentrale deel van hierdie studie.

Daar word hoofsaaklik gekyk na die rol wat mate soos die minimum moduli speel ten opsigte van Fredholm teorie. Onder andere word 'n oop probleem opgelos, naamlik dat die essensiële minimum modulus van 'n element in 'n von Neumann algebra relatief tot

'n geslote ideaal gelyk is aan die minimum modulus van die gesteurde element deur 'n element van die ideaal. Hieruit volg belangrike veralgemenings van resultate wat handel oor die steuringsteorie van Fredholm operatore deur kompakte operatore na 'n von Neumann algebra raamwerk. Daar word aangetoon dat die punte van kontinuïteit van die essensiële minimum modulus volledig gekarakteriseer kan word in terme van Fredholm eienskappe. Asimptotiese eienskappe van die essensiële minimum modulus word ook bestudeer.

Ander mate van belang in die studie is die ondergrens, en essensiële ondergrens van 'n element in 'n von Neumann algebra. Hierdie mate word gebruik om die afstand van 'n element na die groep van inverteerbare elemente te meet. Verder karakteriseer hierdie mate die elemente wat topologiese nuldelers is in die gegewe algebra. Verbande word afgelei tussen hierdie elemente en elemente in die rand van die groep van inverteerbare elemente. Ten einde word nodige en voldoende voorwaardes gegee vir 'n reguliere element van die algebra om in die rand van die groep van inverteerbare elemente te wees.