FREDHOLM THEORY IN VON NEUMANN ALGEBRAS

by

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INTRODUCTION

In his remarkable two papers [1] and [2], Manfred Breuer laid the foundations of a generalized theory of compact and Fredholm operators relative to a von Neumann algebra $\mathcal{A}$. Classical results as :

(i) "The Fredholm alternatives" due to F. Riesz ([13] p 87), which states that I - T is Fredholm of index zero if T is compact ($T \in L(H)$);

(ii) a well known decomposition theorem for compact operators also proved by Riesz ([11], p 431); and

(iii) a theorem due to Atkinson ([13], p 90) which states that if $A(H) = L(H)/C(H)$ ($C(H)$ the two-sided ideal of all compact operators on a Hilbert space H) is the Calkin algebra; then the set of all Fredholm operators in L(H) is exactly the inverse image of the group of all invertable elemets in A(H) under the canonical quotient mapping $\Pi : L(H) \longrightarrow A(H)$,

is generalized to a von Neumann algebra $\mathcal{A}$ ((ii) only to a certain extent). The main goal of this study is to prove these three theorems which are included in Chapter 4.

Since the projections of a von Neumann algebra form a fundamental structure of the algebra, these generalizations depend heavily on the study of the projection lattice existing in a von Neumann algebra $\mathcal{A}$. Therefore, Chapter 1 contains a comprehensive amount of standard material concerning the geometry of projections in a von Neumann algebra which will be used in the chapters that follow. This Chapter may thus be considered as an Appendix.

Once we introduced the notion of a finite projection relative to $\mathcal{A}$ we proof in the concluding section of Chapter 1 one of the deepest and most important theorems in the theory of von Neumann algebras. It characterizes finite von Neumann algebras in terms of traces defined on the algebra. We put this result in the first chapter since it will not include our final goal, but will only be used as an important tool for the main results appearing in Chapter 4.

In Chapter 2 we use the equivalence classes of finite projections in a von Neumann algebra, to construct a commutative monoid M. By considering the Grothendieck group Γ of M, we canonically define an order relation on Γ. This commutative ordered group plays an important role in Fredholm theory since it contains the so called indices of the Fredholm elements defined on a von Neumann algebra. We conlude Chapter 2 by defining a dimension function on the set of all finite projections in $\mathcal{A}$ with values in Γ.

In Chapter 3 the concept of finite, compact and Fredholm elements are introduced and the index defined. We show that the set of compact elements is the smallest closed two-sided ideal containing the finite projections relative to a von Neumann algebra $\mathcal{A}$.

For $T \in \mathcal{A}$ to be Fredholm we shall require that

(i) the null-projection N_T of T is finite relative to $\mathcal{A}$.

(ii) There exists a finite projection $E \in \mathcal{A}$ such that the range of $I - E$ is contained in the range of T. By the use of properties (i) and (ii) an index mapping is defined on the set of all Fredholm elements relative to $\mathcal{A}$, with values in the Grothendieck group Γ. These values are called the indices of the Fredholm operators and the group Γ is referred to as the index group of $\mathcal{A}$.

Chapter 4 is devoted to the generalizations of the three classical theorems mentioned earlier. We conclude this chapter with a number of important corollaries obtained from the generalized Atkinson theorem: For example we obtain, by composition, that the set of all Fredholm elements in $\mathcal{A}$ is a self-adjoint monoid, which is open in the norm topology on $\mathcal{A}$.

We conclude this thesis with an Appendix where we mention several basic results on some useful locally convex topologies defined on $\mathcal{A}$. As far as the references are concerned, the main sources used in this work are [1], [2], [5], [17] and [18]. More detailed references are given throughout the chapters. The notations and conventions used are also defined at the beginning of each section.

CHAPTER 1

COMPARISON OF PROJECTIONS IN A VON NEUMANN ALGEBRA $\mathcal{A}$

This chapter is devoted to a variety of background material. The principal tool for the study of von Neumann algebras is the technique of "comparison" of the projections in a von Neumann algebra. In the first section, we shall define an equivalence relation together with an order relation on the set of all projections in a von Neumann algebra. We shall also define what we mean by a finite projection in a von Neumann algebra. The proofs of the main results appearing in Chapter 4 depend largely on the notion of the finiteness of a projection relative to a von Neumann algebra $\mathcal{A}$. Once we have defined what we mean by a finite von Neumann algebra, we can give a useful characterization of finite von Neumann algebras in terms of traces.

1.1 COMPARISON OF PROJECTIONS

Let H be a Hilbert space over the complex field $\mathbb{C}$. By L(H) we shall denote the C^*-algebra of all bounded linear operators on H. If M is a subset of L(H), we define its commutant M' as the set of all $T \in L(H)$ such that TS = ST for all $S \in M$. A von Neumann algebra on H is a unital $*$-subalgebra $\mathcal{A}$ of L(H) such that $\mathcal{A} = \mathcal{A}''$. By the fundamental theorem of operator algebras (the double commutation theorem), due to J von Neumann ([17], 3.2), one can also define a von Neumann algebra $\mathcal{A}$ as a $*$-subalgebra of L(H) which is closed in the weak operator topology on $\mathcal{A}$.

In this section we make use of several locally convex topologies defined on $\mathcal{A}$. For definitions and well-known results concerning these topologies on $\mathcal{A}$, see Appendix 5.1 where a few properties of these topologies are stated.

The set of all projections of $\mathcal{A}$ is defined as the hermitian operators in $\mathcal{A}$ which are idempotent. This set is denoted by $\mathcal{P}(\mathcal{A})$.

It is easy to see that the order relation $\leq$, defined by $E \leq F$ if and only if EF = E gives a partial order on $\mathcal{P}(\mathcal{A})$.

1.1.1 *LEMMA*

Let E, $F \in \mathcal{P}(\mathcal{A})$ be projections with closed range spaces $E(H)$ and $F(H)$. The following conditions are equivalent :

(i) $E \leq F$

(ii) $E(H) \subseteq F(H)$

Proof :

Let $E \leq F$. By definition this means $EF = E$. Taking adjoints on both sides one has $FE = E$. Thus $E(H) = FE(H) \subseteq F(H)$, which gives the implication (i) implies (ii).
Suppose $E(H) \subseteq F(H)$. Since F is the identity on $F(H)$ and $E(H) \subseteq F(H)$ it follows that F is the identity on $E(H)$. This implies $FE = E$ which implies $EF = E$. Thus (ii) implies (i).

Together with this order relation we have the following lemma.

1.1.2 *LEMMA* ([18], p. 290)

If $\mathcal{A}$ is a von Neumann algebra, then the set of all projections $\mathcal{P}(\mathcal{A})$ is a complete lattice.

Proof :

To see this we must show that if $\{E_i\}_{i\in I}$ is a family of projections in $\mathcal{A}$, the greatest lower bound $\inf_{i\in I} E_i$ and the least upper bound $\sup_{i\in I} E_i$ are elements of $\mathcal{P}(\mathcal{A})$.

Let E_o be the projection of H onto the closed subspace $\bigcap_{i\in I} E_i(H)$ of H. Clearly $UE_i = E_iU$ for every unitary operator $U \in \mathcal{A}'$ and $i \in I$. By definition a unitary operator is onto and so we have
$UE_i(H) = E_iU(H) = E_i(H)$ for all $i \in I$. Thus every unitary in $\mathcal{A}'$ leaves each $E_i(H)$ invariant; it therefore leaves the intersection $\bigcap_{i\in I} E_i(H)$ invariant as well. Since $U(\bigcap_{i\in I} E_i(H)) \subseteq \bigcap_{i\in I} E_i(H)$ we have $UE_o = E_oUE_o$. If we repeat the process for $U^* = U^{-1} \in \mathcal{A}'$ we find $U^*E_o = E_oU^*E_o$. By taking adjoints on both sides we get

$E_oU = E_oUE_o$. Thus, $E_oU = UE_o$ for every $U \in \mathcal{A}'$, unitary. Since every element in $\mathcal{A}'$ is a linear combination of four unitary elements ([17], p 20), we have $E_oT = TE_o$ for all $T \in \mathcal{A}'$. This implies $E_o \in \mathcal{A}'' = \mathcal{A}$ ($\mathcal{A}$ is a von Neumann algebra). Thus $E_o \in \mathcal{P}(\mathcal{A})$. It is clear that E_o is a lower bound of $\{E_i\}_{i \in I}$. Suppose $E' \leq E_i$ for all $i \in I$ lemma 1.1.1 implies $E'(H) \subseteq E_i(H)$ for all $i \in I$ which implies that $E'(H) \subseteq \bigcap_{i \in I} E_i(H)$. Thus $E' \leq E_o$ by lemma 1.1.1, and consequently E_o is the greatest lower bound of $\{E_i\}_{i \in I}$. Since the mapping $E \to I-E$ ($E \in \mathcal{P}(\mathcal{A})$) reverses the ordering of projections we have $\inf_i (I-E_i) = I - \sup_i E_i$. Thus $\sup_i E_i = I - \inf_i (I - E_i)$. Since $\inf_i (I - E_i) \in \mathcal{P}(\mathcal{A})$ by the above argument, we have that $\sup_i E_i \in \mathcal{P}(\mathcal{A})$.

■

1.1.3 *REMARKS*

In the proof of lemma 1.1.2 we have seen that the range space of $\inf_i E_i \in \mathcal{P}(\mathcal{A})$ is $\bigcap_{i \in I} E_i(H)$. We have also seen that $\sup_i E_i = I - \inf_i (I - E_i)$. Hence the range space of $\sup_i E_i \in \mathcal{P}(\mathcal{A})$

is $(\bigcap_{i \in I} E_i(H)^{\perp})^{\perp} = (\bigcup_{i \in I} E_i(H))^{\perp\perp} = [\bigcup_{i \in I} E_i(H)]$ (the closed subspace

of H generated by $\bigcup_{i \in I} E_i(H)$].

1.1.4 *DEFINITION* ([1])

Two projections E and F in $\mathcal{A}$ are said to be equivalent (relative to $\mathcal{A}$) if and only if there exists a $V \in \mathcal{A}$ such that $E = V^* V$ and $F = VV^*$. We write $E \sim F$, and say $E \sim F$ by V.

An order relation $\precsim$ in $\mathcal{P}(\mathcal{A})$ is defined as follows : We say $E \precsim F$ if and only if there is an $E' \in \mathcal{P}(\mathcal{A})$ such that $E \sim E' \leq F$.

1.1.5 *DEFINITION* ([5], p.52)

A partial isometry $V \in \mathcal{A}$ with initial projection E and final projection F is an operator such that

$$\|Vx\| = \|x\| \qquad (x \in E(H))$$
$$Vy = 0 \qquad (y \in (I-E)(H))$$

and $$F(H) = V(H)$$

1.1.6 *LEMMA* ([5], p. 52)

Let V, E, F $\in \mathcal{A}$. Then the following conditions are equivalent :

(i) V is a partial isometry with initial projection E and final projection F.

(ii) V^* is a partial isometry with initial projection F and final projection E

(iii) $V^*V = E$ is a projection and $F = VV^*$

(iv) $VV^* = F$ is a projection and $E = V^*V$.

<u>Proof</u> :

We first show that (iii) implies (iv) :
Suppose $E = V^*V$ is a projection in $\mathcal{A}$. Then we have $[V(I-E)]^*[V(I-E)] = (I-E)V^*V(I-E) = (I-E)E(I-E) = 0$. Thus $\|V(I-E)\|^2 = \|[V(I-E)][V(I-E)]^*\| = 0$. So $V(I-E) = 0$. This implies that $F - F^2 = VV^* - VV^*VV^* = V(I-V^*V)V^* = V(I-E)V^* = 0$. Consequently $F^2 = F$ and $F^* = (VV^*)^* = VV^* = F$. This proves condition (iv).

We now show (iii) implies (i) :
If condition (iii) holds, then for every $x \in H$ we have
$\|Vx\|^2 = (Vx,Vx) = (V^*Vx,x) = (Ex,x) = \|Ex\|^2$.
Thus $\|Vx\| = \|x\|$ for all $x \in E(H)$; and $Vy = 0$ if $y \in (I-E)(H)$. This shows that V is a partial isometry with initial projection E.

We now show that V has F as final projection. Since $F = VV^*$ we have $F(H) = V(V^*(H)) \subset V(H)$. Conversely we know $V(I-E)x = 0$ for all $x \in H$. Thus $V = VE = VV^*V = FV$, which implies $V(H) \subseteq F(H)$. Consequently $V(H) = F(H)$.

(i) implies (iii) :
If V is a partial isometry from E to F, it follows that $\|Vx\| = \|Ex\|$ for all $x \in E(H)$ and since $Vy = 0$, $y \in (I-E)(H)$ we have $\|Vx\|^2 = \|Ex\|^2$ $(x \in H)$.

This implies that $(V^* Vx, x) = (Ex, x)$ $(x \in H)$

From the polarization identity
$(x,y) = \frac{1}{4}\{(x+y,x+y)-(x-y,x-y)+i(x+iy,x+iy)-i(x-iy,x-iy)\}$ $(x,y \in H)$,
we conclude that $(V^*Vx, y) = (Ex, y)$ for all $x, y \in H$.

Thus $V^*V = E$; and E is a projection. According to the final paragraph of the proof : (iii) implies (i), it is clear that VV^* is the projection onto $V(H)$; and $V(H) = F(H)$, so $VV^* = F$.

The implications : (iv) implies (iii), (iv) implies (ii) and (ii) implies (iv) follow easily by interchanging V and V^*, and E and F in the above.

■

1.1.7 *REMARKS* ([5], p. 55)

(1) Due to lemma 1.1.6 two projections E and F in $\mathcal{P}(\mathcal{A})$ are equivalent if there exists a partial isometry in $\mathcal{A}$ with initial projection E and final projection F.

(2) "$\sim$" is indeed an equivalence relation :
<u>Reflexive</u> : $E \sim E$ by partial isometry E

Symmetric : Suppose $E \sim F$ by partial isometry V, then $E = V^*V$ and $F = VV^*$. Thus $F = (V^*)^*V^*$ and $E = V^*(V^*)^*$ which imply $F \sim E$ by partial isometry V^*.

Transitive : Suppose $E \sim F$ by V_1 and $F \sim G$ by V_2, then $E = V_1^*V_1$; $F = V_1V_1^*$ and $F = V_2^*V_2$, $G = V_2V_2^*$. Let $V = V_2V_1$. Then $V^*V = V_1^* F V_1 = V_1^* V_1V_1^* V_1 = V_1^*V_1 = E$. Likewise $VV^* = G$. Thus $E \sim G$ by V.

■

(3) We call two projections $E, F \in \mathcal{P}(\mathcal{A})$ disjoint if $EF = 0$.

1.1.8 *LEMMA* ([8], p. 111)

If $E, F \in \mathcal{P}(\mathcal{A})$ are commuting with corresponding range spaces $E(H)$ and $F(H)$, then $\sup(E,F) = E + F - EF$, $\inf(E,F) = EF$. Moreover $EF \in \mathcal{P}(\mathcal{A})$.

Proof :

We first show that $\inf(E,F) = EF$. Clearly EF is a projection in $\mathcal{A}$ since $(EF)^* = FE = EF$ (E and F commutes) and $(EF)^2 = (EF)(EF) = E^2F^2 = EF$. Since $\inf(E,F) \leq E$ and $\inf(E,F) \leq F$ we have $EF\inf(E,F) = E\inf(E,F) = \inf(E,F)$. Thus $\inf(E,F) \leq EF$. Conversely, let $x \in EF(H)$. Then $EFx = x$, so $Ex = E(EFx) = EFx = x$ and $Fx = F(EFx) = F(FEx) = FEx = EFx = x$. Thus $x \in E(H)$ and $x \in F(H)$ which implies $x \in E(H) \cap F(H)$. We have seen in the proof of lemma 1.1.1 that the range of $\inf(E,F)$ is $E(H) \cap F(H)$. Thus $EF(H) \subseteq \inf(E,F)(H)$, so lemma 1.1.1 implies $EF \leq \inf(E,F)$. Thus $EF = \inf(E,F)$.

By applying the same result to the commuting projections $I-E$ and $I-F$, we have

$$\inf(I-E, I-F) = (I-E)(I-F).$$

Then $\sup(E,F) = \sup[I-(I-E), I-(I-F)] = I - \inf(I-E,I-F)$ $= I - (I-E)(I-F) = E+F - EF$.

■

1.1.9 *COROLLARY* ([8], p. 112)

Suppose that $E, F \in \mathcal{P}(\mathcal{A})$ are projections onto $E(H)$ and $F(H)$, respectively. (1) If E and F are disjoint $\sup(E,F) = E+F$ (2) If $E \leq F$, then $F-E$ is a projection in $\mathcal{A}$ onto $F(H) \cap (I-E)(H)$.

Proof :

(1) follows directly from lemma 1.1.8. Since $E \leq F$ we have $EF = FE = E$. Thus $(I-E)$ and F commute. From lemma 1.1.8 $\inf(F, I-E) = F(I-E)$ $(= F-E)$ is a projection in $\mathcal{A}$ onto $F(H) \cap (I-E)(H)$. This shows (2).

■

1.1.10 *LEMMA* ([8], p. 112)

If $\{E_i\}$ is an increasing (resp. decreasing) net of projections in $\mathcal{A}$, and if $E = \sup_{i \in I} E_i$ (resp. $\inf_{i \in I} E_i$), then $Ex = \lim_i E_i x$ for each $x \in H$. The limit is taken in the norm topology on H.

Proof :

Since $\{E_i(H)\}$ is an increasing set of closed subspaces of H, $\cup_i E_i(H)$ is a linear subspace of H and has norm closure $E(H)$ by remark 1.1.3. Suppose $x \in H$ and $\epsilon > 0$. Since $Ex \in E(H)$, we can choose an element y in one of the subspaces $E_i(H)$ so that $\|Ex-y\| < \epsilon$.

When $i \leq j$, we have $E_i \leq E_j \leq E$, $y \in E_i(H) \subseteq E_j(H) \subseteq E(H)$, and thus

$$\|Ex - E_j x\| = \|E(Ex - y) - E_j(Ex - y)\|$$

$$\leq \|E - E_j\| \, \|Ex - y\| < \epsilon$$

Thus $\{E_j\}_{j \in I}$ converges to E in the strong operator topology on $\mathcal{A}$. The parts in brackets follow by applying the result just proved to $\{I - E_i\}_{i \in I}$.

■

1.1.11 *LEMMA* ([8], p 113)

If $\{E_i\}_{i\in J}$ is a disjoint family of projections in $\mathcal{A}$, $E = \sup_i E_i$ and $x\in H$, then $Ex = \sum_i E_i x$; the sum converges in the norm topology on H.

Proof :

If J is a finite set, it follows from corollary 1.1.9 (1), together with a straightforward argument by induction on the number of elements in J that $E = \sum_{i\in J} E_i$.

When J is an infinite set, let R denote the class of all finite subsets of J; for each $S \in R$, define $G_S = \sum_{i\in S} E_i$. By the preceding paragraph $G_S = \sup_{i\in S} E_i$, so $(G_S, S\in R, \supseteq)$ is an increasing net of projections, and

$$\sup_{S\in R} G_S = \sup\{\sup_{i\in S} E_i : S \in R\} = \sup_{i\in J} E_i = E$$

By lemma 1.1.10, Ex is the limit, in norm, of the net $(G_S x, S\in R, \subseteq)$. Thus since $G_S x = \sum_{i\in S} E_i x$, $\sum_{i\in J} E_i x$ converges in norm to Ex $(x\in H)$.

■

1.1.12 *PROPOSITION* ([5], p 56)

Let $\{E_i\}_{i\in I}$ (resp. $\{F_i\}_{i\in I}$) be a pairwise disjoint family of projections in $\mathcal{A}$. If $E_i \sim F_i$ for all $i\in I$, then $\sum_i E_i \sim \sum_i F_i$ where this sum converges in the strong operator topology on $\mathcal{A}$.

Proof :

Since $E_i \sim F_i$ for all $i\in I$, there exist partial isometries $V_i \in \mathcal{A}$ such that $E_i = V_i^* V_i$ and $F_i = V_i V_i^*$. Then, for all

$x \in H$, $V_i x = F_i V_i E_i x$, since $F_i V_i E_i x = V_i V_i^* V_i V_i^* V_i x = F_i^2 V_i x = F_i V_i x = V_i x$ ($V_i(H) = F_i(H)$, thus F_i is the identity on $V_i(H)$). Therefore $(V_i x, V_j x) = (F_i V_i E_i x, F_j V_j E_j x) = 0$ ($F_i F_j = 0$ $i \neq j$). Thus $V_i x \perp V_j x$ for all $i \neq j$.

Together with $\|V_i x\|^2 = (V_i x, V_i x) = (V_i^* V_i x, x) = (E_i x, x)$ $= \|E_i x\|^2$,

one has

$$\|\sum_i V_i x\|^2 = \sum_i \|V_i x\|^2 = \sum_i \|E_i x\|^2 = \|\sum_i E_i x\|^2 ,$$

where the sum is taken over any finite subset of I. Thus $\sum_i E_i$ is strong operator convergent if and only if $\sum_i V_i$ is strong operator convergent on H. But, from lemma 1.1.11 $\sum_i E_i$ is strong operator convergent to $E = \sup_i E_i \in \mathcal{P}(\mathcal{A})$. Thus $\sum_i V_i$ is strong operator convergent to V, say.

It is clear that $V \in \mathcal{A}$ since $\mathcal{A}$ is strong operator closed. The above equation gives $\|Vx\| = \|\sum_i E_i x\|$.

Thus V is a partial isometry with initial projection $E = \sum_i E_i$ (see the argument in the last paragraph of lemma 1.1.6,(iii) implies (i)).

Similarly, $\sum_i V_i^*$ is strong operator convergent to a partial isometry $W \in \mathcal{A}$ with initial projection $\sum_i F_i$. Thus $\sum_i V_i$ is weak operator convergent to V (the strong operator topology is finer than the weak operator topology on $\mathcal{A}$) and since the $*$-operation is weak operator continuous on $\mathcal{A}$, $\sum_i V_i^*$ is weak operator convergent to V^*; but $\sum_i V_i^*$ is also weak operator convergent to W. Thus $V^* = W$. We have seen that W has $\sum_i F_i$ as initial projection. Lemma 1.1.6 implies that V has final projection $F = \sum_i F_i$. Therefore, lemma 1.1.6 implies that $V^*V = E$ and $VV^* = F$. Thus $E \sim F$.

■

1.1.13 *COROLLARY* ([5], p 56)

If $\{E_i\}_{i\in I}$ (resp. $\{F_i\}_{i\in I}$) is a disjoint family of projections in $\mathcal{A}$ such that $E_i \precsim F_i$ for all $i\in I$, then $\sum_i E_i \precsim \sum_i F_i$.

Proof :

Since $E_i \precsim F_i$ for all i, there exist $E_i' \in \mathcal{P}(\mathcal{A})$ such that

$E_i \sim E_i' \leq F_i$. Then $\{E_i'\}_{i\in I}$ is a disjoint family since $E_i' F_i = E_i'$

and $F_j E_j' = E_j'$ imply $E_i'E_j' = E_i'F_iF_jE_j' = 0$

for all $i\neq j$ $(F_iF_j = 0)$. From lemma 1.1.12 we have that

$\sum_i E_i \sim \sum_i E_i'$. Clearly $\sum_i E_i' \leq \sum_i F_i$; thus $\sum_i E_i \precsim \sum_i F_i$

■

Notice $E \sim F$ $(E, F \in \mathcal{P}(\mathcal{A}))$ implies that $E \precsim F$ and $F \precsim E$. We now show the converse. Moreover we show that "$\precsim$" is a partial order on the set of equivalence classes in $\mathcal{P}(\mathcal{A})$.

1.1.14 *LEMMA* ([5], p 57)

Let $\mathcal{A}$ be a von Neumann algebra, then "$\precsim$" is a partial order on the equivalence classes of projections in $\mathcal{A}$.

Proof :

Reflexive : $E \precsim E$ since $E \sim E \leq E$; $E \in \mathcal{P}(\mathcal{A})$.

Transitive : Suppose $E \sim E' \leq F$ by partial isometry $U \in \mathcal{A}$ and

$F \sim F' \leq G$ by $V \in \mathcal{A}$. Then $U^*U = E$, $V^*V = F$, so $(VU)^* (VU) = E$ and $(VU)(VU)^* = VE'V^*$. Also since $VE'V^* (H) \subseteq V(H) = F'(H)$, one has $E \sim VE'V^* \leq F' \leq G$. Thus $E \precsim G$.

Antisymmetric : Suppose $E \precsim F$ and $F \precsim E$. Choose partial isometries U and $V \in \mathcal{A}$ such that $E = U^*U$, $UU^* = F_1 \leq F$ and

$F = V^*V$, $VV^* = E' \leq E$. (1.1)

Thus $E \overset{U}{\sim} F_1 \leq F \overset{V}{\sim} E' \leq E$. Consider $W = VU$. Then, for all projections, $G \leq E$, $(G \in \mathcal{P}(\mathcal{A}))$ WG is a partial isometry from G to WGW^* :

If $G \leq E$ and $W = VU$, we have $GE = G = EG$; $(WG)(WG)^* = WG^2W^* = WGW^*$ and

$(WG)^*(WG) = G^*W^*WG = GU^* V^* VUG = GU^*FUG = GU^*UG = GEG = G^2 = G$, because $FU = U$. Thus WG is a partial isometry with initial projection G and final projection WGW^* (It is easy to verify that WGW^* is a projection).

Define a sequence $\{E_n\}_{n=o}^{\infty}$ as follows :

$E_o = E$, $E_1 = E'$ and $E_{n+2} = WE_nW^*$. We now show by induction that

$E_{n+1} \leq E_n$ for all n. By (1.1) $E_1 \leq E$. Also $E_2 \leq E_1$, since

$E_2E_1 = WE_oW^*E_1 = WE_o U^*V^*E_1 = WE_o U^*V^* = WE_oW^* = E_2$. Suppose

$E_r \leq E_{r-1}$ $(r=1,\ldots,n)$. Then $E_{n+1} = WE_{n-1}W^* \leq WE_{n-2}W^* = E_n$, since

$E_{n+1}E_n = W E_{n-1} U^*FU E_{n-2} W^* = WE_{n-1} U^* UE_{n-2}W^* = W E_{n-1} W^* = E_n$.

Let $E_\infty = \inf_{n \in N} E_n$. Clearly $E_\infty \in \mathcal{P}(\mathcal{A})$ (lemma 1.1.2).

Since $E_n \leq E$ for all n, WE_n is a partial isometry from E_n to WE_nW^*. Likewise, since $E_n - E_{n+1} \leq E$ for all n we have

$E_n - E_{n+1} \sim W(E_n - E_{n+1})\, W^* = E_{n+2} - E_{n+3}$ for all n. By proposition 1.1.12 and lemma 1.1.10 we have

$$E = E_o = E_\infty + \sum_{n=o}^{\infty} (E_{2n} - E_{2n+1}) + \sum_{n=o}^{\infty} (E_{2n+1} - E_{2n+2})$$

$$\sim E_\infty + \sum_{n=o}^{\infty} (E_{2n+2} - E_{2n+3}) + \sum_{n=o}^{\infty} (E_{2n+1} - E_{2n+2})$$

$= E_1 \sim F$. (Note that all the above series are strong operator convergent).

■

1.1.15 *DEFINITION* ([18], p 291)

Let $T \in \mathcal{A}$. The smallest projection $E \in \mathcal{P}(\mathcal{A})$ such that $ET = T$ is called the left support of T and denoted by $S_\ell(T)$. The right support $S_r(T)$ is the smallest projection $F \in \mathcal{P}(\mathcal{A})$ with $TF = T$. We define the support of T as the smallest projection $E \in \mathcal{P}(\mathcal{A})$ such that $ET = TE = T$ and denote it by $S(T)$.

It is clear that $S_\ell(T)$ and $S_r(T)$ are well-defined elements of $\mathcal{P}(\mathcal{A})$ (lemma 1.1.2) and if $S_\ell(T) = S_r(T)$, then $S(T) = S_\ell(T) = S_r(T)$.

1.1.16 *REMARK*

We claim that $S_\ell(T)$ (resp. $S_r(T)$) is the projection onto $\overline{T(H)}$ (resp. $\overline{T^*(H)}$).

<u>Proof</u> :

Since $S_\ell(T)T = T$ and $S_\ell(T)$ is continuous as element of L(H), we have that $S_\ell(T)(\overline{T(H)}) = \overline{T(H)}$. Thus $S_\ell(T)(H) \supseteq \overline{T(H)}$. Let $[T(H)]$ be the projection onto $\overline{T(H)}$. If we can show that $[T(H)] \in \mathcal{P}(\mathcal{A})$ we have $S_\ell(T) \geq [T(H)]$. Since $[T(H)]T = T$, it follows by definition of $S_\ell(T)$ that $S_\ell(T) = [T(H)]$. Take note that $TU = UT$ for all unitary $U \in \mathcal{A}'$.

Since $U(\overline{T(H)}) \subseteq \overline{UT(H)} = \overline{TU(H)} = \overline{T(H)}$ and $[T(H)]$ is the identity on $\overline{T(H)}$ we have $U[T(H)] = [T(H)]U[T(H)]$. This also holds for $U^* \in \mathcal{A}$.

Thus $[T(H)]U^* [T(H)] = U^*[T(H)]$. By taking adjoints on both sides one gets $[T(H)]U = [T(H)]U[T(H)] = U[T(H)]$. This implies $[T(H)] \in \mathcal{A}'' = \mathcal{A}$. Using definition 1.1.15 it is clear that $S_r(T) = S_\ell(T^*)$. Then the above argument shows that $S_r(T)$ is the projection onto $\overline{T^*(H)}$

■

1.1.17 *LEMMA* ([5], p 53)

Let $T \in \mathcal{A}$, and let $T = VR$ be the polar decomposition of T. Then V is a partial isometry with $V^*V = [T^*(H)] = S_r(T)$ and $VV^* = [T(H)] = S_\ell(T)$. Moreover $S_r(T) \sim S_\ell(T)$.

Proof :

It is clear by the existence proof of such a polar decomposition of $T \in \mathcal{A}$, that $R = (T^*T)^{1/2}$. Also since a von Neumann algebra is a C^*-algebra $R \in \mathcal{A}$. R is called the positive square root of T. Then, for all $x \in H$,

$$\|Rx\|^2 = (R^2x,x) = (T^*Tx,x) = \|Tx\|^2, \qquad (1.2)$$

since R is hermitian.

We may therefore define an isometry

$$V_o: R(H) \to T(H) \text{ by } V_o(Rx) = Tx \ (x \in H).$$

V_o is well defined since if $Rx = Ry$ then $R(x - y) = 0$. This implies that $T(x - y) = 0$ by (1.2). Extend V_o by continuity to an isometry V_1 from $\overline{R(H)}$ onto $\overline{T(H)}$. Define $V' = V_1E$, where $E = [R(H)] \in \mathcal{A}$ is the projection onto $\overline{R(H)}$ (that $E \in \mathcal{A}$ can be seen from remark 1.1.16 and the fact that $R \in \mathcal{A}$). Since $V'Rx = V_1ERx = V_1Rx = V_oRx = Tx$ for all $x \in H$ we have by the uniqueness of this polar decomposition that $V' = V$. We now show that V is a partial isometry with initial projection $[R(H)]$ and final projection $[T(H)]$. Let $F = [T(H)]$. If $x = Ey \in E(H) = \overline{R(H)}$ $(y \in H)$, then $\|x\| = \|Ey\| = \|V_1 Ey\| = \|V_1 E^2 y\| = \|VEy\| = \|Vx\|$ and $V(I-E) = V_1 E(I-E) = 0$. Also

$F(H) = [T(H)](H) = \overline{T(H)} = V_1(\overline{R(H)}) = V_1 E(H) = V(H)$.

Thus $VV^* = [T(H)]$ and $V^*V = [R(H)]$. To conclude the proof we must show that $[R(H)] = [T^*(H)]$. To do this we show that

$[T^*T(H)] = [T^*(H)]$. It is clear that $\overline{T^*T(H)} \subseteq \overline{T^*(H)}$. Thus $[T^*T(H)] \leq [T^*(H)]$. Conversely, we have for all $x \in H$ with $x \perp [T^*T(H)](H)$ that $(x, T^*Ty) = 0$ for all $y \in H$. Thus $(T^*Tx, y) = 0$ $(y \in H)$, which implies $T^*Tx = 0$. From $0 = (T^*Tx, x) = \|Tx\|^2$ we have $Tx = 0$. Thus $0 = (Tx, y) = (x, T^*y)$ for all $y \in H$. Hence $x \perp [T^*(H)](H)$. This implies $[T^*(H)](H) \leq [T^*T(H)](H)$. Consequently, $[T^*(H)] = [T^*T(H)]$.

Then $[R(H)] = [R^*(H)] = [R^*R(H)] = [R^2(H)] = [T^*T(H)] = [T^*(H)]$ (R is hermitian).

■

1.1.18 *COROLLARY* ([5], p 55)

If $T = VR$ is the polar decomposition of $T \in \mathcal{A}$ then $V, R \in \mathcal{A}$.

<u>Proof</u> :

$T \in \mathcal{A}$ implies $R = (T^*T)^{1/2} \in \mathcal{A}$ because $\mathcal{A}$ is a C^*-algebra ($\mathcal{A}$ is a a^*-subalgebra of $L(H)$ and $\bar{\mathcal{A}}^{\|\cdot\|} \subset \bar{\mathcal{A}}^{\text{weak}} = \mathcal{A}$, so $\mathcal{A}$ is norm closed).

To show that $V \in \mathcal{A}$, we show that $VS = SV$ for every $S \in \mathcal{A}'$. This implies $V \in \mathcal{A}'' = \mathcal{A}$. Since $R \in \mathcal{A}$ and $S \in \mathcal{A}'$, $SR = RS$. So $SVRx = STx = TSx = VRSx$ $(x \in H)$. So $VS = SV$ on $R(H)$ and by continuity, on $\overline{R(H)}$.

Now $S \in \mathcal{A}'$ implies $R^*S = SR^*$, which implies $S^*R = RS^*$ (taking adjoints).

So $y \in R(H)^\perp$ implies $(Rx,y) = 0$ $(x \in H)$.

Therefore $\quad (Rx, Sy) = (S^*Rx,y)$

$$= (R(S^*x),y) = 0 \quad (x \in H).$$

Hence $Sy \in R(H)^\perp$

Then clearly $SVy = 0 = VSy$ (In the proof of lemma 1.1.17 we have seen that $V = V_1[R(H)]$, so $Vy = 0$ for all $y \in R(H)^\perp$).

We have shown that $VS = SV$ both on $\overline{R(H)}$ and on $R(H)^\perp$, so $VS = SV$ for all $S \in \mathcal{A}'$.

■

1.1.19 *PROPOSITION* (Parallelogram law [18], p 292)

Let $E, F \in \mathcal{P}(\mathcal{A})$, then $E - \inf(E, I-F) \sim F - \inf(F, I-E)$.

<u>Proof</u> :

Consider $FE \in \mathcal{A}$. We are going to show that

$S_r(FE) = E - \inf(E, I-F)$. Since $\overline{(FE)^*(H)} = [\mathrm{Ker}(FE)]^\perp$ we have from remark 1.1.16 that $S_r(FE)$ is the projection onto $\mathrm{Ker}(FE)^\perp$.
If $x \in H$ and $FEx = 0$, we have $Ex = (I-F)Ex \in \inf(E,I-F)(H)$ $(E(Ex) = Ex;\ (I-F)Ex = Ex)$. This implies that
$x = (I-E)x + Ex \in (I-E + \inf(E,I-F))(H)$.
Thus $\mathrm{Ker}(FE) \subseteq (I-E + \inf(E,I-F))\,(H)$. Converseley if $x \in (I-E + \inf(E,I-F))(H)$ we can write $x = y \oplus z$ ($I-E$ and $\inf(E, I-F)$ are disjoint) with $y=(I-E)y$ and $z = Ez = (I-F)z$. Then
$FEx = FEy + FEz = FE(I-E)y + F(I-F)z = 0$.
Thus $(I-E) + \inf(E,I-F)$ is the projection onto $\mathrm{Ker}(FE)$.
Since $S_r(FE)$ is the projection onto

$Ker(FE)^{\perp} = H\ominus[((I-E)(H)\oplus(E(H)\cap(I-F)(H))] = E(H)\ominus(E(H)\cap(I-F)(H))$, we have that $S_r(FE) = E - inf(E,I-F)$
Likewise $S_\ell(FE) = S_r(EF) = F - inf(F,I-E)$. Since $S_r(FE) \sim S_\ell(FE)$, we have $E - inf(E,I-F) \sim F - inf(F,I-E)$ (lemma 1.1.17)

■

1.1.20 *COROLLARY*

If E, F$\in \mathcal{P}(\mathcal{A})$, we have $sup(E,F) - F \sim E - inf(E,F)$

Proof :

By replacing F with I-F in proposition 1.1.19 the result follows.

■

Observing that the centre of a von Neumann algebra is given by $Z = \mathcal{A}\cap\mathcal{A}'$ we define the following :

1.1.21 *DEFINITION* ([5], p 57)

The central support C(T) of $T \in \mathcal{A}$ is the smallest projection $Q \in Z$ such that $QT = T = TQ$.

Note such an projection exists since $\mathcal{P}(Z)$ is a complete lattice.

1.1.22 *LEMMA* ([5], p 56)

Let E, F $\in \mathcal{P}(\mathcal{A})$, then if $E \precsim F$ we have $PE \precsim PF$ for each central projection $P \in \mathcal{A}$. Moreover, $E \sim F$ implies $PE \sim PF$ for all projections $P \in Z$.

Proof :

Suppose $E \overset{V}{\sim} E_1 \leq F$, then we first show that $PE \overset{VP}{\sim} PE_1$. This follows since $(VP)^*(VP) = PV^*VP = P^2 V^*V = PE$ and $(VP)(VP)^* = VP^2V^* = P^2VV^* = PE_1$. Since $(PE_1)(PF) = P^2E_1F = PE_1$ one has that $PE_1 \leq PF$. This implies $PE \sim PE_1 \leq PF$, and so $PE \precsim PF$.

■

1.1.23 *LEMMA* ([5], p 58))

Let $E, F \in \mathcal{P}(\mathcal{A})$; then
(i) $E \precsim F$ implies $C(E) \leq C(F)$.
(ii) $E \sim F$ implies $C(E) = C(F)$.

<u>Proof</u> :

(i) Suppose $E \precsim F$; then there exists a partial isometry $V \in \mathcal{A}$ such that $V^*V = E$ and $VV^* = F_1 \leq F$. Take any $Q \in Z$. If $QF = F$ we have $QF_1 = QFF_1 = FF_1 = F_1$. Thus $QVV^* = VV^*$. Then $QE = QE^2 = QV^*VV^*V = V^*(QVV^*)V = V^*VV^*V = E^2 = E$ $(Q \in Z)$ In particular for $Q = C(F)$ we have $C(F)E = E$, but $C(E)$ is the smallest such central projection. Thus $C(E) \leq C(F)$.

(ii) If $E \sim F$, then $E \precsim F$ and $F \precsim E$. the result follows from (i).

■

1.1.24 *LEMMA* ([5], p 58)

If $T \in \mathcal{A}$, we have that $C(T) = [\mathcal{A}TH]$ (the projection onto $\overline{\mathcal{A}TH}$ where $\mathcal{A}TH = \{STx | S \in \mathcal{A}, x \in H\}$).

<u>Proof</u> :

Let $[\mathcal{A}TH] = Q$. We first show that $Q \in Z$. If $S \in \mathcal{A}$ we have $S(\mathcal{A}TH) \subseteq \mathcal{A}TH$. Since S is continuous in norm

$S(\overline{\mathcal{A}TH}) \subseteq \overline{S(\mathcal{A}TH)} \subseteq \overline{\mathcal{A}TH}$. Thus $SQ = QSQ$. Since $S^* \in \mathcal{A}$ we can repeat the above argument for S^* to get $S^*Q = QS^*Q$. By taking adjoints on both sides we conclude that $SQ = QSQ = QS$. Thus $Q \in \mathcal{A}'$ which implies $S(\mathcal{A}TH) = \mathcal{A}TSH \subseteq \mathcal{A}TH$ $(S \in \mathcal{A}')$ and as before we have $SQ = QS$. Thus $Q \in \mathcal{A}'' = \mathcal{A}$, so $Q \in A \cap \mathcal{A}' = Z$.

Next we show that $Q = C(T)$. Since $I \in \mathcal{A}$ we have $Tx \in \mathcal{A}TH$ $(x \in H)$, so $QTx = Tx$ (Q is the identity on $\mathcal{A}TH$). By definition of $C(T)$ we have

$Q \geq C(T)$. Conversely we have $C(T) \in \mathcal{A}'$ and $C(T)\,STx = SC(T)Tx = STx$ ($S \in \mathcal{A}$, $x \in H$). This implies $\mathcal{A}TH \subseteq C(T)(H)$, and since $C(T)(H)$ is closed $\overline{\mathcal{A}TH} \subseteq C(T)(H)$. Thus $Q \leq C(T)$.

■

Before we prove one of the most powerful tools in the study of the projection lattice we need the following lemma.

1.1.25 *LEMMA* ([5], p 58 and p 59)

For the two projections E and F in $\mathcal{A}$, consider the following statements

(i) $C(E)\,C(F) \neq 0$ ($C(E)$ and $C(F)$ are not disjoint)

(ii) $E\mathcal{A}F \neq \{0\}$

(iii) There exist non-zero projections $E_1 \leq E$ and $F_1 \leq F$ in $\mathcal{A}$ such that $E_1 \sim F_1$.

Then we have (i) implies (ii) and (i) implies (iii).

<u>Proof</u> :

(i) implies (ii) :

We know that $C(E) = [\mathcal{A}EH]$ and $C(F) = [\mathcal{A}FH]$. Since $C(E)C(F) \neq 0$ there exist $R, S \in \mathcal{A}$ and $x, y \in H$ such that $(REx, SFy) \neq 0$, so $(FS^*REx, y) \neq 0$ which implies $FS^*RE \neq 0$. Consequently $(FS^*\,RE)^* = ER^*SF \neq 0$, so $E\,\mathcal{A}\,F \neq \{0\}$.

(i) implies (iii) :

Let $T \in E\mathcal{A}F$, $T \neq 0$ (from (i) implies (ii)). Since $\overline{T(H)} = \overline{ESFH} \subseteq E(H)$ (for some $S \in \mathcal{A}$), we have $S_\ell(T) = [T(H)] \leq E$. Similarly $S_r(T) = [T^*(H)] \leq F$, and from lemma 1.1.17 we have $S_\ell(T) \sim S_r(T)$ ($S_\ell(T)$ and $S_r(T)$ are non-zero since $T \neq 0$.)

■

1.1.26 *PROPOSITION* (comparability, [5], p 59)

For all E, F $\in \mathcal{P}(\mathcal{A})$ there exists a Q $\in \mathcal{P}(Z)$ satisfying

$$QE \precsim QF \text{ and } (I-Q)E \succsim (I-Q)F$$

Proof :

If $C(E)C(F) = 0$, let $Q = C(F)$. Then we have $QE = C(F)E = C(F)C(E)E = 0 \leq F$. Thus $QE \sim 0 \leq QF(=F)$ and thus $QE \precsim QF$. Also $(I-Q)F = 0 \leq E = (I-Q)E$, so $(I-Q)F \precsim (I-Q)E$ and the result follows. From lemma 1.1.25, if $C(F)C(E) \neq 0$, there exists a pair (E_1,F_1) of non-zero projections in $\mathcal{A}$ such that $E_1 \leq E$ and $F_1 \leq F$ with $E_1 \sim F_1$. Let $\mathcal{K}$ be the class of all families $\{(E_\lambda, F_\lambda)\}_{\lambda\in\Lambda}$ of pairs of projections in $\mathcal{A}$ with the following properties :

(1) $0 < E_\lambda \leq E$ for all $\lambda\in\Lambda$
(2) $\{E_\lambda\}$ is a disjoint family
(3) $0 < F_\lambda \leq F$ for all $\lambda\in\Lambda$
(4) $\{F_\lambda\}$ is a disjoint family
(5) $E_\lambda \sim F_\lambda$ for all $\lambda\in\Lambda$

Clearly $\mathcal{K} \neq \phi$ since $\{(E_1, F_1)\} \in\mathcal{K}$. Then $\mathcal{K}$ is partial ordered by $\subseteq$ and Zorn's lemma implies that a maximal such family in $\mathcal{K}$, say $\{(E_\lambda,F_\lambda)\}$ exists.

Let $E_o = \sum_\lambda E_\lambda$, $F_o = \sum_\lambda F_\lambda$. Then $E_o \sim F_o$ by proposition 1.1.12.

Let $E_2 = E - E_o$, $F_2 = F - F_o$. Then $C(E_2)\ C(F_2) = 0$ since if $C(E_2)C(F_2) \neq 0$ there exist projections E_3, $F_3 \in \mathcal{A}$ with $0 < E_3 \leq E_2$; $0 < F_3 \leq F_2$ and $E_3 \sim F_3$ (see lemma 1.1.25). We know that $0 < E_3 \leq E_2 = E-E_o < E$,
so $E_3E_\lambda = E_3E_2E_\lambda = E_3\ (E-E_o)\ E_\lambda = E_3E\ E_\lambda - E_3\ (\sum_\mu E_\mu)E_\lambda = 0$ for all $\lambda \in \Lambda$.

Similarly $F_3F_\lambda = 0$ for all λ. Thus

$\{(E_2,F_2);(E_\lambda,F_\lambda)\}_{\lambda\in\Lambda} \in \mathcal{K}$ which contradicts the maximality of $\{(E_\lambda,F_\lambda)\}$ in $\mathcal{K}$.

Let $Q = C(F_2)$, then we have that
$QE_2 = QC(E_2)\ E_2 = C(E_2)\ C(F_2)E_2 = 0$. But $E_2 = E-E_o$, so
$QE = QE_2 + QE_o = QE_o \sim Q\ F_o \leq QF$ (lemma 1.1.22), so $QE \precsim QF$.
Similarly, $(I-Q)F_2 = (I-C(F_2))\ C(F_2)F_2 = 0$.

Thus $(I-Q)F = (I-Q)F_o \sim (I-Q)E_o \leq (I-Q)E$; which means
$(I-Q)F \precsim (I-Q)E$.

■

1.1.27 *REMARKS* ([5], p 59)

(1) We define a factor as a von Neumann algebra with trivial centre, that means $Z = \mathcal{A}\cap\mathcal{A}' = \mathbb{C}I$.

(2) If $\mathcal{A}$ is a factor the order relation "$\precsim$" on $\mathcal{P}(\mathcal{A})$ is a total order. To see this we take any two projections $E, F \in \mathcal{P}(\mathcal{A})$. By proposition 1.1.26 there exists a $Q \in Z$ such that $QE \precsim QF$ and $(I-Q)E \succsim (I-Q)F$; but since $\mathcal{A}$ is a factor Q can either be 0 or I (these are the only projections in the centre). Thus $E \precsim F$ or $F \precsim E$.

■

1.2 FINITE AND INFINITE PROJECTIONS

In $L(H)$ we say an operator T is finite (resp. infinite) if $T(H)$ is finite (resp. infinite) dimensional in the usual sense. So a projection $E \in L(H)$ is finite if and only if $\dim(E(H)) < \infty$. We now want to generalize this idea of a finite (resp. infinite) projection to a general von Neumann algebra $\mathcal{A}$.

1.2.1 *DEFINITION* ([5], p 61)

A projection $E \in \mathcal{P}(\mathcal{A})$ is said to be finite if there is no projection E_1 in $\mathcal{A}$ with $E \sim E_1 < E$. A projection is infinite if it is not finite.

1.2.2 *NOTE* :

In L(H) a projection E is finite in the sense of definition 1.2.1 if and only if E(H) is finite dimensional. In fact if E(H) is finite dimensional, then for all $E_1 < E$ $E_1(H) \subsetneq E(H)$ and so $\dim E_1(H) < \dim E(H)$. Suppose $E \sim E_1 < E$. Then there exists a partial isometry $V \in L(H)$ from E to E_1. From $\|Vy\| = \|y\|$ $(y \in E(H))$ we have that V is one-to-one and since $V(H) = E_1(H)$, V is onto $E_1(H)$, so V is a isomorphism from E(H) onto $E_1(H)$. Thus $\dim E(H) = \dim E_1(H)$, but we have seen $\dim E_1(H) < \dim E(H)$ which contains a contradiction. Converseley, let E be finite in the sense of definition 1.2.1 and suppose E(H) is infinite dimensional. Then E(H) is isomorphic to a closed subspace $K \subsetneq E(H)$. Let E_1 be the projection in L(H) onto K. Then $E(H) \cong E_1(H)$ and there exists an isomorphism V from E(H) onto $E_1(H)$. Define $V \equiv 0$ on (I-E)(H). Then V is a partial isometry from E to E_1, so $E \sim E_1$ and $E_1 < E$ $(E_1(H) \subsetneq E(H))$, so $E \sim E_1 < E$ which contradicts the finiteness of E in the sense of definition 1.2.1. In a general von Neumann algebra this result is not always true. In a general von Neumann algebra $\mathcal{A}$ the projection E_1 with range $K \subseteq E(H)$ (see proof above) need not be in $\mathcal{P}(\mathcal{A})$.

■

1.2.3 *DEFINITION* ([5], p 61)

(1) If $E \in \mathcal{P}(\mathcal{A})$ is infinite and PE is either 0 or infinite for each central projection $P \in \mathcal{P}(\mathcal{A})$, E is said to be properly infinite.

(2) A von Neumann algebra $\mathcal{A}$ is said to be finite, infinite, properly infinite according to the property of the identity projection $I \in \mathcal{A}$.

1.2.4 *LEMMA* ([5], p 61))

If E is a finite projection in $\mathcal{A}$, then each subprojection of E is finite; $0 \in \mathcal{A}$ is finite and if $E \sim F$ with F finite then E is finite.

Proof :

Let $E_o \in \mathcal{P}(\mathcal{A})$ with $E_o \leq E$. Suppose E_o is infinite. Then there exists a $F_1 \in \mathcal{P}(\mathcal{A})$ such that $E_o \sim F_1 < E_o$. Since E_o and $E - E_o$ (resp F_1 and $E - E_o$) are disjoint it follows from proposition 1.1.12 that $E = (E - E_o) + E_o \sim (E - E_o) + F_1$ and since $F_1 < E_o$ we have $(E - E_o) + F_1 < (E - E_o) + E_o = E$, so $E \sim (E - E_o) + F_1 < E$ which contradicts the fact that E is finite. Since 0 has no proper subprojection it is finite. Let $E \sim F$. Suppose E is infinite, then there exists a $E_1 \in \mathcal{P}(\mathcal{A})$ with $E \sim E_1 < E$. Let $V \in \mathcal{A}$ be a partial isometry such that $E = V^*V$ and $F = VV^*$. We show that if F_1 is the projection onto $VE_1(H)$ then $E_1 \sim F_1$ by VE_1. Since $E_1(H) \subsetneq E(H)$ and $\|Vy\| = \|y\|$ for all $y \in E(H)$ we have $\|VE_1(E_1x)\| = \|VE_1x\| = \|E_1x\|$ and $\|VE_1(I-E_1)x\| = 0$ $(x \in H)$

Thus $VE_1y = 0$ $(y \in (I-E_1)(H))$. By definition of F_1, $VE_1(H) = F_1(H)$, so $E_1 \sim F_1$. Also, since $E_1 < E$, we have $F_1(H) = VE_1(H) \subseteq VE(H) \subseteq V(H) = F(H)$. Moreover $F_1 < F$, for if not it will follow that $E_1 = E$. Consequently $F \sim E \sim E_1 \sim F_1 < F$ which implies F is infinite, so F finite implies E finite.

■

1.2.5 *LEMMA* (Halving [9] p 412)

If E is a properly infinite projection in $\mathcal{A}$, then there is a projection $F \in \mathcal{P}(\mathcal{A})$ with $F \leq E$ and $F \sim E-F \sim E$.

Proof :

Since E is infinite there exists an $E_1 \in \mathcal{P}(\mathcal{A})$ with $E \sim E_1 < E$. Let $V \in \mathcal{A}$ be a partial isometry such that $E = V^*V$ and $E_1 = VV^*$. Then $E_2 = VE_1V^* < E_1$; in fact, since $VE_1V^*(H) \subseteq V(H) = E_1(H)$ we have $VE_1V^* \leq E_1$; also $VE_1V^* \neq E_1$, for if $VE_1V^* = E_1 = VV^*$, one gets $V^*VE_1V^* = V^*VV^*$ which implies $E_1V^* = EE_1V^* = EV^*$. Thus $VE_1 = VE$ which gives $V^*VE_1 = V^*VE$, so by noticing that $V^*V = E$ we have $E_1 = EE_1 = E^2 = E$ - a contradiction. It follows from

$E - E_1 = V^*VV^*V - V^*VVV^*V^*V = V^*(E_1 - VE_1V^*)V = V^*(E_1-E_2)V$ $(E_1 \leq E)$, that $E - E_1 \sim E_1 - E_2$ by the partial isometry $V^*(E_1-E_2)$.

Continuing in this way ($VE_2V^* = E_3 < E_2$ and $E_1 - E_2 \sim E_2 - E_3$), we construct a countable infinite family $\{E_n - E_{n+1}\}$ of equivalent non-zero subprojections of E. We show that this family is disjoint : Let $E_n - E_{n+1}$ and $E_m - E_{m+1}$ be two elements iin this family with $n \neq m$. We may assume that $n < m$.
Then

$$\begin{aligned}(E_n - E_{n+1})(E_m - E_{m+1}) &= E_nE_m - E_nE_{m+1} - E_{n+1}E_m + E_{n+1}E_{m+1}\\ &= E_m - E_{m+1} - E_m + E_{m+1}\\ &= 0\end{aligned}$$

By Zorn this family is contained in a maximal such family $\{F_i\}_{i\in I}$. We cannot have that $F_i \precsim E - \sum_i F_i$ $(= E_o)$ for some $i \in I$, for then there exists a $F_o \in \mathcal{P}(\mathcal{A})$ with

$F_i \sim F_o \leq E_o$ and since $F_oF_i = F_oE_oF_i = F_o(E - \sum_i F_i)F_i = F_o(F_i - F_i) = 0$

$(F_i \leq E)$, we have that $\{F_i, F_o\}_{i\in I}$ is a disjoint family of equivalent non-zero subprojection of E. This contradicts the maximality of the family $\{F_i\}_{i\in I}$. From proposition 1.1.26 there is for any fixed $i\in I$ a non-zero central projection P_i with $P_iE_o \precsim P_iF_i$. Let $P = \inf_i P_i$. Then $PE_o \precsim P\,F_i$ $i\in I$ by lemma 1.1.22.

Since I is an infinite set, there is a subset I_o of I such that if $i_o \in I_o$, then $I\backslash I_o (=I_1)$; I_o and $I_o\backslash\{i_o\}(= I_2)$ can each be put into one-to-one correspondence with I (if card(I) = α. Then it is known from set theory that $\alpha^2 = \alpha$ where $\alpha^2 = \text{card}(I\text{x}I)$, since I is an infinite set. This means that there exists a bijection $f: I\text{x}I \longrightarrow I$. Define $I_o' = I\text{x}\{i\}$ for a fixed $i \in I$ and $I_1' = I\text{x}I\backslash I_o'$.

Then since $I \times \{i'\} \subset I_1' \subset I \times I \simeq I$ $(i' \neq i)$, one has $\mathrm{card}(I) \leq \mathrm{card}(I_1') \leq \mathrm{card}(I)$. Thus if we let $I_o = f(I_o')$ and $I_1 = f(I_1')$ one has that $I_o \simeq I_1 \simeq I$. That $I_2 \simeq I_o$ follows similarly).

From lemma 1.1.22, we have $PF_i \sim PF_j$ where $i,j \in I$, and from proposition 1.1.12 and its corollary one has

$$PE = \sum_{i \in I} PF_i + PE_o \lesssim \sum_{i \in I_2} PF + PF_{i_o}$$
$$= \sum_{i \in I_o} PF_i \sim \sum_{i \in I_1} PF_i \lesssim \sum_{i \in I_1} PF_i + PE_o \lesssim PE$$

Thus $PE \lesssim \sum_{i \in I_o} PF_i \lesssim G \lesssim PE$ where $G = \sum_{i \in I_1} PF_i + PE_o$; so

$$PE \sim G \sim \sum_{i \in I_o} PF_i \quad \text{(lemma 1.1.14).}$$

Since $\sum_{i \in I_o} PF_i = \sum_{i \in I} PF_i + PE_o - \sum_{i \in I_1} PF_i - PE_o = PE - G$ (where the sums are taken in the strong operator topology on $\mathcal{A}$), one has $PE \sim G \sim PE - G$.

Up to this point, we have proved that if E is a properly infinite projection in $\mathcal{A}$, there is a non-zero central projection P in $\mathcal{A}$ such that PE can be "halved" - that is there is a subprojection G of PE in $\mathcal{A}$ with $G \sim PE-G \sim PE \neq 0$. Also, as seen in the first part of the proof $F_i \leq E$ $(i \in I)$; $P \leq P_i$ $(P = \inf_{i \in I} P_i)$ where $P_i \in Z$ was chosen so that $P_i \leq C(F_i)$ holds for each $i \in I$. From lemma 1.1.23 one has $C(F_i) \leq C(E)$. Hence $P \leq C(E)$. Using Zorn, there exists a maximal family $\{Q_a\}_{a \in A}$ of non-zero, disjoint central subprojections of C(E) such that each Q_aE can be halved.

Thus, let G_a be a subprojection of Q_aE in $\mathcal{A}$ such that $G_a \sim Q_aE - G_a \sim Q_aE$. We want to show that $C(E) = \sum_a Q_a$. If $C(E) - \sum_a Q_a \neq 0$ then it follows that $(C(E) - \sum_a Q_a)E$ is properly infinite, since for every non-zero central projection P, either $P(C(E) - \sum_a Q_a)E$ is infinite or zero (if P is a central projection

$P(C(E) - \sum_a Q_a)$ is also one, and E is properly infinite).

Since $(C(E) - \sum_a Q_a)E$ is properly infinite the first part of our proof states that there exists a non-zero central subprojection Q_o of $C((C(E) - \sum_a Q_a)E)$ $(= (C(E) - \sum_a Q_a)C(E) = C(E) - \sum_a Q_a)$, such that $Q_o(C(E) - \sum_a Q_a)E = Q_oE$ can be halved. Let $Q = C(E) - \sum_a Q_a$.

Now the first equality in brackets holds since $C(QE) = QC(E)$ $(QE \leq Q$, so $C(QE) \leq C(Q) = Q$ by 1.1.23). Thus $C(QE) \leq Q = QC(E)$. Converseley, $QC(E)QE = Q^2C(E)E = QE$, so $C(QE) \leq QC(E)$ (1.1.23). Hence $C(QE) = QC(E)$. Since $Q_oQ_a = Q_o(C(E) - \sum_a Q_a)Q_a = 0$ for each $a \in A$ we have that $\{Q_a, Q_o : a \in A\}$ is a disjoint family of non-zero central subprojections of $C(E)$ such that each Q_aE and Q_oE can be halved. This contradicts the maximality of $\{Q_a\}$. Consequently $C(E) = \sum_a Q_a$. Letting $F = \sum_a G_a$, proposition 1.1.12 implies that.

$$F \sim \sum_a (Q_aE - G_a) = E - F \sim \sum_a Q_aE = E$$

■

1.2.6 *LEMMA* ([9], p 414)

If $\{P_i\}_{i \in I}$ is a family of central projections in $\mathcal{A}$, and $E \in \mathcal{P}(\mathcal{A})$ is such that P_iE is finite for each $i \in I$, then PE is finite, where $P = \sup_i P_i$.

<u>Proof</u> :

Suppose PE is infinite, then an $F \in \mathcal{P}(\mathcal{A})$ exists such that $PE \sim F < PE$. Then $0 \neq PE - F \leq PE \leq P$. If $(PE - F)P_i = 0$ for

each i, $(PE - F)P_i(H) = \{0\}$ for each i; so $(PE - F)[\cup P_i(H)] = \{0\}$ where $[\cup P_i(H)]$ is the closed subspace of H generated by $\cup P_i(H)$. Thus $0 = (PE - F)P = PE - FP = PE - F$ which is a contradiction. Thus $(PE - F)P_{i_o} \neq 0$ for some i_o, so $P_{i_o}PE = P_{i_o}E$ ($P_{i_o} \leq P$). From lemma 1.1.22 one has $P_{i_o}F \sim P_{i_o}PE = P_{i_o}E$ ($F \sim PE$). Hence $P_{i_o}E$ is infinite in $\mathcal{A}$ - contrary to the hypothesis.
Thus PE is finite.

■

1.2.7 *LEMMA* ([9], p 414)

Suppose E is an infinite element in $\mathcal{P}(\mathcal{A})$, then a central projection P in $\mathcal{A}$ exists with $P \leq C(E)$; PE is properly infinite, and $(I - P)E$ finite. If E is properly infinite and $F \sim E$, then F is properly infinite.

<u>Proof</u> :

Let $\{Q_i\}_{i \in I}$ be a maximal disjoint family of central projections in $\mathcal{A}$ such that Q_iE is finite for each i ({0} is such family, so the result follows by Zorn). From 1.2.6 QE is finite where $Q = \sum_i Q_i$ (= $\sup_i Q_i$ in the strong operator topology on $\mathcal{A}$, by lemma 1.1.11). Moreover, PE is properly infinite ($P = I - Q$) for if not, there exists by definition a central projection Q_o with $0 < Q_o \leq I - Q$ and $Q_o(I - Q)E = Q_oE$ be finite ($Q_oQ = Q_o(I - Q)Q = 0$). Thus, by adjoining Q_o to $\{Q_i\}$ the maximality of $\{Q_i\}$ will be contradicted.

If E is properly infinite, $F \sim E$, and P is a central projection with $PF \neq 0$ we want to prove that PF is infinite. From lemma 1.1.22 $PF \sim PE \neq 0$. Since E is properly infinite, PE is infinite (by definition), so lemma 1.2.4 implies that PF is infinite. Thus F properly infinite.

■

1.2.8 *PROPOSITION* ([9], p 414)

If E, F are finite element of $\mathcal{P}(\mathcal{A})$, then sup(E,F) is a finite element of $\mathcal{P}(\mathcal{A})$.

<u>Proof</u> :

By corollary 1.1.20 sup(E,F) - F ~ E - inf(E,F). Since E is finite and E - inf(E,F) $\leq$ E we have from lemma 1.2.4 that E - inf(E,F) is finite, and again lemma 1.2.4 implies that sup(E,F) - F is finite. As sup(E,F) = F + (sup(E,F) - F); F and sup(E,F) - F are disjoint, it suffices to show that the sum of two disjoint finite projections in $\mathcal{A}$ is finite.

We assume thus that EF = 0. Suppose E + F is infinite. Then lemma 1.2.7 states that a central projection P in $\mathcal{A}$ exists, such that P(E + F) is properly infinite. Since E and F are finite lemma 1.2.4 implies PE and PF are finite (PE $\leq$ E, PF $\leq$ F). Clearly, PE and PF are disjoint. Thus if we have proved the proposition for PE + PF, then E + F must be finite; otherwise if E + F is infinite, P(E + F) is properly infinite and thus infinite. We may assume, thus, that E + F is properly infinite.

Lemma 1.2.5 shows that there is a subprojection G of E + F such that G ~ E + F - G (= G') ~ E + F. From proposition 1.1.26 there is a central projection Q such that Qinf(G,E) $\precsim$ Qinf(G',F) and (I - Q)inf(G',F) $\precsim$ (I - Q)inf(G,E). Since E + F $\neq$ 0 it follows that either Q(E + F) or (I - Q)(E + F), or both are not equal to zero. If, say, Q(E + F) $\neq$ 0 then Q(E + F) is infinite by definition of properly infinite; while QE and QF are finite and disjoint (lemma 1.2.4). Moreover QG ~ QG' ~ Q(E + F); also by lemma 1.2.4. Since Q and G (resp. Q and inf(G,E)) commutes; lemma 1.1.8 together with Qinf(G,E) $\precsim$ Qinf(G',F) implies that

$$\begin{aligned}\inf(QG,QE) &= \inf(\inf(Q,G),\inf(Q,E)) = \inf(Q,\inf(G,E)) \\ &= Q\inf(G,E) \precsim Q\inf(G',F) = \inf(QG',QF)\end{aligned}$$

Since QE and QF are disjoint and finite and $Q(E + F)$ is infinite it follows that if we have proved the proposition for $Q(E + F)$, it will also hold for $E + F$, otherwise if $E + F$ is properly infinite, then $Q(E + F) \neq 0$ is infinite.

We may assume, thus, that $\inf(G,E) \precsim \inf(G',F)$.

If $(I - Q)(E + F) \neq 0$ and $Q(E + F) = 0$, we have that $(I - Q)(E + F)$ is infinite, while $(I - Q)E$ and $(I - Q)F$ are finite and disjoint. By reversing the roles of E and F (resp. G and G') we may, by using the same argument as above, assume that $\inf(G,E) \precsim \inf(G',F)$.

Since $G - \inf(G,E) \sim \sup(E,G) - E$; $\inf(G,E) \precsim \inf(G',F)$ and the pairs $(G-\inf(G,E), \inf(G,E))$(resp. $(\sup(E,G) - E, \inf(G',F))$ are disjoint we have from corollary 1.1.13 that

$$G = G - \inf(G,E) + \inf(G,E) \precsim \sup(G,E)-E + \inf(G',F) \leq F.$$

We show that $\sup(E,G)-E$ and $\inf(G',F)$ are disjoint subprojections of F. Then $F \geq \sup(\sup(E,G)-E,\inf(G',F)) = \sup(E,G)-E + \inf(G',F)$ (the equality holds by corollary 1.1.9).

Take any vector z in the range of $\inf(G',F)$. Then $z = G'z$ and $z = Fz$. Hence for all $y \in H$ we have that $(G'z,Gy) = (GG'z,y)$ $= (G(E + F - G)z,y) = ((G - G)z,y) = 0$ and $(Fz,Ey) = (EFz,y) = 0$. Thus, every element in the range of $\inf(G', F)$ is orthogonal to both the range of G and of E - hence, to the range of $\sup(E,G)$. Observing that $G \leq E + F$ and $E \leq E + F$, we have $\sup(E,G) \leq E + F$, so $\sup(E,G)-E \leq F$.

We have seen that $G \precsim F$. Hence G is finite by lemma 1.2.4. But $G \sim E + F$ and $E + F$ was assumed to be infinite - contrary to lemma 1.2.4, so $E + F$ is finite.

∎

1.3 INDUCED AND REDUCED VON NEUMANN ALGEBRAS

We shall use reduced algebras to set up a correspondence between properties of algebras and properties of projections in the algebra. Thus if P is a property of projections, we say that an algebra $\mathcal{A}$ has the property P if and only if $I \in \mathcal{A}$ has P. If Q is a property of algebras, we say that a projection $E \in \mathcal{P}(\mathcal{A})$ has property Q if and only if the reduced algebra $\mathcal{A}_E$ has Q.

1.3.1 *DEFINITION* ([5], p 62)

Let $E \in \mathcal{P}(\mathcal{A})$ ($\mathcal{A}$ as von Neumann algebra). Then $E\mathcal{A}E$ is called the reduced algebra of $\mathcal{A}$, and $\mathcal{A}'E$ is called an induced algebra. We shall write T_E for the restriction of ETE to E(H) ($T\in\mathcal{A}$), $\mathcal{A}_E$ for the restriction of the algebra $E\mathcal{A}E$ to E(H).

1.3.2 *PROPOSITION* ([5], p 62)

Let $E \in \mathcal{P}(\mathcal{A})$. Then $E\mathcal{A}E$ and $\mathcal{A}'E$ are von Neumann algebras on E(H) and $\mathcal{A}'E = (E\mathcal{A}E)'$

<u>Proof</u> :

It is clear that $\mathcal{A}_E$ and $(\mathcal{A}')_E = E\mathcal{A}'E = \mathcal{A}'E$ are *-subalgebras of L(E(H)). If we show (i) $(\mathcal{A}')_E = (\mathcal{A}_E)'$ and (ii) $\mathcal{A}_E = ((\mathcal{A}')_E)'$, then it will follow that $\mathcal{A}_E$ and $(\mathcal{A}')_E$ are von Neumann algebras on E(H). This follows because (a) $(\mathcal{A}_E)'' = ((\mathcal{A}_E)')' = ((\mathcal{A}')_E)' = \mathcal{A}_E$ (by (i) and (ii)) and (b) $((\mathcal{A}')_E)'' = ((\mathcal{A}'_E)')' = (\mathcal{A}_E)' = (\mathcal{A}')_E$. Also $(E\mathcal{A}E)' = (\mathcal{A}E)' = (\mathcal{A}')_E = E\mathcal{A}'E = \mathcal{A}'E$ (by (i) and the fact that $T \in \mathcal{A}'$ commutes with E).

Two of the inclusions are easy, namely :

i(a) $(\mathcal{A}')_E \subseteq (\mathcal{A}_E)'$ and ii(a) $\mathcal{A}_E \subseteq ((\mathcal{A}')_E)'$

i(a) The equation $ETET'E = T'ETE^2 = T'EE^2TE = T'EETE$ ($T' \in \mathcal{A}'$, $T \in \mathcal{A}$ and $E \in \mathcal{P}(\mathcal{A})$), implies

$ETE\ ET'E = ETET'E = T'E\ ETE = T'E^2\ ETE = ET'E\ ETE$. Thus $ET'E \in (\mathcal{A}_E)'$ and $(\mathcal{A}')_E \subseteq (\mathcal{A}_E)'$

ii(a) The above equation also implies

$ETE\ ET'E = ETET'E = T'EETE = T'E^2ETE = ET'E\ ETE$, so $ETE \in ((\mathcal{A}')_E)'$. Consequently $\mathcal{A}_E \subseteq ((\mathcal{A}')_E)'$

The converse of ii(a) : Suppose $S_o \in ((\mathcal{A}')_E)'$, then $S_o \in L(E(H))$ and we define $S \in L(H)$ by $Sx = S_oEx$ $(x \in H)$. Since $S_o \in ((\mathcal{A}')_E)'$, S_o commutes with all $ET'E$ $(T' \in \mathcal{A}')$, and in particular with $E = E^2 = EIE$ $(I \in \mathcal{A}')$. Thus if $y = Ex \in E(H)$, $S_Ey = ESE(Ex) = ESEx = ES_oE^2x = S_oE^3x = S_oy$. Thus $S_o = S_E$.

Now $S \in \mathcal{A}$, since for all $T' \in \mathcal{A}'$, $x \in H$, $ST'x = S_oET'x = S_oT'Ex = S_oT'_EEx = T'_ES_oEx = T'ES_oEx = T'Sx$; so $S \in \mathcal{A}'' = \mathcal{A}$. Thus $S_o = S_E \in \mathcal{A}_E$ which implies $((\mathcal{A}')_E)' \subseteq \mathcal{A}_E$

The converse of i(a) : We want to show that $(\mathcal{A}_E)' \subseteq (\mathcal{A}')_E$. Since $\mathcal{A}_E$ is a $*$-subalgebra of $L(E(H))$, $(\mathcal{A}_E)'$ is a $*$-subalgebra of $L(E(H))$ and $(\mathcal{A}_E)''' = (\mathcal{A}_E)'$, so $(\mathcal{A}_E)'$ is a von Neumann algebra, and since any von Neumann algebra is norm-closed in $L(E(H))$, $(\mathcal{A}_E)'$ is a $*$-subalgebra which is norm-closed in $L(E(H))$.

Thus $(\mathcal{A}_E)'$ is a C^*-algebra. It is sufficient to show that if S_o is a unitary element in $(\mathcal{A}_E)'$ then $S_o \in (\mathcal{A}')_E$ (every element in a C^*-algebra is a linear combination of four unitary elements). So if S_o is a unitary element of $(\mathcal{A}_E)'$, then S_o commutes with every $ETE_{|E(H)}$ $(T \in \mathcal{A})$. We wish to find $T' \in \mathcal{A}'$ such that $T'Ex = S_oEx$ $(x \in H)$. Then for $y = Ex \in E(H)$, $S_oy = S_oEx = T'Ex = T'E^3x = ET'Ey = (T')_Ey$, so $S_o \in (\mathcal{A}')_E$. For such a T' we should have

$$T'\left(\sum_{j=1}^{n} T_jEx_j\right) = \sum_{j=1}^{n} T_jT'Ex_j = \sum_{j=1}^{n} T_jS_oEx_j$$

for all $T_1, \ldots T_n \in \mathcal{A}$ $x_1, \ldots x_n \in H$. This defines what T' has to be on the subspace $\mathcal{A}E(H)$ of H. We shall now show that it is possible to define a continuous operator T' in this way on $\mathcal{A}E(H)$. We then extend T' to $\overline{\mathcal{A}E(H)}$ by continuity, and on the whole of H by T': = T'C(E) (we have seen from lemma 1.1.24 that C(E) = $[\mathcal{A}EH]$). Now

$$\|\sum_{j=1}^{n} T_j S_o E x_j\|^2$$
$$= \sum_{i,j=1}^{n} (T_i S_o E x_i \ , \ T_j S_o E x_j)$$
$$= \sum_{i,j=1}^{n} (T_i E S_o E x_i \ , \ T_j E S_o E x_j) \quad \text{(since } S_o(H) \subseteq E(H))$$
$$= \sum_{i,j=1}^{n} (E T_j^* \ T_i S_o E x_i \ , \ S_o E x_j)$$
$$= \sum_{i,j=1}^{n} (S_o E T_j^* T_i E x_i \ , \ S_o E x_j) \quad (S_o \in (\mathcal{A}_E)')$$
$$= \sum_{i,j=1}^{n} (E T_j^* T_i E x_i \ , \ E x_j) \quad (S_o \text{ is unitary, so } S_o^* \ S_o = I)$$
$$= \sum_{i,j=1}^{n} (T_i E x_i \ , \ T_j E x_j)$$
$$= \|\sum_{j=1}^{n} T_j E x_j\|^2 \text{ for } T_1, \ldots T_n \in \mathcal{A}; \ x_1, \ldots x_n \in H.$$

Thus we can define T' on $\mathcal{A}E(H)$ by the equation

$$T'(\sum_{j=1}^{n} T_j E x_j) = \sum_{j=1}^{n} T_j S_o E x_j .$$

By the above argument T' so defined is an isometry and thus continuous. Thus we can define T' on H as described above (For T' = T'C(E) on H, we have T'(Ex) = T'C(E) Ex = T'Ex = $S_o E(x)$ on E(H)).

To prove that T' $\in \mathcal{A}'$, it suffices to show for all R $\in \mathcal{A}$, T'Rx = RT'x (x$\in$H). Now for x = TEy (T$\in\mathcal{A}$, y$\in$H) we have RT'(TEy) = RTS_oEy (by definition of T' on $\mathcal{A}E(H)$) = T'RTEy (by definition of T', n=1 and T_1 = RT) Thus RT' = T'R on $\mathcal{A}E(H)$.
If x$\in$ $(\mathcal{A}E(H))^\perp$, RT'x = RT'C(E)x = 0 = T'RC(E)x (C(E) = $[\mathcal{A}E(H)]$ and x $\in$ $(\mathcal{A}E(H))^\perp$). So we have found a T'$\in\mathcal{A}'$ such that S_o = $T'E_{|E(H)}$. Hence $S_o \in \mathcal{A}'E = E\mathcal{A}'E = (\mathcal{A}')_E$. This completes the proof.

■

1.3.3 *NOTATION*

We write $E \lesssim F$ for $E \lesssim F$ and $E \sim F$.

1.3.4 *PROPOSITION* ([15], p 90)

Let $\mathcal{A}$ be a finite von Neumann algebra, and let E, E_1; F, F_1 be projections in $\mathcal{A}$ satisfying the following conditions : $E_1 \leq E$, $F_1 \leq F$, $E_1 \sim F_1$ and $E \sim F$. Then $E - E_1 \sim F - F_1$.

<u>Proof</u> :

By the comparability proposition (1.1.26), a central projection Q in $\mathcal{A}$ exists such that $(E - E_1)Q \lesssim (F - F_1)Q$ and $(E - E_1)(I-Q) \gtrsim (F - F_1)(I - Q)$. Suppose $(E - E_1)Q \lesssim (F - F_1)Q$; then a $F' \in \mathcal{P}(\mathcal{A})$ exists with $(E - E_1)Q \sim F' < (F - F_1)Q$. Since $E_1 \sim F_1$ lemma 1.1.22 implies $QE_1 \sim QF_1$ and since E_1Q and $(E - E_1)Q$ (resp F' and F_1Q) are disjoint, proposition 1.1.12 and its corollary imply that
$EQ = (E - E_1)Q + E_1Q \sim F' + F_1Q < (F - F_1)Q + F_1Q = FQ$;
Again, from lemma 1.1.22 $EQ \sim FQ$ $(E \sim F)$.

Thus $FQ \sim EQ \sim F' + F_1Q < FQ$, which contradicts the fact that $FQ \leq F$ is finite ($\mathcal{A}$ is finite and $F \leq I$). Hence $(E - E_1)Q \sim (F - F_1)Q$. Similarly $(E - E_1)(I-Q) \sim (F - F_1)(I - Q)$. By applying proposition 1.1.12 on the disjoint pairs $((E - E_1)Q; (E - E_1)(I - Q))$ and $((F - F_1)Q; (F - F_1)(I - Q))$ we have $E - E_1 \sim F - F_1$.

■

1.3.5 *PROPOSITION* ([4], p 261)

Let E, F be finite elements of $\mathcal{P}(\mathcal{A})$. Then

(i) $E \sim F$ if and only if there is a unitary element U of $\mathcal{A}$ such that $UEU^* = F$.

(ii) If $E \sim F$ and if $G \geq \sup(E,F)$, then a unitary element of U of $\mathcal{A}$ exists such that $U^*EU = F$ and $U^*GU = G$.

<u>Proof</u> :

(i) Let $G_1 = \sup(E,F)$. Then G_1 is finite by proposition 1.2.8. Now $E \leq G_1$, $F \leq G_1$ and $E \sim F$, thus proposition 1.3.4 implies $G_1 - E \sim G_1 - F$ (Consider $\mathcal{A}_{G_1}$ if $\mathcal{A}$ is not finite).

Let V and W be partial isometries of $\mathcal{A}$ with $V^*V = E$, $VV^* = F$ and $W^*W = G_1 - E$, $WW^* = G_1 - F$.

Define U to be the operator which agrees with V on E(H), with W on $(G_1 - E)(H)$ and with I on $(I - G_1)(H)$.

We show that $U \in \mathcal{A}$. If $x \in E(H)$, $T' \in \mathcal{A}'$ we have from $Ex = x$ that $T'Ux = T'Vx = VT'x = VT'Ex = VET'x = UET'x = UT'Ex = UT'x$.

If $x \in (G_1 - E)(H)$ or $x \in (I - G_1)(H)$ it follows similarly that $T'U = UT'$ $(T' \in \mathcal{A}')$, and since $H = E(H) \oplus (G_1 - E)(H) \oplus (I - G_1(H))$ we have that $T'Ux = UT'x$ for all $x \in H$. This implies $U \in \mathcal{A}'' = \mathcal{A}$.

It is also clear that $\|Ux\| = \|x\|$ for all $x \in H$ and U is surjective $(U(H) = F(H) \oplus (G_1 - F)(H) \oplus (I - G_1)(H) = H)$.

So, U is a unitary element of $\mathcal{A}$.

If $x \in E(H)$ one has $UEx = VEx = VV^*Vx = FVx = FUx$.
If $x \in (G_1 - E)(H)$: $UEx = 0 = FWx = FUx$. If $x \in (I - G_1)(H)$, then $UEx = 0 = FIx = FUx$.
Thus $UE = FU$ or $UEU^* = F$ $(U^* = U^{-1})$

Conversely, if a unitary element $U \in \mathcal{A}$ exists such that $UEU^* = F$ we want to show that $E \sim F$. We have that

$(UE)^*(UE) = EU^*UE = EIE = E$; and $(UE)(UE)^* = UEU^* = F$. Thus $E \sim F$ by partial isometry UE (see lemma 1.1.6).

(ii) Suppose $G \geq G_1 = \sup(E,F)$. Since $E \sim F$ a unitary element U of $\mathcal{A}$, as constructed in (i), exists such that $UEU^* = F$. We use the same notation as in (i) to show that $UG = GU$.

If $x \in E(H)$, $UGx = UGEx = UEx = FUx = FVx$

$= GFVx \qquad (G \geq F)$

$= GVx \qquad (F(H) = V(H)$, so $FV = V)$

$= GUx$

If $x \in (G_1 - E)(H)$, $UGx = U(G_1 - E)x = Ux = Wx$

Since $Wx \in (G_1 - F)(H)$ we have $(G_1 - F)Wx = Wx$;

so $GWx = G(G_1 - F)Wx = (G_1 - F)Wx = Wx = UGx$.

Thus $GUx = UGx$. If $x \in (I - G_1)(H)$, $UGx = U(G - G_1)x$

$= I\,(G - G_1)x = (G - G_1)x = GIx = GUx$. So $UGU^* = G$.

■

1.3.6 *COROLLARY* ([1])

If E, F are finite elements of $\mathcal{P}(\mathcal{A})$, then $E \sim F$ implies

$$I - E \sim I - F$$

<u>Proof</u> :

Since $E \sim F$, there exists a unitary $U \in \mathcal{A}$ with $UEU^* = F$. Then $U\,(I - E)U^* = UU^* - UEU^* = I - F$. Since

$[U(I - E)][U(I - E)]^* = U(I - E)U^* = I - F$ and

$[U(I - E)]^*\,[U(I - E)] = (I - E)I(I - E) = I - E$, $I - E \sim I - F$ by partial isometry $U(I - E)$.

■

1.3.7 *COROLLARY* ([9], p 448)

If E, $F \in \mathcal{P}(\mathcal{A})$ are finite and E_1, F_1 are subprojections of E, F with $E_1 \sim F_1$; then $E \precsim F$ implies $E - E_1 \precsim F - F_1$.

<u>Proof</u> :

Since $E_1 \sim F_1$ there exists by proposition 1.3.5 a unitary operator $U \in \mathcal{A}$ with $E_1 = UF_1U^*$.

Since $E \precsim F$ we can choose an $E' \in \mathcal{P}(\mathcal{A})$ with $E \sim E' \leq F$. We claim that $F_1 \precsim E'$. This follows from $F_1 \sim E_1 \leq E \sim E'$. Thus $F_1 \precsim E \precsim E'$, which implies $F_1 \precsim E'$. This means that there exists an F' such that $F_1 \sim F' \leq E'$. From $E_1 = UF_1U^* \sim F_1 \sim F'$ and $E \sim E'$ proposition 1.3.4 implies that
$E - E_1 \sim E' - F' \leq F - F' \sim F - F_1$ ($E' - F' \leq F - F'$ since
$(E' - F')(F - F') = E'F - E'F' - F'F + F' = E' - F'$)
Hence $E - E_1 \precsim F - F_1$.

■

The following proposition is of great importance in the next chapter where we will construct the so-called index group of a von Neumann algebra $\mathcal{A}$.

1.3.8 *PROPOSITION* (Cancellation law, [1])

Let (E_1, E_2); (F_1, F_2) be two pairs of finite projections of $\mathcal{A}$, and let $E_1E_2 = F_1F_2 = 0$. Then $E_1 \sim F_1$ and $E_1 + E_2 \sim F_1 + F_2$ imply $E_2 \sim F_2$.

<u>Proof</u> :

Since $E_1 + E_2 = \sup(E_1,E_2)$ and $F_1 + F_2 = \sup(F_1,F_2)$ (Corollary 1.1.9), proposition 1.2.8 implies that $E_1 + E_2$ and $F_1 + F_2$ are finite.

Part (i) of proposition 1.3.5 implies that there is a unitary element U of $\mathcal{A}$ with $U^*(E_1 + E_2)U = F_1 + F_2$ $(E_1 + E_2 \sim F_1 + F_2)$. Since $(E_1U)^*(E_1U) = U^*E_1U$ and $(E_1U)(E_1U)^* = E_1UU^*E_1 = E_1$, we have that $U^*E_1\ U \sim E_1$ by the partial isometry $E_1U \in \mathcal{A}$. From $E_1 \sim F_1$ we have $U^*E_1U \sim F_1$ and by part (ii) of 1.3.5 there exists a unitary element $V \in \mathcal{A}$ with $V^*F_1V = U^*E_1U$, and $V^*(F_1 + F_2)V = F_1 + F_2$ (choose $F_1 + F_2$ to be G in 1.3.5 (ii)).

Recalling that $U^*(E_1 + E_2)U = F_1 + F_2$, we have $U^*(E_1 + E_2)U = V^*(F_1 + F_2)V$, which implies that $U^*E_2U = V^*F_2V$.

Using part (i) of proposition 1.3.5 on this relation, one gets $E_2 \sim F_2$.

■

1.4 CHARACTERIZATION OF A FINITE VON NEUMANN ALGEBRA IN TERMS OF TRACES

As defined before, a von Neumann algebra $\mathcal{A}$ is called finite if its unit element is a finite projection of $\mathcal{A}$. After we have defined what we mean by a finite normal trace on $\mathcal{A}$, we will show that a finite von Neumann algebra can also be defined in terms of traces on $\mathcal{A}$. It is well known that a von Neumann algebra $\mathcal{A}$ can be considered as the dual space of a Banach space $\mathcal{A}_*$. For the benefit of the reader, an appendix, in which a few basic properties of several useful locally convex topologies defined on $\mathcal{A}$ are summarized, is included. (Chapter 5, 5.1). These results will be used without additional reference.

The concept of a trace on a von Neumann algebra $\mathcal{A}$ and in particular the existence of a finite normal trace in a finite von Neumann algebra $\mathcal{A}$ is developed by F.J. Murray and J. von Neumann. The recent proof, due to Yeadon, can be found in [18].

We begin with the following definitions :

1.4.1 *DEFINITION* ([8], p 338)

Let $\mathcal{A}$ be a von Neumann algebra. Then $\mathcal{A}$ is called countably decomposable if every family of pairwise disjoint projections in $\mathcal{A}$ is countable.

1.4.2 *DEFINITION* ([18], p 309)

Let $\mathcal{A}^+ = \{S^*S;\ S \in \mathcal{A}\}$ be the positive part of a von Neumann algebra $\mathcal{A}$.

A trace on $\mathcal{A}$ is a function ϕ defined on $\mathcal{A}^+$, taking non-negative, extended real values, possessing the following properties :

(i) If $S \in \mathcal{A}^+$ and $T \in \mathcal{A}^+$, we have $\phi(S+T) = \phi(S) + \phi(T)$

(ii) If $S \in \mathcal{A}^+$ and λ a non-negative real number we have $\phi(\lambda S) = \lambda\phi(S)$ (with the usual convention that $0(+\infty) = 0$).

(iii) If $T \in \mathcal{A}$ we have $\phi(T^*T) = \phi(TT^*)$

We say that ϕ is faithful if the conditions $S \in \mathcal{A}^+$, $\phi(S) = 0$ imply that $S = 0$; finite if $\phi(I) < +\infty$, semifinite if for every non-zero $T \in \mathcal{A}^+$, there exists a non-zero element S in $\mathcal{A}^+$ with $\phi(S) < +\infty$ and $S \leq T$.

We say that ϕ is normal if $\phi\,(\sup_i T_i) = \sup_i \phi(T_i)$ for every uniformly bounded increasing net $\{T_i\}_{i \in I}$ in $\mathcal{A}^+$.

1.4.3 *PROPOSITION* (Monotone convergence, [8], p 307)

If $\{T_i\}_{i \in I}$ is a monotone increasing net of self-adjoint operators in $\mathcal{A}$ and $T_i \leq kI$ for all $i \in I$ and k a constant, then $\{T_i\}$ is strong operator convergent to a self-adjoint operator T, thus $T \in \mathcal{A}$ and T is the least upper bound of $\{T_i\}$.

Proof :

Since the convergence of $\{T_i\}_{i\in I}$ and that of $\{T_i, i \geq i_o\}$ are equivalent we may assume that $\{T_i\}$ is bounded below (by T_{i_o}) as well as above. Thus $-\|T_{i_o}\|I \leq T_i \leq kI$, and so $\{T_i\}$ is a bounded set of operators. Since a closed ball S in L(H) is weak-operator compact (Banalch Aloaglu, [6]), and $\mathcal{A}$ is weak-operator closed one has $\mathcal{A}\cap S$ is weak-operator closed in S and thus weak-operator compact. If $\{T_i\} \subseteq \mathcal{A}\cap S$ a subnet which we again denote by $\{T_i\}$ exist which is weak operator convergent to a T in L(H). Since $\mathcal{A}$ is weak-operator closed, $T\in\mathcal{A}$. As $\{T_i\}$ is monotone increasing $(T_\ell x,x) \geq (T_m x,x)$ when $\ell \geq m$ and $x\in H$. Since $(Tx,x) = \lim_\ell (T_\ell x,x) \geq (T_m x,x)$ for all $x\in H$ we have that $T \geq T_\ell$ for all ℓ (the order relation is to be interpreted in the operator sense). If $i \geq \ell$ then $0 \leq T - T_i \leq T - T_\ell$, and
$0 \leq ((T - T_i)x,x) = \|(T - T_i)^{1/2} x\|^2 \leq ((T - T_\ell)x,x)$.
Hence $\{(T - T_i)^{1/2}\}$ is strong operator convergent to zero. The strong operator continuity of multiplication on bounded sets of operators allows us to conclude that $\{T - T_i\}$ is strong operator convergent to 0. We have noted that T is an upper bound for $\{T_i\}$. If $S \geq T_i$ for all i, then $(Sx,x) \geq (T_i x,x) \xrightarrow{i} (Tx,x)$. Hence $(Sx,x) \geq (Tx,x)$ for all $x\in H$ so $S \geq T$. Therefore T is the least upper bound of $\{T_i\}$.

■

1.4.4 *COROLLARY*

If $\{T_i\}$ is a monotone increasing net of self-adjoint operators in $\mathcal{A}$ which is uniformly bounded and T is the least upper bound of $\{T_i\}$. Then S^*TS is the least upperbound for $\{S^*T_iS\}$ $(S\in\mathcal{A})$

Proof :

Since $\{T_i\}$ is a monotone increasing net of self-adjoint operators in $\mathcal{A}$ which is uniformly bounded, $\{S^*T_iS\}$ is a monotone increasing, self-adjoint, uniformly bounded net of operators in $\mathcal{A}$. By proposition 1.4.3 $\{S^* T_i S\}$ has a least upper bound, say $P \in \mathcal{A}$. In the proof of 1.4.3 $\{S^* T_i S\}$ is weak-operator convergent to P. From $(S^*(T_i - T)Sx,x) = ((T_i - T)Sx,Sx) \to 0$ $(x \in H)$, we have that $\{S^* T_i S\}$ converges weakly to S^*TS. Hence $P = S^*TS$.

■

1.4.5 *DEFINITION* ([5], p 36 and p 42)

1. A positive linear functional ϕ on a von Neumann algebra $\mathcal{A}$ is said to be normal if it satisfies $\phi(\sup_\alpha T_\alpha) = \sup_\alpha \phi(T_\alpha)$ for every uniformly bounded increasing directed set $\{T_\alpha\}$ of positive elements in $\mathcal{A}$.

2. Let $\{E_\alpha\}$ be any family of mutually disjoint projections in $\mathcal{A}$. If ϕ is a norm-continuous linear functional on $\mathcal{A}$, then ϕ is said to be completely additive if $\phi(\sum_\alpha E_\alpha) = \sum_\alpha \phi(E_\alpha)$ where $\sum_\alpha E_\alpha$ converges in the strong-operator topology on $\mathcal{A}$.

1.4.6 *REMARK*

It is well known that the σ-weakly continuous functionals on $\mathcal{A}$ are precisely those which are completely additive; and for a state (a positive linear functional with norm 1) σ-weak continuity, normality and complete additivity are equivalent. (This is proved in chapter 5 paragraph 5.2). An important consequence of this fact is the characterization of the σ-weakly relative compact subsets of the predual $\mathcal{A}_*$ of a von Neumann algebra $\mathcal{A}$.

1.4.7 *LEMMA* ([17], p 117)

Consider a von Neumann algebra $\mathcal{A}$ with predual $\mathcal{A}_*$ and let $F \subset \mathcal{A}_*$ be a norm bounded subset. The following assertions are equivalent :

(i) F is $\sigma(\mathcal{A}_*,\mathcal{A})$-relatively compact (i.e. $\bar{F}$, taken in the $\sigma(\mathcal{A}_*,\mathcal{A})$ topology, is $\sigma(\mathcal{A}_*,\mathcal{A})$ compact).

(ii) For any countable family $\{E_n\}$ of mutually disjoint projections in $\mathcal{A}$, one has that $\phi(E_n) \to 0$ uniformly for $\phi \in F$ (i.e. for every $\epsilon > 0$ an n_o exists such that $|\phi(E_n)| < \epsilon$ for every $n \geq n_o$ and $\phi \in F$).

Proof :

We show that (ii) implies (i) :

Since F is a bounded subset of $\mathcal{A}_* \subseteq \mathcal{A}^*$, it follows that $F \subseteq B_r$, B_r a norm-closed ball in $\mathcal{A}^*$ which is weak*-compact by Banach-Aloaglu. So $\bar{F} \subseteq B_r$ (the closure is taken in the $\sigma(\mathcal{A}^*,\mathcal{A})$ topology). Hence $\bar{F}$ is a $\sigma(\mathcal{A}^*,\mathcal{A})$-closed subset of the $\sigma(\mathcal{A}^*,\mathcal{A})$-compact ball and is therefore $\sigma(\mathcal{A}^*,\mathcal{A})$-compact. If we can show that $\bar{F} \subseteq \mathcal{A}_*$, it will follow that $\bar{F}$ is $\sigma(\mathcal{A}_*,\mathcal{A})$-compact since the $\sigma(\mathcal{A}_*,\mathcal{A})$ topology on $\mathcal{A}_*$ is simply the restriction of the $\sigma(\mathcal{A}^*,\mathcal{A})$ topology to $\mathcal{A}_*$.

Therefore, let $\phi \in \bar{F}$, then a net $\{\phi_k\}_{k \in K} \subseteq F$ which is $\sigma(\mathcal{A}^*,\mathcal{A})$ convergent to ϕ exists.

Let $\{E_i\}_{i \in I}$ be a family of mutually disjoint projections in $\mathcal{A}$ and let $E = \sum_{i \in I} E_i$. Since $\phi_k \to \phi$ in the $\sigma(\mathcal{A}^*,\mathcal{A})$-topology on $\mathcal{A}^*$ if and only if $\phi_k(T) \to \phi(T)$ for every $T \in \mathcal{A}$ it follows that :

$$\phi(E) = \lim_{k \in K} \phi_k(E)$$

and

$$\phi(E_i) = \lim_{k \in K} \phi_k(E_i) \text{ for any } i \in I$$

Since each $\phi_k \in F \subset \mathcal{A}_*$ is σ-weakly continuous it is completely additive by remark 1.4.6. Thus $\phi_k(E) = \sum_{i \in I} \phi_k(E_i)$ uniformly for $k \in K$. In fact we have $\psi(E) = \sum_{i \in I} \psi(E_i)$ uniformly for $\psi \in F$. If this is not true, a $\delta > 0$ exists such that for any finite subset $J \subset I$ we can find a $\psi_J \in F$ such that

$|\sum_{I\setminus J}\varphi_J(E_i)| = |\sum_I \varphi_J(E_i) - \sum_J \varphi_J(E_i)| \geq 2\delta$. Since for every $\varphi \in F$

$\sum_{I\setminus J}\varphi(E_i)$ converges, there exists a finite subset $H \subset I\setminus J$ such that

$|\sum_{I\setminus(J\cup H)}\varphi_J(E_i)| \leq \delta$. Hence

$$2\delta \leq |\sum_{I\setminus J}\varphi_J(E_i)| = |\sum_H \varphi_J(E_i) + \sum_{I\setminus(J\cup H)}\varphi_J(E_i)| \leq |\sum_H \varphi_J(E_i)| + \delta$$

So $|\sum_H \varphi_J(E_i)| \geq \delta$

Consider the $\delta > 0$ as above : We have seen that there exists a finite subset $J_1 \subset I$ and a $\varphi_1 \in F$ with $|\sum_{J_1}\varphi_1(E_i)| \geq \delta$. By considering $I\setminus J_1$ instead of I we can similarly find a finite subset $J_2 \subset I\setminus J_1$ and a $\varphi_2 \in F$ such that $|\sum_{J_2}\varphi_2(E_i)| \geq \delta$. Thus, proceeding in this way, one can construct, for the given $\delta > 0$, a sequence $\{\varphi_n\} \subset F$ and a sequence (J_n) of finite mutually disjoint subsets $J_n \subset I$ such that for every n we have

$$|\sum_{i\in J_n}\varphi_n(E_i)| \geq \delta.$$

Define $F_n = \sum_{i\in J_n} E_i$. Since the subsets J_n are mutually disjoint we have that $\{F_n\}$ is a countable family of mutually disjoint projections in $\mathcal{A}$, and for every n we have

$|\varphi_n(F_n)| = |\sum_{i\in J_n}\varphi_n(E_i)| \geq \delta$. Thus for the sequence $\{F_n\}_n$ a $\delta \geq 0$ exists such that for every n we can find a $\varphi_n \in F$ with $|\varphi_n(F_n)| \geq \delta$; which, in view of (ii), is a contradiction.

Thus $\phi_k(E) = \sum_{i\in I}\phi_k(E_i)$ uniformly for $k\in K$, which implies that for every $\epsilon > 0$ a finite subset $J \subset I$ exists, such that for every finite subset $H \supset J$ of I, $|\phi_k(E) - \sum_{i\in H}\phi_k(E_i)| < \epsilon$ for every $k \in K$. It

therefore follows that $|\phi_k(\sum_{i\in I\setminus H} E_i)| < \epsilon$ for every finite subset

$H \supset J$ and $k \in K$. Since $\phi = \lim_k \phi_k$ in the $\sigma(\mathcal{A}^*, \mathcal{A})$-topology if and only if $\phi(T) = \lim_k \phi_k(T)$ $(T \in \mathcal{A})$, it follows that

$\epsilon > |\phi(\sum_{i \in I \setminus H} E_i)| = |\phi(E) - \sum_{i \in H} \phi(E_i)|$ for every finite subset

$H \supset J$ of I. Hence $\phi(E) = \sum_{i \in I} \phi(E_i)$.

This shows that ϕ is completely additive and therefore remark 1.4.6 implies that ϕ is σ-weakly continuous, thus $\phi \in \mathcal{A}_*$.

We now prove the converse. If condition (ii) holds, one has for any countable family $\{E_n\}$ of mutually disjoint projections in $\mathcal{A}$ and for any $\epsilon > 0$, that a N_o exists such that for every $\phi \in F$ and $n > N_o$ $|\phi(E_n)| < \epsilon$. The proof is by contradiction : If condition (ii) is not true, a $\delta > o$ and a sequence $\{E_n\}$ of mutually disjoint projections in $\mathcal{A}$ exist, such that for every $n \in N$ one can choose a $\phi_n \in F$ with $|\phi_n(E_n)| \geq 4\delta$.

Since F is $\sigma(\mathcal{A}_*, \mathcal{A})$-relatively compact in $\mathcal{A}_*$, the sequence $\{\phi_n\}$ has a convergent subsequence with limit $\phi \in \mathcal{A}_*$ (see appendix 5.1, the Eberlein-Smulain theorem (5.1.1). We denote this subsequence again by $\{\phi_n\}$ and the corresponding subsequence of (E_n) by (E_n). (This convergence takes place in the $\sigma(\mathcal{A}_*, \mathcal{A})$ topology on $\mathcal{A}_*$). If we define $P_n = \sum_{k=n}^{\infty} E_k$ we have $E_n \leq P_n$ for all n. Clearly (P_n) is a decreasing sequence of projections and since $E = P_n + \sum_{k=1}^{n-1} E_k$ we

have, by taking limits in the strong-operator topology, that (P_n) converges to zero. Since $E_n \leq P_n$ for all n, $E_n \to 0$ strongly, hence weakly (the strong-operator topology is finer than the weak-operator topology on $\mathcal{A}$). Since the weak-operator topology and the σ-weak operator topology are the same on bounded parts of $\mathcal{A}$, one has that $E_n \to 0$ σ-weakly. Since $\phi \in \mathcal{A}_*$, it is σ-weakly continuous and we have that $\phi(E_n) \to 0$. Observing that $|\phi(E_n)| \leq \delta$ for every n, except for a finite number, we may assume that $|\phi(E_n)| \leq \delta$ for every n. The sequence of forms $\psi_n = \phi_n - \phi \in \mathcal{A}_*$ is $\sigma(\mathcal{A}_*, \mathcal{A})$-convergent to 0 since for every $T \in A$ $\psi_n(T) = \phi_n(T) - \phi(T) \underset{\infty}{\overset{n}{\to}} 0$.

From this it follows that $|\varphi_n(E_n)| = |\phi_n(E_n) - \phi(E_n)|$

$$\geq |\phi_n(E_n)| - |\phi(E_n)|$$

$$\geq 4\delta - \delta = 3\delta \qquad (1.3)$$

We shall now construct an increasing sequence $\{n(1);\ n(2);\ \ldots\}$ of natural numbers with the following properties.

$$|\sum_{j=1}^{k-1} \varphi_{n(k)}(E_{n(j)})| < \delta \text{ for any } k = 2,3,4,\ldots \qquad (1.4)$$

$$\sum_{j=n(k+1)}^{\infty} |\varphi_{n(k)}(E_j)| < \delta \text{ for any } k = 1,2,3,\ldots \qquad (1.5)$$

In order to do this, let us first observe that for any $\varphi \in \mathcal{A}_*$ we have

$$\sum_{n=1}^{\infty} |\varphi(E_n)| < +\infty \qquad (1.6)$$

To see this, let $|\varphi(E_n)| = \lambda_n \varphi(E_n)$ where λ_n is a scalar with absolute value one. We claim that $\sum_{n=1}^{\infty} \lambda_n E_n$ is σ-weakly convergent.

For any $x \in H$, consider

$$\|\sum_{n=1}^{\infty} \lambda_n E_n x\|^2 \qquad (x \in H)$$

$$= (\sum_{n=1}^{\infty} \lambda_n E_n x,\ \sum_{k=1}^{\infty} \lambda_k E_k x)$$

$$= \sum_{n=1}^{\infty} (\lambda_n E_n x, \lambda_n E_n x) \qquad (E_n E_k = 0 \text{ for all } k \neq n)$$

$$= \sum_{n=1}^{\infty} |\lambda_n|^2 (E_n x, x)$$

$$= \sum_{n=1}^{\infty} (E_n x, x) = \|\sum_{n=1}^{\infty} E_n x\|^2$$

Since $\sum_{n=1}^{\infty} E_n$ is strong-operator convergent by lemma 1.1.11, it follows that $\sum_{n=1}^{\infty}\lambda_n E_n$ converges strongly, hence weakly and since the convergence take place on bounded parts of $\mathcal{A}$, σ-weakly. Thus for every $\varphi\in\mathcal{A}_*$ we get

$$\lim_{n\to\infty} \varphi(\sum_{k=1}^{n}\lambda_k E_k) = \varphi(\sum_{k=1}^{\infty}\lambda_k E_k) < \infty$$

Hence $\sum_{k=1}^{\infty}\lambda_k\varphi(E_k) = \varphi(\sum_{k=1}^{\infty}\lambda_k E_k) < \infty$. So, $\sum_{k=1}^{\infty}|\varphi(E_k)| < +\infty$

We begin the construction of the relations (1.4) and (1.5) by taking n(1) = 1 and we assume that n(1), ..., n(p-1) have already been constructed, such that condition (1.4) be satisfied for k = 2, ..., p-1; whereas condition (1.5) be satisfied for k = 1, ... (p-2).

We now show that relation (1.4) is satisfied for k = p whereas relation (1.5) is satisfied for k = p-1. Since $\{\varphi_n\}$ is $\sigma(\mathcal{A}_*,\mathcal{A})$-convergent to 0 and since $\sum_{j=1}^{\infty}|\varphi_{n(p-1)}(E_j)| < +\infty$ by (1.6) we have, for a sufficiently great n_o, the following inequalities:

$$|\sum_{j=1}^{p-1}\varphi_{n_o}(E_{n(j)})| < \delta$$

and

$$\sum_{j=n_o}^{\infty}|\varphi_{n(p-1)}(Ej)| < \delta$$

(remember $\varphi_n \to 0$ in the $\sigma(\mathcal{A}_*,\mathcal{A})$ topology implies

$$\varphi_n(\sum_{j=1}^{p-1}E_{n(j)}) \underset{\infty}{\overset{n}{\to}} 0)$$

Hence, by choosing $n_o = n(p) > n(p-1)$ to be sufficiently great, relation (1.4) is satisfied for k = p, whereas (1.5) is satisfied for k = p - 1.

The required construction is thus possible by induction. From relation (1.5) it follows that

$$\sum_{j=k+1}^{\infty}|\varphi_{n(k)}(E_{n(j)})| < \delta,\ k = 1,2,3... \qquad (1.7)$$

Consider the projection $F = \sum_{j=1}^{\infty} E_{n(j)} \in \mathcal{P}(\mathcal{A})$. Since $(E_{n(j)})_{j=1}^{\infty}$ is a disjoint sequence of projections and each $\varphi_{n(k)}$ is completely additive we have

$$\varphi_{n(k)}(F) = \sum_{j=1}^{\infty} \varphi_{n(k)}(E_{n(j)}) \quad k = 1,2,...$$

From relations (1.3), (1.4) and (1.7) we get

$$|\varphi_{n(k)}(F)|$$

$$= |\sum_{j=1}^{\infty} \varphi_{n(k)}(E_{n(j)})|$$

$$= |\sum_{j=1}^{k-1} \varphi_{n(k)}(E_{n(j)}) + \varphi_{n(k)}(E_{n(k)}) + \sum_{j=k+1}^{\infty} \varphi_{n(k)}(E_{n(j)})|$$

$$\geq -|\sum_{j=1}^{k-1} \varphi_{n(k)}(E_{n(j)})| + |\varphi_{n(k)}(E_{n(k)}) + \sum_{j=k+1}^{\infty} \varphi_{n(k)}(E_{n(j)})|$$

$$> - \delta + |\varphi_{n(k)}(E_{n(k)})| - |\sum_{j=k+1}^{\infty} \varphi_{n(k)}(E_{n(j)})|$$

$$\geq - \delta + 3\delta - \delta = \delta \; ; \quad k = 1,2,3...$$

This contradicts the fact that the sequence $\{\varphi_n\}$ is $\sigma(\mathcal{A}_*,\mathcal{A})$-convergent to 0.

■

Consider the von Neumann algebra $\mathcal{A}$. Let T be a hermitian element of $\mathcal{A}$. The spectral decomposition theorem ([11], p 505) then states that a family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of projections commuting with T exists where each E_λ is defined as

$E_\lambda = N_{T_\lambda^+}$ ($T_\lambda^+ = (T-\lambda I)^+ = \frac{1}{2}([(T-\lambda I)(T-\lambda I)^*]^{1/2} - (T-\lambda I))$ and $N_{T_\lambda^+}$ is the projection onto the null space of T_λ^+). This family has the following properties :

(i) If $\lambda \leq \lambda'$ $\quad E_\lambda \leq E_{\lambda'}$

(ii) $E_\lambda = 0$ if $\lambda < m = \inf\{\lambda | \lambda \in Sp(T)\}$ and $E_\lambda = I$ if $\lambda \geq M = \sup\{\lambda | \lambda \in Sp(T)\}$, where $Sp(T)$ denotes the spectrum of the operator T.

(iii) $\mu \to \lambda+0$ then $E_\mu x \to E_\lambda x$ $(x \in H)$. Hence (i) implies that $E_\lambda = \inf_{\lambda' > \lambda} E_{\lambda'}$.

(iv) For each λ, $TE_\lambda \leq \lambda E_\lambda$ and $\lambda(I-E_\lambda) \leq T(I - E_\lambda)$

(v) $T = \int_{m-o}^{M} \lambda dE_\lambda$ where the integral is to be understood in the sense of uniform operator convergence. Since each E_λ commutes with every S commuting with T we have $E_\lambda S = SE_\lambda$ for all $S \in \mathcal{A}'$. Thus $E_\lambda \in \mathcal{A}'' = \mathcal{A}$ for each λ. If $T \in \mathcal{A}^+$, $N_{T_o^+} = N_T = E_o$. So (iii) and remark 1.1.16 imply that $S(T) = S_\ell(T) = S_r(T) = I-N_T = \sup_{\lambda' > 0}(I-E_{\lambda'})$. So we can choose an increasing sequence $(E_n') \subset \mathcal{P}(\mathcal{A})$ with $S(T) = \sup_n E_n'$ and $TE_n' \geq \frac{1}{n} E_n'$ (Let $E_n' = I-E_{1/n}$).

Let ϕ be a normal positive ($\phi(T) \geq 0$ if $T \geq 0$) linear functional on $\mathcal{A}$ then $\phi \in \mathcal{A}_*^+$ by remark 1.4.6. Since $[E(H) \cup F(H)] = \overline{(E+F)(H)}$ where $[K]$ is the closed subspace of H generated by $K \subset H$, one has $S(E+F) = \sup(E,F)$ (see also remark 1.1.3). If $T \geq 0$ and $\phi(T) = 0$ we claim that $\phi(S(T)) = 0$. Indeed, as we have seen from the above, an increasing sequence $\{E_n\} \subset \mathcal{P}(\mathcal{A})$ exists with $\sup_n E_n = S(T)$ and $TE_n \geq \frac{1}{n} E_n$ for any n. Since $T-TE_n = T(I-E_n) \geq 0$ (T and $I-E_n$ commutes), one has $\phi(\frac{1}{n} E_n) \leq \phi(TE_n) \leq \phi(T) = 0$. So $\phi(E_n) = 0$ for all n. Thus $\phi(S(T)) = 0$ (ϕ is normal). Observing that $S(E+F) = \sup(E,F)$ it follows that $\phi(\sup(E,F)) = 0$, if $\phi(E)$ and $\phi(F)$ are zero, since then $\phi(E+F) = 0$, hence $\phi(S(E+F)) = 0$. Consequently, the family $\{E \in \mathcal{P}(\mathcal{A});\ \phi(E) = 0\}$ is increasingly directed and, therefore, by denoting by $I - S(\phi)$ the supremum of this family we have $\phi(I - S(\phi)) = 0$ (ϕ is normal).

1.4.8 *DEFINITION* ([17], p 119)

(i) The projection $S(\phi)\in\mathcal{P}(\mathcal{A})$ is called the support of ϕ.
(ii) One says that ϕ is faithful if $S(\phi) = I$.

1.4.9 *REMARKS* ([17], p 119)

(i) Using the Schwarz inequality for positive linear functionals we obtain $\phi(T - TS(\phi)) \leq \phi(TT^*)^{1/2}\ \phi(I - S(\phi)) = 0$. Similarly $\phi(T - S(\phi)T) = 0$. Thus $\phi(T) = \phi(TS(\phi)) = \phi(S(\phi)T)$. If $T \in \mathcal{A}^+$ and $\phi(T) = 0$, one has $\phi(S(T)) = 0$ (as we have seen before), thus $S(T) \leq I - S(\phi)$ which implies $I-S(T) \geq S(\phi)$ (see definition of $I - S(\phi)$ above). Hence $\phi(T) = 0$ implies $S(\phi)TS(\phi) = S(\phi)(I - S(T))TS(\phi) = 0$. So, if $S(\phi) = I$ we have $\phi(T) = 0$ implies $T = 0$. Conversely if $\phi(T) = 0$ implies $T = 0$ $(T \geq 0)$ we have that $\phi(I - S(\phi)) = 0$ implies $S(\phi) = I$. Our definition of the 'faithfulness' of ϕ can therefore be seen to correspond with definition 1.4.2.

(ii) One says that a family $\{\phi_k\}_{k\in K}$ of positive normal functionals on $\mathcal{A}$ is sufficient if for any $T\in\mathcal{A}^+$, $T\neq 0$, a $k\in K$ exists such that $\phi_k(T) \neq 0$. As in (i) we can show that a family $\{\phi_k\}_{k\in K}$ of normal positive linear functionals is sufficient if and only if $\sup_{k\in K} S(\phi_k) = I$.

(iii) If τ is a finite normal trace on $\mathcal{A}$, we claim that $S(\tau)$ is a central projection in $\mathcal{A}$:
To this end we first show that the left kernel $N_\tau = \{T\in\mathcal{A};\ \tau(T^*T) = 0\}$ is a two-sided $*$-ideal of $\mathcal{A}$. Suppose $T, S \in N_\tau$, then since $(T + S)^*(T + S) + (T - S)^*(T - S) = 2T^*T + 2S^*S$ we have $(T + S)^*(T + S) \leq 2T^*T + 2S^*S$ and so $\tau((T + S)^*(T + S)) \leq 2\tau(T^*T) + 2\tau(S^*S) = 0$. Hence $T + S \in N_\tau$. Clearly $\alpha T\in N_\tau$ (α a scalar and $T\in N_\tau$), since

$(\alpha T)^*(\alpha T) = |\alpha|^2 T^*T$. Let $S \in \mathcal{A}$ and $T \in N_\tau$, then from $(ST)^*(ST) = T^* S^*ST \leq \|S\|^2 T^*T$ we have $ST \in N_\tau$. We also have $T^* \in N_\tau$ if $T \in N_\tau$, since $\tau((T^*)^*T^*) = \tau(TT^*) = \tau(T^*T) = 0$ (τ is tracial i.e. $\tau(T^*T) = \tau(TT^*)$). Then, if $T \in N_\tau$ and $S \in \mathcal{A}$ we have $TS = (S^*T^*)^* \in N_\tau$. So N_τ is a two-sided $*$-ideal in $\mathcal{A}$. Since $\mathcal{A}^+$ spans $\mathcal{A}$ linearly ($T = T_1 + iT_2$, T_1 and T_2 hermitian and $T_i = T_i^+ - T_i^-$ $(i = 1,2)$ where $T_i^+ = \frac{1}{2}(|T_i| + T_i)$, $T_i^- = \frac{1}{2}(|T_i| - T_i)$, $|T_i| = (T_i^2)^{1/2}$ makes sense via the methods of functional calculus), a finite trace is extended uniquely to a positive linear functional in $\mathcal{A}$, denoted by τ'. Then if τ is normal we know that τ' is σ-weakly continuous. We now claim that N_τ is σ-weakly closed. Suppose $\{T_\alpha\}$ is a net in N_τ with $T_\alpha \to T$ σ-weakly. Since the $*$-operation and multiplication on $\mathcal{A}$ are σ-weakly continuous, it follows that $T_\alpha^*T_\alpha \longrightarrow T^*T$ σ-weakly. Hence $\tau(T_\alpha^*T_\alpha) = \tau'(T_\alpha^*T_\alpha) \longrightarrow \tau'(T^*T) = \tau(T^*T)$, and since $\tau(T_\alpha^*T_\alpha) = 0$ for all α one has that $\tau(T^*T) = 0$. Hence N_τ is σ-weakly closed. By the Banach Alaoglu theorem, $N_\tau \cap S$ is σ-weakly compact (S the unit ball in $\mathcal{A}$), and has an extremal point by the Krein-Milman theorem ([8], p 32). From a well-known theorem in the theory of C^*-algebras, N_τ has an identity, say E. ([18], theorem 10.2 Chapter 1). Since E is the greatest projection in N_τ, $E = I - S(\tau)$ ($I - S(\tau)$ is the greatest projection F in $\mathcal{A}$ with $\tau(F) = 0$). Since N_τ is a two sided $*$-ideal we have $(I - S(\tau))T \in N_\tau$ $(T \in \mathcal{A})$. Hence $(I - S(\tau))T = (I - S(\tau))T(I - S(\tau))$. We also have $T(I - S(\tau)) \in N_\tau$, so $(I - S(\tau))T(I-S(\tau)) = T(I - S(\tau))$ $(T \in \mathcal{A})$. Thus $T(I - S(\tau)) = (I - S(\tau))T$ for all $T \in \mathcal{A}$ so $I - S(\tau)$ is a central projection. Thus $S(\tau)$ is central.

■

As stated earlier, our aim is to characterize finite von Neumann algebras in terms of traces. Before we can give this characterization we need the following two lemmas :

1.4.10 *LEMMA* ([18], p 310)

Let $\{E_n\}$ be an increasing sequence of finite projections in $\mathcal{A}$. If $F \in \mathcal{P}(\mathcal{A})$ with $E_n \precsim F$ for every n, then $E = \sup_{n \in N} E_n \precsim F$.

<u>Proof</u> :

Let $P_n = E_{n+1} - E_n$, n=1,2,... and $P_o = E_1$. Then clearly $\{P_n\}_{n=1}^{\infty}$ is a disjoint sequence of projections and since $E = \lim_{n\to\infty} E_n$ in the strong-operator topology, we have $E = \lim_n E_n = \sum_{n=0}^{\infty} P_n$ (see lemma 1.1.10 and 1.1.11). We shall construct a disjoint sequence $\{Q_n\} \subset \mathcal{P}(\mathcal{A})$ with $Q_n \sim P_n$; n=0,1,2... and $Q_n \leq F$. By assumption $P_o = E_1 \precsim F$. Hence a projection Q_o in $\mathcal{A}$ exists with $Q_o \leq F$ and $Q_o \sim P_o$. Suppose $\{Q_o,...,Q_{n-1}\}$ have been defined. It follows from proposition 1.11.12 that

$$E_n = P_o + P_1 + \ldots + P_{n-1} \sim Q_o + Q_1 + \ldots + Q_{n-1} = F_n \leq F.$$

Since $E_{n+1} \precsim F$ there exists a $F'_{n+1} \in \mathcal{P}(\mathcal{A})$ with $F'_{n+1} \leq F$ and $E_{n+1} \sim F'_{n+1}$. Since $E_n \leq E_{n+1}$ it follows that

$F_n \sim E_n \leq E_{n+1} \sim F'_{n+1}$ which implies that $F_n \precsim F'_{n+1}$. In other words, there exists a $F'_n \leq F'_{n+1}$ with $F_n \sim F'_n$. Since E_n is finite, lemma 1.2.4 implies that F_n is finite too $(E_n \sim F_n)$. By proposition 1.3.4 we have $F - F_n \sim F - F'_n \geq F'_{n+1} - F'_n$. So $F'_{n+1} - F'_n \precsim F - F_n$ which means that there exists a projection

$Q_n \leq F - F_n$ with $Q_n \sim F'_{n+1} - F'_n \sim E_{n+1} - E_n = P_n$ (by proposition 1.3.4). Since $Q_n \leq F - F_n$, $Q_nQ_k = 0$ for every $k = 1,\ldots n-1$. Hence we can construct $\{Q_n\}_{n=1}^{\infty}$ by induction and $E = \sum_{n=0}^{\infty} P_n \sim \sum_{n=0}^{\infty} Q_n \leq F$ by proposition 1.1.12.

■

1.4.11 *LEMMA* ([18], p 310)

If $\{E_n\}$ is a disjoint sequence of projections in a finite von Neumann algebra $\mathcal{A}$, then any sequence $\{F_n\}$ of projections in $\mathcal{A}$ with $E_n \sim F_n$, $n = 1,2,\ldots$ converges to zero σ-strongly.

Proof :

For any P_1, P_2 and Q_1, Q_2 projections in $\mathcal{A}$ with

$P_1 \precsim Q_1$, $P_2 \precsim Q_2$ and $Q_1Q_2 = 0$ we have $\sup(P_1,P_2) \precsim Q_1 + Q_2$ because

$\sup(P_1,P_2) - P_2 \sim P_1 - \inf(P_1,P_2) \precsim Q_1$ $(P_1 \precsim Q_1)$ (corollary

1.1.20), and corollary 1.1.13 implies

$$\sup(P_1,P_2) = (\sup(P_1,P_2)-P_2) + P_2 \precsim Q_1 + Q_2.$$

We therefore find by induction, that for any $m \leq n$

$$\sup_{m\leq k\leq n} F_k \precsim E_m + E_{m+1} + \ldots + E_n \leq \sum_{k=m}^{\infty} E_k$$

If we define $P_m = \sup_{k\geq m} F_m$, lemma 1.4.10 implies $P_m \precsim \sum_{k=m}^{\infty} E_k$.

Then $P_m \sim Q' \leq \sum_{k=m}^{\infty} E_k$, $Q' \in \mathcal{P}(\mathcal{A})$, and so $I-P_m \sim I - Q' \geq I - \sum_{k=m}^{\infty} E_k$

(see corollary 1.3.6). This shows that

$$I - P_m \succsim I - \sum_{k=m}^{\infty} E_k = E_o + E_1 + \ldots + E_{m-1} \text{ where } E_o = I - \sum_{k=1}^{\infty} E_k.$$

Clearly P_m is a decreasing sequence of projections in $\mathcal{A}$, and by putting $P = \inf_{m\in N} P_m$, we have $I - P \geq I - P_m \succsim E_o + E_1 + \ldots + E_{m-1}$.

Using lemma 1.4.10 again, we find that

$$I - P \gtrsim \sum_{j=0}^{\infty} E_j = I \quad (E_o = I - \sum_{j=1}^{\infty} E_j).$$

Thus $I \sim Q'' \lesssim I - P \lesssim I \ (Q'' \in \mathcal{P}(\mathcal{A}))$. Since $\mathcal{A}$ is a finite von Neumann algebra we have $I = I - P$ which implies that $P = 0$. Since $P_m \geq P_{m+1} \geq F_{m+1}$ we have $0 = P = \lim_{m\to\infty} P_m \geq \lim_{n\to\infty} F_n$ in the strong-operator topology on $\mathcal{A}$. Thus $\{F_n\}$ converges to zero strongly, hence σ-strongly (the two topologies coincide on bounded parts of $\mathcal{A}$).

■

Let $\mathcal{A}_*$ be the predual of a von Neumann Algebra $\mathcal{A}$. For any unitary $U \in \mathcal{A}$ we define $T_U : \mathcal{A}_* \longrightarrow \mathcal{A}_*$ such that for any $\varphi \in \mathcal{A}_*$ $(T_U\varphi)(T) = \varphi(U^*TU^*)$. We show that $T_U\varphi \in \mathcal{A}_*$: Since multiplication in the σ-weak topology is separately continuous i.e. $T\in\mathcal{A} \longrightarrow ST\in\mathcal{A}$ and $T \longrightarrow TS\in\mathcal{A}$ are σ-weakly continuous, one has that $T \longrightarrow U^*TU$ is σ-weakly continuous ($U\in\mathcal{A}$, unitary). Hence $T\in\mathcal{A} \longrightarrow \varphi(U^*TU)$ is σ-weakly continuous. Let $\phi \in \mathcal{A}_*$ and consider the set $L_\phi = \{T_U\phi \ ; \ U \in \mathcal{A}$ unitary$\}$. Then $L_\phi \subset \mathcal{A}_*$. Let K_ϕ be the norm closed convex hull of L_ϕ in $\mathcal{A}_*$. Since the predual $\mathcal{A}_*$ of $\mathcal{A}$ is norm closed in $\mathcal{A}^*$, one has that $K_\phi \subseteq \mathcal{A}_*$.

In the following proposition we will use the so-called Ryll-Nardzewski fixed point theorem (a result in the theory of locally convex spaces; see [17], p 351). It states, if X is a locally convex Haussdorff space, $K\subset X$, a non-empty, weakly compact, convex subset and J a non-contracting semi-group of weakly continuous affine mappings of K into K, then an $x_o \in K$ exists such that $Tx_o = x_o$ $(T\in J)$. (J is a non-contracting on K if for any $x,y\in K$, $x\neq y$, a continuous seminorm p on X exists with $\inf_{T\in J} p(Tx-Ty) > 0$, and
$T:K \to K$. (K convex, $K\subset X$) is called affine if for any $x_1,x_2\in K$ and any $\lambda\in[0,1]$, $T(\lambda x_1 + (1-\lambda)x_2) = \lambda T(x_1) + (1-\lambda)T(x_2)$).

1.4.12 *PROPOSITION* ([18], p 311)

Let $\mathcal{A}$ be a von Neumann algebra, then the following conditions are equivalent :

(i) $\mathcal{A}$ is finite

(ii) $\mathcal{A}$ admits sufficiently many finite normal traces.

<u>Proof</u> :

Suppose $\{\tau_i\}_{i\in I}$ is a family of sufficiently many finite traces. To show that $\mathcal{A}$ is finite we must show if $I \sim E \leq I$ (I the identity element of $\mathcal{A}$ and $E\in\mathcal{P}(\mathcal{A})$, then $I = E$. Since $I \sim E$ there exists a partial isometry $U\in\mathcal{A}$ such that $I = U^*U$ and $E = UU^*$. Hence, $\tau_i(I - E + E) = \tau_i(I - E) + \tau_i(E)$ implies $\tau_i(I - E) = \tau_i(I) - \tau_i(E) = \tau_i(U^*U) - \tau_i(UU^*) = 0$ for all $i\in I$ (see (iii) of definition 1.4.2). Since the family $\{T_i\}_{i\in I}$ is sufficient, remark 1.4.9(ii) implies that $E = I$. So $\mathcal{A}$ is finite. This proves condition (i).

Conversely, suppose $\mathcal{A}$ is finite. Since the positive normal linear functionals on $\mathcal{A}$ are precisely the elements in $\mathcal{A}_*^+$, a positive normal linear functional on $\mathcal{A}$ exists. Let ϕ be a positive normal linear functional on $\mathcal{A}$. Consider $L_\phi \subset \mathcal{A}_*$ and K_ϕ the convex norm-closure of L_ϕ in $\mathcal{A}_*$. We claim that K_ϕ is $\sigma(\mathcal{A}_*,\mathcal{A})$-compact. Since for every $T_U\phi\in L_\phi$ we have that

$$\|T_U\phi\| = \sup_{\|T\|=1} |\phi(U^*TU)| \leq \sup_{\|T\|=1} \|\phi\|\|T\| = \|\phi\| \quad (\|T\|=1),$$

it follows that L_ϕ is a norm-bounded subset of $\mathcal{A}_*$. Hence K_ϕ is a norm-bounded subset of $\mathcal{A}_*$. Thus, to show that K_ϕ is $\sigma(\mathcal{A}_*,\mathcal{A})$-compact we may use lemma 1.4.7. By this lemma it suffices to show that for any sequence $\{E_n\}$ of disjoint projections in $\mathcal{A}$, $\{\psi(E_n)\}$ converges to zero uniformly for $\psi\in K_\phi$. Since the convex hull of L_ϕ is norm dense in K_ϕ, we have only to show that $\lim_{n\to\infty} \phi(UE_nU^*)=0$ uniformly for $U\in\mathcal{A}$, unitary. Suppose this is not true. Then a $\delta > 0$, a subsequence $\{F_n\}$ of $\{E_n\}$ and a sequence $\{U_n\}$ of unitary elements in $\mathcal{A}$ exist such that $\phi(U_nF_nU_n^*) \geq \delta$ $n = 1,2,\ldots$ By proposition 1.3.5(i) we know that $U_nF_nU_n^* \sim F_n$ and $\{F_n\}$ is a disjoint sequence ($\{E_n\}$ is one). Hence lemma 1.4.11 implies that $U_nF_nU_n^*$ converges to zero σ-strongly, thus σ-weakly (the σ-strong topology on $\mathcal{A}$ is finer than the σ-weak topology). Since ϕ is σ-weakly continuous one has that $\phi(U_nF_nU_n^*) \overset{n}{\underset{\infty}{\to}} 0$, contradicting the choice of $\{U_n\}$, $\{F_n\}$, and δ. Thus K_ϕ is $\sigma(\mathcal{A}_*,\mathcal{A})$-compact by lemma 1.4.7 (notice that since $\mathcal{A}$ is the dual of the Banach space $\mathcal{A}_*$, K_ϕ norm closed and convex, we have that K_ϕ is $\sigma(\mathcal{A}_*,\mathcal{A})$-closed. This follows since $\bar{K}_\phi$ is the same in all the locally convex topologies on $\mathcal{A}_*$, which is compatable with the dual pair $(\mathcal{A}_*,\mathcal{A})$; see [13], proposition 8, p 34). Consider $J = \{T_U : U\in\mathcal{A}, \text{ unitary}\}$. We claim that J is a group of isometries on $\mathcal{A}_*$. For any T_U, T_V, we have $T_UT_V\phi(T) = \phi((UV)^*TUV) = T_{UV}\,\phi(T)$ $(T\in\mathcal{A})$. Hence $T_UT_V = T_{UV}\in J$ since $UV\in\mathcal{A}$, unitary. Also $T_I\in J$ is the identity element and for each T_U, $T_U*\in J$ is the inverse element of T_U in J. Clearly $\|T_U\phi\|$ $= \sup_{\|T\|=1} |\phi(U^*TU)| \leq \|\phi\|$, and since $T_U* = T_U-1$ we have $\|T_U*(T_U\phi)\|$ $\leq \|T_U\phi\|$ (T_U* is also bounded). Hence $\|T_U\phi\| = \|\phi\|$ for every $\phi\in\mathcal{A}_*$. This shows that J is a group of isometries from $\mathcal{A}_*$ onto $\mathcal{A}_*$.

We now have the following particular case for the Ryll-Nardzewski fixed points theorem :

(i) $\mathcal{A}_*$ in the norm topology is a separated locally convex vector space, whose dual is $\mathcal{A}$.

(ii) K_ϕ is a $\sigma(\mathcal{A}_*,\mathcal{A})$-compact, convex non-empty subset of $\mathcal{A}_*$.

(iii) Let V be any unitary operator in $\mathcal{A}$. Since $T_V(T_U\phi)$ $= T_{VU}\phi \in L_\phi$ for every $U\in\mathcal{A}$ unitary, $T_VL_\phi \subset L_\phi$ for every $T_V \in J$. Thus $T_V\,K_\phi \subset K_\phi$ for every $T_V \in J$. Hence J is a group of isometries from K_ϕ into K_ϕ. Each mapping T_U is $\sigma(\mathcal{A}_*,\mathcal{A})$- continuous since if $\varphi_\alpha \to 0$ in the $\sigma(\mathcal{A}_*,\mathcal{A})$-topology on $\mathcal{A}_*$, one has that $\varphi_\alpha(T) \to 0$ for every $T\in\mathcal{A}$.

Hence in particular $\Psi_\alpha(U^*TU) \to 0$ which implies that $T_U\Psi$ is $\sigma(\mathcal{A}_*,\mathcal{A})$-continuous. Since each T_U is linear on $\mathcal{A}_*$ it is an affine mapping on K_ϕ. It is clear that J is non- contracting since T_U is an isometry for all $U\in\mathcal{A}$, unitary. So the Ryll-Nardzewski fixed point theorem states that a $\tau_\phi \in K_\phi$ exists which is a fixed point under J (i.e. $T_U\tau_\phi = \tau_\phi$ for every $U\in\mathcal{A}$, unitary).

By definition, any fixed point τ under J in $\mathcal{A}_*^+$ is a normal finite trace because $\tau(UTU^*) = \tau(T)$, for every $T\in\mathcal{A}$, implies $\tau(U(TU)U^*) = \tau(TU)$ ($TU\in\mathcal{A}$ for every $T\in\mathcal{A}$). Hence $\tau(UT) = \tau(TU)$ for every $T\in\mathcal{A}$ and $U\in\mathcal{A}$, unitary. Since every element in $\mathcal{A}$ is a linear combination of four unitary elements we have that $\tau(ST) = \tau(TS)$ for every T, $S\in\mathcal{A}$. In particular $\tau(T^*T) = \tau(TT^*)$ for every $T\in\mathcal{A}$. From remark 1.4.6 we have that $\tau\in\mathcal{A}_*^+$ implies τ normal (notice that $\mathcal{A}_*^+$ is the set of all positive elements in $\mathcal{A}_*$). The properties (i) and (ii) in definition 1.4.2 follow (the proofs are trivial) since τ is linear. This shows the existence of a finite normal trace on $\mathcal{A}$.

Thus, for any normal finite trace τ on $\mathcal{A}$, it follows from remark 1.4.9 (iii) that the support of τ is a central projection in $\mathcal{A}$.

Now if ϕ if a positive element in $\mathcal{A}_*$ (i.e. $\phi\in\mathcal{A}_*^+$), we have $\phi(T) = \phi(UTU^*)$ for every central element T of $\mathcal{A}(U\in\mathcal{A}$, unitary).

Hence $T_U\phi|_Z = \phi|_Z$ for every $U\in\mathcal{A}$, unitary ($Z = \mathcal{A}\cap\mathcal{A}'$ is the center

of $\mathcal{A}$). Since τ_ϕ is the norm limit of a sequence of convex combinations of $T_U\,\phi$, we conclude that $\tau_\phi(T) = \phi(T)$ for every $T\in Z$. As seen above, the support of τ_ϕ is central and since ϕ and τ_ϕ coincide on Z we have that $S(\phi|_Z) = S(\tau_\phi)$.

Hence sup $\{S(\tau_\phi);\ \phi\in\mathcal{A}_*^+\} = \sup\{S(\phi|_Z);\ \phi\in\mathcal{A}_*^+\}$. We now show that $\{\phi|_Z;\ \phi\in\mathcal{A}_*^+\}$ is a sufficient set : Let $T > 0$ an element of Z^+. Then $T\in\mathcal{A}^+$, so there exists a $\phi\in\mathcal{A}_*$ such that $\phi(T)\neq 0$ ($(\mathcal{A}_*,\mathcal{A})$ is a dual pair). For every $\phi\in\mathcal{A}_*$, $\phi = \phi_1 + i\,\phi_2$ where ϕ_1 and ϕ_2 are hermitian functionals, i.e. $\phi_i^*(T) = \overline{\phi_i(T^*)}$ ($\phi_1 = \frac{1}{2}(\phi + \phi^*)$ and $\phi_2 = \frac{1}{2i}(\phi - \phi^*)$). Let $\phi_i = \phi_i^+ - \phi_i^-$ be the Jordan decomposition (see [18], p 140) for ϕ_1 and ϕ_2 (ϕ_i^+ and ϕ_i^- are positive functionals ($i = 1,2,$)). So $\mathcal{A}_*^+$ spans $\mathcal{A}_*$ linearly. Therefore a positive functional $\phi\in\mathcal{A}_*$ exists with $\phi(T) > 0$. Hence $\{\phi|_Z,\ \phi\in\mathcal{A}_*^+\}$ is sufficient and by remark 1.4.9 (ii) we have
$\sup\{S(\tau_\phi);\ \phi\in\mathcal{A}_*^+\} = \sup\{S(\phi|_Z);\ \phi\in\mathcal{A}_*^+\} = I$. Hence remark 1.4.9 (ii) implies that $\{\tau_\phi,\ \phi\in\mathcal{A}_*^+\}$ is a sufficient family of finite normal traces.

■

This proposition above characterizes finite von Neumann algebras in terms of finite normal traces. As defined before, a von Neumann algebra $\mathcal{A}$ is countably decomposable if every family of pairwise disjoint projections in $\mathcal{A}$ is countable we have the following characterizations.

1.4.13 *PROPOSITION* ([4], p 111, proposition 9)

Let $\mathcal{A}$ be a von Neumann algebra. Then the following conditions are equivalent

(i) There exists a faithful finite normal trace on $\mathcal{A}$

(ii) $\mathcal{A}$ is finite and countably decomposable

(iii) $\mathcal{A}$ is finite and the center Z is countably decomposable.

<u>Proof</u> :

Suppose that condition (i) holds. Let ϕ be a faithful finite normal trace on $\mathcal{A}$. We show that $\mathcal{A}$ is finite and countably decomposable. Since $\{\phi\}$ is a sufficient family of finite normal traces on $\mathcal{A}$, it follows that $\mathcal{A}$ is finite by proposition 1.4.12. Let $\{E_i\}_{i\in I}$ be a family of disjoint projections in $\mathcal{A}$. Let $I_n = \{i\in I \mid \phi(E_i) \geq 1/n\}$. It is clear that $\bigcup_{n=1}^{\infty} I_n \subset I$. Conversely, since ϕ is faithful for all $i\in I$, $\phi(E_i)\neq 0$, therefore there exists, for each $i \in I$, an n with

$\phi(E_i) \geq \frac{1}{n}$. Thus $I = \bigcup_{n=1}^{\infty} I_n$. Since $\sum_{i\in I_n} E_i \leq I$ we have $\phi(I) - \phi(\sum_{i\in I_n} E_i) = \phi(I - \sum_{i\in I_n} E_i) \geq 0$. Normality of ϕ implies that

$$\phi(I) \geq \phi(\textstyle\sum_{i\in I_n} E_i) = \textstyle\sum_{i\in I_n} \phi(E_i) \geq \frac{1}{n} \text{ card } I_n$$

Since ϕ is finite one has that card I_n is finite. Thus I is countable ($I = \bigcup_{n=1}^{\infty} I_n$)

This proves condition (ii). Suppose (ii) holds. Since the center $Z \subset \mathcal{A}$ (iii), follows trivially.

We now suppose that condition (iii) holds and prove condition (i). Since $\mathcal{A}$ is finite proposition 1.4.12 implies that a finite normal trace ϕ on $\mathcal{A}$ exists. As seen in remark 1.4.9 (iii) the central support of this trace is a central projection. Let $\{\phi_i\}_{i\in I}$ be a maximal family of non-zero finite normal traces on $\mathcal{A}$, whose supports E_i, which are non-zero projections in Z, are pairwise disjoint (this family exists by Zorn's lemma).

Let $E = \sum_{i\in I} E_i$. We show that $E = I$; if $E \neq I$ then $I - E > 0$ and by proposition 1.4.12 a finite normal trace ϕ on $\mathcal{A}$ exists such that $\phi(I-E) \neq 0$ ($\mathcal{A}$ is finite). Consider the trace $\varphi : T\in\mathcal{A}^+ \longrightarrow \phi(T(I-E))$. Then since $\varphi(I) = \phi(I-E) \neq 0$ we have that ϕ is non-zero. It is obvious that the requirements in definition 1.4.2 are met since ϕ is a trace on $\mathcal{A}$. Clearly φ is finite, since $\varphi(I) = \phi(I-E) \leq \phi(I) < +\infty$ (ϕ is finite). Also φ is normal on

$\mathcal{A}$: If $\{T_\alpha\}_{\alpha\in J}$ is an uniformly bounded increasing net in $\mathcal{A}^+$, $\{T_\alpha(I-E)\} = \{(I-E)^* T_\alpha(I-E)\}$ is also one in $\mathcal{A}^+$ (each $E_i \in \mathcal{P}(Z)$ thus $E \in \mathcal{P}(Z)$ so $I-E$ commutes with every T_α, so $T_\alpha(I-E) = (I-E)^* T_\alpha(I-E))$. Thus $\varphi(\sup_\alpha T_\alpha) = \phi((\sup_\alpha T_\alpha)(I-E))$

$$= \phi((I-E)^* \sup_\alpha T_\alpha \ (I-E)) = \phi(\sup_\alpha((I-E^*)T_\alpha(I-E)) = \sup_\alpha \phi\ (T_\alpha(I-E))$$
$$= \sup_\alpha \varphi(T_\alpha). \quad \text{(corollary 1.4.4).}$$

Thus φ is normal. Since $\varphi(E) = \varphi(E(I-E)) = 0$ one has $I - S(\varphi) \geq E$ which implies that $S(\varphi) \leq I-E$. Thus $\{S(\varphi);E_i\}$ is a mutually disjoint family of projections in Z, and they are the supports of $\{\varphi, \phi_i\}_{i\in I}$. This contradicts the maximality of $\{\phi_i\}_{i\in I}$. Thus $E = I$. Since Z is countably decomposable, the family $\{E_i\}_{i\in I}$ is countable, say $I = \sum_{n=1}^{\infty} E_n$ $(E_n \neq 0)$. Define $\tau = \sum_{n=1}^{\infty} 2^{-n}\ \phi_n / \phi_n(I)$

It is clear that each ϕ_n is faithful on the reduced algebra $\mathcal{A}_{E_n}$ because E_n is the identity element of $\mathcal{A}_{E_n}$ and $S(\phi_n) = E_n$(see definition 1.4.8). Hence $\phi_n(E_n) \neq 0$ by remark 1.4.9(i). Since $I \geq E_n$, $\phi_n(I)\neq 0$. Thus τ is well-defined. It is clear that since $\tau(I - S(\tau)) = 0$ implies

$\sum_{n=1}^{\infty} 2^{-n}\ \phi_n(I - S(\tau))/\phi_n(I) = 0$, we have $\phi_n(I-S(\tau) = 0$ for all $n = 1,2,\ldots$. Thus $I - S(\phi_n) \geq I - S(\tau)$ for all n, which implies $S(\phi_n) \leq S(\tau)$ for all n. Thus, by defining $P_k = \sum_{n=1}^{k} S(\phi_n)$, $\{P_k\}$ is an increasing sequence of projections in $\mathcal{A}$ with $P_k \leq S(\tau)$ $(k=1,2,\ldots)$(Notice that $S(\phi_n)S(\phi_m) = 0$ for all $1\leq n,m\leq k$). Lemma 1.1.10 implies that $I = \sum_{n=1}^{\infty} S(\phi_n) \leq S(\tau)$. Since $S(\tau) \leq I$ (trivial) we have $S(\tau) = I$. So τ is faithful. That τ is finite follows from $\tau(I) = \sum_{n=1}^{\infty} 2^{-n}\ \phi_n(I)/\phi_n(I) = \sum_{n=1}^{\infty} 2^{-n} < \infty$. The fact that τ is a trace follows directly since each ϕ_n is one. Finally, we show that τ is normal.

Let $\{T_\alpha\}$ be an increasing uniformly bounded set of elements in $\mathcal{A}^+$.

$$\text{Then } \tau(\sup_\alpha T_\alpha) = \sum_{n=1}^{\infty} 2^{-n} \phi_n(\sup_\alpha T_\alpha)/\phi_n(I)$$

$$= \sum_{n=1}^{\infty} 2^{-n} \sup_\alpha \phi_n(T_\alpha)/\phi_n(I) \quad \text{(each } \phi_n \text{ is normal)}$$

$$= \sup_\alpha \sum_{n=1}^{\infty} 2^{-n} \phi_n(T_\alpha)/\phi_n(I) \quad \text{(all terms are positive)}$$

$$= \sup_\alpha \tau(T_\alpha)$$

Thus τ is normal. So τ is a faithful finite normal trace on $\mathcal{A}$. This concludes the proof.

■

1.4.14 *PROPOSITION* ([1])

Let $\mathcal{A}$ be a finite von Neumann algebra and consider E, F$\in \mathcal{P}(\mathcal{A})$. Then the following conditions hold.

(i) If $E \precsim F$, then $\phi(E) \leq \phi(F)$ for every trace ϕ of $\mathcal{A}$.

(ii) If $\phi(E) \leq \phi(F)$ for every finite normal trace ϕ on $\mathcal{A}$, then $E \precsim F$.

Proof

(i) Let $E' \in \mathcal{P}(\mathcal{A})$ such that $E \sim E' \leq F$. A partial isometry $U \in \mathcal{A}$ exists, such that $E = U^*U$ and $E' = UU^*$. Thus
$\phi(E) = \phi(U^*U) = \phi(UU^*) = \phi(E')$ (ϕ is tracial). Since
$E' \leq F$, $F - E' \in \mathcal{P}(\mathcal{A})$ and $\phi(F - E' + E') = \phi(F - E') + \phi(E')$
Thus $\phi(F) - \phi(E') = \phi(F - E') \geq 0$ which implies $\phi(F) \geq \phi(E')$.
Since $\phi(E) = \phi(E')$, we have $\phi(F) \geq \phi(E)$. This holds for any trace ϕ on $\mathcal{A}$.

(ii) Since $\mathcal{A}$ is finite, proposition 1.4.12 implies a finite normal trace ϕ on $\mathcal{A}$ exists. Thus by using Zorn's lemma a maximal family $\{\phi_i\}_{i\in I}$ of non-zero finite normal traces on $\mathcal{A}^+$ exists, whose supports $H_i \in Z$ are mutually disjoint. We claim that $I = \sum_i H_i = H$. This follows exactly as in the proof of proposition 1.4.13, (iii) implies (i). It is clear that for each $i \in I$ ϕ is finite on the reduced algebra $\mathcal{A}_i = \mathcal{A}_{H_i}$, since it is finite on $\mathcal{A}$. Since each H_i is the identity element of $\mathcal{A}_i = \mathcal{A}_{H_i}$ and $S(\phi_i) = H_i$ for each $i\in I$ $(H_i \in Z)$ one has that the support of each ϕ_i restricted to $\mathcal{A}_{H_i}$, equals the identity of $\mathcal{A}_{H_i}$. So definition 1.4.8 implies that each ϕ_i restricted to $\mathcal{A}_i$ is a faithful finite normal trace on $\mathcal{A}_i$. Thus 1.4.13 implies that each $\mathcal{A}_i$ is finite and countably decomposable. Consider EH_i and FH_i; elements of $\mathcal{A}_i$, for each $i\in I$. According to proposition 1.1.26 a G_i $(i\in I)$ exists in the centre of $\mathcal{A}_i$ such that
$EG_i \gtrsim FG_i$ and $E(H_i - G_i) \lesssim F(H_i - G_i)$ $(H_iG_i = G_i$ since H_i is the identity in $\mathcal{A}_i)$.

For each $i\in I$, define $\psi_i(T) = \phi_i(TG_i)(T\in\mathcal{A}^+)$. Clearly ψ_i is well-defined since $TG_i \in \mathcal{A}_{H_i}(G_i = G_iH_i)$. We show that ψ_i is finite and normal on $\mathcal{A}$. Since ϕ_i is finite on $\mathcal{A}_i$ we have $\phi_i(H_i) < +\infty$, so $\psi_i(I) = \phi_i(G_i) \leq \phi_i(H_i) < +\infty$ (the inequality follows by part (i)). Hence ψ_i is finite for each $i\in I$. We can prove that ψ_i is normal for each $i\in I$ in exactly the same way that we proved that $\psi: T\in\mathcal{A}^+ \to \phi(T(I-H))$ is normal in proposition 1.4.13, (iii) implies (i).

Part (i) of this proposition and $EG_i \gtrsim FG_i$ imply $\psi_i(E) \geq \psi_i(F)$. On the other hand $\psi_i(E) \leq \psi_i(F)$ by hypothesis. Hence $\phi_i(EG_i) = \phi_i(FG_i)$. Since $FG_i \lesssim EG_i$ an $F_i \in \mathcal{P}(\mathcal{A}_i)$ exists for each $i\in I$, with $FG_i \sim F_i \leq EG_i$.

Then $\phi_i(EG_i - F_i) = \phi_i(EG_i) - \phi_i(F_i) = 0$ (by part (i), since $FG_i \sim F_i$ implies $FG_i \lesssim F_i$ and $F_i \lesssim FG_i$). Since ϕ_i is faithful on $\mathcal{A}_i$ we have $EG_i = F_i$ for all $i \in I$. Thus

$$EG_i = F_i \sim FG_i.$$

Since $EH_i - EG_i \lesssim FH_i - FG_i$; $EG_i \sim FG_i$ and the pair $(EH_i - EG_i;\ EG_i)$ (resp. $(FH_i - FG_i;\ FG_i)$) is disjoint, corollary 1.1.13 implies that $EH_i \lesssim FH_i$ for all $i \in H$.

Using corollary 1.1.13 again, one gets

$$E = \sum_i H_i E \lesssim \sum_i H_i F = F.$$

■

1.4.15 *PROPOSITION* ([1])

Let $E_1 \leq E_2 \leq E_3 \leq \ldots$ be a non-decreasing sequence in $\mathcal{P}(\mathcal{A})$. If the supremum E_∞ of this sequence is finite, then
$\inf(E_\infty, F) = \sup_n(\inf(E_n, F))$ for all $F \in \mathcal{P}(\mathcal{A})$

<u>Proof</u> :

Proposition 1.1.19 implies

$$F - \inf(F, I-E_n) \sim E_n - \inf(E_n, I-F)$$

and

$$F - \inf(F, I-E_\infty) \sim E_\infty - \inf(E_\infty, I-F)$$

The fact that $E_\infty \in \mathcal{P}(\mathcal{A})$ follows since $\mathcal{P}(\mathcal{A})$ is a complete lattice.
Since $E_n \leq E_\infty$ one has $I - E_n \geq I - E_\infty$, thus

$$F - \inf(F, I-E_n) \leq F - \inf(F, I-E_\infty)$$

Hence $E_n - \inf(E_n, I-F) \sim F - \inf(F, I-E_n) \leq F - \inf(F, I-E_\infty)$
So $E_n - \inf(E_n, I-F) \precsim E_\infty - \inf(E_\infty, I-F)$

Clearly $E_n \leq E_\infty$ implies $E_\infty \geq \inf(E_\infty, I-F) \geq \inf(E_n, I-F)$. From this and lemma 1.2.4 one has for all $F \in \mathcal{P}(\mathcal{A})$ that $\inf(E_\infty, I-F)$ and $\inf(E_n, I-F)$ are finite projections in the reduced algebra $\mathcal{A}_{E_\infty}$, which is a finite von Neumann algebra.

Since $\inf(E_\infty, I-F) \geq \inf(E_n, I-F)$ one has for every finite normal trace ϕ on $\mathcal{A}_{E_\infty}$ that $0 \leq \phi(\inf(E_\infty, I-F) - \inf(E_n, I-F))$. From $E_n - \inf(E_n, I-F) \precsim E_\infty - \inf(E_\infty, I-F)$ together with proposition 1.4.14 one has $\phi(E_n - \inf(E_n, I-F) \leq \phi(E_\infty - \inf(E_\infty, I-F))$ for every finite normal trace ϕ on $\mathcal{A}_{E_\infty}$. Using the trace properties one gets

$$\phi(E_n) = \phi(E_n - \inf(E_n, I-F)) + \phi(\inf (E_n, I-F)) \text{ and}$$

$$\phi(E_\infty) = \phi(E_\infty - \inf(E_\infty, I-F)) + \phi(\inf(E_\infty, I-F). \text{ So}$$

$$\phi(E_\infty - E_n) - \phi(\inf(E_\infty, I-F) - \inf(E_n, I-F))$$

$$= \phi(E_\infty - \inf(E_\infty, I-F)) - \phi(E_n - \inf(E_n, I-F)) \geq 0. \text{ Thus}$$

$$\phi(E_\infty - E_n) \geq \phi(\inf(E_\infty, I-F) - \inf(E_n, I-F)) \geq 0$$

Observing that (E_n) is an increasing sequence of projections, it follows that $\{\phi(E_n)\}$ is an increasing sequence of positive real numbers that is bounded above by $\phi(E_\infty)$. Since ϕ is normal, $\phi(E_\infty) = \lim_n \phi(E_n)$. This implies

$$\lim_{n\to\infty} \phi(E_\infty - E_n) = \lim_{n\to\infty} [\phi(E_\infty) - \phi(E_n)] = \phi(E_\infty) - \phi(E_\infty) = 0.$$

Hence $\lim_{n\to\infty} \phi(\inf(E_\infty, I-F)) - \inf(E_n, I-F)) = 0$. Consequently

$\phi(\inf(E_\infty, I-F)) = \lim_{n\to\infty} \phi(\inf(E_n, I-F))$ for any finite normal trace ϕ

on $\mathcal{A}_{E_\infty}$. Clearly $\inf(E_1, I-F) \leq \inf(E_2, I-F) \leq \dots$, so $\{\phi(\inf(E_n, I-F))\}$ is an increasing sequence of real numbers which is bounded above by $\phi(\inf(E_\infty, I-F))$, thus $\{\phi(\inf(E_n, I-F)\}_{n=1}^{\infty}$ converges towards its supremum.

Hence $\phi(\inf(E_\infty, I-F)) = \lim\limits_{n\to\infty} \phi(\inf(E_n, I-F)) = \sup\limits_n \phi(\inf(E_n, I-F))$

$= \phi(\sup\limits_n \inf(E_n, I-F))$ for every finite normal trace ϕ on $\mathcal{A}_{E_\infty}$.

It is clear that $\inf(E_\infty, I-F)$, $\inf(E_n, I-F) \in \mathcal{P}(\mathcal{A}_{E_\infty})$ for all $n \in N$; $F \in \mathcal{P}(\mathcal{A})$.

Since $\mathcal{P}(\mathcal{A}_{E_\infty})$ is a complete lattice $\sup\limits_n \inf(E_n, I-F) \in \mathcal{P}(\mathcal{A}_{E_\infty})$ and

$$\phi(\inf(E_\infty, I-F) - \sup_n \inf(E_n, I-F)) = 0$$

for every finite normal trace on $\mathcal{A}_{E_\infty}$. Part (ii) of proposition 1.4.14 implies that $\inf(E_\infty, I-F) - \sup\limits_n \inf(E_n, I-F) \sim 0$ which holds only if

$$\inf(E_\infty, I-F) - \sup_n \inf(E_n, I-F) = 0$$

for any $F \in \mathcal{P}(\mathcal{A})$.

■

CHAPTER 2

THE INDEX GROUP OF A VON NEUMANN ALGEBRA $\mathcal{A}$

In this chapter we shall consider a von Neumann algebra $\mathcal{A}$ with its commutant $\mathcal{A}'$ and shall construct, by using representation theory of a $*$-algebra in some $L(H)$, a certain abelian monoid $M(\mathcal{A},\mathcal{A}')$. This construction depends largely on some of the results in the first chapter. The Grothendieck group $\Gamma(\mathcal{A},\mathcal{A}')$ of $M(\mathcal{A},\mathcal{A}')$ can canonically be equiped with an order relation $\leq$ such that $(\Gamma(\mathcal{A},\mathcal{A}'), \leq)$ is an ordered commutative group. This group $\Gamma(\mathcal{A},\mathcal{A}')$ will be called the index group of the operator algebra $\mathcal{A}$ because it contains the indices of the Fredholm elements of $\mathcal{A}$, which will be defined in the next chapter.

We conclude this chapter by defining a dimension function on the set of all finite projections of $\mathcal{A}$. This function will be used to define the indices of the Fredholm elements of $\mathcal{A}$ in the next chapter.

2.1 <u>THE INDEX GROUP OF A VON NEUMANN ALGEBRA</u>

Let B be an involutive algebra and let K be a complex Hilbert space. A representation of B in K is a $*$-homomorphism ρ of B into $L(K)$. K is called the representation space of ρ and is denoted by H_ρ. Two representations ρ and σ are said to be unitarily equivalent or just equivalent, and we write $\rho \simeq \sigma$, if an isometry U of H_ρ onto H_σ exists such that the following diagram commutes for all $x \in B$.

$$\begin{array}{ccc} H_\rho & \xrightarrow{\quad U \quad} & H_\sigma \\ {\scriptstyle \rho(x)}\downarrow & & \downarrow{\scriptstyle \sigma(x)} \\ H_\rho & \xrightarrow[\quad U \quad]{} & H_\sigma \end{array}$$

This means, $U\rho(x) = \sigma(x)U$ for all $x \in B$.

2.1.1 *LEMMA*

The relation $\simeq$ is an equivalence relation on the set of all representations of B.

Proof :

Reflexive : $\rho \simeq \rho$ since the identity $I : H_\rho \xrightarrow{\text{onto}} H_\rho$ is an isometry with $I\rho(x) = \rho(x)I$ for all $x \in B$.

Symmetric : if $\rho \simeq \sigma$ an isometry $U : H_\rho \xrightarrow{\text{onto}} H_\sigma$ exists such that $U\rho(x) = \sigma(x)U$ for all $x \in B$. Then $U^* = U^{-1}$ maps H_σ isometric onto H_ρ and $U^*\sigma(x) = \rho(x)U^*$ $(x \in B)$. Thus $\sigma \simeq \rho$.

Transitive : If $\rho \simeq \sigma$ and $\sigma \simeq \mu$, then unitary operators $U: H_\rho \xrightarrow{\text{onto}} H_\sigma$ and $V: H_\sigma \xrightarrow{\text{onto}} H_\mu$ exists with $U\rho(x) = \sigma(x)U$ and $V\sigma(x) = \mu(x)V$ for all $x \in B$. Consider the unitary operator VU from H_ρ onto H_μ. Then for all $x \in B$ $VU\rho(x) = V\sigma(x)U = \mu(x)VU$; so $\rho \simeq \mu$ by VU.

■

This lemma shows that the set of all representations of B divides into so called equivalence classes modulo $\simeq$. We denote the equivalence class which contains the representation ρ by $[\rho]$.

Consider two representations σ and ρ of B in H_σ and H_ρ. Let H be the direct sum Hilbert space $H_\sigma \oplus H_\rho$. For each vector $z = z_\sigma \oplus z_\rho \in H$ and $x \in B$ put

$$\mu(x)z = \sigma(x)z_\sigma \oplus \rho(x)z_\rho$$

Since $\|\sigma(x)z_\sigma\| \leq \|z_\sigma\|$ and $\|\rho(x)z_\rho\| \leq \|z_\rho\|$ (by [18], p 21) we have

$$\|\mu(x)z\| := (\|\sigma(x)z_\sigma\|^2 + \|\rho(x)z_\rho\|^2)^{1/2}$$

$$\leq (\|z_\sigma\|^2 + \|z_\rho\|^2)^{1/2} := \|z\|$$

Thus $\mu(x)$ is a bounded operator on H. It is clear that μ is linear; $\mu(xy) = \mu(x)\mu(y)$ $x,y \in B$, and

$$\mu(x^*)z = \sigma(x^*)z_\sigma \oplus \rho(x^*)z_\rho$$
$$= \sigma^*(x)\, z_\sigma \oplus \rho^*(x)z_\rho$$

$$= \mu^*(x)\ (z_\sigma \oplus z_\rho)$$
$$= \mu^*(x) z$$

The penultimate equality follows, since for every $z_\sigma \oplus z_\rho$ and $z'_\sigma \oplus z'_\rho$ in H one has that

$$(\mu^*(x)\ z_\sigma \oplus z_\rho\ ,\ z'_\sigma \oplus z'_\rho)$$
$$= (z_\sigma \oplus z_\rho\ ,\ \mu(x) z'_\sigma \oplus z'_\rho)$$
$$:= (z_\sigma\ ,\ \sigma(x) z'_\sigma) + (z_\rho\ ,\ \rho(x) z'_\rho)$$
$$= (\sigma^*(x) z_\sigma\ ,\ z'_\rho) + (\rho^*(x) z_\rho\ ,\ z_\rho')$$
$$= (\sigma^*(x) z_\sigma \oplus \rho^*(x) z_\rho\ ,\ z_\sigma' \oplus z_\rho')$$

Since it holds for every $z_\sigma' \oplus z_\rho' \in H$ one has $\mu^*(x)\ z = \sigma^*(x) z_\sigma \oplus \rho^*(x) z_\rho$ $(x \in B)$. Thus μ is a $*$-homomorphism from B in H. The representation μ is called the direct sum of σ and ρ and we write $\mu = \sigma \oplus \rho$.

Suppose $\rho_1 \simeq \sigma_1$ and $\rho_2 \simeq \sigma_2$ and let $U : H_{\rho_1} \xrightarrow{\text{onto}} H_{\sigma_1}$ and $V : H_{\rho_2} \xrightarrow{\text{onto}} H_{\sigma_2}$ be isomorphisms such that $U\rho_1(x) = \sigma_1(x) U$ and $V\rho_2(x) = \sigma_2(x) V$ for all $x \in B$. Consider
$U \oplus V : H_{\rho_1} \oplus H_{\rho_2} \to H_{\sigma_1} \oplus H_{\sigma_2} : (x_{\rho_1}, x_{\rho_2}) \to (Ux_{\rho_1}, Vx_{\rho_2})$.
Clearly $U \oplus V$ is an isometric ismorphism from $H_{\rho_1} \oplus H_{\rho_2}$ onto $H_{\sigma_2} \oplus H_{\rho_2}$ and $(U \oplus V)(\rho_1 \oplus \rho_2)(x) = (\sigma_1 \oplus \sigma_2)(x)(U \oplus V)$ for each $x \in B$; so $\rho_1 \oplus \rho_2 \simeq \sigma_1 \oplus \sigma_2$. Thus if we define an addition operation, "+", on the set of all equivalence classes of representations of B by $[\rho_1] + [\rho_2] := [\rho_1 \oplus \rho_2]$, the above argument shows that "+" is well-defined (i.e. if $\sigma_1 \in [\rho_1]$ and $\sigma_2 \in [\rho_2]$ then
$[\sigma_1] + [\sigma_2] = [\sigma_1 \oplus \sigma_2] = [\rho_1 \oplus \rho_2]$).

2.1.2 *DEFINITION* ([12], p 5)

Let M be a set. Consider the mapping $M \times M \to M$ that associates with each pair $(x,y) \in M \times M$ an element $x + y \in M$ (the sum of x and y). Then M is called a monoid if;

(i) there exists such a mapping on $M \times M$ which is associative (i.e. $x + (y + z) = (x + y) + z$ for all $x, y, z \in M$)

(ii) there exists a $0 \in M$ with $0 + x = x + 0 = x$ for all $x \in M$.

M is called abelian if $x + y = y + x$ for all $x, y \in M$.

2.1.3 *LEMMA* ([1])

The set M(B) of all equivalence classes of representation of B, equiped with + , is an abelian monoid.

<u>Proof</u>

If $[\rho_1], [\rho_2] \in M(B)$, then $[\rho_1] + [\rho_2] = [\rho_1 \oplus \rho_2]$, which is a well-defined element of M(B). Consider the zero representation θ. This is a $*$-homomorphism of B in the trivial Hilbert space $\{0\}$. If ρ is any representation of B in H_ρ, it follows directly that $\rho \simeq \rho \oplus \theta$; for $U : H_\rho \oplus \{0\} \to H_\rho : (x_\rho, 0) \longrightarrow x_\rho$ is trivially an isomorphism with $U(\rho \oplus \theta)(x) = \rho(x)U$ for all $x \in B$. Thus $[\rho] = [\rho] + [\theta]$. Similarly $[\theta] + [\rho] = [\rho]$. Hence $[\theta]$ is the zero element of M(B). Let ρ_1, ρ_2, ρ_3 be representation of B in H_{ρ_1}, H_{ρ_2}, H_{ρ_3}. Since $(H_{\rho_1} \oplus H_{\rho_2}) \oplus H_{\rho_3} \overset{U}{\simeq} H_{\rho_1} \oplus (H_{\rho_2} \oplus H_{\rho_3})$ canonically, and $U(\rho_1 \oplus \rho_2) \oplus \rho_3(x) = \rho_1 \oplus (\rho_2 \oplus \rho_3)(x)U$ for all $x \in B$, one has

$$\begin{aligned} ([\rho_1] + [\rho_2]) + [\rho_3] &= [(\rho_1 \oplus \rho_2) \oplus \rho_3] \\ &= [\rho_1 \oplus (\rho_2 \oplus \rho_3)] \\ &= [\rho_1] + ([\rho_2] + [\rho_3]) \end{aligned}$$

Likewise $[\rho_1] + [\rho_2] = [\rho_2] + [\rho_1]$ for all $[\rho_1], [\rho_2] \in M(B)$

■

Consider a von Neumann algebra $\mathcal{A}$ of continuous linear operators of the complex Hilbert space H. Let $E \in \mathcal{P}(\mathcal{A})$ and let H_E be the range of E. Consider the restriction map

$$\pi_E : \mathcal{A}' \longrightarrow L(H_E) : T \to T_{|H_E}$$

It is clear if $T \in \mathcal{A}'$ then $\pi_E T \in L(H_E)$, in fact $\|\pi_E T\| = \|T_{|H_E}\| \leq \|T\|$, so $\pi_E T$ is bounded. Linearity follows directly from that of T. Thus π_E is well defined.

2.1.4 *LEMMA* ([1])

The mapping π_E is a representation of $\mathcal{A}'$ (the commutant of $\mathcal{A}$) in H_E.

Proof :
Choose T, $S \in \mathcal{A}'$, $\alpha \in \mathbb{C}$ arbitary, then

$$\pi_E(\alpha T+S) = (\alpha T+S)_{|H_E} = \alpha T_{|H_E} + S_{|H_E} = \alpha \pi_E(T) + \pi_E(S).$$

Similarly $\pi_E(ST) = \pi_E(S)\pi_E(T)$. Since for all $x,y \in H_E$ $(T_{|H_E}x,y) = (Tx,y) = (x,T^*y) = (x,(T^*)_{|H_E}y)$ we have $(T_{|H_E})^* = T^*_{|H_E}$. Thus π_E is a $*$-homomorphism from $\mathcal{A}'$ into $L(H_E)$ and therefore a representation of $\mathcal{A}'$.

■

2.1.5 *PROPOSITION* ([1])

Let $E,F \in \mathcal{P}(\mathcal{A})$. Then $E \sim F$ if and only if $\pi_E \simeq \pi_F$.

Proof :
Suppose that $E \sim F$. Then a partial isometry $U \in \mathcal{A}$ exists with $E = U^*U$ and $F = UU^*$. By the definition of a partial isometry with initial projection E and final projection F; U is an isometry on H_E and $U(H) = H_F$. Since $U(H_{I-E}) = 0$ one has $U = UE$, so $U(H_E) = U(H) = H_F$. Hence U is an isomorphism from H_E onto H_F. Observing that $UE = UU^*U = FU$, we have $UTE = TUE = TFU$ for all $T \in \mathcal{A}'$, and since $TE = T_{|H_E}$ it follows that

$$UT_{|H_E} = T_{|H_F}U \quad \text{for all } T \in \mathcal{A}'$$

Hence $\pi_E \simeq \pi_F$. Conversely, suppose $\pi_E \simeq \pi_F$. Then there is an isomorphism U' of H_E onto H_F such that $U'(\pi_E T) = (\pi_F T)U'$ for all $T \in \mathcal{A}'$.

Define U on H by U' on H_E and zero on H_{I-E}. Then $\|Ux\| = \|x\|$ for all $x \in H_E$. $U = 0$ on H_{I-E} and $U(H) = U'(H_E) = H_F$. Thus U is a partial isometry with initial projection E and final projection F such that

$$U(I-E) = 0 \text{ and } (I-F)U = 0$$

The first relation follows by definition of U and the second since $U(H) = H_F$, so $(I-F)U(H) = (I-F)(H_F) = \{0\}$. Let $T \in \mathcal{A}'$. Then $U(\pi_E T) = (\pi_F T)U$ implies $UTE = TFU$. By using the two relations above one gets $UT = UET = UTE = TFU = TU$ $(T \in \mathcal{A}')$. Hence $U \in \mathcal{A}'' = \mathcal{A}$ and so $E \sim F$.

■

As we shall see later the construction of the index group depends largely on the following proposition.

2.1.6 *PROPOSITION* (cancellation law, [1])

Let E_1, E_2, F_1, F_2 be finite projections in $\mathcal{A}$. Then $\pi_{E_1} \simeq \pi_{F_1}$ and $\pi_{E_1} \oplus \pi_{E_2} \simeq \pi_{F_1} \oplus \pi_{F_2}$ imply $\pi_{E_2} \simeq \pi_{F_2}$.

<u>Proof</u> :

It is not difficult to show that $L(H \oplus H) \cong M_2(L(H))$ where $M_2(L(H))$ is the $*$-algebra of all (2x2) matrices with entries, elements of $L(H)$. Thus we can write

$$L(H \oplus H) = \{\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \ | T_{ij} \in L(H)\}$$

$$\text{Let} \quad \mathcal{B} = \{\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \ | T_{ij} \in \mathcal{A}\}$$

Let $\tilde{T} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}$ We show that $\mathcal{B}' = \{\tilde{T} | T \in \mathcal{A}'\}$ Let $\mathcal{C} = \{\tilde{T} | T \in \mathcal{A}'\}$.

Clearly $\mathcal{C} \subseteq \mathcal{B}'$. In fact if $T \in \mathcal{A}'$ and if $(S_{ij}) = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ is any element of $\mathcal{B}$, we have $TS_{ij} = S_{ij}T$ ($T \in \mathcal{A}'$ and $S_{ij} \in \mathcal{A}$). This implies

$$\tilde{T}\ (S_{ij}) = (S_{ij})\ \tilde{T}$$

Hence $\tilde{T} \in \mathcal{B}'$. Conversely let $T = (T_{ij}) \in \mathcal{B}'$. For any $(S_{ij}) \in \mathcal{B}$ we have $(T_{ij})(S_{ij}) = (S_{ij})(T_{ij})$ which holds if and only if

(1) $T_{11}S_{11} + T_{12}S_{21} = S_{11}T_{11} + S_{12}T_{21}$

(2) $T_{11}S_{12} + T_{12}S_{22} = S_{11}T_{12} + S_{12}T_{22}$

(3) $T_{21}T_{11} + T_{22}S_{21} = S_{21}T_{11} + S_{22}T_{21}$

(4) $T_{21}S_{12} + T_{22}S_{22} = S_{21}T_{12} + S_{22}T_{22}$

for all $S_{ij} \in \mathcal{A}$.

Consider the following cases :

(a) Choose $S_{12} = S_{21} = 0$:
From (1) $T_{11}S_{11} = S_{11}T_{11}$ for all $S_{11} \in \mathcal{A}$. Hence $T_{11} \in \mathcal{A}'$
By considering (4) one has $T_{22} \in \mathcal{A}'$

(b) Choose $S_{12} = I$ and $S_{22} = 0 = S_{11}$:
From (2) $T_{11} = T_{22}$ and (4) implies $T_{21} = T_{12}$

(c) Choose $S_{11} = I$ and $S_{21} = 0 = S_{22}$
From (3) $T_{21} = T_{12} = 0$

Thus $T = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{11} \end{bmatrix}$ with $T_{11} \in \mathcal{A}'$

Thus $\mathcal{B}' \subseteq \mathcal{C}$ and we have $\mathcal{B}' = \mathcal{C} = \{(\tilde{T}) \mid T \in \mathcal{A}'\}$. Then

$$(\mathcal{B}')' = \{(S_{ij}) \mid (S_{ij})\tilde{T} = \tilde{T}(S_{ij}) \text{ for all } T \in \mathcal{A}\}$$

$$= \{(S_{ij}) \mid S_{ij}\ T = T\ S_{ij} \text{ for all } T \in \mathcal{A}'\}$$

$$= \{(S_{ij}) \mid S_{ij} \in \mathcal{A}'' = \mathcal{A}\} = \mathcal{B}$$

Thus $\mathcal{B} = \mathcal{B}''$ and since $\mathcal{B}$ is a $*$-subalgebra of $L(H \oplus H)$ ($\mathcal{A}$ is a $*$-subalgebra of $L(H)$) with identity $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ we have that $\mathcal{B}$ is a von Neumann algebra of bounded linear operators on $H \oplus H$.

For any $G \in \mathcal{P}(\mathcal{A})$ define

$$\tilde{G} = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}, \ \tilde{\tilde{G}} = \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}$$

Since $\tilde{G}$ and $\tilde{\tilde{G}}$ are both self-adjoint and idempotent we have that $\tilde{G}$, $\tilde{\tilde{G}} \in \mathcal{P}(\mathcal{B})$. Moreover $\tilde{G}\ \tilde{\tilde{G}} = 0$.

Let $\alpha : (H \oplus H)_{\tilde{E}_1 + \tilde{\tilde{E}}_2} \to H_{E_1} \oplus H_{E_2}$ be the canonical isomorphism defined by $\alpha\ (\tilde{E}_1 + \tilde{\tilde{E}}_2)\binom{x}{y} = \begin{bmatrix} E_1 x \\ E_2 y \end{bmatrix}$. $(x, y \in H)$. Clearly α is an isometry, in fact

$$\|\alpha(\tilde{E}_1 + \tilde{\tilde{E}}_2)\binom{x}{y}\|$$

$$= \left\| \begin{bmatrix} E_1 x \\ E_2 y \end{bmatrix} \right\| := (\|E_1 x\|^2 + \|E_2 y\|^2)^{1/2}$$

$$= \|(\tilde{E}_1 + \tilde{\tilde{E}}_2)\binom{x}{y}\|$$

That α is linear follows by a straightforward calculation. Since α is an isometry it is one-to-one. For every

$\begin{bmatrix} E_1 x \\ E_2 y \end{bmatrix} \in H_{E_1} \oplus H_{E_2}$ we have that

$(\tilde{E}_1 + \tilde{\tilde{E}}_2)\binom{x}{y} \in (H \oplus H)_{\tilde{E}_1 + \tilde{\tilde{E}}_2}$ and $\alpha(\tilde{E}_1 + \tilde{\tilde{E}}_2)\binom{x}{y} = \begin{bmatrix} E_1x \\ E_2y \end{bmatrix}$. Thus α is an isometric isomorphism from $(H \oplus H)_{\tilde{E}_1+\tilde{\tilde{E}}_2}$ onto $H_{E_1} \oplus H_{E_2}$

Likewise $\beta : (H \oplus H)_{\tilde{F}_1+\tilde{\tilde{F}}_2} \to H_{F_1} \oplus H_{F_2}$ is a canonical isomorphism.

We show that $\Pi_{\tilde{E}_1 + \tilde{\tilde{E}}_2} \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} = \alpha^{-1} \begin{bmatrix} \Pi_{E_1}T & 0 \\ 0 & \Pi_{E_2}T \end{bmatrix} \alpha \quad (T \in \mathcal{A}')$

As defined above $\Pi_{\tilde{E}_1+\tilde{\tilde{E}}_2}$ is the restriction map from $\mathfrak{B}'$ into $L((H \oplus H)_{\tilde{E}_1 + \tilde{\tilde{E}}_2})$

Thus $\Pi_{\tilde{E}_1 + \tilde{\tilde{E}}_2} \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \in L((H \oplus H)_{\tilde{E}_1 + \tilde{\tilde{E}}_2})$. Take any element

$(\tilde{E}_1 + \tilde{\tilde{E}}_2)\binom{x}{y} \in (H \oplus H)_{\tilde{E}_1 + \tilde{\tilde{E}}_2}$, then

$$\alpha^{-1}\begin{bmatrix} TE_1 & 0 \\ 0 & TE_2 \end{bmatrix} \alpha\, (\tilde{E}_1 + \tilde{\tilde{E}}_2)\binom{x}{y}$$

$$= \alpha^{-1} \begin{bmatrix} TE_1 & 0 \\ 0 & TE_2 \end{bmatrix} \begin{bmatrix} E_1x \\ E_2y \end{bmatrix}$$

$$= \alpha^{-1} \begin{bmatrix} TE_1x \\ TE_2y \end{bmatrix}$$

$$= (TE_1 + TE_2)\binom{x}{y}$$

$$= \begin{bmatrix} TE_1 & 0 \\ 0 & TE_2 \end{bmatrix} \binom{x}{y}$$

$$= \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \binom{x}{y}$$

$$= \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} (\tilde{E}_1 + \tilde{\tilde{E}}_2)\binom{x}{y}$$

$$= \pi_{\tilde{E}_1 + \tilde{\tilde{E}}_2}\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}(\tilde{E}_1 + \tilde{\tilde{E}}_2)\binom{x}{y}$$

Similarly we have $\pi_{\tilde{F}_1 + \tilde{\tilde{F}}_2}\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} = \beta^{-1}\begin{bmatrix} \pi_{F_1}T & 0 \\ 0 & \pi_{F_2}T \end{bmatrix}\beta$ for every $T \in \mathcal{A}'$.

Since $\pi_{E_1} \oplus \pi_{E_2} \simeq \pi_{F_1} \oplus \pi_{F_2}$, an unitary operator $U: H_{E_1} \oplus H_{E_2} \to H_{F_1} \oplus H_{F_2}$ exists, satisfying $U(\pi_{E_1} \oplus \pi_{E_2})(T) = (\pi_{F_1} \oplus \pi_{F_2})(T)U$ for every $T \in \mathcal{A}'$.

Then we have that $V = \beta^{-1}U\alpha : (H \oplus H)_{\tilde{E}_1 + \tilde{\tilde{E}}_2} \xrightarrow{\text{onto}} (H \oplus H)_{\tilde{F}_1 + \tilde{\tilde{F}}_2}$ is an isomorphism (β^{-1}, U and α are isomorphisms).

For every $T \in \mathcal{A}'$ one has $\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \in \mathcal{B}'$ and

$$V\pi_{\tilde{E}_1 + \tilde{\tilde{E}}_2}\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} = V\,\alpha^{-1}\begin{bmatrix} \pi_{E_1}T & 0 \\ 0 & \pi_{E_2}T \end{bmatrix}\alpha$$

$$= \beta^{-1}\,U\begin{bmatrix} \pi_{E_1}T & 0 \\ 0 & \pi_{E_2}T \end{bmatrix}\alpha$$

Since

$$U\begin{bmatrix} \pi_{E_1}T & 0 \\ 0 & \pi_{E_2}T \end{bmatrix}\alpha\,(\tilde{E}_1 + \tilde{\tilde{E}}_2)\binom{x}{y}$$

$$= U\begin{bmatrix} \pi_{E_1}T & 0 \\ 0 & \pi_{E_2}T \end{bmatrix}\begin{bmatrix} E_1x \\ E_2y \end{bmatrix}$$

$$= U\begin{bmatrix} \pi_{E_1}T\,E_1x \\ \pi_{E_2}T\,E_2y \end{bmatrix}$$

$$= U(\pi_{E_1} \oplus \pi_{E_2})(T)\begin{bmatrix} E_1x \\ E_2y \end{bmatrix}$$

$$= (\pi_{F_1} \oplus \pi_{F_2})(T)\ U\begin{bmatrix} E_1x \\ E_2y \end{bmatrix}$$

$$= (\pi_{F_1} \oplus \pi_{F_2})\begin{bmatrix} F_1x' \\ F_2y' \end{bmatrix}$$

$$= \begin{bmatrix} \pi_{F_1}T\ F_1x' \\ \pi_{F_2}T\ F_2y' \end{bmatrix}$$

$$= \begin{bmatrix} \pi_{F_1}T & 0 \\ 0 & \pi_{F_2}T \end{bmatrix}\begin{bmatrix} F_1x' \\ F_2y' \end{bmatrix}$$

$$= \begin{bmatrix} \pi_{F_1}T & 0 \\ 0 & \pi_{F_2}T \end{bmatrix}\ U\begin{bmatrix} E_1x \\ E_2y \end{bmatrix}$$

$$= \begin{bmatrix} \pi_{F_1}T & 0 \\ 0 & \pi_{F_2}T \end{bmatrix}\ U\ \alpha(\tilde{E}_1 + \tilde{\tilde{E}}_2)\binom{x}{y} \qquad \binom{x}{y} \in H \oplus H,$$

one has $V\pi_{\tilde{E}_1 + \tilde{\tilde{E}}_2}\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} = \beta^{-1}\begin{bmatrix} \pi_{F_1}T & 0 \\ 0 & \pi_{F_2}T \end{bmatrix}\ U\alpha$

$$= \pi_{\tilde{F}_1 + \tilde{\tilde{F}}_2}\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}\ V$$

Hence $\pi_{\tilde{E}_1 + \tilde{\tilde{E}}_2} \simeq \pi_{\tilde{F}_1 + \tilde{\tilde{F}}_2}$. Proposition 2.1.5 implies $\tilde{E}_1 + \tilde{\tilde{E}}_2 \sim \tilde{F}_1 + \tilde{\tilde{F}}_2$. Since $\pi_{E_1} \simeq \pi_{F_2}$ we have $E_1 \sim F_1$ (proposition 2.1.5). Therefore, a partial isometry $U \in \mathcal{A}$ with $U^*U = E_1$ and $UU^* = F_1$ exists.

Then $\tilde{U} \in \mathcal{B}$, $\tilde{U}^*\tilde{U} = \begin{bmatrix} U^*U & 0 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} = \tilde{E}_1$$

and $\tilde{U}\tilde{U}^* = \tilde{F}_1$, thus $\tilde{E}_1 \sim \tilde{F}_1$.

We claim that if E is a finite projection in $\mathcal{A}$, then $\tilde{E}$ is finite in $\mathcal{B}$. In fact, if

$$\tilde{E} \sim F' \leq \tilde{E} \ ; \ F' = \begin{bmatrix} F_{11} F_{12} \\ F_{21} F_{22} \end{bmatrix}$$

then $\begin{bmatrix} F_{11} F_{12} \\ F_{21} F_{22} \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} F_{11} F_{12} \\ F_{21} F_{22} \end{bmatrix}$ implies

$F_{12} = F_{22} = 0$ and $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_{11} & 0 \\ F_{21} & 0 \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ F_{21} & 0 \end{bmatrix}$

implies $F_{21} = 0$ and $F_{11} \leq E$. Thus $\tilde{E} \sim \tilde{F}_{11} \leq \tilde{E}$. Let

$W = \begin{bmatrix} U_{11} U_{12} \\ U_{21} U_{22} \end{bmatrix} \in \mathcal{B}$ be a partial isometry with $\tilde{E} = W^*W$ and $\tilde{F}_{11} = WW^*$

where $W^* = \begin{bmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{bmatrix}$

An easy calculation shows that $U_{12} = U_{21} = U_{22} = 0$ and $E = U_{11}^*U_{11}$, $F_{11} = U_{11}U_{11}^*$ $(U_{11} \in \mathcal{A})$ thus $E \sim F_{11} \leq E$

Since E is finite, $E = F_{11}$. Therefore $\tilde{E} = \tilde{F}_{11}$, which shows that $\tilde{E}$ is a finite projection in $\mathcal{B}$.

Since $(\tilde{E}_1, \tilde{\tilde{E}}_2)$ and $(\tilde{F}_1, \tilde{\tilde{F}}_2)$ are disjoint pairs of finite projections in $\mathcal{P}(\mathcal{B})$; $\tilde{E}_1 + \tilde{\tilde{E}}_2 \sim \tilde{F}_1 + \tilde{\tilde{F}}_2$ and $\tilde{E}_1 \sim \tilde{F}_1$. Proposition

1.3.8 then implies that $\tilde{\tilde{E}}_2 \sim \tilde{\tilde{F}}_2$. Choose a $W = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in \mathcal{B}$ with

$W^*W = \tilde{\tilde{E}}_2$ and $WW^* = \tilde{\tilde{F}}_2$.

It follows easily that $E_2 = U_{22}^* U_{22}$ and $F_2 = U_{22} U_{22}^*$

Hence $E_2 \sim F_2$ and proposition 2.1.5 implies $\Pi_{E_2} \simeq \Pi_{F_2}$. ∎

Let $M(\mathcal{A}')$ be the abelian monoid consisting of all equivalence classes of representations of $\mathcal{A}'$.

Consider the submonoid $M(\mathcal{A},\mathcal{A}')$ of $M(\mathcal{A}')$ generated by the set of all $[\Pi_E]$, where $E \in \mathcal{P}(\mathcal{A})$ is finite relative to $\mathcal{A}$. We now show the construction of the Grothendieck group $\Gamma(\mathcal{A},\mathcal{A}')$ from the abelian monoid $M(\mathcal{A},\mathcal{A}')$. Since this construction is standard we will do it for a general abelian monoid.

Let $(M,+)$ be a abelian monoid which has the cancellation property, that is if $m + n = m + n'$ $m,n,n' \in M$, then $n = n'$.

Consider the product $M \times M = \{(m,n) | m,n \in M\}$. We define an equivalence relation on $M \times M$ as follows :

$$(m_1,n_1) \sim (m_2,n_2)$$

if and only if $m_1 + n_2 = m_2 + n_1$

2.1.7 *LEMMA* ([13])

"~" is an equivalence relation.

<u>Proof</u> :

Since $m + n = m + n$ one has $(m,n) \sim (m,n)$. Thus "~" is reflexive.

Suppose $(m_1,n_1) \sim (m_2,n_2)$ then $m_1 + n_2 = m_2 + n_1$. This implies $m_2 + n_1 = m_1 + n_2$, so $(m_2,n_2) \sim (m_1,n_1)$ which shows that "~" is symmetric.

If $(m_1,n_1) \sim (m_2,n_2)$; $(m_2,n_2) \sim (m_3,n_3)$ it follows that $m_1 + n_2 = m_2 + n_1$ and $m_2 + n_3 = m_3 + n_2$. Hence
$m_1 + n_2 + m_2 + n_3 = m_2 + n_1 + m_3 + n_2$

Since M is commutative and the cancellation property holds in M, we have $m_1 + n_3 = m_3 + n_1$ and consequently $(m_1,n_1) \sim (m_3,n_3)$ This proves that "~" is transitive and thus an equivalence relation.

■

This equivalence relation gives rise to a partition of M x M into so called equivalence classes. We now define the Grothendieck group as $K(M) = MxM/\sim = \{[(m,n)] \mid (m,n) \in M \times M\}$, where $[(m,n)] = \{(m',n') \in MxM \mid (m',n') \sim (m,n)\}$.

To show that this is a group we first have to define an operation $+_K$ in K(M) and show that K(M) is a group under this operation.

2.1.8 *LEMMA* ([13])

Let $[(m_1,n_1)]$ and $[(m_2,n_2)]$ be two arbitrary elements in K(M). Then the equation

$$[(m_1,n_1)] +_K [(m_2,n_2)] = [(m_1 + m_2, n_1 + n_2)]$$

give a well-defined operation of addition on K(M).

<u>Proof</u> :

Note first that if $[(m_1,n_1)]$ and $[(m_2,n_2)]$ are in K(M), then (m_1,n_1) and (m_2,n_2) are in M x M. Since M is a monoid $(m_1 + m_2, n_1 + n_2) \in M \times M$, so $[(m_1 + m_2, n_1 + n_2)] \in K(M)$. This shows that the right-hand side of the defining equation is at least in K(M). We now have to show that this operation of addition is well defined. We must show that if different representatives in MxM are chosen, the same element of K(M), will result. To this end, suppose that $(m_1',n_1') \in [(m_1,n_1)]$ and $(m_2', n_2') \in [(m_2,n_2)]$. We must show that

$$(m_1' + m_2', n_1' + n_2') \in [(m_1 + m_2, n_1 + n_2)].$$

Since $(m_1', n_1') \in [(m_1, n_1)]$ we have that $(m_1', n_1') \sim (m_1, n_1)$. This means $m_1' + n_1 = m_1 + n_1'$. Similarly, $(m_2', n_2') \in [(m_2, n_2)]$ implies that $m_2' + n_2 = m_2 + n_2'$

By adding the above equations and using the fact that M is a commutative monoid, we obtain

$$(m_1' + m_2') + (n_1 + n_2) = (m_1 + m_2) + (n_1' + n_2')$$

Hence $(m_1' + m_2', n_1' + n_2') \sim (m_1 + m_2, n_1 + n_2)$

Thus $(m_1' + m_2', n_1' + n_2') \in [(m_1 + m_2, n_1 + n_2)]$, which completes the proof.

■

It remains to show that K(M) is a commutative group with addition $+_K$. It follows trivially by observing that $[(0,0)]$ is the identity where 0 is the identity of M and for every $[(m,n)] \in K(M)$ the inverse is given by $-[(m,n)] = [(n,m)] \in K(M)$.

Consider the canonical monoid homomorphism

$$\gamma : M \to K(M) : m \to [(m,0)]$$

That γ is a monoid homomorphism follows from the relation

$$\gamma(m+n) = [(m+n,0)] = [(m,0)] +_K [(n,0)]$$
$$= \gamma(m) +_K \gamma(n)$$

Since the cancellation law holds in M we have the following

$$\gamma(m) = \gamma(n)$$

if and only if $[(m,0)] = [(n,0)]$

if and only if $(m,0) \sim (n,0)$

if and only if $m + 0 = n + 0$

if and only if $m = n$

Hence γ is one to one and thus a monomorphism.

2.1.9 *PROPOSITION* ([1])

Let $\Gamma(\mathcal{A},\mathcal{A}')$ be the Grothendieck group of the commutative monoid

$M(\mathcal{A},\mathcal{A}')$. Then the canonical monoid homomorphism $\gamma:M(\mathcal{A},\mathcal{A}') \to \Gamma(\mathcal{A},\mathcal{A}')$ is a monomorphism.

Proof :
Proposition 2.1.6 says exactly that the cancellation law holds in $M(\mathcal{A},\mathcal{A}')$, so γ is a monomorphism.

As we said earlier, this group $\Gamma(\mathcal{A},\mathcal{A}')$ plays an important role in the theory of Fredholm elements in a von Neumann algebra $\mathcal{A}$. For any finite projection $E \in L(H)$ we define the dimension of E as the dimension of H_E in the usual sense. We want to generalize this concept of dimension of a finite projection to a general von Neumann algebra $\mathcal{A}$.

■

2.1.10 *DEFINITION* ([1])

The dimension Dim_E of a finite projection E of $\mathcal{A}$ is defined by the formula

$$Dim\ E = \gamma[\Pi_E] \in \Gamma(\mathcal{A},\mathcal{A}')$$

Our aim now is to define a certain order relation in $\Gamma(\mathcal{A},\mathcal{A}')$ so that we can compare finite projections in $\mathcal{A}$ by means of their dimensions.

Consider again the general case where we have a commutative monoid $(M,+)$. We claim that if M has a partial ordering "$\leq$" with the following property P:

$m \leq n$ if and only if for all $\ell \in M$ one has
$m + \ell \leq n + \ell$;

then the Grothendieck group K(M) can canonically be equipped with an order relation "$\leq$" such that P holds.

2.1.11 *LEMMA*

The relation "$\leq$" in K(M) defined by $(m_1,n_1) \leq (m_2,n_2)$ if and only if $m_1 + n_2 \leq m_2 + n_1$ in M, gives a partial order on K(M) with property P.

<u>Proof</u> :

<u>Reflexive</u> : $(m,n) \leq (m,n)$ since $m + n \leq m + n$ in M.

<u>Antisymmetric</u> : If $(m_1,n_1) \leq (m_2,n_2)$ and $(m_2,n_2) \leq (m_1,n_1)$ we have $m_1 + n_2 \leq m_2 + n_1$ and $m_2 + n_1 \leq m_1 + n_2$ in M, so
$m_1 + n_2 = m_2 + n_1$ in M
Hence $(m_1,n_1) = (m_2,n_2)$

<u>Transitive</u> : If $(m_1,n_1) \leq (m_2,n_2)$ and $(m_2,n_2) \leq (m_3,n_3)$ one has $m_1 + n_2 \leq m_2 + n_1$ and $m_2 + n_3 \leq m_3 + n_2$. Since the property P holds in M we have $m_1 + n_2 + n_3 \leq m_2 + n_1 + n_3$ and
$m_2 + n_3 + n_1 \leq m_3 + n_2 + n_1$, which implies
$m_1 + n_2 + n_3 \leq m_3 + n_2 + n_1$ in M.

Hence $m_1 + n_3 \leq m_3 + n_1$ by property P again, thus $(m_1,n_1) \leq (m_3, n_3)$. Let $(m_3,n_3) \in K(M)$. Then

$$
\begin{aligned}
& (m_1,n_1) \leq (m_2,n_2) \\
\text{iff}\quad & m_1 + n_2 \leq m_2 + n_1 \text{ in M} \\
\text{iff}\quad & m_1 + m_3 + n_2 + n_3 \leq m_2 + m_3 + n_1 + n_3 \text{ (P holds in M)} \\
\text{iff}\quad & (m_1 + m_3, n_1 + n_3) \leq (m_2 + m_3, n_2 + n_3) \\
\text{iff}\quad & (m_1,n_1) +_K (m_3,n_3) \leq (m_2,n_2) +_K (m_3,n_3).
\end{aligned}
$$

■

Consider the abelian monoid $M(\mathcal{A},\mathcal{A}')$. We define an order relation "$\leq$" on $M(\mathcal{A},\mathcal{A}')$ by $[\pi_E] \leq [\pi_F]$ if and only if $E \precsim F$. Since $\precsim$ is a partial order on $\mathcal{P}(\mathcal{A})$ by lemma 1.1.14 "$\leq$" is a partial order on $M(\mathcal{A},\mathcal{A}')$. We now show that property P holds in $M(\mathcal{A},\mathcal{A}')$.

Let $[\pi_G] \in M(\mathcal{A},\mathcal{A}')$ and suppose $[\pi_E] \leq [\pi_F]$. We want to show that

$$[\pi_E] + [\pi_G] \leq [\pi_F] + [\pi_G]$$

By considering the representation $\pi_E \oplus \pi_G$ we assume that E and G are disjoint. Thus $\sup(E,G) = E + G$ is a finite projection in $\mathcal{A}$ and $[\pi_{E+G}] \in M(\mathcal{A},\mathcal{A}')$. Moreover, since
$H_{E+G} = (E \oplus G)(H) = E(H) \oplus G(H)$ one has that $[\pi_{E+G}] = [\pi_E \oplus \pi_G]$, and since $E + G \precsim F + G$ by corollary 1.1.13, $[\pi_{E+G}] \leq [\pi_{F+G}]$.
Hence $[\pi_E] + [\pi_G] \leq [\pi_F] + [\pi_G]$

Conversely, if $[\Pi_E] + [\Pi_G] \leq [\Pi_F] + [\Pi_G]$ one has $[\Pi_{E+G}] \leq [\Pi_{F+G}]$, which implies $E + G \lesssim F + G$ and by corollary 1.3.7 we have $E \lesssim F$. Thus $[\Pi_E] \leq [\Pi_F]$

Therefore lemma 2.1.11 shows that $\Gamma(\mathcal{A},\mathcal{A}')$ can be equipped canonically with an order relation $\leq$ such that $(\Gamma(\mathcal{A},\mathcal{A}'), \leq)$ is an ordered commutative group and that Dim E $\leq$ Dim F if and only if $E \lesssim F$ for any pair E,F of finite projections in $\mathcal{A}$. We call $\Gamma(\mathcal{A},\mathcal{A}')$ the index group of the operator algebra $\mathcal{A}$.

CHAPTER 3

DEFINITION OF FINITE, COMPACT AND FREDHOLM ELEMENTS RELATIVE TO $\mathcal{A}$

In this chapter we generalize the theory of compact and Fredholm operators on a complex Hilbert space to von Neumann algebras. This generalization depends to a large extent on the notion of the finiteness of a projection relative to $\mathcal{A}$.

In the first section we introduce the ideal of finite elements in $\mathcal{A}$ and define the compact elements relative to $\mathcal{A}$ as the norm closure of the set of all finite elements in $\mathcal{A}$. After that, the concept of a Fredholm element relative to $\mathcal{A}$ is introduced and the index defined.

3.1 <u>FINITE AND COMPACT ELEMENTS RELATIVE TO A VON NEUMANN ALGEBRA</u>

We begin this section by defining the null projection and the range projection of an element in $\mathcal{A}$.

Once we have defined what we mean by a compact element in $\mathcal{A}$, we will show that the set of all compact elements relative to $\mathcal{A}$ is a norm-closed two-sided *-ideal in $\mathcal{A}$. Moreover, we will show that this set is the smallest closed two-sided ideal containing the finite projections of $\mathcal{A}$.

3.1.1 *DEFINITION* ([1])

Let $T \in \mathcal{A}$. Then $N_T = \sup\{E \in \mathcal{P}(\mathcal{A}) \mid TE = 0\}$ is called the null projection of T, and $R_T = \inf\{E \in \mathcal{P}(\mathcal{A}) \mid ET = T\}$ is called the range projection of T.

3.1.2 *REMARKS*

(i) It is clear that N_T and R_T exist since $T0 = 0$ and

$IT = T$, where 0 (resp. I) is the zero projection (resp. identity projection) in $\mathcal{A}$. That N_T and R_T are elements of $\mathcal{A}$ follows since $\mathcal{P}(\mathcal{A})$ is a complete lattice.

(ii) By definition 1.1.15 we have that $R_T = S_\ell(T)$ (the left support of $T\in\mathcal{A}$). Hence remark 1.1.16 shows that R_T is the projection onto $\overline{T(H)}$. So R_T maps $\overline{T(H)}$ onto $\overline{T(H)}$, which implies $R_T T = T$.

By definition 3.1.1 $R_{T^*} = \inf\{E\in\mathcal{P}(\mathcal{A}) | ET^* = T^*\}$

$$= \inf\{E\in\mathcal{P}(\mathcal{A}) | TE = T\}$$

$$= S_r(T) \quad \text{(see def 1.1.15)}$$

Hence $R_T \sim R_{T^*}$ by lemma 1.1.17.

(iii) We claim that N_T is the projection onto the closed subspace $\{x\in H | Tx = 0\}$ of H.

If $N_T(H) \supsetneqq \{x\in H | Tx = 0\}$ there exists a $x\in N_T(H)$ with $Tx \neq 0$. Since $x\in N_T(H)$ one has $x = N_T x$. Hence $TN_T x\neq 0$ and by def 3.1.1 an $E\in\mathcal{R}$ exists such that $TEx \neq 0$-contrary to the fact that $TE = 0$ for every $E\in\mathcal{R}$ ($\mathcal{R} = \{E\in\mathcal{P}(\mathcal{A}) | TE = 0\}$). If $N_T(H) \subsetneqq \{x\in H | Tx = 0\}$ then the projection E' that corresponds to the closed subspace $\{x\in H | Tx = 0\}$ is such that $TE' = 0$ and $E' > N_T$ by lemma 1.1.1. If $E' \in\mathcal{P}(\mathcal{A})$ we have a contradiction with definition 3.1.1. Thus $N_T(H) = \{x\in H | Tx = 0\}$. We show that $E'\in\mathcal{A}$. Since every $T\in\mathcal{A}'$ is a linear combination of four unitary elements it is sufficient to show that $UE' = E'U$ for all unitary elements in $\mathcal{A}'$. Since $UT = TU$, one has $TUE'x = UTE'x = 0$ for all $x\in H$. Hence $UE'x\in E'(H)$, which implies $E'UE'x = UE'x$ ($x\in H$). So $E'UE' = UE'$. The same holds for the unitary element U^*. Thus $E'U^*E' = U^*E'$. By taking adjoints on both sides one has $E'UE' = E'U$. Thus $E'U = UE'$. This holds for every unitary element $U\in\mathcal{A}'$. So $E' \in \mathcal{A}'' = \mathcal{A}$.

Since $N_T(H) = \{x\in H | Tx = 0\}$ one has $TN_T = 0$.

■

3.1.3 *LEMMA* ([8], p 118)

If $T \in \mathcal{A}$ we have $N_T = I-R_{T^*}$ and $N_{T^*} = I-R_T$

Proof :

Since

$\{x \in H | Tx = 0\} = \{x \in H | (Tx,y) = 0 \text{ for all } y \in H\} = \{x \in H | (x,T^*y) = 0 \text{ for all } y \in H\} = T^*(H)^{\perp} = \overline{T^*(H)}^{\perp}$, it follows that $N_T = I-R_{T^*}$ (by remark 3.1.2 we have $R_{T^*} = S_r(T) = [T^*(H)]$) If we replace T by T^* we obtain $N_{T^*} = I-R_T$

■

3.1.4 *NOTE*

We could prove 3.1.3 directly from definition 3.1.1 and the fact that the mapping $E \rightarrow I-E$ reverses the ordering of projections in $\mathcal{A}$.

3.1.5 *DEFINITION* ([1])

The element $T \in \mathcal{A}$ is called finite (or of finite rank) relative to $\mathcal{A}$, if R_T is finite.

Let M_o be the set of all finite elements of $\mathcal{A}$. Then we have the following lemma.

3.1.6 *LEMMA* ([9], p 442)

The set M_o is a two-sided $*$-ideal of $\mathcal{A}$.

Proof :

Let $S \in \mathcal{A}$ and $T \in M_o$ arbitrary. Since $R_{TS}(H) = \overline{TS(H)} \subseteq \overline{T(H)} = R_T(H)$ lemma 1.1.1 implies that $R_{TS} \leq R_T$ and since R_T is finite relative to $\mathcal{A}$ lemma 1.2.4 implies that R_{TS} is finite. Hence $TS \in M_o$. Thus $M_o = M_o\mathcal{A}$.

Suppose $T \in M_o$ and $S \in M_o$. Then $R_{T+S}(H) = \overline{T+S(H)} = \overline{T(H) + S(H)}$ $\subseteq [T(H) \cup S(H)] \subseteq [\overline{T(H)} \cup \overline{S(H)}] = \sup(R_T, R_S)(H)$ by remark 1.1.3. Hence $R_{T+S} \leq \sup(R_T, R_S)$. Since $\sup(R_T, R_S)$ is finite by proposition 3.3.1, lemma 1.2.4 implies that R_{T+S} is finite. Hence $T+S \in M_o$.

If $a \neq 0$ is a scalar we have $R_{aT}(H) = \overline{aT(H)} = \overline{T(H)} = R_T(H)$. Thus $R_{aT} = R_T$. So R_{aT} is finite, which implies $aT \in M_o$. By remark 3.1.2(ii) $R_T \sim R_{T^*}$, so $T^* \in M_o$ if $T \in M_o$ (lemma 1.2.4). As $ST = (T^*S^*)^*$ and $T^*S^* \in M_o$ for all $T \in M_o$, $S \in \mathcal{A}$ one has $R_{ST} \sim R_{T^*S^*}$ and $ST \in M_o$. Thus M_o is a two-sided *-ideal of $\mathcal{A}$.

■

3.1.7 *DEFINITION* ([1])

Let M be the norm-closure of M_o. The elements of M are called compact (relative to $\mathcal{A}$). Clearly $M \subseteq \mathcal{A}$, since $\mathcal{A}$ is norm-closed and $M_o \subseteq \mathcal{A}$.

3.1.8 *LEMMA* ([1])
M is the smallest norm-closed two-sided *-ideal of $\mathcal{A}$ containing the finite projections of $\mathcal{A}$.

<u>Proof</u> :

Note that since $M = \bar{M}_o$ and M_o is a two-sided *-ideal in $\mathcal{A}$ we have that M is a closed two sided *-ideal of $\mathcal{A}$. Since $R_E = E$ for every projection E in $\mathcal{A}$ one has $E \in M_o$ for every finite projection E in $\mathcal{A}$. Hence M is a closed two-sided *-ideal of $\mathcal{A}$ containing the finite projections of $\mathcal{A}$.

We now show that M is the smallest such ideal. Let I be the set of all finite projections of $\mathcal{A}$ and let M' be the two-sided *-ideal in $\mathcal{A}$ generated by I. We want to show that $M_o = M'$. Then clearly M_o will be the smallest two-sided $*$-ideal in $\mathcal{A}$ that contains the finite projections of $\mathcal{A}$. If $T \in M_o$ we have $R_T \in I$. From $T = R_T T$ we have $T \in M'$. Thus $M_o \subseteq M'$. Conversely, since $I \subseteq M_o$ we have $M' \subseteq M_o$. Hence $M' = M_o$. Thus M is the smallest closed two-sided $*$-ideal in $\mathcal{A}$ containing I. If $M_o \subset M_1 \subset M$ and M_1 is a closed two-sided $*$-ideal in $\mathcal{A}$ then $\bar{M}_o \subset M_1$. Thus $M \subset M_1$ which implies $M = M_1$.

■

3.2 FREDHOLM ELEMENTS RELATIVE TO A VON NEUMANN ALGEBRA $\mathcal{A}$

Our aim in this section is to define a Fredholm element relative to $\mathcal{A}$ and the index of a Fredholm element in $\mathcal{A}$. We will also show that if finiteness of a projection $E \in \mathcal{A}$ implies finite dimensionality of E(H), then the following definition implies the classical definition for a bounded linear operator on H to be Fredholm.

3.2.1 *DEFINITION* ([1])

The element $T \in \mathcal{A}$ is called Fredholm (relative to $\mathcal{A}$), if the following two conditions hold

(i) N_T is finite

(ii) There is a finite projection $E \in \mathcal{P}(\mathcal{A})$ such that $(I-E)(H) \subseteq T(H)$.

We denote the set of all Fredholm elements in $\mathcal{A}$ by $\mathcal{F}(\mathcal{A})$.

3.2.2 *LEMMA* ([6], p 128)

If M is a closed subspace of the Hilbert space H, an N is a finite-dimensional subspace of H. Then the direct sum $M \oplus N$ is a closed subspace of H.

Proof : Let $a \in \overline{M \oplus N}$, then there exists a sequence $\{x_n + y_n\}_{n=1}^{\infty}$ in $M \oplus N$ such that $x_n + y_n \underset{\infty}{\overset{n}{\to}} a$.

We now show that $\{y_n\}_{n=1}^{\infty}$ is bounded. If it were not, there would exist a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ and a unit vector z in N such that

$$\lim_{k\to\infty} \|y_{n_k}\| = \infty \text{ and } \lim_{k\to\infty} y_{n_k}/\|y_{n_k}\| = z$$

(since N is finite dimensional its unit ball is compact).
However, since the sequence $\{(1/\|y_{n_k}\|)(x_{n_k} + y_{n_k})\}_{k=1}^{\infty}$ converges to 0, we have $\lim_{k\to\infty} x_{n_k}/\|y_{n_k}\| = -z$. This would imply that z is in both M and N - contrary to the fact that $z \neq 0$. Since the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded we may extract a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} y_{n_k} = y$ for some $y \in N$. Therefore, since $\{x_{n_k} + y_{n_k}\}_{k=1}^{\infty}$ converges, it is a Cauchy sequence and we obtain from

$$\|x_{n_k} - x_{n_m}\| = \|(x_{n_k} + y_{n_k}) - (x_{n_m} + y_{n_m}) - (y_{n_k} - y_{n_m})\|$$

$$\leq \|(x_{n_k} + y_{n_k}) - (x_{n_m} + y_{n_m})\| + \|y_{n_k} - y_{n_m}\|$$

$$\underset{\to\infty}{\overset{k,m}{}} 0,$$

that (x_{n_k}) is a Cauchy sequence in M and hence converges to a vector $x \in M$. Therefore $a = x+y \in M \oplus N$ which implies that $M \oplus N$ is closed.

■

3.2.3 *REMARKS*

It is well known that a Fredholm operator in L(H) is defined as an operator for which T(H) is closed, dim (Ker T) is finite and dim(Ker T^*) is finite in the usual sense.

We claim that if the finiteness of a projection relative to $\mathcal{A}$ implies finite dimensionality of its range space in the usual sense (The Note 1.2.2 shows that this is the case when $\mathcal{A} = L(H)$), definition 3.2.1 implies the above definition for $T \in \mathcal{A}$ to be Fredholm. Since $N_T(H) = \text{Ker}T$ by remark 3.1.2 (iii), condition (i) in 3.2.1 implies that dim (KerT) $< \infty$. If condition (ii) in 3.2.1 holds, a finite projection $E \in \mathcal{P}(\mathcal{A})$ with $(I-E)(H) \subseteq T(H)$ exists. We also have $(I-R_T)(H) = T(H)^{\perp}$. Since $H = E(H) \oplus (I-E)(H)$ we have $T(H)^{\perp} \subseteq E(H)$. Hence $I-R_T \leq E$ and since E is finite, $I-R_T$ is finite. Thus $N_{T^*} = I-R_T$ is finite. So $\text{Dim}(\text{Ker } T^*) < \infty$. Finally we show that condition (ii) in definition 3.2.1 implies that T(H) is closed :

Since T is Fredholm a finite projection E of $\mathcal{A}$ exists with $(I-E)(H) \subseteq T(H)$. By hypothesis E(H) is finite dimensional, so $E(H) \cap T(H)$ is finite dimensional. It is clear that $T(H) = (I-E)(H) \oplus E(H) \cap T(H)$. Since E is a projection we have that (I-E)(H) is closed. Hence T(H) is closed by lemma 3.2.2.

■

We have seen in the previous remark that condition (ii) of defintion 3.2.1 implies that N_{T^*} is finite, if T is Fredholm. This allows us to define the following :

3.2.4 *DEFINITION* ([1])

Let T be a Fredholm element of $\mathcal{A}$. We define the index of T as an element of the index group $\Gamma(\mathcal{A},\mathcal{A}')$ by the formula

$$\text{Index}(T) = \text{Dim}N_T - \text{Dim}N_{T^*}$$

where $\text{Dim}N_T$ was defined in chapter 2.

3.2.5 *PROPOSITION* ([1])

For every $T \in \mathcal{A}$ there is a non-decreasing sequence $F_1 \leq F_2 \leq \ldots$ in $\mathcal{P}(\mathcal{A})$ satisfying the following two conditions :

(i) For $k = 1,2,\ldots$ the range of F_k is contained in the range of T.

(ii) R_T is the supremum of the sequence $(F_k)_{k=1,2,\ldots}$

<u>Proof</u> :

Let $T \in \mathcal{A}$. Suppose $T = W|T|$ is the polar decomposition of T. Then W, $|T| \in \mathcal{A}$ by Corollary 1.1.18 and $R_T = WW^*$, $R_T* = W^*W$ by lemma 1.1.17. Suppose the proposition holds for $|T|$. Then a non-decreasing sequence $(E_k)_{k=1}^{\infty}$ of projections in $\mathcal{A}$ exists such that (i) and (ii) hold. Let $E_k' = WE_kW^*$. Since $E_k'E_\ell' = WE_kW^*WE_\ell W^*$ $= WE_kR_T*E_\ell W^* = WE_kR_{|T|}E_\ell W^*$ ($R_T* = [T^*(H)] = [|T|(H)]$ by the proof of lemma 1..1.17).

$$\begin{aligned}
\text{Hence } E_k'E_\ell' &= WE_kR_{|T|}E_\ell W^* \\
&= WE_kE_\ell W^* \quad (R_{|T|} = \sup_k E_k \geq E_k) \\
&= WE_kW^* \quad (\text{for all } k \leq \ell) \\
&= E_k'
\end{aligned}$$

Thus (E_k') is non-decreasing. Clearly $E_k'^* = E_k'$ and $E_k'^2 = WE_kW^*WE_kW^* = WE_k\ R_{|T|}E_kW^* = WE_kW^* = E_k'$ for all k. We also have $E_k'(H) = WE_kW^*(H) \subseteq WE_k(H) \subseteq W|T|(H) = T(H)$ for all k. Thus if $E \leq F$ $E,F \in \mathcal{P}(\mathcal{A})$ we have for all $x \in H$, that $(Ex,x) = \|Ex\|^2$ $= \|EFx\|^2 \leq \|Fx\|^2 = (Fx,x)$. So $E \leq F$ in the operator sense. Hence $\sup_k E_k' = \sup_k WE_kW^* = WR_{|T|}W^* = WR_T*W^* = WW^* = R_T$ (apply corollary 1.4.4).

Thus it suffices to show the theorem for T a positive element of A. So let T be given by its spectral decomposition $T = \int_0^\infty \lambda\ dE_\lambda$ where $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is the spectral family of projections of T. As seen in Chapter 1; section 1.4 where we stated the spectral decomposition theorem, each $E_\lambda \in \mathcal{A}$. We have also seen that E_λ is defined by $E_\lambda = N_{(T-\lambda I)^+}$ (see notes just after lemma 1.4.7), and $E_\lambda = \inf_{\lambda'>\lambda} E_{\lambda'}$ (the upper continuity property).

Hence $E_o = N_{T^+} = N_T$ (T is positive), and $R_T = I-N_{T^*} = I-N_T = I-E_o$. Since $E_o = \inf_{\lambda'>0} E_{\lambda'}$ we have

$$R_T = I-E_o = I - \inf_{\lambda'>0} E_{\lambda'} = \sup(I-E_{\lambda'}).$$

Consider any $\epsilon>0$. Since each E_ϵ commutes with T and each E_λ we have

$$T(E_\epsilon(H)) = E_\epsilon(T(H)) \subseteq E_\epsilon(H)$$

and

$$T(I-E_\epsilon)(H) = (I-E_\epsilon)T(H) \subseteq (I-E_\epsilon)(H).$$

Hence the pair $(E_\epsilon(H),(I-E_\epsilon)(H)$ of subspaces of H reduces T. Similarly we can show that the pair $(E_\epsilon(H),(I-E_\epsilon)(H))$ of subspaces of H reduces each E_λ, $(\lambda\in\mathbb{R})$. Denote the restrictions of T and E_λ to the space $(I-E_\epsilon)(H)$ by T_ϵ and $E_{\lambda,\epsilon}$. Consider the reduced algebra $\mathcal{A}_{I-E_\epsilon}$. Then $T_\epsilon \in \mathcal{A}_{1-E_\epsilon}$, i.e. $T_\epsilon:(I-E_\epsilon)(H) \to (I-E_\epsilon)(H)$ is a positive operator which is bounded. Consider $\{E_{\lambda,\epsilon}\}_{\lambda\in\mathbb{R}}$. Then clearly

(i) $E_{\lambda,\epsilon} \leq E_{\mu,\epsilon}$ for every $\lambda \leq \mu$ (since $E_\lambda(I-E_\epsilon)(H) \subseteq E_\mu(I-E_\epsilon)(H)$)

(ii) $E_{\lambda,\epsilon}\,(I-E_\epsilon)(H) = E_\lambda(I-E_\epsilon)(H) = 0$ if $\lambda \leq \epsilon$, so $E_{\lambda,\epsilon} = 0$ for all $\lambda \leq \epsilon$.

(iii) If $\mu \to \lambda+0$ we have $E_\mu x \to E_\lambda x$ for all $x\in H$. So $E_\mu(I-E_\epsilon)x \to E_\lambda(I-E_\epsilon)x$ for every $(I-E_\epsilon)x \in (I-E_\epsilon)(H)$. Thus $E_{\mu,\epsilon}x \to E_{\lambda,\epsilon}x$ if $\mu \to \lambda+0$.

Since $T = \int_{-\infty}^{\infty} \lambda dE_\lambda = \int_{0}^{\infty} \lambda dE_\lambda$ we have for every $\delta > 0$ that there exist $\lambda_1, \ldots \lambda_n \in Sp(T) \subset [0, \|T\|]$ (say $0 \leq \lambda_1 < \lambda_2 < \ldots\ldots < \lambda_n \leq \|T\|$) and projections $E_{\lambda_1}, \ldots, E_{\lambda_n} \in \mathcal{P}(\mathcal{A})$ such that

$$\|T - \sum_{j=1}^{n} \lambda_j (E_{\lambda_j} - E_{\lambda_{j-1}})\| < \delta \text{ (clearly } n \text{ depends on } \delta)$$

This holds if and only if

$$\sup_{\substack{\|x\| \leq 1 \\ x \in H}} \|(T - \sum_{j=1}^{n} \lambda_j (E_{\lambda_j} - E_{\lambda_{j-1}}))x\| < \delta$$

Thus $\sup_{\substack{\|x\| \leq 1 \\ x \in (I-E)(H)}} \|(T - \sum_{j=1}^{n} \lambda_j (E_{\lambda_j} - E_{\lambda_{j-1}}))x\| < \delta$, which implies

$\|T_\epsilon - \sum_{j=1}^{n} \lambda_j (E_{\lambda_j, \epsilon} - E_{\lambda_{j-1}, \epsilon})\| < \delta$. Hence

$T_\epsilon = \int_{-\infty}^{\infty} \lambda dE_{\lambda,\epsilon} = \int_{\epsilon}^{\infty} \lambda dE_{\lambda,\epsilon}$ (Notice from (ii) above that $dE_{\lambda,\epsilon} = 0$ for every $\lambda \leq \epsilon$)

From the representation $T_\epsilon = \int_{\epsilon}^{\infty} \lambda dE_{\lambda,\epsilon}$, $0 \notin Sp(T_\epsilon)$, which means that T_ϵ is regular (i.e. T_ϵ has an inverse in $\mathcal{A}_{I-E_\epsilon}$). Since $f : [\epsilon, \infty] \to \mathbb{R} : \lambda \to \lambda^{-1}$ is a continuous function on $Sp(T_\epsilon)$ one has $f(T_\epsilon) = \int_{\epsilon}^{\infty} f(\lambda)\, dE_{\lambda,\epsilon}$ ([11], theorem 9.10-1). Hence $T_\epsilon T_\epsilon^{-1} = T_\epsilon^{-1} T_\epsilon = I - E_\epsilon$ the identity of $\mathcal{A}_{I-E_\epsilon}$. Thus

$$(I-E_\epsilon)(H) = T_\epsilon T_\epsilon^{-1}(I-E_\epsilon)(H) \subseteq T_\epsilon(I-E_\epsilon)(H) \subseteq T(H)$$

Now let $\epsilon = 1/k$ $k = 1,2,3,\ldots$. If we define $F_k = I-E_{1/k}$ we have from $R_T = \sup_{\epsilon>0}(I-E_\epsilon)$ that $R_T = \sup_{k\in N} F_k$ and $F_k(H) = (I-E_{1/k})(H) \subseteq T(H)$ for all $k = 1,2,3,\ldots$ Clearly $F_1 \leq F_2 \leq F_3 \leq F_4 \ldots$ Thus the sequence $\{E_{1/k}\}$ is non-increasing. This completes the proposition.

■

3.2.6 COROLLARY ([1])

For every Fredholm element T of $\mathcal{A}$ there is a non-decreasing sequence $E_1 \leq E_2 \leq \ldots$ in $\mathcal{P}(\mathcal{A})$ such that conditions (i) and (ii) of proposition 3.2.5 are satisfied and $I-E_k$ is finite, relative to $\mathcal{A}$ (k=1,,2,...).

Proof :

Since $T\in\mathcal{A}$ is Fredholm there is an $E\in \mathcal{P}(\mathcal{A})$ such that $E(H) \subseteq T(H)$ and I-E is finite. From $E(H) \subseteq T(H) \subseteq \overline{T(H)}$ we have $E \leq R_T$. Thus $F = R_T - E$ is a projection in $\mathcal{A}$. Consider $FT\in\mathcal{A}$. From proposition 3.2.5 a sequence $E_1' \leq E_2' \leq \ldots$ of projections in $\mathcal{A}$ exists with $E_k'(H) \subseteq R_{FT}(H)$ and $\sup_k E_k' = R_{FT}$

Let $E_k = E + E_k'$. We show that $\{E_k\}_{k=1}^{\infty}$ is a non-decreasing sequence of projections in $\mathcal{A}$ such that the conditions of the corollary are satisfied :

Since $E_k'(H) \subseteq FT(H) \subseteq F(H)$ and $FE = 0$ one has $E_k' E = E_k'FE = 0$ for all k. Thus $E_k = E + E_k'$ is a projection in $\mathcal{A}$ for all k by corollary 1.1.9. Also $E_1 \leq E_2 \leq \ldots$ since $\{E_k'\}_{k=1}^{\infty}$ is a non-decreasing sequence of projections in $\mathcal{A}$. Moreover,

$$\begin{aligned} E_k(H) = (E + E_k')(H) &\subseteq E(H) + FT(H) \\ &\subseteq T(H) + (R_T-E)T(H) \\ &\subseteq T(H) \end{aligned}$$

because R_T is the identity on T(H) and $E(T(H)) \subseteq T(H)$.

Since $\sup E_k' = R_{FT}$ one has $\sup_k E_k = \sup_k (E + E_k') = E + \sup_k E_k'$ $= E + R_{FT}$. By definition 3.1.1 $R_{FT} = \inf\{G \in \mathcal{P}(\mathcal{A}) \mid GFT = FT\}$. Since $F(T(H)^{\perp}) = \{0\}$ and $F(\overline{T(H)}) \subseteq \overline{T(H)}$, $GFT = FT$ if and only if $GF = F$. Hence $R_{FT} = \inf\{G \in \mathcal{P}(\mathcal{A}) \mid GF = F\} = R_F = F$. So $\sup_k E_k = E + R_{FT} = E + F = R_T$. We also have that $I - E_k = I - (E + E_k') \leq I - E$ and since $I - E$ is finite lemma 1.2.4 implies that $I - E_k$ is finite relative to $\mathcal{A}$.

■

CHAPTER 4

GENERALIZATION OF THEOREMS IN FREDHOLM THEORY TO A VON NEUMANN ALGEBRA $\mathcal{A}$

We conclude this study with the generalization of several classical theorems on Fredholm operators to Fredholm elements in a von Neumann algebra. The main differences in the proofs of these classical theorems and the generalized ones are :

1. compact elements relative to $\mathcal{A}$ are not necessarily compact operators in the usual sense, and

2. the range of a Fredholm element in $\mathcal{A}$ is not necessarily closed.

The first theorem, due to F Riesz ([13], p 87), which will be generalized (the generalized Fredholm alternatives) says that I-T is Fredholm of index zero (relative to $\mathcal{A}$) if T is compact (relative to $\mathcal{A}$). This theorem will be used in the remaining two theorems : a decomposition theorem of F Riesz for compact operators and a theorem which characterizes the relative Fredholm elements modulo the relative compact elements, due to Atkinson ([13], p 90). The Theorem states that the Fredholm elements in $\mathcal{A}$ are exactly the inverse image of the group $G(\mathcal{A}/M)$ of regular elements of the quotient algebra $\mathcal{A}/M$ (M the compact elements) under the canonical quotient mapping $\Pi:\mathcal{A} \to \mathcal{A}/M$. From this theorem a number of important corollaries can be deduced, for example, the set of all Fredholm elements denoted by $F(\mathcal{A})$ is open in the norm topology on $\mathcal{A}$, and $F(\mathcal{A})$ is an involutive monoid with respect to multiplication in $\mathcal{A}$ etc.

4.1 <u>GENERALIZATION OF THEOREMS IN FREDHOLM THEORY TO A VON NEUMANN ALGEBRA $\mathcal{A}$</u>

We begin this section with the following theorem :

4.1.1 *THEOREM* (Generalized Fredholm alternatives, [1])

Consider the von Neumann algebra $\mathcal{A}$. If $T \in \mathcal{A}$ is compact relative to $\mathcal{A}$, I-T is Fredholm relative to $\mathcal{A}$ with index zero.

Proof :
The theorem is proved in two steps

(i) Suppose T is finite relative to $\mathcal{A}$. Then R_T is finite, and since $R_T \sim R_{T^*}$ from remark 3.1.2(ii), lemma 1.2.4 implies that R_{T^*} is finite. Hence $E = \sup (R_T, R_{T^*})$ is finite by proposition 1.2.8.

Clearly $I - E = I - \sup(R_T, R_{T^*}) = \inf(I-R_T, I-R_{T^*})$ and $(I-E)(I-T) = (I-E) - (I-E)T = I-E - (T-ET)$
Remark 1.1.3 implies that E is the identity on

$[\overline{T(H)} \cup \overline{T^*(H)}]$, thus also on T(H). Hence ET = T, so

$$(I-E)(I-T) = I-E$$

Similarly

$$(I-E)(I-T^*) = I-E.$$

By taking adjoints left and right of the two equations we have $(I-T^*)(I-E) = I-E$ and $(I-T)(I-E) = I-E$

Since $\overline{(I-T^*)(H)} \supseteq \overline{(I-T^*)(I-E)(H)}$ one has

$$R_{I-T^*} \geq R_{(I-T^*)(I-E)} = R_{I-E} = I-E$$

Hence lemma 3.1.3 implies that
$N_{I-T} = I-R_{I-T^*} \leq I-(I-E) = E$. Similarly $N_{I-T^*} \leq E$
Hence N_{I-T}, N_{I-T^*} are finite projections in $\mathcal{A}$ (lemma 1.2.4)

Since $R_S \sim R_{S^*}$ for every $S \in \mathcal{A}$, one has $R_{E-T} \sim R_{E-T^*}$. By using lemma 3.1.3 we show that

$$R_{E-T} = E - N_{I-T^*} \quad , \quad R_{E-T^*} = E - N_{I-T}$$

We prove the first relation. The second one follows by interchanging the roles of T and T^*. By definition 3.1.1 we have

$$E - N_{I-T^*} = E - \sup\{F\in\mathcal{P}(\mathcal{A}) \mid (I-T^*)F = 0\}$$
$$= \inf\{E - F \in\mathcal{P}(\mathcal{A}) \mid (I-T^*)F = 0\}$$

Note that E − F is a projection for every $F\in\mathcal{P}(\mathcal{A})$ for which $(I-T^*)F = 0$ since $F \leq N_{I-T^*} \leq E$. Thus

$$E - N_{I-T^*} = \inf\{G\in\mathcal{P}(\mathcal{A}) \mid (I-T^*)(E-G) = 0\} \quad (G = E-F)$$

Since $T(H) \subseteq E(H)$ we have ET = T, hence

$$(I-T^*)(E-G) = E-G - (ET)^* + T^*G = E-G - T^* + T^*G$$
$$= E-EG - T^* + T^*G = (E-T^*)(I-G) \quad (G\leq E)$$

Therefore,
$$E - N_{I-T^*} = \inf\ \{G\in\mathcal{P}(\mathcal{A}) \mid (E-T^*)(I-G) = 0\}$$
$$= \inf\ \{I-F'\in\mathcal{P}(\mathcal{A}) \mid (E-T^*)F' = 0\}$$
$$= I - \sup\ \{F'\in\mathcal{P}(\mathcal{A}) \mid (E-T^*)F' = 0\}$$
$$= I - N_{E-T^*}$$
$$= R_{E-T}$$

Then since E is finite and $R_{E-T} \leq E$ and $R_{E-T^*} \leq E$ we have that R_{E-T} and R_{E-T^*} are finite (lemma 1.2.4) with $R_{E-T} \sim R_{E-T^*}$. By considering the reduced algebra $\mathcal{A}_E$ with identity element E we have $N_{I-T^*}, N_{I-T}\in\mathcal{A}_E$. Hence $R_{E-T}, R_{E-T^*} \in \mathcal{A}_E$ ($\mathcal{A}_E$ a finite von Neumann algebra)

By proposition 1.3.4 $E - R_{E-T} \sim E - R_{E-T^*}$. Together with $R_{E-T} = E - N_{I-T^*}$ and $R_{E-T^*} = E - N_{I-T^*}$ we have that $N_{I-T^*} \sim N_{I-T}$. Since N_{I-T} is finite and E is a finite projection in $\mathcal{A}$ with $(I-E)(H) = (I-T)(I-E)(H) \subseteq (I-T)(H)$ and

$$\text{Index } (I-T) = \dim N_{I-T} - \dim N_{I-T^*}$$
$$= 0 \quad (N_{I-T^*} \sim N_{I-T});$$

we have from definition 3.2.1 that I-T is a Fredholm operator with index zero.

(ii) Suppose T is a compact operator relative to $\mathcal{A}$. Since $M = \bar{M}_o$ a $T_o \in M_o$ exists with

$$\| T - T_o \| < 1$$

Thus $\| I-(I-(T-T_o)) \| < 1$. Let $S = I-(T-T_o)$, then S is regular (This well known fact can be found in [11], p 398).

By definition 3.1.1 $N_{SF} = \sup\{E \in \mathcal{P}(\mathcal{A}) \mid ESF = 0\}$. Since S^{-1} exists $SFx = 0$ if and only if $S^{-1}(SFx) = 0$ if and only if $Fx = 0$ $(x \in H, F \in \mathcal{P}(\mathcal{A}))$. Hence N_{SF} is the projection onto $\{x \in H \mid Fx = 0\} = N_F(H)$. Consequently $N_{SF} = N_F = I-R_F = I-F$ for every $F \in \mathcal{P}(\mathcal{A})$. Together with lemma 3.1.3 and remark 3.1.2(i) this implies $R_{SF} \sim R_{FS^*} = I-N_{SF} = I-(I-F) = F$ for every $F \in \mathcal{P}(\mathcal{A})$.

We want to prove the following equivalences :

$$(S-T_o)F = 0 \text{ iff } (I-T_oS^{-1})SF = 0 \text{ iff } (I-T_oS^{-1})R_{SF} = 0, \quad (4.1)$$

and

$$(I-T_oS^{-1})F = 0 \text{ iff } (I-T_oS^{-1})SS^{-1}F = 0 \text{ iff } (S-T_o)R_{S^{-1}F} = 0. \quad (4.2)$$

The first equivalences in the two relations (4.1 and 4.2) follow directly since $(S-T_o)F = (I-T_oS^{-1})SF$ and $(I-T_oS^{-1})F = (I-T_oS^{-1})SS^{-1}F$ $(SS^{-1} = I)$

We show the second equivalence in relation 4.1 (the second equivalence in relation 4.2 follows similarly)

$$(I-T_oS^{-1})SF(H) = \{0\}$$

if and only if $(I-T_oS^{-1})R_{SF}(H) = \{0\}$ $(R_{SF}(H) = \overline{SF(H)}$, and $(I-T_oS^{-1})(\overline{SF(H)}) \subseteq \overline{(I-T_oS^{-1})SF(H)}$, since $I-T_oS^{-1}$ is continuous).

Substitute F by N_{S-T_o} (resp. $N_{I-T_oS^{-1}}$) in relation 4.1 (resp. relation 4.2). We have seen that $R_{SF} \sim F$ for all $F \in \mathcal{P}(\mathcal{A})$; thus from $F = N_{S-T_o}$, we obtain

$$R_{SN_{S-T_o}} \sim N_{S-T_o}.$$

Observing that $(S-T_o)N_{S-T_o} = 0$, relation 4.1 above implies that

$$(I-T_oS^{-1})R_{SN_{S-T_o}} = 0$$

Hence from definition 3.1.1 one has that $R_{SN_{S-T_o}} \leq N_{I-T_oS^{-1}}$.

So $$N_{S-T_o} \sim R_{SN_{S-T_o}} \leq N_{I-T_oS^{-1}},$$

or equivalently

$$N_{S-T_o} \lesssim N_{I-T_oS^{-1}}$$

Similarly, the second relation above, together with the fact that $R_{SF} \sim F$ for all $F \in \mathcal{P}(\mathcal{A})$ imply that

$$N_{I-T_oS^{-1}} \lesssim N_{S-T_o} \qquad (F = N_{I-T_oS^{-1}})$$

Thus $$N_{S-T_o} \sim N_{I-T_oS^{-1}}$$ by lemma 1.1.14

If $(S^* - T_o^*)x = 0$ $(x \in H)$, then $(S^*)^{-1}(S^*-T_o^*)x = 0$ where $(S^*)^{-1} = (S^{-1})^*$ and $x \in H$. Thus $N_{S^*-T_o^*}(H) \subseteq N_{I-(S^*)^{-1}T_o^*}(H)$ so lemma 1.1.1 implies that $N_{S^*-T_o^*} \leq N_{I-(S^*)^{-1}T_o^*}$

Likewise $N_{I-(S^*)^{-1}T_o^*} \leq N_{S^*(I-(S^*)^{-1}T_o^*)} = N_{S^*-T_o^*}$

Thus $N_{S^*-T_o^*} = N_{I-(S^*)^{-1}T_o^*}$

So, the relation $N_{S^*-T_o^*} \sim N_{I-(S^*)^{-1}T_o^*}$ is trivial. Since $T_o \in M_o$, and M_o is a two-sided $*$-ideal, $T_oS^{-1} \in M_o$ which means that T_oS^{-1} and $(S^*)^{-1}T_o^*$ are finite.

Part (i) of this theorem implies that $N_{I-T_oS^{-1}}$ and $N_{I-(S^*)^{-1}T_o^*}$ are finite projections in $\mathcal{A}$. From $N_{I-T} = N_{S-T_o} \sim N_{I-T_oS^{-1}}$; $N_{I-T^*} = N_{S^*-T_o^*} \sim N_{I-(S^*)^{-1}T_o^*}$ and lemma 1.2.4 we conclude that N_{I-T} and N_{I-T^*} are finite. Part (i) also implies that $N_{I-T_oS^{-1}} \sim N_{I-(S^*)^{-1}T_o^*}$, which gives

$$N_{I-T} \sim N_{I-T^*}.$$

Define $F = \sup(R_{T_oS^{-1}}, R_{(S^*)^{-1}T_o^*})$ and $F_S = S^{-1}FS$

Then F is finite by proposition 1.2.8 and

$I-F_S = S^{-1}(I-F)S = S^{-1}\inf(I-R_{T_oS^{-1}}, I-R_{(S^*)^{-1}T_o^*})S$

Since $I-T = S-T_o$ we have by using lemma 3.1.3 that

$$\begin{aligned}
&(I-T)(I-F_S)S^{-1} \\
&= (S-T_o)S^{-1}\inf(N_{T_oS^{-1}}, N_{(S^*)^{-1}T_o^*}) \\
&= \inf(N_{T_oS^{-1}}, N_{(S^*)^{-1}T_o^*}) - T_oS^{-1}\inf(N_{T_oS^{-1}}, N_{(S^*)^{-1}T_o^*}) \\
&= \inf(N_{T_oS^{-1}}, N_{(S^*)^{-1}T_o^*}) = I-F
\end{aligned}$$

($T_oS^{-1}\inf(N_{T_oS^{-1}}, N_{(S^*)^{-1}T_o^*}) = 0$ since $T_oS^{-1}N_{T_oS^{-1}} = 0$ and $\inf(N_{T_oS^{-1}}, N_{(S^*)^{-1}T_o^*}) \leq N_{T_oS^{-1}}$)

Hence $(I-F)(H) = (I-T)(I-F_S)S^{-1}(H) \subseteq (I-T)(H)$ and F is a finite projection in $\mathcal{A}$. Together with $N_{I-T} \sim N_{I-T^*}$ and N_{I-T} finite, we conclude that I-T is Fredholm with index zero.

■

4.1.2 *COROLLARY*

If $T \in \mathcal{A}$ is compact relative to $\mathcal{A}$ then $(I-T)^n$ is Fredholm with index zero for every n = 1,2,3,...

Proof :

Consider $(I-T)^n$, $T \in \mathcal{A}$ compact. Then $I-(I-T)^n = Tp(T)$ where p(T) is a polinomial in T of degree (n-1). Since T is compact and M is a closed two-sided ideal in $\mathcal{A}$ one has $Tp(T) \in M$. Thus $I-(I-T)^n$ is compact relative to $\mathcal{A}$. From theorem 4.1.1 one has $(I-T)^n$ is Fredholm relative to $\mathcal{A}$ with index zero.

■

4.1.3 *DEFINITION*

Let A, B and C be vector spaces. Consider the maps $\alpha: A \to B$ and $\beta: B \to C$. The sequence $0 \xrightarrow{j} A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} 0$ is called a short exact sequence if it is exact at A, B and C, i.e. Imf j = Ker α, Imf α = Ker β and Imf β = Ker γ.

Clearly exactness at A is equivalent to α being injective ($\{0\}$ = Imf j and Ker $\alpha = \alpha^{-1}(\{0\})$), while exactness at C is equivalent to β being surjective (Ker γ = C).

Before we can proof the remaining two theorems we need the following lemma

4.1.4 *LEMMA* ([2])

Let S,T be elements of $F(\mathcal{A})$ (the Fredholm elements relative to $\mathcal{A}$). Then $N_{ST} - N_T \sim \inf(R_T, N_S)$

Proof :

We claim that the sequence

$$0 \rightarrow \text{Ker } T \xrightarrow{i} \text{Ker } ST \xrightarrow{T} \text{range } T \cap \text{Ker } S \rightarrow 0 \text{ is exact}$$

It is exact at Ker T since i is the inclusion mapping which is injective. Since i(Ker T) = Ker T the sequence is exact at Ker ST. To show exactness at range $T \cap$ Ker S one has only to show that T is onto range $T \cap$ Ker S. Take $y \in$ range $T \cap$ Ker S. Then $y = Tx$ for some $x \in H$ and $Sy = 0$. Thus $STx = 0$ which implies $x \in$ Ker ST and $Tx = y$. Thus T is onto.

It is well known that every short exact sequence splits i.e.

$$0 \rightarrow \text{Ker } ST \ominus \text{Ker } T \xrightarrow{T(N_{ST}-N_T)} \text{range } T \cap \text{Ker } S \rightarrow 0$$

is exact or equivalently

$$\text{Ker } (ST) \ominus \text{Ker } T \overset{T(N_{ST}-N_T)}{\simeq} \text{range } T \cap \text{Ker } S$$

We now show that $R_{(N_{ST}-N_T)T^*} = N_{ST} - N_T$

Lemma 3.1.3 implies $R_{(N_{ST}-N_T)T^*} = I - N_{T(N_{ST}-N_T)}$ and from remark 3.1.2(iii) we have

$$N_{T(N_{ST}-N_T)}(H) = \{x \in H \mid T(N_{ST}-N_T)x = 0\}$$

But $T(N_{ST} - N_T)\, x = 0$ if and only if $(N_{ST} - N_T)\, x \in \text{Ker }(T) = N_T(H)$ Since $N_T(N_{ST} - N_T) = 0$ we also have $N_T(H) \cap (N_{ST}-N_T)(H) = \{0\}$. Thus since $(N_{ST} - N_T)x \in (N_{ST} - N_T)(H)$, we have $T(N_{ST} - N_T)x = 0$ if and only if $(N_{ST} - N_T)x \in N_T(H) \cap (N_{ST} - N_T)(H) = \{0\}$

So $N_{T(N_{ST}-N_T)}$ is the projection onto

$$\{x \in H \mid (N_{ST} - N_T)x = 0\} = N_{N_{ST}-N_T}(H)$$

Hence $N_{N_{ST}-N_T} = I - R_{(N_{ST}-N_T)} = I - (N_{ST} - N_T)$, and consequently $R_{(N_{ST}-N_T)T^*} = I - N_{T(N_{ST}-N_T)} = I - (I - N_{ST} - N_T) = N_{ST} - N_T$

Since $R_{T(N_{ST}-N_T)}(H) = \overline{T(N_{ST} - N_T)(H)} \subseteq \overline{T(H)}$, $R_{T(N_{ST}-N_T)} \leq R_T$.

If $y \in T(N_{ST} - N_T)(H)$ then $x \in H$ exists with $y = T(N_{ST} - N_T)x$ and $Sy = ST(N_{ST} - N_T)x = 0$. Thus $T(N_{ST} - N_T)(H) \subseteq N_S(H)$ which implies $\overline{T(N_{ST} - N_T)(H)} \subseteq N_S(H)$ ($N_S(H)$ is closed). Thus $R_{T(N_{ST}-N_T)} \leq N_S$ by lemma 1.1.1. Together with $R_{T(N_{ST}-N_T)} \leq R_T$ we get

$$R_{T(N_{ST}-N_T)} \leq \inf(R_T, N_S) \tag{4.3}$$

Choose a sequence $E_1 \leq E_2 \leq \ldots$ of projections in $\mathcal{A}$ according to corollary 3.2.6 such that $E_n(H) \subseteq T(H)$; $\sup_n E_n = R_T$ and $I-E_n$ is finite for each $n = 1,2,\ldots$

From the exactness of the sequence

$$0 \longrightarrow \text{Ker } ST \ominus \text{Ker } T \xrightarrow{T(N_{ST}-N_T)} \text{range } T \cap \text{Ker } S \longrightarrow 0$$

one has $\inf(E_n, N_S)(H) = E_n(H) \cap N_S(H) \subseteq T(H) \cap N_S(H) = T(N_{ST} - N_T)(H)$ $\subseteq \overline{T(N_{ST} - N_T)(H)}$

Thus $\inf(E_n, N_S) \leq R_{T(N_{ST}-N_T)}$ for every n (lemma 1.1.1). (4.4)
Define $E_o = \inf(E_1, I-N_S)$. Then $I - E_o = I - \inf(E_1, I-N_S)$ $= \sup(I-E_1, N_S)$ which is finite since $I-E_1$ and N_S are finite ($S \in \mathcal{F}(\mathcal{A})$).

$N_S(I-N_S) = 0$ and $E_1 \leq E_n$ imply $E_oN_S = E_o(I-N_S)N_S = 0$ and $E_o \leq E_1 \leq E_n$. Since

$$(E_n - E_o)(H) \cap N_S(H) = E_n(H) \cap (I-E_o)(H) \cap N_S(H)$$

from corollary 1.1.9 we have

$$(E_n - E_o)(H) \cap N_S(H) = E_n(H) \cap N_S(H) \quad (N_S(H) \subseteq (I-E_o)(H).$$

Together with $R_T \geq E_o$ we have that

$$\inf(E_n, N_S) = \inf(E_n - E_o, N_S),$$

and

$$\inf(R_T, N_S) = \inf(R_T - E_o, N_S) \qquad (E_o N_S = 0) \qquad (4.5)$$

Then $E_1 - E_o \leq E_2 - E_o \leq E_3 - E_o \leq \ldots$ and $E_n - E_o \in \mathcal{P}(\mathcal{A})$ with

$\sup_n (E_n - E_o) = R_T - E_o \leq I - E_o$. Hence lemma 1.2.4 implies that

$R_T - E_o$ is a finite projection in $\mathcal{A}$.

From proposition 1.4.15 one has

$$\inf(R_T - E_o, N_S) = \sup_n\{\inf(E_n - E_o, N_S)\}.$$

Thus by (4.5) $\inf(R_T, N_S) = \sup_n \{\inf(E_n, N_S)\}$ (4.6)

The relation (4.4) and (4.6) imply that $\inf(R_T, N_S) \leq R_{T(N_{ST} - N_T)}$

Together with (4.3) we have $\inf(R_T, N_S) = R_{T(N_{ST} - N_T)}$

Since $R_{(N_{ST} - N_T)T^*} = N_{ST} - N_T$ and $R_{(N_{ST} - N_T)T^*} \sim R_{T(N_{ST} - N_T)}$ (remark 3.1.2(ii)), it follows that $R_{T(N_{ST} - N_T)} \sim N_{ST} - N_T$.

The result follows since $R_{T(N_{ST} - N_T)} = \inf(R_T, N_S)$ ■

4.1.5 *THEOREM* (decomposition theorem, [1])

Let $T \in \mathcal{A}$ be a compact element relative to $\mathcal{A}$. Let N_∞ be the supremum of the non-decreasing sequence

$$N_{I-T} \leq N_{(I-T)^2} \leq N_{(I-T)^3} \leq \ldots$$

and let R_∞ be the infimum of the non-increasing sequence

$$R_{I-T} \geq R_{(I-T)^2} \geq R_{(I-T)^3} \geq \ldots$$

Then

(i) $N_\infty \sim I - R_\infty$

(ii) $N_\infty TN_\infty = TN_\infty$, $R_\infty TR_\infty = TR_\infty$

(iii) If T is finite, then N_∞ is finite.

(iv) If N_∞ is finite, then $\inf(N_\infty, R_\infty) = 0$, $\sup(N_\infty, R_\infty) = I$.

Proof :

Define $N_n = N_{(I-T)^n}$, $R_n = R_{(I-T)^n}$

(i) Corollary 4.1.2 implies that $(I-T)^n$ is Fredholm with index zero. Thus $N_{(I-T)^n} \sim N_{(I-T)^{n*}} = I-R_{(I-T)^n}$ (lemma 3.1.3). So $N_n \sim I-R_n$. Since $(I-T)^n \in F(\mathcal{A})$ for all n, N_n is a finite projection in $\mathcal{A}$ for each $n = 1,2,\ldots$. From the relations $N_n \sim I - R_n$ and $N_{n+1} \sim I-R_{n+1}$, and proposition 1.3.4 one has $N_{n+1} - N_n \sim (I-R_{n+1}) - (I-R_n) = R_n - R_{n+1}$

Since $\{N_n\}_{n=1}^{\infty}$ is an increasing sequence of projections in $\mathcal{A}$ lemma 1.1.10 and 1.1.11 imply that

$$N_\infty = \lim_n N_n = N_1 + \sum_{n=1}^{\infty} (N_{n+1} - N_n),$$

where the limit and sum are taken in the strong operator topology on $\mathcal{A}$. Similarly

$$I - R_\infty = I-R_1 + \sum_{n=1}^{\infty} (R_n - R_{n+1}).$$

Taking note of the fact that the sequence $\{N_1, N_{n+1} - N_n\}_{n=1}^{\infty}$ (resp. $\{I - R_1, R_n - R_{n+1}\}_{n=1}^{\infty}$) is mutually disjoint, proposition 1.1.12 together with the relation $N_{n+1} - N_n \sim R_n - R_{n+1}$ $(n \in N)$ imply that

$$N_\infty = N_1 + \sum_{n=1}^{\infty}(N_{n+1} - N_n) \sim I - R_1 + \sum_{n=1}^{\infty}(R_n - R_{n+1})$$

$$= I - R_\infty$$

(ii) From remark 3.1.2(iii) it follows that

$$(I - T)^n (I - T)N_{n+1} = 0$$

Therefore $(I - T)^n R_{(I-T)N_{n+1}}(H) = (I - T)^n \overline{(I-T)N_{n+1}(H)}$

$\subseteq \overline{(I-T)^n(I-T)N_{n+1}(H)} = \{0\}$

Thus, from the relation $(I - T)^n R_{(I-T)N_{n+1}} = 0$ and definition 3.1.1 we obtain $R_{(I-T)N_{n+1}} \leq N_{(I-T)^n} = N_n$

Hence N_n is the identity on $(I - T)N_{n+1}(H)$, which implies that $N_n(I - T)N_{n+1} = (I - T)N_{n+1}$ or $(I - N_n)(I - T)N_{n+1} = 0$. In view of lemma 1.1.10 it follows that $(I - N_\infty)(I - T)N_\infty = 0$ (the limits are taken in the strong operator topology on $\mathcal{A}$).

Thus $((I - N_\infty) - (I - N_\infty)T)N_\infty = 0$, which implies that $(I - N_\infty)TN_\infty = 0$, or equivalently $TN_\infty = N_\infty TN_\infty$. This proves the first relation (ii). Consider the relation

$$R_n (I - T)^n = (I - T)^n$$

or

$$(I - R_n)(I - T)(I - T)^{n-1} = 0$$

Then

$$(I - R_n)(I - T)R_{n-1}(H)$$

$$= (I - R_n)(I - T)\overline{(I - T)^{n-1}(H)}$$

$$\subseteq \overline{I - R_n(I - T)(I - T)^{n-1}(H)} = \{0\}$$

Thus

$$(I - R_n)(I - T)R_{n-1} = 0$$

Taking the limits in the strong operator topology on $\mathcal{A}$ one gets

$$(I - R_\infty)(I - T)R_\infty = 0$$

Hence $(I - R_\infty)R_\infty - (I - R_\infty)TR_\infty = 0$, which implies $TR_\infty = R_\infty TR_\infty$. This proves the second relation (ii).

(iii) Suppose that T is finite. Let $E = \sup(R_T, R_{T^*})$. Then E is finite from proposition 1.2.8. We have seen in part (i) of theorem 4.1.1 that $(I - E)(I - T) = I - E$. Then

$$\begin{aligned}(I - E)(I - T)^2 &= [(I - E)(I - T)](I - T) \\ &= (I - E)(I - T) = I - E\end{aligned}$$

By induction we get $(I - E)(I - T)^n = (I - E)$ for all $n = 1,2,\ldots$

Hence $N_n \leq E$ for all $n = 1,2,\ldots$(see part (i) of theorem 4.1.1). This implies that $N_\infty \leq E$ $(N_\infty = \sup_n N_n \leq E)$. Since E is a finite projection in $\mathcal{A}$ lemma 1.2.4 implies that N_∞ is finite.

(iv) We have seen in the proof of Corollary 4.1.2 that $I - (I - T)^k \in M$ for all $k = 1,2,\ldots$ Let $T_{(k)} = I - (I - T)^k$. Then $I - T_{(k)} = (I - T)^k \in \mathcal{F}(\mathcal{A})$ from corollary 4.1.2. We have that $N_k = N_{I-T_{(k)}}$ and $R_k = R_{I-T_{(k)}}$.

Define $R_n^{(k)} = \inf(R_{nk}, N_k)$. We apply lemma 4.1.4 : Let $S = I - T_{(k)}$ and $T = (I - T_{(k)})^n = I - T_{(nk)}$

Then by lemma 4.1.4 if $S,T \in \mathcal{F}(\mathcal{A})$, then

$$N_{ST} - N_T \sim \inf(R_T, N_S)$$

In the above notation we have that

$$N_{(n+1)k} - N_{nk} \sim \inf(R_{nk}, N_k) = R_n^{(k)}$$

Since N_∞ is finite by hypothesis, the reduced algebra $\mathcal{A}_{N_\infty}$ is finite. For every finite normal trace ϕ on $\mathcal{A}_{N_\infty}$ we have :

$$\lim_{n\to\infty} \phi(N_{(n+1)k} - N_{nk})$$

$$= \lim_{n\to\infty} [\phi(N_{(n+1)k}) - \phi(N_{nk})]$$

$$= \lim_{n\to\infty} \phi(N_{(n+1)k}) - \lim_{n\to\infty} \phi(N_{nk})$$

$$= \phi(N_\infty) - \phi(N_\infty) = 0 \quad (\phi \text{ is normal})$$

Since $R_n^{(k)} \sim N_{(n+1)k} - N_{nk}$ proposition 1.4.14 implies that $\phi(R_n^{(k)}) = \phi(N_{(n+1)k} - N_{nk})$ for every finite normal trace ϕ on $\mathcal{A}_{N_\infty}$

Thus

$$\lim_{n\to\infty} \phi(R_n^{(k)}) = \lim_{n\to\infty} \phi(N_{(n+1)k} - N_{nk}) = 0$$

or

$$\phi(R_\infty^{(k)}) = 0 \qquad (\phi \text{ is normal})$$

Hence proposition 1.4.14 implies that $R_\infty^{(k)} = 0$, and consequently $\inf(R_\infty,N_k) = R_\infty^{(k)} = 0$ for all k.

The finiteness of the projection N_∞ and proposition 1.4.15 imply that

$$\inf(R_\infty,N_\infty) = \sup_k\{\inf(R_\infty,N_k)\} = 0$$

From corollary 1.1.20 one has that

$$\sup(R_\infty,N_\infty) - R_\infty \sim N_\infty - \inf(R_\infty,N_\infty);$$

but $\inf(R_\infty,N_\infty) = 0$, so

$$\sup(R_\infty,N_\infty) - R_\infty \sim N_\infty .$$

Since $I - R_\infty \sim N_\infty$ (by (i)), we have

$$I - R_\infty \sim N_\infty \sim \sup(R_\infty, N_\infty) - R_\infty$$

Also observe that $I - R_\infty \geq \sup(R_\infty, N_\infty) - R_\infty$. Thus

$$I - R_\infty \sim \sup(R_\infty, N_\infty) - R_\infty \leq I - R_\infty$$

Taking note of the fact that $N_\infty \sim I - R_\infty$ and N_∞ is finite, lemma 1.2.4 implies that $I - R_\infty$ is finite. Hence $I - R_\infty = \sup(R_\infty, N_\infty) - R_\infty$ (definition 1.2.1). Therefore, $\sup(R_\infty, N_\infty) = I$

■

In the proof of the last theorem we will use the following notation : As before, M denotes the two-sided $*$-ideal of compact elements of $\mathcal{A}$. The quotient algebra $\mathcal{A}/M$ is denoted by $\bar{\mathcal{A}}$; $\pi : \mathcal{A} \to \bar{\mathcal{A}} : T \to T + M$ is the canonical homomorphism and $G(\mathcal{A})$ (resp. $G(\bar{\mathcal{A}})$) denotes the group of regular elements of $\mathcal{A}$ (resp. $\bar{\mathcal{A}}$).

4.1.6 *THEOREM* ([2])

(i) If $\mathcal{A}$ is a finite von Neumann algebra, then $M = \mathcal{A}$ and $\mathcal{F}(\mathcal{A}) = \mathcal{A}$

(ii) If $\mathcal{A}$ is not of finite type, then $M \neq \mathcal{A}$ and $\mathcal{F}(\mathcal{A}) = \pi^{-1} G(\bar{\mathcal{A}})$

<u>Proof</u> :

(i) Let $T \in \mathcal{A}$ then R_T is a finite projection ($\mathcal{A}$ is finite if and only if $I \in \mathcal{A}$ is finite; and $R_T \leq I$). Thus $T \in M_o \subseteq M$. Hence $\mathcal{A} \subseteq M$. It is clear that $M \subseteq \mathcal{A}$. So we concluded that $M = \mathcal{A}$. To show that $\mathcal{F}(\mathcal{A}) = \mathcal{A}$, let $T \in \mathcal{A}$. Then $T - I = -S \in \mathcal{A}$ ($S := I - T$). Thus $T = I - S$ is Fredholm, since $S \in \mathcal{A} = M$ (Theorem 4.1.1). This shows that $\mathcal{A} \subseteq \mathcal{F}(\mathcal{A})$. Clearly $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{A}$. Thus $\mathcal{A} = \mathcal{F}(\mathcal{A})$.

(ii) We claim that M contains no infinite projections. Let $E \in \mathcal{P}(\mathcal{A})$ be an infinite projection with $E \in M$. Then $I - E$ is Fredholm by theorem 4.1.1, and so definition 3.2.1(i) implies that $N_{I-E} = I - R_{I-E} = I - (I-E) = E$ is finite. This contradicts the fact that E was chosen to be infinite. Hence M contains no infinite projections. So if $\mathcal{A}$ is infinite $I \notin M$. Therefore $M \neq \mathcal{A}$. Let $T \in \Pi^{-1}(G(\bar{\mathcal{A}}))$. Then $\Pi(T) \in G(\bar{\mathcal{A}})$. Therefore there exists an $S \in \mathcal{A}$ such that $\Pi(T)\Pi(S) = \Pi(S)\Pi(T) = \Pi(I)$ ($\Pi(I)$ is the identity element in $\bar{\mathcal{A}}$). Since Π is a homomorphism, $\Pi(T)\Pi(S) = \Pi(TS)$. Hence $\Pi(TS) = \Pi(ST) = \Pi(I)$. This implies that $\Pi(TS - I) = \bar{0}$ and $\Pi(ST - I) = \bar{0}$ where $\bar{0}$ (= M) is the identity element of $\bar{\mathcal{A}}$. Thus $ST - I \in M$ and $TS - I \in M$. Let $-C$ and $-D$ be elements in M with $ST - I = -C$ and $TS - I = -D$.
So

$$ST = I - C \quad \text{and} \quad TS = I - D. \tag{4.7}$$

Since $N_T \leq N_{ST}$ ($Tx = 0$ implies $STx = 0$ $x \in H$, and so $N_T(H) \subseteq N_{ST}(H)$), the first relation above implies that $N_T \leq N_{I-C}$. Theorem 4.1.1 clearly implies that $I - C$ is Fredholm relative to $\mathcal{A}$. Therefore N_{I-C} is a finite projection in $\mathcal{A}$. Observing that N_T is a subprojection of N_{I-C} we conclude by lemma 1.2.4 that N_T is finite. This proves the first axiom for T to be Fredholm.

The second equality in (4.7) implies that

$$\text{range } (I-D) = \text{range}(TS) \subseteq \text{range}(T).$$

Since I-D is Fredholm (theorem 4.1.1), there exists a finite projection E of A satisfying

$$\text{range } (I - E) \subseteq \text{range } (I - D).$$

Thus range $(I - E) \subseteq$ range $(I - D) \subseteq$ range(T).

This proves the second and last axiom for T to be Fredholm. Hence $\pi^{-1}(G(\tilde{\mathcal{A}})) \subseteq \mathcal{F}(\mathcal{A})$. Since E is a finite projection and thus an element of $M_o \subseteq M$ we have that I - E is Fredholm. So by applying lemma 4.1.4 to I - E, $T \in \mathcal{F}(\mathcal{A})$, one gets

$$N_{(I-E)T} - N_T \sim \inf(R_T, N_{I-E}).$$

Since $N_{I-E} = \sup\{F \in \mathcal{P}(\mathcal{A}) \mid F(I - E) = 0\}$
$= \sup\{F \in \mathcal{P}(\mathcal{A}) \mid F = FE\}$
$= \sup\{F \in \mathcal{P}(\mathcal{A}) \mid F \leq E\} = E,$

we conclude that

$N_{(I-E)T} - N_T \sim \inf(R_T, E) \leq E$, and so $N_{(I-E)T} - N_T$ is a finite projection (E is finite, see lemma 1.2.4). Recalling that N_T is finite (T is Fredholm), we have that $(N_{(I-E)T} - N_T) + N_T = N_{(I-E)T}$ is finite ($N_{(I-E)T} - N_T$ and N_T are disjoint finite projections, so their sum is finite by corollary 1.1.9(i) and proposition 1.2.8). Let $F := N_{(I-E)T}$

We claim that the sequence

$$0 \longrightarrow \text{range }(I-F) \xrightarrow{(I-E)T(I-F)} \text{range }(I-E) \longrightarrow 0$$

is exact. To show this it is sufficient to show that $(I-E)T(I-F)$ is a bijection from range $(I - F)$ onto range $(I - E)$. Suppose $(I - E)T(I - F)x = 0$, $x \in (I - F)(H)$. Since $(I - F)x = x$ we have that $(I - E)Tx = 0$ which implies $x \in N_{(I-E)T}(H)$. By hypothesis $x \in (I - F)(H) = (I - N_{(I-E)T})(H)$.

Thus $x = 0$ ($N_{(I-E)T}$ and $I - N_{(I-E)T}$ are disjoint). This

proves that $(I - E)T(I - F)$ is one to one. Take any $y \in$ range $(I - E)$. Since range $(I - E) \subseteq$ range T an $x \in H$ exists with $y = Tx$. Then $(I - E)y = (I - E)Tx = Tx = y$. Let $x' = (I - F)x \in$ range $(I - F)$. Then

$$(I - E)T(I - F)x' = (I - E)T(I - F)x$$
$$= (I - E)T(I - N_{(I-E)T})x = (I - E)Tx = Tx = y.$$

This proves that $(I - E)T(I - F)$ is onto range $(I - E)$. The fact that $(I - E)T(I - F)$ is onto range $(I - E)$ implies that $(I - F)T^*(I - E)$ is a one to one mapping from range $(I - E)$ into range $(I - F)$ and since $(I - E)T(I - F)$ has a bounded inverse (by the open mapping theorem) we have that

$(I - F)T^*(I - E)$ is onto range $(I - F)$.

$$\text{Hence } 0 \to \text{range } (I - E) \xrightarrow{(I - F)T^*(I - E)} \text{range } (I - F) \to 0$$

is exact. It follows that $(I - E)T(I - F)(I - F)T^*(I - E)$ is a bijection from range $(I - E)$ onto range $(I - E)$ and $(I - F)T^*(I - E)T(I - F)$ s a bijection from range $(I - F)$ onto range $(I - F)$. Thus if we consider the reduced algebras $\mathcal{A}_{I-E}$ and $\mathcal{A}_{I-F}$(the element in $\mathcal{A}_{I-E}$ resp. $\mathcal{A}_{I-F}$ are operators on range $(I - E)$ resp. range$(I - F)$),

$(I - E)T(I - F)T^*(I - E)$ and $(I - F)T^*(I - E)T(I - F)$ are regular elements of $\mathcal{A}_{I-E}$ and $\mathcal{A}_{I-F}$. Hence there are elements T' and T'' in $\mathcal{A}_{I-E}$ and $\mathcal{A}_{I-F}$ such that

$$(I - E)T(I - F)T^*(I - E)T' = I - E,$$

$$T''(I - F)T^*(I - E)T(I - F) = I - F$$

From the first relation one has

$T(T^* T') + p(T,E,F,T^*,T') = I - E$ where p contains in each term an E or an F. Since E and F are finite projections in $\mathcal{A}$ and thus elements of M_o (the two-sided $*$-ideal generated by the finite projections in $\mathcal{A}$) one has that $-p(T,E,F,T^*,T')$ and $-E$ are elements of M ($M_o \subseteq M$) . Hence

$$T(T^*T') - I \in M$$

Similarly, the second relation implies

$$(T''T^*)T - I \in M$$

Consequently T is regular modulo M.

■

A consequence of this theorem is that the ideal of compact elements in $\mathcal{A}$ is a proper subset of $\mathcal{A}$ if and only if $\mathcal{A}$ is an infinite von Neumann algebra. We now prove a number of important corollaries. The first one is only a reformulation of the theorem :

4.1.7 *COROLLARY*

Let $T \in \mathcal{A}$, $\mathcal{A}$ not of finite type. Then $T \in \mathcal{F}(\mathcal{A})$ if and only if there exist compact elements C and D in $\mathcal{A}$ and an operator $S \in \mathcal{A}$ with $TS = I - C$ and $ST = I - D$.

Since $G(\bar{\mathcal{A}})$ is an open set in $\bar{\mathcal{A}}$([11], p 399) and the canonical quotient mapping $\Pi : \mathcal{A} \to \bar{\mathcal{A}}$ is continuous ($\|\Pi(T)\| \leq \|T\|$) for all $T \in \mathcal{A}$ we have

4.1.8 *COROLLARY* ([2])

$\mathcal{F}(\mathcal{A})$ is open in the norm topology on $\mathcal{A}$.

4.1.9 *COROLLARY* ([2])

$\mathcal{F}(\mathcal{A})$ is an involutive monoid, i.e.

(i) $I \in \mathcal{F}(\mathcal{A})$

(ii) $S, T \in \mathcal{F}(\mathcal{A})$ implies $ST \in \mathcal{F}(\mathcal{A})$

(iii) $S \in \mathcal{F}(\mathcal{A})$ implies $S^* \in \mathcal{F}(\mathcal{A})$

<u>Proof</u> :

Condition

(i) follows since $\Pi(I)$ is the identity element of $\bar{\mathcal{A}}$ which is clearly regular, so $\Pi(I) \in G(\bar{\mathcal{A}})$. Thus $I \in \Pi^{-1} G(\bar{\mathcal{A}}) = \mathcal{F}(\mathcal{A})$

(ii) If $S, T \in \mathcal{G}(\mathcal{A})$ we have $\pi(S), \pi(T) \in G(\bar{\mathcal{A}})$ and since $G(\bar{\mathcal{A}})$ is a group with respect to the multiplication in $\bar{\mathcal{A}}$ one has $\pi(S)\pi(T) = \pi(ST) \in G(\bar{\mathcal{A}})$. Thus $ST \in \pi^{-1}\, G(\mathcal{A}) = \mathcal{G}(\mathcal{A})$

(iii) If $S \in \mathcal{G}(\mathcal{A})$, $\pi(S) \in G(\bar{\mathcal{A}})$. Thus $\pi^*(S) \in G(\bar{\mathcal{A}})$ $([\pi^*(S)]^{-1} = [(\pi(S))^{-1}]^*)$. Since $\pi^*(S) = (S + M)^* = S^* + M = \pi(S^*)$ (M is a $*$-ideal in $\mathcal{A}$), $S^* \in \mathcal{G}(\mathcal{A})$.

■

CHAPTER 5

APPENDIX

5.1 LOCALLY CONVEX TOPOLOGIES ON A VON NEUMANN ALGEBRA

Let $\mathcal{A}$ be a von Neumann algebra i.e. $\mathcal{A}$ is a $*$-subalgebra of L(H), containing an identity $I\in\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}''$. As stated in Chapter 1, this is the equivalent of saying that $\mathcal{A}$ is a $*$-subalgebra of L(H) which is closed in the weak-operator topology on L(H) (the double commutation theorem). The weak-operator topology on $\mathcal{A}$ is the topology generated by the family of seminorms

$$T\in \mathcal{A} \longrightarrow |(Tx,y)| \qquad x,y\in H.$$

If $\mathcal{A}_{\sim}$ is the linear hull of the set of all weak operator continuous functionals on $\mathcal{A}$, then this weak operator topology is nothing but the $\sigma(\mathcal{A},\mathcal{A}_{\sim})$-topology. The strong operator topology on $\mathcal{A}$ is the locally convex topology determined by the family of seminorms

$$T \in \mathcal{A} \longrightarrow \|Tx\| \qquad x\in H.$$

The σ-weak-operator topology on $\mathcal{A}$ is the locally convex topology determined by the family of seminorms

$$T\in\mathcal{A} \longrightarrow \sum_{n=1}^{\infty}(Tx_n, y_n) \text{ where } \sum_{n=1}^{\infty}\|x_n\|^2 < +\infty \text{ and } \sum_{n=1}^{\infty}\|y_n\|^2 < +\infty.$$

Let $\mathcal{A}_*$ be the set of all σ-weak continuous linear functionals on $\mathcal{A}$. it can be shown that every $f\in\mathcal{A}_*$ is of the form

$f(T) = \sum_{n=1}^{\infty}(Tx_n,y_n)$ for some sequences (x_n), $(y_n) \subseteq H$ with $\sum_{n=1}^{\infty}\|x_n\|^2 < +\infty$ and $\sum_{n=1}^{\infty}\|y_n\|^2 < +\infty$ and that the σ-weak-operator topology on $\mathcal{A}$ is exactly the $\sigma(\mathcal{A},\mathcal{A}_*)$ topology on $\mathcal{A}$. The locally convex topology determined by the family of seminorms

$$T\in\mathcal{A} \longrightarrow (\sum_{n=1}^{\infty}\|Tx_n\|^2)^{1/2}, \ \sum_{n=1}^{\infty}\|x_n\|^2 < +\infty$$

where (x_n) is a sequence in H, is called the σ-strong operator topology on $\mathcal{A}$. The topology given by the norm $\|T\|$ is called the norm topology on $\mathcal{A}$. If "<" means the left-hand side is finer than the right-hand side, the relation between these various topologies defined on $\mathcal{A}$ is as follows :

$$\begin{array}{ccccc} \text{norm} & < & \sigma\text{-strong} & < & \sigma\text{-weak} \\ & & \wedge & & \wedge \\ & & \text{strong} & < & \text{weak} \end{array}$$

It can be shown that the σ-strong and strong (resp. σ-weak and weak)operator topologies coincide on bounded parts of $\mathcal{A}$. Consider $\mathcal{A}_*$ and $\mathcal{A}_\sim$ as defined above. Then, by using the general duality theory of Banach spaces it can be shown that $\mathcal{A}_*$ is a closed subspace of the conjugate space $\mathcal{A}^*$ of $\mathcal{A}$ and $\mathcal{A}_\sim$ is dense in $\mathcal{A}_*$ with respect to the norm topology. Furthermore, $\mathcal{A}$ is isometrically isomorphic to the conjugate space of the Banach space $\mathcal{A}_*$ under the natural correspondence $T\in\mathcal{A} \longrightarrow \hat{T} \in(\mathcal{A}_*)^*$ where $\hat{T}(\omega) = \omega(T)$ for every $\omega\in\mathcal{A}_*$. We call $\mathcal{A}_*$ the predual of $\mathcal{A}$. If $\mathcal{A}$ is a $*$-subalgebra of L(H), then $\mathcal{A}$ has the same closure in each of the topologies weak, strong, σ-strong and σ-weak ([5], corollary 3.6.2). Hence since a von Neumann algebra $\mathcal{A}$ is weakly-closed, it is closed in all these locally convex topologies on $\mathcal{A}$. For the proofs of all these statements we refer to [5] p18 to 31.

One merit of all the locally convex topologies defined above, is that multiplication is separately continuous. This means that the mappings $T\in\mathcal{A} \longrightarrow TS\in\mathcal{A}$, $T\in\mathcal{A} \longrightarrow ST\in\mathcal{A}$ are continuous for every $S\in\mathcal{A}$. We show this for the weak-operator topology on $\mathcal{A}$ (the proofs for the others are similar). If $T_\alpha \longrightarrow 0$ weakly, one has that $|(T_\alpha x,y)| \longrightarrow 0$ for every $x,y\in H$ ($\{T_\alpha\}$ a net in $\mathcal{A}$). Thus $|(T_\alpha x,S^*y)| \longrightarrow 0$ for every $x,S^*y\in H$. Hence $|(ST_\alpha x,y)| \to 0$ for every $x,y\in H$. This proves that $ST_\alpha \longrightarrow 0$ weakly. The same

procedure is used to show that $T \in \mathcal{A} \longrightarrow TS \in \mathcal{A}$ is weak operator continuous. Another merit of the weak and σ-weak topology on $\mathcal{A}$ is that the mapping $T \in \mathcal{A} \longrightarrow T^* \in \mathcal{A}$ is continuous. The proof of this proceeds as above. This is not true in the strong and σ-strong operator topologies. The following result was also needed : Multiplication is jointly continuous on bounded parts in the strong operator topology on $\mathcal{A}$. Moreover if $T_\lambda \longrightarrow T$, $S_\lambda \longrightarrow S$ and $\|S_\lambda\| \leq k$ for all λ then the relation
$\|(S_\lambda T_\lambda - ST)x\| \leq k \, \|(T_\lambda - T)x\| + \|(S_\lambda - S)Tx\|$ implies that
$(T,S) \in \mathcal{A}x\mathcal{A}^b \longrightarrow TS \in \mathcal{A}$ is continuous where $\mathcal{A}^b$ is a uniformly bounded subset of $\mathcal{A}$.

We conclude this section by stating the so called Eberlein-Smulian theorem which is used in Chapter 1.

5.1.1 *THEOREM* ([7], p 430)

Let F be a subset of a Banach space X. Consider the weak topology on X (i.e. the $\sigma(X,X^*)$ topology, where X^* is the conjugate space of X). Then the following statements are equivalent :

(i) F is relatively weakly sequentially compact - i.e. every sequence in F has a subsequence which converges weakly to an element of X;

(ii) every countably infinite subset of F has a weak limit point in X - i.e. a point such that every weak neighborhood contains an element in the infinite set;

(iii) the closure of F in the weak topology on X (the smallest topology on X that makes each $f \in X^*$ continuous) is weak-compact. (Remember that a weak neighborhood of an $x_o \in X$ is of the form $V(x_o, X, \epsilon) = \{x \in X \mid |(f(x) - f(x_o)| < \epsilon, f \in A\}$, where $\epsilon > 0$ and A is a finite subset of X^*)

5.2 COMPLETE ADDITIVITY AND σ-WEAK CONTINUITY OF FUNCTIONALS ON A VON NEUMANN ALGEBRA $\mathcal{A}$

Let $\mathcal{A}$ be a von Neumann algebra with predual $\mathcal{A}_*$. Consider the $\sigma(\mathcal{A},\mathcal{A}_*)$-topology on $\mathcal{A}$ (i.e. the σ-weak topology on $\mathcal{A}$). Our aim in this section is to show that the σ-weak continuous linear functionals on $\mathcal{A}$ (the elements of $\mathcal{A}_*$) are precisely the completely additive ones, (see Chapter 1, 1.4.5, for the definition of a completely additive linear functional on $\mathcal{A}$).

5.2.1 *LEMMA* ([5], p 41)

Let f be a norm-continuous hermitian (i.e. $\overline{f(T^*)} = f(T)$, $T\in\mathcal{A}$) functional on $\mathcal{A}$. Let $\lambda\in\mathbb{R}$.

(i) $T\in\mathcal{A}_1^+$ (the positive part of the unit ball of $\mathcal{A}$), and $f(T) > \lambda$ implies $f(E) > \lambda$ for some $E\in\mathcal{P}(\mathcal{A})$ (Note, since f is hermitian f(T) is real for every T a hermitian element of $\mathcal{A}$).

(ii) If $|f(E)| \leq \lambda$ for all $E\in\mathcal{P}(\mathcal{A})$, one has $\|f\| \leq 4\lambda$.

Proof :

(i) $T\in\mathcal{A}_1^+$ implies that the spectrum $Sp(T) \subset [0,\|T\|]$, $\|T\| \leq 1$. Hence $Sp(T) \subseteq [0,1]$. From the spectral decomposition theorem $dE_\lambda = 0$ if $\lambda \notin Sp(T)$, so $T = \int_0^1 \lambda dE_\lambda$. If we put $\epsilon = \|f\|^{-1}(f(T) - \lambda)$, then $\epsilon > 0$ and there exist by the spectral theorem projections E_j with $\|T - \sum k_jE_j\| < \epsilon$,

$k_j\in[0,1]$ and $j = 1,\dots n$. Hence $\|\epsilon^{-1}(T - \sum k_jE_j)\| < 1$. So

$|f(\epsilon^{-1}(T - \sum k_jE_j))| < \sup_{\|S\|\leq 1} |f(S)| = \|f\|$. This implies that

$f(T) - \sum k_jf(E_j) \leq |f(T) - \sum k_jf(E_j)| < \epsilon\,\|f\| = f(T) - \lambda$.

Thus $\sum k_j f(E_j) > \lambda$. Since E_j is hermitian, $f(E_j)$ is real for each j. By rearranging the E_j, we may suppose that $f(E_j) > 0$ $(1 \leq j \leq m)$ and $f(E_j) \leq 0$ $(m < j \leq n)$. Let $E = E_1 + \ldots + E_m$. Then

$$f(E) = \sum_{j=1}^{m} f(E_j) \geq \sum_{j=1}^{m} k_j \; f(E_j) \geq \sum_{j=1}^{n} k_j \; f(E_j) > \lambda.$$

(ii) If $|f(E)| \leq \lambda$ for all $E \in \mathcal{P}(\mathcal{A})$, then, by applying (i) to f (the contrapositive) $f(E) \leq \lambda$ for all $E \in \mathcal{P}(\mathcal{A})$ implies $f(T) \leq \lambda$ for all $T \in \mathcal{A}_1^+$. Applying (i) to $-f$, similarly, gives $-f(E) \leq \lambda$ for all $E \in \mathcal{P}(\mathcal{A})$, which implies $-f(T) \leq \lambda$ for all $T \in \mathcal{A}_1^+$. Hence

$$|f(T)| \leq \lambda \text{ for all } T \in \mathcal{A}_1^+ \qquad (5.1)$$

For any $T \in \mathcal{A}_1$ we get that $T = H + i\,K$ ($H, K \in \mathcal{A}_1^h$, i.e. the hermitian elements in $\mathcal{A}_1$), where $H = \frac{1}{2}(T + T^*)$ and $K = \frac{1}{2i}(T - T)^*$. Let $\mathcal{A}\{H,I\}$ be the commutative C^*-subalgebra of $\mathcal{A}$ containing H and I. Then the Gelfard Naimark theorem states that $\mathcal{A}\{H,I\} = C(X)$, X compact and Haussdorf and C(X) all real-valued continuous functions on X. If $\tau : \mathcal{A}(H,I) \to C(X)$ is the Gelfard mapping, then $\tau(H)$ is a real valued function on X and can thus be written as $\tau(H) = \tau(H)^+ - \tau(H)^-$ where $\tau(H)^+$ and $\tau(H)^-$ are positive elements in C(X). Thus H can be written as $H = H^+ - H^-$ in $\mathcal{A}$ where $H^+, H^- \in \mathcal{A}_1^+$. Similarly, for $K \in \mathcal{A}_1^h$. Hence $T = H + i\,K = H^+ - H^- + i\,K^+ - iK^-$. So $|f(T)| \leq |f(H^+)| + |f(H^-)| + f|(K^+)| + |f(K^-)| \leq 4\lambda$ by (5.1). So

$$\|f\| = \sup_{T \in \mathcal{A}_1} |f(T)| \leq 4\lambda.$$

∎

5.2.2 *LEMMA* ([5], p 42)

Let f be a norm-continuous linear functional on $\mathcal{A}$. Then f is positive (i.e. $f(T) \geq 0$ for all $T \in A^+$) if and only if $f(E) \geq 0$ for all $E \in \mathcal{P}(\mathcal{A})$.

Proof :

Suppose f is positive. Since $E \in \mathcal{A}^+$ we have $f(E) \geq 0$ for all $E \in \mathcal{P}(\mathcal{A})$. Conversely, if $T \in \mathcal{A}^+$ we have $Sp(T) \subset [0,\infty)$ and T can be approximated by a positive linear combination of projections (by the spectral decomposition theorem). Since f is norm-continuous and T is contained in the norm closure $\bar{\Sigma}$, where Σ is the set of all positive linear combinations of projections we have

$f(\bar{\Sigma}) \subset \overline{f(\Sigma)} \subset \overline{\mathbb{R}^+} = \mathbb{R}^+$. Hence $f(T) \geq 0$.

■

5.2.3 *LEMMA* ([5], p 42)

Let f be a non-zero norm-continuous, hermitian, completely additive functional on $\mathcal{A}$ and $E \in \mathcal{P}(\mathcal{A})$. Then a projection $F \in \mathcal{A}$, $F \leq E$ exists, such that $f(F) \geq f(E)$ and $f_{|\mathcal{A}_F}$ is a positive functional on the reduced algebra $\mathcal{A}_F$.

Proof :

Since f is hermitian f(T) is real for every $T \in \mathcal{A}_1^+$. Let $T \in \mathcal{A}_1^+$ with $-f(T) > 0$. Since f is non-zero such a T exists, for if $f(T) = 0$ for all $T \in \mathcal{A}_1^+$ it follows that $f(T) = 0$ for all $T \in \mathcal{A}$. Then by lemma 5.2.1 an $E' \in P(\mathcal{A})$ with $-f(E') > 0$ exists. So $f(E') < 0$ for some $E' \in \mathcal{P}(\mathcal{A})$. Let $F' = \inf(E',E)$. Then $f(F') < 0$ and $F' \leq E$. Let $\{E_\lambda\}$ be a maximal family of disjoint projections in $\mathcal{A}$, with $E_\lambda \leq E$ such that $f(E_\lambda) < 0$ for all λ (use Zorn).

Let $F = E - \sum E_\lambda$. If $G \leq F$, then $f(G) \geq 0$, otherwise if $f(G) < 0$, then $\{E_\lambda, G\}$ is a disjoint family of subprojections of E with $f(E_\lambda) < 0$ and $f(G) < 0$. This contradicts the maximality of the family $\{E_\lambda\}$.

So, $f(G) \geq 0$ for all projections $G \in \mathcal{A}_F$. Lemma 5.2.2 implies that $f_{|\mathcal{A}_F} \geq 0$ and

$$f(F) = f(E) - f(\sum E_\lambda)$$

$$= f(E) - \sum f(E_\lambda) \qquad \text{(f is completely additive)}$$

$$\geq f(E)$$

■

5.2.4 *LEMMA* ([5], p 43)

If f is a completely additive norm-continuous, positive functional on $\mathcal{A}$, and E is a non-zero projection of $\mathcal{A}$, then a non-zero projection $F \leq E$ in $\mathcal{A}$ and a vector $x \in H$ exist such that $|f(T)| \leq \|Tx\|$ $(T \in \mathcal{A}F)$.

<u>Proof</u> :

Since $E \neq 0$ an $y' \in H$ exists with $Ey' \neq 0$. Hence $\|Ey'\|^2 > 0$. If $f(E) = 0$, then $\|Ey'\|^2 > f(E)$. If $f(E) \neq 0$, let $y = (f(E)/\epsilon)^{1/2}y'$ where $\|Ey'\|^2 > \epsilon > 0$. Then

$$\|Ey\|^2 = f(E)/\epsilon \; \|Ey'\|^2 > f(E) \; \epsilon/\epsilon = f(E).$$

Hence

$$(\omega_{y,y} - f)(E) = \|Ey\|^2 - f(E) > 0 \qquad (5.2)$$

where $\omega_{y,y}(E) = (Ey,y)$. We now apply lemma 5.2.3 to $\omega_{y,y} - f$:

$$|\omega_{y,y}(T)| = |(Ty,y)| \leq \|T\|\|y\|^2,$$

so $\omega_{y,y}$ is norm-continuous. Since f is positive it is hermitian. Also $\omega_{y,y}(T) = (Ty,y) = \overline{(T^*y,y)} = \overline{\omega_{y,y}(T^*)}$ $(T \in \mathcal{A})$. Hence $\omega_{y,y}$ is hermitian. f is completely additive and clearly $\omega_{y,y}$ is completely additive. So $\omega_{y,y} - f$ is a norm-continuous, hermitian and completely additive functional on $\mathcal{A}$. Hence lemma 5.2.3 implies that a projection $F \leq E$ exists with $(\omega_{y,y} - f)(F) \geq (\omega_{y,y} - f)(E) > 0$ by (5.2) and $(\omega_{y,y} - f)_{|\mathcal{A}_F} \geq 0$.

Whence $F \neq 0$. If $T \in \mathcal{A}F$ we have $T^*T \in \mathcal{A}_F = F\mathcal{A}F$ ($T \in \mathcal{A}F$ implies $T^* \in F\mathcal{A}$, so $T^*T \in F\mathcal{A}F = \mathcal{A}_F$).

So

$$0 \leq (\omega_{y,y} - f)(T^*T) = \|Ty\|^2 - f(T^*T),$$

and

$$|f(T)|^2 \leq f(I)\ f(T^*T) \text{ (Cauchy-Schwarz)}$$
$$\leq f(I)\ \|Ty\|^2$$

Therefore, if we take $x = (f(I))^{1/2}y$, one has that

$$|f(T)| \leq \|Tx\| \quad (T \in \mathcal{A}_F)$$

■

5.2.5 *LEMMA* ([5], p 42)

If f is a completely additive, positive functional on $\mathcal{A}$, then f is σ-weakly continuous.

<u>Proof</u> :

Let $E = I$. Using lemma 5.2.4, let $F \leq E$ be a non-zero projection and $x \in H$ be an vector such that $|f(T)| \leq \|Tx\|$ ($T \in \mathcal{A}F$). By Zorn we can extend $\{F\}$ to a maximal disjoint family $\{F_i\}_{i \in I}$ of projections in $\mathcal{A}$ such that for each $i \in I$ there is a vector $x_i \in H$ with

$$|f(T)| \leq \|Tx_i\| \quad (T \in \mathcal{A}F_i) \qquad (5.3)$$

If $\sum_{i \in I} F_i \neq I$ (the identity in $\mathcal{A}$), we could apply lemma 5.2.4 to $I - \sum_i F_i$ and get a non-zero subprojection F of $I - \sum_i F_i$ which, when added to $\{F_i\}$, would contradict the maximality of $\{F_i\}$. Therefore,

$$\sum_i F_i = I.$$

By complete additivity of f, $\sum_{i\in I} f(F_i) = f(I) \leq \|f\|$. So, given $\epsilon > 0$, a finite set $K \subseteq I$ exists such that $f(\sum_{i\in I\setminus K} F_i) < \epsilon^2 \|f\|^{-1}$ (this is direct, since the family of reals $(f(F_i))_{i\in I}$ is summable).

Let $E = \sum_{I\setminus K} F_i$. Then $I = \sum_K Fi + E$, and so

$$f(T) = \sum_K f(TF_i) + f(TE) \qquad (T\cdot I = T)$$

$$= f_1(T) + f_2(T) \quad (T\in\mathcal{A}), \text{ where}$$

$f_1(T) := \sum_K f(TF_i)$ and $f_2(T) := f(TE)$.

Since $TF_i \in \mathcal{A}F_i$, relation (5.3) gives $|f(TF_i)| \leq \|TF_i x_i\|$ $(i\in K)$ and $|f_1(T)| \leq \sum_K \|TF_i x_i\|$. Hence f_1 is strong operator continuous on $\mathcal{A}$ (if $T_\alpha \to 0$ strongly i.e. $\|T_\alpha x\| \to 0$ for all $x\in H$, hence $\|T_\alpha(F_i x_i)\| \to 0$ which implies that $|f_1(T_\alpha)| \to 0$).

Since the strong operator topology is finer than the weak-operator topology on $\mathcal{A}$ we have that f_1 is weak-operator continuous. Hence $f\in\mathcal{A}_\sim$, where $\mathcal{A}_\sim$ is the set of all weak continuous functionals on $\mathcal{A}$. Since

$$\begin{aligned}|f_2(T)|^2 &= |f(TE)|^2 \\ &= |f(ET^*)^*E)|^2 \\ &\leq f(ET^*TE)\, f(E) \qquad \text{(Cauchy-Schwarz)} \\ &\leq \|f\|\, \|T\|^2\, f(E) \\ &\leq \epsilon^2 \|T\|^2 \quad (f(E) < \epsilon^2 \|f\|^{-1})\end{aligned}$$

So

$$\|f_2\| < \epsilon.$$

Since $f(T) = f_1(T) + f_2(T)$ $(T \in \mathcal{A})$, and $\|f_2\| < \epsilon$, $f_2 \in \mathcal{A}_\sim$ we have that f can be approximated, in norm, by elements of $\mathcal{A}_\sim$, so $f \in \bar{\mathcal{A}}_\sim = \mathcal{A}_*$. Thus f is σ-weak continuous.

■

5.2.6 *TECHNICAL LEMMA* ([5], p 44)

Suppose $a,b,c, \in \mathbb{R}$ such that $a,b,ab-c^2 \geq 0$. Further let $E,T \in L(H)$, E a projection and $\|T\| \leq 1$. Then

$$aE + b(I-E) + c[ET(I-E) + (I-E)T^*E] \geq 0$$

<u>Proof</u> :

For $x \in H$ $(ET(I-E)x,x) = (T(I-E)x,Ex)$ and

$$((I-E)T^*Ex,x) = (Ex,T(I-E)x) = \overline{(T(I-E)x,Ex)}$$

Thus

$$\begin{aligned} & a(Ex,x) + b(I-E)x,x) + c(ET(I-E)x,x) + c((I-E)T^*Ex,x) \\ &= a\|Ex\|^2 + b\|(I-E)x\|^2 + 2\,c\,\mathrm{Re}(T(I-E)x,Ex) \\ &\geq a\|Ex\|^2 + b\|(I-E)x\|^2 - 2|c||(T(I-E)x,Ex)| \\ &\geq a\|Ex\|^2 + b\|(I-E)x\|^2 - 2|c|\|(I-E)x\|\ \|Ex\| \qquad (\|T\| \leq 1) \end{aligned}$$

If we set $s = \|Ex\|$ and $t = \|(I-E)x\|$. Then $as^2 + bt^2 - 2|c|st$ is a quadratic form and we associate with it, the matrix

$$C = \begin{bmatrix} a & -|c| \\ -|c| & b \end{bmatrix}$$

Since $a \geq 0$, $b \geq 0$ and $\det(C) = ab - c^2 \geq 0$ one has that the quadratic form is positive semi-definite which implies that $as^2 + bt^2 - 2|c|st \geq 0$ for all $s, t \in R^+$ (a result in Linear Algebra). Hence

$$(aE + b(I-E) + c[ET(I-E) + (I-E)T^*E]x,x) \geq 0 \text{ for all } x \in H.$$

■

5.2.7 *THEOREM* ([5], p 42)

Let f be a norm-continuous linear functional on $\mathcal{A}$. Then f is completely additive if and only if it is σ-weakly continuous.

<u>Proof</u> :

Let $\{E_\lambda\}$ be a disjoint family of projections in $\mathcal{A}$. The sum $\sum E_\lambda$ converges in the strong operator topology by lemma 1.1.11. Since the strong operator topology is finer than the weak operator topology on $\mathcal{A}$, and the weak operator topology equals the σ-weak operator topology on bounded parts of $\mathcal{A}$ one has that the sum converges in the σ-weak operator topology on $\mathcal{A}$. Therefore if f is σ-weak continuous, then $f(\sum E_\lambda) = \sum f(E_\lambda)$. Thus f is completely additive.

Conversely, suppose f is completely additive. We must show that $f \in \mathcal{A}_*$. Since for every $f \in \mathcal{A}^*$ one has $f = f_1 + if_2$ where f_1 and f_2 are hermitian ($f_1 = \frac{1}{2}(f+f^*)$ and $f_2 = \frac{1}{2i}(f-f^*)$, where $f^*(T) = \overline{f(T^*)}$), we may assume that f is hermitian. We may also assume that $\|f\| \leq 1$.

Let $\mu = \sup\{f(T);\ \ T = T^* \in \mathcal{A},\ 0 \leq T \leq I\}$. So $0 \leq \mu \leq \|f\| \leq 1$. Since $\bar{\mathcal{A}}_\sim = \mathcal{A}_*$, it is sufficient to show that there are elements of $\mathcal{A}_\sim$ arbitrarily close to f in $\mathcal{A}^*$. Thus, suppose we are given $\epsilon > 0$, and for, convenience, assume $\epsilon \leq 3/4$. By definition of μ an $E_1 = E_1^* \in \mathcal{A}$ exists, $0 \leq E_1 \leq I$, such that $f(E_1) > \mu - \epsilon$. Lemma 5.2.1(i) allows us to assume that E_1 is a projection and from lemma 5.2.3 we may assume that $f_{|\mathcal{A}_{E_1}}$ is positive.

Therefore $f_{|\mathcal{A}_{E_1}}$ is a positive, completely additive functional and lemma 5.2.5 implies that $f_{|\mathcal{A}_{E_1}}$ is σ-weakly continuous. Since $T \in \mathcal{A} \longrightarrow E_1TE_1 \in \mathcal{A}_{E_1}$ is σ-weakly continuous (multiplication in the σ-weak topology on $\mathcal{A}$ is separately continuous), and since $f_{|\mathcal{A}_{E_1}}$ is σ-weak continuous we have that $T \in \mathcal{A} \longrightarrow f(E_1TE_1)$ is σ-weak continuous. Let $E_2 = I - E_1$. Then

$E_1TE_1 + E_1TE_2 + E_2TE_1 + E_2TE_2 = E_1T(E_1 + (I-E_1) + E_2T(E_1+(I-E_1))$
$= (E_1 + E_2)\ T = T.$

So,

$$f(T) = f(E_1TE_1) + f(E_1TE_2) + f(E_2TE_1) + f(E_2TE_2)$$
$$:= f_{11}(T) + f_{12}(T) + f_{21}(T) + f_{22}(T)$$

We have already seen that $f_{11} \in \mathcal{A}_*$. We now show that f_{12} and f_{21} are of small norm.

For $T \in \mathcal{A}_1$ we define :

$$S = (1-\epsilon)E_1 + \epsilon E_2 + \epsilon^{1/2}(1-\epsilon)^{1/2}(E_1TE_2 + E_2T^*E_1).$$

Let $a = (1-\epsilon)$, $b = \epsilon$ and $c = \epsilon^{1/2}(1-\epsilon)^{1/2}$. Then $a \geq 0$, $b \geq 0$ and $ab - c^2 = 0$.

Also, $I - S = \epsilon E_1 + (1-\epsilon)E_2 - \epsilon^{1/2}(1-\epsilon)^{1/2}(E_1TE_2 + E_2T^*E_1)$

The above equation follows since $I = E_1 + E_2$ and $E_1 + E_2 - (1-\epsilon)E_1 - \epsilon E_2 = \epsilon E_1 + (1-\epsilon)E_2$. Putting $a_1 = \epsilon$, $b_1 = 1 - \epsilon$ and $c_1 = -(\epsilon)^{1/2}(1 - \epsilon)^{1/2}$ we have that $a_1 \geq 0$, $b \geq 0$ and $a_1b_1 - c_1^2 = 0$.

Thus lemma 5.2.6 shows that $S \geq 0$ and $I - S \geq 0$. Therefore, $0 \leq S \leq I$, $S^* = S$ which implies that

$$\mu \geq f(S) = (1 - \epsilon)f(E_1) + \epsilon f(E_2) + \epsilon^{1/2}(1-\epsilon)^{1/2}f(E_1TE_2 + E_2T^*E_1).$$

Since f is hermitian one has $f(E_2T^*E_1) = f((E_1TE_2)^*) = \overline{f(E_1TE_2)}$.

Observing that $f(E_1) > \mu-\epsilon$ and $\|E_2\| \leq 1$ implies that $|f(E_2)| \leq \|f\| \leq 1$ which implies that $f(E_2) \geq -1$, we get

$$\mu \geq (1-\epsilon)(\mu-\epsilon) - \epsilon + 2\ \epsilon^{1/2}\ (1-\epsilon)^{1/2}\ \mathrm{Re}[f(E_1TE_2)]$$

Thus

$$\begin{aligned}\mathrm{Re}(f_{12}(T)) &= \mathrm{Re}[f(E_1 T E_2)] \\ &\leq \frac{1}{2}[\epsilon(1-\epsilon)]^{-1/2}[\mu + \epsilon - \mu + \epsilon + \epsilon\mu - \epsilon^2] \\ &= \frac{1}{2}[\mu + 2 - \epsilon]\sqrt{\epsilon}\,(1-\epsilon)^{-1/2} \\ &\leq \frac{1}{2}(1 + 2)\sqrt{\epsilon}\,2 \qquad (0<\mu<1;\quad 0 < \epsilon \leq 3/4) \\ &= 3\sqrt{\epsilon} \qquad (5.4)\end{aligned}$$

Applying this for $T_1 = |f_{12}(T)|\cdot[f_{12}(T)]^{-1}T$, in place of T we have $\|T_1\| = \|T\| \leq 1$, so $T\in\mathcal{A}_1$ and

$$\mathrm{Re}(|f_{12}(T)|\cdot[f_{12}(T)]^{-1} f_{12}(T)) \leq 3\sqrt{\epsilon}.$$

Thus

$$|f_{12}(T)| \leq 3\sqrt{\epsilon}$$

which implies that

$$\|f_{12}\| = \sup_{\|T\|\leq 1}|f_{12}(T)| \leq 3\sqrt{\epsilon}.$$

Similarly, we can show that

$\|f_{21}\| \leq 3\sqrt{\epsilon}$. We must show that f_{22} is near some $f_o\in\mathcal{A}_*$ where $f_{22}(T) = f(E_2 T E_2)$. If $F\in \mathcal{P}(\mathcal{A}_{E_2})$, then $E_1 + F \in \mathcal{P}(\mathcal{A})$, so $0 \leq E_1 + F \leq I$. Thus $\mu \geq f(E_1 + F) = f(E_1) + f(F) > \mu - \epsilon + f(F)$ This implies that

$$f(F) < \epsilon. \qquad (5.5)$$

Now, let $g = -f_{|\mathcal{A}_{E_2}}$. Then g is a completely additive, norm-continuous, hermitian functional on $\mathcal{A}_{E_2}$ (f is one on $\mathcal{A}$) with $\|g\| \leq 1$, and since $f(F) < \epsilon$ for all $F\in \mathcal{P}(\mathcal{A}_{E_2})$ we have $g(F) > -\epsilon$ for all $F\in \mathcal{P}(\mathcal{A}_{E_2})$. As before, we can find projections F_1, $F_2 \in \mathcal{P}(\mathcal{A}_{E_2})$ with sum E_2 (the identity of $\mathcal{A}_{E_2}$) such that if we define
$g_{ij}(T) = g(F_i T F_j)$ $(i,j = 1,2)$, then $g = g_{11} + g_{12} + g_{21} + g_{22}$ and g_{11} is σ-weak continuous. Also $\|g_{12}\| \leq 3\sqrt{\epsilon}$, $\|g_{21}\| \leq 3\sqrt{\epsilon}$ and $g(F) < \epsilon$ for every projection $F\in \mathcal{A}_{F_2}$. Thus

$$-\epsilon < g(F) < \epsilon \quad (g(F) > -\epsilon \text{ follows since } \mathcal{A}_{F_2} \subseteq \mathcal{A}_{E_2}).$$

Thus $|g(F)| < \epsilon$ for every $F \in \mathcal{A}_{F_2}$. Hence lemma 5.2.1(ii) implies that

$$\|g|_{\mathcal{A}_{F_2}}\| \leq 4\epsilon.$$

Therefore, $|g_{22}(S)| = |g(F_2SF_2)| \leq 4\epsilon\|F_2SF_2\| \leq 4\epsilon\ \|S\|\ (S\in\mathcal{A})$.
Hence

$$\|g_{22}\| \leq 4\epsilon.$$

Let $f_o(T) := -g_{11}(E_2TE_2)\ (T\in\mathcal{A})$.
Then

$$\begin{aligned}|(f_{22} - f_o)(T)| &= |\ f(E_2TE_2) + g_{11}\ (E_2TE_2)| \\ &= |(g_{11} - g)(E_2TE_2)| \\ &\leq \|g_{11} - g\|\ \|E_2TE_2\| \\ &\leq (4\epsilon + 6\sqrt{\epsilon})\ \|T\| \quad (g_{11} - g = -g_{12} - g_{21} - g_{22}).\end{aligned}$$

Thus $\|f_{22} - f_o\| \leq 4\epsilon + 6\sqrt{\epsilon}$ and $f_o \in \mathcal{A}_*$. This proves the theorem.

5.2.8 *COROLLARY* ([5], p 44)

For a state on $\mathcal{A}$ (i.e. a positive linear functional with norm one), one has that σ-weak continuity, normality and complete additivity are equivalent.

<u>Proof</u> :
Suppose f is normal (see definition 1.4.5(1)). Let $\{E_i\}_{i\in I}$ be a family of disjoint projections in $\mathcal{A}$. Take any finite subset

$J \subset I$. Then $E_J = \sum_{i\in J}E \in \mathcal{P}(\mathcal{A})$. If H is a finite subset of I with

$H \subseteq J$, we have $E_H \leq E_J$. Let $\mathcal{R}$ be the class of all finite subsets of I. Then the net $(E_J,\ J\in\mathcal{R},\ \subseteq)$ is increasing, uniformly bounded and for each $J\in\mathcal{R}$, $E_J \geq 0$. Hence

$$f(\textstyle\sum_{i\in I}E_i) = f(\sup_{J\in\mathcal{R}} E_J) = \lim_J f(E_J) = \lim_J \sum_{i\in J} f(E_i) = \sum_{i\in I} f(E_i).$$

Thus f is completely additive.

If f is completely additive, lemma 5.2.5 implies that f is σ-weakly continuous. Suppose f is σ-weakly continuous on $\mathcal{A}$. Let $\{T_\lambda\}$ be a increasing net of elements in $\mathcal{A}^+$ (with supremum T), which is uniformly bounded. Then the monotone convergence proposition 1.4.3 states that $T_\lambda \longrightarrow T$ in the strong-operator topology on $\mathcal{A}$. Since the strong topology on $\mathcal{A}$ is finer than the weak-operator topology, and the weak-operator topology equals the σ-weak topology on bounded parts, $T_\lambda \longrightarrow T$ σ-weakly. The σ-weak continuity of f implies that $f(T_\lambda) \longrightarrow f(T)$. Clearly $T_\lambda \le T_\mu$ $(\lambda \le \mu)$ implies $f(T_\lambda) \le f(T_\mu)$ (f is positive). Hence $\sup_\lambda f(T_\lambda) = f(T) = f(\sup_\lambda T_\lambda)$, which means that f is normal.

■

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FREDHOLM THEORY IN VON NEUMANN ALGEBRAS

by

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SUMMARY

The main goal of this study is to generalize the theory of compact and of Fredholm operators defined on a complex Hilbert space H to von Neumann algebras. Since this generalization depend heavily on the study of the projection lattice existing on a von Neumann algebra, the first chapter contains a comprehensive amount of standard material concerning the geometry of projections in a von Neumann algebra $\mathcal{A}$.

If we consider the commutant $\mathcal{A}'$ of a von Neumann algebra and a projection E in $\mathcal{A}$ then the restriction of each element of $\mathcal{A}'$ to E(H) defines a representation π_E of $\mathcal{A}'$ into the C^*-algebra of all bounded linear operators on E(H) (E(H) is the range space of the projection E). In Chapter 2 we consider all these representations of $\mathcal{A}'$ into E(H) (where E is assumed to be finite relative to $\mathcal{A}$), to construct a commutative monoid M. The Grothendieck group Γ of M can canonically be equipped with an order relation. This group is important in the Chapters that follow, since it contains the so called indices of the Fredholm elements defined on a von Neumann algebra $\mathcal{A}$.

In Chapter 3 the concept of finite, compact and Fredholm elements are introduced. On the set of all Fredholm elements relative to $\mathcal{A}$ an index mapping is defined with values in the Grothendieck group Γ. These values are called the indices of the Fredholm elements relative to $\mathcal{A}$.

The main theorems of this study are obtained in Chapter 4. These results generalize theorems, obtained by F. Riesz and Atkinson :

(i) "The generalized Fredholm alternatives say that $I - T$ is Fredholm with index zero (relative to $\mathcal{A}$) if T is compact (relative to $\mathcal{A}$).

(ii) In the second theorem we study properties that hold for the increasing sequence of null projections of the elements $(I - T)^n$, $n = 1,2,\ldots$, where T is compact.

(iii) If $\bar{\mathcal{A}}$ is the Calkin algebra of $\mathcal{A}$. Then the set of all Fredholm elements relative to $\mathcal{A}$, is exactly the inverse image of the group of all invertable elements in $\bar{\mathcal{A}}$ under the canonical quotient mapping $\Pi : \mathcal{A} \longrightarrow \bar{\mathcal{A}}$.

--- o0o ---

FREDHOLM-TEORIE IN VON NEUMANN ALGEBRAS

deur

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Departement : Wiskunde en Toegepaste Wiskunde
Graad : MSc

OPSOMMING

Die doel van hierdie verhandeling is om die teorie van kompakte en Fredholm-operatore, wat gedefinieer is op 'n komplekse Hilbert-ruimte H, na von Neumann-algebras te veralgemeen. Aangesien hierdie verhandeling berus op die studie van die projeksierooster wat op 'n von Neumann-algebra bestaan, gee ons in die eerste hoofstuk die nodige agtergrond omtrent die geometrie van projeksies in von Neumann-algebras.

In hoofstuk 2 konstrueer ons 'n sekere kommutatiewe monoïed M deur te gaan kyk na alle representasies π_E vanaf $\mathcal{A}'$ in die C^*-algebra bestaande uit alle kontinue lineêre operatore op E(H) (E(H) is die beeldruimte van die projeksie E), waar $\mathcal{A}'$ die kommutant van $\mathcal{A}$ is en π_E die afbeelding wat elke element van $\mathcal{A}'$ beperk tot E(H). Ons definieer verder 'n natuurlike ordening op die Grothendieck-groep Γ van M. Die feit dat hierdie groep die sogenaamde indekse van die Fredholm-elemente relatief tot 'n von Neumann-algebra $\mathcal{A}$ bevat, is van essensiële belang in die daaropvolgende hoofstukke.

In hoofstuk 3 definieer ons die begrippe eindige, kompakte en Fredholm-elemente, relatief tot 'n von Neumann-algebra $\mathcal{A}$. Ons definieer ook die indeksafbeelding op die versameling van alle Fredholm-elemente met waardes in die groep Γ. Hierdie waardes word die indekse van Fredholm-elemente relatief tot $\mathcal{A}$ genoem.

Die hoofresultate in hierdie verhandeling word in hoofstuk 4 bewys. Drie stellings bewys deur F Riesz en Atkinson is veralgemeen na von Neumann-algebras. Die veralgemeende stellings behels die volgende :

(i) As T kompak relatief tot $\mathcal{A}$ is, dan is I - T Fredholm relatief tot $\mathcal{A}$ met indeks nul.

(ii) In die tweede stelling word sekere eienskappe ondersoek wat geld vir die stygende ry van nul projeksies van die elemente $(I - T)^n$, n = 1,2,..., waar T kompak relatief tot $\mathcal{A}$ is.

(iii) Laat $\bar{\mathcal{A}}$ die Calkin algebra van $\mathcal{A}$ wees en gestel $\pi : \mathcal{A} \longrightarrow \bar{\mathcal{A}}$ is die kanoniese kwosiëntafbeelding. Dan is die versameling van alle Fredholm-elemente relatief tot $\mathcal{A}$ presies die inverse beeld van die groep van alle inverteerbare elemente in $\bar{\mathcal{A}}$ onder die afbeelding π.

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