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YAMAMOTO TYPE THEOREMS IN BANACH ALGEBRAS

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Yamamoto type theorems in Banach algebras

by

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CONTENTS

1. INTRODUCTION

In 1967, T. Yamamoto [16], stated and proved an asymptotic relation between the singular values and eigenvalues of a matrix. As mentioned in [12], it was C.R. Loesner who rediscovered this result in 1976. The formulation of this result involving approximation numbers of matrices, see Definition 5.1, suggested an extention of this result to $B(X)$, where X is a Banach space. Nylen and Rodman [12] proved this relation for Banach algebras satisfying the spectral radius property, see Definition 6.4. They also showed that C^*-algebras and finite dimensional algebras do have the spectral radius property and stated that $B(X)$, where X is a Banach space, also has the property. In Theorem 6.10 of this paper we give a complete proof of this property for $B(X)$.

In [12] Nylen and Rodman conjectured that every Banach algebra with a unit element has the spectral radius property. We prove their conjecture in the affirmative in Theorem 6.12.

Other interesting results proved are found in Sections 5 and 6 as Lemma 5.2 and Theorem 6.1, both results on some properties of approximation numbers.

Let us state Yamamoto's Theorem for the algebra of matrices over the complex field $\mathbb{C}$. Let $M_k(\mathbb{C})$ denote the set of all k by k matrices over $\mathbb{C}$. Given a matrix $A \in M_k(\mathbb{C})$, we denote the eigenvalues of A by

$$\alpha_1(A), \alpha_2(A), \ldots, \alpha_k(A),$$

with the convention that multiple eigenvalues are repeated according to their multiplicities and indexed so that

$$|\alpha_k(A)| \leq \ldots \leq |\alpha_1(A)|.$$

The singular values of A are denoted by

$$\sigma_1(A), \ldots, \sigma_k(A),$$

where $\sigma_i(A)$ is the non-negative square root of $\alpha_i(A^*A)$, A^* being the adjoint of A.

Proposition 1.1 ([16], Theorem 1). *Let $A \in M_k(\mathbb{C})$. For each i,*

$$\lim_{n\to\infty}(\sigma_i(A^n))^{\frac{1}{n}} = |\alpha_i(A)|. \tag{1.1}$$

It is worth noting that $\sigma_1(A) = \|A\|$, with $\|.\|$ being the underlying algebra norm. So $(\sigma_1(A^n))^{\frac{1}{n}} = \|A^n\|^{\frac{1}{n}}$ and $|\alpha_1(A)| = r(A)$, the spectral radius of A. For $i = 1$, equation (1.1) is just the spectral radius formula, that is Beurling's formula.

Hence, for Proposition 1.1 to make sense in $B(X)$, where X is a Banach space, the generalization of singular values, namely the notion of approximation numbers, had to be introduced. This notion has been widely explored in the literature, see for instance monographs [6], [9], and [13].

With the notion of approximation numbers, D. E. Edmunds and W. D. Evans [6] extended Proposition 1.1 to elements of $B(X)$, where X is a Banach space.

To get a meaningful extension of Proposition 1.1 to the elements of a general Banach algebra A, we need the notion of rank of elements of A, so as to define approximation numbers for elements of A.

In Section 2 we introduce the notion of rank in a general Banach algebra and it turns out that this notion corresponds to the classical notion of rank for the Banach algebra $B(X)$. This correspondence is shown in Theorem 2.6. We provide some properties of the rank function which will be needed in the sequel. In Section 3 we compare the notion of rank defined in this paper, see Definition 2.1, with those appearing in the literature. In Section 4 we introduce the notion of spectral multiplicity of an isolated spectral point. We prove the main result of this paper in Sections 5 and 6. At the end of Section 6 we give a proof of the conjecture of Nylen and Rodman mentioned earlier.

Throughout this paper we assume that the reader is familiar with definitions and standard results on Banach algebras and C*-algebras. Functionals and operators are assumed linear and bounded unless otherwise stated.

2. RANKS OF ELEMENTS IN A BANACH ALGEBRA

Throughout this section, a Banach algebra A is assumed to be over the complex field $\mathbb{C}$ and is also assumed to contain a unit element 1 unless otherwise stated. We will often write λ in place of $\lambda 1$, $\lambda \in \mathbb{C}$.

Definition 2.1. *An element $a \in A$ is said to be of rank one if $a \neq 0$ and for every $b \in A$, there is a scalar $\lambda \in \mathbb{C}$ such that $aba = \lambda a$. An element $x \in A$ is said to be of rank n if x can be expressed as a sum of n elements of rank one but cannot be expressed as a sum of less than n elements of rank one.*

We say $x \in A$ is of finite rank if rank(x) $= k$, for some non-negative integer k and then write rank(x) $< \infty$.

An element a of A is of infinite rank if it is not of finite rank and we write $\mathrm{rank}(a) = \infty$.

Proposition 2.2 ([12], Proposition 2.2). *Let $a,b \in A$. The rank function has the following properties:*

(a) rank($a+b$) $\leq$ rank(a) + rank(b),

(b) if rank(a) $= 1$, then for every $b \in A$ either $ab = 0$ or rank(ab) $= 1$. Moreover rank(ba) $= 1$ unless $ba = 0$,

(c) rank(ab) $\leq$ min$\{$rank(a), rank(b)$\}$,

(d) the set F_A defined by $F_A = \{a \in A$: rank(a) $< \infty\}$ is a two-sided ideal in A, and

(e) the subalgebra $A(a_1, \ldots, a_s)$ of A, generated by 1 and a finite number of finite rank elements $a_1, \ldots, a_s$ of A, is finite dimensional, as a vector space over $\mathbb{C}$.

Proof. (a) For the case where at least one of a and b is of infinite rank the result follows trivially. Hence, suppose both $\mathrm{rank}(a)$ and $\mathrm{rank}(b)$ are finite, that is $\mathrm{rank}(a) = n_1$ and $\mathrm{rank}(b) = n_2$. So, $a = a_1 + \ldots + a_{n_1}$, where $\mathrm{rank}(a_i) = 1$, $(i = 1, \ldots, n_1)$. Also $b = b_1 + \ldots + b_{n_2}$, where $\mathrm{rank}(b_j) = 1$, $(j = 1, \ldots, n_2)$. Then $a + b = a_1 + \ldots + a_{n_1} + b_1 + \ldots + b_{n_2}$, from which we deduce that $\mathrm{rank}(a + b) \leq n_1 + n_2 = \mathrm{rank}(a) + \mathrm{rank}(b)$.

(b) Since $\mathrm{rank}(a) = 1$, for every $c \in A$ there is a $\lambda \in \mathbb{C}$ such that $aca = \lambda a$. Arbitrarily choose $d \in A$ and assume $ab \neq 0$. Then $abdab = (\lambda_0)ab$, for some $\lambda_0 \in \mathbb{C}$. That is $(ab)d(ab) = \lambda_0(ab)$ which shows that $\mathrm{rank}(ab) = 1$. By a similar argument, it can be shown that $\mathrm{rank}(ba) = 1$ unless $ba = 0$.

(c) If both $\mathrm{rank}(a)$ and $\mathrm{rank}(b)$ are infinite we then have nothing to prove. Suppose $\mathrm{rank}(a) = n \leq \mathrm{rank}(b)$. So, $ab = (a_1 + \ldots + a_n)b$ with $\mathrm{rank}(a_i) = 1$, $(i = 1, \ldots, n)$. That is, $ab = a_1 b + \ldots + a_n b$. Thus, from (a) $\mathrm{rank}(ab) \leq \mathrm{rank}(a_1 b) + \ldots + \mathrm{rank}(a_n b)$. By (b) $\mathrm{rank}(ab) \leq n = \min\{\mathrm{rank}(a), \mathrm{rank}(b)\}$. The analogous argument works for the case where $\mathrm{rank}(b) = n \leq \mathrm{rank}(a)$.

(d) Let $a, b \in F_A$ and $d \in A$. We apply (a) and (c) to get the following: $\operatorname{rank}(\alpha a + \beta b) \leq \operatorname{rank}(\alpha a) + \operatorname{rank}(\beta b) \leq \operatorname{rank}(a) + \operatorname{rank}(b)$, $(\alpha, \beta \in \mathbb{C})$. That is, $\operatorname{rank}(\alpha a + \beta b) < \infty$. Hence, F_A is a subspace of A.

Also from part (b), $\operatorname{rank}(db) \leq \min\{\operatorname{rank}(d), \operatorname{rank}(b)\} \leq \operatorname{rank}(b) < \infty$. Thus, $db \in F_A$ and by a similar argument, $bd \in F_A$. Whence, F_A is a two-sided ideal in A.

(e) Since any finite rank element of A is a sum of a finite number of rank one elements of A, we assume that $\operatorname{rank}(a_i) = 1$, $(i = 1, \ldots, s)$.

The algebra $A(a_1, a_2, \ldots, a_s)$ consists of all polynomials in $1, a_1, a_2, \ldots, a_s$. Let $a_0 = 1$. For all $1 \leq i \leq s$ and $0 \leq k \leq s$ it follows that $a_i a_k a_i = \lambda_{ik} a_i$.

Thus all powers of finite products of elements of $A(a_1, a_2, \ldots, a_s)$ will reduce to scalar multiples of products of distinct elements from $\{a_0, a_1, \ldots, a_s\}$.

Hence $A(a_1, a_2, \ldots, a_s)$ will be spanned by all possible finite products of distinct elements from $\{a_0, a_1, a_2, \ldots, a_s\}$ which can only be finite in number.

Therefore $\dim(A(a_1, a_2, \ldots, a_s)) < \infty$ and the theorem is established.

□

Corollary 2.3 ([12], Corollary 2.3). *Every $x \in A$ with rank$(x) < \infty$ is algebraic. That is, there is a non-zero polynomial $P(t)$ such that $P(x) = 0$. In particular, the spectrum $\sigma_A(x)$ is a finite set.*

Proof. The subalgebra $A(x)$, generated by the identity 1 and x is finite dimensional, which follows from Proposition 2.2 (e). Since $x^n \in A(x)$, $(n = 1, 2, \ldots)$, the x^n's cannot all be linearly independent over $\mathbb{C}$. This says that scalars $\lambda_1, \ldots, \lambda_k$ exist, not all zero, such that

$$\lambda_k x^{n_k} + \ldots + \lambda_1 x^{n_1} = 0.$$

Whence,

$$P(t) = \lambda_k t^{n_k} + \ldots + \lambda_1 t^{n_1}$$

is the required polynomial for which $P(x) = 0$.

Lastly, by the Spectral Mapping Theorem, it follows that

$$\sigma(P(x)) = P(\sigma(x)).$$

But,

$$\begin{aligned} \sigma(P(x)) &= \sigma(0) \\ &= \{0\}. \end{aligned}$$

So, $P(\sigma(x)) = \{0\}$. Since $P(t)$ has a finite number of zeros, it follows that $\sigma(x)$ is a finite set and we are done.

□

We next show that if $A = B(X)$, then the rank notion coincides with the classical notion of finite dimensional range. We will need the following lemma to prove that.

Lemma 2.4. *Let $T \in B(X)$, with X a Banach space and dim$(T(X)) = n < \infty$. Then, T has a representation of the form*

$$Tx = f_1(x)y_1 + \ldots + f_n(x)y_n,$$

where $\{y_1, \ldots, y_n\}$ and $\{f_1, \ldots, f_n\}$ are sets in X and X' respectively.

Proof. There is an independent set $\{y_1, \ldots, y_n\}$ in Y, such that

$$\text{span}\{y_1, \ldots, y_n\} = T(X).$$

Then for each $x \in X$,

$$Tx = \textstyle\sum_{i=1}^n f_i(x)y_i. \tag{2.1}$$

Since this representation is unique the coefficients $f_i(x)$ are clearly seen to define linear functionals on X.

Since $T(X)$ has a finite dimension, all norms on $T(X)$ are equivalent, hence there exists a constant $K > 0$ such that

$$\sum_{i=1}^n |f_i(x)| \leq K \| \sum_{i=1}^n f_i(x)y_i \| \text{ for any } x \in X.$$

Hence

$$\begin{aligned} \sum_{i=1}^n |f_i(x)| &\leq K\|Tx\|, \\ &\leq K\|T\|\|x\|. \end{aligned}$$

This shows that all the f_i's are bounded.

□

The $y_1, \ldots, y_n$ in representation (2.1) are chosen to be linearly independent. We could arrange it so that $f_1, \ldots, f_n$ are also linearly independent but we will not need this fact for our purpose.

Remark 2.5. We will use the following notation. If $y \in X$ and $f \in X'$, X a Banach space, we define

$$T = f \otimes y \text{ on } X$$

as

$$Tx := f(x)y.$$

Then it is clear from Lemma 2.4 that if $T \in B(X)$ then $\dim(T(X)) = 1$ if and only if there exists $y \in X$ and $f \in X'$ such that

$$T = f \otimes y.$$

For the case where X is a Hilbert space, it follows from the Riesz representation theorem on bounded linear functionals that there exists a unique vector $z \in X$ such that $f =< , z >$ where $< ., . >$ denotes the inner product on X. In this case we will write $T = z \otimes y$ meaning

$$(z \otimes y)(x) =< x, z > y.$$

Note that if T is of rank one, we can always arrange either y or z to be a unit vector. Then clearly T is a rank one projection on H if and only if $T = z \otimes z$ for some unit vector $z \in H$. This follows by using the following standard properties:-

$$(z \otimes z')(w \otimes w') =< w', z > w \otimes z',$$

$$(z \otimes y)^* = y \otimes z, \text{ where } * \text{ denotes the adjoint of an operator,}$$

$$\|z \otimes y\| = \|z\|\|y\|.$$

□

We next prove a theorem that confirms that if $A = B(X)$, where X is a Banach space, the rank notion applied to an elment x of A boils down to the rank of x as an operator, that is, the dimension of the range of x.

Theorem 2.6 ([12], Theorem 2.4). *Let $A = B(X)$, where X is a Banach space. Let $T \in A$. Then, rank$(T) = n$ if and only if dim$(T(X)) = n$.*

Proof. By the representation in Lemma 2.4 it is clear that any operator $T \in B(X)$ with $\dim(T(X)) = n$ can be written as a sum of n operators each with one-dimensional range.

Hence it suffices to prove the theorem for $n = 1$.

Now suppose $\mathrm{rank}(T) = 1$. Since $T^2 = \lambda T$ for some scalar λ, we can assume, if $\lambda \neq 0$, by rescaling, that $T^2 = T$. The rescaling can be done by considering $T_0 = \frac{T}{\lambda}$, if $\lambda \neq 0$. So that, $T_0^2 = \frac{T^2}{\lambda^2} = \frac{\lambda T}{\lambda^2} = \frac{T}{\lambda} = T_0$. Hence either $T^2 = T$ or $T^2 = 0$.

Case 1: Suppose $T^2 = T$. That is, T is a projection. Thus, X is a direct sum of T-invariant subspaces X_0 and X_1 such that $Ta = a$, for all $a \in X_1$ and $Tb = 0$, for all $b \in X_0$.

We now assume $\dim(X_1) > 1$. So, there are $a_1, a_2 \in X_1$, which are linearly independent. Let X_2 be a direct complement of span$\{a_1\}$ in X_1. Clearly $a_2 \in X_2$. Define $S \in A$ by

$$Sb = 0, \quad b \in X_0 \oplus X_2$$

and

$$Sa_1 = a_1.$$

So,

$$\begin{aligned} TSTa_1 &= TSa_1 \\ &= Ta_1 \\ &= a_1. \end{aligned}$$

Also,

$$\begin{aligned} TSTa_2 &= TSa_2 \\ &= T0 \\ &= 0. \end{aligned}$$

So, there is no $\lambda \in \mathbb{C}$ such that $TST = \lambda T$, which contradicts the assumption that $\operatorname{rank}(T) = 1$. So, we must have

$$\begin{aligned} \dim(T(X)) &= \dim(X_1) \\ &= 1 \\ &= \operatorname{rank}(T). \end{aligned}$$

Case 2: Suppose $T^2 = 0$. We also carry out the argument by contradiction. We suppose that $\dim(T(X)) > 1$. Let $a_1 = Tc_1$ and $a_2 = Tc_2$ be linearly independent for some c_1 and $c_2 \in X$. It then follows from the linearity of T that c_1 and c_2 are also linearly independent. Since $T^2 = 0$, it follows that

$$\{a_1, a_2, c_1, c_2\}$$

is linearly independent, because

$$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 c_1 + \lambda_4 c_2 = 0$$

implies that

$$T(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 c_1 + \lambda_4 c_2) = 0,$$

which implies that

$$\lambda_1 Ta_1 + \lambda_2 Ta_2 + \lambda_3 Tc_1 + \lambda_4 Tc_2 = 0.$$

Hence

$$\lambda_1 T^2 c_1 + \lambda_2 T^2 c_2 + \lambda_3 Tc_1 + \lambda_4 Tc_2 = 0.$$

So,

$$\lambda_3 Tc_1 + \lambda_4 Tc_2 = 0, \text{ because } T^2 = 0.$$

That is,

$$\lambda_3 a_1 + \lambda_4 a_2 = 0,$$

which means that

$$\lambda_3 = \lambda_4 = 0, \text{ because } a_1 \text{ and } a_2 \text{ are linearly independent.}$$

Since
$$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 c_1 + \lambda_4 c_2 = 0,$$
it follows that
$$\lambda_1 a_1 + \lambda_2 a_2 = 0,$$
Therefore,
$$\lambda_1 = \lambda_2 = 0, \text{ because } a_1 \text{and } a_2 \text{ are linearly independent.}$$
We then have
$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.$$

We now let
$$M = \text{span}\{a_1, a_2, c_1, c_2\}.$$
Clearly, M is T-invariant. Restricting T to the subspace M, we can represent T by the following matrix:
$$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$
Consider
$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
Clearly
$$TST = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
Thus, there is no scalar $\lambda \in \mathbb{C}$ such that
$$TST = \lambda T.$$
This contradicts the assumption that $\text{rank}(T) = 1$. We therefore have
$$\dim(T(X)) = 1 = \text{rank}(T).$$

For the converse, assume that $\dim(T(X)) = 1$. So, by using Lemma 2.4 it follows that

$$Ta = \psi(a)b, \ (a \in X), \ b \text{ a fixed vector and } \psi \text{ fixed in } X'. \tag{2.2}$$

Note that from (2.2) we have

$$b = \frac{Ta}{\psi(a)}, \ (a \neq 0).$$

So, for any $S \in B(X)$, we get

$$TSb = \psi(Sb)b.$$

Therefore,

$$\frac{TSTa}{\psi(a)} = \psi(Sb)b.$$

That is,

$$TSTa = \psi(Sb)\psi(a)b.$$

That is,

$$\begin{aligned} TSTa &= \frac{Ta}{\psi(a)}\psi(a)\psi(Sb) \\ &= \psi(Sb)Ta. \end{aligned}$$

Therefore,

$$TSTa = \psi(Sb)Ta, \text{ for all } a \in X.$$

So,

$$TST = \lambda T \text{ for all } S \in B(X), \text{ where } \lambda = \psi(Sb).$$

Whence, $\text{rank}(T) = 1$.

□

Remark 2.7. Let $A = M_n(\mathbb{C})$, the algebra of all n by n matrices over $\mathbb{C}$. The space A, with the usual matrix multiplication, can be viewed as the algebra $B(\mathbb{C}^n)$ with multiplication being the composition of operators and the classical matrix rank of an n by n matrix T is exactly the dimension of $T(\mathbb{C}^n)$ which by Theorem 2.6 equals $\text{rank}(T)$.

□

We now illustrate by an example that if we define another product on the vector space $A = M_n(\mathbb{C})$, the matrix rank differs from our notion of rank.

Example 2.8. *Let $A = M_n(\mathbb{C})$ with Hadamard multiplication. Then, in general*

$$rank(x) \neq rank_M(x),$$

where $x \in A$ and $rank_M$ denotes the matrix rank. Recall that Hadamard multiplication of matrices is carried out entrywise.

Let us consider the case where $n = 3$. Let

$$x = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$y = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $\text{rank}_M(x) = 1$. But

$$xyx = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, there is no scalar $\lambda \in \mathbb{C}$ such that

$$xyx = \lambda x.$$

This implies that

$$\text{rank}(x) \neq \text{rank}_M(x).$$

□

Example 2.9. Let $A = l_\infty$, the algebra of all bounded sequences of complex numbers. Multiplication is pointwise. Then $x \in l_\infty$ is of rank one if and only if x has only one non-zero entry. The ideal F_A consists of all sequences with only a finite number of non-zero entries.

□

We now give a natural example of an algebra $A \neq \mathbb{C}$ which illustrates that if an element is of finite rank, then it must be of rank one. Hence, F_A consists only of rank one elements and F_A is not equal to all of A.

Definition 2.10. *An element $x \in A$ such that $x^k = 0$ but $x^{k-1} \neq 0$ for some $k \geq 2$ is said to be a nilpotent element of order k.*

Example 2.11. *Let A be the algebra generated by the identity matrix I and the powers of an n by n nilpotent matrix M of order k. Let $x \in A$. So,*

$$x = z_s M^s + \ldots + z_{k-1} M^{k-1}, \ (z_i \in \mathbb{C}), \tag{2.3}$$

where $s \geq 0$ and $z_s \neq 0$.

We observe that any non-zero element of A is of the form (2.3). Suppose $\text{rank}(x) = 1$. Then, for any $y \in A$ there is a scalar λ such that

$$xyx = \lambda x.$$

Let y be the identity I. So, $x^2 = \lambda x$. for some $\lambda \in \mathbb{C}$.

Clearly, since x is nilpotent, λ must be zero. Note that multiplying x by itself, the smallest exponent is $2s$ and for the product x^2 to be zero, the smallest exponent, $2s$, must be at least k. That is,

$$2s \geq k,$$

which implies that $s \geq \lceil \frac{k}{2} \rceil$. Of course, if $s \geq \lceil \frac{k}{2} \rceil$ we have, for any $y \in A$ that

$$xyx = 0.$$

So, $\text{rank}(x) = 1$.

Whence, $\text{rank}(x) = 1$ if and only if $s \geq \lceil \frac{k}{2} \rceil$. Almost the same argument shows that the finite rank elements of A are exactly the rank one elements.

□

For the rest of this section we concentrate on specific finite dimensional algebras and show that in this case the notion of rank is determined by the classical matrix rank. We will consider finite dimensional C^*-algebras.

Lemma 2.12. *Let A be a finite dimensional C^*-algebra. Then A is the linear span of its projections.*

Proof. Since any element in A is a linear combination of self-adjoint elements and a self-adjoint element generates a commutative C^*-algebra, we may assume, without loss of generality, that A is commutative. Hence, by the Gelfand-Naimark Theorem, $A \cong C(K)$, for some compact Hausdorff space K.

Since A is finite dimensional, K must be a finite set. This follows from the following considerations:-

For $k \in K$, let $\tau_k \in C(K)$ be defined by

$$\tau_k(f) = f(k), \quad (f \in C(K)).$$

Then the set $\{\tau_k \; : \; k \in K\}$ is linearly independent, for suppose $\lambda_1 \tau_{k_1} + \ldots + \lambda_n \tau_{k_n} = 0$. By Urysohn's Lemma, if we fix i, there is an $f \in C(K)$ such that $f(k_i) = 1$ and $f(k_j) = 0$ for $j \neq i$. This implies that $\lambda_i = 0$. Say $K = \{k_1, \ldots, k_n\}$. The lemma follows by letting $p_i = \chi_{\{k_i\}}$, where χ denotes the characteristic function.

□

Definition 2.13. *Let A be a C^*-algebra. An irreducible representation of A on a Hilbert space H is a *-homomorphism π of A into $B(H)$ such that the* *only subspaces invariant under $\pi(T)$, for all $T \in A$, are the trivial subspaces $\{0\}$ and H.*

It is clear from the Gelfand-Naimark-Segal construction pertaining to representations of C^*-algebras ([15], Theorems 9.18 and 9.22) that irreducible representations of C^*-algebras always exist.

Definition 2.14. *For a Hilbert space H let $K(H)$ denote the compact operators on H. A C^*-algebra is said to be liminal if for every non-zero irreducible representation $(H\ ,\ \phi)$ of A we have*

$$\phi(A) = K(H).$$

We prove that every finite dimensional C^*-algebra is liminal. As an overture to that, we prove the following theorem.

Theorem 2.15 ([11], Theorem 2.4.9). *Let A be a C^*-algebra acting irreducibly on a Hilbert space H and having non-zero intersection with $K(H)$. Then, $K(H) \subset A$.*

Proof. Since $A \bigcap K(H)$ is not $\{0\}$ and the adjoint of any element in $A \bigcap K(H)$ is again in $A \bigcap K(H)$, it follows that $A \bigcap K(H)$ contains a non-zero self-adjoint element a. It follows from the Beurling formula for the spectral radius ([11], Theorem 1.2.7) that $r(a) = \|a\| > 0$. So, $\sigma(a)$ has a non-zero element. Let $0 \neq \lambda \in \sigma(a)$. So, $\sigma(a)$ consists only of isolated eigenvalues, because $a \in K(H)$. Let

$$f(z) = \begin{cases} 1, & \text{if } z = \lambda; \\ 0, & \text{if } z \in \sigma(a)\backslash\{\lambda\}. \end{cases}$$

Then $f \in C(\sigma(a))$. The Gelfand Naimark theorem implies that the commutative C^*-algebra generated by 1 and a, which we denote by $A(1, a)$, is isomorphic to $C(\sigma(a))$. Hence, this isomorphism gives a projection $f(a) \in A(1, a) \subset A$ and we denote it by p_λ. Let $g(z) = z,\ (z \in \sigma(a))$. So,

$$\begin{aligned}(g(z) - \lambda)f(z) &= g(z)f(z) - \lambda f(z) \\ &= 0.\end{aligned}$$

Therefore,

$$(g(a) - \lambda)f(a) = 0.$$

That is,

$$(a - \lambda)p_\lambda = 0.$$

Therefore, $p_\lambda(H) \subseteq \ker(a - \lambda)$. Since $a \in K(H)$, the restriction of a to the subspace $\ker(a-\lambda)$ is also compact. But since this restriction is just λ times the identity element of $B(\ker(a - \lambda))$, the identity element of $B(\ker(a - \lambda))$ is compact, implying that $\ker(a - \lambda)$ is finite dimensional. We thus have $\dim(p_\lambda(H)) < \infty$. Hence we may choose a non-zero projection p in A of minimal finite dimensional range. Then pAp is finite dimensional and, by Lemma 2.12, pAp is spanned by its projections. Since p is of minimal finite dimensional range, the only projections in pAp are 0 and p. Hence, $pAp = \mathbb{C}p$. Let $0 \neq x \in p(H)$ be a unit vector. Let $M = \overline{\{a(x) : a \in A\}}$. Then, M is a closed vector subspace of H invariant for A and $M \neq \{0\}$, because $x = p(x) \in M$. Since A is irreducible, $M = H$. So, if y is an arbitrary element of $p(H)$, then $y = \lim_{n\to\infty} a_n(x)$, with $\{a_n\}_{n=1}^{\infty} \subset A$. So, $y = \lim_{n\to\infty} pa_np(x)$, because $y = p(y)$ and $x = p(x)$. Since $pa_np = \lambda_n p$ for some $\lambda_n \in \mathbb{C}$, it follows that $y \in \mathbb{C}x$. So, $p(H) = \mathbb{C}x$. Therefore, from Remark 2.5 and the fact that p is a projection, we have that $p = x \otimes x$. We now suppose y is an arbitrary unit vector of H. So, there is an $a_n \in A$ such that $y = \lim_{n\to\infty} a_n(x)$. Since $\|y \otimes y - a_n(x) \otimes a_n(x)\| \leq \|y - a_n(x)\|\|y\| + \|a_n(x)\|\|y - a_n(x)\|$ and $\{a_n(x)\}$ is bounded, we get

$$y \otimes y = \lim_{n\to\infty} a_n(x) \otimes a_n(x).$$

That is,

$$\begin{aligned} y \otimes y &= \lim_{n\to\infty} a_n(x \otimes x)a_n^* \\ &= \lim_{n\to\infty} a_npa_n^* \in A. \end{aligned}$$

That is, all rank one projections are in A, which, by ([11], Theorem 2.4.6), implies that $F(H) \subseteq A$. So, $K(H) \subseteq A$.

□

Lemma 2.16 ([11], Example 5.62). *Let A be a finite dimensional C^*- algebra. Then A is liminal.*

Proof. Let (H, π) be a non-zero irreducible representation of A. For some non-zero vector $x \in H$, $\pi(A)x$ is dense in H. But, $\pi(A)x$ is finite dimensional. So, $H = \pi(A)x$ is also finite dimensional. We thus have $\pi(A) \subseteq K(H)$. But from Theorem 2.15, we have $K(H) \subseteq \pi(A)$. Therefore, $\pi(A) = K(H)$. So, A is liminal.

□

Definition 2.17 *We call a C^*-algebra A simple if $\{0\}$ and A are the only closed ideals in A.*

Theorem 2.18 ([11], Remark 6.2.1). *If A is a finite dimensional, simple C^*-algebra, then*

$$A \overset{*}{\cong} M_n(\mathbb{C})$$

for some positive integer n.

Proof. Let (H, π) be as in the proof of Lemma 2.16. Since A is simple and $\ker(\pi)$ is a closed ideal in A, we have $\ker(\pi) = \{0\}$. It follows that π is faithful. This says that

$$A \overset{*}{\cong} \pi(A) = K(H).$$

The last equality follows from Lemma 2.16. So, we have $A \overset{*}{\cong} B(H)$, because $\dim(H) < \infty$. If $\dim(H) = n$, then $A \overset{*}{\cong} M_n(\mathbb{C})$, as wanted.

□

We now prove the structure theorem for finite dimensional C^*-algebras.

Proposition 2.19. *Let A be a finite dimensional C^*-algebra. Then,*

$$A \overset{*}{\cong} M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_m}(\mathbb{C}),$$

for some integers $n_1, \ldots, n_m$.

Proof. If A is simple, the result follows from Theorem 2.18. We prove the general result by induction on the dimension n of A. The case $n = 1$ is obvious. Suppose the result holds for all dimensions less than n. We may suppose that A is not simple, and so contains a non-zero proper closed ideal I, and we may take I to be of minimum dimension. In this case I has no non-trivial ideals, so I is *-isomorphic to $M_{n_1}(\mathbb{C})$ for some integer n_1. Hence, I has a unit element and we denote it by p. So $I = Ap$ and p is in the centre of A. To see this, note that for any $x \in A$, we have $xp \in I$. Hence, $pxp = xp$. Similarly, for x^* one has $px^*p = x^*p$. This implies that $px = px^*p^* = pxp = xp$. Also, $A(1-p)$ is a C^*-subalgebra of A and the map

$$A \to Ap \oplus A(1-p), a \mapsto (ap, a(1-p)),$$

is a *-isomorphism. Since the algebra $A(1-p)$ has dimension less than n, it is, by the inductive hypothesis, *-isomorphic to $M_{n_2}(\mathbb{C}) \oplus \ldots \oplus M_{n_m}(\mathbb{C})$ for some integers $n_2, \ldots, n_m$. Thus, A is *-isomorphic to $M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_m}(\mathbb{C})$.

□

Remark 2.20. The structure theorem for finite dimensional C^*-algebras now enables one to view the rank notion for elements of C^*-algebras in terms of the matrix rank. The structure theorem for Banach algebras is in the appendix as Proposition A.19. If

$$A \cong M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_m}(\mathbb{C}),$$

for some integers $n_1, \ldots, n_m$, then $a \in A$ implies that

$$a = a^{(1)} \oplus \ldots \oplus a^{(1)} \oplus \ldots \oplus a^{(p)} \oplus \ldots \oplus a^{(p)},$$

where $a^{(k)}$ is a d_k by d_k matrix repeated r_k times in the direct sum. Hence,

$$\operatorname{rank}(a) = r_1 \operatorname{rank}(a^{(1)}) + \ldots + r_p \operatorname{rank}(a^{(p)}).$$

□

3. COMPARISON OF RANK CONCEPTS

Throughout this section we bear in mind the notion of rank introduced in Definition 2.1 and compare it with others appearing in the literature.

In the appendix we prove some standard results on semi-simple and semi-prime Banach algebras which will be needed in this section.

Puhl [14] introduced a notion of rank for semi-prime algebras, see Definition A.7 in the appendix for the definition of semi-prime algebra. According to Puhl, an element x of a semi-prime algebra A is of rank one if there is a bounded linear functional $\psi \in A'$, such that $xyx = \psi(y)x$, for all $y \in A$, that is, the wedge operator $T_x : A \mapsto A$ defined by $T_x(y) = xyx$ has one dimensional range. Clearly, our notion of rank one implies the notion used by Puhl. Since a detailed study of the comparison of the rank concept in semi-prime algebras appears in an MSc thesis of Essman [8], we will not continue any further discussion on it.

Ylinen [17] defined an element x of an algebra A to be of finite rank if the wedge operator $T_x : A \to A$ defined by $T_x(y) = xyx$ has a finite dimensional range. It turns out that an element of a semi-simple Banach algebra, see Definition A.1 in the appendix, is of finite rank in the sense of Ylinen's definition if and only if it is of finite rank in the sense of Definition 2.1. By Proposition A.13 any semi-simple algebra is semi-prime and again the comparison can be found in an MSc thesis of Essmann [8]. In Theorem 3.3 we obtain a stronger comparison by giving exact estimates.

Definition 3.1. *We call an element q of an algebra A quasi-nilpotent if the spectral radius of q is zero and we denote the set of all quasi-nilpotent elements of A by $Q(A)$.*

Lemma 3.2. *Let A be a Banach algebra and $Rad(A)$ the radical of A. If $Aq \subset Q(A)$, then $q \in Rad(A)$. Similarly, if $qA \subset Q(A)$, then $q \in Rad(A)$.*

Proof. Suppose $q \notin Rad(A)$. Then there is a continuous irreducible representation $\pi : A \to B(X)$ of A on a linear space X, such that $\pi(q) \neq 0$. So, there is an $x \in X$ such that $\pi(q)x \neq 0$. Since, by hypothesis, $q \in Q(A)$, it follows that $\pi(q)x$ and x are linearly independent. By the Jacobson Density Theorem, Theorem A.17, there is an $a \in A$ such that $\pi(a)x = 0$ and $\pi(a)\pi(q)x = x$. That is $\pi(aq)x = x$. Hence, $\sigma(\pi(aq)) \neq \{0\}$ and by using Remark A.5, $\sigma(\pi(aq)) \subset \sigma(aq)$. We thus have $r(aq) \neq 0$. That is $aq \notin Q(A)$. Since $\sigma(aq)\backslash\{0\} = \sigma(qa)\backslash\{0\}$, the second statement will follow from the first.

□

Theorem 3.3 ([12], Proposition 7.1). *Let A be a semi-simple Banach algebra and let $x \in A$ with rank$(x) = k < \infty$. Then $dim(xAx) \leq k^2$.*

Proof. We first observe, by Definition 2.1, that xAx is a sum of k^2 terms of the form nAm, where $n, m \in A$ and $\text{rank}(n) = \text{rank}(m) = 1$. Hence, if we choose elements $a, b \in A$ such that $\text{rank}(a) = \text{rank}(b) = 1$, it will be sufficient to show that $\dim(aAb) \leq 1$. Assume $aAb \neq \{0\}$. So, there is a $c \in A$ such that $acb \neq 0$. Since $Rad(A) = \{0\}$, we have $acb \notin Rad(A)$. So, by Lemma 3.2, there is a $d \in A$ such that $r(acbd) \neq 0$. Therefore, by the spectral mapping theorem, we have $r((acbd)^2) \neq 0$. Whence, $(acbd)^2 \neq 0$. Suppose $acbda = 0$, then $(acbd)^2 = 0$, which yields a contradiction. Therefore, d satisfies $acbda \neq 0$.

We next let $f \in A$ and consider functionals λ_a and λ_b on A given by

$$aza = \lambda_a(z)a$$

and

$$bzb = \lambda_b(z)b.$$

We then have

$$acbdafb = \lambda_a(cbd)afb,$$

and

$$acbdafb = \lambda_b(daf)acb.$$

Since $acbda \neq 0$, it follows that $\lambda_a(cbd) \neq 0$. Thus

$$\begin{aligned} afb &= \frac{acbdafb}{\lambda_a(cbd)} \\ &= \frac{\lambda_b(daf)acb}{\lambda_a(cbd)}. \end{aligned}$$

So, $\dim(aAb) \leq 1$, as wanted.

□

Remark 3.4 Suppose A is the semi-simple Banach algebra consisting of all 2 by 2 matrices and let

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\operatorname{rank}(x) = 2$ and clearly $\dim(xAx) = dim(A) = 4$. Hence, $\operatorname{rank}(x) < \dim(xAx)$.

□

Remark 3.5. *If an algebra is not semi-simple, Theorem 3.3 need not hold.* To see this, consider the algebra A of 3 by 3 upper triangular matrices with entries from $l_\infty(\mathbb{N})$. The algebra operations of addition and multiplication are the usual matrix operations and operations within the entries are exactly the algebra operations of $l_\infty(\mathbb{N})$. Let

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

According to Definition 2.1, $\operatorname{rank}(x) = 2$. But

$$xAx = \left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : y \in l_\infty(\mathbb{N}) \right\}.$$

Since $l_\infty(\mathbb{N})$ is infinite dimensional, it follows that $\dim(xAx) = \infty$. So, Theorem 3.3 does not hold in this algebra.

$\square$

Definition 3.6. *An element $y \in A$, A any algebra, is said to be single if $xyz = 0$ for some x and $z \in A$ implies that $xy = 0$ or $yz = 0$.*

Definition 3.7. *An element $x \in A$, A any algebra, is said to be compactly acting if and only if the wedge operator*

$$T_x : A \to A$$

defined by

$$T_x(y) = xyx$$

is compact.

We next introduce another notion of rank, the primitive-rank, and compare it with that of Definition 2.1. Prior to introducing this notion, we establish some facts pertaining to primitive Banach algebras as the primitive-rank notion makes sense only in primitive Banach algebras.

Lemma 3.8 ([3], page 29). *Let A be a primitive Banach algebra. Let $x, y \in A$ with*

$$xAy = \{0\}.$$

Then, either $x = 0$ or $y = 0$.

Proof. Suppose $x \neq 0$ and $y \neq 0$ and let ψ be a faithful irreducible representation of A on a linear space X. Then there are $a, b \in X$ such that

$$\psi(x)a \neq 0 \text{ and } \psi(y)b \neq 0.$$

$\psi(Ay)a$ is a subspace of X which is invariant under each element of $\psi(A)$. Hence,

$$\overline{\psi(Ay)a} = X.$$

Therefore there is a sequence z_n in A such that

$$\lim_{n\to\infty} \psi(z_n y)a = b.$$

We thus have

$$\lim_{n\to\infty} \psi(xz_ny)a = \psi(x)a$$
$$\neq 0.$$

Whence, there exists a z_n such that $xz_ny \neq 0$.

$\square$

Lemma 3.9 ([7], Lemma 1). *Let a be any element of a Banach algebra A.*
(i) If s is a single element of A, so are as and sa.

(ii) If s acts compactly on A, so do as and sa.

Proof. (i) If s is single and $x(as)y = 0$ for some $x, y \in A$, then either $xas = 0$ or $sy = 0$. So, either $xas = 0$ or $asy = 0$. Thus, as is single. By a similar argument, sa can be shown to be single and (i) is established.

(ii) Since the map

$$x \mapsto asxas$$

is a composition of the maps

$$x \mapsto xa \mapsto sxas \mapsto asxas,$$

if s acts compactly then so does as. Similarly sa acts compactly and this completes the proof.

$\square$

Theorem 3.10 ([7], Theorem 4). *Let s be a non-zero compactly acting single element of a semi-simple Banach algebra A. Then, there is an $e \in Min(A)$ such that $s = se$.*

Proof. Since A is assumed semi-simple, using Lemma 3.2, the left ideal As contains an element $b = xs$ which is not quasi-nilpotent and is of unit norm. From Lemma 3.9, b is single and acts compactly on A. Let T be the compact operator on A defined by

$$Ta = bab.$$

Since $T^n b = b^{2n+1}$, we have the following:

$$\|T^n\|^{\frac{1}{n}} \geq \|b^{2n+1}\|^{\frac{1}{n}},$$

and this shows that T is not quasi-nilpotent as b is not quasi-nilpotent. Since T is compact, it follows that T has a non-zero eigenvalue λ and a corresponding finite dimensional eigenspace S_λ. Let $0 \neq l \in S_\lambda$. For any positive integer n,

$$Tlb^n = blb^{n+1} = \lambda lb^n.$$

So, we have $lb^n \in S_\lambda$. Since $\dim(S_\lambda) < \infty$, the set $\{lb^n : n = 1, 2, \dots\}$ is linearly dependent and hence, there is a polynomial $P(.)$ of minimal degree such that $lbP(b) = 0$. Since b is single, either $lb = 0$ or $bP(b) = 0$. Let us suppose P can be factored as a product of non-constant polynomials $P = P_1P_2$. Then, $P_1(b)bP_2(b) = bP(b) = 0$ and since b is single, either $bP_1(b) = 0$ or $bP_2(b) = 0$ contradicting the minimality of P. Therefore, P is of degree one and hence, $b(b - \alpha) = 0$, for some scalar α. Then, $e = \frac{b}{\alpha}$ is an idempotent element of A. From Lemma 3.9, e is single and acts compactly on A. Since the map $a \mapsto eae$ is an identity on eAe, it follows that $\dim(eAe) < \infty$. Also, if $eaeebe = eaebe = 0$, then since e is single, either $eae = 0$ or $ebe = 0$, showing that eAe is an integral domain. But eAe being an integral domain and finite dimensional implies that eAe is a division algebra. So, e is a minimal idempotent, see Definition A.9 in the appendix. We now let $a \in A$ and recall that $b = xs$. Then,

$$\begin{aligned} xs(a - ea) &= \alpha e(a - ea) \\ &= 0. \end{aligned}$$

Since s is single with $xs = \alpha e \neq 0$, we have

$$(s - se)a = 0.$$

Since A is semi-simple and a is arbitrarily chosen from A, we have $s = se$ as wanted.

□

We next prove a theorem that relates compactly acting single elements to rank one elements of a primitive Banach algebra.

Theorem 3.11 ([12], page 747). *Let A be a primitive Banach algebra. An element $0 \neq x \in A$ is single and compactly acting if and only if* $\operatorname{rank}(x) = 1$.

Proof. ($\Rightarrow$) Suppose $0 \neq x$ is single and acts compactly. So, by Theorem 3.10, there is an $e \in Min(A)$ such that

$$x = xe.$$

Therefore, since a minimal idempotent (see Definition A.9), is clearly of rank one and $x \neq 0$, it follows from Proposition 2.2 (b) that

$$\operatorname{rank}(x) = 1, \text{ because } x \neq 0.$$

($\Leftarrow$) Suppose $\operatorname{rank}(x) = 1$. So, x is compactly acting. Thus, we should only show that x is single. Suppose are $a, b \in A$ such that $ax \neq 0$ and $xb \neq 0$. We show that $axb \neq 0$. By the contraposition of Lemma 3.8,

$$axyxb \neq 0 \text{ for some } y \in A\ . \tag{3.1}$$

We have $axyxb = a\lambda xb$, for some $\lambda \in \mathbb{C}$, because $\text{rank}(x) = 1$. Of course $\lambda \neq 0$, otherwise we would have

$$\begin{aligned} axyxb &= a\lambda xb \\ &= 0, \end{aligned}$$

which would then contradict (3.1). So, we have

$$\begin{aligned} axb &= \frac{axyxb}{\lambda} \\ &\neq 0. \end{aligned}$$

Thus, x is single as wanted.

□

We will need the following facts to introduce the notion of primitive-rank for elements of a primitive Banach algebra A. We assume that A contains single compactly acting elements. Hence, from Theorem 3.10, A possesses minimal idempotents.

Lemma 3.12 ([3], Lemma F.2.1). *Let A be a primitive algebra and let e, f $\in Min(A)$ and R be a right ideal of A. Then,*

(i) there exist u, $v \in A$ such that $f = uev$;

(ii) $dim(eAf) = 1$;

(iii) $dim(Re) = dim(Rf)$.

Proof. (i) From Lemma 3.8 we have $eAf \neq \{0\}$. Let v be a non-zero element of eAf. Since Af is a minimal left ideal, we have $Af = Av$, and so $f = uv$ for some $u \in A$. Also, $v = ev$, hence $f = uev$.

(ii) From (i), since R is a right ideal, we have $eAf = eAuev \subset eAev = \mathbb{C}ev$. Whence, (ii) follows.

(iii) From (i) we have $Rf = Ruev \subset Rev$. So if $\dim(Re) < \infty$, so is $\dim(Rev)$ and $\dim(Rf) \leq \dim(Rev) \leq \dim(Re)$. Similarly, if $\dim(Rf) < \infty$, then $\dim(Re) \leq \dim(Rf)$. Also, if Rf is infinite dimensional so is Rev and therefore, so is Re and the analogous argument gives the converse.

□

Theorem 3.13. *Let A be a primitive Banach algebra. If $e \in Min(A)$, then,*

$$\begin{aligned} \pi : A &\to B(Ae) \\ a &\mapsto \pi(a) \end{aligned}$$

defined by

$$\pi(a)x = ax, \ (x \in Ae)$$

is a continuous, faithful, irreducible representation of A on Ae.

Proof. Since $\pi(a)$ is in $B(Ae)$, we have

$$\begin{aligned}\|\pi(a)\| &= \sup_{\|xe\|=1} \|\pi(a)(xe)\| \\ &= \sup_{\|xe\|=1} \|axe\| \\ &\leq \|a\|.\end{aligned}$$

So, π is norm reducing and so it is continuous.

Since, from Lemma 3.8, $xAe = \{0\}$ implies that $x = 0$, π is faithful. Lastly, if B is a subspace of Ae which is invariant under $\pi(x)$, for each $x \in A$, it follows that B is a left ideal of A. It follows from Lemma A.11 that either $B = \{0\}$ or $B = Ae$, because Ae is a minimal ideal. So, π is irreducible.

$\square$

Remark 3.14. With π as in Theorem 3.13 above, we observe that $\pi(x) = xAe$. It follows from Lemma 3.12 (iii) that the rank of the operator $\pi(x) \in B(Ae)$ is independent of the particular choice of $e \in Min(A)$. Thus, the rank notion we introduce in the next definition is well defined.

$\square$

Definition 3.15. *Let A be a primitive Banach algebra such that Min(A) is non-empty. Let $x \in A$, $e \in Min(A)$. Let $\pi : A \to Ae$ be a continuous, faithful, irreducible representation of A on Ae. We define the primitive-rank(x) to be the rank of $\pi(x)$. We denote it by*

$$\text{primitive} - \text{rank}(x).$$

Proposition 3.16. *Let A be a primitive Banach algebra. Let $x \in A$. Then x has a rank one image under some continuous faithful representation if and only if x is single and compactly acting.*

Proof. ($\Leftarrow$) Let s be any non-zero compactly acting single element of A. From Theorem 3.10, there exists $e_0 \in Min(A)$ such that $s = se_0$. Let s be any non-zero compactly acting single element of A. Then from Theorem 3.13, the representation π of A on Ae_0 is faithful, continuous, and irreducible.

We should show that $\text{rank}(\pi(s)) = 1$. But,

$$\pi(s)(Ae) = sAe.$$

So,

$$\text{rank}(\pi(s)) = \dim(sAe).$$

Since

$$sAe_0 \subseteq e_0 A e_0,$$

and from the proof of Theorem 3.10, $e_0 A e_0$ has been shown to be a division algebra, by the Gelfand-Mazur Theorem,

$$\dim(e_0 A e_0) = 1.$$

But,

$$sAe_0 \subseteq e_0 A e_0.$$

Therefore,

$$\dim(sAe_0) \leq \dim(e_0 A e_0) = 1.$$

That is,

$$\mathrm{rank}(\pi(s)) = \dim(sAe) \leq 1.$$

But, $\pi(s) \neq 0$. Therefore, $\mathrm{rank}(\pi(s)) = 1$. Recall from Theorem 2.6 that in $B(Ae)$ the rank notion coincides with the dimension of the range.

($\Leftarrow$) For the converse, suppose $\mathrm{rank}(\pi(s)) = 1$, for some $s \in A$. So,

$$\{\pi(s)\pi(a)\pi(s) : a \in A\} = \{\pi(sas) : a \in A\}$$

has dimension at most one. So, $\dim(sAs) \leq 1$, by faithfulness of π. So, s acts compactly. Also, an operator of rank one is single, for if $T \in B(X)$ is of rank one, from Lemma 2.4, $T = (f \otimes y)$, for some $y \in X$ and some $f \in X'$. So $RTSx = 0$ for all $x \in X$ will imply that $((f \circ S) \otimes Ry)x = 0$ for all $x \in X$ where $f \circ S$ denotes the composition of f with S. This will imply that either $f \circ S = 0$ or $Ry = 0$. Which will mean that either $f \otimes Ry = 0$ or $f \circ S \otimes y = 0$. But $RT = f \otimes Ry$ and $TS = f \circ S \otimes y$. So we will have either $RT = 0$ or $TS = 0$ which will mean that T is single as claimed. Now if $asb = 0$, for some elements $a, b \in A$, then

$$\pi(a)\pi(s)\pi(b) = \pi(asb) = 0.$$

So, either $\pi(as) = 0$ or $\pi(sb) = 0$, which implies that $as = 0$ or $sb = 0$, because π is faithful. Whence, s is single.

□

We are now ready to state and prove the fact that primitive-rank as in Definition 3.15 coincides exactly with the rank concept introduced in Definition 2.1.

Theorem 3.17. *Let A be a primitive Banach algebra and $x \in A$. If $rank(x) < \infty$ or* $\mathrm{primitive} - \mathrm{rank}(x) < \infty$, *then*

$$\mathrm{rank}(x) = \mathrm{primitive} - \mathrm{rank}(x).$$

Proof. Suppose $\text{rank}(x) = 1$. By Theorem 3.11, the single compactly acting elements of A are exactly the rank one elements of A. So by Proposition 3.16, x has a rank one image in $B(Ae)$, $(e \in Min(A))$. Thus,

$$\text{rank}(x) = \text{primitive} - \text{rank}(x).$$

Conversely, suppose $\text{primitive} - \text{rank}(x) = 1$. Again, by Proposition 3.16, x is single and compactly acting. By Theorem 3.11, $\text{rank}(x) = 1$. Therefore,

$$\text{primitive} - \text{rank}(x) = \text{rank}(x).$$

Suppose $\text{rank}(x) = k$. That is,

$$x = x_1 + \ldots + x_k, \text{ with } \text{rank}(x_i) = 1, \ (i = 1, \ldots, k).$$

This implies that

$$\pi(x) = \pi(x_1) + \ldots + \pi(x_k),$$

where $\pi : A \to B(Ae)$ is defined as before by $\pi(x)be = xbe$. Therefore,

$$\begin{aligned}\text{rank}(\pi(x)) &\leq \text{rank}(\pi(x_1)) + \ldots + \text{rank}(\pi(x_k)) \\ &\leq k, \text{because } \text{rank}(\pi(x_i)) = 1 \text{ by the first part of the proof.}\end{aligned}$$

Therefore, $\text{primitive} - \text{rank}(x) \leq k = \text{rank}(x)$.

Suppose now that $\text{primitive} - \text{rank}(x) = k$. Let $a_1e, \ldots, a_ke$ be the basis elements of range of $\pi(x)$. By the Jacobson density theorem, Theorem A.9, there are $b_1, \ldots, b_k \in$ A such that

$$b_ia_je = a_je, \ (i = j),$$

and

$$b_ia_je = 0, \text{ otherwise.}$$

So,

$$\pi(x) = \pi(b_1)\pi(x) + \ldots + \pi(b_k)\pi(x), \text{ with primitive} - \text{rank}(b_ix) = 1.$$

By the faithfulness of the representation of A on Ae, $x = b_1x + \ldots + b_kx$. Therefore $\text{rank}(x) \leq k = \text{primitive} - \text{rank}(x)$. Whence, $\text{rank}(x) = \text{primitive} - \text{rank}(x)$. $\square$

4. FUNCTIONAL CALCULUS AND MULTIPLICITIES OF SPECTRAL POINTS

Throughout this section, a Banach algebra A is assumed to have a unit element 1 unless otherwise stated. We use the notion of rank as in Definition 2.1 to introduce the concept of spectral points with finite multiplicity. Since the notion of multiplicity we introduce relies heavily on the analytic functional calculus, we include it in this section. In fact the functional calculus we establish can be found in [10].

To establish the functional calculus we will make use of the following two formulas.

Let $a \in A$ and $\sigma(a)$ denote the spectrum of a. For $\lambda \in \rho(a) := \mathbb{C} \backslash \sigma(a)$ with $|\lambda| > r(x)$ we have

$$r_\lambda := (\lambda - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}. \tag{4.1}$$

For $\lambda, \mu \in \rho(a)$ it follows that

$$r_\lambda - r_\mu = (\mu - \lambda) r_\lambda r_\mu \tag{4.2}$$

The above equations are well-known in spectral theory and can be found in [10].

Proposition 4.1. *Let $\sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series with the radius of convergence r. Let A be a Banach algebra and let $x \in A$ with the spectral radius $r(x)$. Then the series $\sum_{n=0}^{\infty} \alpha_n x^n$ converges if $r(x) < r$ and diverges if $r(x) > r$.*

Proof. Suppose $r(x) < r$. The case where $r(x) > r$ will follow exactly the same. In fact we show that $\sum_{n=0}^{\infty} \alpha_n x^n$ is absolutely convergent by using the root test and the spectral radius formula.

$$\begin{aligned} \limsup_{n\to\infty} \|\alpha_n x^n\|^{\frac{1}{n}} &\leq \limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}} \limsup_{n\to\infty} \|x\|^{\frac{1}{n}} \\ &= \frac{1}{r} r(x) \\ &< 1. \end{aligned}$$

□

Let A be a Banach algebra and let f be a complex valued function which is analytic for $|\lambda| < r$. That is, for $|\lambda| < r$ we have $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$. If $x \in A$ with $r(x) < r$ then we define

$$f(x) := \sum_{n=0}^{\infty} \alpha_n x^n.$$

If Γ is an oriented, closed, rectifiable curve in $\mathbb{C}$ then Γ is said to be an *integration path.* If

$$f : \Gamma \to X$$

where X denotes a Banach space, we define a *path integral* to be

$$\int_\Gamma f(\lambda) d\lambda$$

as the limit of the Riemann sums (taken in the usual sense)

$$\sum f(\lambda_k)(\lambda_k - \lambda_{k-1}).$$

The existence of this integral is verified as in the complex function theory. For instance, if f is continuous on Γ, then $\int_\Gamma f(\lambda)d\lambda$ exists. From the above definition the following properties follow:

(a) $\int_\Gamma \alpha f(\lambda)d\lambda = \alpha \int_\Gamma f(\lambda)d\lambda.$;

(b) $\int_\Gamma (f(\lambda) + g(\lambda))d\lambda = \int_\Gamma f(\lambda)d\lambda + \int_\Gamma g(\lambda)d\lambda$;

(c) $\| \int_\Gamma f(\lambda)d\lambda\| \leq \max_{\lambda \in \Gamma} \|f(\lambda)\|$(length of Γ);

(d) $x'(\int_\Gamma f(\lambda)d\lambda) = \int_\Gamma x'(f(\lambda))d\lambda$ for every $x' \in X'$; and

(e) $T \int_\Gamma f(\lambda)d\lambda = \int_\Gamma Tf(\lambda)d\lambda$ for every $T \in B(X)$.

Definition 4.2. *Let $\Delta \subseteq \mathbb{C}$ be any region, that is, a connected set in a complex plane. We say a function*

$$f : \Delta \to A$$

is differentiable at a point $\lambda_0 \in \Delta$ if there is $f'(\lambda_0) \in A$ such that

$$\|\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0)\| \to 0$$

as $\lambda \to \lambda_0$ and we call $f'(\lambda_0)$ the derivative of f at λ_0. We say f is differentiable in $\Omega \subseteq \Delta$ if f is differentiable at every $\lambda \in \Omega$.

Proposition 4.3 (Cauchy's Integral Theorem.) *Let X be a Banach space, and let Δ be a non-empty open subset of $\mathbb{C}$. If $f : \Delta \to A$ is differentiable in the region Δ and if Γ_1, Γ_2 are two integration paths each with the same initial and final points which can be deformed into each other continuously in Δ, then*

$$\int_{\Gamma_1} f(\lambda)d\lambda = \int_{\Gamma_2} f(\lambda)d\lambda.$$

In particular,

$$\int_\Gamma f(\lambda)d\lambda = 0,$$

if Γ is a closed curve whose interior contains only points on Δ.

Proof. Let $x' \in X'$. Then

$$x'(\int_{\Gamma_1} f(\lambda)d\lambda) = \int_{\Gamma_1} x'(f(\lambda))d\lambda$$

$$= \int_{\Gamma_2} x'(f(\lambda))d\lambda, \text{ by Cauchy's Integral Theorem for}$$

complex valued analytic functions,

$$= x'(\int_{\Gamma_2} f(\lambda)d\lambda).$$

The result follows as x' was arbitrarily chosen from X'.

□

Proposition 4.4. *Let A be a Banach algebra. If the complex valued function f is analytic for $|\lambda| < r$ and if $x \in A$ with $r(x) < r$, then the equation*

$$f(x) = \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)r_{\lambda}d\lambda$$

is valid, where Γ is a positively oriented circle whose radius lies strictly between $r(x)$ and r and $r_{\lambda} = (\lambda - x)^{-1}$ is the resolvent of x. Of course the integral on the right makes sense as r_{λ} is continuous on Γ.

Proof. Since f is analytic for $|\lambda| < r$, we can write

$$f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n.$$

So,

$$\frac{1}{2\pi i}\int_{\Gamma} f(\lambda)r_{\lambda}d\lambda = \frac{1}{2\pi i}\int_{\Gamma}((\sum_{n=0}^{\infty}\alpha_n)\lambda^n r_{\lambda}d\lambda$$

$$= \sum_{n=0}^{\infty}\alpha_n \frac{1}{2\pi i}\int_{\Gamma}\lambda^n r_{\lambda}d\lambda.$$

Interchanging the sum and the integral is justified by the fact that the radius of Γ is less than r, that is the sum $\sum_{n=0}^{\infty}\alpha_n\lambda^n$. converges uniformly on Γ. From equation (4.1) we have that

$$r_{\lambda} = \sum_{k=0}^{\infty}\frac{x^k}{\lambda^{k+1}}$$

for $\lambda \in \Gamma$. Since the series in (4.1) converges uniformly on Γ, using the integral formula below

$$\int_{\Gamma}\frac{1}{\lambda^k} = \begin{cases} 2\pi i, & \text{if } k = 1; \\ 0, & \text{otherwise,} \end{cases}$$

and integrating termwise, it follows that.

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma}\lambda^n r_\lambda d\lambda &= \frac{1}{2\pi i}\int_{\Gamma}\lambda^n(\sum_{k=0}^{\infty}\frac{x^k}{\lambda^{k+1}})d\lambda \\ &= \sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_{\Gamma}\frac{x^k}{\lambda^{k+1-n}}d\lambda \\ &= x^n.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma}f(\lambda)r_\lambda d\lambda &= \sum_{n=0}^{\infty}\alpha_n\frac{1}{2\pi i}\int_{\Gamma}\lambda^n r_\lambda d\lambda \\ &= \sum_{n=0}^{\infty}\alpha_n x^n \\ &= f(x).\end{aligned}$$

□

For $x \in A$, let $H(x) = \{f : \Delta(f) \to \mathbb{C}$ such that $\sigma(x) \subseteq \Delta(f)$, and $\Delta(f)$ is an open set on which f is analytic.$\}$

Definition 4.5. *A region $B \subseteq \mathbb{C}$ is said to be admissible with respect to $x \in A$ if*

(a) $\sigma(x) \subseteq B$;

(b) *B is open and bounded*;

(c) *the boundary, ∂B, of B consists of finitely many closed rectifiable Jordan curves, $C_1, \ldots, C_n$, which are all pairwise disjoint*;

(d) *the orientation of ∂B is given by the orientation of each C_i where the orientation of each C_i is described as in the complex function theory.*

If $f \in H(x)$, then there is an admissible region B with

$$\sigma(x) \subseteq B \subseteq \overline{B} \subseteq \Delta(f).$$

If B' is another admissible region then, since $\lambda \to r_\lambda$ is by standard spectral theory holomorphic on $\rho(x)$, by Cauchy's Integral Theorem, Proposition 4.3, we have

$$\int_{\partial B} f(\lambda)r_\lambda d\lambda = \int_{\partial B'} f(\lambda)r_\lambda d\lambda.$$

We now define $f(x)$, for all $f \in H(x)$. Let B be an admissible region with $\sigma(x) \subseteq B \subseteq \overline{B} \subseteq \Delta(f)$. Then

$$f(x) := \frac{1}{2\pi i}\int_{\partial B} f(\lambda)r_\lambda d\lambda.$$

We define the following operations on $H(x)$:

$(\alpha f)(\lambda) := \alpha f(\lambda)$ for all $\lambda \in \Delta(f)$;

$(f+g)(\lambda) := f(\lambda) + g(\lambda)$; and

$(fg)(\lambda) := f(\lambda)g(\lambda)$ for all $\lambda \in \Delta(f) \bigcap \Delta(g)$.

These operations are well-defined and make $H(x)$ into an algebra. The following theorem serves as the foundation for an analytic functional calculus:

Theorem 4.6. *The mapping $\phi : H(x) \to A$ defined by $\phi : f \mapsto f(x)$ has the following properties:*

(a) $(\alpha f)(x) = \alpha f(x)$;

(b) $(f+g)(x) = f(x) + g(x)$;

(c) $(fg)(x) = f(x)g(x)$, hence $f(x)$ commutes with $g(x)$. In particular, $f(x)$ commutes with x;

(d) for $f(\lambda) = \lambda^n$, we have $f(x) = x^n$, $(n = 0, 1, 2, \dots)$;

(e) if $f(\lambda) \neq 0$ for all $\lambda \in \sigma(x)$, then $f(x)$ has an inverse $f(x)^{-1} = (\frac{1}{f})(x)$.

Proof. Clearly, (a) and (b) follow from the properties of the integral. From the proof of Proposition 4.4 we have

$$\frac{1}{2\pi i}\int_\Gamma \lambda^n r_\lambda d\lambda = x^n.$$

Hence (d) follows.

For the proof of (c), let B_f and B_g be admissible regions of f and g respectively with

$$\overline{B_f} \subseteq B_g \subseteq \overline{B_g} \subseteq \Delta(f) \bigcap \Delta(g).$$

So,

$$\begin{aligned} f(x)g(x) &= (\frac{1}{2\pi i})^2 \int_{\partial B_f} f(\lambda)(\lambda - x)^{-1} d\lambda \int_{\partial B_g} g(\mu)(\mu - x)^{-1} d\mu \\ &= (\frac{1}{2\pi i})^2 \int_{\partial B_f}\int_{\partial B_g} f(\lambda)g(\mu)(\lambda - x)^{-1}(\mu - x)^{-1} d\mu d\lambda \\ &= (\frac{1}{2\pi i})^2 \int_{\partial B_f}\int_{\partial B_g} f(\lambda)g(\mu)(\mu - \lambda)^{-1}((\lambda - x)^{-1} - (\mu - x)^{-1}) d\mu d\lambda, \end{aligned}$$

which follows from equation (4.2).

Therefore,

$$
\begin{aligned}
f(x)g(x) &= (\frac{1}{2\pi i})^2 \int_{\partial B_f} \int_{\partial B_g} f(\lambda)g(\mu)(\lambda - x)^{-1}(\mu - \lambda)^{-1} d\mu d\lambda \\
&\quad - (\frac{1}{2\pi i})^2 \int_{\partial B_f} \int_{\partial B_g} f(\lambda)g(\mu)(\mu - x)^{-1}(\mu - \lambda)^{-1} d\mu d\lambda \\
&= (\frac{1}{2\pi i})^2 \int_{\partial B_f} f(\lambda)(\lambda - x)^{-1}(\int_{\partial B_g} \frac{g(\mu)}{(\mu - \lambda)} d\mu) d\lambda \\
&\quad - (\frac{1}{2\pi i})^2 \int_{\partial B_g} g(\mu)(\mu - x)^{-1}(\int_{\partial B_f} \frac{f(\lambda)}{(\mu - \lambda)} d\mu) d\lambda \\
&= (\frac{1}{2\pi i})^2 \int_{\partial B_f} f(\lambda)(\mu - x)^{-1}(2\pi i)g(\lambda) d\lambda + 0,
\end{aligned}
$$

which follows from Cauchy's Integral Theorem, Proposition 4.3. Thus,

$$
f(x)g(x) = \frac{1}{2\pi i} \int_{\partial B_f} f(\lambda)g(\lambda)d\lambda(\lambda - x)^{-1} = (fg)(x).
$$

Part (e) follows from (c) , (d) and the equation

$$
(f\frac{1}{f})(\lambda) = 1 \text{ for } \lambda \in \Delta(f).
$$

□

We next proof the Spectral Mapping Theorem which will be needed often in this thesis.

Theorem 4.7 ([10], Proposition 47.1.) *Let A be a Banach algebra with a unit element 1 and let $x \in A$. For every $f \in H(x)$, we have $\sigma(f(x)) = f(\sigma(x))$.*

Proof. Let $\mu \in \sigma(f(x))$. Suppose $\mu \notin f(\sigma(x))$, that is $\mu - f(\lambda)$ for all $\lambda \in \sigma(x)$. Then $\mu - f(x)$ is, by Theorem 4.6(e), invertible. We thus have a contradiction as μ is assumed to be in the spectrum of $f(x)$. So, $\sigma(f(x)) \subseteq f(\sigma(x))$.

For the converse, we suppose that $\mu \in f(\sigma(x))$. That is $\mu = f(\omega)$ for some $\omega \in \sigma(x)$. We define a function g on $\Delta(f)$ by

$$
g(\lambda) := \frac{f(\lambda) - f(\omega)}{\lambda - \omega}, \text{ for } \lambda \neq \omega, \quad g(\omega) := f'(\omega).
$$

Clearly, g lies in $H(x)$. Since

$$
g(\lambda)(\omega - \lambda) = f(\omega) - f(\lambda),
$$

it follows, with the aid of Theorem 4.6(c), that

$$g(x)(\omega - x) = f(\omega) - f(x) = \mu - f(x).$$

Suppose $\mu \in \rho(f(x))$, then

$$((\mu - f(x))^{-1} g(x))(\omega - x) = (\omega - x)((\mu - f(x))^{-1} g(x)) = 1.$$

It follows from the above equation that $\omega - x$ is invertible which cannot be the case because ω lies in $\sigma(x)$. So, $\mu \notin \rho(f(x))$ as supposed. We thus have $f(\sigma(x)) \subseteq \sigma(f(x))$. Therefore

$$\sigma(f(x)) = f(\sigma(x)).$$

□

We will need to establish some background material in order to define spectral projections in a Banach algebra A.

Definition 4.8. *Let A be a Banach algebra and let $x \in A$. A subset σ of $\sigma(x)$ is called a spectral set of x if σ and $\sigma(x) \backslash \sigma$ are closed. We call $\sigma(x) \backslash \sigma$ the spectral set complementary to σ.*

If σ_1 and σ_2 are complementary spectral sets of x, and if Δ_1 and Δ_2 are open disjoint sets that cover σ_1 and σ_2 respectively, then we define on $\Delta := \Delta_1 \bigcup \Delta_2$ the functions f_1 and f_2 by

$$f_1(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \Delta_1 \\ 0, & \text{if } \lambda \in \Delta_2 \end{cases}$$

$$f_2(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \Delta_2 \\ 0, & \text{if } \lambda \in \Delta_1. \end{cases}$$

Clearly $f_1, f_2 \in H(x)$. Define

$$p_1 := f_1(x) \text{ and } p_2 := f_2(x).$$

We have $p_1^2 = f_1(x) f_1(x) = f_1(x) = p_1$ and, analogously, we have $p_2^2 = p_2$. Since $f_1 f_2 = 0$, it follows from Theorem 4.6 (c) that $p_1 p_2 = 0$. This establishes the following proposition:

Proposition 4.9. *Let A be a Banach algebra, and let σ_1 and σ_2 be complementary spectral sets of $x \in A$. If Γ_1, Γ_2 are simple, closed integration paths oriented counter-clockwise which lie in $\rho(x)$ and contain σ_1 and σ_2 in their interior respectively but no other parts of the spectrum, $\sigma(x)$, of x, then the elements*

$$p_i := \frac{1}{2\pi i} \int_{\Gamma_i} (\lambda - x)^{-1} d\lambda, \quad (i = 1, 2)$$

lie in A, $p_i^2 = p_i$ and $p_1 p_2 = 0$. It is permitted that one of the spectral sets be empty.

□

Definition 4.10. *Let A be a Banach algebra with a unit. Let $x \in A$. A point $\lambda \in \sigma_A(x)$ will be called a spectral point with finite multiplicity if λ is an isolated point in $\sigma_A(x)$ and the spectral projection*

$$e = \frac{1}{2\pi i} \int_{\Gamma} (z - x)^{-1} dz,$$

is of finite rank, where $\Gamma \subset \rho(x)$ is a closed simple contour, with λ being the only spectral point of x in Γ. The rank of e will be called the multiplicity of λ.

Proposition 4.11 ([12], Proposition 3.2). *Let A be a Banach algebra with a unit element 1 and let $a \in A$. Suppose that $\sigma_1, \ldots, \sigma_k$ are disjoint, closed and non-empty subsets of $\sigma(a)$ such that $\sigma(a) \backslash \sigma_i$ is also closed, $(i = 1, \ldots, k)$. Let Γ_i be a simple rectifiable curve enclosing σ_i but not σ_j, $(i \neq j)$ and $\Gamma_i \bigcap \sigma(a) = \emptyset$. Define e_i by*

$$e_i = \frac{1}{2\pi i} \int_{\Gamma_i} (z - a)^{-1} dz.$$

Then,

(a) e_i is independent of the choice of Γ_i, provided Γ_i satisfies the stated criteria, $(i = 1, \ldots, k)$,

(b) e_i is an idempotent and $e_i e_j = 0$, $(i \neq j)$,

(c) the sum $e_1 + \ldots + e_k$ is an idempotent,

(d) the spectrum $\sigma_A(ae_i) = \sigma_i \cup \{0\}$ and the spectrum $\sigma_{e_i A e_i}(ae_i) = \sigma_i$, $(i = 1, \ldots, k)$,

(e) if, in addition, $\sigma_A(a) = \sigma_1 \cup \ldots \cup \sigma_k$, then

$$1 = e_1 + \ldots + e_k.$$

Proof. (a) Follows directly from Proposition 4.3.

(b) Follows from Proposition 4.9.

(c) From part (a) and part(b),

$$\begin{aligned}(e_1 + \ldots + e_k)^2 &= e_1^2 + \ldots + e_k^2 + \sum_{i \neq j} e_i e_j \\ &= e_1 + \ldots + e_k + 0.\end{aligned}$$

Therefore, $e_1 + \ldots + e_k$ is an idempotent.

(d) Let

$$f(z) = z, \ (z \in \sigma(a))$$

and let

$$g_i(z) = \begin{cases} 1, & \text{if } z \in \sigma_i; \\ 0, & \text{otherwise.} \end{cases}$$

By functional calculus,

$$a = f(a),$$
$$e_i = g_i(a).$$

Consider

$$h(z) = f(z)g_i(z) = \begin{cases} z, & z \in \sigma_i; \\ 0, & \text{otherwise.} \end{cases}$$

So,

$$\begin{aligned} ae_i &= h(a) \\ &= f(a)g_i(a). \end{aligned}$$

Whence, $\sigma_A(ae_i) = \sigma_i \cup \{0\}$, by the Spectral Mapping Theorem.

We show that $\sigma_{e_iAe_i}(ae_i) = \sigma_i$. Suppose $\lambda \notin \sigma_i$. Let

$$f_i(z) = \begin{cases} 1, & \text{if } z \in \sigma_i; \\ 0, & \text{otherwise,} \end{cases}$$

$$g_i(z) = \begin{cases} \frac{f_i(z)}{(\lambda - z)}, & \text{if } z \in \sigma_i; \\ 0, & \text{otherwise.} \end{cases}$$

Then $g_i \in H(a)$ and from the functional calculus,

$$(\lambda - a)g_i(a) = e_i,$$

and

$$e_ig(a) = g(a)e_i = g(a).$$

Thus

$$(\lambda e_i - ae_i)g(a)e_i = e_i$$

from which it follows that $\lambda \notin \rho_{e_iAe_i}(ae_i)$. Hence

$$\sigma_{e_iAe_i}(ae_i) \subset \sigma_i.$$

Now the following argument will finish the proof.

Suppose $\lambda \in \bigcap_{i=1}^{k} \rho_{e_iAe_i}(ae_i)$, then by an easy compution we have

$$(\lambda - a)^{-1} = (\lambda e_1 - ae_1)^{-1} e_1 + \ldots + (\lambda e_k - ae_k)^{-1} e_k.$$

Hence

$$\lambda \in \rho(a) = (\sigma_1 \cup \ldots \cup \sigma_k)^c.$$

That is,

$$\sigma_1 \cup \ldots \cup \sigma_k \subset \sigma_{e_1Ae_1}(ae_1) \cup \ldots \cup \sigma_{e_kAe_k}(ae_k).$$

Since σ_i are mutually disjoint, it follows that

$$\sigma_i \subset \sigma_{e_i A e_i}(ae_i).$$

Whence,

$$\sigma_{e_i A e_i}(ae_i) = \sigma_i.$$

(e) Let Γ be a closed contour arround $\sigma(a)$. So,

$$1 = \frac{1}{2\pi i}\int_{\Gamma}(z-a)^{-1}dz.$$

But,

$$\frac{1}{2\pi i}\int_{\Gamma}(z-a)^{-1}dz = \frac{1}{2\pi i}\int_{\Gamma_1}(z-a)^{-1}dz + \ldots + \frac{1}{2\pi i}\int_{\Gamma_k}(z-a)^{-1}dz.$$ Thus

$$1 = e_1 + \ldots + e_k.$$

□

We are now going to look into the algebra pAp, where p is a non-trivial idempotent. We think of pAp not as a subalgebra of A, but as an algebra equipped with p as its unit element and $\|.\|_p$, to be defined below, as its, norm equivalent to $\|.\|$. The formula for the alleged norm is as follows:

$$\|pap\|_p = \sup\{\|(pap)(pxp)\| : x \in A, \|pxp\| = 1\}.$$

Theorem 4.12 ([12], page 734). *Let A be a Banach algebra with a unit element 1. Then,*

(a) $\|.\|_p$ is a norm on pAp,

(b) $\|pap\|.\|p\|^{-1} \leq \|pap\|_p \leq \|pap\|$,

(c) $\|p\|_p = 1$,

(d) $\|(pap)(pbp)\|_p \leq \|pap\|_p\|pbp\|_p$,

(e) pAp is a norm-closed subset of A.

Proof. (a) Suppose $pap = 0$. Then, $\|pap\|_p$ is clearly 0. For the converse of this, if $pap \neq 0$, let $x = \frac{p}{\|p\|}$. Then $\|pxp\| = 1$ and since $\|.\|$ is a norm on A, we have

$$\|(pap)(pxp)\| \neq 0.$$

So $\|pap\|_p \neq 0$.

That $\|pap\|_p \geq 0$, follows from the fact that

$$\|(pap)(pxp)\| \geq 0 \ (x \in A).$$

In what follows we will often use the fact that $\|.\|$ is an algebra norm on A.

Let $c \in \mathbb{C}$. So,

$$\begin{aligned}
\|c(pap)\|_p &= \sup\{\|c(pap)(pxp)\| : x \in A, \|pxp\| = 1\} \\
&= |c| \sup\{\|(pap)(pxp)\| : x \in A, \|pxp\| = 1\} \\
&= |c| \|pap\|_p.
\end{aligned}$$

For the triangle inequality, we note the following:

$$\begin{aligned}
\|pap + pbp\|_p &= \sup\{\|(pap + pbp)(pxp)\| : x \in A, \|pxp\| = 1\} \\
&= \sup\{\|(pap)(pxp) + (pbp)(pxp)\| : x \in A, \|pxp\| = 1\} \\
&\leq \sup\{\|(pap)(pxp)\| : x \in A, \|pxp\| = 1\} \\
&\quad + \sup\{\|(pbp)(pxp)\| : x \in A, \|pxp\| = 1\} \\
&= \|pap\|_p + \|pbp\|_p.
\end{aligned}$$

Whence, $\|.\|_p$ is a norm of pAp and part (a) is established.

(b) From the definition of $\|.\|_p$, we have

$$\begin{aligned}
\|pap\|_p &= \sup\{\|(pap)(pxp)\| : x \in A, \|pxp\| = 1\} \\
&\leq \sup\{\|pap\|\|pxp\| : x \in A, \|pxp\| = 1\} \\
&= \|pap\|.
\end{aligned}$$

That is,

$$\|pap\|_p \leq \|pap\|.$$

We only need to show that

$$\|pap\|\|p\|^{-1} \leq \|pap\|_p.$$

Since

$$\|\frac{p}{\|p\|}\| = 1,$$

we have

$$\begin{aligned}
\|pap\|_p &\geq \|(pap)(\frac{p}{\|p\|})\| \\
&= \frac{\|pap\|}{\|p\|}.
\end{aligned}$$

That is,

$$\|pap\|\|p\|^{-1} \leq \|pap\|_p.$$

Hence we have

$$\|pap\|\|p\|^{-1} \leq \|pap\|_p \leq \|pap\|.$$

(c) From the definition of $\|.\|_p$, we have

$$\begin{aligned}\|p\|_p &= \sup\{\|(ppp)(pxp)\| : x \in A, \|pxp\| = 1\} \\ &= \sup\{\|pxp\| : x \in A, \|pxp\| = 1\} \\ &= 1.\end{aligned}$$

(d) Consider

$$L_{pap} \in B(pAp),$$

defined by

$$L_{pap}(pbp) = (pap)(pbp), \ (b \in A).$$

Note that

$$\begin{aligned}\|L_{pap}\| &= \sup_{\|pxp\|=1} \|L_{pap}(pxp)\| \\ &= \sup\{\|(pap)(pxp)_{\|} : x \in A, \|pxp\| = 1\} \\ &= \|pap\|_p.\end{aligned}$$

But,

$$\|L_{pap}L_{pbp}\| \leq \|L_{pap}\|\|L_{pbp}\|.$$

Therefore,

$$\|(pap)(pbp)\|_p \leq \|pap\|_p\|pbp\|_p.$$

(e) Remember that $pAp = \{a \in A : pap = a\}$. Now, observe that

$$px_np \to x \text{ implies that } p^2x_np^2 \to pxp.$$

This follows from the fact that

$$p^2x_np^2 = p(px_np)p.$$

So,

$$px_np \to pxp.$$

Therefore,

$$pxp = x.$$

Whence,

$$x \in pAp.$$

Therefore, pAp is indeed a norm closed subset of A.

□

Proposition 4.13 ([12], Proposition 3.3). *Let A be a Banach algebra. Let $a \in A$ be an idempotent of rank n. Then, there are elements $a_1, \ldots, a_n \in A$ such that*

$$a = a_1 + \ldots + a_n, \tag{4.3}$$

where rank(a_i) = 1, i = 1, ... ,n and $a_1, \ldots, a_n$ are mutually orthogonal idempotents. Conversely, if (4.3) *holds for mutually orthogonal rank one idempotents $a_1, \ldots, a_n$, then a is a rank n idempotent.*

Proof. Let $a \in A$ be an idempotent with $\operatorname{rank}(a) = n$. So,

$$a = x_1 + \ldots + x_n,$$

where $x_i \in A$ and $\operatorname{rank}(x_i) = 1$, $(i = 1, \ldots, n)$. Let

$$b_i = ax_ia, \ (i = 1, \ldots, n).$$

Then,

$$b_1 + \ldots + b_n = a(x_1 + \ldots + x_n)a.$$

That is,

$$b_1 + \ldots + b_n = a, \ \text{because } a \text{ is an idempotent.}$$

Also,

$$\begin{aligned} \operatorname{rank}(b_i) &= \operatorname{rank}(ax_ia), \ (i = 1, \ldots, n), \\ &\leq \operatorname{rank}(x_i), \ (i = 1, \ldots, n), \\ &= 1. \end{aligned}$$

Also, $\operatorname{rank}(b_i) \neq 0$, for, $\operatorname{rank}(b_i) = 0$ for some i will contradict $\operatorname{rank}(a) = n$. Therefore, $\operatorname{rank}(b_i) = 1$, $(i = 1, \ldots, n)$. We next consider B, the algebra generated by $\{b_1, \ldots, b_n\}$. Note that

$$b_i = ax_ia \text{ implies that } ab_i = a^2x_ia = b_i,$$

and

$$b_ia = ax_ia^2 = b_i.$$

It follows that a is the identity for the algebra B. Suppose $(a - b_i)$ is invertible in B. Then, there is an $x \in B$ such that

$$\begin{aligned} (a - b_i)x &= x(a - b_i) \\ &= a. \end{aligned}$$

Clearly, $x \in A$ as well. So,

$$\begin{aligned} n = \ & \operatorname{rank}(a) \\ &= \operatorname{rank}((a - b_i)x) \\ &\leq \operatorname{rank}(a - b_i) \\ &\leq \operatorname{rank}(b_1) + \ldots + \operatorname{rank}(b_{i-1}) + \operatorname{rank}(b_{i+1}) + \ldots + \operatorname{rank}(b_n) \\ &= n - 1, \ \text{which yields a contradiction.} \end{aligned}$$

Thus, $a - b_i$ is not invertible in B. That is, $1 \in \sigma_B(b_i)$.

Since $\text{rank}(b_i) = 1$, $(i = 1, \ldots, n)$, it follows that

$$b_i^2 = \lambda_i b_i, \text{ for some } \lambda_i \in \mathbb{C}.$$

Note that this formula holds in B as well. Now consider the polynomial

$$P(t) = t^2 - \lambda_i t.$$

We thus have

$$P(b_i) = 0.$$

So,

$$\{0\} = \sigma(P(b_i)) = P(\sigma(b_i)),$$

that is,

$$P(\sigma(b_i)) = \{0\}.$$

Since $1 \in \sigma_B(b_i)$, it follows that $P(1) = 0$. But,

$$P(1) = (1 - \lambda).$$

Therefore, $1 - \lambda = 0$ which implies that $\lambda = 1$. So, we have $b_i^2 = b_i$ and b_i is an idempotent, $(i = 1, \ldots, n)$. We complete the proof by induction. The statement for $n = 2, 3, \ldots$ to be proven is P_n as follows: Suppose $a \in A$ is of rank n and $a^2 = a$. If there exists $b_1, \ldots, b_n$ such that $b_i^2 = b_i$, $\text{rank}(b_i) = 1$, $ab_i = b_i a = b_i$ for all $i = 1, \ldots, n$ and $a = b_1 + \ldots + b_n$, then there exists $a_1, \ldots, a_n$ enjoying all of the above properties listed for $b_1, \ldots, b_n$ and the additional property that $a_i a_j = 0$ when $i \neq j$.

For P_2. Given $a = b_1 + b_2$ with the hypothesis of P_2, we have

$$\begin{aligned} b_1 &= b_1^2 \\ &= b_1(a - b_2) \\ &= b_1 a - b_1 b_2. \end{aligned}$$

Therefore,

$$\begin{aligned} b_1 b_2 &= b_1 - b_1 a \\ &= 0, \text{ because } b_1 a = b_1. \end{aligned}$$

Now suppose P_n is true for $n < k$. Suppose now that $\text{rank}(a) = k$ and $b_1, \ldots, b_k$ satisfy the hypotheses of P_k. Now let

$$a' = b_2 + b_3 + \ldots + b_k.$$

Therefore,

$$a' = a - b_1. \tag{4.4}$$

Note that

$$\begin{aligned}(a')^2 &= (a - b_1)^2 \\ &= a - ab_1 - b_1 a + b_1^2 \\ &= a - b_1 - b_1 + b_1 \\ &= a'.\end{aligned}$$

So, a' is an idempotent. Define $b_i' = a'b_ia'$ for $i = 2, \dots, k$ and let $x \in$ A. We then have,

$$\begin{aligned}b_i'xb_i' &= (a'b_ia')x(a'b_ia'), \text{ following from the definition of } b_i', \\ &= a'(b_i(a'xa')b_i)a'.\end{aligned}$$

Therefore,

$$\begin{aligned}b_i'xb_i' &= a'\lambda b_ia', \text{ because } \operatorname{rank}(b_i) = 1, \ (i = 2, 3, \dots, k), \\ &= \lambda a'b_ia' \\ &= \lambda b_i', \text{ following from the definition of } b_i'.\end{aligned}$$

Therefore,

$$\operatorname{rank}(b_i') = 1, \ (i = 2, 3, \dots, k).$$

Also,

$$\begin{aligned}a'b_i' &= a'a'b_ia' \\ &= a'b_ia' \\ &= a'b_ia'a' \\ &= b_i'a' = b_i',\end{aligned}$$

with

$$a' = b_2' + \dots + b_k'.$$

By the induction hypothesis, there are rank one elements $a_2, \dots, a_k$ such that

$$\begin{aligned}a_i^2 &= a_i, \\ a_ia' &= a'a_i = a_i, \\ a' &= a_2 + \dots + a_k, \text{ and} \\ a_ia_j &= 0, \ (i \neq j).\end{aligned}$$

We define

$$a_1 = b_1.$$

So that,

$$a' = a - a_1. \tag{4.5}$$

So,

$$\begin{aligned} a &= a' + a_1 \\ &= (a_2 + \ldots + a_k) + a_1. \end{aligned}$$

Thus, we only need to show that $a_1 a_i = a_i a_1 = 0$, $(i \neq 1)$. Since $a_i a' = a' a_i = a_i$, by the inductive hypothesis, we have

$$\begin{aligned} a_1 a_i &= a_1 a' a_i \\ &= a_1(a - a_1)a_i, \text{ which follows from equation (4.4),} \\ &= (a_1 a - a_1^2)a_i \\ &= (a_1 - a_1)a_i, \text{ by hypothesis,} \\ &= 0. \end{aligned}$$

Similarly, $a_i a_1 = 0$.

Conversely, Suppose $a_1, \ldots, a_n$ are mutually orthogonal rank one idempotents, with

$$a = a_1 + \ldots + a_n.$$

We want to show that $\operatorname{rank}(a) = n$ and that a is an idempotent. That a is an idempotent, is easy to see, for,

$$\begin{aligned} a^2 &= (a_1 + \ldots + a_n)^2 \\ &= a_1^2 + \ldots + a_n^2 + \sum_{i \neq j} a_i a_j \\ &= a_1 + \ldots + a_n + 0, \text{ 0 emanating from mutual orthogonality of the } a_i's, \\ &= a. \end{aligned}$$

For the rest of this proof we establish that $\operatorname{rank}(a) = n$. Clearly,

$$\begin{aligned} \operatorname{rank}(a) &\leq \operatorname{rank}(a_1) + \ldots + \operatorname{rank}(a_n) \\ &= n. \end{aligned}$$

Suppose $\operatorname{rank}(a) < n$. It will then follow that

$$a = b_1 + \ldots + b_k, k < n, \text{ with } \operatorname{rank}(b_i) = 1, \ (i = 1, \ldots, k).$$

Let S be a finite dimensional Banach algebra generated by

$$\{1, a_1, \ldots, a_n, b_1, \ldots, b_k.\}$$

From Remark A.20 we see that S is a direct sum of its radical, $Rad(S)$ and a semi-simple Banach algebra A_s. That is,

$$S = Rad(S) \oplus A_s.$$

We thus have

$$a_i = a_{ir} \oplus a_{is}, \ (a_{ir} \in Rad(S)\ , a_{is} \in A_s), \ (i = 1, \ldots, n).$$

and

$$b_i = b_{ir} \oplus b_{is}, \ (b_{ir} \in Rad(S)\ , b_{is} \in A_s), \ (i = 1, \ldots, n).$$

We claim that

$$a_{1s}, \ldots, a_{ns}$$

are mutually orthogonal idempotents, for, $a_i^2 = a_i$ implies, by using the fact that $Rad(S)$ is an ideal and the uniqueness of the direct sum decomposition, that $a_{is}^2 = a_{is}$. So, $a_{1s}, \ldots, a_{ns}$ are idempotents.

For mutual orthogonality we use exactly the same argument. So, $a_{is}a_{js} = 0$. Note that $a_{is} \neq 0$, otherwise $a_i = a_{ir}$ and that will mean, by Remark A.4, that $r(a_{ir}) = 0$, because $a_{ir} \in Rad(S)$. Since a_{ir} is an idempotent, this will mean that $a_{ir} = 0$.

That $\text{rank}(a_{is}) = 1$, is just a routine check.

Let $b_{1s}, \ldots, b_{qs}, \ (q \leq k)$ be non-zero elements among the $b_{1s}, \ldots, b_{ks}$. Similarly, $\text{rank}(b_{js}) = 1, \ (j = 1, \ldots, q)$ as elements of A_s.

Lastly, we also have

$$\begin{aligned} a_s &= a_{1s} + \ldots + a_{ns} \\ &= b_{1s} + \ldots + b_{qs}. \end{aligned} \tag{4.6}$$

This is motivated by the fact that

$$a = b_1 + \ldots + b_q = (b_{1r} + \ldots + b_{qr}) \oplus (b_{1s} + \ldots + b_{qs}),$$

and

$$a = (a_{1r} + \ldots + a_{nr}) \oplus (a_{1s} + \ldots + a_{ns}).$$

Since A_s is a finite dimensional semi-simple algebra,

$$a_{js} = a_{js}^{(1)} \oplus \ldots \oplus a_{js}^{(1)} \oplus \ldots \oplus a_{js}^{(p)} \oplus \ldots \oplus a_{js}^{(p)},$$

where $a_{js}^{(k)}$ is a d_k by d_k matrix repeated r_k times in the direct sum. Hence, Remark 2.20 gives

$$\text{rank}(a_{js}) = r_1\text{rank}(a_{js}^{(1)}) + \ldots + r_p\text{rank}(a_{js}^{(p)}).$$

Since a_{js}'s are rank one mutually orthogonal matrices, it follows that

$$\operatorname{rank}(a_s) = \sum_{i=1}^{n} \operatorname{rank}(a_{js}) = n.$$

Thus, we have contradicted the assumption that $q \leq k < n$ and equation (4.6). Therefore, $\operatorname{rank}(a) = n$ as required and this completes the proof of the theorem.

□

Corollary 4.14. *Let $a_1, \ldots, a_r$ be mutually orthogonal idempotents in A of finite ranks. Then,*

$$\operatorname{rank}(a_1 + \ldots + a_r) = \operatorname{rank}(a_1) + \ldots + \operatorname{rank}(a_r).$$

Proof. Using Proposition 4.13, we can write each a_i as a sum of rank one mutually orthogonal idempotents a_{ij}'s. Equation (4.5) of Proposition 4.13 tells us that $\operatorname{rank}(a_1 + \ldots + a_r) = \operatorname{rank}(a_1) + \ldots + \operatorname{rank}(a_r)$.

□

5. THE SPECTRAL POINT SEQUENCE, APPROXIMATION NUMBERS AND THEIR PROPERTIES

Throughout this section A denotes a Banach algebra endowed with a unit element 1 unless otherwise stated.

To generalise Yamamoto's Theorem to a general Banach algebra setting, we need to make an extensive study of the properties that the spectrum of an element in a Banach algebra possesses regarding the finite multiplicity property points. We now introduce the notion of a spectral point sequence.

Let $x \in A$. Let $r(x)$ be the spectral radius of x. Consider the following set:

$$S_1 = \{z : |z| = r(x)\} \bigcap \sigma(x). \tag{5.1}$$

If the set S_1 is comprised of only finite multiplicity spectral points of x, we then set

$$\mu_i(x) = r(x), \ (i = 1, \dots, n),$$

where n is the sum of the multiplicities of the finite multiplicity spectral points of x in S_1. If S_1 has some spectral points of x with infinite multiplicities, we set

$$\mu_i(x) = r(x), \ (i = 1, 2, \dots).$$

In the former case we continue by considering $x_1 = (1-e)x(1-e)$ as an element of the algebra $(1-e)A(1-e)$, where e is the sum of the Riesz idempotents corresponding to the points in S_1. It is worth noting that $r(x_1) < r(x)$, because

$$\sigma(x)\backslash S_1 = \sigma((1-e)x(1-e)),$$

because $\sigma_{eAe}(exe) = S_1$, by part (d) of Proposition 4.11. We consider the set

$$S_2 = \{z : |z| = r(x_1)\} \bigcap \sigma(x_1).$$

If S_2 consists only of finite multiplicity spectral points of x_1, we set

$$\mu_{n+i}(x) = r(x_1), \ (i = 1, \dots, n_1),$$

where n_1 is the sum of the multiplicities of the points of S_2. Otherwise, we set

$$\mu_{n+i}(x) = r(x_1), \ (i = 1, 2, \dots).$$

Continuing like this, we obtain the sequence

$$\mu_1(x) \geq \mu_2(x) \geq \dots \tag{5.2}$$

of non-negative numbers called the spectral point sequence of x.

The sequence (5.2) can either be infinite or finite. It can be finite if $\sigma(x)$ is comprised of only a finite number of points of finite multiplicity and this is the case when the Banach algebra A is finite dimensional and semi-simple, see Proposition A.19.

We use $n(x)$ to denote the length of the sequence (5.2).

Definition 5.1. *Let $x \in A$. The j^{th} approximation number of x is given by*

$$a_j(x) = \inf\{\|x - y\| : y \in A, rank(y) < j\}.$$

Properties of approximation numbers for elements of $B(X)$, where X is a Banach space have been thoroughly investigated in the literature, see for instance monographs [6] and [13].

In the next lemma we state and prove some properties of approximation numbers for elements of A, necessary for the proofs of the main result in this thesis.

Lemma 5.2. *Let x, $y \in A$.*
(a) If $m \geq n$ then $a_m(x) \leq a_n(x)$, and $a_1(x) = \|x\|$;
(b) $a_{m+n-1}(xy) \leq a_m(x)a_n(y)$;
(c) $a_{m+n-1}(x+y) \leq a_m(x) + a_n(y)$;
(d) $a_m(xy) \leq a_m(x)\|y\|$ and $a_n(xy) \leq a_n(y)\|x\|$;
(e) if x is an idempotent of rank at least m then $a_m(x) \geq 1$.

Proof. (a) Follows trivially from Definition 5.1.

(b) Let $\epsilon > 0$ and $\tilde{x}, \tilde{y} \in A$ be given such that $\text{rank}(\tilde{x}) < m$ and $\text{rank}(\tilde{y}) < n$ with $\|x - \tilde{x}\| < a_m(x) + \epsilon$ and $\|y - \tilde{y}\| < a_n(y) + \epsilon$. Using Proposition 2.2 it follows that

$$\begin{aligned} \text{rank}(\tilde{x}(x - \tilde{y}) + x\tilde{y}) &\leq \text{rank}(\tilde{x}(y - \tilde{y})) + \text{rank}(x\tilde{y}) \\ &\leq \text{rank}(\tilde{x}) + \text{rank}(\tilde{y}) \\ &< m + n - 1. \end{aligned}$$

We thus have,

$$\begin{aligned} a_{m+n-1}(xy) &= \inf\{\|xy - z\| : z \in A,\ \text{rank}(z) < m + n - 1\} \\ &\leq \|xy - (\tilde{x}(y - \tilde{y}) + x\tilde{y})\| \\ &= \|xy - \tilde{x}y + \tilde{x}\tilde{y} - x\tilde{y}\| \\ &= \|x(y - \tilde{y}) - \tilde{x}(y - \tilde{y})\| \\ &= \|(x - \tilde{x})(y - \tilde{y})\| \\ &\leq \|x - \tilde{x}\|.\|y - \tilde{y}\| \\ &< (a_m(x) + \epsilon)(a_n(y) + \epsilon). \end{aligned}$$

Therefore, since $\epsilon > 0$ was chosen arbitrarily,

$$a_{m+n-1}(xy) \leq a_m(x)a_n(y).$$

(c) Let $\epsilon > 0$ and let $\tilde{x}, \tilde{y} \in A$ be such that $\operatorname{rank}(\tilde{x}) < m$ and $\operatorname{rank}(\tilde{y}) < n$ and $\|x - \tilde{x}\| < a_m(x) + \frac{\epsilon}{2}$ and $\|y - \tilde{y}\| < a_n(y) + \frac{\epsilon}{2}$.

Since $\operatorname{rank}(\tilde{x} + \tilde{y}) < m + n - 1$, it follows that

$$\begin{aligned} a_{m+n-1}(x+y) &\leq \|x + y - (\tilde{x} + \tilde{y})\| \\ &= \|x + y - \tilde{x} - \tilde{y}\| \\ &= \|(x - \tilde{x}) + (y - \tilde{y})\| \\ &\leq \|x - \tilde{x}\| + \|y - \tilde{y}\| \\ &< a_m(x) + a_n(y) + \epsilon. \end{aligned}$$

Hence,

$$a_{m+n-1}(x+y) \leq a_m(x) + a_n(y).$$

(d) From (b) with $n = 1$ and using (a), we get

$$\begin{aligned} a_m(xy) &\leq a_m(x)a_1(y) \\ &= a_m(x)\|y\|. \end{aligned}$$

Similarly, with $m = 1$, we get,

$$a_n(xy) \leq a_n(y)\|x\|.$$

(e) We prove this by contradiction. Suppose x is an idempotent of rank at least m and $a_m(x) < 1$. Then, there is a $y \in A$ with $\operatorname{rank}(y) < m$ such that $\|x - y\| < 1$. It then follows that $1 - x + y$ is invertible. So,

$$\begin{aligned} m &\leq \operatorname{rank}(x) \\ &= \operatorname{rank}(x(1 - x + y)(1 - x + y)^{-1}) \\ &\leq \operatorname{rank}(x(1 - x + y)) \\ &= \operatorname{rank}(xy) \\ &\leq \operatorname{rank}(y) \\ &< m, \end{aligned}$$

which yields a contradiction.

□

Lemma 5.3. *Let $x \in A$ and define the number*

$$e_k(x) = \sup\{\|y\|^{-1} : k \leq \operatorname{rank}(xy) \leq \infty, (yx)^2 = yx, x \in A\}, \ (k \geq 1).$$

If the set of all $y \in A$ such that $k \leq \text{rank}(xy) \leq \infty$ and $(yx)^2 = yx$ is empty, we leave e_k undefined. Note that this happens when rank$(x) < k$. If $e_k(x)$ exists, then $e_k(x) \leq a_k(x)$.

Proof. Note that

$$\begin{aligned} \|yx\| &= \|(yx)(yx)\| \\ &\leq \|yx\|\|yx\|. \end{aligned}$$

Therefore, $\|yx\| \geq 1$. Thus, $\|y\| \geq \|x\|^{-1}$. So, if the set of such y's is non-empty, $e_k(x) \leq \|x\| < \infty$. Let $\epsilon > 0$ be given and let y satisfy

$$e_k(x) - \epsilon \leq \|y\|^{-1}, \text{ rank}(yx) \geq k \text{ and } (yx)^2 = yx.$$

Then, by Lemma 5.2,

$$\begin{aligned} 1 &\leq a_k(xy) \\ &\leq a_k(x)\|y\|. \end{aligned}$$

So,

$$\begin{aligned} e_k(x) - \epsilon &\leq \|y\|^{-1} \\ &\leq a_k(x). \end{aligned}$$

Therefore, $e_k(x) \leq a_k(x)$.

□

The following sequence of lemmas paves a way for the proofs of the main result, Theorem 5.9, to be stated later in the section.

Lemma 5.4 ([12], Lemma 4.5). *Let $x \in A$ and let $\{\mu_n(x)\}_{n=1}^{\infty}$ be the spectral point sequence of x where $n(x) \leq \infty$. Let n be a positive integer not greater than $n(x)$. Then,*

$$\limsup_{m\to\infty} (a_n(x^m))^{\frac{1}{m}} \leq \mu_n(x).$$

Proof. Let e be the sum of the Riesz idempotents corresponding to the spectral points of x with absolute value strictly greater than $\mu_n(x)$. Since each such a spectral point is of finite multiplicity, rank$(e) < \infty$. Let $k = \text{rank}(e)$. By definition of $\mu_n(x)$ and part (a) of Lemma 5.2, $k < n$. So, for any non-negative integer m,

$$\begin{aligned} a_n(x^m) &\leq a_{k+1}(x^m), \text{ because } k \leq k-1 \text{ implies } k+1 \leq n \\ &\leq \|x^m - ex^m\|, \text{ because rank}(ex^m) \leq k < k+1 \\ &= \|(1-e)x^m\|. \end{aligned}$$

By the spectral radius formula and properties of the Riesz idempotents, we have

$$\begin{aligned} \lim_{m\to\infty} \|(1-e)x^m\|^{\frac{1}{m}} &= \lim_{m\to\infty} \|(1-e)x(1-e)^m\|^{\frac{1}{m}} \\ &= \mu_n(x). \end{aligned}$$

Therefore,

$$\lim_{m\to\infty}(a_n(x^m))^{\frac{1}{m}} \le \lim_{m\to\infty}\|(1-e)x^m\|^{\frac{1}{m}}$$
$$= \mu_n(x).$$

□

Theorem 5.5 ([2], page 46). *Let $x \in A$. Suppose $\alpha \notin \sigma(x)$. Then,*
$$\mathrm{dist}(\alpha, \sigma(x)) = \frac{1}{r((\alpha-x)^{-1})}.$$

Proof. Let Ω be an open set containing $\sigma(x)$, but not α. Then $f(\lambda) = \frac{1}{\alpha-\lambda}$ is holomorphic on Ω. So, $\sigma((\alpha-x)^{-1}) = \{\frac{1}{\alpha-\lambda} : \lambda \in \sigma(x)\}$. In particular,

$$r((\alpha-x)^{-1}) = \sup\{\frac{1}{|\alpha-\lambda|} : \lambda \in \sigma(x)\}$$
$$= \frac{1}{\inf\{|\alpha-\lambda| : \lambda \in \sigma(x)\}}$$
$$= \frac{1}{dist(\alpha, \sigma(x))}.$$

□

Corollary 5.6. *If 0 is not in $\sigma(x)$ and $\mu_0 = \min\{|\lambda| : \lambda \in \partial\sigma(x)\}$, then*

$$r(x^{-1}) = \frac{1}{\mu_0}.$$

Proof. In Theorem 5.5, let $\alpha = 0$. Thus,

$$r(x^{-1}) = \frac{1}{dist(0, \sigma(x))}$$
$$= \frac{1}{\mu_0}.$$

□

Lemma 5.7. *Let $y \in A$. Let $\beta \in \mathbb{R}^+$. Then,*

$$\lim_{m\to\infty}\|y^m\|^{\frac{1}{m}} = \frac{1}{\beta} \quad \textit{implies that} \quad \lim_{m\to\infty}(\|y^m\|^{-1})^{\frac{1}{m}} = \beta.$$

Proof. We have

$$\lim_{m\to\infty}(\|y^m\|^{-1})^{\frac{1}{m}} = \lim_{m\to\infty}\left(\frac{1}{\|y^m\|}\right)^{\frac{1}{m}}$$
$$= \frac{1}{\lim_{m\to\infty}\|y^m\|^{\frac{1}{m}}}$$
$$= \beta.$$

$\square$

Lemma 5.8 ([12], Lemma 4.6). *Let $a \in A$, $\{\mu_n(a)\}_{n=1}^{\infty}$ be the spectral point sequence of a and $n(x) \leq \infty$. Let $n \leq n(a)$ be given and suppose $\mu_n(a) > 0$. Let k denote the sum of the multiplicities of the spectral points of a with absolute value at least $\mu_n(a)$. Then,*

$$e_k(a^m) \text{ exists, } \textit{for all } m \in \mathbb{Z}^+ \text{ and } \liminf_{m\to\infty}(e_k(a^m))^{\frac{1}{m}} \geq \mu_k(a).$$

Proof. Let e be the sum of the Riesz idempotents corresponding to the spectral points of a with absolute value greater than or equal to $\mu_n(a)$. By Corollary 4.14, rank$(e) = k$. By Theorem 4.12, eAe is a Banach algebra with norm $\|.\|_e$, equivalent to $\|.\|$ on A. Moreover, $\sigma_{eAe}(eae)$ consists of all $\lambda \in \sigma(a)$ such that $|\lambda| \geq \mu_k(a)$, the inequality following from the nature of the sequence (5.2). So, $0 \notin \sigma_{eAe}(eae)$. This says that eae is invertible in eAe, that is, there is a $b \in A$ such that

$$(eae)(ebe) = (ebe)(eae) = e.$$

By part (c) of Theorem 4.6,

$$eae = ea.$$

Thus,

$$\begin{aligned}(ebe)(eae) &= (ebe)(ea)\\ &= (ebe)a.\end{aligned}$$

Therefore, for every $m \in \mathbb{Z}^+$, $(ebe)^m a^m = e$. To see that $e_k(a^m)$ is defined, let $y = (ebe)^m$ then

$$((y)a^m)^2 = (y)a^m \text{ and rank}(a^m(y)) \geq k.$$

In this case rank$(a^m(y)) =$ rank$(e) = k$, so $e_k(a^m)$ is well defined. In fact,

$$e_k(a^m) \geq \|(ebe)^m\|^{-1}, \text{ by the definition of } e_k(a^m).$$

Since

$$ebe = (eae)^{-1} \text{ in } eAe,$$

it follows that

$$r(ebe) = \frac{1}{\mu_k(a)}, \text{ by Corollary 5.6.}$$

Thus,

$$\lim_{m\to\infty} \|(ebe)^m\|_e^{\frac{1}{m}} = \frac{1}{\mu_k(a)}, \text{ which is Beurling's formula.}$$

So,

$$\lim_{m\to\infty} \|(ebe)^m\|^{\frac{1}{m}} = \frac{1}{\mu_k(a)}, \text{ because } \|.\|_e \text{ is equivalent to } \|.\|.$$

Therefore,

$$\mu_k(a) = \lim_{m\to\infty} \|(ebe)^m\|^{-1})^{\frac{1}{m}}, \text{ by Lemma 5.7.}$$

So,

$$\liminf_{m\to\infty} (\|(ebe)^m\|^{-1})^{\frac{1}{m}} \leq \liminf_{m\to\infty} (e_k(a^m))^{\frac{1}{m}}, \text{ because } \|(ebe)^m\|^{-1} \leq e_k(a^m),$$

that is,

$$\lim_{m\to\infty} (\|(ebe)^m\|^{-1})^{\frac{1}{m}} \leq \liminf_{m\to\infty} (e_k(a^m))^{\frac{1}{m}}.$$

Thus,

$$\mu_k(a) \leq \liminf_{m\to\infty} (e_k(a^m))^{\frac{1}{m}}.$$

□

We are now in a position to state the main result of this paper and prove a special case thereof. A proof for the general case is relegated to the next section.

Theorem 5.9 ([12], Theorem 4.2). *Let $a \in A$, and let $\{\mu_n(a)\}_{n=1}^{\infty}$ be the spectral point sequence of a, where $n(a) \leq \infty$. Define $\mu(a) = \lim_{j\to\infty} \mu_j(a)$ if $n(a) = \infty$, otherwise, $\mu(a) = \mu_{n(a)}(a)$.*

(a) Assume $n(a) = \infty$. Then, for every n such that $\mu_n(a) > \mu(a)$, we have

$$\lim_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} = \mu_n(a), \tag{5.3}$$

and if $\mu_n(a) = \mu(a)$ then,

$$\limsup_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} \leq \mu(a). \tag{5.4}$$

(b) Assume $n(a) < \infty$. Then, equation (5.3) holds for every $n \leq n(a)$.

Proof. (a) Let $n \in \mathbb{Z}^+$ be given.

$$\lim_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} \leq \mu_n(a), \text{ by Lemma 5.4.}$$

So, the inquality (5.4) always hold. If $\mu_n(a) > \mu(a)$, then let k be the sum of the multiplicities of the spectral points with absolute value at least $\mu_n(a)$. We observe that $\mu_k(a) = \mu_n(a)$ and $k \geq n$. So,

$$\begin{aligned}\mu_n(a) &\leq \liminf_{m\to\infty} (e_k(a^m))^{\frac{1}{m}}, \text{ just by Lemma 5.8,}\\ &\leq \liminf_{m\to\infty} (a_n(a^m))^{\frac{1}{m}}, \text{ from Lemma 5.3,}\\ &\leq \limsup_{m\to\infty} (a_n(a^m))^{\frac{1}{m}}\\ &\leq \mu_n(a), \text{ by Lemma 5.4.}\end{aligned}$$

That is,

$$\liminf_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} = \limsup_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} = \mu_n(a).$$

So,

$$\lim_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} = \mu_n(a),$$

and so, part (a) is proved.

(b) If $\mu_n(a) > 0$, then the above argument applies just fine. If $\mu_n(a) = 0$, Lemma 5.4 says that

$$\limsup_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} \leq \mu_n(a) = 0.$$

Therefore,

$$\lim_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} = \limsup_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} = 0.$$

Whence,

$$\lim_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} = 0 = \mu_n(a).$$

□

To explore the inequality (5.4) further, some work will need to be done and in the next section we look into that.

6. THE SPECTRAL RADIUS PROPERTY FOR BANACH ALGEBRAS

So far we have explored the limiting behaviour of $(a_n(a^m))^{\frac{1}{m}}$, for a fixed n as m increases, for the case $n(a) < \infty$ and for the case $n(a) = \infty$ with $\mu_n(a) > \mu(a)$. For the case $\mu_n(a) = \mu(a)$, with $n(a) = \infty$, we do the work in this section. As mentioned in the introduction, Nylen and Rodman conjectured that in this case we have equality with *limsup* replaced by *limit*. We prove the truth of their conjecture in this section.

To formulate the general result, we first investigate $\bar{F}_A^{\|\cdot\|}$, the norm closure of finite rank elements of a Banach algebra A. We will consider the factor algebra $A/\bar{F}_A^{\|\cdot\|}$. We will use $\sigma_K(a)$ to denote the spectrum of $a + \bar{F}_A^{\|\cdot\|}$ and $r_K(a)$ to denote its spectral radius, where $a + \bar{F}_A^{\|\cdot\|}$ is an element of $A/\bar{F}_A^{\|\cdot\|}$. Also let $\|a\|_K$ denote the norm of $a + \bar{F}_A^{\|\cdot\|}$.

Theorem 6.1. *Let $a \in A$ and let $a_\infty(a) = \lim_{k\to\infty} a_k(a)$. Then,*
(a) $a_\infty(a) = \inf\{\|a - y\| : y \in F_A\}$, the distance from a to F_A.
(b) Let $K_A = \{a \in A : a_\infty(a) = 0\}$. Then,

$$K_A \text{ is a norm closed two} - \text{sided ideal in } A \quad \textit{and}$$
$$\bar{F}_A^{\|\cdot\|} = K_A.$$

Proof. (a) Obviously

$$\inf\{\|a - y\| : rank(y) < \infty\} \leq a_k(a) \text{ for each } k.$$

Hence

$$\begin{aligned}\inf\{\|a - y\| : rank(y) < \infty\} &\leq \inf a_k(a)\\ &= a_\infty(a).\end{aligned}$$

For the opposite inequality, let $y \in A$ be such that rank$(y) < \infty$. Suppose rank$(y) < k$. Then

$$\|a - y\| \geq a_k(a) \geq a_\infty(a).$$

Hence

$$\inf\{\|a - y\| : rank(y) < \infty\} \geq a_\infty(a).$$

Whence

$$dist(a, F_A) = a_\infty(a).$$

(b) We first show that K_A is norm closed.

Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence with $x_n \in K_A$, for all n. Say $x_n \to x$. So, given $\epsilon > 0$, there is an $N > 0$ such that $\|x_n - x\| < \epsilon$ whenever $n \geq N$. But,

$$a_{2k-1}(x - x_n + x_n) \leq a_k(x - x_n) + a_k(x_n), \text{by Lemma 5.2(c)}.$$

That is,

$$\begin{aligned}a_{2k-1}(x) &\le a_k(x-x_n)+a_k(x_n)\\ &\le a_1(x-x_n)+a_k(x_n), \text{ by Lemma 5.2(a),}\\ &= \|x-x_n\|+a_k(x_n), \text{ by Lemma 5.2(a).}\end{aligned}$$

So,

$$\lim_{k\to\infty} a_{2k-1}(x) \le \|x-x_n\| + \lim_{k\to\infty} a_k(x_n).$$

That is,

$$\begin{aligned}a_\infty(x) &\le \|x-x_n\| + a_\infty(x_n)\\ &= \|x-x_n\| \text{ because } x_n \in K_A \text{ implies that } a_\infty(x_n)=0.\end{aligned}$$

So,

$$\begin{aligned}\lim_{n\to\infty} a_\infty(x) &\le \lim_{n\to\infty} \|x-x_n\|\\ &= 0.\end{aligned}$$

Therefore, $a_\infty(x) \le 0$, which implies that $a_\infty(x)=0$. Hence, $x \in K_A$. Whence, K_A is norm closed.

We next show that K_A is a two-sided ideal in A. Let $x, y \in K_A$. So,

$$a_{2n-1}(x-y) \le a_n(x) + a_n(y).$$

Therefore,

$$\lim_{n\to\infty} a_{2n-1}(x-y) \le \lim_{n\to\infty} a_n(x) + \lim_{n\to\infty} a_n(y).$$

That is,

$$\begin{aligned}a_\infty(x-y) &\le a_\infty(x)+a_\infty(y),\\ &= 0.\end{aligned}$$

So,

$$x-y \in K_A.$$

Let $z \in K_A$ and $x \in A$. So, $a_n(xz) \le a_n(z)\|x\|$, by Lemma 5.2(d). Therefore,

$$\begin{aligned}a_\infty(xz) &\le a_\infty(z)\|x\|\\ &= 0.\end{aligned}$$

Therefore,

$$xz \in K_A.$$

Similarly,

$$zx \in K_A.$$

Therefore, K_A is a two-sided ideal of A.

If $x \in F_A$, say $\operatorname{rank}(x) < k$. Then $a_k(x) = 0$. Hence $a_\infty(x) = \inf_k a_k(x) = 0$. Thus $F_A \subseteq K_A$. Let $x \in \bar{F}_A^{\|\cdot\|}$. Since K_A is closed, it follows that $\bar{F}_A^{\|\cdot\|} \subseteq K_A$.

For the reverse inclusion, let $x \in K_A$. So, $a_\infty(x) = 0$. It then follows from (a) that

$$\inf\{\|x - y\| : y \in F_A\} = 0.$$

Whence,

$$K_A = \bar{F}_A^{\|\cdot\|}.$$

□

Remark 6.2. In the proof of Theorem 6.1 (a), it is also clear that $a_\infty(a) = \inf\{\|a - y\| : y \in K_A\}$, which is the norm of the Banach algebra A/K_A. Also, if $A = B(X)$, with X a Banach space having the approximation property, that is $\overline{F(X)}^{\|\cdot\|} = K(X)$, then K_A will be the ideal of the compact operators on X.

□

In the following theorem we will provide conditions on a Banach algebra such that the inequality (5.4) becomes equality, with *limsup* replaced by *limit*.

Theorem 6.3 ([12], Theorem 5.1). *Let $a \in A$ be such that the spectral point sequence $\{\mu_n(a)\}_{n=1}^\infty$ of a has an infinite length, with limit $\mu(a)$ and that for some integer n,*

$\mu_n(a) = \mu(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)$ *and λ is not an f.m. spectral point.*$\}$ (6.1)

(Herein, f.m. stands for finite multiplicity.)

Assume that

$$r_K(a) = \mu(a). \tag{6.1}$$

Then,

$$\lim_{m \to \infty} (a_n(a^m))^{\frac{1}{m}} = \mu_n(a).$$

Proof. For any $n \in \mathbb{Z}^+$,

$$\begin{aligned}
\liminf_{m \to \infty} (a_n(a^m))^{\frac{1}{m}} &= \liminf_{m \to \infty} (\inf\{\|a^m - x\| : \operatorname{rank}(x) < n\})^{\frac{1}{m}} \\
&\geq \liminf_{m \to \infty} (\inf\{\|a^m - x\|^{\frac{1}{m}} : \operatorname{rank}(x) < \infty\}) \\
&= \liminf_{m \to \infty} (a_n(a^m))^{\frac{1}{m}} \\
&\geq \liminf_{m \to \infty} (\inf\{\|a^m - x\|^{\frac{1}{m}} : x \in K_A\}) \\
&= \liminf_{m \to \infty} \|a^m + K_A\|^{\frac{1}{m}} \\
&= r_K(a) \\
&= \mu_n(a).
\end{aligned}$$

But,

$$\limsup_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} \leq \mu_n(a).$$

So,

$$\begin{aligned}\liminf_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} &\leq \limsup_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} \\ &\leq \mu_n(a).\end{aligned}$$

Whence,

$$\lim_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} = \mu_n(a).$$

□

Note that from the above proof of Theorem 6.3, and Lemma 5.4, we have

$$r_K(a) \leq \liminf_{m\to\infty} (a_n(a^m))^{\frac{1}{m}} \leq \mu_n(a).$$

So, the following inequality always hold:

$$r_K(a) \leq \mu(a), \tag{6.2}$$

where $\mu(a)$ is as in Theorem 6.2.

Definition 6.4. *A Banach algebra A is said to have the spectral radius property if*

$$r_K(a) = \mu(a),$$

for every $a \in A$ for which $\{\mu_n(a)\}_{n=1}^{\infty}$ is an infinite length spectral point sequence with limit $\mu(a)$ attained by $\mu_k(a)$, for some integer k.

We will need the following lemma to prove that finite dimensional algebras have the spectral radius property.

Lemma 6.5 ([12], Lemma 5.5). *Suppose for $a \in A$ there is an element $b \in A$ such that ab is an idempotent of infinite rank. Then,*

$$a_\infty(a) = \|a\|_K \geq \|b\|^{-1}.$$

Proof. We show that if e is an idempotent of infinite rank, then $\|e\|_K \geq 1$. Suppose $\|e\|_K < 1$. There is a finite rank element x such that $\|e - x\| < 1$. So, $1 - e + x$ is invertible. As such,

$$\begin{aligned}e &= e(1-e+x)(1-e+x)^{-1} \\ &= ex(1-e+x)^{-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{rank}(e) &\leq \operatorname{rank}(x)\\ &< \infty.\end{aligned}$$

Thus, we have a contradiction, because, by assumption, $\operatorname{rank}(e) = \infty$. Since ab is an idempotent of infinite rank, it follows that

$$\|ab\|_K \geq 1.$$

So,

$$\begin{aligned}1 &\leq \|ab\|_K\\ &\leq \|a\|_K \|b\|_K\\ &\leq \|a\|_K \|b\|.\end{aligned}$$

Therefore,

$$\|a\|_K \geq \|b\|^{-1}.$$

□

Theorem 6.6 ([12], Theorem 5.4). *Suppose the spectral point sequence of $a \in A$ is infinite, and for some n, $\mu_n(a) = \mu(a)$. Suppose further that there is an isolated point $\lambda \in \sigma(a)$ such that $|\lambda| = \mu(a)$ and the corresponding Riesz idempotent e is of infinite rank, then*

$$r_K(a) = \mu(a).$$

Proof. Suppose $\mu(a) = 0$. Obviously, since $r_K(a) \leq \mu(a)$, no work needs to be done in this case.

We now Suppose $\mu(a) > 0$. Since $0 \notin \sigma_{eAe}(aea)$, it follows that eae is invertible in eAe. Let ede be the inverse of eae in eAe.

By Corollary 5.6, since $\sigma_{eAe}(aea) = \{\lambda\}$, we have

$$|\lambda|^{-1} = \lim_{m\to\infty} \|(ede)^m\|^{\frac{1}{m}}. \tag{6.3}$$

Recall that e commutes with a. So $e = aede = a^m(ede)^m$, which is an idempotent for any $m \in \mathbb{Z}^+$. By Lemma 6.5, with a^m playing the role of a and $(ede)^m$ playing the role of b, we get

$$\|a^m\|_K \geq \|(ede)^m\|^{-1}. \tag{6.4}$$

Using equality (6.3) and inequality (6.4) and Beurling's formula we have,

$$\begin{aligned} r_K(a) &= \lim_{n\to\infty} \|a^n\|_K^{\frac{1}{n}} \\ &\geq \lim_{n\to\infty} (\|(ede)^n\|^{-1})^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \|(ede)^n\|^{-\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{1}{\|(ede)^n\|^{\frac{1}{n}}} \\ &= |\lambda|. \end{aligned}$$

That is,

$$\begin{aligned} r_K(a) &\geq |\lambda| \\ &= \mu(a). \end{aligned}$$

So, we have

$$r_K(a) = \mu(a),$$

because, from inequality (6.2), we always have $r_K(a) \leq \mu(a)$.

□

Corollary 6.7. *Let A be a finite dimensional Banach algebra. Then, A has the spectral radius property.*

Proof. We first claim that if A is finite dimensional, then $x \in A$ implies that x is algebraic.

Let $a \in A$ be such that $\{\mu_n(a)\}_{n=1}^{\infty}$ is an infinite length spectral point sequence with limit $\mu(a)$ attained by $\mu_k(a)$, for some integer k. Consider $B = \{a^n : n = 1, 2, \ldots\}$. Note that $a^n \in A$, $(n = 1, 2, \ldots)$. Since $\dim(A) < \infty$, only a finite number of elements of B are linearly independent over $\mathbb{C}$. So, there is a polynomial $P(z)$ such that $P(a) = 0$. Therefore, a is algebraic and the claim is established. By Corollary 2.3, $\sigma(a)$ is a finite set. Whence, applying Theorem 6.6, the result follows.

□

In the rest of this section we will use our own techniques to show that $B(X)$ has the spectral radius property and exploit the same idea to show that any Banach algebra with a unit element also has the spectral radius property.

To accomplish this we will make use of the standard results on Riesz theory and Fredholm theory for Banach algebras. Since this theory is involved and well established in the literature, see [3] and [5], we will use the results without proofs.

We call an element $x \in A$ *inessential* if its spectrum $\sigma_A(x)$ is at most countable with zero being the only possible accumulation point. An ideal consisting only of inessential elements is called an *inessential ideal.* It follows from Corollary 2.3 that F_A is an inessential ideal. Clearly the norm closure of any inessential ideal will also be inessential. As such, K_A is an inessential norm closed ideal of A.

From [3], Section R, we see that Fredholm theory and Riesz theory in A is carried out relative any inessential two-sided ideal of A. For our purpose we consider Fredholm theory and Riesz theory relative to K_A. We call $x \in A$ *Fredholm* relative to K_A if $x + K_A$ is invertible in A/K_A. We will denote the class of Fredholm elements of A by $\Phi(A)$. We call $x \in A$ *Riesz* if $r_K(x) = 0$ and denote the class of Riesz elements of A by $R(A)$.

It is a direct consequence of the above definitions that $x \in R(A)$ if and only if $\sigma_K(x) = \{0\}$ if and only if $\lambda - x \in \Phi(A)$ for all $\lambda \neq 0$.

Remark 6.8. If $A = B(X)$, then it is clear from [3], Theorem O.2.2 that invertibility modulo $F_A = F(X)$ is equivalent to invertibility modulo $K(X)$. Since $F(X) \subseteq K_A \subseteq K(X)$, in general, it will follow that the Riesz and Fredholm elements relative K_A as we defined them, coincide with the classical Riesz and Fredholm operators. In fact $r_K(x)$ and $\sigma_K(x)$ are exactly the essential spectral radius and the essential spectrum respectively.

We call $\lambda \in \mathbb{C}$ a *Fredholm point* of $x \in A$ if $\lambda - x \in \Phi(A)$ and we call $\lambda \in \mathbb{C}$ a *Riesz point* if either $\lambda \notin \sigma(x)$ or if λ is an isolated Fredholm point of $\sigma(x)$, see [3] Section R.

We will need the following theorem to prove that $B(X)$, where X is a Banach space, has the spectral radius property.

Theorem 6.9 ([3], Theorem R.2.4). *Let $x \in A$. Then, every Fredholm point of x lying in $\partial\sigma(x)$ is isolated.*

□

Theorem 6.10. *Let $A = B(X)$, where X is a Banach space. Then A has the spectral radius property.*

Proof. Let $T \in B(X)$, with the spectral point sequence $\{\mu_j(T)\}_{j=1}^{\infty}$ of infinite length, satisfying $\mu_m(T) = \lim_{n\to\infty} \mu_n(T)$, for some integer m. The essential spectral radius of T has the following property:

$$r_K(T) \leq \mu_m(T), \text{ which is just inequality (6.2).}$$

So, we only need to show that $\mu_m(T) \leq r_K(T)$.

Recall that the essential spectrum of T is given by

$$\begin{aligned}\sigma_K(T) &= \sigma(T + K(X)) \\ &= \sigma(T)\backslash\{\lambda \in \sigma(T) : \lambda - T \in \Phi(A)\}.\end{aligned}$$

Consider the set

$$S = \{\lambda \in \sigma(T) : |\lambda| > r_K(T)\}.$$

Suppose $S = \emptyset$. In this case we have

$$|\lambda| \leq r_K(T), \text{ for all } \lambda \in \sigma(T).$$

Therefore, $\mu_m(T) \leq r_K(T)$. We thus have $\mu_m(T) = r_K(T)$.

Now we suppose $S \neq \emptyset$. We show that all elements of S are contained in $\partial\sigma_{B(X)}(T)$. Suppose not. Then there exists a point $\lambda \in S \backslash \partial\sigma_{B(X)}(T)$. Hence we can find a neighbourhood $U \subset \sigma_{B(X)}(T)$ which contains λ. (We can even assume that $U \not\subset \sigma_K(T)$). Now we let

$$t_0 = \sup\{t | (1+\epsilon)\lambda \in \sigma(T) \text{ for all } \epsilon \leq t\}.$$

Then $(1+t_0)\lambda \in \sigma(T)$ because $\sigma(T)$ is closed. Moreover $(1+t_0)\lambda \in \partial\sigma(T) \bigcap S$. Hence $(1+t_0)\lambda$ is isolated by Theorem 6.9 which gives a contradiction by the definition of t_0. It follows from Theorem 6.9 that S consists only of isolated points. Let $\lambda_0 \in S$ and consider $p(\lambda_0, T)$, the spectral projection corresponding to λ_0. Let Γ be a rectifiable curve enclosing no other point of $\sigma(T)$ other than λ_0. We thus have

$$\begin{aligned} p(\lambda_0, T) + K(X) &= p(\lambda_0, T + K(X)) \\ &= \frac{1}{2\pi i} \int_\Gamma (\lambda - T + K(X))^{-1} d\lambda \\ &= 0, \text{ by the Cauchy's Integral Theorem, as } \lambda_0 \notin \sigma_K(x). \end{aligned}$$

So, $p(\lambda_0, T) \in K(X)$, which means that $p(\lambda_0, T) \in F(X)$, since any compact projection is of finite rank. Whence, λ_0 is a finite multiplicity point. Therefore, $\lambda \in S$ implies that λ is a finite multiplicity point. So, $\mu_m(T) = \mu(T) \leq r_K(T)$ and the theorem is established.

□

The idea of the proof of Theorem 6.10 will be used to show that any Banach algebra with a unit element has the spectral radius property.

Lemma 6.11. *Let I be a two-sided ideal of A and let $x \in A$. Suppose $\mu \neq 0$ is isolated in $\sigma_A(x)$. If the Riesz idempotent e_μ corresponding to μ is in $\bar{I}^{\|\cdot\|}$, then $e_\mu \in I$.*

Proof. The subalgebra $e_\mu \bar{I}^{\|\cdot\|} e_\mu$ of A is closed. So, it is a Banach algebra with identity e_μ. Also, $e_\mu I e_\mu$ is a dense two-sided ideal in $e_\mu \bar{I}^{\|\cdot\|} e_\mu$. Clearly,

$$e_\mu \bar{I}^{\|\cdot\|} e_\mu = e_\mu I e_\mu,$$

for if not then we can find a proper maximal ideal J containing $e_\mu I e_\mu$. Since J is closed and dense in $e_\mu \bar{I}^{\|\cdot\|} e_\mu$, it has to be all of $e_\mu \bar{I}^{\|\cdot\|} e_\mu$, which yields a contradiction. So,

$$\begin{aligned} e_\mu &\in e_\mu \bar{I}^{\|\cdot\|} e_\mu \\ &= e_\mu I e_\mu \\ &\subset I. \end{aligned}$$

That is, $e_\mu \in I$ as wanted.

□

Theorem 6.12. *Let A be a Banach algebra with a unit element 1. Then, A has the spectral radius property.*

Proof. Let $a \in A$, with the spectral point sequence given by $\{\mu_n(a)\}_{n=1}^{\infty}$ with $n(a) = \infty$, and $\mu_m(a) = \lim_{n\to\infty} \mu_n(a)$, for some integer m. From inequality (6.2), we always have

$$r_k(a) \leq \mu_m(a). \tag{6.5}$$

So, we only need to show that $\mu_m(a) \leq r_k(a)$. Remember that

$$\sigma_K(a) = \sigma_A(a) \backslash \{\lambda \in \sigma_A(a) : (\lambda - a) \in \Phi(A)\}.$$

Consider the set

$$S = \{\lambda \in \sigma_A(a) : |\lambda| > r_k(a)\}.$$

If $S = \emptyset$, then $|\lambda| \leq r_k(a)$ for all $\lambda \in \sigma_A(a)$. Therefore, $\mu_m(a) \leq r_k(a)$.

Now we suppose $S \neq \emptyset$. We show that all elements of S are contained in $\partial\sigma_A(a)$. Suppose not. Then there exists a point $\lambda \in S\backslash\partial\sigma_A(a)$. Hence we can find a neighbourhood $U \subset \sigma_A(a)$ which contains λ. (We can even assume that $U \not\subset \sigma_K(a)$). Now we let

$$t_0 = \sup\{t \mid (1+\epsilon)\lambda \in \sigma(a) \text{ for all } \epsilon \leq t\}.$$

Then $(1+t_0)\lambda \in \sigma(a)$ because $\sigma(a)$ is closed. Moreover $(1+t_0)\lambda \in \partial\sigma(a) \bigcap S$. Hence $(1+t_0)\lambda$ is isolated by Theorem 6.9 which gives a contradiction by the definition of t_0. It follows from Theorem 6.9 that S consists only of isolated points. Let $\lambda_0 \in S$ and consider $p(\lambda_0, a)$, the spectral idempotent in A corresponding to λ_0. Let Γ be a simple rectifiable curve enclosing no other point of $\sigma_A(a)$ other than λ_0. Thus,

$$\begin{aligned} p(\lambda_0, a) + K_A &= p(\lambda_0, a + K_A) \\ &= \frac{1}{2\pi i}\int_\Gamma (\lambda - a + K_A)^{-1} d\lambda \\ &= 0, \text{ by the Cauchy's Integral Theorem, as } \lambda_0 \notin \sigma_K(a). \end{aligned}$$

Therefore,

$$p(\lambda_0, a) \in K_A = \bar{F}_A^{\|\cdot\|}.$$

So, by Lemma 6.11, $p(\lambda_0, a) \in F_A$. Therefore, λ_0 is a finite multiplicity point. We thus have $\lambda \in S$ implies that λ is a finite multiplicity point of $\sigma_A(a)$. Therefore,

$$\mu_m(a) \leq r_k(a). \tag{6.6}$$

Inequalities (6.5) and (6.6) yield,

$$r_K(a) = \mu(a).$$

Whence, A has the spectral radius property.

$\square$

We have shown that any Banach algebra with unit has the spectral radius property. Hence, C^*-algebras do have the property. Nylen and Rodman in [2] proved that any C^*-algebra with unit has the spectral radius property. Their proof makes use of the *reduced atomic* representation of C^*-algebras. To conclude this thesis we would like to mention that the representation used in [3], Theorem C^*.4.3, gives a much more elegant proof that a C^*-algebra with a unit element has the spectral radius property. We state Theorem C^*.4.3 without proof. It was shown in [14], that $F_A = Soc(A)$, where $Soc(A) = \{0\}$ if A has no minimal left or right ideals. Otherwise if A has minimal ideals and the smallest left ideal containing all minimal left ideals coincides with the smallest right ideal containing all minimal right ideals and we call this two-sided ideal the socle of A, denoted by $Soc(A)$.

Theorem 6.13 ([3], Theorem C*.4.3). *Let A be a C^*-algebra. there is a faithful *-(and therefore isometric) representation (φ, H) of A on a Hilbert space H with the following properties:*

(a) $\varphi(Soc(\mathrm{A})) = F(H) \bigcap \varphi(A)$,

(b) $\varphi(\overline{Soc(A)}) = K(H) \bigcap \varphi(A)$,

(c) $\varphi(R(A)) = R(H) \bigcap \varphi(A)$,

(d) $\varphi(\Phi(A)) = \Phi(H) \bigcap \varphi(A)$ *if A is unital.*

$\square$

Theorem 6.14. *Let A be a C^*-algebra with a unit element. Then, A has the spectral radius property.*

Proof. We may assume that A is a C^*-subalgebra of $B(H)$ and that

$$K_A = K(H) \bigcap A.$$

The map

$$\psi : A/K_A \to B(H)/K(H)$$

defined by

$$\psi(a + K_A) = a + K(H)$$

is a *-isomorphism. Hence $\psi(A/K_A)$ is a C^*-subalgebra of $B(H)/K(H)$. That

$$\sigma_{A/K_A}(a+K_A) = \sigma_{B(H)/K(H)}(\varphi(a)+K(H))$$

follows from a well-known fact in C^*-algebra theory, namely that the spectrum of an element in a C^*-algebra is the same as the spectrum with respect to any C^*-subalgebra containing that element.

From Theorem 6.9, $B(H)$ has the spectral radius property. It follows from Theorem 6.13(a) and the fact that $\sigma_A(a) = \sigma_{B(H)}(\varphi(a))$ that A has the spectral radius property as well: Suppose λ_0 is an isolated point of σ_A with finite multiplicity. Then, since φ is a C^*-algebra isomorphism,

$$\varphi(p(\lambda_0, a)) = p(\lambda_0, \varphi(a)),$$

where in each case $p(\lambda_0, .)$ is the spectral projection corresponding to λ_0. From Theorem 6.13(a) $p(\lambda_0, a)$ is finite rank if and only if $p(\lambda_0, \varphi(a))$ is finite rank. □

APPENDIX

The following standard facts and definitions have been relegated to this section, the appendix, merely to facilitate the flow of the presentation of the main results.

Definition A.1. *Let A be a unital Banach algebra over $\mathbb{C}$. Let X be a Banach space over $\mathbb{C}$.*

(i) We define a non-zero irreducible representation of A on X namely π to be a homomorphism from A into $B(X)$ such that $\{0\}$ and X are the only invariant subspaces under $\pi(a)$ for all $a \in A$. We say π is continuous if there is a constant C such that $\|\pi(a)\| \leq C\|a\|$.

(ii) We define a primitive ideal to be a kernel of some continuous irreducible representation of A and we denote the set of such ideals by Π_A.

(iii) An algebra A is primitive if $\{0\}$ is a primitive ideal of A.

(iv) The radical of an algebra A, denoted by $Rad(A)$, is defined to be the intersection of the kernels of all continuous irreducible representations of A. If there are no primitive ideals we define $Rad(A) = A$.

(v) An algebra A is semi-simple if $Rad(A) = \{0\}$.

Theorem A.2. *Let A be an algebra with unit 1. Then the following are identical:*

(a) the intersection of all maximal left ideals of A,

(b) the intersection of all maximal right ideals of A,

(c) the set of x such that $1 - zx$ is invertible in A, for all $z \in A$,

(d) the set of x such that $1 - xz$ is invertible in A, for all $z \in A$.

Proof. The equivalence of (c) and (d) follows from the fact that $\sigma_A(zx) \cup \{0\} = \sigma_A(xz) \cup \{0\}$. We only show the equivalence of (a) and (c) as that of (b) and (d) is shown analogously.

Let x be in the intersection of all maximal left ideals of A. If $1 - zx$ is not invertible then $A(1 - zx)$ is a left ideal of A, so it is contained in some maximal left ideal I_0. Then $zx \in I_0$ and $1 - zx \in I_0$. So $1 \in I_0$. That is $I_0 = A$, which yields a contradiction.

For the converse, suppose that $1 - zx$ is invertible for all $z \in A$. If x is not in the intersection of all maximal left ideals, it means that there is a maximal left ideal J_0 such that $x \notin J_0$. Then $I_0 + Ax = A$ and as a result $1 - zx \in J_0$, which yields a contradiction because it will imply that $1 \in J_0$.

$\square$

Let I be a maximal left ideal of A. It is closed , so

$$X = A/I$$

is a Banach space for the norm defined by

$$|||\overline{a}||| = \inf_{u \in I} \|a + u\|,$$

where $\overline{a} = a + I$ is an element of A/I. There is a natural representation π of A on X defined by

$$\pi(x)\bar{a} = \overline{xa}.$$

This continuous representation is called the *left regular representation* associated to I. It is irreducible because I is a maximal left ideal. The kernel of the representation is

$$ker(\pi) = (I : A) = \{x : x \in A, xA \subset I\}.$$

The following theorem provides the characterization of the radical in terms of irreducible representations for Banach algebras.

Theorem A.3. *Let A be a Banach algebra with unit 1. Then,*

(a) for every irreducible representation π of A there is a maximal left ideal I such that $ker(\pi) = (I : A)$, as a result, $ker(\pi)$ is a closed two-sided ideal of A,

(b) the radical of A, $Rad(A)$, equals any of the sets described in Theorem A.2

Proof. (a) Let π be an irreducible representation of A on a complex vector space X and let $\alpha \neq 0$ be in X. Then

$$F = \{\pi(x)\alpha : x \in A\}$$

contains α and it is invariant under π, so $F = X$. Let

$$I = \{x : x \in A, \pi(x)\alpha = 0\}.$$

Because $F = X$, we have $I \neq A$, so I is a left ideal of A. Let J be a proper left ideal of A containing I. Then either $J = I$ or $\{\pi(x)\alpha : x \in J\}$ is different from $\{0\}$ and is invariant under π. So $\{\pi(x)\alpha : x \in J\} = X$. As a result, there is an $e \in J$ such that $\pi(e)\alpha = \alpha$, so $xe - x \in I$ for all $x \in A$. Then for any $x \in A$ we have

$$x = (x - xe) + xe \in I + J \subset J,$$

so that $A = J$ and that yields a contradiction. As a result I is a maximal left ideal. Clearly $ker(\pi) \subset (I : A)$. If $x \in (I : A)$, then $xA \subset I$ and thus $\pi(x)\pi(y)\alpha = 0$ for every $y \in A$. As a result we have $\pi(x)X = \{0\}$, whence $\pi(x) = 0$. That is $x \in ker(\pi)$.

(b) If $x \in Rad(A)$ then $xA \subset Rad(A) \subset I$ for all maximal left ideals I. As a result thereof, $x \in (I : A)$ for all maximal left ideals I. It follows from (a) that x

is in the intersection of the kernels of all continuous irreducible representations of A.

For the converse, if x is in the intersection of the kernels of all continuous irreducible representations of A then $x \in (I : A)$ for all maximal left ideals I. Whence, $x = x1 \in I$ for all such I, and thus $x \in Rad(A)$.

□

Remark A.4. Let A be an algebra with a unit 1. Parts (c) and (d) of Theorem 2 yields that the radical, $Rad(A)$, of A is contained in the set of *quasi-nilpotent* elements of A.

□

Remark A.5. We note that if $\pi : A \to B(X)$ is a continuous irreducible representation of A, then $\pi(1)$ is the identity operator I in $B(X)$ for if this is not true, we will have $\pi(1)X$ to be a closed subspace of X which is invariant under $\pi(a)$ for every $a \in A$. To see closedness, suppose $\pi(1)x_n \to y$. So $\pi(1)x_n = \pi(1)\pi(1)x_n \to \pi(1)y$. So $y \in \pi(1)X$. This contradicts the irreducibility of π.

□

Remark A.6. It is worth noting that if an algebra A is primitive then it is semi-simple. This follows from the fact that $\{0\} \in \Pi_A$ which implies that $Rad(A) = \bigcap_{P \in \Pi_A} P = \{0\}$. Whence, by Definition A.1 (v), A is semi-simple.

□

Definition A.7 *We call an algebra A semi-prime if $uxu = 0$ for all $x \in A$ implies that $u = 0$.*

Remark A.8. Let A be a semi-prime algebra. Let J be a left ideal or right ideal of A. Then $J^2 = \{0\}$ implies that $J = \{0\}$. To see this let $y \in J$. So $yxy \in J^2$ for all $x \in A$. That is $yxy = 0$ for all $x \in A$. From Definition A.7 $y = 0$. Since y was arbitrarily chosen from J, we conclude that $J = \{0\}$.

□

Definition A.9. *Let A be any algebra.*

(i) An idempotent in A is an element p such that $p^2 = p$.

(ii) A minimal idempotent in A is a non-zero idempotent p such that pAp is a division algebra and we denote the set of all such idempotents by $Min(A)$

Definition A.10. *A right ideal R of an algebra A is minimal if $R \neq \{0\}$ and if for any right ideal $R_1 \subset R$, either $R_1 = \{0\}$ or $R_1 = R$. Analogously, a left ideal L of A is minimal if $L \neq \{0\}$ and if for any left ideal $L_1 \subset L$, either $L_1 = \{0\}$ or $L_1 = L$.*

Lemma A.11 ([4], Lemma 6). *Let A be a semi-prime algebra.*

(i) If $e \in Min(A)$ then Ae is a minimal left ideal of A.

(ii) If $e \in Min(A)$ then eA is a minimal rigth ideal of A.

Proof. (i) Suppose that $e \in Min(A)$. Let J be a non-zero left ideal of A with $J \subset Ae$. By the contraposition of Remark A.8, $J^2 \neq \{0\}$. So there are elements ae and be of J such that $aebe \neq 0$. It follows that $ebe \neq 0$. Since eAe is a division algebra, there exists a $c \in eAe$ such that $cebe = e$. Then $Ae = Acebe \subset Abe \subset J$, and so Ae is a minimal left ideal.

(ii) This part of the lemma follows by an analogous argument as in (i).

□

Lemma A.12 ([3], Theorem BA.3.5). *Let A be a semi-simple algebra. If $e \in Min(A)$, then there is a unique $P_e \in \Pi_A$ such that $e \notin P_e$.*

Proof. Let $e \in Min(A)$. So,

$$P_e = \{x \in A \quad : \quad xA \subset A(1-e)\}$$

is a primitive ideal because,

$$\pi : A \rightarrow B(Ae)$$

defined by

$$\pi(a)be = abe$$

is an irreducible representation of A on Ae with kernel P_e. The irreducibility follows since e is minimal. Thus, $e \notin P_e$, because $e \in P_e$ implies that

$$\pi(e)Ae = eAe = \{0\},$$

which will yield a contradiction.

If $P \in \Pi_A$ and $e \notin P$, then $P \bigcap Ae = \{0\}$, because by Lemma A.11 Ae is a minimal left ideal. Hence, $Pe = \{0\}$, because $Pe \subseteq P \bigcap Ae = \{0\}$. If $x \in P$, then, obviously, $xA \subset P$. So, $xAe = \{0\}$. It then follows that $x \in P_e$ and thus $P \subset P_e$. But $P_eAe = \{0\} \subset P$. So, either $Ae \subset P$ or $P_e \subset P$, again this follows from the fact that both P_e and Ae are left ideals, see for instance Lemma A.11. But $P \bigcap Ae = \{0\}$. Therefore, $P_e \subset P$ which yields that $P_e = P$.

□

Proposition A.13 *Let A be a semi-simple algebra with a unit element 1. Then A is semi-prime.*

Proof. Let $x \in A$. Suppose $xyx = 0$ for all $y \in A$. So $yxyx = 0$. That is $r((yx)^2) = 0$. This implies that $(r(yx))^2 = 0$. So we have $r(yx) = 0$. That is $yx \in Q(A)$ for all $y \in A$. By Lemma 3.1 we have $x \in Rad(A) = \{0\}$. That is $x = 0$. Thus A is semi-prime.

□

Lemma A.14 ([2], Lemma 4.2.2). *Let A be a Banach algebra and π be a continuous irreducible representation of A on a Banach space X. Then, $C = \{T \ : \ T \in \mathrm{B(X)},\ T\pi(x) \ = \pi(x)T,$ for all $x \in A\}$ is isomorphic to $\mathbb{C}$.*

Proof. Obviously C is a closed subalgebra of $B(X)$ containing the identity. Let $T \neq 0$ be in C. Then,

$$T\pi(x)(X) = \pi(x)T(X) \subset T(X), \text{ for every } x \in A.$$

This says $T(X)$ is invariant under π. Thus, $T(X) = X$. Also, $\ker(T)$ is invariant under π, but $\ker(T) = X$ is not possible. So, T is an invertible linear operator. By the Open Mapping Theorem, T is invertible in $B(X)$ and its inverse satisfies

$$T^{-1}\pi(x) = \pi(x)T^{-1}, \ (x \in A).$$

So, $T^{-1} \in C$. Therefore, C is a division algebra and the result follows from the Gelfand-Mazur Theorem.

□

Lemma A.15 ([2], Lemma 4.2.3). *Let π be a continuous irreducible representation of A on a Banach space X. If x_1 and x_2 are linearly independent in X, then there is an element $a \in A$ such that $\pi(a)x_1 = 0$ and $\pi(a)x_2 \neq 0$.*

Proof. Suppose that $\pi(x)x_1 = 0$ implies that $\pi(a)x_2 = 0$. Let

$$L_i = \{x \ : \ x \in A, \pi(x)x_i = 0\}, \ (i = 1, 2).$$

Clearly, L_1 and L_2 are both maximal left ideals and $L_1 \subset L_2$. So, $L_1 = L_2 = L$. The linear mappings

$$T_i : A/L \to X,$$

defined by

$$T(\bar{a}) = \pi(a)x_i, \ (i = 1, 2),$$

are bounded and bijective. Let $D = T_2T_1^{-1}$, which is a bounded linear operator on X. Let $y \in X$ and suppose that $y = \pi(b)x_1$. Then, we have

$$\begin{aligned}\pi(a)Dy &= \pi(a)T_2T_1^{-1}(y) \\ &= \pi(a)T_2(\bar{b}) \\ &= \pi(a)\pi(b)x_2 \\ &= \pi(ab)x_2,\end{aligned}$$

and

$$\begin{aligned}D\pi(a)y &= T_2T_1^{-1}\pi(ab)x_1 \\ &= T_2(\bar{ab}) \\ &= \pi(ab)x_2.\end{aligned}$$

So,

$$D\pi(a)y = \pi(a)Dy, \ (a \in A).$$

By Lemma A.14, there is a scalar $\lambda \neq 0$ such that $D = \lambda I$. Then, $T_2 = \lambda T_1$. So, taking $a = 1$, we get $x_2 = \lambda x_1$, contradicting the independence of x_1 and x_2.

□

Lemma A.16 ([2], Lemma 4.2.4). *Let A be a Banach algebra. Let π be a continuous irreducible representation of A on a Banach space X. If $x_1, \ldots, x_n$ are linearly independent in X then there is an element $a \in A$ such that*

$$\pi(a)x_i = 0, \ (i = 1, \ldots, n-1), \ \textit{and } \pi(a)x_n \neq 0.$$

Proof. Lemma A.15 tells us that this Lemma holds true for $n = 2$. We thus carry out the proof by induction. Suppose $n > 2$ and the result holds for n - 1. So, there is an $a_1 \in A$ such that $\pi(a_1)x_i = 0, (i = 1, \ldots, n-2)$, and $\pi(a_1)x_n \neq 0$.

If $\pi(a_1)x_{n-1} = 0$, then we are done.

If $\pi(a_1)x_{n-1}$ and $\pi(a_1)x_n$ are linearly independent. Then, by Lemma 3.17, there is an $a_2 \in A$ such that

$$\pi(a_2)\pi(a_1)x_{n-1} = 0,$$

and

$$\pi(a_2)\pi(a_1)x_n \neq 0.$$

So,

$$a = a_2 a_1 \text{ will do }.$$

We now suppose that $\lambda\pi(a_1)x_{n-1} = \pi(a_1)x_n$, for some $0 \neq \lambda \in \mathbb{C}$. The vectors $x_1, \ldots, x_{n-1}, \lambda x_{n-1} - x_n$ are linearly independent. So, there is an $a_3 \in A$ such that $\pi(a_3)x_i = 0$, $(i = 1, \ldots, n-2)$, and $\pi(a_3)(\lambda x_{n-1} - x_n) \neq 0$. If $\pi(a_3)x_{n-1} = 0$ we are done, we thus suppose it is not. If $\pi(a_3)x_{n-1}$ and $\pi(a_3)x_n$ are linearly independent, there is an $a_4 \in A$ such that $\pi(a_4)\pi(a_3)x_{n-1} = 0$ and $\pi(a_4)\pi(a_3)x_n \neq 0$. Then, $a = a_4 a_3$ does it. So, suppose to the contrary that $\alpha\pi(a_3)x_{n-1} = \pi(a_3)x_n$. By assumption, $\lambda\pi(a_3)x_{n-1} \neq \pi(a_3)x_n$. So, $\lambda \neq \alpha$. Since $\pi(a_3)x_{n-1} \neq 0$, there is an $a_5 \in A$ such that $\pi(a_5)\pi(a_3)x_{n-1} = \pi(a_1)x_{n-1}$.

By taking $a = a_1 - a_5 a_3$, we get $\pi(a)x_i = 0, (i = 1, \ldots, n-1)$, and

$$\begin{aligned}
\pi(a)x_n &= \pi(a_1)x_n - \pi(a_5)\pi(a_3)x_n \\
&= \lambda\pi(a_1)x_{n-1} - \alpha\pi(a_5 a_3)x_{n-1} \\
&= (\lambda - \alpha)\pi(a_1)x_{n-1} \\
&\neq 0, \text{ as } \lambda - \alpha \neq 0 \text{ and } \pi(a_1)x_{n-1} \neq 0.
\end{aligned}$$

□

We next use the sequence of Lemmas we have just established, to prove the Jacobson Density Theorem.

Theorem A.17 ([2], Theorem 4.2.5). *Let A be a Banach algebra. Let π be a continuous irreducible representation of A on a Banach space X. If $x_1, \dots, x_n$ are linearly independent in X and if $y_1, \dots, y_n$ are in X, then there is $a \in A$ such that*

$$\pi(a)x_i = y_i, \quad (i = 1, \dots, n).$$

Proof. By Lemma A.16, there is a $b_k \in A$ such that $\pi(b_k)x_i = 0$ if $i \neq k$ and $\pi(b_k)x_k \neq 0$. So there is a $c_k \in A$ such that $\pi(c_k)\pi(b_k)x_k \neq y_k$. We then take

$$a = c_1b_1 + \dots + c_nb_n$$

and the theorem follows.

□

Corollary A.18. *Let A be a finite dimensional Banach algebra. Let π be a continuous irreducible representation of A on a Banach space X. Then π is onto.*

Proof. Let $T \in B(X)$. Let $\{x_1, \dots, x_n\}$ be a basis for X. By Theorem A.17, there exists an $a \in A$ such that $\pi(a)x_i = Tx_i$ for $i = 1, \dots, n$. Whence $\pi(a) = T$.

□

Proposition A.19. *Let A be a finite dimensional semi-simple Banach algebra over $\mathbb{C}$. Then*

$$A \cong M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_m}(\mathbb{C}),$$

for some integers $n_1, \dots, n_m$.

Proof. We recall that Π denotes the set of all primitive ideals of A. Let $P \in \Pi_A$. The algebra A/P has a faithful continuous irreducible representation on a Banach space X over $\mathbb{C}$. Clearly the dimension of X is finite. By Corollary A.18, A/P is isomorphic to $B(X)$. But $B(X) \cong M_{n_k}(\mathbb{C})$ as the dimension of X is finite. Particularly, A/P has a unit element and has no bi-ideals other than $\{0\}$ and A/P. Thus P is maximal.

Since A is finite dimensional and A is semi-simple, there is a finite subset $\{P_1, \dots, P_m\}$ of Π_A with $\bigcap_{i=1}^{m} P_i = \{0\}$ and we suppose that this set is chosen so that m is as small as possible.

Let $J_i = \bigcap_{k \neq i} P_k$. Then J_i is a non-zero bi-ideal and $P_i \bigcap J_i = \{0\}$. Since P_i is maximal, we have

$$A = P_i \oplus J_i.$$

This implies that J_i is isomorphic to A/P_i. Therefore J_i is a minimal bi-ideal of A and has a unit element e_i. We prove that

$$A = J_1 \oplus \ldots \oplus J_m.$$

Let $a \in A$. There exist $j_1 \in J_1$ and $p_1 \in P_1$ such that $a = j_1 + p_1$. Then there exist $j_2 \in J_2$ and $p_2 \in P_2$ such that $p_1 = j_2 + p_2$. Since $J_2 \subset P_1$, we have $p_2 = p_1 - j_2 \in P_1$, $p_2 \in P_1 \bigcap P_2$. We next have $p_2 = j_3 + p_3$ with $j_3 \in J_3$ and $p_3 \in P_1 \bigcap P_2 \bigcap P_3$. Continuing this process, we arrive at $p_{m-1} = j_m + p_m$ with $j_m \in J_m$ and $p_m \in P_1 \bigcap \ldots \bigcap P_m = \{0\}$. That is $a = j_1 + \ldots j_m$. That this sum is direct follows from the equation $A = J_i \oplus P_i$. Since a was arbitrarily chosen from A, we have

$$A = J_1 \oplus \ldots \oplus J_m.$$

Since each J_i is isomorphic to $A/P_i \cong M_{n_i}(\mathbb{C})$, we conclude that

$$A \cong M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_m}(\mathbb{C}).$$

□

Remark A.20. If the Banach algebra in Proposition A.19 is finite dimensinal but not necessarily semi-simple but with a non-empty set of primitive ideals, we have

$$A = J_1 \oplus \ldots \oplus J_m \oplus Rad(A).$$

But $A_s = J_1 \oplus \ldots \oplus J_m$, is semi-simple. So

$$A = A_s \oplus Rad(A),$$

where A_s is a semi-simple algebra. If the set of primitive ideals is empty we have $Rad(A) = A$, hence we still have the decomposition of A into a direct sum of a semi-simple algebra and its radical, where the semi-simple algebra in this case is just $A_s = \{0\}$.

□

NOTATIONS

Let X, Y be Banach spaces. Let A be a Banach algebra with $B \subset A$. Let Z be a subset of a vector space V.

$\mathrm{rank}(x)$	The rank of $x \in A$
primitive-rank(x)	The primitive-rank of $x \in A$
$\dim(X)$	The dimension of X
X'	The dual space of X
$L(X,Y)$	Linear operators on X into Y
$L(X)$	Linear operators on X into X
$B(X,Y)$	Bounded operators on X into Y
$B(X)$	Bounded operators on X into X
$T(X)$	The range of $T \in B(X,Y)$
$F(X)$	Finite rank operators in $B(X)$
$K(X)$	Compact operators in $B(X)$
$\Phi(X)$	Fredholm operators in $B(X)$
$R(X)$	Riesz operators in $B(X)$
$\Phi(A)$	Fredholm elements of A relative to the closure of the ideal I in A
$R(A)$	Riesz elements of A relative to the closure of the ideal I of A
$Soc(A)$	The socle of A
$r(a)$	The spectral radius of $a \in A$
$\sigma_A(a)$	The spectrum of $a \in A$
$\rho(a)$	The resolvent set of $a \in A$
$Min(A)$	The set of all minimal idempotents of A
Π_A	The set of all primitive ideals of A
$\bar{B}^{\Vert\cdot\Vert}$	The norm closure of B in A
$\overset{*}{\cong}$	A star isomorphism
$\cong$	An algebra isomorphism
$\oplus$	Direct sum
$\ker(\psi)$	The kernel of ψ
$M_n(\mathbb{C})$	The algebra of all $n \times n$ matrices with entries in $\mathbb{C}$
$l_\infty(X)$	Bounded sequences $\{x_n\}$, where $x_n \in X$
∂Z	The boundary of Z
$dist(x,Z)$	The distance from $x \in V$ to Z
$Inv(A)$	Invertible elements of A
F_A	Finite rank elements of A
K_A	The norm closure of $\mathbb{F}_A$ in A
$\mathbb{C}$	The field of complex numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{Z}^+$	The positive integers
$\mathbb{N}$	The Natural numbers
$Rad(A)$	The radical of A

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Yamamoto type theorems in Banach algebras

by

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SUMMARY

The aim of this thesis is to study the asymptotic relation between the approximation numbers and isolated spectral points with finite multiplicity in a general Banach algebra setting.

In 1967 T. Yamamoto was the first to show that such asymptotic results hold for the algebra of n by n matrices with entries in the complex field. About twenty years later Edmunds and Evans found a meaningful extension of Yamamoto's Theorem for bounded operators on a Banach space.

After an extensive study of the notion of finite rank elements, we extend Yamamoto's Theorem to a general Banach algebra setting. Recently, Nylen and Rodman proved a special case of the result by showing that Yamamoto's Theorem holds for Banach algebras with the spectral radius property and conjectured that any Banach algebra possesses this property. In this thesis we prove their conjecture in the affirmative.

Yamamoto type theorems in Banach algebras

deur

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OPSOMMING

Die doel van hierdie verhandeling is om in 'n Banach-algebra die asimptotiese verbande tussen die approksimasiegetalle en die geisoleerde eiewaardes met eindige multiplisiteit te bestudeer.

Hierdie verbande vir die algebra van n by n-matrikse is in 1967 deur T. Yamamoto bewys en is twintig jaar later deur Edmunds en Evans veralgemeen na operatore op 'n Banach-ruimte.

Nadat 'n volledige studie van elemente met eindige rang in 'n Banach-algebra gemaak word, bewys ons dat Yamamoto se stelling veralgemeen kan word na enige Banach-algebra. Hierdie studie los onder andere 'n oop probleem van Nylen en Rodman, op naamlik dat elke Banach-algebra die spektraal-radius eienskap het.