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PERTURBATIONS OF EVOLUTION EQUATIONS

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PERTURBATIONS OF EVOLUTION EQUATIONS

by

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My wish is that my thesis will bring honour to those whose names I bear:

- my grandmother, after whom I was named;
- my father, whose memory I cherish;
- the Lord Jesus Christ, to the praise and glory of whose name I live and work.

Title: Perturbations of evolution equations
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SYNOPSIS

Abstract evolution problems of type

$$\begin{aligned} &\frac{d}{dt}(B_\epsilon u) = A_\epsilon u, \quad t > 0 \\ &\lim_{t\to 0^+} B_\epsilon u = y_\epsilon \end{aligned} \tag{1}$$

are considered in this study. The ϵ-subscript is used to indicate that these problems are seen as perturbations of an evolution problem

$$\begin{aligned} &\frac{d}{dt}(Bu) = Au, \quad t > 0 \\ &\lim_{t\to 0^+} Bu = y, \end{aligned} \tag{2}$$

where the pair of operators $\langle A_\epsilon, B_\epsilon \rangle$ tends, in some sense, to the pair $\langle A, B \rangle$ as ϵ tends to zero. All operators are linear and have domains in a common Banach space X. A_ϵ and B_ϵ map into a Hilbert space Y_ϵ and A and B into a Hilbert space Y.

One aspect of this study is to identify sets of initial conditions for which unique solutions for (1) and (2) exist. If solutions for (1) and (2) exist, both map $(0,\infty)$ into the Banach space X and the convergence, with respect to some norm in X, of the solution of (1) to that of (2) as ϵ tends to zero, is feasible.

As Y_ϵ and Y need not necessarily be the same, another aspect of the study is to identify initial values $y_\epsilon \in Y_\epsilon$ for (1) and $y \in Y$ for (2) such that the above convergence can be established.

To show that unique solutions exist the notion of a B-evolution, as developed by Sauer in [S1] is used. This is a generalization of the concept of a semigroup. A B-evolution of bounded operators $\{S(t) : t > 0\}$, with generating pair $\langle A,B\rangle$ can be interpreted as a solution operator for the evolution problem (2) as the solution for (2) is given by

$$u(t) = S(t)y, \quad t > 0.$$

In [S2] Sauer constructed a Friedrichs extension $\langle A,B\rangle$ for a pair $\langle A_0,B_0\rangle$, with respect to a third operator C_0. For the special case $C_0 = B_0$ the closed pair $\langle -A,B\rangle$ is shown to be the generating pair of a holomorphic B-evolution in Y.

In this study this procedure is adapted for more general situations. The two special cases $C_0 = B_0$ and $C_0 = A_0 + \omega B_0$ are considered separately and sufficient conditions are given for the Friedrichs extension to be the generating pair of a holomorphic B-evolution.

These results are then applied to prove existence results for

(i) Sobolev equations, including pseudo-parabolic and generalized biharmonic equations, and

(ii) dynamic boundary value problems.

For all these examples the solutions are given by holomorphic B-evolutions for which contour integral representation exist [S1]. These will be used to establish convergence results for the evolution equations (1) and (2).

For strongly continuous semigroups a corollary of the Trotter-Kato Theorem states that the pointwise convergence of semigroups is equivalent to the pointwise convergence of the resolvent operators for the infinitesimal generators.

For a B-evolution $S(t)$ the concept of a generalized resolvent operator $P(\lambda)$ for the generating pair $\langle A,B\rangle$ was introduced in [S1] as $P(\lambda) = (\lambda B - A)^{-1}$.

In this study is shown that for holomorphic B-evolutions the convergence of B-evolutions may be linked to the convergence of generalized resolvent operators. As Y_ϵ and Y need not necessarily be the same space, this result lacks the simplicity of the Trotter-Kato Theorem.

This result is applied to three examples. In all three cases equations (1) and (2) are of different types. Also, in the first example $Y = Y_\epsilon$ for all ϵ. In the second example $Y_\epsilon \subseteq Y$ for all ϵ with $Y_{\epsilon_1} \neq Y_{\epsilon_2}$ if $\epsilon_1 \neq \epsilon_2$. In the final example $Y_\epsilon \subset Y$ with Y_ϵ independent of ϵ.

In this study is shown that for holomorphic B-evolutions the convergence of B-evolutions may be linked to the convergence of generalized resolvent operators. As Y_ϵ and Y need not necessarily be the same space, this result lacks the simplicity of the Trotter-Kato Theorem.

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Titel: Steurings van evolusie vergelykings

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SAMEVATTING

Abstrakte evolusie vergelykings van die volgende tipe word in hierdie studie ondersoek:

$$\begin{aligned} &\frac{d}{dt}(B_\epsilon u) = A_\epsilon u, \quad t > 0 \\ &\lim_{t\to 0^+} B_\epsilon u = y_\epsilon . \end{aligned} \qquad (1)$$

Die ϵ-onderskrif dui aan dat die probleem beskou word as 'n steuring van 'n evolusie vergelyking

$$\begin{aligned} &\frac{d}{dt}(Bu) = Au, \quad t > 0 \\ &\lim_{t\to 0^+} Bu = y \end{aligned} \qquad (2)$$

waar die paar operatore $\langle A_\epsilon, B_\epsilon\rangle$ neig na die paar $\langle A,B\rangle$ as ϵ neig na 0. Alle operatore is lineêr en het definisieversamelings in 'n gemeenskaplike Banachruimte X. A_ϵ en B_ϵ het waardes in 'n Hilbertruimte Y_ϵ en A en B het waardes in 'n Hilbertruimte Y.

Een aspek van hierdie studie is om beginwaardes te bepaal, waarvoor eenduidige oplossings vir (1) en (2) bestaan. Oplossings van (1) en (2) beeld $(0,\infty)$ af in die Banachruimte X en dit is dus sinvol om die konvergensie van die oplossings van (1) na die oplossing van (2) te ondersoek. Aangesien Y_ϵ en Y nie noodwendig dieselfde ruimte is nie, is deel van die ondersoek om beginwaardes $y_\epsilon \in Y_\epsilon$ vir (1) en $y \in Y$ vir (2) te bepaal, waarvoor die konvergensie aangetoon kan word.

Om aan te toon dat eenduidige oplossings bestaan, word die begrip van 'n B-evolusie, soos ontwikkel deur Sauer in [S1], gebruik. Dit is 'n veralgemening van die semigroep-begrip. 'n B-evolusie van begrensde operatore $\{S(t) : t > 0\}$, met genererende paar $\langle A,B\rangle$, word geïnterpreteer as 'n oplossingsoperator vir die evolusie vergelyking (2) want die oplossing vir (2) word gegee deur

$$u(t) = S(t)y, \quad t > 0.$$

In [S2] word 'n Friedrichs-uitbreiding $\langle A,B\rangle$ vir 'n paar operatore $\langle A_0,B_0\rangle$ met betrekking tot 'n derde operator C_0 gekonstrueer. Vir die spesiale geval $C_0 = B_0$ word aangetoon dat die geslote paar $\langle -A,B\rangle$ die genererende paar vir 'n holomorfe B-evolusie in Y is.

In hierdie studie word die prosedure in [S2] aangepas vir algemener toepassings. Die twee spesiale gevalle $C_0 = B_0$ en $C_0 = A_0 + \omega B_0$ word afsonderlik beskou en in elke geval word voorwaardes gegee waaronder $\langle -A,B\rangle$ die genererende paar vir 'n holomorfe B-evolusie is.

Hierdie resultate word dan toegepas om bestaanstellings te bewys vir
(i) Sobolev-vergelykings, wat pseudo-paraboliese en veralgemeende biharmoniese vergelykings insluit, en
(ii) dinamiese randwaarde probleme.

Vir 'n holomorfe B-evolusie bestaan 'n kontoerintegraalvoorstelling [S1]. Hierdie voorstellings word gebruik om die konvergensie van oplossings van (1) na die van (2) aan te toon.

Vir sterk kontinue semigroepe volg uit die Trotter-Kato-stelling dat die puntsgewyse konvergensie van semigroepe ekwivalent is aan die puntgewyse konvergensie van die resolventoperatore van die infinitesimale generatore van die semigroepe.

Vir 'n B-evolusie met genererende paar $\langle A,B\rangle$ word die veralgemeende resolventoperator $P(\lambda)$ in [S1] gedefinieer deur $P(\lambda) = (\lambda B - A)^{-1}$. Vir holomorfe B-evolusies word in hierdie studie aangetoon dat die konvergensie van B-evolusies verband hou met die konvergensie van die ooreenstemmende veralgemeende resolventoperatore van die genererende pare. Omdat Y_ϵ en Y nie noodwendig dieselfde is nie, is die resultaat minder eenvoudig as die Trotter-Kato-stelling.

Die konvergensie-stelling word op drie voorbeelde toegepas waar (1) en (2) in elke geval van verskillende tipes is. Ook is $Y = Y_\epsilon$ vir alle ϵ, in die eerste voorbeeld; $Y_\epsilon \subseteq Y$ vir alle ϵ met $Y_{\epsilon_1} \neq Y_{\epsilon_2}$ vir $\epsilon_1 \neq \epsilon_2$, in die tweede voorbeeld; en, $Y_\epsilon \subset Y$ vir alle ϵ, in die laaste voorbeeld.

CONTENTS

CHAPTER 1
INTRODUCTION

Abstract evolution problems of type

$$\frac{d}{dt}(B_\epsilon u) = A_\epsilon u, \quad t > 0$$
$$\lim_{t\to 0^+} B_\epsilon u = y_\epsilon \tag{1.1}$$

are considered in this study. The ϵ-subscript is used to indicate that these problems are seen as perturbations of an evolution problem

$$\frac{d}{dt}(Bu) = Au, \quad t > 0$$
$$\lim_{t\to 0^+} Bu = y, \tag{1.2}$$

where the pair of operators $\langle A_\epsilon, B_\epsilon\rangle$ tends, in some sense, to the pair $\langle A,B\rangle$ as ϵ tends to zero. All operators are linear and have domains in a common Banach space X. A_ϵ and B_ϵ map into a Hilbert space Y_ϵ and A and B into a Hilbert space Y.

One aspect of this study is to identify sets of initial conditions for which unique solutions for (1.1) and (1.2) exist. If solutions for (1.1) and (1.2) exist, both map $(0,\infty)$ into the Banach space X and the convergence, with respect to some norm in X, of the solution of (1.1) to that of (1.2) as ϵ tends to zero, is feasible.

As Y_ϵ and Y need not necessarily be the same, another aspect of the study is to identify initial values $y_\epsilon \in Y_\epsilon$ for (1.1) and $y \in Y$ for (1.2) such that the above convergence can be established.

The existence of solutions is dealt with in Chapter 2. The notion of a B-evolution, as developed by Sauer in [S1] is used. This is a generalization of the concept of a semigroup. A B-evolution of bounded operators $\{S(t) : t > 0\}$, with generating pair $\langle A,B\rangle$, from a Banach space Y into a Banach space X can be interpreted as a solution operator for the evolution problem (1.2) as the solution for (1.2) is given by

$$u(t) = S(t)y, \quad t > 0.$$

A summary of results from [S1] that are used in this study is provided in Section 2.1. An important property of the generating pair $\langle A,B\rangle$ is that the operators are jointly closed. This joint closedness does not imply that both operators are closed.

In [S2] Sauer constructed a Friedrichs extension $\langle A,B\rangle$ for a pair $\langle A_0,B_0\rangle$. The procedure includes introducing a third operator C_0 for which the expression

$$(A_0u,C_0u) + \omega(B_0u,C_0u) \tag{1.3}$$

yields a norm $|[\ ]|$ in a subspace of the Banach space X. The inner product in Y is denoted by $(,)$ and ω is a non-negative constant. If (1.3) yields a strong enough norm a closed extension $\langle A,B\rangle$ of $\langle A_0,B_0\rangle$, with respect to C_0, exists.

For the special case $C_0 = B_0$ the closed pair $\langle -A,B\rangle$ is shown to be the generating pair of a holomorphic B-evolution in Y. A review of these results is provided in Section 2.2.1.

In Sections 2.2.2 - 2.2.4 a generalization of [S2] is presented. For the expression in (1.3) to be a norm certain conditions of symmetry have to be satisfied. Using ideas from [LM] the procedure in [S2] is adapted for situations with less symmetry. The two special cases $C_0 = B_0$ and $C_0 = A_0 + \omega B_0$ are considered separately and sufficient conditions are given for the Friedrichs extension to be the generating pair of a holomorphic B-evolution.

These results are then applied to prove existence results for
(i) Sobolev equations and
(ii) dynamic boundary value problems.

Evolution equations of Sobolev type

$$\begin{aligned} &Mu' + Lu = 0, \quad t > 0 \\ &\lim_{t\to 0^+} u = a \end{aligned} \tag{1.4}$$

were first introduced by Sobolev [So]. Except for the difference in initial conditions, evolution equations of type (1.2) are equivalent to Sobolev equations of type (1.4) if M is a closed operator.

A systematic treatment of the Sobolev equations is presented by Showalter ([Sh 1], [Sh 2]). His approach is to find conditions for the operator $-M^{-1}L$ to be the infinitesimal generator of a semigroup, after having extended the operators L and M separately.

In our approach L and M are extended jointly. For these extended operators the properties of the operator LM^{-1} yield stronger results than those in [Sh 2]. This fact is illustrated in Section 2.3.3 with an example supplied by Showalter [Sh 3].

Existence and regularity results for the Dirichlet problem for an elliptic partial differential operator of arbitary order are needed for these examples. [P] and [F] are used for reference purposes.

In [S2] the dynamic boundary value problem

$$\begin{aligned} \partial_t u &= Lu \quad \text{in} \quad \Omega \times (0,\infty) \\ \partial_t(\gamma_0 u) &= -L_\nu u \quad \text{on} \quad \partial\Omega \times (0,\infty) \\ \lim_{t\to 0^+} \langle u,\gamma_0 u\rangle &= y \in L^2(\Omega) \times L^2(\partial\Omega) \end{aligned} \tag{1.5}$$

is considered. Ω is a bounded open set in R^n with smooth boundary $\partial\Omega$. L is a second order symmetric differential operator. L_ν is the co-normal derivative associated with L at the boundary $\partial\Omega$, and γ_0 is the trace operator. In [S2] is shown that a unique solution for (1.5) exists for every $y \in L^2(\Omega) \times L^2(\partial\Omega)$; $L^2(\Omega)$ and $L^2(\partial\Omega)$ being spaces of square integrable functions on Ω and $\partial\Omega$ respectively.

In Sections 2.4.2 and 2.4.3 evolution problems of type (1.5) are considered, with L

(i) not necessarily symmetric and
(ii) having additional lower order terms.

In both these cases it is proved that a unique solution for (1.5) exists for every $y \in L^2(\Omega) \times L^2(\partial\Omega)$.

Finally, in Section 2.4.4, a dynamic boundary value problem of type

$$\begin{aligned} \partial_t u &= Lu \quad \text{in} \quad \Omega \times (0,\infty) \\ \partial_t(\gamma_0 u + k^2(x) L_\nu u) &= -L_\nu u \quad \text{on} \quad \partial\Omega \times (0,\infty) \end{aligned} \tag{1.6}$$

is considered. This type of boundary condition results from a contact condition between the domain Ω and the boundary $\partial\Omega$. A one-dimensional example was solved in [Fu] by means of eigenfunction expansions of boundary value problems involving a spectral parameter which also occurs in the boundary condition.

In Section 2.2.4 we apply results from Section 2.2.2 to show that (1.6) has a unique solution for every initial condition in $L^2(\Omega) \times H^{1/2}(\partial\Omega)$. The space $H^{1/2}(\partial\Omega)$ is a Sobolev space defined in [L]. Existence and regularity results for regular elliptic boundary value problems as developed in [L] are used.

In Section 2.2 several special cases are considered depending on the boundedness and symmetry properties of the different operators. The examples that are presented in Section 2.3 and 2.4 cover the whole spectrum of these cases.

The question of convergence is the next issue to be considered.

For all the evolution equations that will be presented in Sections 2.3 and 2.4 the solutions are given by holomorphic B-evolutions. From [S1] follows that contour integral representation exist for those B-evolutions. These will be used to establish convergence results for the evolution equations (1.1) and (1.2).

For strongly continuous semigroups (C_0-semigroups) a corollary of the Trotter-Kato Theorem [P, p 87] states that if $E_n(t)$ is a sequence of equi-continuous C_0-semigroups in a Banach space X and $E(t)$ is a C_0-semigroup in the same Banach space X, then the pointwise convergence of $E_n(t)$ to $E(t)$ for $t > 0$ is equivalent to the pointwise convergence of $R(\lambda,A_n)$ to $R(\lambda,A)$; $R(\lambda,A_n)$ and $R(\lambda,A)$ denote the resolvent operators for the infinitesimal generators A_n and A of $E_n(t)$ and $E(t)$ respectively.

For a B-evolution $S(t)$ the concept of a generalized resolvent operator $P(\lambda)$ for the generating pair $\langle A,B\rangle$ was introduced in [S1] as $P(\lambda) = (\lambda B - A)^{-1}$.

The Trotter-Kato Theorem suggests the possibility that the convergence of B-evolutions may be linked to the convergence of generalized resolvent operators.

For holomorphic B-evolutions a result of this nature is presented in Section 3.2. As Y_ϵ and Y need not necessarily be the same space, this result lacks the simplicity of the Trotter-Kato Theorem.

The result in Section 3.2 is applied to three examples:

(i) the convergence of the solutions of pseudo-parabolic equations to the solution of a related parabolic equation;

(ii) the convergence of the solutions of generalized "biharmonic" equations to the solution of a related parabolic equation; and

(iii) the convergence of the solutions for dynamic boundary value problems with imperfect contact to the solution of the related problem with perfect contact.

In the first example $Y = Y_\epsilon$ for all ϵ. In the second example $Y_\epsilon \subseteq Y$ for all ϵ with $Y_{\epsilon_1} \neq Y_{\epsilon_2}$ if $\epsilon_1 \neq \epsilon_2$. In the final example $Y_\epsilon \subset Y$ with Y_ϵ independent of ϵ.

A special case of the first example has been solved in [T]. The Yosida-approximation for the semigroup generated by the elliptic operator was used and only self-adjoint operators are considered.

CHAPTER 2
EXISTENCE OF SOLUTIONS

2.1 Evolution equations and holomorphic B-evolutions.

In this section results on B-evolutions from Sauer [S1] are quoted. Only the holomorphic case is dealt with as this is sufficient for the examples we consider later.

Let X and Y be complex Banach spaces and let B be a linear operator with domain $D(B) \subset X$ and values in Y. A family $\{S(t) : t > 0\}$ of bounded linear operators defined on Y is called a B-evolution [S1, p 298] if

$$S(t)[Y] \subset D(B) \quad \text{for all} \quad t > 0$$

and

$$S(t + s) = S(t)BS(s) \quad \text{for all} \quad s,t > 0.$$

Associated with any B-evolution is a semigroup $\{E(t) : t > 0\}$ of linear operators in Y defined by

$$E(t) = BS(t), \quad t > 0.$$

$S(t)$ is called *strongly continuous* if $E(t)$ is a semigroup of class C_0.

$S(t)$ is *uniformly bounded* if there is a constant $M > 0$ such that

$$\|S(t)\| \leq M \quad \text{for} \quad t > 0.$$

The Laplace transform $P(\lambda)$ of a strongly continuous, uniformly bounded B-evolution $S(t)$ is defined for λ with $\mathrm{Re}\,\lambda > 0$ by

$$P(\lambda)y = \int_0^\infty e^{-\lambda t}S(t)y \, dt \quad \text{for all} \quad y \in Y.$$

A strongly continuous, uniformly bounded B-evolution is called *of type* L if

$$P(\lambda)y \in D(B) \quad \text{for all} \quad y \in Y \quad \text{and all} \quad \lambda \quad \text{with} \quad \operatorname{Re} \lambda > 0$$

and

$$BP(\lambda)y = \int_0^\infty e^{-\lambda t} BS(t)y \, dt = \int_0^\infty e^{-\lambda t} E(t)y \, dt.$$

The *infinitesimal generator* A of a B-evolution S(t) is defined as follows:

$$A_h x := h^{-1}(BS(h)B - B)x \quad \text{for} \quad x \in D(B) \quad \text{and} \quad h > 0$$

$$x \in D(A) \quad \text{if} \quad Ax := \lim_{h \to 0} A_h x \quad \text{exists.}$$

A B-evolution S(t) of type L is determined uniquely by the pair of linear operators $\langle A_0, B_0 \rangle$ with A_0 and B_0 the restrictions of A and B to $D = Rg(P(\lambda))$, the range of $P(\lambda)$. D does not depend on λ with $\operatorname{Re} \lambda > 0$. [S1, p 292 & 293]

$\langle A_0, B_0 \rangle$ is called the *generating pair* of the B-evolution S(t) and $A_0 B_0^{-1}$ is the infinitesimal generator of the associated semigroup E(t).

B-evolutions yield solutions to certain evolution equations.

Theorem 1 [S1, p 293] *Let* $y \in Rg(B_0)$ *and* $u(t) = S(t)y$. *Then* $u(t) \in D$ *and*

$$\frac{d}{dt}(B_0 u) = A_0 u$$

$$\lim_{t \to 0^+} B_0 u = y.$$

Also, if $y \in Y$ *such that* $S(t)y \in D$ *and* $u(t)$ *is any solution of the initial value problem then* $u(t) = S(t)y$. *By a solution we mean a function* u *with values* $u(t) \in D$ *for all* $t > 0$, *and* $B_0 u$ *differentiable for all* $t > 0$ *with respect to the norm topology in* X.

A B-evolution is *holomorphic* if the associated semi-group $E(t)$ is holomorphic [P, p 61 & 62].

For a holomorphic B-evolution Theorem 1 remains valid for any $y \in Y$ and the B-evolution is also a B_0-evolution. [S1, p 298]

It is important to know if a given pair of operators $\langle A_0, B_0 \rangle$ is the generating pair of a holomorphic B-evolution of type L.

Theorem 2 [S1, p 296] *Let* $\langle A_0, B_0 \rangle$ *be a pair of operators with a common domain* $D \subset X$ *and values in* Y. $\langle A_0, B_0 \rangle$ *is the generating pair of a holomorphic* B_0-*evolution of type* L *in* Y *if and only if*

B_0 *has a bounded inverse*

$A_0 B_0^{-1}$ *generates a uniformly bounded holomorphic semigroup in* Y.

Remark

The uniform boundedness of the semigroup is not a serious restriction. If the semigroup $E(t)$ is not uniformly bounded there still are $\omega > 0$ and $M > 0$ such that

$$\|E(t)\| \leq Me^{\omega t}, \quad t > 0 \qquad [P, p 4, Th 2.2]$$

and then

$$A_0 B_0^{-1} - \omega I_Y = (A_0 - \omega B_0) B_0^{-1}$$

is the infinitesimal generator of a uniformly bounded semigroup. I_Y is the identity map in Y.

From Theorem 2 it is clear that $\langle A_0 - \omega B_0, B_0 \rangle$ is the generating pair of a holomorphic B_0-evolution $S(t)$ of type L.

Let $y \in Y$ and $u(t) = S(t)y$. From Theorem 1

$$\frac{d}{dt}(B_0u(t)) = (A_0 - \omega B_0)u(t), \quad t > 0$$
$$\lim_{t\to 0^+} B_0u(t) = y.$$

Let $v(t) = e^{\omega t}u(t)$. Then

$$\frac{d}{dt}(B_0v(t)) = A_0v(t), \quad t > 0$$
$$\lim_{t\to 0^+} B_0v(t) = y.$$

2.2 Holomorphic B-evolutions and the Friedrichs extension of a pair of operators.

2.2.1 Introduction

Sauer [S2] presented another set of sufficient conditions for a pair of operators $\langle A,B\rangle$ to be the generating pair of a holomorphic B-evolution of type L. The conditions were obtained by constructing a Friedrichs extension of a pair of operators $\langle A_0,B_0\rangle$.

For reference purposes the main results of [S2] are quoted.

Let X be a complex Banach space and Y a complex Hilbert space. A_0, B_0 and C_0 are linear operators with a common domain $D_0 \subset X$ and map into Y. Define the bilinear forms R_0 and S_0 as follows:

$$R_0(u,v) := (A_0u,\ C_0v)$$
$$S_0(u,v) := (B_0u,\ C_0v)\ ; \quad u,v \in D_0$$

with $(\cdot,\cdot)$ denoting the inner product in Y.

$\| \ \|$ and $\| \ \|_X$ will denote the norms in Y and X respectively.

Assume that:

1. R_0 *and* S_0 *are Hermitian.*

2. R_0 *and* S_0 *are nonnegative.*

3. *For some* $\omega \geq 0$ *the form* $[\ ,\ ]$ *with*
$$[u,v] := R_0(u,v) + \omega S_0(u,v)$$
is positive definite on D_0 *in the sense that there exists* $c > 0$ *such that*
$$|[u]|^2 := [u,u] \geq c\|u\|_X^2 \quad \textit{for all} \quad u \in D_0 .$$

4. *The mapping*
$$x \in \langle D_0 ,\ \|\ \|_X\rangle \longrightarrow x \in \langle D_0 ,\ |[\ ]|\rangle$$
is injective in the sense that if $\{x_n\} \subset D_0$ *is a Cauchy-sequence in* $|[\ ]|$ *and* $\|x_n\|_X \longrightarrow 0$, *then* $|[x_n]| \longrightarrow 0$.

5. $C_0[D_0]$ *is dense in* Y.

6. B_0 *and* C_0 *are bounded in the norm* $|[\ ]|$.

Under these conditions a closed extension $\langle A,B\rangle$ of $\langle A_0,B_0\rangle$ is constructed on a subspace D with $D_0 \subset D \subset X$.

A pair of operators $\langle A,B\rangle$ is called *closed* if for a convergent sequence $\{x_n\}$ in D with $x_n \longrightarrow x$ in X, $Ax_n \longrightarrow y$ and $Bx_n \longrightarrow z$ in Y it follows that $x \in D$, $Ax = y$ and $Bx = z$. The operators A and B are also called *jointly closed.*

In the special case $C_0 = B_0$ it was shown that $\langle A,B\rangle$ is the generating pair of a holomorphic B-evolution of type L if the following additional assumption is satisfied.

7. *For some* $k > 0$

$$S_0(u,u) = \|B_0 u\|^2 \geq k^2 \|u\|_X^2 \quad \textit{for all} \quad u \in D_0 .$$

This technique will be extended to situations with less symmetry. An approach similar to that of Lax and Milgram [LM] will be used. The special cases $C_0 = B_0$ and $C_0 = A_0 + \omega B_0$ will be considered.

2.2.2 Construction

Let X be a complex Banach space and Y a complex Hilbert space. A_0, B_0, C_0 and N_0 are linear operators with a common domain $D_0 \subset X$ which map into Y. Define bilinear forms R_0, S_0 and T_0 as follows:

$$\begin{aligned} R_0(u,v) &:= (A_0 u,\ C_0 v) \\ S_0(u,v) &:= (B_0 u,\ C_0 v) \\ T_0(u,v) &:= (N_0 u,\ C_0 v) \ ; \quad u,v \in D_0 . \end{aligned}$$

In this case a closed extension for the pair $\langle A_0 + N_0,\ B_0\rangle$ is constructed. A_0 is regarded as the "symmetric" part of an operator in the sense of Assumption F2 below. N_0 is regarded as the "anti-symmetric" part of the same operator. The example in Section 2.4.2 serves as an illustration.

This extension for $\langle A_0 + N_0,\ B_0\rangle$ is constructed under a boundedness condition on T_0. See Assumption F8 below. A closed extension for $\langle A_0, B_0\rangle$ can be constructed under the same condition with these extensions not necessarily defined on the same subspace of X.

If a stricter boundedness condition on N_0 (see Assumption F9 later on) is assumed the closed extensions for $\langle A_0 + N_0,\ B_0\rangle$ and $\langle A_0, B_0\rangle$ are defined on the same subspace of X. Also, A_0 and N_0 can be extended separately and the sum of these extensions equals the extension of $A_0 + N_0$. See Theorem 4 below.

The example in Section 2.4.3 serves as an illustration. In that case the operator N_0 is used for some lower order terms of a differential operator.

Assume that:

F1. *For some* $\omega \geq 0$ *the form* $[\cdot,\cdot]$ *with*

$$[u,v] := R_0(u,v) + \omega S_0(u,v) \;; \quad u,v \in D_0$$

is Hermitian.

F2. $\text{Re } S_0(u,u) \geq 0$ *for all* $u \in D_0$.

F3. *The form* $[\ ,\ ]$ *is positive definite on* D_0 *in the sense that there is some* $c > 0$ *such that*

$$|[u]|^2 := [u,u] \geq c\|u\|_X^2 \text{ for all } u \in D_0.$$

F4. *The mapping* $x \in \langle D_0, \|\ \|_X\rangle \longrightarrow x \in \langle D_0, |[\ ]|\rangle$ *is injective in the sense that if* $\{x_n\} \subset D_0$ *is a Cauchy-sequence in* $|[\ ]|$ *and* $\|x_n\|_X \longrightarrow 0$, *then* $|[x_n]| \longrightarrow 0$.

F5. $C_0[D_0]$ *is dense in* Y.

F6. B_0 *and* C_0 *are bounded in* $|[\ ]|$.

F7. *For some* $\delta \geq 0$ *and* $0 \leq \epsilon < 1$

$$\text{Re } T_0(u,u) \geq -\epsilon|[u]|^2 - \delta \text{ Re } S_0(u,u) \text{ for all } u \in D_0.$$

F8. T_0 *is bounded in* $|[\ ]|$.

Theorem 3

3.1 *For some subspace* D *with* $D_0 \subset D \subset X$ *there exists a closed extension* $\langle A,B\rangle$ *on* D *of* $\langle A_0, B_0\rangle$.

3.2 *For some subspace* D' *with* $D_0 \subset D' \subset X$ *there exists a closed extension* $\langle M,B'\rangle$ *on* D' *of* $\langle A_0 + N_0, B_0\rangle$.

The proof of this theorem follows from the following lemmas.

Let D_1 denote the completion of D_0 with respect to $|[\ ]|$.

Lemma 1 *The bilinear forms* R_0, S_0, T_0 *and the operators* B_0 *and* C_0 *may be extended by continuity to* R, S, T *and* B_1 *and* C_1 *on* D_1.

Proof $R_0(u,v) = [u,v] - \omega S_0(u,v) = [u,v] - \omega(B_0u,\ C_0v)$.

F6 and F8 imply that R_0, S_0 and T_0 are bounded forms with respect to $|[\cdot]|$ and B_0 and C_0 are bounded operators.

For complex λ let

$$Q(u,v;\lambda) := R(u,v) + \lambda S(u,v)$$
$$W(u,v;\lambda) := R(u,v) + T(u,v) + \lambda S(u,v); \quad u,v \in D_1.$$

For λ real

$$\operatorname{Re} Q(u,u;\lambda) = |[u]|^2 + (\lambda - \omega)\operatorname{Re} S(u,u)$$

and

$$\operatorname{Re} W(u,u;\lambda) = |[u]|^2 + (\lambda - \omega)\operatorname{Re} S(u,u) + \operatorname{Re} T(u,u) \quad \text{for all} \quad u \in D_1.$$

Using F7

$$\operatorname{Re} W(u,u;\lambda) \geq (1 - \epsilon)|[u]|^2 + (\lambda - \omega - \delta)\operatorname{Re} S(u,u) \quad \text{for all} \quad u \in D_1.$$

For $\lambda \geq \omega$

$$\operatorname{Re} Q(u,u;\lambda) \geq |[u]|^2 \quad \text{for all} \quad u \in D_1 \tag{2-1}$$

and for $\lambda \geq \omega_1 = \omega + \delta$

$$\operatorname{Re} W(u,u;\lambda) \geq (1 - \epsilon)|[u]|^2 \quad \text{for all} \quad u \in D_1. \tag{2-2}$$

By the Lax-Milgram lemma [LM] for $\lambda \geq \omega$ there exists for given $y \in Y$ a unique $q_y \in D_1$ such that

$$Q(q_y,v;\lambda) = (y,C_1v) \quad \text{for all} \quad v \in D_1 \tag{2-3}$$

Also for $\lambda \geq \omega_1$ and given $y \in Y$ there exists a unique $w_y \in D_1$ such that

$$W(w_y,v;\lambda) = (y,C_1v) \quad \text{for all} \quad v \in D_1. \tag{2-4}$$

Define for $\lambda \geq \omega_1$ the linear operators $P_Q(\lambda) : Y \longrightarrow D_1$ and $P_W(\lambda) : Y \longrightarrow D_1$ by

$$P_Q(\lambda)y := q_y \quad \text{and} \quad P_W(\lambda)y := w_y.$$

Lemma 2 *For* $\lambda \geq \omega_1$ *and* $\mu \geq \omega_1$

$$P_Q(\lambda) = P_Q(\mu) + (\mu - \lambda)P_Q(\mu)B_1P_Q(\lambda)$$

and

$$P_W(\lambda) = P_W(\mu) + (\mu - \lambda)P_W(\mu)B_1P_W(\lambda).$$

Proof From the identities

$$\begin{aligned}(y,C_1v) = Q(q_y,v;\lambda) &= R(q_y,v) + \lambda S(q_y,v) \\ &= Q(q_y,v;\mu) - (\mu - \lambda)S(q_y,v) \\ &= Q(q_y,v;\mu) - (\mu - \lambda)(B_1q_y,C_1v)\end{aligned}$$

we conclude that

$$\begin{aligned}Q(q_y,v;\mu) &= (y + (\mu - \lambda)B_1q_y,C_1v) \\ &= (y + (\mu - \lambda)B_1P_Q(\lambda)y,C_1v).\end{aligned}$$

From the definition of $P_Q(\mu)$

$$q_y = P_Q(\mu)(y + (\mu - \lambda)B_1P_Q(\lambda)y)$$

or

$$P_Q(\lambda)y = P_Q(\mu)y + (\mu - \lambda)P_Q(\mu)B_1P_Q(\lambda)y \quad \text{for all} \quad y \in Y.$$

Similarly, for $y \in Y$,

$$\begin{aligned}(y,C_1v) &= W(w_y,v;\lambda)\\ &= R(w_y,v) + T(w_y,v) + \lambda S(w_y,v)\\ &= W(w_y,v;\mu) - (\mu - \lambda)S(w_y,v)\\ &= W(w_y,v;\mu) - (\mu - \lambda)(B_1w_y,C_1v)\end{aligned}$$

and therefore

$$\begin{aligned}W(w_y,v;\mu) &= (y + (\mu - \lambda)B_1w_y,C_1v)\\ &= (y + (\mu - \lambda)B_1P_W(\lambda)y,C_1v).\end{aligned}$$

From the definition of $P_W(\mu)$

$$w_y = P_W(\mu)(y + (\mu - \lambda)B_1P_W(\lambda)y)$$

or

$$P_W(\lambda)y = P_W(\mu)y + (\mu - \lambda)P_W(\mu)B_1P_W(\lambda)y \quad \text{for all} \quad y \in Y.$$

Lemma 3 *For $\lambda \geq \omega_1$ the operators $P_Q(\lambda)$ and $P_W(\lambda)$ are invertible and the ranges $Rg(P_Q(\lambda))$ and $Rg(P_W(\lambda))$ are independent of λ.*

Proof If $P_Q(\lambda)y = u_y = 0$ for some $y \in Y$ it follows from the definition of $P_Q(\lambda)$ that $(y,C_1v) = 0$ for all $v \in D_1$. From Assumption F5 that $C_0[D_0]$ is dense in Y it follows that $C_1[D_1]$ is also dense in Y as $C_1[D_1] \supset C_0[D_0]$. Therefore $y = 0$ and $P_Q(\lambda)$ is invertible for $\lambda \geq \omega_1$.

In a similar way it follows that $P_W(\lambda)$ is invertible for $\lambda \geq \omega_1$.

To prove the second statement let $u \in Rg(P_Q(\lambda))$ for some $\lambda \geq \omega_1$. Then

$$u = P_Q(\lambda)y \text{ for some } y \in Y.$$

From Lemma 2 for $\mu \geq \omega_1$

$$\begin{aligned} u &= P_Q(\mu)y + (\mu - \lambda)P_Q(\mu)B_1P_Q(\lambda)y \\ &= P_Q(\mu)(y + (\mu - \lambda)B_1P_Q(\lambda)y) \\ &= P_Q(\mu)y' \end{aligned}$$

for

$$y' = y + (\mu - \lambda)B_1u \in Y.$$

Similarly, $Rg(P_W(\lambda))$ is independent of λ for $\lambda \geq \omega_1$.

Define $D = Rg(P_Q(\lambda))$ for $\lambda \geq \omega_1$, and

$D' = Rg(P_W(\lambda))$ for $\lambda \geq \omega_1$.

For $\lambda \geq \omega_1$, $\mu \geq \omega_1$ define the operators A and B on D as

$$\begin{aligned} A &:= (\mu - \lambda)^{-1}(\mu P_Q^{-1}(\lambda) - \lambda P_Q^{-1}(\mu)) \\ B &:= (\lambda - \mu)^{-1}(P_Q^{-1}(\lambda) - P_Q^{-1}(\mu)). \end{aligned}$$

Define, also, the operators M and B' on D' as

$$\begin{aligned} M &:= (\mu - \lambda)^{-1}(\mu P_W^{-1}(\lambda) - \lambda P_W^{-1}(\mu)) \\ B' &:= (\lambda - \mu)^{-1}(P_W^{-1}(\lambda) - P_W^{-1}(\mu)). \end{aligned}$$

Lemma 4 *The operators* A, B, M *and* B' *are independent of* λ *and* μ.

$P_Q^{-1}(\nu) = \nu B + A$ *for* $\nu \geq \omega_1$ *and*

$P_W^{-1}(\nu) = \nu B' + M$ *for* $\nu \geq \omega_1$.

Proof For $u \in D$, $v \in D$ and λ and μ as in the definition of A and B

$$Q(u,v;\lambda) = ((\lambda B + A)u, C_1 v)$$

and

$$Q(u,v;\mu) = ((\mu B + A)u, C_1 v).$$

The identity

$$(\mu - \nu)Q(u,v;\lambda) + (\nu - \lambda)Q(u,v;\mu) + (\lambda - \mu)Q(u,v;\nu) = 0$$

yields

$$P_Q^{-1}(\nu) = \nu B + A \quad \text{for} \quad \nu \geq \omega_1.$$

Also, for $u \in D'$, $v \in D'$ and λ and μ as in the definition of M and B′

$$W(u,v;\lambda) = ((\lambda B' + M)u, C_1 v)$$

and

$$W(u,v;\mu) = ((\mu B' + M)u, C_1 v).$$

The identity

$$(\mu - \nu)W(u,v;\lambda) + (\nu - \lambda)W(u,v;\mu) + (\lambda - \mu)W(u,v;\nu) = 0$$

yields

$$W(u,v;\nu) = ((\nu B' + M)u, C_1 v) \quad \text{for all} \quad v \in D_1$$

and thus

$$P_W^{-1}(\nu) = \nu B' + M \quad \text{for} \quad v \geq \omega_1.$$

Lemma 5 $B_0 \subset B \subset B_1$, $A_0 \subset A$, $B_0 \subset B' \subset B_1$, $A_0 + N_0 \subset M$.

Proof For $u,v \in D_0$

$$Q(u,v;\lambda) = ((\lambda B_0 + A_0)u, C_0 v)$$

and thus

$$P_Q(\lambda)(\lambda B_0 + A_0)u = u \in D.$$

This proves that $D_0 \subset D$ or $B_0 \subset B$ and $A_0 \subset A$. $D \subset D_1$ follows from the definition of D. From Lemma 4 $P_Q^{-1}(\nu) = \nu B + A$ for all $\nu \geq \omega_1$.

For $u \in D_0$, $\nu_1 \geq \omega_1$, $\nu_2 \geq \omega_1$, $\nu_1 \neq \nu_2$

$$P_Q^{-1}(\nu_1)u = \nu_1 Bu + Au = \nu_1 B_0 u + A_0 u$$

and $P_Q^{-1}(\nu_2)u = \nu_2 Bu + Au = \nu_2 B_0 u + A_0 u.$

It follows that $(\nu_2 - \nu_1)Bu = (\nu_2 - \nu_1)B_0 u.$

Hence $Bu = B_0 u$

and $Au = A_0 u.$

A and B extend A_0 and B_0 respectively.

That B_1 extends B follows from Lemma 2.

For any $u \in D$, $u = P_Q(\lambda)y$ for some $y \in Y$.

$$P_Q(\mu)B_1 u = P_Q(\mu)B_1 P_Q(\lambda)y = (\mu - \lambda)^{-1}(P_Q(\lambda)y - P_Q(\mu)y)$$

or

$$B_1 u = (\mu - \lambda)^{-1}(P_Q^{-1}(\mu)u - P_Q^{-1}(\lambda)u) = Bu.$$

Similarly, for $u,v \in D_0$

$$W(u,v;\lambda) = ((\lambda B_0 + N_0 + A_0)u, C_0 v)$$

or

$$P_W(\lambda)(\lambda B_0 + N_0 + A_0)u = u$$

and therefore $D_0 \subset D'$ or $B_0 \subset B'$ and $A_0 + N_0 \subset M$.

$D' \subset D_0$ follows from the definition of D'.

From Lemma 4 $P_W^{-1}(\nu) = \nu B' + M$ for all $\nu \geq \omega_1$.

B' and M extend B_0 and $A_0 + N_0$ respectively.

$B' \subset B_1$ follows from Lemma 2.

Lemma 6 *The pairs of operators* $\langle A,B \rangle$ *and* $\langle M,B' \rangle$ *are both closed.*

Proof For $\lambda \geq \omega_1$ the operators $P_Q(\lambda)$ and $P_W(\lambda)$ are all bounded. We prove this for $P_W(\lambda)$. The proof for $P_Q(\lambda)$ is similar. [S2, p 244].

For $\lambda \geq \omega_1$ and $y \in Y$, from (2-2) and (2-4),

$$\begin{aligned} |[P_W(\lambda)y]|^2 &\leq \operatorname{Re} W(P_W(\lambda)y, P_W(\lambda)y;\lambda) \\ &= \operatorname{Re}(y, C_1 P_W(\lambda)y) \\ &\leq |(y, C_1 P_W(\lambda)y)| \\ &\leq \|y\| \, \|C_1 P_W(\lambda)y\| \end{aligned}$$

From Assumption F6

$$|[P_W(\lambda)y]|^2 \leq M\|y\| \, |[P_W(\lambda)y]|$$

and from Assumption F3 we conclude that

$$\|P_W(\lambda)y\|_X^2 \leq c^{-1}M^2\|y\|^2 \quad \text{for all} \quad y \in Y.$$

For $\lambda \geq \omega_1$ the inverse operators $P_Q^{-1}(\lambda)$ and $P_W^{-1}(\lambda)$ are closed. [Y, p 79].

From Lemma 4 this implies $\lambda B + A$ and $\lambda B' + M$ are closed for $\lambda \geq \omega_1$. But if $\lambda B + A$ is closed for at least two non-zero complex numbers the pair $\langle A,B\rangle$ is closed [S1, p 295]. Similarly for $\langle M,B'\rangle$.

Proof of Theorem 3 In Lemmas 1, 2, 3 and 4 the validity of the definitions of the subspaces D and D′ and the operators A, B, M and B′ is shown.

In Lemmas 5 and 6 it is shown that $\langle A,B\rangle$ and $\langle M,B'\rangle$ are closed extensions of $\langle A_0,B_0\rangle$ and $\langle A_0 + N_0, B_0\rangle$ respectively. This completes the proof.

Under the additional assumption:

F9 N_0 *is bounded in* $|[\ ]|$,

stronger results for these extensions are obtained.

Note that Assumptions F6 and F9 imply F8 which now becomes superfluous.

The operator N_0 may be extended by continuity to N_1 on D_1 and T_0 may be extended to T with

$$T(u,v) = (N_1 u, C_1 v) \quad \text{for all} \quad u,v \in D_1 .$$

The following result now holds.

Theorem 4 *If* F9 *is satisfied*

$B' = B$ *and* $M = A + N$

with $N_0 \subset N \subset N_1$.

Proof We first show that $D = D'$. For any $u \in D'$ and $\lambda \geq \omega_1$ there is a $y \in Y$ with $P_W(\lambda)y = u$.

Therefore from (2-4)

$$W(u,v;\lambda) = R(u,v) + T(u,v) + \lambda S(u,v) = (y, C_1 v) \quad \text{for all} \quad v \in D_1 ,$$

and

$$\begin{aligned} Q(u,v;\lambda) = R(u,v) + \lambda S(u,v) &= (y, C_1 v) - T(u,v) \\ &= (y - N_1 u, C_1 v) \quad \text{for all} \quad v \in D_1 . \end{aligned}$$

From (2-3)

$$P_Q(\lambda)(y - N_1 u) = u$$

and

$$u \in \mathrm{Rg}\ (P_Q(\lambda)) = D.$$

Similarly $D \subset D'$ and thus $D = D'$. As $B \subset B_1$ and $B' \subset B_1$ this shows that $B' = B$.

For any $u \in D_0$

$$Nu = Mu - Au = (A_0 + N_0)u - A_0u = N_0u$$

and

$$N_0 \subset N.$$

For $u \in D = D'$

$$W(u,v;\lambda) = ((\lambda B + M)u, C_1v) \quad \text{for all} \quad v \in D_1$$

and

$$Q(u,v;\lambda) = ((\lambda B + A)u, C_1v) \quad \text{for all} \quad v \in D_1.$$

This yields

$$W(u,v;\lambda) - Q(u,v;\lambda) = (Nu, C_1v) \quad \text{for all} \quad v \in D_1.$$

But also

$$\begin{aligned} W(u,v;\lambda) &= R(u,v) + \lambda S(u,v) + T(u,v) \\ &= Q(u,v;\lambda) + (N_1u, C_1v) \end{aligned}$$

or

$$W(u,v;\lambda) - Q(u,v;\lambda) = (N_1u, C_1v) \quad \text{for all} \quad v \in D_1.$$

As $C_1[D_1]$ is dense in Y from Assumption F5

$$(N_1u - Nu, C_1v) = 0 \quad \text{for all} \quad v \in D_1$$

yields

$$N_1u = Nu \quad \text{for all} \quad u \in D = D'$$

or

$$N \subset N_1.$$

For the closed pairs $\langle A,B\rangle$ or $\langle M,B'\rangle$ to be generating pairs of B- or B'-evolutions B or B' must have a bounded inverse. The following assumption is sufficient:

F10 *For some* $k > 0$, $\|B_0u\| \geq k\|u\|_X$ *for all* $u \in D_0$.

Theorem 5 *If* F10 *is satisfied*
$$\|B_1 u\| \geq k\|u\|_X \quad \textit{for all} \quad u \in D_1.$$

Proof For any $u \in D_1$ there is a sequence $\{u_n\} \subset D_0$ with $|[u_n - u]| \longrightarrow 0$ as $n \longrightarrow \infty$.

From Assumption F3
$$c\|u_n - u\|_X^2 \leq |[u_n - u]|^2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$
and from Assumption F6 for some $b > 0$
$$\|B_0 u_n - B_1 u\| = \|B_1(u_n - u)\| \leq b|[u_n - u]| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

From Assumption F10
$$0 \leq \|B_0 u\|^2 - k^2\|u_n\|_X^2 \longrightarrow \|B_1 u\|^2 - k^2\|u\|_X^2 \quad \text{as} \quad n \longrightarrow \infty.$$

This implies $\|B_1 u\|^2 \geq k^2\|u\|_X^2$ for all $u \in D_1$.

Corollary 1 B *and* B' *have bounded inverses.*

Proof $B \subset B_1$ and $B' \subset B_1$.

In order to proceed the operator C_0 will have to be specified.

2.2.3 Special case 1 : $C_0 = B_0$

In this case
$$\begin{aligned} R_0(u,v) &= (A_0 u, B_0 v) \\ S_0(u,v) &= (B_0 u, B_0 v) \\ T_0(u,v) &= (N_0 u, B_0 v). \end{aligned}$$

Assumption F2 is satisfied as
$$\text{Re } S_0(u,u) = \|B_0 u\|^2 \geq 0 \quad \text{for all} \quad u \in D_0.$$

Assumption F1 is satisfied if and only if R_0 is Hermitian.

Theorem 6 *Let* $C_0 = B_0$ *and assume that* F1 - F8 *and* F10 *are satisfied. Then* $-M(B')^{-1}$ *is the infinitesimal generator of a holomorphic semigroup in* Y.

Proof Let $L = -M(B')^{-1} - \omega_1 I_Y$. The operator L is well-defined as Corollary 1 yields that $(B')^{-1}$ exists.

For the domain D(L) of L is known that $D(L) = Rg(B') \supset B_0[D_0] = C_0[D_0]$, and from Assumption F5 then follows that D(L) is dense in Y.

For $\lambda > 0$

$$\begin{aligned} \lambda I_Y - L &= (\lambda + \omega_1)I_Y + M(B')^{-1} \\ &= ((\lambda + \omega_1)B' + M)(B')^{-1} \\ &= P_W^{-1}(\lambda + \omega_1)(B')^{-1} \end{aligned}$$

and hence

$$(\lambda I_Y - L)^{-1} = B'P_W(\lambda + \omega_1)$$

is defined on Y.

We will now show that this resolvent operator is bounded.

For $u \in D'$ and $f = B'u$ let

$$\begin{aligned} p &= ((\lambda + \omega_1)B'u + Mu, B'u) \\ &= \lambda\|B'u\|^2 + \omega_1 S(u,u) + R(u,u) + T(u,u) \\ &= \lambda\|f\|^2 + q \end{aligned}$$

with

$$q = R(u,u) + \omega_1 S(u,u) + T(u,u), \quad \omega_1 = \omega + \delta.$$

Then

$$\mathrm{Re}\; q = |[u]|^2 + \delta S(u,u) + \mathrm{Re}\; T(u,u)$$

and from Assumption F7

$$\mathrm{Re}\; q \geq (1 - \epsilon)|[u]|^2.$$

From Assumption F8

$$|\mathrm{Im}\; q| = |\mathrm{Im}\; T(u,u)| \leq |T(u,u)| \leq m|[u]|^2$$

for some $m > 0$.

Define $K(\theta) := \{\nu \in C : |\arg \nu| \le \theta\}$.

For $\theta = \arctan(\frac{m}{1-\epsilon}) < \frac{\pi}{2}$ we have shown $q \in K(\theta)$ for all $u \in D'$.

The lemma in the Appendix now implies that for any ψ, $0 < \psi < \frac{\pi}{2} - \theta$ there is some $c(\psi) > 0$ with

$$|p|^2 \ge c(\psi)|\lambda|^2\|f\|^4 \quad \text{for all} \quad \lambda \in K(\tfrac{\pi}{2} + \psi).$$

But $|p|^2 \le \|(\lambda + \omega_1)B'u + Mu\|^2\|f\|^2$ and this yields

$$\|(\lambda + \omega_1)B'u + Mu\|^2 \ge c(\psi)|\lambda|^2\|f\|^2$$

or, equivalently,

$$\|(\lambda I_Y - L)f\| \ge c(\psi)^{1/2}\|\lambda f\| \quad \text{for all} \quad f \in D(L) \quad \text{and all} \quad \lambda \in K(\tfrac{\pi}{2} + \psi).$$

This implies for the resolvent set $\rho(L)$ of L that

$$\rho(L) \supseteq \{\lambda : \lambda > 0\}$$

and then from a well-known result in spectral theory ([F, p 73, problem 4])

$$\rho(L) \supseteq K(\tfrac{\pi}{2} + \psi).$$

This proves that L is the infinitesimal generator of a uniformly bounded holomorphic semigroup in Y. [P, p 61 & 62].

$-M(B')^{-1} = L + \omega_1 I_Y$ is the infinitesimal generator of a holomorphic semigroup in Y.

Theorem 7 *Let* $C_0 = B_0$ *and assume that* F1 - F8 *and* F10 *are satisfied. Then* $<-M - \omega_1 B', B'>$ *is the generating pair of a holomorphic* B-*evolution* $S(t)$ *of type* L *in* Y.

Proof From Corollary 1 B' has a bounded inverse and from the proof of Theorem 6 $L = -(M + \omega_1 B')(B')^{-1}$ is the infinitesimal generator of a uniformly bounded holomorphic semigroup in Y. The result follows from Theorem 2.

Theorem 8 *Let* $C_0 = B_0$ *and assume that* F1 - F8 *and* F10 *are satisfied. For any* $y \in Y$ *let*

$$u(t) = e^{\omega_1 t}S(t)y.$$

Then u *is the unique solution to*

$$\frac{d}{dt}(B'u(t)) = -Mu(t)$$

$$\lim_{t\to 0^+} B'u(t) = y.$$

Proof See the remark in Section 2.1.

Remarks

1. Let $C_0 = B_0$ and assume F1 - F10. Then $M(B')^{-1} = (A + N)B^{-1}$ and $\langle -A - N - \omega_1 B, B\rangle$ will be the generating pair of a holomorphic B-evolution $S(t)$ of type L. For $y \in Y$ and $u(t) = e^{\omega_1 t}S(t)y$

$$\frac{d}{dt}(Bu(t)) = -(A + N)u(t)$$

$$\lim_{t\to 0^+} Bu(t) = y.$$

2. For $N_0 = 0$ Assumption F7 is satisfied for $\delta = 0$. In the proof of Theorem 6 $m = 0$ and $\theta = 0$ imply that any value for ψ, $0 < \psi < \frac{\pi}{2}$ may be used. In this case these results specialize to the results of Sauer [S2].

3. In the proof of Theorem 6 is shown that $\rho(L) \supseteq K(\frac{\pi}{2} + \psi)$. This implies that $\rho(-M(B')^{-1}) \supseteq \{\lambda : \lambda - \omega_1 \in K(\frac{\pi}{2} + \psi)\}$.

2.2.4 Special case 2 : $C_0 = A_0 + \omega B_0$

In this case

$$R_0(u,v) = (A_0u, A_0v + \omega B_0v)$$

$$S_0(u,v) = (B_0u, A_0v + \omega B_0v)$$

$$T_0(u,v) = (N_0u, A_0v + \omega B_0v)$$

Assumption F1 is satisfied for any $\omega \geq 0$ as

$$[u,v] = (C_0u, C_0v).$$

Assumption F2 is satisfied if and only if for some $\omega \geq 0$

$$\text{Re } (B_0u, A_0u) \geq -\omega\|B_0u\|^2 \quad \text{for all} \quad u \in D_0.$$

C_0 is bounded in $|[\;]|$ as $\|C_0u\|^2 = |[u]|^2$.

$A_0 = C_0 - \omega B_0$ and from Assumption F6 follows that A_0 is bounded in $|[\;]|$ and may be extended by continuity to A_1 on D_1. In this case $R(u,v) = (A_1u, C_1v)$ for all $u,v \in D_1$.

Lemma 7 $A \subset A_1$.

Proof $Q(u,v;\lambda) = (\lambda Bu + Au, C_1v)$ for all $u \in D$, $v \in D_1$ and

$$\begin{aligned} Q(u,v;\lambda) &= R(u,v) + \lambda S(u,v) \\ &= (A_1u + \lambda B_1u, C_1v) \quad \text{for all} \quad u,v \in D_1. \end{aligned}$$

For $u \in D$

$$(A_1u - Au, C_1v) = 0 \quad \text{for all} \quad v \in D_1$$

and as $C_1[D_1]$ is dense in Y, this yields $A_1u = Au$ for all $u \in D$.

The following two additional assumptions are needed.

F11 *For some* θ, $0 \leq \theta < \frac{\pi}{2}$

$S_0(u,u) \in K(\theta)$ *for all* $u \in D_0$.

F12 $B_0[D_0]$ *is dense in* Y.

Note that Assumption F11 implies F2 which now becomes superfluous.

In the results that follow we assume that $N_0 = 0$. This is done because the bilinear form $[\ ,\ ]$ remains Hermitian even for nonsymmetrical operators.

Theorem 9 *Let* $C_0 = A_0 + \omega B_0$, $N_0 = 0$ *and assume that* F3 - F6, F11 - F12 *are satisfied. Then* $-AB^{-1}$ *is the infinitesimal generator of a holomorphic semigroup in* Y.

Proof Let $L = -AB^{-1} - \omega I_Y$.
The operator L is well-defined as Corollary 1 yields that B^{-1} exists.

For the domain D(L) of L it is known that
$$D(L) = Rg(B) \supset Rg(B_0)$$
and from Assumption F12 follows that D(L) is dense in Y.

For $\lambda > 0$
$$\begin{aligned} \lambda I_Y - L &= (\lambda + \omega)I_Y + AB^{-1} \\ &= ((\lambda + \omega)B + A)B^{-1} \\ &= P_Q^{-1}(\lambda + \omega)B^{-1} \end{aligned}$$
and hence
$$(\lambda I_Y - L)^{-1} = BP_Q(\lambda + \omega)$$
is defined on Y.

We now show that this resolvent operator is bounded.

For $u \in D$ and $f = Bu$ let
$$\begin{aligned} p &= ((\lambda + \omega)Bu + Au, Bu) \\ &= \lambda\|Bu\|^2 + (Au + \omega Bu, Bu) \\ &= \lambda\|f\|^2 + q \end{aligned}$$
with
$$q = \overline{S(u,u)},\ \text{the complex conjugate of } S(u,u).$$

Assumption F11 implies that $q \in K(\theta)$ for all $u \in D$ and the lemma in the Appendix that for any ψ, $0 < \psi < \frac{\pi}{2} - \theta$ there is some $c(\psi) > 0$ with

$$|p|^2 \geq c(\psi)|\lambda|^2\|f\|^4 \quad \text{for all} \quad \lambda \in K(\tfrac{\pi}{2} + \psi).$$

As in the proof of Theorem 6 this implies that L is the infinitesimal generator of a uniformly bounded holomorphic semigroup in Y. [P, p 61 & 62].

$-AB^{-1} = L + \omega I_Y$ is the infinitesimal generator of a holomorphic semigroup in Y.

Theorem 10 *Let* $C_0 = A_0 + \omega B_0$, $N_0 = 0$ *and assume that* F3 - F6, F10 - F12 *are satisfied. Then* $\langle -A - \omega B, B\rangle$ *is the generating pair of a holomorphic* B-*evolution* $S(t)$ *of type* L *in* Y.

Proof From Corollary 1, B has a bounded inverse and from the proof of Theorem 8 $L = -(A + \omega B)B^{-1}$ is the infinitesimal generator of a uniformly bounded holomorphic semigroup in Y. The result follows from Theorem 2.

Theorem 11 *Let* $C_0 = A_0 + \omega B_0$, $N_0 = 0$ *and* F3 - F6, F10 - F12 *are satisfied. For any* $y \in Y$ *let* $u(t) = e^{\omega t}S(t)y$. *Then* u *is the unique solution to*

$$\frac{d}{dt}(Bu(t)) = -Au(t)$$
$$\lim_{t \to 0^+} Bu(t) = y.$$

Proof The result follows from Theorem 10 and the remark in Section 2.1.

Remark

In the proof of Theorem 9

$$\rho(L) \supseteq K(\tfrac{\pi}{2} + \psi).$$

This implies that $\rho(-AB^{-1}) \supseteq \{\lambda : \lambda - \omega \in K(\frac{\pi}{2} + \psi)\}$.

2.3 Sobolev equations

2.3.1 Formulation of the problem

In this section the results of Section 2.2 will be applied to a Sobolev equation

$$\begin{gathered}\partial_t (M(x,D)u) = -L(x,D)u, \quad u = u(x,t), \quad x \in \Omega, \quad t > 0, \\ \partial^{\alpha} u = 0, \quad 0 \le |\alpha| < \max \{m,\ell\}, \quad x \in \partial\Omega, \quad t > 0, \\ M(x,D)u(x,t)\big|_{t=0} = y(x), \quad x \in \Omega.\end{gathered} \tag{2.5}$$

Ω is a bounded domain in R^n with a C^{∞}-boundary $\partial\Omega$. M and L are uniformly strongly elliptic partial differential operators of order $2m$ and 2ℓ respectively with coefficients sufficiently smooth. [F, p 1,2].

In order to formulate the initial boundary value problem as an evolution equation of the type discussed in Sections 2.1 and 2.2 we need the following function spaces.

$H^k(\Omega)$ denotes the Sobolev space of complex-valued functions with all derivatives up to order k square integrable. [F, p 33].

$H_0^k(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ in $H^k(\Omega)$, $C_0^{\infty}(\Omega)$ being the space of infinitely differentiable functions with compact support in Ω. The norm in $H^k(\Omega)$ is denoted by $\| \ \|_k$.

In the space $L^2(\Omega)$ of square integrable functions the norm is denoted by $\| \ \|_{\Omega}$ and the inner product by $(\ , \ )_{\Omega}$.

Define a linear operator M with domain $D(M) \subset H^{2m}(\Omega)$ and range $Rg(M) \subset L^2(\Omega)$ by

$$(Mu)(x) = M(x,D)u(x).$$

Likewise, the linear operator L with $D(L) \subset H^{2\ell}(\Omega)$ and $Rg(L) \subset L^2(\Omega)$ is defined by

$$(Lu)(x) = L(x,D)u(x) \qquad [F, p\ 70].$$

The initial boundary value problem (2.5) is then formulated as

$$\begin{aligned} &\frac{d}{dt}(Mu(t)) = -Lu(t), \quad t > 0 \\ &\lim_{t\to 0^+} Mu(t) = y \end{aligned} \tag{2.6}$$

The boundary condition in generalized form is incorporated by the requirement that $u(t) \in H_0^k(\Omega)$, $k = \max\{m,\ell\}$.

The two cases $m \geq \ell$ and $m < \ell$ are treated separately.

In both cases we assume that M is positive definite in the sense that for some $c_1 > 0$

$$\text{Re } (Mu,u)_\Omega \geq c_1\|u\|_m^2 \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega). \tag{2.7}$$

For M and L the following a priori inequalities are well-known [F, p 68].

For some $k_1 > 0$ and $k_2 > 0$

$$\|u\|_{2m} \leq k_1(\|Mu\|_\Omega + \|u\|_\Omega) \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega) \tag{2.8}$$

and

$$\|u\|_{2\ell} \leq k_2(\|Lu\|_\Omega + \|u\|_\Omega) \quad \text{for all} \quad u \in H_0^\ell(\Omega) \cap H^{2\ell}(\Omega). \tag{2.9}$$

From (2.7) and (2.8)

$$\|u\|_{2m} \leq k_1(\|Mu\|_\Omega + \frac{1}{c_1}\|Mu\|_\Omega)$$

or

$$\|u\|_{2m} \leq k_3\|Mu\|_\Omega, \; k_3 = \frac{k_1(1 + c_1)}{c_1} \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega). \qquad (2.10)$$

It is also clear that for some $m_1 > 0$ and $\ell_1 > 0$

$$\|Mu\|_\Omega \leq m_1\|u\|_{2m} \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega) \qquad (2.11)$$

and

$$\|Lu\|_\Omega \leq \ell_1\|u\|_{2\ell} \quad \text{for all} \quad u \in H_0^\ell(\Omega) \cap H^{2\ell}(\Omega). \qquad (2.12)$$

Also, for some $m_2 > 0$ and $\ell_2 > 0$,

$$|(Mu,u)_\Omega| \leq m_2\|u\|_m^2 \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega) \qquad (2.13)$$

and

$$|(Lu,u)_\Omega| \leq \ell_2\|u\|_\ell^2 \quad \text{for all} \quad u \in H_0^\ell(\Omega) \cap H^{2\ell}(\Omega). \qquad (2.14)$$

2.3.2 The case $m \geq \ell$

Note that the pseudo-parabolic equations $(m = \ell)$ are included as a special case.

We use the results of Sections 2.1 and 2.2 to prove the existence of a unique solution to the initial value problem (2.6).

Using the notation of Section 2.2, let

$$X = H^{2m}(\Omega), \quad Y = L^2(\Omega) \quad \text{and} \quad D_0 = H_0^m(\Omega) \cap H^{2m}(\Omega)$$

and

$$B_0 = M_0 = M|_{D_0}, \quad \text{the restriction of } M \text{ to } D_0,$$

$$A_0 = L_0 = L|_{D_0}, \quad \text{the restriction of } L \text{ to } D_0.$$

For $C_0 = A_0 + \omega B_0$ and $N_0 = 0$, we show that Assumptions F3 - F6 and F10 - F12 are satisfied and that Theorem 11 applies.

Assumption F3

We have to show that for some $\omega \geq 0$ there is a $c > 0$ such that

$$|[u]|^2 = \|L_0 u + \omega M_0 u\|_\Omega^2 \geq c\|u\|_{2m}^2 \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega).$$

For any $\omega > 0$, $\omega M_0 + L_0$ is a uniformly strongly elliptic operator of order 2m. From the a priori inequality [F, p 68] there is some $k_4 = k_4(\omega) > 0$ such that

$$\|u\|_{2m} \leq k_4(\omega)(\|\omega M_0 u + L_0 u\|_\Omega + \|u\|_\Omega) \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega). \tag{2.15}$$

From (2.7) and (2.14)

$$\begin{aligned} \operatorname{Re}\,(\omega M_0 u + L_0 u, u)_\Omega \geq \omega c_1 \|u\|_m^2 - \ell_2\|u\|_\ell^2 \geq (\omega c_1 - \ell_2)\|u\|_m^2 \\ \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega). \end{aligned} \tag{2.16}$$

From (2.15) and (2.16) it easily follows that the assumption is satisfied for $\omega > \ell_2/c_1$ with $c = c(\omega) = \delta_1/k_4(1 + \delta_1)$, $\delta_1 = \omega c_1 - \ell_2 > 0$.

Assumption F4

We have to show that if $\{u_n\} \subset H_0^m(\Omega) \cap H^{2m}(\Omega)$ is a Cauchy-sequence in $|[\ ]|$ and $\|u_n\|_{2m} \to 0$, then $|[u_n]| \to 0$.

Note that from Assumption F3, (2.11) and (2.12) for $\omega > \ell_2/c_1$

$$c(\omega)\|u\|_{2m}^2 \leq |[u]|^2 \leq 2(\ell_1^2 + \omega^2 m_1^2)\|u\|_{2m}^2 \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega).$$

Hence, $|[\ ]|$ is equivalent to the $H^{2m}(\Omega)$-norm and the assumption is satisfied.

Assumption F5

We have to show that

$$C_0[D_0] = \{\omega M_0 u + L_0 u : u \in H_0^m(\Omega) \cap H^{2m}(\Omega)\}$$

is dense in $L^2(\Omega)$.

Indeed, for $\omega > \ell_2/c_1$, $\omega M_0 + L_0$ is positive definite and $\omega M_0 + L_0$ is a bijection from $H_0^m(\Omega) \cap H^{2m}(\Omega)$ onto $L^2(\Omega)$. [F, p 71]. Hence $C_0[D_0] = L^2(\Omega)$.

Assumption F6

We have to show that $B_0 = M_0$ is bounded in $|[\ ]|$.

From (2.11) and Assumption F3 for $\omega > \ell_2/c_1$

$$\|M_0 u\|_\Omega \leq m_1 \|u\|_{2m} \leq m_1 c(\omega)^{-1/2} |[u]| \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega).$$

Assumption F10

We have to show that for some $k > 0$,

$$\|B_0 u\| = \|M_0 u\|_\Omega \geq k\|u\|_{2m} \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega).$$

It follows directly from (2.10).

Assumption F11

We have to show that for some θ, $0 \leq \theta < \frac{\pi}{2}$

$$S_0(u,u) = (M_0 u, L_0 u + \omega M_0 u)_\Omega \in K(\theta) \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega).$$

From (2.11) and (2.12)

$$\begin{aligned}|\mathrm{Im}\ (M_0u, L_0u + \omega M_0u)_\Omega| &= |\mathrm{Im}\ (M_0u, L_0u)_\Omega| \\ &\leq |(M_0u, L_0u)_\Omega| \\ &\leq m_1\ell_1\|u\|_{2m}^2 \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega).\end{aligned}$$

From (2.10), (2.11) and (2.12)

$$\begin{aligned}\mathrm{Re}(M_0u, L_0u + \omega M_0u)_\Omega &\geq -|(M_0u, L_0u)_\Omega| + \omega\|M_0u\|_\Omega^2 \\ &\geq -m_1\ell_1\|u\|_{2m}^2 + \omega/k_3^2\|u\|_{2m}^2 \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega).\end{aligned}$$

For $\omega > m_1\ell_1k_3^2$ let $\delta_2 = \omega/k_3^2 - m_1\ell_1 > 0$. The assumption is satisfied for $\omega > m_1\ell_1k_3^2$ with $\theta = \arctan\ (m_1\ell_1/\delta_2)$.

Assumption F12

We have to show that $Rg(M_0)$ is dense in $L^2(\Omega)$.

$Rg(M_0) = \{M_0u : u \in H_0^m(\Omega) \cap H^{2m}(\Omega)\}$. From (2.7) M_0 is positive definite and therefore a bijection from $H_0^m(\Omega) \cap H^{2m}(\Omega)$ onto $L^2(\Omega)$. [F, p 71].

Remarks

1. In this case $|[\cdot]|$ is equivalent to the $H^{2m}(\Omega)$-norm. $D_0 = H_0^m(\Omega) \cap H^{2m}(\Omega)$ is a closed subspace in $H^{2m}(\Omega)$ and in the construction in Section 2.2 $D_1 = D_0 = D$. Hence $A = A_0 = L_0$ and $B = B_0 = M_0$.

2. From Theorem 10 $\langle -L_0 - \omega M_0, M_0\rangle$ is the generating pair of a holomorphic M_0-evolution $S(t)$ of type L on $L^2(\Omega)$.

3. Note that for Assumption F3 to hold, we need

$$\omega > \ell_2/c_1 \tag{2.17}$$

and for Assumption F11 to hold, we need

$$\omega > m_1\ell_1 k_3^2 \ . \tag{2.18}$$

4. If L is positive definite in the sense that for some $c_2 > 0$

$$\mathrm{Re}\ (Lu,u)_\Omega \geq c_2\|u\|_\ell^2 \quad \text{for all} \quad u \in H_0^\ell(\Omega) \cap H^{2\ell}(\Omega), \tag{2.19}$$

condition (2.17) is superfluous since then

$$\mathrm{Re}\ (\omega M_0 u + L_0 u,u)_\Omega \geq \omega c_1\|u\|_m^2 + c_2\|u\|_\ell^2 \geq \omega c_1\|u\|_m^2$$
$$\text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega)$$

and Assumption F3 is satisfied for all $\omega > 0$.

Theorem 12 *For* $m \geq \ell$ *and any* $y \in L^2(\Omega)$ *the unique solution to*

$$\frac{d}{dt}\,(Mu(t)) = -Lu(t),\ t > 0$$
$$\lim_{t\to 0^+} Mu(t) = y$$

is given by

$$u(t) = e^{\omega t}S(t)y$$

with $S(t)$ *the holomorphic* M_0*-evolution of type* L *with generating pair* $\langle -L_0 - \omega M_0, M_0\rangle$.

Proof. The result follows directly from Theorem 11.

Remark

In Theorem 12 the solution u may also be written as

$$u(t) = S_1(t)y$$

with $S_1(t)$ the holomorphic M_0-evolution with generating pair $\langle -L_0, M_0\rangle$. From the remark on p 23

$$\rho(-L_0M_0^{-1}) \supseteq \{\lambda : \lambda - \omega \in K(\tfrac{\pi}{2} + \psi)\}.$$

2.3.3 The case $m < \ell$

In this case we assume that L is positive definite in the sense that (2.19) is satisfied. Choosing X, Y and D_0 is more involved than in the previous case.

M is a bijection from $H_0^m(\Omega) \cap H^{2m}(\Omega)$ onto $L^2(\Omega)$ and L a bijection from $H_0^\ell(\Omega) \cap H^{2\ell}(\Omega)$ onto $L^2(\Omega)$. Inequality (2.10) is used to show that Assumption F10 is satisfied and hence we choose $X = H^{2m}(\Omega)$. The boundary condition implies that $D_0 \subseteq H_0^\ell(\Omega)$. Hence $D_0 \subseteq H_0^\ell(\Omega) \cap H^{2\ell}(\Omega)$. Note that $V = M[H_0^\ell(\Omega) \cap H^{2\ell}(\Omega)]$ need not be dense in $L^2(\Omega)$ as the orthogonal complement

$$V^\perp = \{f \in L^2(\Omega) : (f,Mu)_\Omega = 0 \text{ for all } u \in H_0^\ell(\Omega) \cap H^{2\ell}(\Omega)\}$$

need not be trivial.

Let $Y = \mathrm{Cl}(M[H_0^\ell(\Omega) \cap H^{2\ell}(\Omega)])$, the closure being with respect to the $L^2(\Omega)$-norm. Let $D_0 = L^{-1}[Y]$.

For this choice of Y and D_0 it has been shown in [V] that:

$$D_0 \subset H_0^{\ell}(\Omega) \cap H^{2\ell}(\Omega) \subset X$$
$$Y = M[H_0^{2m}(\Omega)] \quad \text{for} \quad \ell \geq 2m$$
$$Y = M[H_0^{\ell}(\Omega) \cap H^{2m}(\Omega)] \quad \text{for} \quad m < \ell < 2m$$
$$Cl(Rg(M_0)) = Y$$
$$Rg(\omega M_0 + L_0) = Y \quad \text{for} \quad \omega > 0.$$

We assume that for some θ, $0 \leq \theta < \frac{\pi}{2}$,

$$(L_0 u, M_0 u)_\Omega \in K(\theta) \quad \text{for all} \quad u \in D_0. \tag{2.20}$$

We will present examples of pairs of operators that satisfy this condition at the end of this section and also in Chapter 3.

For $A_0 = L_0$, $B_0 = M_0$, $C_0 = L_0 + \omega M_0$ and $N_0 = 0$, we show that Assumptions F3 - F6 and F10 - F12 are satisfied and that Theorem 11 applies.

Assumption F3

We have to show that for some $\omega \geq 0$ there is a $c > 0$ with

$$|[u]|^2 = \|L_0 u + \omega M_0 u\|_\Omega^2 \geq c\|u\|_{2m}^2 \quad \text{for all} \quad u \in D_0.$$

For any $\omega \geq 0$, $\omega M_0 + L_0$ is a uniformly strongly elliptic operator of order 2ℓ. From the a priori inequality [F, p 68] there is some $k_4 = k_4(\omega) > 0$ such that

$$\|u\|_{2\ell} \leq k_4(\omega)(\|\omega M_0 u + L_0 u\|_\Omega + \|u\|_\Omega) \quad \text{for all} \quad u \in D_0. \tag{2.21}$$

From (2.7) and (2.19), as $\|u\|_m \leq \|u\|_\ell$

$$\begin{aligned} \operatorname{Re}(\omega M_0 u + L_0 u, u)_\Omega &\geq (\omega c_1 + c_2)\|u\|_m^2 \\ &\geq c_2\|u\|_m^2 \quad \text{for all} \quad u \in D_0. \end{aligned} \tag{2.22}$$

From (2.22)

$$\begin{aligned} c_2\|u\|_\Omega^2 \le c_2\|u\|_m^2 &\le \operatorname{Re}(\omega M_0 u + L_0 u, u)_\Omega \\ &\le |(\omega M_0 u + L_0 u, u)_\Omega| \\ &\le \|\omega M_0 u + L_0 u\|_\Omega \|u\|_\Omega \quad \text{for all} \quad u \in D_0. \end{aligned}$$

Combining this with (2.21)

$$\|u\|_{2\ell} \le k_4(1 + 1/c_1)\|\omega M_0 u + L_0 u\|_\Omega \quad \text{for all} \quad u \in D_0.$$

The assumption is satisfied for any $\omega \ge 0$ with $c = c_2/k_4(1 + c_2)$, as $\|u\|_{2m} \le \|u\|_{2\ell}$ for all $u \in D_0$.

Assumption F4

We have to show that if $\{u_n\} \subset D_0$ is a Cauchy-sequence in $|[\ ]|$ and $\|u_n\|_{2m} \longrightarrow 0$, then $|[u_n]| \longrightarrow 0$.

Note that from Assumption F3 and (2.11) and (2.12) for $\omega \ge 0$

$$c(\omega)\|u\|_{2\ell}^2 \le |[u]|^2 \le 2(\ell_1^2 + \omega^2 m_1^2)\|u\|_{2\ell}^2 \quad \text{for all} \quad u \in D_0.$$

Hence $|[\ ]|$ is equivalent to the $H^{2\ell}(\Omega)$-norm and the assumption is satisfied as $\|u\|_{2m} \le \|u\|_{2\ell}$ for all $u \in D_0$.

Assumption F5

We have to show that

$$C_0[D_0] = \{\omega M_0 u + L_0 u : u \in D_0\}$$

is dense in Y.

The proof of this result is given in [V] and is repeated here.

For $\omega \geq 0$, from (2.7) and (2.19)

$$\operatorname{Re}((\omega M + L)u,u) \geq (\omega c_1 + c_2)\|u\|_\ell^2 \quad \text{for all} \quad u \in H_0^\ell(\Omega) \cap H^{2\ell}(\Omega).$$

As $\omega M + L$ is a uniformly strongly elliptic operator on $H_0^\ell(\Omega) \cap H^{2\ell}(\Omega)$, this implies that $\omega M + L$ is a closed bijection from $H_0^\ell(\Omega) \cap H^{2\ell}(\Omega) \subset L^2(\Omega)$ onto $L^2(\Omega)$ ([F, p 71]). Hence for any $y \in Y \subset L^2(\Omega)$ there exists a $u \in H_0^\ell(\Omega) \cap H^{2\ell}(\Omega)$ with

$$\omega Mu + Lu = y$$

or

$$Lu = y - \omega Mu.$$

As $Mu \in Y$ for $u \in H_0^\ell(\Omega) \cap H^{2\ell}(\Omega)$ this yields $Lu \in Y$, i.e. $u \in D_0$. This shows that $\omega M_0 + L_0$ is a bijection from D_0 onto Y.

Assumption F6

We have to show that $B_0 = M_0$ is bounded in $|[\ ]|$.

From (2.11) and Assumption F3, for $\omega \geq 0$

$$\|M_0 u\|_\Omega \leq m_1 \|u\|_{2m} \leq m_1 c(\omega)^{-1/2} |[u]| \quad \text{for all} \quad u \in D_0.$$

Assumption F10

We have to show that for some $k > 0$

$$\|B_0 u\| = \|M_0 u\|_\Omega \geq k\|u\|_{2m} \quad \text{for all} \quad u \in D_0.$$

This follows directly from (2.10) as

$$D_0 \subset H_0^\ell(\Omega) \cap H^{2\ell}(\Omega) \subset H_0^m(\Omega) \cap H^{2m}(\Omega).$$

Assumption F11

We have to show that for some θ, $0 \leq \theta < \frac{\pi}{2}$

$$S_0(u,u) = (M_0u, L_0u + \omega M_0u)_\Omega \in K(\theta) \quad \text{for all} \quad u \in D_0.$$

For any $\omega \geq 0$ and θ as in (2.20)

$$(M_0u, L_0u + \omega M_0u)_\Omega = (M_0u, L_0u)_\Omega + \omega\|M_0u\|_\Omega^2$$

is contained in $K(\theta)$ for all $u \in D_0$ as $(M_0u, L_0u)_\Omega = \overline{(L_0u, M_0u)_\Omega}$, the complex conjugate of $(L_0u, M_0u)_\Omega$.

Assumption F12

We have to show that $Rg(M_0)$ is dense in Y.

The proof of this result is given in [V] and is repeated here.

It is sufficient to proof that the orthogonal complement of $Rg(M_0)$,

$$Rg(M_0)^\perp := \{f \in Y : (f, M_0u)_\Omega = 0 \quad \text{for all} \quad u \in D_0\}$$

is trivial.

For $\omega \geq 0$ and any $f \in Rg(M_0)^\perp \subset Y$ there is some $u \in D_0$ with $(\omega M_0 + L_0)u = f$. Then $(f, M_0u) = 0$ reduces to

$$0 \leq \omega\|M_0u\|_\Omega^2 = -(L_0u, M_0u)_\Omega \leq 0.$$

The last inequality follows from (2.20).

From Assumption F10 $M_0u = 0$, implies $u = 0$ and hence $f = \omega M_0 u + L_0 u = 0$.

Remarks

1. In this case $|[\ ]|$ is equivalent to the $H^{2\ell}(\Omega)$-norm. $D_0 = L^{-1}[Y]$ is a closed subspace in $H^{2\ell}(\Omega)$ and in the construction in Section 2.2 $D_1 = D_0 = D$. Hence $A = A_0 = L_0$ and $B = B_0 = M_0$.

2. The choice $\omega = 0$ is allowed and from Theorem 10 $\langle -L_0, M_0 \rangle$ is the generating pair of holomorphic M_0-evolution $S(t)$ of type L on Y.

3. Note that the Hilbert space Y depends on the operator M.

4. All the assumptions are also satisfied for $X = L^2(\Omega)$.

Theorem 13 *For* $m < \ell$ *and any* $y \in Y$ *the unique solution to*

$$\frac{d}{dt}(Mu(t)) = -Lu(t), \quad t > 0$$
$$\lim_{t \to 0^+} Mu(t) = y$$

is given by

$$u(t) = S(t)y$$

with $S(t)$ *the holomorphic* M_0-*evolution of type* L *with generating pair* $\langle -L_0, M_0 \rangle$.

Proof. The result follows directly from Theorem 11.

We conclude this section with an example, supplied by Showalter [Sh3], which illustrates that the joint extension of the operators leads to stronger results than those in [Sh 1] where the operators are extended separately.

In [Sh 1] the evolution problem

$$Mu' = -Lu, \quad t > 0$$
$$\lim_{t \to 0^+} u = a$$

with M and L uniformly strongly elliptic operators, of orders $2m$ and 2ℓ respectively, are considered. For the main existence result [Sh 1, p 89, Th 3] one of the conditions is that there exists a θ, $0 < \theta < \frac{\pi}{2}$ such that

$$(Lu, Mu)_\Omega \in K(\theta) \quad \text{for all} \quad u \in V$$

with

$$V = \{u \in C^\infty(\bar{\Omega}) : \partial_\nu^{j-1} u = 0 \quad \text{on} \quad \partial\Omega, \quad 0 < j \le \ell\}.$$

∂_ν is the normal derivative on $\partial\Omega$.

For the Laplace operator Δ, consider $M = -\Delta$ and $L = \Delta^2$.

Then $V = \{u \in C^\infty(\bar{\Omega}) : u = \partial_\nu u = 0 \text{ on } \partial\Omega\}$ and

$$(Lu, Mu)_\Omega = \int_\Omega |\nabla(\Delta u)|^2 - \int_{\partial\Omega} \partial_\nu(\Delta u)\Delta u.$$

∇ denotes the gradient operator.

As there is no control on the last term the above condition cannot be satisfied.

In (2.20) the same type of assumption is used but on the smaller set D_0. Consider the special case with M formally self-adjoint and positive definite and $L = M^2$. This includes the above example. We show that Condition (2.20) is satisfied for this example.

As $\ell = 2m$, $D_0 \subset H_0^{2m}(\Omega) \cap H^{4m}(\Omega)$. Also, for $u \in D_0$ it follows that $L_0u = M_0^2u \in Y$ and there exists a sequence $\{w_n\} \subset H_0^{2m}(\Omega) \cap H^{4m}(\Omega)$ with

$$\|Mw_n - M_0^2u\|_\Omega \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

As M is positive definite this implies that $\{w_n\}$ converges to a $w \in H_0^{2m}(\Omega)$ with $M_0^2u = Mw$.

For $u \in D_0$,

$$(L_0u,M_0u)_\Omega = (M_0^2u,M_0u)_\Omega = (Mw,M_0u)_\Omega.$$

As M is self-adjoint

$$(L_0u,M_0u)_\Omega = (w,M_0^2u)_\Omega = (w,Mw)_\Omega.$$

This yields, from (2.7),

$$\mathrm{Re}(L_0u,M_0u)_\Omega = \mathrm{Re}(w,Mw)_\Omega \geq c_1\|w\|_m^2 .$$

As $\mathrm{Re}(w,Mw)_\Omega = (w,Mw)_\Omega$ this yields that Condition (2.2) is satisfied with $\theta = 0$.

Finally we remark that in [Fi] the biharmonic case was solved by means of eigenfunction expansions. The same set of permissible initial conditions was found ([Fi, p 254]).

2.4 Dynamical boundary value problems

2.4.1 Introduction

In [S2] the following dynamic boundary value problem is discussed.

$$\begin{aligned} &\partial_t u = Lu, \quad u = u(x,t), \quad x \in \Omega, \quad t > 0 \\ &\partial_t(\gamma_0 u) = -L_\nu u, \qquad\qquad x \in \partial\Omega, \quad t > 0 \end{aligned} \tag{2.23}$$

Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$. L is a symmetric second order differential operator

$$Lu := \sum_{i,j=1}^{n} \partial_i (a_{ij}(x)\partial_j u)$$

with the coefficients a_{ij} real-valued,

$$a_{ij} = a_{ji} \tag{2.24}$$

and

$$a_{ij} \in C^\infty(\bar{\Omega}), \quad i,j = 1,2,\ldots,n.$$

L is uniformly strongly elliptic in the sense that for some $c_1 > 0$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c_1|\xi|^2 \quad \text{for all} \quad \xi \in R^n \quad \text{and} \quad x \in \bar{\Omega}. \tag{2.25}$$

L_ν is the co-normal derivative associated with L at the boundary $\partial\Omega$. ν is the unit outward normal vector on $\partial\Omega$.

$$L_\nu u := \sum_{i,j=1}^{n} a_{ij}(x)\nu_i(x)\partial_j u.$$

γ_0 is the trace operator. [L, p 41]

The problem is formulated as an abstract evolution equation by choosing

$$X = L^2(\Omega), \quad Y = L^2(\Omega) \times L^2(\Gamma), \quad D_0 = C^2(\bar{\Omega})$$

and defining A_0 and B_0 on D_0 by

$$A_0 u = \langle -Lu, L_\nu u \rangle,$$

and

$$B_0 u = \langle u, \gamma_0 u \rangle.$$

It is then shown that for $C_0 = B_0$ Assumptions 1 - 7 of Section 2.2 are satisfied, and hence for any

$$y \in L^2(\Omega) \times L^2(\partial\Omega)$$

the unique solution to

$$\frac{d}{dt}(Bu(t)) = Au(t), \quad t > 0$$
$$\lim_{t \to 0^+} Bu(t) = y$$

is given by

$$u(t) = S(t)y$$

with $S(t)$ the holomorphic B-evolution of type L generated by $\langle -A, B \rangle$, the Friedrichs extension of $\langle -A_0, B_0 \rangle$.

In this section various generalizations of problem (2.23) are presented.

2.4.2 Non-symmetric case

The boundary value problem (2.23) is considered with smooth complex-valued coefficients a_{ij} for the differential operator L. The symmetry condition (2.24) is dropped and we assume that L is uniformly strongly elliptic in the sense that for some $c_1 > 0$

$$\operatorname{Re} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c_1|\xi|^2 \text{ for all } \xi \in R^n \text{ and } x \in \bar{\Omega}. \tag{2.26}$$

Let L^* denote the formal adjoint of L [F, p 3];

$$L^*u := \sum_{i,j=1}^{n} \partial_i (\bar{a}_{ji}(x)\partial_j u).$$

The co-normal derivative L_ν^* associated with L^* at the boundary $\partial\Omega$ is given by

$$L_\nu^* u := \sum_{i,j=1}^{n} \bar{a}_{ji}(x)\nu_i(x)\partial_j u.$$

As in Section 2.4.1 choose

$$X = L^2(\Omega), \quad Y = L^2(\Omega) \times L^2(\partial\Omega) \quad \text{and} \quad D_0 = C^2(\bar{\Omega}),$$

and define operators A_0, B_0, C_0 and N_0 by

$$\begin{aligned} 2A_0 u &:= \langle -(L + L^*)u, (L_\nu + L_\nu^*)u\rangle \\ B_0 u = C_0 u &:= \langle u, \gamma_0 u\rangle \\ 2N_0 u &:= \langle -(L - L^*)u, (L_\nu - L_\nu^*)u\rangle \quad \text{for all} \quad u \in D_0. \end{aligned}$$

Note that for $u,v \in D_0 = C^2(\bar{\Omega})$ integration by parts is valid and if $(,)_{\partial\Omega}$ denotes the inner product in $L^2(\partial\Omega)$,

$$(-Lu,v)_\Omega + (L_\nu u, \gamma_0 v)_{\partial\Omega} = \sum_{i,j=1}^{n} (a_{ij}\partial_i u, \partial_j v)_\Omega \tag{2.27}$$

and

$$(-L^*u,v)_\Omega + (L_\nu^* u, \gamma_0 v)_{\partial\Omega} = \sum_{i,j=1}^{n} (\bar{a}_{ji}\partial_i u, \partial_j v)_\Omega. \tag{2.28}$$

We show that Assumptions F1 - F8 and F10 are satisfied and hence that Theorems 7 and 8 apply.

Assumption F1

We have to show that for some $\omega \geq 0$, $[u,v] = S_0(u,v) + \omega R_0(u,v)$ is Hermitian.

$$S_0(u,v) = (B_0u,B_0v) \text{ is Hermitian.}$$
$$\begin{aligned} 2R_0(u,v) &= (2A_0u,B_0v) \\ &= (-Lu - L^*u,v)_\Omega + (L_nu + L_n^*u,\gamma_0 v)_{\partial\Omega}. \end{aligned}$$

From (2.27) and (2.28)

$$2R_0(u,v) = \sum_{i,j=1}^{n} ((a_{ij}\partial_i u,\partial_j v)_\Omega + (\bar{a}_{ji}\partial_i u,\partial_j v)_\Omega$$

which is easily shown to be Hermitian.

Assumption F2

We have to show that $\operatorname{Re} S_0(u,u) \geq 0$ for all $u \in D_0$.

This is satisfied trivially as $S_0(u,u) = \|B_0u\|^2$.

Assumption F3

We have to show that for some $\omega \geq 0$ there is a $c > 0$ with

$$|[u]|^2 = R_0(u,u) + \omega S_0(u,u) \geq c\|u\|_\Omega^2 \text{ for all } u \in C^2(\bar{\Omega}).$$

For $u \in D_0$ and $\omega \geq 0$

$$\begin{aligned} &R_0(u,u) + \omega S_0(u,u) \\ &= \frac{1}{2}\sum_{i,j=1}^{n} ((a_{ij}\partial_i u,\partial_j u)_\Omega + (\bar{a}_{ji}\partial_i u,\partial_j u)_\Omega) + \omega\|B_0u\|^2 \\ &= \operatorname{Re}\sum_{i,j=1}^{n} (a_{ij}\partial_i u,\partial_j u) + \omega\|u\|_\Omega^2 + \omega\|\gamma_0 u\|_{\partial\Omega}^2. \end{aligned}$$

From (2.26)

$$R_0(u,u) + \omega S_0(u,u) \geq c\|\nabla u\|_\Omega^2 + \omega\|u\|_\Omega^2 + \omega\|\gamma_0 u\|_{\partial\Omega}^2$$

with $\nabla u = (\partial_1 u, \partial_2 u, \ldots, \partial_n u)$ the gradient of u.

For any $\omega > 0$ this yields

$$|[u]|^2 \geq \omega\|u\|_\Omega^2 \quad \text{for all} \quad u \in D_0 = C^2(\bar{\Omega}).$$

Assumption F4

We have to show that if $\{u_n\} \subset D_0$ is a Cauchy-sequence in $|[\ ]|$ and $\|u_n\|_\Omega \longrightarrow 0$, then $|[u_n]| \longrightarrow 0$.

From the proof of Assumption F3, for $c = \min\{c_1, \omega\}$,

$$|[u]|^2 \geq c\|u\|_1^2 \quad \text{for all} \quad u \in D_0. \tag{2.29}$$

Also, for some $k_1 > 0$

$$\|\gamma_0 u\|_{\partial\Omega} \leq k_1\|u\|_1 \quad \text{for all} \quad u \in H^1(\Omega) \tag{2.30}$$

as the trace operator γ_0 is a bounded operator from the Sobolev space $H^1(\Omega)$ into $L^2(\partial\Omega)$. [L, p 41].

Hence, for some $k_2 > 0$,

$$|[u]|^2 \leq k_2\|u\|_1^2 \quad \text{for all} \quad u \in D_0.$$

This shows that $|[\ ]|$ is equivalent to the $H^1(\Omega)$-norm on D_0 and if $\{u_n\}$ is a Cauchy sequence in $|[\ ]|$ there exists some $u \in H^1(\Omega)$ with

$$|[u_n - u]|^2 \le k_2 \|u_n - u\|_1^2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

If $\|u_n\|_\Omega \longrightarrow 0$ this implies that $u = 0$ and $|[u_n]| \longrightarrow 0$ as $n \longrightarrow \infty$.

Assumption F5

We have to show that

$$C_0[D_0] = B_0[D_0] = \{\langle u, \gamma_0 u\rangle : u \in C^2(\bar{\Omega})\}$$

is dense in $L^2(\Omega) \times L^2(\partial\Omega)$.

This is a special case of a result in [VR, p 58].

Assumption F6

We have to show that B_0 is bounded in $|[\ ]|$.

For $u \in D_0 = C^2(\bar{\Omega})$

$$\|B_0 u\|^2 = \|u\|_\Omega^2 + \|\gamma_0 u\|_{\partial\Omega}^2$$

and, from (2.30) and (2.29),

$$\|B_0 u\|^2 \le (1 + k_1^2)\|u\|_1^2 \le (1 + k_1^2)/c_1 |[u]|^2.$$

Assumption F7

We have to show that for some $\delta \ge 0$ and $0 \le \epsilon < 1$

$$\text{Re } T_0(u,u) \ge -\epsilon|[u]|^2 - \delta \text{ Re } S_0(u,u) \quad \text{for all} \quad u \in D_0.$$

For $u \in D_0$

$$\begin{aligned} 2T_0(u,u) &= (2N_0u, B_0u) \\ &= (-Lu + L^*u, u)_\Omega + (L_\nu u - L_\nu^* u, \gamma_0 u)_{\partial\Omega} \\ &= \sum_{i,j=1}^{n} ((a_{ij}\partial_i u, \partial_j u)_\Omega - (\bar{a}_{ji}\partial_i u, \partial_j u)_\Omega) \\ &= i \operatorname{Im} \sum_{i,j=1}^{n} (a_{ij}\partial_i u, \partial_j u)_\Omega . \end{aligned}$$

This shows that $\operatorname{Re} T_0(u,u) = 0$ for all $u \in D_0$ and that the assumption is satisfied with $\delta = 0$ and $\epsilon = 0$.

Assumption F8

We have to show that T_0 is a bounded form with respect to $|[\ ]|$.

From

$$2T_0(u,u) = i \operatorname{Im} \sum_{i,j=1}^{n} (a_{ij}\partial_i u, \partial_j u)_\Omega$$

it is clear that T_0 is bounded with respect to the $H^1(\Omega)$-norm which is equivalent to $|[\ ]|$.

Assumption F10

We have to show that for some $k > 0$

$$\|B_0 u\| \geq k\|u\|_\Omega \quad \text{for all} \quad u \in D_0 .$$

This follows directly as

$$\|B_0 u\|^2 = \|u\|_\Omega^2 + \|\gamma_0 u\|_{\partial\Omega}^2 \geq \|u\|_\Omega^2$$

for all $u \in D_0$.

Remarks

1. As $\delta = 0$ in Assumption F7, it follows that $\omega_1 = \omega + \delta = \omega$.

2. As $|[\ ]|$ is equivalent to the $H^1(\Omega)$-norm, it follows that $D_1 = H^1(\Omega)$ and $D' \subset H^1(\Omega)$ in the construction of the Friedrichs extension $\langle M,B'\rangle$ of $\langle A_0 + N_0, B_0\rangle$.

3. Note that $A_0 + N_0 = \langle -L, L_\nu\rangle$ and that this operator is extended to M.

4. As $D_1 = H^1(\Omega)$ it follows that

$$B_1 u = \langle u, \gamma_0 u\rangle \quad \text{for all} \quad u \in H^1(\Omega)$$

and, as $B' \subset B_1$, also that

$$B' u = \langle u, \gamma_0 u\rangle \quad \text{for all} \quad u \in D'.$$

5. As $N_0 = \langle \frac{1}{2}(L^* - L), \frac{1}{2}(L_\nu - L_\nu^*)\rangle$ and $|[\ ]|$ is equivalent to the $H^1(\Omega)$-norm the stricter boundedness assumption F9 will usually not be satisfied.

6. If L is symmetric, in the sense that $a_{ij} = \bar{a}_{ji}$ for all i and j, it follows that $N_0 = 0$ and hence $M = A$. Also, as Assumption F7 is satisfied for $\delta = 0$ and $\epsilon = 0$, it follows from the proof of Theorem 6 that

$$\rho(-AB^{-1}) \supseteq \{\lambda : \lambda - \omega \in K(\tfrac{\pi}{2} + \psi)\}$$

for any ψ with $0 < \psi < \frac{\pi}{2}$.

Theorem 14 *For* L *and* L_ν *as above and any* $y \in L^2(\Omega) \times L^2(\partial\Omega)$ *the unique solution to*

$$\frac{d}{dt}\langle u,\gamma_0 u\rangle = -Mu, \quad t > 0$$
$$\lim_{t\to 0^+} \langle u,\gamma_0 u\rangle = y$$

is given by

$$u(t) = e^{\omega t}S(t)y$$

with $S(t)$ *the holomorphic* B'*-evolution of type* L *with generating pair* $\langle -M - \omega B', B'\rangle$. *The pair* $\langle M,B'\rangle$ *is the Friedrichs extension of* $\langle A_0 + N_0, B_0\rangle$.

Proof The result follows directly from Theorem 8.

2.4.3 Lower order terms

In this section the boundary value problem (2.23) is considered with some lower order terms added to the differential operator.

Let $L = L_1 + L_2$ with

$$L_1 u := \sum_{i,j=1}^{n} \partial_i (a_{ij}(x)\partial_j u) \quad \text{and} \quad L_2 u := \sum_{i=1}^{n} a_i(x)\partial_i u + a(x)u.$$

Assume that all coefficients are smooth real-valued functions on $\bar{\Omega}$ and that the symmetry condition (2.24) and the uniform strong ellipticity condition (2.25) hold.

The co-normal derivative L_ν associated with L at the boundary $\partial\Omega$ remains unchanged.

As in Section 2.4.1 choose

$$X = L^2(\Omega), \quad Y = L^2(\Omega) \times L^2(\partial\Omega) \quad \text{and} \quad D_0 = C^2(\bar{\Omega}),$$

and define operators A_0, B_0, C_0 and N_0 by

$$A_0 u := \langle -L_1 u, L_\nu u\rangle$$
$$B_0 u = C_0 u := \langle u, \gamma_0 u\rangle$$
$$N_0 u := \langle -L_2 u, 0\rangle \quad \text{for all} \quad u \in D_0 .$$

Again, integration by parts is valid for $u,v \in D_0 = C^2(\bar{\Omega})$ and

$$(-L_1 u, v)_\Omega + (L_\nu u, \gamma_0 v)_{\partial\Omega} = \sum_{i,j=1}^{n} (a_{ij}\partial_i u, \partial_j v)_\Omega . \tag{2.31}$$

The operator A_0 is a special case of the operator A_0 defined in Section 2.4.2. This is clear by noting that $L = L^*$ in Section 2.4.2 if a_{ij} is real-valued and $a_{ij} = a_{ji}$. As the same operators B_0 and C_0 are used in Section 2.4.2 this implies that Assumptions F1 - F6 and F10 are satisfied in this case.

We proceed to show that Assumption F7 and the stricter boundedness Assumption F9 are also satisfied.

Assumption F7

We have to show that for some $\delta \geq 0$ and $0 \leq \epsilon < 1$

$$\text{Re } T_0(u,u) \geq -\epsilon|[u]|^2 - \delta \text{ Re } S_0(u,u) \quad \text{for all} \quad u \in D_0 .$$

For $u \in D_0$

$$T_0(u,u) = (-L_2 u, u)_\Omega$$

and

$$|T_0(u,u)| \leq \|L_2 u\|_\Omega \, \|u\|_\Omega .$$

L_2 is a bounded operator in the $H^1(\Omega)$-norm and as $|[\ ]|$ is equivalent to the $H^1(\Omega)$-norm, for some $c_2 > 0$

$$|T_0(u,u)| \le c_2 |[u]| \; \|u\|_\Omega \quad \text{for all} \quad u \in D_0 .$$

The well-known inequality for real numbers a, b and $\eta > 0$

$$2ab \le \eta a^2 + \frac{1}{\eta} b^2 ,$$

yields

$$|T_0(u,u)| \le \eta c_2/2 \; |[u]|^2 + c_2/2\eta \; \|u\|_\Omega^2 .$$

Choose $\epsilon = \eta c_2/2 < 1$ and $\delta = c_2/2\eta$ and note that

$$\|u\|_\Omega^2 \le \|u\|_\Omega^2 + \|\gamma_0 u\|_\Omega^2 = S_0(u,u) .$$

Then

$$\mathrm{Re}\; T_0(u,u) \ge -|T_0(u,u)| \ge -\epsilon |[u]|^2 - \delta S_0(u,u) \quad \text{for all} \quad u \in D_0$$

and the assumption is satisfied.

Assumption F9

We have to show that N_0 is bounded in $|[\ ]|$.

For $u \in D_0$, $N_0 u = \langle L_2 u, 0\rangle$ and L_2 is bounded in the $H^1(\Omega)$-norm. Hence the assumption is satisfied as $|[\ ]|$ is equivalent to the $H^1(\Omega)$-norm.

Remarks

1. As $|[\ ]|$ is equivalent to the $H^1(\Omega)$-norm it follows that $D_1 = H^1(\Omega)$ in the construction of the Friedrichs extension $\langle M,B'\rangle$ of $\langle A_0 + N_0, B_0\rangle$. As Assumption F9 is satisfied, from Theorem 4 $D = D'$ and the extension $\langle M,B'\rangle = \langle A + N,B\rangle$.

This extension is determined by the extension $\langle A,B\rangle$ of $\langle A_0,B_0\rangle$. To find N the operator N_0 is extended by continuity to N_1 on D_1 and then N_1 is restricted to D, the domain of A and B.

2. As $D_1 = H^1(\Omega)$ it follows as in Section 2.4.2 that $B_1u = \langle u,\gamma_0 u\rangle$ for all $u \in D_1 = H^1(\Omega)$ and hence that $Bu = \langle u,\gamma_0,u\rangle$ for all $u \in D$.

3. Note that $A_0u + N_0u = \langle -Lu, L_\nu u\rangle$.

Theorem 15 *For* L *and* L_ν *as above and any* $y \in L^2(\Omega) \times L^2(\partial\Omega)$ *the unique solution to*

$$\frac{d}{dt}\langle u,\gamma_0 u\rangle = -(A + N)u, \quad t > 0$$
$$\lim_{t\to 0^+} \langle u,\gamma_0 u\rangle = y$$

is given by

$$u(t) = e^{\omega_1 t}S(t)y$$

with $S(t)$ *the holomorphic* B-*evolution of type* L *with generating pair* $\langle -(A + N) - \omega_1 B,B\rangle$. *The pair* $\langle A + N,B\rangle$ *is the Friedrichs extension of* $\langle A_0 + N_0,B_0\rangle$.

Proof The result follows directly from Theorem 8 and the first remark in Section 2.2.3.

2.4.4 Dynamic boundary condition for imperfect contact

Consider the following boundary value problem:

$$\begin{aligned} \partial_t u &= Lu, \quad u = u(x,t), \quad x \in \Omega, \quad t > 0 \\ \partial_t U &= -L_\nu u, \quad U = U(x,t), \quad x \in \partial\Omega, \quad t > 0. \end{aligned} \tag{2.32}$$

The domain Ω and the operators L and L_ν are taken as in Section 2.4.1.

Note that $U(\cdot,t)$ is defined only on $\partial\Omega$ and that (2.32) reduces to (2.23) if $U = \gamma_0 u$.

The dynamic boundary value problem (2.32) can be derived by using a conservation law approach. The domain Ω and the boundary $\partial\Omega$ are treated separately with u and U representing the conserved quantity in Ω and $\partial\Omega$ respectively.

A diffusion process in Ω gives rise to the first equation in (2.32).

In the boundary condition $-L_\nu u$ represents the flux of the preserved quantity into the boundary $\partial\Omega$.

The trace $\gamma_0 u$ of u on the boundary $\partial\Omega$ and the value of U are related through some contact condition. We assume that the flux into the boundary is proportional to the difference between $\gamma_0 u$ and U;

$$U - \gamma_0 u = k^2(x) L_\nu u, \quad x \in \partial\Omega, \quad t > 0 \tag{2.33}$$

with k a smooth function on $\partial\Omega$ such that for some $\delta > 0$, $k(x) \geq \delta > 0$ for all $x \in \partial\Omega$.

Intuitively, $k \longrightarrow 0$ implies $\gamma_0 u \longrightarrow U$ on $\partial\Omega$ which may be interpreted as perfect contact between Ω and $\partial\Omega$. On the other hand $k \longrightarrow \infty$ implies $L_\nu u \longrightarrow 0$ on $\partial\Omega$ and therefore that $\gamma_0 u$ and U are independent and that there is no interaction between the processes in Ω and $\partial\Omega$. The case $k \longrightarrow 0$ is dealt with in Section 3.5.

This contact condition (2.33) yields the dynamic boundary value problem (2.32) as

$$\partial_t \begin{bmatrix} u \\ \gamma_0 u + k^2 L_\nu u \end{bmatrix} = \begin{bmatrix} Lu \\ -L_\nu u \end{bmatrix} .$$

Choose $X = L^2(\Omega)$, $Y = L^2(\Omega) \times H^{1/2}(\partial\Omega)$ and $D_0 = C^2(\bar{\Omega})$.

The Sobolev space $H^{1/2}(\partial\Omega)$ is defined in [L, p 34]. The norm in $H^{1/2}(\partial\Omega)$ is denoted by $\| \;\|_{1/2}$ and the inner product by $(\;, \;)_{1/2}$.

Define operators A_0, B_0, N_0 and C_0 by

$A_0 u := \langle -Lu, L_\nu u\rangle$

$B_0 u := \langle u, U\rangle = \langle u, \gamma_0 u + k^2 L_\nu u\rangle$

$N_0 u := 0$

$C_0 u := A_0 u + \omega B_0 u = \langle -Lu + \omega u, L_\nu u + \omega U\rangle$ for all $u \in D_0 = C^2(\bar{\Omega})$.

Note that the notation $U := \gamma_0 u + k^2 L_\nu u$ is used as it simplifies the presentation of the calculations below.

We show that Assumption F3 to F6 and F10 to F12 are satisfied and that Theorem 11 applies.

These proofs are based on standard results for the elliptic boundary value problem

$$\begin{aligned} -Lu + \omega u &= f \quad \text{in } \Omega \\ L_\nu u + \omega U = (1 + \omega k^2)L_\nu u + \omega\gamma_0 u &= g \quad \text{on } \partial\Omega \end{aligned} \tag{2.34}$$

with $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$.

Results from [L, p 148-165] may be used if the operator $-L + \omega$ is properly elliptic in $\bar{\Omega}$ and the boundary operator $(1 + \omega k^2)L_\nu + \omega\gamma_0$ is normal on $\partial\Omega$ and covers the operator $-L + \omega$ on $\partial\Omega$.

The operator $-L + \omega$ is a uniformly strongly elliptic operator on $\bar{\Omega}$ and hence also properly elliptic, [L, p 110-111].

A boundary operator

$$Mu = \sum_{j=1}^{n} m_j \partial_j u + m_0 \gamma_0 u$$

with $m_j \in C^\infty(\partial\Omega)$ for $j = 0,1,\ldots,n$ is *normal* on $\partial\Omega$ if

$$\sum_{j=1}^{n} m_j(x)\xi_j \neq 0 \quad \text{for all} \quad x \in \partial\Omega$$

and all $\xi \neq 0$ and normal to $\partial\Omega$ at x. [L, p 113]

For the boundary operator $(1 + \omega k^2)L_\nu + \omega\gamma_0$

$$m_j(x) = (1 + \omega k^2(x)) \sum_{i=1}^{n} a_{ij}(x)\nu_i(x), \quad j = 1,2,\ldots,n$$

and

$$\sum_{j=1}^{n} m_j(x)\xi_j = (1 + \omega k^2(x)) \sum_{i,j=1}^{n} a_{ij}(x)\nu_i(x)\xi_j .$$

If ξ is normal to $\partial\Omega$ at x

$$\xi = \pm |\xi|\nu(x)$$

and

$$\sum_{i,j=1}^{n} m_j(x)\xi_j = \pm(1 + \omega k^2(x))|\xi|^{-1} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j .$$

The ellipticity condition (2.25) now yields that the boundary operator is normal on $\partial\Omega$.

From a result (formulated as a problem) in [F, p 76] follows that if μ is a nontangential smoothly varying direction on $\partial\Omega$ and

$$Mu = \frac{\partial u}{\partial \mu} + \omega\gamma_0 u$$

then M covers any second order properly elliptic operator on $\partial\Omega$. (In [F] if a boundary operator M covers a differential operator it is said to be complimentary.)

For the boundary operator $(1 + \omega k^2)L_\nu + \omega\gamma_0$ we choose the direction $\mu(x)$ at $x \in \partial\Omega$ by

$$\mu_j(x) = m_j(x) = (1 + \omega k^2(x)) \sum_{i=1}^{n} a_{ij}(x)\nu_i(x)$$

μ is a nontangential direction as

$$\begin{aligned} \mu(x)\cdot\nu(x) &= \sum_{i,j=1}^{n} (1 + \omega k^2(x))a_{ij}(x)\nu_i(x)\nu_j(x) \\ &\geq (1 + \omega k^2(x))c_1 > 0. \end{aligned}$$

This follows from (2.25).

The a priori estimate [L, p 149] yields that there exists some $c = c(\omega) > 0$ such that

$$\|u\|_2 \leq c(\omega)(\|-Lu + \omega u\|_\Omega + \|L_\nu u + \omega U\|_{1/2} + \|u\|_\Omega) \quad \text{for all} \quad u \in H^2(\Omega). \quad (2.35)$$

Consider the set

$$N = \{u \in C^\infty(\bar{\Omega}) : -Lu + \omega u = 0, \quad L_\nu u + \omega U = 0\}.$$

Proposition $N = \{0\}$.

Proof For any $u \in C^\infty(\bar{\Omega})$

$$(-Lu + \omega u, u)_\Omega = \sum_{i,j=1}^{n} (a_{ij}\partial_j u, \partial_i u)_\Omega - (L_\nu u, \gamma_0 u)_{\partial\Omega}.$$

For $u \in N$

$$(1 + \omega k^2)L_\nu u + \omega\gamma_0 u = 0$$

and hence

$$L_\nu u = - \frac{\omega}{1 + \omega k^2} \gamma_0 u .$$

This yields that for $u \in N$

$$\sum_{i,j=1}^{n} (a_{ij}\partial_j u, \partial_i u)_\Omega + (\frac{\omega}{1 + \omega k^2} \gamma_0 u, \gamma_0 u)_{\partial\Omega} = 0. \qquad (2.36)$$

From (2.25)

$$\sum_{i,j=1}^{n} (a_{ij}\partial_j u, \partial_i u)_\Omega \geq c_1 \sum_{i=1}^{n} \|\partial_i u\|_\Omega^2 \geq 0.$$

As $u \in C^\infty(\bar{\Omega})$, and $k^2(x) > 0$ for all $x \in \partial\Omega$ equation (2.36) implies that if $\omega > 0$

$$\gamma_0 u = 0 \quad \text{on} \quad \partial\Omega \qquad (2.37)$$

and

$$\sum_{i=1}^{n} \|\partial_i u\|_\Omega^2 = 0. \qquad (2.38)$$

From (2.37) and a result in [F, p 39] we conclude that $u \in H_0^1(\Omega)$.

But on $H_0^1(\Omega)$ the expression

$$(\sum_{i=1}^{n} \|\partial_i u\|_\Omega^2)^{1/2} \quad \text{for} \quad u \in H_0^1(\Omega)$$

is a norm equivalent to the usual Sobolev space norm. This follows easily from the Poincaré inequality. [GT, p 157].

From (2.38) we conclude that $u = 0$ and hence $N = \{0\}$. Note that $\omega > 0$.

This proposition has some important implications. Firstly, the a priori inequality (2.35) may be improved ([L, p 161]) to

$$\|u\|_2 \le c(\omega)(\|-Lu + \omega u\|_\Omega + \|L_\nu u + \omega U\|_{1/2}) \quad \text{for all} \quad u \in H^2(\Omega). \tag{2.39}$$

Concerning the existence of solutions for the boundary value problem (2.34) one must note that because of the symmetry condition (2.24) the same problem may be chosen as a formal adjoint problem.

As $N = \{0\}$ we conclude from the existence theorem [L, Th 5.3, p 164] that the boundary value problem (2.34) has a unique solution in $H^2(\Omega)$ for every $\langle f,g\rangle \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$.

Assumption F3

We have to show that for some $\omega \geq 0$ there is a $c > 0$ with

$$\begin{aligned} |[u]|^2 &= \|A_0 u + \omega B_0 u\|^2 \\ &= \|-Lu + \omega u\|_\Omega^2 + \|L_\nu u + \omega U\|_{1/2}^2 \\ &\geq c\|u\|_2^2 \quad \text{for all} \quad u \in C^2(\bar{\Omega}). \end{aligned}$$

This follows directly from the a priori inequality (2.39) for any $\omega > 0$.

Assumption F4

We have to show that if $\{u_n\} \subset D_0$ is a Cauchy sequence in $|[\ ]|$ and $\|u_n\|_\Omega \longrightarrow 0$, then $|[u_n]| \longrightarrow 0$.

From Assumption F3 $\{u_n\}$ is also a Cauchy sequence in $H^2(\Omega)$ and as $H^2(\Omega)$ is complete there is some $u \in H^2(\Omega)$ with

$$\|u_n - u\|_2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

From $\|u_n\|_\Omega \longrightarrow 0$ we know that $u = 0$.

Finally, there is a $c_2 > 0$ such that

$$|[u]^2| = \|-Lu + \omega u\|_\Omega^2 + \|L_\nu u + \omega U\|_{1/2}^2 \leq c_2\|u\|_2^2 \quad \text{for all} \quad u \in H^2(\Omega) \qquad (2.40)$$

as the mapping from $u \in H^2(\Omega)$ to $\langle -Lu + \omega u, L_\nu u + \omega U\rangle \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ is continuous. [L, p 148]

Hence $|[u_n]|^2 \leq c_2\|u_n\|_2^2 \longrightarrow 0$ as $n \longrightarrow \infty$.

Assumption F5

We have to show that

$$C_0[D_0] = \{\langle -Lu + \omega u, L_\nu u + \omega U\rangle : u \in C^2(\bar{\Omega})\}$$

is dense in $Y = L^2(\Omega) \times H^{1/2}(\partial\Omega)$.

For any $\langle f,g\rangle \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ it has been shown that there exists a unique $v \in H^2(\Omega)$ with

$$-Lv + \omega v = f$$
$$L_\nu v + \omega V = g.$$

As $D_0 = C^2(\bar{\Omega})$ is dense in $H^2(\Omega)$ there exists a sequence $\{u_n\} \subset C^2(\bar{\Omega})$ with

$$\|u_n - v\|_2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

This yields

$$\|C_0 u_n - \langle f,g\rangle\|^2 = \|\omega(u_n - v) - L(u_n - v)\|_\Omega^2 + \|L_\nu(u_n - v) + \omega(U_n - V)\|_{1/2}^2$$

and from (2.40)

$$\|C_0u_n - \langle f,g\rangle\|^2 \le c_2\|u_n - v\|_2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Assumption F6

We have to show that B_0 is bounded in $|[\cdot]|$.

For $u \in D_0 = C^2(\bar{\Omega})$

$$\begin{aligned}\|B_0u\|^2 &= \|u\|_\Omega^2 + \|U\|_{1/2}^2 \\ &\le c_2\|u\|_2^2\end{aligned}$$

and as $|[\ ]|$ is equivalent to the $H^2(\Omega)$-norm this shows that the assumption is satisfied.

Assumption F10

We have to show that for some $k_1 > 0$

$$\|B_0u\| \ge k_1\|u\|_\Omega \quad \text{for all} \quad u \in D_0 = C^2(\bar{\Omega}).$$

This follows directly as

$$\|B_0u\|^2 = \|u\|_\Omega^2 + \|U\|_{1/2}^2 \ge \|u\|_\Omega^2 \quad \text{for all} \quad u \in D_0.$$

Assumption F11

We have to show that for some θ, $0 \le \theta < \frac{\pi}{2}$,

$$S_0(u,u) = (B_0u,\ A_0u + \omega B_0u) \in K(\theta) \quad \text{for all} \quad u \in D_0.$$

Consider $(A_0u,\ B_0u) = (-Lu,u)_\Omega + (L_\nu u,U)_{1/2}$

$$= \sum_{i,j=1}^{n} (a^{ij}\partial_i u,\partial_j u)_\Omega - (L_\nu u,\gamma_0 u)_{\partial\Omega} + (L_\nu u,U)_{1/2}$$

$$= \sum_{i,j=1}^{n} (a^{ij}\partial_i u,\partial_j u)_\Omega - (\frac{1}{k^2}(U - \gamma_0 u),\gamma_0 u)_{\partial\Omega} + (\frac{1}{k^2}(U - \gamma_0 u),U)_{1/2}$$

$$= \sum_{i,j=1}^{n} (a^{ij}\partial_i u,\partial_j u)_\Omega + \|\tfrac{1}{k}\gamma_0 u\|^2_{\partial\Omega} + \|\tfrac{1}{k} U\|^2_{1/2}$$

$$- (\frac{1}{k^2} U,\gamma_0 u)_{\partial\Omega} - (\frac{1}{k^2}\gamma_0 u,U)_{1/2} \quad \text{for all} \quad u \in D_0 = C^2(\bar{\Omega}).$$

For any $\eta > 0$

$$|(\frac{1}{k^2} U,\gamma_0 u)_{\partial\Omega} + (\frac{1}{k^2}\gamma_0 u,U)_{1/2}| \le \tfrac{1}{2}(\|\tfrac{1}{k} U\|^2_{\partial\Omega} + \|\tfrac{1}{k}\gamma_0 u\|^2_{\partial\Omega} + \eta^2\|\tfrac{1}{k}\gamma_0 u\|^2_{1/2} + \frac{1}{\eta^2}\|\tfrac{1}{k} U\|^2_{1/2})$$

and this yields

$$\mathrm{Re}(A_0u,B_0u) \ge c_1 \sum_{j=1}^{n} \|\partial_j u\|^2_\Omega + \tfrac{1}{2}\|\tfrac{1}{k}\gamma_0 u\|^2_{\partial\Omega} + \tfrac{1}{2}(1 - \frac{1}{\eta^2})\|\tfrac{1}{k} U\|^2_{1/2} - \tfrac{1}{2}\eta^2\|\tfrac{1}{k}\gamma_0 u\|^2_{1/2} .$$

The trace operator γ_0 is a bounded operator from the Sobolev space $H^1(\Omega)$ into $H^{1/2}(\partial\Omega)$ [L, p 41] and hence for some $k_1 > 0$

$$\|\gamma_0 u\|_{1/2} \le k_1\|u\|_1 \quad \text{for all} \quad u \in H^1(\Omega). \tag{2.41}$$

As $k(x) \ge \delta > 0$ for all $x \in \partial\Omega$,

$$\|\tfrac{1}{k}\gamma_0 u\|_{1/2} \le \frac{k_1}{\delta}\|u\|_1 \quad \text{for all} \quad u \in D_0 = C^2(\bar{\Omega}).$$

This yields

$$\begin{aligned}\mathrm{Re}(A_0u,B_0u) \geq \Big(c_1 - \frac{1}{2}\Big(\frac{k_1\eta}{\delta}\Big)^2\Big)\, \|u\|_1^2 - c_1\|u\|_\Omega^2 \\ + \frac{1}{2}\, \Big\|\frac{1}{k}\,\gamma_0 u\Big\|_{\partial\Omega}^2 + \frac{1}{2}\,\Big(1 - \frac{1}{\eta^2}\Big)\, \Big\|\frac{1}{k}\,U\Big\|_{1/2}^2 .\end{aligned}$$

Choose $\eta > 0$ such that $m = c_1 - \frac{1}{2}\,\big(\frac{k_1\eta}{\delta}\big)^2 > 0$. Then

$$\mathrm{Re}(A_0u,\ B_0u) \geq -\ c_1\|u\|_\Omega^2 - \frac{1}{2\eta^2\delta^2}\,\|U\|_{1/2}^2 \quad \text{for all} \quad u \in D_0 = C^2(\bar{\Omega}).$$

Choose $\omega > \max\,\{c_1,\ \frac{1}{2\eta^2\delta^2}\}$ then

$$\begin{aligned}\mathrm{Re}\ S_0(u,u) &= \mathrm{Re}(B_0u,\ A_0u + \omega B_0u) \\ &= \omega\|B_0u\|_2^2 + \mathrm{Re}(A_0u,B_0u) \\ &= \omega(\|u\|_\Omega^2 + \|U\|_{1/2}^2) + \mathrm{Re}(A_0u,B_0u) \\ &\geq m(\|u\|_1^2 + \Big\|\frac{1}{k}\,U\Big\|_{1/2}^2) \quad \text{for all} \quad u \in D_0 .\end{aligned} \tag{2.42}$$

Also

$$\begin{aligned}|\mathrm{Im}\ S_0(u,u)| &= |\mathrm{Im}\ (A_0u,B_0u)| \\ &= \Big|\Big(\frac{1}{k^2}\,U,\gamma_0u\Big)_{\partial\Omega} + \Big(\frac{1}{k^2}\,\gamma_0u,U\Big)_{1/2}\Big| \\ &\leq m_1\Big(\Big\|\frac{1}{k}\,U\Big\|_{1/2}^2 + \Big\|\frac{1}{k}\,\gamma_0u\Big\|_{1/2}^2\Big)\end{aligned}$$

and from (2.41)

$$|\mathrm{Im}\ S_0(u,u)| \leq m_2(\|u\|_1^2 + \Big\|\frac{1}{k}\,U\Big\|_{1/2}^2) \quad \text{for all} \quad u \in D_0 . \tag{2.43}$$

Let $\theta = \arctan\,(\frac{m_2}{m})$, then (2.42) and (2.43) yield

$$S_0(u,u) \in K(\theta) \quad \text{for all} \quad u \in D_0 = C^2(\bar{\Omega}).$$

Assumption F11

We have to show that

$$B_0[D_0] = \{\langle u,U\rangle : u \in C^2(\bar{\Omega})\}$$

is dense in $Y = L^2(\Omega) \times H^{1/2}(\partial\Omega)$.

Assume that for some $\langle f,\alpha\rangle \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$

$$(\langle f,\alpha\rangle, B_0 u) = 0$$

or

$$(f,u)_\Omega + (\alpha,U)_{1/2} = 0 \quad \text{for all} \quad u \in D_0 = C^2(\bar{\Omega}). \tag{2.44}$$

As $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ there exists a sequence $\{w_n\} \subset C_0^\infty(\Omega) \subset D_0$ such that

$$\|w_n - f\|_\Omega \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

From (2.44)

$$(f,w_n)_\Omega = 0 \quad \text{for all} \quad n \quad \text{as} \quad W_n = 0.$$

This implies

$$\begin{aligned} \|f\|_\Omega^2 &= (f,f - w_n)_\Omega \\ &\leq \|f\|_\Omega \, \|f - \omega_n\|_\Omega \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \end{aligned}$$

and hence

$$f = 0.$$

Equation (2.44) is reduced to

$$(\alpha,U)_{1/2} = 0 \quad \text{for all} \quad u \in D_0 = C^2(\bar{\Omega}). \tag{2.45}$$

The boundary value problem

$$-Lu + u = f \quad \text{in} \quad \Omega$$
$$k^2 L_\nu u + \gamma_0 u = g \quad \text{on} \quad \partial\Omega$$

has a unique solution $u \in H^2(\Omega)$ for any $\langle f,g\rangle \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$. This can be shown exactly as it was done for the boundary value problem (2.34).

Hence, for any $\alpha \in H^{1/2}(\partial\Omega)$ there exists a $u \in H^2(\Omega)$ with

$$-Lu + u = 0$$
$$U = k^2 L_\nu u + \gamma_0 u = \alpha.$$

As $C^2(\bar{\Omega})$ is dense in $H^2(\Omega)$ there exists a sequence $\{w_n\} \subset D_0$ with

$$\|w_n - u\|_2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

and then also

$$\|W_n - \alpha\|_{1/2} = \|W_n - U\|_{1/2} \le c_2 \|w_n - u\|_2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

From (2.45)

$$\begin{aligned}\|\alpha\|_{1/2}^2 &= (\alpha,\ \alpha - W_n)_{1/2} \\ &\le \|\alpha\|_{1/2}\ \|\alpha - W_n\|_{1/2} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty\end{aligned}$$

and hence

$$\alpha = 0.$$

We conclude that $B_0[D_0] = \{\langle u,U\rangle : u \in C^2(\Omega)\}$ is dense in $L^2(\Omega) \times H^{1/2}(\partial\Omega)$.

Remarks

1. $|[\ ]|$ is equivalent to the $H^2(\Omega)$-norm and therefore $D_1 = H^2(\Omega)$. The regularity of solutions of elliptic boundary value problems yields that $D = D_1 = H^2(\Omega)$.

2. Assumption F10 is satisfied for

$$\omega > \max\ \{c_1,\ \frac{1}{2\eta^2\delta^2}\}$$

with c_1 as in (2.25)

δ such that $k(x) \geq \delta > 0$ for all $x \in \partial\Omega$

and $\eta^2 < 2\delta^2 c_1/k_1^2$, k_1 as in (2.41).

Theorem 16 *For* L *and* L_ν *as above and any* $y \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ *the unique solution to*

$$\frac{d}{dt}\ \langle u, \gamma_0 u + k^2 L_\nu u\rangle = \langle Lu, -L_\nu u\rangle$$

$$\lim_{t\to 0^+}\ \langle u, \gamma_0 u + k^2 L_\nu u\rangle = y$$

is given by

$$u(t) = e^{\omega t} S(t) y$$

with $S(t)$ *the holomorphic* B-*evolution of type* L *with generating pair* $\langle -A - \omega B, B\rangle$. *The pair* $\langle A,B\rangle$ *is the Friedrichs extension of* $\langle A_0, B_0\rangle$.

CHAPTER 3
CONVERGENCE OF SOLUTIONS

3.1 B-evolutions and the convergence of solutions of evolution equations

Consider the following initial value problem for an evolution equation

$$\begin{aligned} &\frac{d}{dt}(Bu) = Au \\ &\lim_{t\to 0^+} Bu = y. \end{aligned} \tag{3.1}$$

Assume that $\langle A,B\rangle$ is the generating pair of a holomorphic B-evolution $S(t)$ in a Banach space Y. For any $y \in Y$ the unique solution to (3.1) is given by

$$u(t) = S(t)y, \quad t > 0.$$

Consider also a sequence of initial value problems

$$\begin{aligned} &\frac{d}{dt}(B_n u) = A_n u \\ &\lim_{t\to 0^+} B_n u = y_n. \end{aligned} \tag{3.2}$$

For $n \in \mathbb{N}$ (3.2) is regarded as a perturbation of (3.1). Assume that the pairs $\langle A_n,B_n\rangle$, $n \in \mathbb{N}$ are generating pairs of holomorphic B_n-evolutions $S_n(t)$ in the Banach spaces Y_n. For any $y_n \in Y_n$ the unique solution to (3.2) is given by

$$u_n(t) = S_n(t)y_n, \quad t > 0.$$

In the examples the domains D_n of $\langle A_n,B_n\rangle$, $n \in \mathbb{N}$ and D of $\langle A,B\rangle$ are subspaces of the same Banach space X while $\langle A_n,B_n\rangle$, $n \in \mathbb{N}$ and $\langle A,B\rangle$ may map into different Hilbert spaces Y_n, $n \in \mathbb{N}$ and Y respectively.

Because the B_n-evolutions $S_n(t)$, $n \in \mathbb{N}$ and the B-evolution $S(t)$ all map into X the question of the convergence of the solutions $u_n(t)$ of (3.2) to the solution $u(t)$ of (3.1) may be studied with respect to some norm in X.

If $Y_n \neq Y$ part of the question is to identify initial values $y_n \in Y_n$, $n \in \mathbb{N}$ for (3.2) such that

$$u_n(t) = S_n(t)y_n \longrightarrow S(t)y = u(t) \quad \text{as} \quad n \longrightarrow \infty.$$

3.2 Convergence of holomorphic B-evolutions and generalized resolvent operators

For C_0-semigroups the convergence of a sequence of semigroups may be obtained from the convergence of the resolvent operators of the infinitesimal generators.

We quote the relevant results from Pazy [P].

We write $A \in G(M,\omega)$ if A is the infinitesimal generator of a C_0-semigroup $\{E(t) : t \geq 0\}$ with $\|E(t)\| \leq Me^{\omega t}$, $t \geq 0$.

For λ complex, $R(\lambda,A) = (\lambda I - A)^{-1}$ is the resolvent operator for A.

Theorem 1 (Trotter-Kato theorem [P, p 87]) *Let* X *be a complex Banach space with* $E_n(t)$, $n \in \mathbb{N}$ *a sequence of* C_0*-semigroups in* X. A_n *is the infinitesimal generator of* $E_n(t)$ *and* $A_n \in G(M,\omega)$, $n \in \mathbb{N}$. *If for some* λ_0 *with* $\text{Re}\, \lambda_0 > \omega$

(a) $R(\lambda_0,A_n)x \longrightarrow R(\lambda_0)x$ *as* $n \longrightarrow \infty$ *for all* $x \in X$

and

(b) $Rg(R(\lambda_0))$ *is dense in* X, *then there is a unique* $A \in G(M,\omega)$ *with* $R(\lambda_0) = R(\lambda_0, A)$. *For the semigroup* $E(t)$ *generated by* A

$$E_n(t)x \longrightarrow E(t)x \quad as \quad n \longrightarrow \infty$$

for all $x \in X$ *and* $t \geq 0$.

This convergence is uniform in t *on bounded intervals.*

Theorem 2 [P, p 85] *Let* $A, A_n \in G(M,\omega)$ *and* $E(t)$ *and* $E_n(t)$ *be the semigroups generated by* A *and* A_n *respectively. The following two properties are equivalent:*

(a) *For every* $x \in X$ *and* λ *with* $\operatorname{Re} \lambda > \omega$,
$$R(\lambda, A_n)x \longrightarrow R(\lambda, A)x \quad as \quad n \longrightarrow \infty.$$

(b) *For every* $x \in X$ *and* $t \geq 0$
$$E_n(t)x \longrightarrow E(t)x \quad as \quad n \longrightarrow \infty.$$

These results suggest that the convergence of B-evolutions may be linked to the convergence of the generalized resolvent operators.

The examples which will be presented in the rest of this chapter fit into the following abstract formulation.

Let X be a complex Banach space. Y and Y_n, $n \in \mathbb{N}$ are complex Hilbert spaces. $\langle A_n, B_n \rangle$ is the generating pair of a holomorphic B_n-evolution $S_n(t)$ of type L on Y_n. A_n and B_n are defined on $D_n \subset X$. $\langle A, B \rangle$ is the generating pair of a holomorphic B-evolution $S(t)$ of type L on Y. A and B are defined on $D \subset X$.

The Laplace transform $P_n(\lambda)$ of $S_n(t)$ is defined for all $\lambda \in \rho(A_n B_n^{-1})$, the resolvent of $A_n B_n^{-1}$, by

$$P_n(\lambda)y = \int_0^\infty e^{-\lambda t} S_n(t)y\,dt \quad \text{for all} \quad y \in Y_n.$$

$P_n(\lambda)$ is also the generalized resolvent operator of the pair $\langle A_n, B_n \rangle$. [S1, p 292]

$$P_n(\lambda)y = (\lambda B_n - A_n)^{-1}y \quad \text{for all} \quad y \in Y_n.$$

Similarly for $\lambda \in \rho(AB^{-1})$

$$P(\lambda)y = \int_0^\infty e^{-\lambda t} S(t)y\,dt = (\lambda B - A)^{-1}y \quad \text{for all} \quad y \in Y.$$

If $S(t)$ is a holomorphic B-evolution of type L in Y there is a contour integral representation for $S(t)$. [S1, p 298]

$$S(t)y = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} P(\lambda)y\,d\lambda \quad \text{for all} \quad y \in Y \ . \tag{3.3}$$

If $\rho(AB^{-1}) \supset K(\theta_1)$ with $\frac{\pi}{2} < \theta_1 < \pi$, Γ may be chosen as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with

$$\begin{aligned} \Gamma_1 &= \{re^{-i\theta_1} : 0 < r_0 \le r < \infty\} \\ \Gamma_2 &= \{r_0 e^{i\theta} : -\theta_1 < \theta < \theta_1\} \\ \Gamma_3 &= \{re^{i\theta_1} : r_0 \le r < \infty\} \quad [\text{P, p 30, 61}]. \end{aligned} \tag{3.4}$$

There is also a constant $m > 0$ such that

$$\|P(\lambda)y\|_X \le \frac{m}{|\lambda|} \|y\| \quad \text{for all} \quad y \in Y \quad \text{and} \quad \lambda \in K(\theta_1) \ . \tag{3.5}$$

See [S3, p 36].

If there is some θ_1 with $\frac{\pi}{2} < \theta_1 < \pi$ such that

$$\rho(AB^{-1}) \supset K(\theta_1)$$

and (3.6)

$$\bigcap_{n=1}^{\infty} \rho(A_nB_n^{-1}) \supset K(\theta_1)$$

the contour integral representation of $S_n(t)$ is

$$S_n(t)y = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}P_n(\lambda)yd\lambda \quad \text{for all} \quad y \in Y \tag{3.7}$$

with Γ as in (3.4) independent of n.

Remark

The integrals in (3.3) and (3.7) are line integrals of vector valued functions along the curve Γ and should be understood in the following sense.

Define a function h with $h : (-\infty,\infty) \longrightarrow \Gamma$ by

$$h(s) = \begin{cases} sr_0e^{-i\theta_1}, & -\infty < s < -1 \\ r_0e^{is\theta_1}, & -1 \leq s \leq 1 \\ sr_0e^{i\theta_1}, & 1 < s < \infty . \end{cases}$$

Then $\Gamma = \{\lambda \in C : \lambda = h(s),\ s \in (-\infty,\infty)\}$ and h is called a parameterization of Γ.

For a function f defined on Γ and with values in some Banach space X the line integral along Γ is defined by

$$\int_{\Gamma} f(\lambda)d\lambda := \int_{-\infty}^{\infty} f(h(s))h'(s)ds. \tag{3.8}$$

See [R, p 217].

The vector valued integral in (3.8) exists if and only if

$$\int_{-\infty}^{\infty} \|f(h(s))h'(s)\|_X ds < \infty$$

and then

$$\left\|\int_\Gamma f(\lambda)d\lambda\right\|_X = \left\|\int_{-\infty}^{\infty} f(h(s))h'(s)ds\right\| \le \int_{-\infty}^{\infty} \|f(h(s))h'(s)\|_X ds . \tag{3.9}$$

See [Y, p 133].

We quote Lebesgue's dominated convergence theorem which will be used to prove Theorem 4 below.

Theorem 3 [R, p 27] *Suppose* f_n *is a sequence of complex measurable functions on a measurable space* X *such that*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$. *If there is a function* G *with* $\int_X |G| < \infty$ *such that*

$$|f_n(x)| \le |G(x)| \quad \textit{for} \quad n = 1,2,\ldots,x \in X$$

then

$$\int_X |f| < \infty \quad \textit{and} \quad \lim_{n\to\infty} \int_X f_n = \int_X f.$$

Remark

In [R] it is required that $|f_n(x)| \le G(x)$ for all $x \in X$. The same proof applies if $|f_n(x)| \le |G(x)|$ is required.

The following theorem is the main result of this section.

Theorem 4 *Let* $S_n(t)$, $n \in \mathbb{N}$ *and* $S(t)$ *be holomorphic* B_n- *and* B-*evolutions of type* L *on Banach spaces* Y_n *and* Y *respectively. Suppose that* $S_n(t)$ *and* $S(t)$ *all map into a Banach space* X *and that* (3.6) *is satisfied for the generating pairs of* $S_n(t)$ *and* $S(t)$. *If there is a sequence* $\{y_n\}$, $y_n \in Y_n$ *and some* $y \in Y$ *such that*

$$\text{(a)} \quad \|P_n(\lambda)y_n - P(\lambda)y\|_X \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty \quad \textit{for all} \quad \lambda \in \Gamma \tag{3.10}$$

and

(b) $\|P_n(\lambda)y_n\|_X \leq g(\lambda)$ *for all* $\lambda \in \Gamma$ *with* g *such that*

$$\int_\Gamma e^{t \text{ Re } \lambda} g(\lambda) d\lambda \quad \textit{exists for all} \quad t > 0 \tag{3.11}$$

then

$$\|S_n(t)y_n - S(t)y\|_X \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty \quad \textit{for all} \quad t > 0.$$

Proof $S_n(t)y_n - S(t)y = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t}(P_n(\lambda)y_n - P(\lambda)y)d\lambda.$

From (3.8) and (3.9)

$$\|S_n(t)y_n - S(t)y\|_X \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{t \text{ Re } h(s)} \|P_n(h(s))y_n - P(h(s))y\|_X |h'(s)| ds$$

$$:= \int_{-\infty}^{\infty} f_n(s) ds. \tag{3.12}$$

For every $s \in (-\infty,\infty)$ Condition (3.10) implies that $\lim\limits_{n \to \infty} f_n(s) = 0.$

Also, from (3.11) and (3.5)

$$|f_n(s)| \leq \frac{1}{2\pi} e^{t \text{ Re } h(s)} |h'(s)| (g(h(s)) + \frac{m}{|h(s)|} \|y\|).$$

Let $G(s) := \frac{1}{2\pi} e^{t \operatorname{Re} h(s)} h'(s)(g(h(s)) + \frac{m}{|h(s)|} \|y\|)$.

This yields

$$|f_n(s)| \le |G(s)| \quad \text{for all} \quad s \in (-\infty,\infty).$$

Finally

$$\int_{-\infty}^{\infty} |G(s)|ds = \int_{-\infty}^{\infty} \frac{|G(s)|}{h'(s)} h'(s)ds = \int_{\Gamma} \frac{|G(h^{-1}(\lambda))|}{h'(h^{-1}(\lambda))} d\lambda$$

with $h^{-1} : \Gamma \longrightarrow (-\infty,\infty)$ the inverse function for h.

Note that

$$h_1 := \min \{r_0, r_0\theta_1\} \le |h'(s)| \le \max \{r_0, r_0\theta_1\} := h_2 \quad \text{for all} \quad s \ne \pm 1,$$

and therefore

$$\frac{|G(h^{-1}(\lambda))|}{|h'(h^{-1}(\lambda))|} \le \frac{1}{2\pi} e^{t \operatorname{Re} \lambda} \frac{h_2}{h_1} (g(\lambda) + \frac{m\|y\|}{|\lambda|}).$$

Condition (3.11) yields that

$$\int_{-\infty}^{\infty} |G(s)|ds < \infty.$$

From Theorem 3 we conclude that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(s)ds = 0$$

and from (3.12) that

$$\lim_{n\to\infty} \|S_n(t)y_n - S(t)y\|_X = 0 \quad \text{for all} \quad t > 0.$$

Remark

If (3.10) and (3.11) are satisfied in another norm on X, the result remains valid with the convergence being in that norm.

Theorem 4 will now be applied to some examples.

3.3 Pseudo-parabolic and parabolic equations

In [T] it is shown that a pseudo-parabolic initial value problem may in a certain sense be regarded as a perturbation of a parabolic initial value problem.

Two second order self-adjoint elliptic partial differential operators M and L are considered. A parabolic and a pseudo-parabolic problem are formulated on the same bounded domain $G \subset \mathbb{R}^n$ in the usual abstract way as

$$\frac{d}{dt} u = Lu, \quad t > 0$$

$$\lim_{t\to 0^+} u = u_0$$

and

$$\frac{d}{dt} u_\lambda - \frac{1}{\lambda} M \frac{d}{dt} u_\lambda = Lu_\lambda, \quad t > 0$$

$$\lim_{t\to 0^+} u_\lambda = u_0 .$$

It is then shown that, under suitable ellipticity and boundedness conditions on M and L, for $t \geq 0$ and $u_0 \in H_0^1(G) \cap H^2(G)$

$$\lim_{\lambda\to\infty} \|u_\lambda(t) - u(t)\|_\Omega = 0.$$

We will show that similar results follow from Theorem 4. We consider elliptic operators of arbitrary order and prescribe initial conditions as needed for B-evolutions.

We use the notation of Section 2.3 and consider the pseudo-parabolic case ($m = \ell$). We assume that the operators M and L satisfy all the conditions in Section 2.3.1. We also assume that L is positive definite in the sense of (2.19). This may be done without loss in generality.

Let $X = L^2(\Omega)$

$D_{0n} = D_0 = H_0^m(\Omega) \cap H^{2m}(\Omega)$

and $Y_n = Y = L^2(\Omega)$ for $n \in \mathbb{N}$.

Choose $A_n = -L_0$

and $B_n = M_n = \frac{1}{n} M_0 + I_0$

with I_0 the restriction to D_0 of the identity operator in $L^2(\Omega)$. M_0 and L_0 are as in Section 2.3.1.

Note the difference in notation in Section 2.2 and Section 3.2. In the construction of the Friedrichs extension the generating pair of the B-evolution is $\langle -A,B\rangle$ whereas in Section 3.2 the typical generating pair is taken as $\langle A,B\rangle$.

From Theorem 12 in Chapter 2 and the subsequent remark $\langle -L_n,M_n\rangle$ is the generating pair of a holomorphic M_n-evolution $S_n(t)$ in $Y_n = L^2(\Omega)$. The solution to

$$\frac{d}{dt}(M_n u) = -L_n u, \quad t > 0$$

$$\lim_{t\to 0^+} M_n u(t) = y$$

is given by

$$u(t) = S_n(t)y$$

for all $y \in Y_n = L^2(\Omega)$.

For the parabolic problem

$$\frac{d}{dt} u = -L_0 u, \quad t > 0$$
$$\lim_{t \to 0^+} u(t) = y$$

the solution is given by

$$u(t) = E(t)y$$

for all $y \in Y = L^2(\Omega)$.

$\{E(t) : t > 0\}$ is the uniformly bounded holomorphic semigroup on Y with infinitesimal generator $-L_0$. [P, p 211]

Intuitively, we expect for $y \in L^2(\Omega)$ and $t > 0$ that

$$\lim_{n \to \infty} S_n(t)y = E(t)y$$

with the convergence in the $L^2(\Omega)$-norm. We prove this result by showing that Theorem 4 applies to this situation.

The semigroup $\{E(t) : t > 0\}$ is a holomorphic I_0-evolution of type L on $Y = L^2(\Omega)$ with generating pair $\langle -L_0, I_0 \rangle$. The generalized resolvent operator $P(\lambda)$ reduces to the resolvent operator $R(\lambda, -L_0) = (\lambda I + L_0)^{-1}$. Also, for some θ_1, with $\frac{\pi}{2} < \theta_1 < \pi$

$$\rho(-L_0) \supset K(\theta_1)$$

and a contour integral representation for $E(t)$ is given by

$$E(t)y = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda, -L_0) y \, d\lambda \quad \text{for all} \quad y \in Y \tag{3.13}$$

with Γ as in (3.4). [P, p 30, p 211]

Next we show that Condition (3.6) is satisfied for $A_nB_n^{-1} = -L_nM_n^{-1}$.

From Remarks 3, 4 and 5 in Section 2.3.2 follows that

$$\rho(-L_nM_n) \supseteq \{\lambda : \lambda - \omega_n \in K(\tfrac{\pi}{2} + \psi_n)\} \tag{3.14}$$

with ω_n the parameter that is used in constructing the Friedrichs extension $<-L_n,M_n>$. This parameter depends on n as can be seen from the following inequalities.

$$\mathrm{Re}(M_nu,u)_\Omega = \frac{1}{n}\,\mathrm{Re}(M_0u,u)_\Omega + (u,u)_\Omega .$$

Hence

$$\mathrm{Re}(M_nu,u)_\Omega \geq \frac{1}{n}\, c_1\|u\|_m^2 \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega) \tag{3.15}$$

and

$$\mathrm{Re}(M_nu,u)_\Omega \geq \|u\|_\Omega^2 \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega). \tag{3.16}$$

From (3.14) and (3.15)

$$\|M_nu\|_\Omega \geq \sqrt{c_1/n}\ \|u\|_m \tag{3.17}$$

and

$$\|M_nu\|_\Omega \geq \|u\|_\Omega \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega). \tag{3.18}$$

From (2.10) and (3.18)

$$\begin{aligned} 1/k_3\ \|u\|_{2m} \leq \|M_0u\|_\Omega &= n\|\tfrac{1}{n}\, M_0u + u - u\|_\Omega \\ &\leq n(\|M_nu\|_\Omega + \|u\|_\Omega) \\ &\leq 2n\|M_nu\|_\Omega \end{aligned}$$

and hence

$$\|M_n u\|_\Omega \geq (2nk_3)^{-1}\|u\|_{2m} \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega). \tag{3.19}$$

From (2.11)

$$\begin{aligned} \|M_n u\|_\Omega &\leq (m_1/n)\|u\|_{2m} + \|u\|_\Omega \\ &\leq ((m_1 + n)/n)\|u\|_{2m} \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega). \end{aligned} \tag{3.20}$$

The parameter ω_n must satisfy Condition (2.17) which follows from (3.19) and (3.20) as

$$\omega_n > (m_1 + n)\ell_1(2nk_3)^2/n = 4(m_1 + n)n\ell_1 k_3^2.$$

It is clear that $\omega_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and that (3.14) is not sufficient to prove that Condition (3.6) is satisfied.

A slightly better result is obtained by noting that from (2.12) and (3.19)

$$\begin{aligned} \|L_n M_n^{-1} y\|_\Omega &\leq \ell_1 \|M_n^{-1} y\|_{2\ell} \\ &\leq \ell_1 \|M_n^{-1} y\|_{2m} \\ &\leq 2nk_3\ell_1 \|y\|_\Omega \quad \text{for all} \quad y \in L^2(\Omega). \end{aligned}$$

This shows that $L_n M_n^{-1}$ is a bounded operator on $L^2(\Omega)$ with $\|L_n M_n^{-1}\| \leq 2nk_3\ell_1$ and hence

$$\rho(L_n M_n^{-1}) \supset \{\lambda : |\lambda| > 2nk_3\ell_1\}. \quad [Y, p 211]$$

As before this is not sufficient to prove that Condition (3.6) is satisfied as $2nk_3\ell_1 \longrightarrow \infty$ as $n \longrightarrow \infty$.

We now show that Condition (3.6) is satisfied in some cases.

From (2.7) and (2.14)

$$(M_0u,u)_\Omega \in K(\varphi) \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega)$$

if $0 \leq \varphi < \frac{\pi}{2}$ with $\cos \varphi = c_1/m_2$.

For $n \in \mathbb{N}$

$$(M_nu,u)_\Omega = \frac{1}{n}(M_0u,u)_\Omega + (u,u)_\Omega \in K(\varphi) \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega). \tag{3.21}$$

Also, from (2.19) and (2.14) for $n \in \mathbb{N}$

$$(L_nu,u)_\Omega = (L_0u,u)_\Omega \in K(\theta) \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega) \tag{3.22}$$

if $0 \leq \theta < \frac{\pi}{2}$ with $\cos \theta = c_2/\ell_2$.

Consider for $n \in \mathbb{N}$, $\lambda \in C$ and $v \in H_0^m(\Omega) \cap H^{2m}(\Omega)$

$$Q = \lambda(M_nv,v)_\Omega + (L_nv,v)_\Omega \ .$$

Assume that $\theta + \varphi < \frac{\pi}{2}$. From the lemma in the Appendix

$$|Q|^2 \geq (1 - \beta^2)|\lambda(M_nv,v)_\Omega|^2 \quad \text{for all} \quad \lambda \in K(\tfrac{\pi}{2} + \psi)$$

and

$$0 < \psi < \frac{\pi}{2} - \theta - \varphi, \quad -\beta = \cos\left(\frac{\pi}{2} + \theta + \varphi + \psi\right).$$

From (3.16)

$$|(M_nv,v)_\Omega| \geq \mathrm{Re}(M_nv,v)_\Omega \geq \|v\|_\Omega^2$$

and this yields

$$\|\lambda M_nv + L_nv\|_\Omega^2\|v\|_\Omega^2 \geq |Q|^2 \geq (1 - \beta^2)|\lambda|^2\|v\|_\Omega^4 \quad \text{for all} \quad \lambda \in K(\tfrac{\pi}{2} + \psi). \tag{3.23}$$

For $\lambda > 0$ the operator $\lambda M_n + L_n$ is uniformly strongly elliptic of order $2m$ and positive definite in the sense that

$$\operatorname{Re}((\lambda M_n + L_n)v,v)_\Omega \geq \lambda c_1 \|v\|_m^2 \quad \text{for all} \quad v \in H_0^m(\Omega) \cap H^{2m}(\Omega)$$

and the operator $\lambda M_n + L_n$ is a bijection from $H_0^m(\Omega) \cap H^{2m}(\Omega)$ onto $L^2(\Omega)$.

From (3.23) it is clear that for $\lambda > 0$, $(\lambda M_n + L_n)^{-1}$ exists and

$$\|(\lambda M_n + L_n)^{-1}\| \leq (|\lambda| \sqrt{1 - \beta^2})^{-1} .$$

For $\lambda \in K(\frac{\pi}{2} + \psi)$

$$\|(\lambda M_n + L_n)v\|_\Omega \geq |\lambda|\sqrt{1 - \beta^2} \, \|v\|_\Omega \quad \text{for all} \quad v \in H_0^m(\Omega) \cap H^{2m}(\Omega)$$

and as in the proof of Theorem 6 in Chapter 2 this yields

$$K(\tfrac{\pi}{2} + \psi) < \rho(L_n M_n^{-1}) \quad \text{for} \quad n \in \mathbb{N}.$$

This means that Condition (3.6) is satisfied for $\theta_1 = \frac{\pi}{2} + \psi$ on condition that $\theta + \varphi < \frac{\pi}{2}$.

A contour integral representation for $S_n(t)$ is given by

$$S_n(t)y = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t}(\lambda M_n + L_n)^{-1} y \, d\lambda \tag{3.24}$$

with Γ as in (3.4) with $\frac{\pi}{2} < \theta_1 < \frac{\pi}{2} + \psi$.

Γ does not depend on n and the same value for θ_1 may be used in (3.24) and (3.13).

We now proceed to show the conditions of Theorem 4 are satisfied for this example.

For any fixed $\lambda \in \Gamma$ and $y \in L^2(\Omega) = Y$, let y_n be a sequence, $y_n \in Y_n = L^2(\Omega)$ with

$$\|y_n - y\|_\Omega \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Let $v_n = (\lambda M_n + L_n)^{-1} y_n = P_n(\lambda) y_n$

and

$$v = (\lambda I + L_0)^{-1} y = R(\lambda, -L_0) y.$$

Then

$$(\lambda M_n + L_n) v_n = \lambda(v_n + \frac{1}{n} M_0 v_n) + L_0 v_n = y_n \tag{3.25}$$

and

$$\lambda v + L_0 v = y. \tag{3.26}$$

The regularity of solutions of elliptic boundary value problems yields that $v \in H_0^m(\Omega) \cap H^{2m}(\Omega)$ and hence that $M_0 v$ is well-defined. [F, p 67]

Let $w_n = v_n - v$. Then

$$\lambda M_n w_n + L_n w_n = y_n - y - \frac{\lambda}{n} M_0 v$$

and from (3.23)

$$|\lambda| \sqrt{1 - \beta^2} \, \|w_n\|_\Omega \le \|\lambda M_n w_n + L_n w_n\|_\Omega$$

and

$$\le \|y_n - y\|_\Omega + \frac{1}{n} \|\lambda M_0 v\|_\Omega \, .$$

This shows that

$$\|P_n(\lambda)y_n - P(\lambda)y\|_\Omega = \|w_n\|_\Omega \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

if

$$\|y_n - y\|_\Omega \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty;$$

or that Condition (3.10) is satisfied if $\|y_n - y\|_\Omega \longrightarrow 0$ as $n \longrightarrow \infty$.

Also, from (3.25) and (3.23) for $\lambda \in \Gamma$

$$\begin{aligned} \|P_n(\lambda)y_n\|_\Omega = \|v_n\|_\Omega &\leq (|\lambda|\sqrt{1 - \beta^2})^{-1}\|\lambda M_n v_n + L_n v_n\|_\Omega \\ &= (|\lambda|\sqrt{1 - \beta^2})^{-1}\|y_n\|_\Omega \\ &\leq c|\lambda|^{-1} \quad \text{for all} \quad n \in \mathbb{N} \end{aligned}$$

as $\{y_n\}$ is a convergent, and hence bounded sequence in $L^2(\Omega)$.

As $\int_\Gamma e^{t\operatorname{Re}\lambda}|\lambda|^{-1}d\lambda < \infty$ for all $t > 0$ this shows that Condition (3.11) is satisfied.

This completes the proof of the following result.

Theorem 5 *Let* M_0 *and* L_0 *be positive definite uniformly strongly elliptic operators of order* $2m$ *and let* $\theta + \varphi < \frac{\pi}{2}$ *with* θ *and* φ *as in* (3.21) *and* (3.22). *For* $y_n \in L^2(\Omega)$ *and* $y \in L^2(\Omega)$ *with* $\|y_n - y\|_\Omega \longrightarrow 0$ *as* $n \longrightarrow \infty$, *and* $t > 0$

$$\|u_n(t) - u(t)\|_\Omega \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty$$

if u_n *is the solution of*

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{n} M_0 u + u\right) = -L_0 u, \quad t > 0 \\ &\lim_{t \to 0^+} \left(\frac{1}{n} M_0 u + u\right) = y_n \end{aligned}$$

and u *is the solution of*

$$\frac{d}{dt} u = -L_0 u, \quad t > 0$$
$$\lim_{t \to 0^+} u = y.$$

For the case $m > \ell$ most of the arguments above remain valid. Note that in Section 2.2.3 the cases $m = \ell$ and $m > \ell$ are not treated separately. The only difference appears when we have to show that Condition (3.10) is satisfied. In this case for v in (3.26) the regularity of the solution only yields that $v \in H_0^{\ell}(\Omega) \cap H^{2\ell}(\Omega)$ and therefore $M_0 v$ is not necessarily well-defined.

If $y \in H^{2m-2\ell}(\Omega)$ in (3.26) we know that $v \in H^{2m}(\Omega)$ [F, p 67] and the same arguments as above apply and the following theorem follows.

Theorem 6 *Let* M_0 *and* L_0 *be positive definite uniformly strongly elliptic operators of order* $2m$ *and* 2ℓ, $2m \geq 2\ell$, *and let* $\theta + \varphi < \frac{\pi}{2}$ *with* θ *and* φ *as in* (3.21) *and* (3.22). *For* $y \in H^{2m-2\ell}(\Omega)$ *and* $y_n \in L^2(\Omega)$ *with* $\|y_n - y\|_\Omega \longrightarrow 0$ *as* $n \longrightarrow \infty$, *and* $t > 0$

$$\|u_n(t) - u(t)\|_\Omega \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

if u_n *is the solution of*

$$\frac{d}{dt}\left(\frac{1}{n} M_0 u + u\right) = -L_0 u, \quad t > 0$$
$$\lim_{t \to 0^+} \left(\frac{1}{n} M_0 u + u\right) = y_n$$

and u *is the solution of*

$$\frac{d}{dt}(u) = -L_0 u, \quad t > 0$$
$$\lim_{t \to 0^+} u = y.$$

3.4 A generalized biharmonic equation

In this section we consider a Sobolev equation of the type discussed in Section 2.3.3 and the convergence of solutions to the solution of a related parabolic equation.

We assume that the operator M satisfies all the conditions in Section 2.3.1 and additionally that M is formally self-adjoint, i.e.

$$(Mu,v)_\Omega = (u,Mv)_\Omega \quad \text{for all} \quad u,v \in H_0^m(\Omega) \cap H^{2m}(\Omega). \tag{3.27}$$

For $n \in \mathbb{N}$, consider the evolution problem

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{n} Mu + u\right) = -M^2u, \quad t > 0 \\ &\lim_{t\to 0^+} \left(\frac{1}{n} Mu + u\right) = y \end{aligned} \tag{3.28}$$

Let $M_n u = \frac{1}{n} Mu + u$

and $L_n u = M^2 u$.

As in Section 2.3.3 we choose

$$X = L^2(\Omega)$$
$$Y_n = M_n[H_0^{2m}(\Omega)] = \{f \in L^2(\Omega) : f = \frac{1}{n} Mu + u,\ u \in H_0^{2m}(\Omega)\}.$$

$$\begin{aligned} D_{0n} &= L^{-1}[Y_n] \\ &= \{u \in H_0^{2m}(\Omega) \cap H^{4m}(\Omega) : M^2u = \frac{1}{n} Mw + w \ \text{ for some } \ w \in H_0^{2m}(\Omega)\} \subset X \end{aligned}$$

$$M_{0n} = M_n\big|_{D_{0n}}$$
$$L_{0n} = M^2\big|_{D_{0n}} = M(M\big|_{D_{0n}})$$

We still have to show that L_n is positive definite in the sense of (2.19) and that the pair $\langle L_{0n}, M_{0n}\rangle$ satisfies Condition (2.20).

From (3.27) and (2.10)

$$\begin{aligned}\mathrm{Re}(L_n u,u)_\Omega &= \mathrm{Re}(M^2u,u)_\Omega \\ &= (Mu,Mu)_\Omega \\ &\geq k_3^{-2}\|u\|_{2m}^2 \quad \text{for all} \quad u \in H_0^{2m}(\Omega) \cap H^{4m}(\Omega).\end{aligned}$$

For $u \in D_{0n} \subset H_0^{2m}(\Omega) \cap H^{4m}(\Omega)$

$$\begin{aligned}(L_{0n}u,M_{0n}u)_\Omega &= (M^2u, \tfrac{1}{n} Mu + u)_\Omega \\ &= \tfrac{1}{n} (M^2u,Mu)_\Omega + (Mu,Mu)_\Omega\end{aligned}$$

because of the self-adjointness of M. Also, for $u \in D_{0n}$ there exists some

$$w \in H_0^{2m}(\Omega) \quad \text{with} \quad M^2u = \frac{1}{n} Mw + w \tag{3.29}$$

which yields

$$(L_{0n}u,M_{0n}u)_\Omega = \frac{1}{n} (\frac{1}{n} Mw,Mu)_\Omega + \frac{1}{n} (w,Mu)_\Omega + \|Mu\|_\Omega^2 .$$

From (3.27)

$$(L_{0n}u,M_{0n}u)_\Omega = \frac{1}{n^2} (w,M^2u)_\Omega + \frac{1}{n} (w,Mu)_\Omega + \|Mu\|_\Omega^2$$

and from (3.29)

$$(L_{0n}u,M_{0n}u)_\Omega = \frac{1}{n^3} (w,Mw)_\Omega + \frac{1}{n^2} \|w\|_\Omega^2 + \frac{1}{n} (w,Mu)_\Omega + \|Mu\|_\Omega^2 . \tag{3.30}$$

From (2.7) and (3.27)

$$(Mu,u)_\Omega = (u,Mu)_\Omega = \mathrm{Re}(Mu,u)_\Omega \geq c_1\|u\|_m^2 \quad \text{for all} \quad u \in H_0^m(\Omega) \cap H^{2m}(\Omega).$$

For $u \in D_{0n}$ from (3.30)

$$\begin{aligned} \mathrm{Re}(L_{0n}u, M_{0n}u)_\Omega &\geq \frac{c_1}{n^3}\|w\|_\Omega^2 + \frac{1}{n^2}\|w\|_\Omega^2 - \|\tfrac{w}{n}\|_\Omega\|Mu\|_\Omega + \|Mu\|_\Omega^2 \\ &\geq \frac{1}{2n^2}\|w\|_\Omega^2 + \frac{1}{2}\|Mu\|_\Omega^2 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Im}(L_{0n}u, M_{0n}u)_\Omega &\leq |(\tfrac{w}{n}, Mu)_\Omega| \\ &\leq \frac{1}{2n^2}\|w\|_\Omega^2 + \frac{1}{2}\|Mu\|_\Omega^2 \end{aligned}$$

and hence

$$(L_{0n}u, M_{0n}u)_\Omega \in K(\tfrac{\pi}{4}) \quad \text{for all} \quad u \in D_{0n}\ , \quad n \in \mathbb{N}. \tag{3.31}$$

This shows that Condition (2.20) is satisfied for the pair of operators $\langle L_{0n}, M_{0n}\rangle$.

Theorem 13 of Chapter 2 yields that $\langle -L_{0n}, M_{0n}\rangle$ is the generating pair of a uniformly bounded M_{0n}-evolution $S_n(t)$ in Y_n.

For any $y \in Y_n$ the unique solution to (3.28) is given by

$$u(t) = S_n(t)y, \quad t > 0.$$

Note that from the Remark on p 23 and Remark 2 of Section 2.3.3

$$\rho(-L_{0n}M_{0n}^{-1}) \supset K(\tfrac{\pi}{2} + \tfrac{\pi}{4}) \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.32}$$

We compare the solution of (3.28) with the solution of the parabolic problem

$$\frac{d}{dt} u = -M^2 u, \quad t > 0$$
$$\lim_{t \to 0^+} u = y. \tag{3.33}$$

Define the operator L by

$$L = M^2 \big|_{H_0^{2m}(\Omega) \cap H^{4m}(\Omega)} .$$

From (3.27) and (2.10) follow, as before that L is positive definite, i.e. $\mathrm{Re}(Lu,u) \geq k_3^{-2}\|u\|_{2m}^2$ for all $u \in H_0^{2m}(\Omega) \cap H^{4m}(\Omega)$.

The theory of elliptic operators then yields that $-L$ is the infinitesimal generator of a uniformly bounded holomorphic semigroup $E(t)$ on $Y = L^2(\Omega)$ [P, p 211] and

$$\rho(-L) \supset K(\theta_1) \tag{3.34}$$

for some θ_1 with $\frac{\pi}{2} < \theta_1 < \pi$.

For any $y \in L^2(\Omega)$ the solution to (3.33) is given by

$$u(t) = E(t)y, \quad t > 0.$$

As in Section 3.3 the semigroup $\{E(t) : t > 0\}$ is a holomorphic I-evolution by type L on $Y = L^2(\Omega)$ with generating pair $<-L,I>$. The generalized resolvent operator $P(\lambda)$ reduces to the resolvent operator $R(\lambda,-L) = (\lambda I + L)^{-1}$.

Note that for the generalized biharmonic problem (3.28) only initial conditions $y \in Y_n \subset Y = L^2(\Omega)$ may be prescribed whereas the parabolic problem (3.33) is solved for all initial conditions $y \in Y = L^2(\Omega)$.

In this case we show that the convergence of solutions of (3.28) to the solution of (3.33) follows from Theorem 4 for some specified initial conditions.

Clearly, from (3.32) and (3.34), Condition (3.6) is satisfied.

For $y \in H_0^{2m}(\Omega) \subset L^2(\Omega) = Y$ choose

$$y_n = M_n y \in Y_n = M_n[H_0^{2m}(\Omega)], \quad n \in \mathbb{N}.$$

For $\lambda \in \Gamma$, let

$$v_n = P_n(\lambda)y_n, \quad n \in \mathbb{N}$$

and

$$v = P(\lambda)y.$$

Then

$$(\lambda M_{0n} + L_{0n})v_n = \frac{\lambda}{n} Mv_n + \lambda v_n + M^2 v_n = y_n$$

and

$$\lambda v + M^2 v = y.$$

Let $w_n = v_n - v \in H_0^{2m}(\Omega) \cap H^{4m}(\Omega)$.

Then

$$M^2 w_n + \lambda w_n + \frac{\lambda}{n} Mw_n = y_n - y - \frac{\lambda}{n} Mv. \tag{3.35}$$

We now proceed to show that $\|w_n\|_\Omega \longrightarrow 0$ as $n \longrightarrow \infty$ which implies that Condition (3.10) is satisfied.

For $w \in H_0^{2m}(\Omega) \cap H^{4m}(\Omega)$ consider

$$Q_\lambda = (M^2 w + \lambda w + \frac{\lambda}{n} Mw, w)_\Omega \ . \tag{3.36}$$

From (3.27)

$$Q_\lambda = \|Mw\|_\Omega^2 + \lambda(\|w\|_\Omega^2 + \frac{1}{n}(Mw,w)_\Omega).$$

As $\|Mw\|_\Omega^2 \geq 0$ and $\|w\|_\Omega^2 + \frac{1}{n}(Mw,w)_\Omega \geq 0$, the lemma in the Appendix yields

$$|Q_\lambda|^2 \geq (1 - \ell^2)|\lambda(\|w\|_\Omega^2 + \frac{1}{n}(Mw,w)_\Omega)|^2$$

with $-\ell = \cos(\frac{\pi}{2} + \psi)$ for some ψ, $0 < \psi < \frac{\pi}{2}$.

Hence

$$|Q_\lambda|^2 \geq c|\lambda|^2\|w\|_\Omega^4$$

and from (3.36)

$$\|M^2w + \lambda w + \frac{\lambda}{n} Mw\|_\Omega \geq c|\lambda|\|w\|_\Omega \quad \text{for all} \quad w \in H_0^{2m}(\Omega) \cap H^{4m}(\Omega). \tag{3.37}$$

Combining (3.37) and (3.35) yields

$$\begin{aligned} c|\lambda|\|w_n\|_\Omega &\leq \|y_n - y\|_\Omega + \frac{|\lambda|}{n}\|Mv\|_\Omega \\ &= \frac{1}{n}(\|My\|_\Omega + |\lambda|\|Mv\|_\Omega) \end{aligned}$$

because of the choice of y_n.

For fixed $\lambda \in \Gamma$ it is clear that $\|w_n\|_\Omega \longrightarrow 0$ as $n \longrightarrow \infty$.

From (3.37) also follows that

$$\begin{aligned} \|P_n(\lambda)y_n\|_\Omega = \|v_n\|_\Omega &\leq (c|\lambda|)^{-1}\|\lambda(v_n + \frac{1}{n} Mv_n) + M^2v_n\|_\Omega \\ &= (c|\lambda|)^{-1}\|y_n\|_\Omega . \end{aligned} \tag{3.38}$$

The sequence $\{y_n\}$ converges in $L^2(\Omega)$ as

$$\|y_n - y\|_\Omega = \frac{1}{n} \|My\|_\Omega$$

and hence $\{y_n\}$ is bounded in $L^2(\Omega)$.

From (3.38)

$$\|P_n(\lambda)y_n\|_\Omega \leq (c|\lambda|)^{-1} \quad \text{for all} \quad n \in \mathbb{N}.$$

As $\int_\Gamma e^{t \operatorname{Re} \lambda}|\lambda|^{-1}d\lambda < \infty$ for all $t > 0$ this shows that Condition (3.11) is satisfied.

This completes the proof of the following result.

Theorem 6 *Let* M *be a positive definite formally self-adjoint uniformly strongly elliptic operator. For any* $y \in H_0^{2m}(\Omega)$ *and* $y_n = \frac{1}{n} My + y$ *and* $t > 0$

$$\|u_n(t) - u(t)\|_\Omega \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

if u_n *is the solution of*

$$\frac{d}{dt} \left(\frac{1}{n} Mu + u\right) = -M^2u, \quad t > 0$$
$$\lim_{t\to 0^+} \left(\frac{1}{n} Mu + u\right) = y_n$$

and u *the solution of*

$$\frac{d}{dt} u = -M^2u, \quad t > 0$$
$$\lim_{t\to 0^+} u = y.$$

3.5 A dynamic boundary value problem for imperfect contact

In Section 2.4.4 it has been shown that for any $y \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ a unique solution exists for

$$\begin{aligned} &\frac{d}{dt} \langle u, \gamma_0 u + k^2 L_\nu u\rangle = \langle Lu, -L_\nu u\rangle, \quad t > 0 \\ &\lim_{t \to 0^+} \langle u, \gamma_0 u + k^2 L_\nu u\rangle = y \ . \end{aligned} \tag{3.39}$$

A condition we need to be able to apply the B-evolution theory, is that $k(x) \geq \delta > 0$ for all $x \in \partial\Omega$ and some real δ.

In Section 2.4.2 was shown that for any $y \in L^2(\Omega) \times L^2(\partial\Omega)$ a unique solution exists for

$$\begin{aligned} &\frac{d}{dt} \langle u, \gamma_0 u\rangle = \langle Lu, -L_\nu u\rangle, \quad t > 0 \\ &\lim_{t \to 0^+} \langle u, \gamma_0 u\rangle = y \ . \end{aligned} \tag{3.40}$$

We assume that the symmetry condition (2.24) is satisfied.

Intuitively, we expect the solution of the initial value problem (3.39) to converge to the solution of (3.40) if $k(x) \longrightarrow 0$ for all $x \in \partial\Omega$.

For a special choice of k we now show that Theorem 4 applies.

Let $k_n^2(x) = \frac{1}{n}$ for all $x \in \partial\Omega$ and $n \in \mathbb{N}$.

From Section 2.4.4 we know that for any $y \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ the unique solution to

$$\begin{aligned} &\frac{d}{dt} \langle u, \gamma_0 u + \frac{1}{n} L_\nu u\rangle = \langle Lu, -L_\nu u\rangle, \quad t > 0 \\ &\lim_{t \to 0^+} \langle u, \gamma_0 u + \frac{1}{n} L_\nu u\rangle = y \end{aligned} \tag{3.41}$$

is given, for all $n \in \mathbb{N}$, by

$$u_n(t) = S_n(t)y, \quad t > 0.$$

$S_n(t)$ is the holomorphic B_n-evolution on $Y_n = L^2(\Omega) \times H^{1/2}(\partial\Omega)$ with generating pair $\langle -A_n - \omega_n B_n, B_n\rangle$. $\langle A_n, B_n\rangle$ is the Friedrich extension of $\langle A_{0n}, B_{0n}\rangle$. A_{0n} and B_{0n} are defined by

$$A_{0n}u := \langle -Lu, L_\nu u\rangle$$
$$B_{0n}u := \langle u, U^n\rangle = \langle u, \gamma_0 u + \frac{1}{n} L_\nu u\rangle$$

for all $u \in D_{0n} = C^2(\bar{\Omega})$.

$\langle A_n, B_n\rangle$ is defined on $D_n = H^2(\Omega) \subset L^2(\Omega) = X$.

ω_n is the parameter used in the construction of the Friedrichs extension.

From Remark 2 in Section 2.4.4

$$\omega_n > \max \{c_1, \frac{1}{2\eta_n^2 \ \delta_n^2}\}$$

with

$$\eta_n^2 < 2\delta_n^2 \ c_1/k_1^2$$

and c_1 as in (2.25), k_1 as in (2.41).

For $k_n^2(x) = \frac{1}{n}$ we have $\delta_n^2 = \frac{1}{n}$ and these conditions reduce to

$$\omega_n > \max \{c_1, \frac{n}{2\eta_n^2}\}$$

with

$$\eta_n^2 < \frac{2c_1}{nk_1^2} .$$

For n large

$$\omega_n > \frac{n^2 k_1^2}{4c_1}$$

and hence

$$\omega_n \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty.$$

As in Section 3.3 the construction of the Friedrichs extension yields

$$\rho(-A_n B_n^{-1}) \supseteq \{\lambda : \lambda - \omega_n \in K(\tfrac{\pi}{2} + \psi_n)\}$$

and this is not sufficient to show that Condition (3.6) is satisfied.

We proceed to show that Condition (3.6) is satisfied.

For $\lambda \in C$ and $u \in D_n = H^2(\Omega) \subset X = L^2(\Omega)$

$$A_n u + \lambda B_n u = y = \langle a, \alpha \rangle \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$$

and

$$\begin{aligned}
&(A_n u + \lambda B_n u, B_n u)_{L^2(\Omega)\times L^2(\partial\Omega)} \\
&= (-Lu + \lambda u, u)_\Omega + (L_\nu u + \lambda U^n, U^n)_{\partial\Omega} \\
&= \sum_{i,j=1}^{n} (a_{ij}\partial_i u, \partial_j u)_\Omega + \lambda(u,u)_\Omega - (L_\nu u, \gamma_0 u)_{\partial\Omega} + (L_\nu u, U^n)_{\partial\Omega} + \lambda(U^n, U^n)_{\partial\Omega} \\
&= \lambda(\|u\|_\Omega^2 + \|U^n\|_{\partial\Omega}^2 + \sum_{i,j=1}^{n} (a_{ij}\partial_i u, \partial_j u)_\Omega + n\|U^n - \gamma_0 u\|_{\partial\Omega}^2 \\
&= \lambda r + q \quad \text{with} \quad q \geq 0, \quad r \geq 0.
\end{aligned}$$

From the lemma in the Appendix for any ψ with $0 < \psi < \frac{\pi}{2}$

$$\begin{aligned}
&\left|(A_n u + \lambda B_n u, B_n u)_{L^2(\Omega)\times L^2(\partial\Omega)}\right| \\
&\geq c(\psi)|\lambda|(\|u\|_\Omega^2 + \|U^n\|_{\partial\Omega}^2) \\
&= c(\psi)|\lambda| \; \|B_n u\|^2_{L^2(\Omega)\times L^2(\partial\Omega)} \qquad \text{for all} \quad \lambda \in K(\tfrac{\pi}{2} + \psi).
\end{aligned}$$

This yields

$$\|A_n u + \lambda B_n u\|_{L^2(\Omega)\times L^2(\partial\Omega)} \geq c(\psi)|\lambda| \; \|B_n u\|_{L^2(\Omega)\times L^2(\partial\Omega)} \tag{3.43}$$
$$\text{for all} \quad u \in H^2(\Omega), \quad \lambda \in K(\tfrac{\pi}{2} + \psi).$$

As $\|A_n u + \lambda B_n u\|_{L^2(\Omega)\times H^{1/2}(\partial\Omega)} \geq \|A_n u + \lambda B_n u\|_{L^2(\Omega)\times L^2(\partial\Omega)}$

and

$$\|B_n u\|_{L^2(\Omega)\times L^2(\partial\Omega)} \geq \|u\|_\Omega \quad \text{for all} \quad u \in H^2(\Omega)$$

we conclude

$$\|A_n u + \lambda B_n u\|_{L^2(\Omega)\times H^{1/2}(\partial\Omega)} \geq c(\psi)|\lambda| \; \|u\|_\Omega \tag{3.44}$$
$$\text{for all} \quad u \in H^2(\Omega), \quad \lambda \in K(\tfrac{\pi}{2} + \psi).$$

For $\lambda > 0$ the regular elliptic boundary value problem

$$-Lu + \lambda u = f \quad \text{in} \quad \Omega$$
$$L_\nu u + \lambda U^n = g \quad \text{on} \quad \partial\Omega$$

has a unique solution $u \in H^2(\Omega)$ for every $\langle f,g\rangle \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$. This implies that $P_n(\lambda) = (A_n + \lambda B_n)^{-1} : Y_n = L^2(\Omega) \times H^{1/2}(\partial\Omega) \longrightarrow L^2(\Omega)$ exists for $\lambda > 0$. From (3.44) follows that $P_n(\lambda)$ is bounded. This yields

$$\rho(-A_n B_n^{-1}) \supseteq \{\lambda : \lambda > 0\}.$$

As in the proof of Theorem 6 of Chapter 2 from (3.44) we also conclude that for all $n \in \mathbb{N}$

$$\rho(-A_n B_n^{-1}) \supseteq K(\tfrac{\pi}{2} + \psi)$$

for any ψ with $0 < \psi < \frac{\pi}{2}$. This shows that Condition (3.6) is satisfied for the pair $\langle A_n, B_n\rangle$.

Turning to problem (3.40), we know from Section 2.4.2 that for any $y \in Y = L^2(\Omega) \times L^2(\partial\Omega)$ the unique solution to (3.40) is given by

$$u(t) = S(t)y, \quad t > 0.$$

$S(t)$ is the holomorphic B-evolution on $Y = L^2(\Omega) \times L^2(\partial\Omega)$ with generating pair $\langle -A - \omega B, B\rangle$. $\langle A,B\rangle$ is the Friedrichs extension of $\langle A_0, B_0\rangle$. A_0 and B_0 are defined by

$$A_0 u = \langle -Lu, L_\nu u\rangle$$

$$B_0 u = \langle u, \gamma_0 u\rangle \quad \text{for all} \quad u \in D_0 = C^2(\bar{\Omega}) \subset L^2(\Omega) = X.$$

From the symmetry condition (2.24) follows that $N_0 = 0$ in the construction of the Friedrichs extension.

Hence from Remark 6 in Section 2.4.2

$$\rho(-AB^{-1}) \supseteq \{\lambda : \lambda - \omega \in K(\tfrac{\pi}{2} + \psi)\} \quad \text{for any} \quad \psi \quad \text{with} \quad 0 < \psi < \tfrac{\pi}{2}.$$

Any $\omega > 0$ may be used in the construction of the Friedrichs extension and

$$\rho(-AB^{-1}) \supseteq K(\tfrac{\pi}{2} + \psi).$$

This shows that Condition (3.6) is satisfied.

We proceed to show that the conditions of Theorem 4 are satisfied.

Choose Γ as in (3.4) with $\frac{\pi}{2} < \theta_1 < \frac{\pi}{2} + \psi$.

In the construction of the Friedrichs extension $\langle A,B\rangle$ the operator $C_0 = B_0$ is used. This implies that for $y = \langle a, \alpha\rangle \in L^2(\Omega) \times L^2(\partial\Omega)$ and $u \in D \subset H^1(\Omega)$

$$P(\lambda)y = (\lambda B + A)^{-1}\langle a, \alpha\rangle = u$$

if and only if (in the notation of Section 2.2)

$$Q(u,v;\lambda) = (y,B_1 v)_{L^2(\Omega)\times L^2(\partial\Omega)} \quad \text{for all} \quad v \in D_1 = H^1(\Omega).$$

In this case this reduces to

$$\sum_{i,j=1}^{n} (a_{ij}\partial_i u, \partial_j v)_\Omega + \lambda(u,v)_\Omega + \lambda(\gamma_0 u, \gamma_0 v)_{\partial\Omega} \tag{3.45}$$
$$= (a,v)_\Omega + (\alpha, \gamma_0 u)_{\partial\Omega} \quad \text{for all} \quad v \in H^1(\Omega).$$

For any $y = \langle a,\alpha\rangle \in L^2(\Omega) \times L^2(\partial\Omega) = Y$ there is a sequence $\{\alpha_n\}$ in $H^{1/2}(\partial\Omega)$ with

$$\|\alpha_n - \alpha\|_{\partial\Omega} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \alpha. \tag{3.46}$$

For $y_n = \langle a,\alpha_n\rangle \in L^2(\Omega) \times H^{1/2}(\partial\Omega) = Y_n$, let $P_n(\lambda)y_n = u_n$.

For this choice of y_n we show that Condition (3.11) is satisfied.

For $n \in \mathbb{N}$, from (3.43), follows

$$\begin{aligned} \|y_n\|_{L^2(\Omega)\times L^2(\partial\Omega)} &= \|A_n u_n + \lambda B_n u_n\|_{L^2(\Omega)\times L^2(\partial\Omega)} \\ &\geq c(\psi)|\lambda| \; \|B_n u_n\|_{L^2(\Omega)\times L^2(\partial\Omega)} \\ &\geq c(\psi)|\lambda| \; \|u_n\|_\Omega \quad \text{for all} \quad \lambda \in K(\tfrac{\pi}{2} + \psi) \end{aligned} \tag{3.47}$$

or

$$\|P_n(\lambda)y_n\|_\Omega \leq \frac{1}{c(\psi)|\lambda|} \|\langle a,\alpha_n\rangle\|_{L^2(\Omega)\times L^2(\partial\Omega)}.$$

The sequence $\{\alpha_n\}$ converges in $L^2(\partial\Omega)$ and is therefore a bounded sequence.

This yields, finally,

$$\|P_n(\lambda)y_n\|_\Omega \leq \frac{M}{c(\psi)|\lambda|} \quad \text{for all} \quad \lambda \in \Gamma, \quad n \in \mathbb{N}.$$

As $\int_\Gamma e^{t \operatorname{Re} \lambda}|\lambda|^{-1}d\lambda < \infty$ for all $t > 0$ this shows that Condition (3.11) is satisfied.

To show that Condition (3.10) is satisfied, let
$w_n = u_n - u = P_n(\lambda)y_n - P(\lambda)y$.

Note that $u_n = P_n(\lambda)y_n$ implies that

$$\langle -Lu_n + \lambda u_n,\ L_\nu u_n + \lambda U_n^n\rangle = \langle a, \alpha_n\rangle$$

if $U_n^n := \frac{1}{n} L_\nu u_n + \gamma_0 u_n$.

Hence

$$\begin{aligned}(-Lu_n + \lambda u_n, v)_\Omega &+ (L_\nu u_n + \lambda U_n^n, \gamma_0 v)_{\partial\Omega} \\ &= (a,v)_\Omega + (\alpha_n, \gamma_0 v)_{\partial\Omega} \quad \text{for all} \quad v \in H^1(\Omega).\end{aligned}$$

This reduces to

$$\begin{aligned}\sum_{i,j=1}^{n} (a_{ij}\partial_i u_n, \partial_j v)_\Omega &+ \lambda(u_n,v)_\Omega + \lambda(U_n^n, \gamma_0 v)_{\partial\Omega} \\ &= (a,v)_\Omega + (\alpha_n, \gamma_0 v)_{\partial\Omega} \quad \text{for all} \quad v \in H^1(\Omega).\end{aligned} \tag{3.48}$$

From (3.45) and (3.48) follows

$$\begin{aligned}\sum_{i,j=1}^{n} (a_{ij}\partial_i w_n, \partial_j v)_\Omega &+ \lambda(w_n,v)_\Omega + \lambda(\gamma_0 w_n, \gamma_0 v)_{\partial\Omega} \\ &+ \lambda(U_n^n - \gamma_0 u_n, \gamma_0 v)_\Omega = (\alpha_n - \alpha, \gamma_0 v)_{\partial\Omega} \quad \text{for all} \quad v \in H^1(\Omega)\end{aligned}$$

and if $v = w_n$, for $n \in \mathbb{N}$,

$$\sum_{i,j=1}^{n} (a_{ij}\partial_i w_n, \partial_j w_n)_\Omega + \lambda(\|w_n\|_\Omega^2 + \|\gamma_0 w_n\|_{\partial\Omega}^2)$$
$$= (\alpha_n - \alpha, \gamma_0 w_n)_{\partial\Omega} - \lambda(U_n^n - \gamma_0 u_n, \gamma_0 w_n)_{\partial\Omega} . \tag{3.49}$$

In order to show that $\|w_n\|_\Omega \longrightarrow 0$ as $n \longrightarrow \infty$ we need a bound on $\|U_n^n - \gamma_0 u_n\|_\Omega$ for $n \in \mathbb{N}$.

For any $v \in H^2(\Omega)$

$$(-Lu_n + \lambda u_n, v)_\Omega + (L_\nu u_n + \lambda U_n^n, V^n)_{\partial\Omega}$$
$$= (a,v)_\Omega + (\alpha_n, V^n)_{\partial\Omega} .$$

For $v = u_n$ this can be written as

$$\lambda(\|u_n\|_\Omega^2 + \|U_n^n\|_{\partial\Omega}^2) + \sum_{i,j=1}^{n} (a_{ij}\partial_i u_n, \partial_j u_n)_\Omega$$
$$+ n\|U_n^n - \gamma_0 u_n\|_{\partial\Omega}^2 = (a, u_n) + (\alpha_n, U_n^n)_{\partial\Omega} .$$

For $\lambda \in C$, $q \geq 0$ and $r \geq 0$

$$|q| \leq |\lambda r + q| + |\lambda r| .$$

Hence

$$\sum_{i,j=1}^{n} (a_{ij}\partial_i u_n, \partial_j u_n)_\Omega + n\|U_n^n - \gamma_0 u_n\|_{\partial\Omega}^2$$
$$\leq \left|(\langle a, \alpha_n\rangle, \langle u_n, U_n^n\rangle)_{L^2(\Omega)\times L^2(\partial\Omega)}\right| + |\lambda|(\|u_n\|_\Omega^2 + \|U_n^n\|_{\partial\Omega}^2)$$
$$= \left|(y_n, B_n u_n)_{L^2(\Omega)\times L^2(\partial\Omega)}\right| + |\lambda|\ \|B_n u_n\|_{L^2(\Omega)\times L^2(\partial\Omega)}^2 .$$

From (3.47)

$$n\|U_n^n - \gamma_0 u_n\|_{\partial\Omega}^2 \le \frac{M}{|\lambda|} \|y_n\|^2_{L^2(\Omega)\times L^2(\partial\Omega)} .$$

The sequence $\{a_n\}$ converges in $L^2(\partial\Omega)$ and hence is a bounded sequence. This yields

$$\|U_n^n - \gamma_0 u_n\|_{\partial\Omega}^2 \le \frac{M}{n|\lambda|} \text{ for all } \lambda \in K(\psi + \tfrac{\pi}{2}). \tag{3.50}$$

Returning to (3.49) we note that the left hand side is of the form $\lambda r + q$ with $r \geq 0$ and $q \geq 0$. From the lemma in the Appendix for any ψ with $0 < \psi < \frac{\pi}{2}$ there is a constant $c(\psi) > 0$ such that for all $\lambda \in K(\frac{\pi}{2} + \gamma)$

$$c(\psi)|\lambda|(\|w_n\|_\Omega^2 + \|\gamma_0 w_n\|_{\partial\Omega}^2) \le |(a_n - a, \gamma_0 w_n)_{\partial\Omega} - \lambda(U_n^n - \gamma_0 u_n, \gamma_0 w_n)_{\partial\Omega}|$$

or

$$c(\psi)|\lambda|(\|w_n\|_\Omega^2 + \|\gamma_0 w_n\|_{\partial\Omega}^2) \le (\|a_n - a\|_{\partial\Omega} + |\lambda| \|U_n^n - \gamma_0 u_n\|)\|\gamma_0 w_n\|_{\partial\Omega} .$$

This yields

$$c(\psi)|\lambda| \|\gamma_0 w_n\|_{\partial\Omega} \le \|a_n - a\|_{\partial\Omega} + |\lambda| \|U_n^n - \gamma_0 u_n|_{\partial\Omega}$$

and hence also

$$\begin{aligned} c(\psi)|\lambda| \|w_n\|_\Omega^2 &\le (\|a_n - a\|_{\partial\Omega} + |\lambda| \|U_n^n - \gamma_0 u_n\|_{\partial\Omega})\|\gamma_0 w_n\|_{\partial\Omega} \\ &\le \frac{1}{c(\psi)|\lambda|} (\|a_n - a\|_{\partial\Omega} + |\lambda| \|U_n^n - \gamma_0 u_n\|_{\partial\Omega})^2 \end{aligned}$$

or

$$\|w_n\|_\Omega \le \frac{1}{c(\psi)|\lambda|} (\|a_n - a\|_{\partial\Omega} + |\lambda| \|U_n^n - \gamma_0 u_n\|_{\partial\Omega}).$$

For $\lambda \in \Gamma$, from (3.46) and (3.50), we conclude that

$$\|w_n\|_\Omega \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

This shows that Condition (3.10) is satisfied for $y_n = \langle a, \alpha_n \rangle$ and $y = \langle a, \alpha \rangle$.

This completes the proof of the following result.

Theorem 7 *Let* L *and* L_ν *be the operators defined in Section* 2.4.4. *For any* $y = \langle a, \alpha \rangle \in L^2(\Omega) \times L^2(\partial\Omega) = Y$ *and* $y_n = \langle a, \alpha_n \rangle \in L^2(\Omega) \times H^{1/2}(\partial\Omega) = Y_n$ *with* $\|\alpha_n - \alpha\|_{\partial\Omega} \longrightarrow 0$ *as* $n \longrightarrow \alpha$, *and* $t > 0$

$$\|u_n(t) - u(t)\|_\Omega \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty$$

if u_n *is the solution of*

$$\frac{d}{dt} \langle u, \gamma_0 u + \frac{1}{n} L_\nu u \rangle = \langle Lu, -L_\nu u \rangle, \quad t > 0$$
$$\lim_{t \to 0^+} \langle u, \gamma_0 u + \frac{1}{n} L_\nu u \rangle = \langle a, \alpha_n \rangle$$

and u *is the solution of*

$$\frac{d}{dt} \langle u, \gamma_0 u \rangle = \langle Lu, -L_\nu u \rangle, \quad t > 0$$
$$\lim_{t \to 0^+} \langle u, \gamma_0 u \rangle = \langle a, \alpha \rangle .$$

APPENDIX
AN INEQUALITY FOR COMPLEX NUMBERS

C denotes the set of complex numbers and $K(\theta) = \{z \in C : |\arg z| < \theta\}$.

Lemma

Let $Q_\lambda = \lambda r + q$ *and* $r \in K(\varphi)$, $q \in K(\theta)$ *with* $\varphi \geq 0$, $\theta \geq 0$ *and* $\theta + \varphi < \frac{\pi}{2}$. *For any* ψ *such that* $0 < \psi < \frac{\pi}{2} - \theta - \varphi$ *let* $-\ell = \cos(\frac{\pi}{2} + \theta + \varphi + \psi)$. *Then*

$$|Q_\lambda|^2 \geq (1 - \ell^2)|\lambda r|^2 \quad \textit{for all} \quad \lambda \in K(\tfrac{\pi}{2} + \psi).$$

Proof. $|Q_\lambda|^2 = |\lambda r|^2 + |q|^2 + 2|\lambda r||q|\cos(\arg(\lambda r) - \arg q)$.

For $\lambda \in K(\frac{\pi}{2} + \psi)$

$$-\frac{\pi}{2} - \psi - \varphi < \arg(\lambda r) < \frac{\pi}{2} + \psi + \varphi,$$
$$-\theta < \arg q < \theta$$

and

$$-\frac{\pi}{2} - \theta - \varphi - \psi < \arg(\lambda r) - \arg q < \frac{\pi}{2} + \theta + \varphi + \psi.$$

Hence

$$\cos(\arg(\lambda r) - \arg q) > \cos(\tfrac{\pi}{2} + \theta + \varphi + \psi) = -\ell,$$

and

$$\begin{aligned} |Q_\lambda|^2 &\geq |\lambda r|^2 + |q|^2 - 2\ell|\lambda r||q| \\ &\geq (\ell|\lambda r| - |q|)^2 + (1 - \ell^2)|\lambda r|^2 \\ &\geq (1 - \ell^2)|\lambda r|^2. \end{aligned}$$

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