Closed two-sided ideals in a von Neumann algebra and applications

by

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CONTENTS

INTRODUCTION

The theory of two-sided ideals in a von Neumann algebra $\mathcal{A}$ has been studied by several authors (Wright [28], Wils [27] and others). One of the most important of these ideals is certainly the closed two-sided ideal generated by the finite projections relative to $\mathcal{A}$. The elements of this ideal are called compact operators relative to $\mathcal{A}$ and it has been shown (cf. [5], [13] and [21]) that these operators behave somewhat like the ideal of compact operators on a Hilbert space. In two papers, [5] and [6], Manfred Breuer laid the foundations of a generalized theory of Fredholm operators relative to a von Neumann algebra where classical results such as

(i) "the Fredholm alternative" due to F. Riesz, which says that $I - T$ is Fredholm of index zero if T is compact,

(ii) a well-known Riesz decomposition theorem for compact operators

and

(iii) a well-known characterization of Fredholm operators due to Atkinson

were generalized to a von Neumann algebra $\mathcal{A}$. He also defined an index map on the set of all Fredholm elements relative to $\mathcal{A}$, with values in a commutative ordered group referred to as the index group of $\mathcal{A}$. Working in von Neumann algebras with Segal measures, Sonis extended several classical properties on the ideal of relatively compact elements including results such as the Calkin theorem and the minimax theorem for singular and characteristic numbers (cf. [21]). He also introduced the notion of the infinite (essential) spectrum of a self-adjoint element in the algebra. Kaftal then showed that the geometric definition for compact operators in [21] is also valid for general von Neumann algebras. In [13] he generalized most of the classical characterizations of compact operators.

Using a notion of finite nullity, Kaftal gave a useful characterization of semi-Fredholm operators and also extended some results on the essential spectra. Kaftal also studied a notion of weak convergence relative to a semifinite von Neumann algebra, which he used to extend the classical Hilbert characterization for relatively compact operators (cf. [14]). By replacing weak convergence with relative weak convergence in a classical theorem due to Wolf, Kaftal obtained a characterization of a more general class than the left Fredholm operators in $\mathcal{A}$ called the (relatively) almost left Fredholm operators (cf. [14] and [15]). In [23], the basic theory of measures of noncompactness and applications to Fredholm operators were extended to von Neumann algebras. In [24] Riesz operators relative to the ideal of compact elements in $\mathcal{A}$ were studied. Characterization theorems as well as a Riesz decomposition theorem for these operators were deduced.

In [17] Olsen developed a complete Fredholm and semi-Fredholm theory relative to an arbitrary closed ideal $\mathcal{I}$. She defined an index function relative to $\mathcal{I}$ by using a relative dimension function defined on the projections of $\mathcal{A}$ (due to Tomiyama, cf. [17], Theorem 5.1). If the ideal is contained in the ideal of relatively compact operators, the index map enjoys all the desired properties, e.g. it is invariant under perturbations by elements of $\mathcal{I}$, locally constant on components of the Fredholm elements, etc. If the ideal is not contained in the relatively compact ideal, however, then these properties fail to hold and Olsen showed how to modify this index map in such a way that these properties were recovered. In this thesis we will continue the study of closed two-sided ideals in a von Neumann algebra, not only by looking into the structure of these ideals, but by using them in several applications to the theory of von Neumann algebras. For example, one of the main objects of this thesis is to develop a Riesz theory relative to any closed two-sided ideal in a von Neumann algebra by proving some characterization theorems of relatively Riesz operators and then use this to prove a Riesz decomposition theorem. Let us now describe the thesis in more detail.

Section 1 contains a summary of our notation as well as those basic facts concerning von Neumann algebras which will be used throughout the thesis. Some proofs of results, scattered throughout the literature, are also given.

In section 2 we give a useful characterization theorem for the elements in any closed two-sided ideal in a von Neumann algebra $\mathcal{A}$. A similar result was proved by Kaftal for the ideal of relatively compact elements in $\mathcal{A}$ (cf. [13], Theorem 1.3). The main purpose of this section is to consider three specific examples of closed two-sided ideals, namely, the ideal of operators compact relative to the von Neumann algebra, the ideal consisting of the compact operators contained in $\mathcal{A}$ and the ideal of the so called Rosenthal operators relative to $\mathcal{A}$. In the Banach space setting the Rosenthal operators play an important role with respect to the geometry of certain Banach spaces. In this situation it is known that every compact operator is a Rosenthal operator and restricted to Hilbert spaces every bounded linear operator is Rosenthal. It is interesting that in our setting of relatively Rosenthal operators, there is no general relationship between the ideals of (relatively) compact operators and (relatively) Rosenthal operators. We shall only consider these three ideals in a semifinite algebra with a non-zero type I direct summand. The reason why this restriction on $\mathcal{A}$ is required, will become clear from the definitions of these ideals. These three ideals will also be used to obtain factorization results as well as a duality theorem.

In the third section we deduce geometrical characterizations as well as a spectral characterization for the quotient norm on $\mathcal{A}/\mathcal{I}$, where $\mathcal{I}$ is any closed ideal in $\mathcal{A}$. We then prove some characterization theorems on the semi-Fredholm elements relative to $\mathcal{I}$. In particular, if $\mathcal{I} = \mathcal{K}$ and $\mathcal{A}$ has a non-large center, we characterize the semi-Fredholm operators in terms of the left and right topological divisors of zero in the Calkin algebra. In doing this we show that in this case the topological and algebraic (left, resp. right) divisors of zero in the Calkin algebra coincide.

As was mentioned before, Olsen extended the Fredholm theory towards an arbitrary closed two-sided ideal in $\mathcal{A}$. In section 4 we use this Fredholm theory to define in a very natural way a class of Riesz operators relative to a closed ideal $\mathcal{I}$. An operator $T \in \mathcal{A}$ will be called Riesz relative to $\mathcal{I}$ if $T - \lambda I$ is Fredholm relative to $\mathcal{I}$ for every $\lambda \neq 0$. We shall see that results similar to those known for the classical case still hold. These results will be used in the sequel to obtain some characterization theorems for Riesz operators and a Riesz decomposition theorem in some special cases. Whereas in the classical case the theory of Riesz operators has an intimate connection with spectral theory, it should be noted that in our representation we do not use spectral theory at all. Actually one can not hope to obtain any results on the spectrum of a Riesz operator, for instance if $\mathcal{A}$ is a finite algebra and $\mathcal{K}$ the ideal of relatively compact elements in $\mathcal{A}$, then it will turn out that all elements of $\mathcal{A}$ will be Riesz relative to the ideal $\mathcal{K}$. One can thus find Riesz operators with spectral properties very different from the classical case. We conclude section 4 by showing that the class of Riesz operators relative to $\mathcal{K}$ behaves well under reduction with respect to central projections as well as under decompositions of the von Neumann algebra. Similar results for the class of compact and Fredholm operators relative to $\mathcal{K}$ were obtained by Kaftal (cf. [15], 2.1, 2.2).

In section 5 we give a few characterization theorems for a Riesz operator relative to a closed two-sided ideal $\mathcal{I}$. In particular it contains a geometrical characterization of the relative Riesz operators. This may be considered as the main result of section 5, since it allows one to obtain a Riesz decomposition theorem for some specific cases in the last section of this thesis.

We conclude this thesis with section 6, proving a Riesz type of decomposition for Riesz operators relative to some specific ideals. A similar theorem was proved (to a certain extent) for compact operators (relative to $\mathcal{A}$) by Breuer (cf. [5], Theorem 2), and a complete decomposition theorem can be found in a paper of Breuer and Butcher (cf. [7], Theorem 3). It should be noted that our decomposition depends heavily on a characterization theorem of Riesz operators (proved in section 5) and the techniques used in [5] and [7].

1. PRELIMINARIES

Unless otherwise stated, we shall use the following notation throughout the thesis: Let H be a complex Hilbert space and $\mathcal{L}(H)$ the algebra of all bounded linear operators on H. If Σ is a subset of $\mathcal{L}(H)$, then its commutant Σ' is defined as the set of all $T \in \mathcal{L}(H)$ satisfying $ST = TS$ for all $S \in \Sigma$. A *-subalgebra $\mathcal{A}$ of $\mathcal{L}(H)$ is called a von Neumann algebra if $\mathcal{A}'' = \mathcal{A}$. Further, let $Z := \mathcal{A} \cap \mathcal{A}'$ be the center of $\mathcal{A}$. We denote by $\mathcal{P}(\mathcal{A})$ the complete lattice of projections in $\mathcal{A}$ with the usual order relation $E \leq F$ iff $EF = E$ $(E,F \in \mathcal{P}(\mathcal{A}))$. The relations $\sim$ and $\precsim$ on $\mathcal{P}(\mathcal{A})$ are defined by $E \sim F$ iff $E = U^*U$, $F = UU^*$ for some $U \in \mathcal{A}$, and $E \precsim F$ (i.e. E is of smaller dimension than F) iff $E \sim G \leq F$ for some $G \in \mathcal{P}(\mathcal{A})$. An element $E \in \mathcal{P}(\mathcal{A})$ is called finite relative to $\mathcal{A}$, if for every $F \in \mathcal{P}(\mathcal{A})$ the relations $E \sim F$ and $F \leq E$ imply $E = F$ (cf. [25], Chapter V for properties of the projection lattice $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$).

We state a lemma on the projection lattice $\mathcal{P}(\mathcal{A})$ which will be used in a Riesz decomposition theorem, and a proof can be found in the literature.

1.1 **Lemma** (cf. [5], Lemma 10)

Let $(E_n)_{n=1,2,\ldots}$ *be a non-decreasing sequence in* $\mathcal{P}(\mathcal{A})$. *If the supremum* E_∞ *of this sequence is finite, it follows that*

$$\inf(E_\infty,F) = \sup\{\inf(E_n,F) \mid n=1,2,\ldots\} \quad \text{for all} \quad F \in \mathcal{P}(\mathcal{A}).$$

If $T \in \mathcal{A}$, let $T = U|T|$ denote the polar decomposition of T in $\mathcal{L}(H)$, let N_T be the projection onto the null space of T in $\mathcal{L}(H)$, and let R_T denote the projection onto the closure of the range of T. Then $|T| := (T^*T)^{1/2}$ and U are in $\mathcal{A}$, thus $R_T = U^*U$ and $R_T^* = UU^* = I - N_T$ are also in $\mathcal{A}$ with $R_T \sim R_T^*$. The following two lemmas will be needed in the sequel.

1.2 **Lemma** (cf. [13], Lemma 2.5)

Let $T \in \mathcal{A}$, $\epsilon > 0$ *and* $P_\epsilon = E[0,\epsilon)$ (*where* E *is the spectral measure of* $|T|$). *Then*:

(a) $\|Tx\| < \epsilon\|x\|$ *for every* $0 \neq x \in P_\epsilon(H)$

(b) $\|Tx\| \geq \epsilon\|x\|$ *for every* $x \in (I - P_\epsilon)(H)$.

Proof We prove (a): If E is the spectral measure of $|T|$, then we pass from this spectral measure to a spectral resolution $\{E_\lambda\}$ of $|T|$ by letting $E_\lambda = E(-\infty,\lambda)$. Then if $x \in P_\epsilon(H)$ it follows that

$$\|Tx\|^2 = \|\,|T|x\|^2 = \int_0^\epsilon \lambda^2 d(E_\lambda x,x)$$

$$< \epsilon^2\|x\|^2$$

(b) follows similarly. □

The following subsets of $\mathcal{P}(\mathcal{A})$ will play an important role in many of the proofs of our results:

1.3 **Definition** ([28], 2.1)

A subset $\mathcal{P}_0$ *of* $\mathcal{P}(\mathcal{A})$ *is called a* p-*ideal if it satisfies the following conditions*:

(a) $E,F \in \mathcal{P}_0$ *implies* $\sup(E,F) \in \mathcal{P}_0$

(b) $E \in \mathcal{P}_0$, $F \in \mathcal{P}(\mathcal{A})$ *with* $F \leq E$ *imply* $F \in \mathcal{P}_0$

(c) $E \in \mathcal{P}_0$, $F \sim E$ *imply* $F \in \mathcal{P}_0$.

We state a lemma (due to F.B. Wright) which gives one great insight into the structure of ideals in von Neumann algebras.

1.4 **Lemma** ([28], 2.1 and 2.2)

Let $\mathcal{I}$ *be a two-sided ideal in* $\mathcal{A}$, *then* $\mathcal{I} \cap \mathcal{P}(\mathcal{A})$ *is a* p-*ideal. Moreover, if* $\mathcal{I}$ *is a closed ideal in* $\mathcal{A}$ *and* $\mathcal{J}$ *is the ideal generated by* $\mathcal{I} \cap \mathcal{P}(\mathcal{A})$, *then* $\overline{\mathcal{J}} = \mathcal{I}$.

It is well-known that if $\mathcal{F}$ is the ideal generated by the finite projections in $\mathcal{A}$, then $T \in \mathcal{F}$ iff R_T is a finite projection and $T \in \mathcal{K}$ ($\mathcal{K}$ is the closed two-sided ideal generated by the finite projections) iff T is the norm limit of a sequence (T_n) for which R_{T_n} is finite. Call $T \in \mathcal{A}$ "finite" relative to a closed two-sided ideal $\mathcal{I}$ when $R_T \in \mathcal{I}$. If $\mathcal{I}_0 = \{T \in \mathcal{A} : R_T \in \mathcal{I}\}$ (i.e. $\mathcal{I}_0$ is the set of finite elements relative to $\mathcal{I}$) then we can use the above results of Wright to prove a similar result for any closed two-sided ideal, such as the one just mentioned for the ideal $\mathcal{K}$ of relative compact operators:

1.5 **Lemma** (cf. [12], 6.9.49(ii))

Let $\mathcal{I}$ *be any closed two-sided ideal in* $\mathcal{A}$, *then* $\mathcal{I}$ *is the norm closure of* $\mathcal{I}_0$.

Proof Since $\mathcal{I} \cap \mathcal{P}(\mathcal{A})$ is a p-ideal it is straightforward to see that $\mathcal{I}_0$ is a two-sided ideal in $\mathcal{A}$ with $\mathcal{I}_0 \subset \mathcal{I}$. Hence $\overline{\mathcal{I}}_0 \subset \mathcal{I}$. On the other hand, since $\mathcal{I} \cap \mathcal{P}(\mathcal{A}) = \mathcal{I}_0 \cap \mathcal{P}(\mathcal{A})$ it follows from Lemma 1.4 that $\mathcal{I} \subset \mathcal{I}_0$. □

Olsen extended the Fredholm theory towards an arbitrary closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$ in a very natural way: Let $\pi_{\mathcal{I}}:\mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the canonical quotient map. An operator $T \in \mathcal{A}$ is called a left Fredholm operator (relative to $\mathcal{I}$) if $\pi_{\mathcal{I}}(T)$ is left invertible in $\mathcal{A}/\mathcal{I}$. We denote this class of operators by Φ^+ and if the reference to $\mathcal{A}$ and $\mathcal{I}$ is necessary we denote this set by $\Phi^+(\mathcal{A},\mathcal{I})$. The class Φ^- of right Fredholm elements is defined in an obvious similar way. An operator $T \in \mathcal{A}$ is called a Fredholm operator (relative to $\mathcal{I}$) if $T \in \Phi := \Phi^- \cap \Phi^+$. In [17] (section 4), Olsen proved several equivalent characterizations of the Fredholm and left (right) Fredholm elements relative to $\mathcal{I}$ which are useful. For $T \in \mathcal{A}$ let $m_{\mathcal{I}}(T) = \inf \sigma(\pi(|T|))$, where $\sigma(\cdot)$ denotes the spectrum.

1.6 **Proposition** (cf [17], Theorem 4.5)

Let $\mathcal{I}$ be a closed ideal in $\mathcal{A}$, with $\pi_{\mathcal{I}}:\mathcal{A} \to \mathcal{A}/\mathcal{I}$. Then the following are equivalent:

(a) $T \in \Phi^+$

(b) $|T| \in \Phi$

(c) $m_{\mathcal{I}}(T) > 0$

(d) *the range of* T^* *contains the range of* $I - E$ *for some projection* E *in* $\mathcal{I}$ *(then also* $N_T \in \mathcal{I}$*).*

In that same paper, Olsen defined an index function relative to an arbitrary closed two-sided ideal $\mathcal{I}$. In doing this she made use of a relative dimension function defined on the projections of $\mathcal{A}$ (due to

Tomiyama, cf. [17], Theorem 5.1) and a characterization of the closed ideals of $\mathcal{A}$ by W. Wils. Since only a few properties of this function will be used (similar to those of the classical index), we shall not deem it necessary to define it. However, a very nice exposition of the index function can be found in [17]. Other definitions and notations will be introduced as needed, primarily at the beginning of each section. The reader is referred to [10] and [25] for basic information concerning von Neumann algebras and to [4], [8] and [11] for a development of the classical Riesz-Schauder theory.

2. ON CLOSED TWO-SIDED IDEALS IN VON NEUMANN ALGEBRAS

In this section we make use of three examples of closed two-sided ideals in a von Neumann algebra $\mathcal{A}$ to obtain factorization theorems as well as a duality theorem. Let $\mathcal{K}(H)$ denote the ideal of compact operators on H, then clearly $\mathcal{A} \cap \mathcal{K}(H)$ is a closed two-sided ideal in $\mathcal{A}$ which is contained in the closed two-sided ideal $\mathcal{K}$ of compact operators relative to $\mathcal{A}$. In order to define the third ideal we need the notion of a relatively weak Cauchy sequence which was introduced by Kaftal for semifinite algebras (cf. [14]). We call a norm-bounded sequence $(x_n) \subset H$ relatively weak Cauchy if for every finite projection P in $\mathcal{A}$ the sequence (Px_n) is Cauchy in norm.

2.1 Definition

An operator $T \in \mathcal{A}$ *will be called relatively Rosenthal if it maps bounded sequences in* H *onto sequences which have relatively weak Cauchy subsequences. We denote the class of all these operators by* $\mathcal{Q}$.

Whenever the reference to $\mathcal{A}$ is necessary, we shall write $\mathcal{Q}(\mathcal{A})$, $\mathcal{K}(\mathcal{A})$, etc. instead of $\mathcal{Q}$, $\mathcal{K}$, etc. It will be shown that $\mathcal{Q}$ is a closed two-sided ideal in $\mathcal{A}$.

2.2 Remarks

1. The notion of a Rosenthal operator between Banach spaces can be found in [19]. These operators were used to prove useful factorization results (cf. [19], 3.2.4).
2. To avoid trivial cases for these ideals, we assume that $\mathcal{A}$ is semifinite and contains a type I direct summand. We

assume semifinite since in the case of a purely infinite algebra $\mathcal{A} \cap \mathcal{K}(H) = \mathcal{K} = \{0\}$. Moreover, by [14], the elements of $\mathcal{Q}$ can only be defined for semifinite algebras. Also, if $\mathcal{A}$ contains no type I direct summand one has that $\mathcal{A} \cap \mathcal{K}(H) = \{0\}$. This follows since in that case $\mathcal{A}$ contains no minimal projections and hence no projections with finite euclidean dimension.

Before we study these ideals in more detail we show that a similar result as the one proved by Kaftal for the ideal $\mathcal{K}$ of relatively compact operators actually holds for any closed two-sided ideal in $\mathcal{A}$ (cf. [13], Theorem 1.3). This proposition will be used several times in section 2 and in section 4. It should be noted that the proof of our result is more or less the same as the one by Kaftal. In proving this we need the following lemma:

2.3 **Lemma** (cf. [13], Lemma 1.2)

Let $T \in \mathcal{A}$ *and* $Q \in \mathcal{P}(\mathcal{A})$ *be such that* $Q(H) \subseteq T(H)$. *Then there exists a* $P \in \mathcal{P}(\mathcal{A})$ *such that* $P \sim Q$ *and* T *is one to one from* $P(H)$ *onto* $Q(H)$.

2.4 **Proposition**

Let $\mathcal{I}$ *be a closed two-sided ideal in* $\mathcal{A}$ *and* U *the unit ball of* H. *Then the following conditions are equivalent.*

(a) $T \in \mathcal{I}$

(b) *For every* $\epsilon > 0$ *there is a projection* $P_\epsilon \in \mathcal{I}$ *and a bounded set* $N_\epsilon \subset P_\epsilon(H)$ *such that for every* $x \in T(U)$ *there is a* $y \in N_\epsilon$ *with* $\|x - y\| < \epsilon$

(c) *If* $Q \in \mathcal{P}(\mathcal{A})$ *and* $Q(H) \subset T(H)$ *then* $Q \in \mathcal{I}$

(d) *If* $P \in \mathcal{P}(\mathcal{A})$ *and* T *is bounded from below on* $P(H)$, *then* $P \in \mathcal{I}$

(e) *For every* $\epsilon > 0$ *there is a* $P \in \mathcal{P}(\mathcal{A})$ *such that* $\|TP\| < \epsilon$ *and* $I-P \in \mathcal{I}$.

Proof Let (a) hold with $T \in \mathcal{I}$ and $\epsilon > 0$ be given. Then by Lemma 1.5 there exists an $S \in \mathcal{I}$ with $R_S \in \mathcal{I}$ and $\|T - S\| < \epsilon$. If we choose $N_\epsilon = S(U)$ and $P_\epsilon = R_S$ condition (b) follows easily.

If (b) holds we show that (c) follows: If $Q \in \mathcal{P}(\mathcal{A})$ and $Q(H) \subset T(H)$, then let $P \in \mathcal{P}(\mathcal{A})$ be the projection, given by Lemma 2.3 such that T is one to one from $P(H)$ onto $Q(H)$. Let $N = Q(H) \cap U$. The restriction $T_{|P}$ of T to $P(H)$ has a bounded inverse, thus $T_{|P}^{-1}(N) \subseteq P(H)$ is bounded. It is clear that in (b) we could have replaced U with any bounded subset of H. As $N = T(T_{|P}^{-1}(N))$, condition (b) implies that for every $\epsilon > 0$ there exist a projection $P_\epsilon \in \mathcal{I}$ and a bounded set $N_\epsilon \subset P_\epsilon(H)$ such that if $x \in N$ there is a $y \in N_\epsilon$ with $\|x-y\| < \epsilon$. Hence for every $x \in U$ there is a $y \in N_\epsilon$ such that $\|Qx - P_\epsilon Qx\| = \inf_{y \in P_\epsilon(H)} \|Qx - y\| < \epsilon$. Since $P_\epsilon Q \in \mathcal{I}$ it follows that Q is in the closed ideal $\mathcal{I}$.

We now show that (c) implies (d): Let $P \in \mathcal{P}(\mathcal{A})$ be such that T is bounded from below on $P(H)$ and let $Q = R_{TP}$. Since $Q(H) = TP(H) \subset T(H)$, $Q \in \mathcal{I}$, and since T is one to one from $P(H)$ onto $Q(H)$, we obtain $N_{TP} = I - P$. Hence $Q = R_{TP} \sim I - N_{TP} = P$ and the result follows by Lemma 1.4.

Let (d) hold and $\epsilon > 0$ be given. If we choose $P = E[0,\epsilon)$, condition (e) follows from Lemma 1.2.

That (e) implies (a) is trivial. □

It is well-known that if $\mathcal{I}$ is a closed two-sided ideal in a C^*-algebra and T is a positive element in $\mathcal{I}$, then $T^{1/2} \in \mathcal{I}$, (cf. [22], E.3.19). In the case where $\mathcal{I}$ is an ideal in a von Neumann algebra we can make use of the proposition mentioned above to prove the same result. In fact if $T \in \mathcal{I}$ is positive, let $\{E_\lambda\}$ be the spectral resolution for T. Then it is clear that T is bounded from below on $(I - E_\lambda)(H)$ for every $\lambda \neq 0$ (use the spectral representation for T). Hence by (d) of the proposition one has that $I-E_\lambda \in \mathcal{I}$ for every $\lambda > 0$. Since $\|T^{1/2} - T^{1/2}(I-E_\lambda)\| < \sqrt{\lambda}$ for every $\lambda > 0$, it follows that $T^{1/2} \in \mathcal{I}$. Hence by using the polar decomposition of an operator we obtain:

2.5 **Corollary**

Every closed two-sided ideal $\mathcal{I}$ *in* $\mathcal{A}$ *is idempotent (i.e.* $\mathcal{I}^2 = \mathcal{I}$).

2.6 **Remark**

In 2.5 we cannot do without the closedness of $\mathcal{I}$. In fact it is easy to find examples of non-closed two-sided ideals for which this corollary fails to hold: If $\mathcal{A} = \mathcal{L}(\ell_2)$ and $\mathcal{N} = \{T \in \mathcal{A}: \mathrm{tr}(T^*T) < \infty\}$ (i.e. $\mathcal{N}$ is the two-sided ideal of Hilbert-Schmidt operators in $\mathcal{A}$), then $\mathcal{N}^2 = \mathcal{M}$ (the trace class of $\mathcal{A}$), and clearly $\mathcal{M} \neq \mathcal{N}$.

Let us continue to consider the class $\mathcal{Q}$ of relatively Rosenthal operators. It is well-known that if $\mathcal{A} = \mathcal{L}(H)$ every operator of $\mathcal{A}$ is Rosenthal, hence $\mathcal{Q} = \mathcal{L}(H)$. It is thus important to ask for which types of von Neumann algebras $\mathcal{Q}$ is strictly contained in $\mathcal{A}$. We illustrate this with an example that shows that such von Neumann algebras do exist.

2.7 Example

Consider ℓ_∞ (the algebra of all bounded sequences) as a concrete von Neumann algebra on ℓ_2. This can be done by means of the representation $\pi : \ell_\infty \to \mathcal{L}(\ell_2) : (x_n) \longmapsto \pi(x_n)$, where $\pi(x_n)(y_n) = (x_n y_n)$. It then follows that $\mathcal{Q} = c_0$ which is certainly strictly contained in ℓ_∞. Moreover, it can be seen from the following lemma that for any finite von Neumann algebra $\mathcal{A}$ one has $\mathcal{Q} = \mathcal{A} \cap \mathcal{K}(H)$.

The following lemma is actually a summary of a few results appearing in [14]. Since we shall use this lemma throughout this section, we state it without any proof. By inspection of the proofs in [14], it is clear that we may replace relatively weak convergence with relatively weak Cauchyness. In doing this one needs the fact that a Hilbert space H is weakly sequentially complete.

2.8 Lemma

(a) *Relatively weak Cauchyness implies weak convergence.*

(b) *Strong convergence and relatively weak Cauchyness coincide iff* $\mathcal{A}$ *is finite.*

(c) *Weak convergence and relatively weak Cauchyness coincide iff all finite projections of* $\mathcal{A}$ *have finite euclidean dimension.*

(d) $T \in \mathcal{K}$ *iff* T *maps relatively weak Cauchy sequences onto strong converging ones.*

Once we have shown that $\mathcal{Q}$ is a closed two-sided ideal, we characterize those von Neumann algebras for which $\mathcal{Q}$ is strictly contained in $\mathcal{A}$.

2.9 **Proposition**

$\mathcal{Q}$ is a closed two-sided ideal in $\mathcal{A}$.

Proof It follows straightforwardly from its definition and Lemma 2.8(d) that $\mathcal{Q}$ is a two-sided ideal in $\mathcal{A}$. To show that $\mathcal{Q}$ is norm-closed, choose an arbitrary sequence (T_m) in $\mathcal{Q}$ converging uniformly to T and let (x_n) be a bounded sequence in H. By using a "diagonal method" we can construct a subsequence $(y_k) = (x_{n_k})$ of (x_n) such that for every fixed positive integer m, $(T_m y_k)$ is relatively weak Cauchy. Let $c = \sup_n \|x_n\|$ and $\epsilon > 0$ be given. Then there exists an $\ell \in \mathbb{N}$ such that $\|T - T_\ell\| < \epsilon/3c$ and for any finite projection P in $\mathcal{A}$ there exists an $n_0 \in \mathbb{N}$ such that $\|PT_\ell y_i - PT_\ell y_j\| < \epsilon/3$ for all $i,j > n_0$. Thus from $\|PTy_i - PTy_j\| < \epsilon$ for all $i,j > n_0$, the result follows. □

2.10 **Proposition**

$\mathcal{Q} = \mathcal{A}$ if and only if $\mathcal{A} \cap \mathcal{K}(H) = \mathcal{K}$, (i.e. iff all the finite projections have finite euclidean dimension).

Proof Suppose $\mathcal{Q} = \mathcal{A}$ and suppose there exists a finite projection P in $\mathcal{A}$ with infinite euclidean dimension, then we can choose in $P(H)$ an infinite orthonormal sequence (x_n) which converges weakly to zero but has no relative weak Cauchy subsequence since

$$\|Px_n - Px_m\| = \|x_n - x_m\| = \sqrt{2} \quad \text{for all} \quad n \neq m.$$

Hence $I \notin \mathcal{Q}$ - a contradiction. The converse follows directly from Lemma 2.8(c). □

2.11 **Remark**

This proposition shows that there is a large class of von Neumann algebras such that $\mathcal{Q} \neq \mathcal{A}$ which thus makes it worthwhile to study the ideal $\mathcal{Q}$ in $\mathcal{A}$. In fact if $\mathcal{A}$ contains a type II direct summand, then clearly $\mathcal{Q} \neq \mathcal{A}$. Comparing the ideals $\mathcal{A} \cap \mathcal{K}(H)$, $\mathcal{K}$ and $\mathcal{Q}$ we obtain the following relations where the arrows point from the smaller ideal to the larger ones.

$$\begin{array}{ccc} \mathcal{K} & & \mathcal{Q} \\ \nwarrow & & \nearrow \\ & \mathcal{A} \cap \mathcal{K}(H) & \end{array}$$

It is easy to find examples which illustrate that the above inclusions are strict. Moreover in general $\mathcal{K}$ and $\mathcal{Q}$ are incomparable. For instance Example 2.7 shows that if we choose $\mathcal{A} = \ell_\infty$ then $\mathcal{Q} \subsetneq \mathcal{K}$ and if $\mathcal{A}$ is a type I_∞ factor $\mathcal{K} \subsetneq \mathcal{Q}$. The following proposition enables one to construct an example for which no inclusion holds.

2.12 **Proposition**

Let $\mathcal{A} = \bigoplus_{i=1}^{n} \mathcal{A}_i$ *be a direct sum of a finite number of von Neumann algebras on the Hilbert space* $H = \bigoplus_{i=1}^{n} H_i$. *The following equalities hold:*

(a) $\mathcal{Q}(\mathcal{A}) = \bigoplus_{i=1}^{n} \mathcal{Q}(\mathcal{A}_i)$

(b) $\mathcal{A} \cap \mathcal{K}(H) = \bigoplus_{i=1}^{n} \mathcal{A}_i \cap \mathcal{K}(H_i)$.

Proof By inspection of the proofs it will be clear that it suffices to prove the result for the case $n = 2$:

(a) Let $T \in \mathcal{Q}(\mathcal{A})$, then $T = T_1 \oplus T_2$ where $T_1 \in \mathcal{A}_1$ and $T_2 \in \mathcal{A}_2$. We show that $T_i \in \mathcal{Q}(\mathcal{A}_i)$, i=1,2. Suppose $(y_n) \subset H_1$ is a bounded sequence and let $x_n := y_n \oplus 0$ for each n. Since $T \in \mathcal{Q}(\mathcal{A})$, the sequence (Tx_n) has a weak Cauchy subsequence relative to $\mathcal{A}$, say (Tx_{n_k}). It is then clear that $(T_1y_{n_k})$ is weak Cauchy relative to $\mathcal{A}_1$. To show that $T_2 \in \mathcal{Q}(\mathcal{A}_2)$ follows in exactly the same way. On the other hand suppose $T = T_1 \oplus T_2$ where $T_i \in \mathcal{Q}(\mathcal{A}_i)$, i=1,2. Let $(x_n) \subset H$ be a bounded sequence. Since $H = H_1 \oplus H_2$ one has that $x_n = y_n \oplus z_n$, where $(y_n) \subset H_1$ and $(z_n) \subset H_2$ are clearly also bounded sequences. Since $T_1 \in \mathcal{Q}(\mathcal{A}_1)$, the sequence (T_1y_n) has a weak Cauchy subsequence relative to $\mathcal{A}_1$, say $(T_1y_{n_k})$. Similarly $T_2 \in \mathcal{Q}(\mathcal{A}_2)$ implies that $(T_2z_{n_k})$ has a weak Cauchy subsequence relative to $\mathcal{A}_2$, say (T_2z_m). Then since $P := P_1 \oplus P_2 \in \mathcal{P}(\mathcal{A})$ is $\mathcal{A}$-finite iff P_i are $\mathcal{A}_i$-finite, it is clear that (Tx_m) is weak Cauchy relative to $\mathcal{A}$.

(b) We prove this result by using Proposition 2.4(c): Suppose $T = T_1 \oplus T_2 \in \mathcal{A} \cap \mathcal{K}(H)$, where $T_i \in \mathcal{A}_i$ (i=1,2). Let $Q_1 \in \mathcal{P}(\mathcal{A}_1)$ with $Q_1(H_1) \subset T_1(H_1)$. Then if we let $Q = Q_1 \oplus 0$ it is clear that $Q \in \mathcal{P}(\mathcal{A})$ with $Q(H) \subset T(H)$. By 2.4(c) Q has finite euclidean dimension, hence Q_1 has finite euclidean dimension. Hence by using 2.4(c) again, we have $T_1 \in \mathcal{A}_1 \cap \mathcal{K}(H_1)$. Similarly $T_2 \in \mathcal{A}_2 \cap \mathcal{K}(H_2)$. The converse inclusion follows by using exactly the same arguments.

□

2.13 **Remarks**

1. A similar result is known for the ideal $\mathcal{K}$ of relatively compact operators (cf. [15], Proposition 2.2). Thus if we let $\mathcal{A}_1 = \ell_\infty$ and let $\mathcal{A}_2$ be a type I_∞ factor, then it follows from Example 2.7 that $\mathcal{Q}(\mathcal{A}) = c_0 \oplus \mathcal{A}_2$ and $\mathcal{K}(\mathcal{A}) = \ell_\infty \oplus \mathcal{K}(\mathcal{A}_2)$. This example shows that there exist von Neumann algebras for which $\mathcal{Q} \subset \mathcal{K}$ and $\mathcal{K} \subset \mathcal{Q}$.
2. It is interesting to ask whether Proposition 2.12 can be extended to infinite direct sums. The answer to this question is no. For if $\mathcal{A}_i = \mathcal{L}(H_i)$, where $H_i = \mathbb{C}$ for each $i \in \mathbb{N}$, then clearly $\mathcal{A} := \bigoplus_{i=1}^{\infty} \mathcal{A}_i = \ell_\infty$ and $H := \bigoplus_{i=1}^{\infty} H_i = \ell_2$. Hence $\mathcal{Q}(\mathcal{A}) = c_0$ and $\bigoplus_{i=1}^{\infty} \mathcal{Q}(\mathcal{A}_i) = \ell_\infty$. Result (b) is also not true for infinite direct sums. Exactly the same example can be used to illustrate this fact.

We have seen that there exist examples of von Neumann algebras for which $\mathcal{Q} \subseteq \mathcal{K}$ and examples for which $\mathcal{K} \subseteq \mathcal{Q}$. The following proposition states necessary as well as sufficient conditions for the inclusions to hold.

2.14 **Proposition**

(a) $\mathcal{Q} \subset \mathcal{K}$ *if and only if* $\mathcal{Q} = \mathcal{A} \cap \mathcal{K}(H)$

(b) $\mathcal{K} \subset \mathcal{Q}$ *if and only if* $\mathcal{K} = \mathcal{A} \cap \mathcal{K}(H)$ (*i.e. if and only if* $\mathcal{Q} = \mathcal{A}$).

Proof If $\mathcal{Q} = \mathcal{A} \cap \mathcal{K}(H)$ it can be seen from our diagram (cf. page 18) that $\mathcal{Q} \subseteq \mathcal{K}$. Conversely suppose $T \in \mathcal{Q}$; then $|T| = U^*T \in \mathcal{Q}$, where

$T = U|T|$ is the polar decomposition for T. Moreover, the arguments used in proving Corollary 2.5 imply that $|T|^{1/2} \in \mathcal{Q}$. Now if we let $(x_n) \subset H$ be bounded, then there exists a subsequence (x_{n_k}) such that $(|T|^{1/2}x_{n_k})$ is relatively weak Cauchy. From our assumption that $\mathcal{Q} \subseteq \mathcal{K}$ and Lemma 2.8(d) it follows that $(|T|x_{n_k})$ is norm convergent. Hence $|T| \in \mathcal{A} \cap \mathcal{K}(H)$ which implies that $T \in \mathcal{A} \cap \mathcal{K}(H)$. Thus $\mathcal{Q} \subseteq \mathcal{A} \cap \mathcal{K}(H)$, and since $\mathcal{A} \cap \mathcal{K}(H) \subset \mathcal{Q}$ always holds, (a) is proved. Result (b) follows similarly. □

2.15 **Remark**

There are non-trivial examples of von Neumann algebras where all three of these ideals coincide. For instance, let $\mathcal{A}$ be a finite dimensional C^*-algebra - i.e. $\mathcal{A} = \bigoplus_{i=1}^{n} \mathcal{L}(H_i)$ where $\dim(H_i) < \infty$ for each i (cf. [12], Proposition 6.6.6). Then $\mathcal{Q}(\mathcal{A}) = \bigoplus_{i=1}^{n} \mathcal{Q}(\mathcal{L}(H_i)) = \mathcal{A}$ and $\mathcal{K}(\mathcal{A}) = \mathcal{A} \cap \mathcal{K}(H) = \mathcal{A}$, where $H = \bigoplus_{i=1}^{n} H_i$.

We can now make use of Corollary 2.5 to obtain another factorization theorem:

2.16 **Theorem**

For the ideals $\mathcal{K}$, $\mathcal{Q}$ *and* $\mathcal{A} \cap \mathcal{K}(H)$, *the following holds*: $\mathcal{K}\cdot\mathcal{Q} = \mathcal{A} \cap \mathcal{K}(H)$.

Proof Let $S \in \mathcal{K}$, $T \in \mathcal{Q}$ and $(x_n) \subset H$ be a bounded sequence. It follows by the definition of $\mathcal{Q}$ and Lemma 2.8(d) that (STx_n) has a norm convergent subsequence, hence $ST \in \mathcal{A} \cap \mathcal{K}(H)$.

Conversely, since $\mathcal{A} \cap \mathcal{K}(H) \subseteq \mathcal{K}$ and $\mathcal{A} \cap \mathcal{K}(H) \subseteq \mathcal{Q}$, it follows from Corollary 2.5 that $\mathcal{A} \cap \mathcal{K}(H) \subseteq \mathcal{K} \cdot \mathcal{Q}$. □

We have seen from our diagram on page 18 that both $\mathcal{Q}$ and $\mathcal{K}$ contain the ideal $\mathcal{A} \cap \mathcal{K}(H)$. An interesting corollary of our factorization theorem is that $\mathcal{A} \cap \mathcal{K}(H) = \mathcal{K} \cap \mathcal{Q}$.

2.17 **Corollary**

Let $\mathcal{Q}$, $\mathcal{K}$ *and* $\mathcal{A} \cap \mathcal{K}(H)$ *be the three ideals under consideration. Then* $\mathcal{Q} \cap \mathcal{K} = \mathcal{A} \cap \mathcal{K}(H)$.

Proof We only need to show that $\mathcal{Q} \cap \mathcal{K} \subseteq \mathcal{A} \cap \mathcal{K}(H)$. This is actually straightforward: Clearly $\mathcal{Q} \cap \mathcal{K}$ is a closed two-sided ideal in $\mathcal{A}$. Thus by Corollary 2.5 and Theorem 2.16 one has
$\mathcal{Q} \cap \mathcal{K} = (\mathcal{Q} \cap \mathcal{K})^2 \subseteq \mathcal{K} \cdot \mathcal{Q} = \mathcal{A} \cap \mathcal{K}(H)$. □

2.18 **Remark**

This corollary actually implies that the only finite projections contained in $\mathcal{Q}$, are those which have finite euclidean dimension. Combining the theorem and its corollary we obtain $\mathcal{K} \cdot \mathcal{Q} = \mathcal{K} \cap \mathcal{Q}$. In the theory of operator ideals on Banach spaces necessary conditions were found for two closed operator ideals $\mathcal{U}$ and $\mathcal{B}$ to satisfy the following equality, $\mathcal{U} \circ \mathcal{B} = \mathcal{U} \cap \mathcal{B}$ (cf. [19], p 407). Now if $\mathcal{I}$ and $\mathcal{J}$ are closed two-sided ideals in $\mathcal{A}$ we easily obtain (as in the proof of Corollary 2.17) that $\mathcal{I} \cdot \mathcal{J} = \mathcal{I} \cap \mathcal{J} = \mathcal{J} \cdot \mathcal{I}$.

Comparing Theorem 2.16 with factorization theorems in the theory of operator ideals on Banach spaces, it seems as if $\mathcal{A} \cap \mathcal{K}(H)$ plays a similar role than that of the compact operators and $\mathcal{K}$ that of the completely continuous ones (cf. [19], Part 1, section 3). If $\mathcal{A} = \mathcal{L}(H)$ it is well-known that the dual of $\mathcal{K}$ is $\mathcal{A}_*$ where $\mathcal{A}_*$ is the predual of $\mathcal{A}$. In general von Neumann algebras this is not true for if $\mathcal{A} = \ell_\infty$, then clearly $\mathcal{K}^* = (\ell_\infty)^* \simeq \ell_1 \oplus \mathbb{C}$ (note that $\mathcal{A}_* = \ell_1$). However, since $\mathcal{A} \cap \mathcal{K}(\ell_2) = c_0$, we know that $(\mathcal{A} \cap \mathcal{K}(\ell_2))^* = \ell_1$. We now show a representation theorem for the dual of $\mathcal{A} \cap \mathcal{K}(H)$ in general.

Let $\mathcal{A}_*$ be the predual of $\mathcal{A}$ and let $\mathcal{I}$ be the $\sigma(\mathcal{L}(H), \mathcal{L}(H)_*)$ closure of $\mathcal{A} \cap \mathcal{K}(H)$. Then since $\mathcal{A}$ is $\sigma(\mathcal{L}(H), \mathcal{L}(H)_*)$ closed, one has that $\mathcal{I}$ is contained in $\mathcal{A}$.

2.19 **Theorem**

The dual of $\mathcal{A} \cap \mathcal{K}(H)$ *is* $\mathcal{M}$, *where* $\mathcal{M} = \{f_{|\mathcal{I}} : f \in \mathcal{L}(H)_*\}$.

Proof We apply the general duality theory of Banach spaces to $\mathcal{A} \cap \mathcal{K}(H)$, considered as a subspace of the Banach space $\mathcal{K}(H)$, to obtain $(\mathcal{A} \cap \mathcal{K}(H))^* = \mathcal{L}(H)_*/(\mathcal{A} \cap \mathcal{K}(H))^0$ (where $(\mathcal{A} \cap \mathcal{K}(H))^0$ is the polar of $\mathcal{A} \cap \mathcal{K}(H)$ in $\mathcal{L}(H)_*$). Consider the dual pair $(\mathcal{L}(H), \mathcal{L}(H)_*)$. It then follows from the bipolar theorem that $\mathcal{I} = (\mathcal{A} \cap \mathcal{K}(H))^{00}$. Hence $\mathcal{I}^0 = (\mathcal{A} \cap \mathcal{K}(H))^0$. Now if we define $\Phi:\mathcal{L}(H)_*/\mathcal{I}^0 \to \mathcal{M}$ by $\Phi(f + \mathcal{I}^0) = f_{|\mathcal{I}}$, it is clear that Φ is a well-defined bounded linear bijection with $\|\Phi(f + \mathcal{I}^0)\| \leq \|f + \mathcal{I}^0\|$. We prove that Φ is an isometry. Let $f \in \mathcal{L}(H)_*$ be such that $\|f + \mathcal{I}^0\| = 1$. Using the Hahn-Banach theorem we can choose a bounded linear functional φ on $\mathcal{L}(H)_*$, such that $\|\varphi\| = \varphi(f) = 1$ and $\varphi(g) = 0$ for any $g \in \mathcal{I}^0$. Since $\mathcal{L}(H) = (\mathcal{L}(H)_*)^*$, there exists a $T \in \mathcal{L}(H)$ such that $\|T\| = f(T) = 1$ and $g(T) = 0$ for any $g \in \mathcal{I}^0$.

It follows that $T \in \mathcal{I}^{00} = \mathcal{I}$, therefore $\|\Phi(f + \mathcal{I}^0)\| = \|f_{|\mathcal{I}}\| \geq f(T) = 1$.

□

2.20 **Remarks**

1. Suppose $\mathcal{A}$ is a factor. In view of Remark 2.2 $\mathcal{A}$ must be of type I. It is well-known that the $\sigma(\mathcal{L}(H),\mathcal{L}(H)_*)$-closed ideals in $\mathcal{A}$ are of the form $\mathcal{A}E$, where E is a projection in $Z := \mathcal{A} \cap \mathcal{A}'$. Hence $\mathcal{I} = \mathcal{A}$ and it can thus be seen from the theorem that $(\mathcal{A} \cap \mathcal{K}(H))^* = \mathcal{A}_*$.
2. An interesting problem would be to characterize those algebras for which $(\mathcal{A} \cap \mathcal{K}(H))^* = \mathcal{A}_*$, and to find a representation theorem for the dual of $\mathcal{K}$ in general von Neumann algebras.

3. ON THE $\mathcal{I}$-ESSENTIAL NORM AND APPLICATIONS TO FREDHOLM OPERATORS RELATIVE TO ANY CLOSED IDEAL $\mathcal{I}$

Let $\mathcal{I}$ be any closed two-sided ideal in $\mathcal{A}$ and let $\tilde{\alpha}(T+\mathcal{I}) := \inf_{S\in\mathcal{I}} \|T-S\|$ be the quotient norm on $\mathcal{A}/\mathcal{I}$. Recall that if $\pi_{\mathcal{I}} : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{I}$ is the canonical quotient map then the $\mathcal{I}$-essential spectrum $\sigma_{\mathcal{I}}(T)$ of $T \in \mathcal{A}$ is defined by the spectrum of $\pi_{\mathcal{I}}(T)$ in $\mathcal{A}/\mathcal{I}$. Then the $\mathcal{I}$-essential spectral radius is the spectral radius of $\pi_{\mathcal{I}}(T)$ in $\mathcal{A}/\mathcal{I}$ (i.e. $r_{\mathcal{I}}(T) = \max\{|\lambda|:\lambda \in \sigma_{\mathcal{I}}(T)\}$). In this section we obtain geometrical characterizations as well as a spectral characterization for the quotient norm $\tilde{\alpha}$. We then apply these results to prove a few characterization theorems on semi-Fredholm operators in $\mathcal{A}$ relative to $\mathcal{I}$. An interesting application is to study the semi-Fredholm operators in case $\mathcal{A}$ has a non-large center with respect to the ideal $\mathcal{K}$. We show that in this case the semi-Fredholm operators relative to $\mathcal{K}$ can be characterized in terms of the left and right topological divisors of zero in $\mathcal{A}/\mathcal{K}$. In doing this we also prove that the topological and algebraic (left, resp. right) divisors of zero in $\mathcal{A}/\mathcal{K}$ coincide.

A part of the results appearing in this section have been published for the case $\mathcal{I} = \mathcal{K}$ (see [23]).

3.1 Definition

Let $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$. The $\mathcal{I}$-essential seminorm α on $\mathcal{A}$ is defined by

$$\alpha(T) := \tilde{\alpha}(T+\mathcal{I})$$

3.2 Remark

It is clear that α measures the degree of non-$\mathcal{I}$-ness of an operator in the sense that it vanishes precisely on the ideal $\mathcal{I}$.

In the classical theory of operators between Banach spaces Lebow and Schechter [16] studied examples of measures of noncompactness on $\mathcal{L}(X,Y)$ (the space of bounded linear operators from a Banach space X to Y). They proved that for any of their measures δ the following conditions hold:

(i) δ is a seminorm on $\mathcal{L}(X,Y)$

(ii) $\delta(T) = 0$ iff T is a compact operator

(iii) $\delta(T) \leq \|T\|$

(iv) $\delta(ST) \leq \delta(S)\delta(T)$ if $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$.

In the following theorem we prove that in the case of a von Neumann algebra $\mathcal{A}$, α is the only seminorm on $\mathcal{A}$ satisfying similar conditions.

3.3 Theorem

Let $\mathcal{I}$ *be any closed two-sided ideal in* $\mathcal{A}$. *The* $\mathcal{I}$-*essential seminorm is the only mapping* $\delta : \mathcal{A} \rightarrow \mathbb{R}$ *satisfying the following conditions :*

(i) δ *is a seminorm on* $\mathcal{A}$

(ii) $\delta(T) = 0$ *iff* $T \in \mathcal{I}$

(iii) $\delta(T) \leq \|T\|$ *for* $T \in \mathcal{A}$

(iv) $\delta(ST) \leq \delta(S)\delta(T)$ *for* $S,T \in \mathcal{A}$.

Proof Since α induces a C^*-algebra norm on $\mathcal{A}/\mathcal{I}$, it is well-known that conditions (i) - (iv) hold.

Conversely, let δ be any mapping on $\mathcal{A}$ satisfying (i) - (iv) and let $T \in \mathcal{A}$. It follows directly from conditions (i) and (ii) that $\delta(T) = \delta(T+S)$ for every $s \in \mathcal{I}$. Hence, by using (iii) it follows that $\delta(T) \leq \alpha(T)$. To prove equality, one needs to show that $r_{\mathcal{I}}(T) \leq \delta(T)$. In order to do so we show that if $\lambda \in \mathbb{C}$ such that $\delta(\frac{1}{\lambda}T) < 1$ then $I - \frac{1}{\lambda}T \in \Phi$:
Suppose $I - \frac{1}{\lambda}T \notin \Phi^+$ and choose $\epsilon > 0$ so small that $\delta(\frac{1}{\lambda}T) + \epsilon < 1$. Then by using Proposition 1.6, Lemma 1.2 and [17], Proposition 4.2 we can find a projection $Q_\epsilon \notin \mathcal{I}$ such that $\|(I - \frac{1}{\lambda}T)Q_\epsilon\| < \epsilon$.
Since $\delta(Q_\epsilon) = \delta((I - \frac{1}{\lambda}T)Q_\epsilon + \frac{1}{\lambda}TQ_\epsilon) < \epsilon + \delta(\frac{1}{\lambda}T) < 1$ and Q_ϵ is a projection it follows that $\delta(Q_\epsilon) = 0$, hence $Q_\epsilon \in \mathcal{I}$ - a contradiction. If $I - \frac{1}{\lambda}T \notin \Phi^-$, then $I - (\frac{1}{\lambda}T)^* \notin \Phi^+$. So by using a similar argument as the one above we obtain a contradiction. Thus $I - \frac{1}{\lambda}T \in \Phi^- \cap \Phi^+ = \Phi$. Hence if $0 \neq \lambda \in \sigma_{\mathcal{I}}(T)$ (i.e. $I - \frac{1}{\lambda}T \notin \Phi$) it follows by the argument above that $|\lambda| \leq \delta(T)$. Hence $r_{\mathcal{I}}(T) \leq \delta(T)$.

By using the polar decomposition of an operator in $\mathcal{A}$ it follows directly from condition (iii) that $\delta(T) = \delta(|T|)$ and $\alpha(T) = \alpha(|T|)$. Hence

$$\alpha(T) = \alpha(|T|) = r_{\mathcal{I}}(|T|) \leq \delta(|T|) = \delta(T) \qquad \square$$

With this uniqueness theorem in hand we can prove a few characterization theorems for the $\mathcal{I}$-essential seminorm on $\mathcal{A}$.

First of all we need to generalize the concept of a finite ϵ-net (suggested in [13] and [21]) to any closed two-sided ideal $\mathcal{I}$:

3.4 Definition

1. *A bounded set* $N_\epsilon \subset H$ *will be called an* ϵ*-net (relative to* $\mathcal{I}$*) for a fixed subset* $\Omega \subseteq H$ *if there exists a projection* P_ϵ *in* $\mathcal{I}$ *such that* $N_\epsilon \subseteq P_\epsilon(H)$ *and for each* $x \in \Omega$ *there exists a* $y \in N_\epsilon$ *with* $\|x - y\| < \epsilon$.

2. *If an* ϵ*-net relative to* $\mathcal{I}$ *exists for a subset* $\Omega \subseteq H$, *we define*

   $q(\Omega) := \inf\{\epsilon > 0 |$ *there exists an* ϵ*-net for* $\Omega\}$, *otherwise we define* $q(\Omega) = \infty$.

3.5 Lemma

A set $\Omega \subseteq H$ *is bounded if and only if* $q(\Omega) < \infty$.

Proof If Ω is bounded, let $\epsilon = \sup\{\|x\| + 1 | x \in \Omega\}$. By taking $N_\epsilon = \{0\}$ and $P_\epsilon = 0$ one trivially has $q(\Omega) \leq \epsilon$.
Conversely, suppose $q(\Omega) = d < \infty$ and let $\delta > 0$ be given. Then there exist an $\epsilon > 0$ and an ϵ-net N_ϵ for Ω such that $d \leq \epsilon < d+\delta$. Now let $x \in \Omega$. Then there exists a $y \in N_\epsilon$ such that $\|x - y\| < \epsilon$, hence $\|x\| \leq d + \sup\{\|y\| | y \in N_\epsilon\}$. □

3.6 Remarks

1. It should be noted that the existence of an ϵ-net for a set Ω which is equivalent to boundedness of Ω is obviously independent of the specific von Neumann algebra as well as the specific ideal $\mathcal{I}$. However the value $q(\Omega)$ depends on the specific von Neumann algebra and ideal $\mathcal{I}$. For example, let $\mathcal{A} = \ell^\infty$ and let $\mathcal{I}_1 = \{(x_i) \in \ell^\infty : x_i = 0$ for all $i \geq 2\}$ and $\mathcal{I}_2 = \{(x_i) \in \ell^\infty : x_i = 0$ for all $i \geq 3\}$.

Let $\Omega = \{(y_i) \in \ell^2 : |y_1| \leq 1, \quad |y_2| \leq 1, \quad y_i = 0$ for all $i \geq 3\}$. Relative to the ideal $\mathcal{I}_j$ we denote $q(\Omega)$ by $q_j(\Omega)$. It is easy to verify that

$$q_1(\Omega) = 1 \text{ and } q_2(\Omega) = 0.$$

2. Let $\mathcal{I} = \mathcal{K}$. Then, in the terminology of [13] Ω is $\mathcal{A}$-relatively compact iff $q(\Omega) = 0$.

3.7 **Theorem**

Let $\mathcal{I}$ *be any closed two-sided ideal in* $\mathcal{A}$. *Then*

$\alpha(T) = \inf\{\epsilon > 0$: *there exists a projection* P *with* $I-P \in \mathcal{I}$ *and* $\|TP\| < \epsilon\}$

Proof Let $\gamma(T) := \inf\{\epsilon > 0$: there exists a projection P with $I-P \in \mathcal{I}$ and $\|TP\| < \epsilon\}$. We show that γ satisfies (i) - (iv) of Theorem 3.3. We first have to show that γ is a seminorm: Since trivially $\gamma(T) \leq \|T\|$, γ is finitely valued. Now let $0 \neq \lambda \in \mathbb{C}$ and $\epsilon > 0$ be given. Then there exists a $P_\epsilon \in \mathcal{P}(\mathcal{A})$ such that $I-P_\epsilon \in \mathcal{I}$ and $\|TP_\epsilon\| \leq \epsilon/|\lambda| + \gamma(T)$. Hence $\|\lambda TP_\epsilon\| \leq \epsilon + |\lambda|\gamma(T)$. This implies that $\gamma(\lambda T) \leq |\lambda|\gamma(T)$. The reverse inequality follows in an exactly similar way. Further let $S,T \in \mathcal{A}$ and $\epsilon > 0$ be given. Then there exist projections P_ϵ and Q_ϵ with both $(I-P_\epsilon)$ and $(I-Q_\epsilon)$ elements of $\mathcal{I}$ such that

$$\|TP_\epsilon\| \leq \gamma(T) + \epsilon/2 \ ; \ \|SQ_\epsilon\| \leq \gamma(S) + \epsilon/2 \qquad (1)$$

Let $R_\epsilon = \inf(P_\epsilon, Q_\epsilon)$, then since $\mathcal{I} \cap \mathcal{P}(\mathcal{A})$ is a p-ideal it follows that $I-R_\epsilon \in \mathcal{I}$ and $\|(T+S)R_\epsilon\| \leq \|TR_\epsilon\| + \|SR_\epsilon\| \leq \gamma(T) + \gamma(S) + \epsilon$. Hence $\gamma(T+S) \leq \gamma(T) + \gamma(S)$. It follows directly from Proposition 2.4 that $\gamma(T) = 0$ if and only if $T \in \mathcal{I}$. Hence it remains only to show property (iv). Let $S,T \in \mathcal{A}$ and $\epsilon > 0$ be given. Then there exist projections P_ϵ, Q_ϵ as above such that (1) holds. Let $F_\epsilon = N_{(I-Q_\epsilon)TP_\epsilon}$ and $R_\epsilon = \inf(P_\epsilon, F_\epsilon)$. Then since $I-F_\epsilon = R_{P_\epsilon T^*(I-Q_\epsilon)} \sim R_{(I-Q_\epsilon)TP_\epsilon} \leq I - Q_\epsilon$, it follows from Lemma 1.4 that $I-F_\epsilon \in \mathcal{I}$. From that same lemma we have that

$$I-R_\epsilon = \sup(I-P_\epsilon, I-F_\epsilon) \in \mathcal{I}.$$

For $x \in R_\epsilon(H)$ we have that $x \in P_\epsilon(H)$ and $Tx = TP_\epsilon F_\epsilon x = Q_\epsilon TP_\epsilon F_\epsilon x = Q_\epsilon Tx$ implies that $Tx \in Q_\epsilon(H)$.

Hence
$$\begin{aligned} \|STR_\epsilon\| &= \|SQ_\epsilon TR_\epsilon\| \\ &\leq \|SQ_\epsilon\|\|TR_\epsilon\| \\ &\leq (\gamma(S) + \epsilon/2)(\gamma(T) + \epsilon/2), \end{aligned}$$

from which property (iv) follows. □

In the following Corollary we give a spectral characterization for the $\mathcal{I}$-essential seminorm α.

3.8 Corollary

Let $T \in \mathcal{A}$ *and* E *be the spectral measure of* $|T|$. *Then* $\alpha(T) = \inf\{\epsilon > 0 : I - E[0,\epsilon) \in \mathcal{I}\}$.

Proof Let $\epsilon > 0$ be such that $I - E[0,\epsilon) \in \mathcal{I}$. If we let $P = E[0,\epsilon)$ it follows from Lemma 1.2 that $\|TP\| \leq \epsilon$. Thus

$$\alpha(T) \leq \inf\{\epsilon > 0 : I-E[0,\epsilon) \in \mathcal{I}\}.$$

Conversely, let $P \in \mathcal{P}(\mathcal{A})$ be such that $I-P \in \mathcal{I}$ and $\|TP\| < \epsilon$. Then one has for $P_\epsilon = I - E[0,\epsilon)$ that $\|Tx\| \geq \epsilon\|x\|$ for every $x \in P_\epsilon(H)$. Hence $\inf(P,P_\epsilon) = 0$ and we obtain from the parallelogram law (cf. [25], Chapter V, 1.6) that $P_\epsilon \lesssim I-P$, thus $P_\epsilon \in \mathcal{I}$ (cf. Lemma 1.4). Hence $\inf\{\epsilon > 0 : I-E[0,\epsilon) \in \mathcal{I}\} \leq \alpha(T)$. □

By using Theorem 3.7 we obtain the following geometrical characterization for α.

3.9 **Theorem**

$\alpha(T) = q(T(U))$, *where* U *is the unit ball in* H.

Proof Let $\beta(T) := q(T(U))$. For $\mu > 0$ given, there exist an $\epsilon > 0$ and an ϵ-net N_ϵ for $T(U)$ such that $\epsilon \leq \beta(T) + \mu$. Then there exists a projection $P_\epsilon \in \mathcal{I}$ such that for each $x \in U$ there is a $y \in N_\epsilon \subseteq P_\epsilon(H)$ with
$\|Tx - y\| < \epsilon$. Hence $\|Tx - P_\epsilon Tx\| = \inf_{y \in P_\epsilon(H)} \|Tx - y\| < \epsilon$ for each $x \in U$. Thus $\|(I-P_\epsilon)T\| \leq \beta(T) + \mu$.
Let $F_\epsilon = N_{P_\epsilon T}$ and $R_\epsilon = \inf(I-P_\epsilon, F_\epsilon)$. Then $TR_\epsilon = (I-P_\epsilon)TR_\epsilon$ and we obtain $\|TR_\epsilon\| \leq \|(I-P_\epsilon)T\| \leq \beta(T) + \mu$. Since $I-R_\epsilon = \sup(P_\epsilon, I-F_\epsilon) \in \mathcal{I}$ it follows from Theorem 3.7 that $\alpha(T) \leq \beta(T)$.
For the converse inequality let $\mu > 0$ be given. Then there exists a projection R_μ such that

$$(I-R_\mu) \in \mathcal{I} \quad \text{and} \quad \|TR_\mu\| < \gamma(T) + \mu \tag{1}$$

Let $\epsilon = \gamma(T) + \mu$. If we then let $P_\epsilon = R_{TQ_\epsilon}$ where $Q_\epsilon = I - R_\mu$ and further let $N_\epsilon = TQ_\epsilon(U)$, we have an ϵ-net N_ϵ for $T(U)$. Thus $\beta(T) \leq \alpha(T)$. □

With this in hand we prove the following proposition which gives an interesting representation of the quotient norm on the algebra $\mathcal{A}/\mathcal{I}$.

3.10 **Proposition**

Let $T \in \mathcal{A}$, *then* $\alpha(T) = \inf_{S \in \mathcal{I}} \|TR_{I-S}\|$.

Proof For $\epsilon > 0$ given, there exists a projection P_ϵ in $\mathcal{A}$ such that $I-P_\epsilon \in \mathcal{I}$ and $\|TP_\epsilon\| \le \alpha(T) + \epsilon$ (cf. Theorem 3.7).
Then clearly $\inf_{S\in\mathcal{I}} \|TR_{I-S}\| \le \alpha(T) + \epsilon$.
Conversely, suppose $S \in \mathcal{I}$ then clearly $I-S \in \Phi$ and by [17], Theorem 4.7 it follows that $I - R_{I-S}$ $(= N_{I-S^*})$ is an element of $\mathcal{I}$.
Thus $\alpha(T) \le \|TR_{I-S}\|$. Hence $\alpha(T) \le \inf_{S\in\mathcal{I}} \|TR_{I-S}\|$. □

By using the quantity q defined in Definition 3.4 we are able to give a characterization theorem for the left Fredholm operators in $\mathcal{A}$. In order to do so we need the following lemma:

3.11 **Lemma**

For bounded sets $\Omega, \psi \subseteq H$ *and* $T \in \mathcal{A}$ *one has*

(i) $q(\Omega + \psi) \le q(\Omega) + q(\psi)$

(ii) $q(T(\Omega)) \le \|T\| q(\Omega)$

Proof (i) Let $\epsilon > 0$ be given. By direct application of the definition of q there exists a μ-net for Ω such that $\mu < q(\Omega) + \epsilon/2$. Similarly we can choose a δ-net for Ψ with $\delta < q(\Psi) + \epsilon/2$. If we let $\alpha = \delta + \mu$, $N_\alpha = N_\delta + N_\mu$ and $P_\alpha = \sup(P_\delta, P_\mu)$ it follows easily by application of Definition 3.4 that N_α is an α-net for $\Omega + \Psi$. Hence

$$q(\Omega + \Psi) < q(\Omega) + q(\Psi) + \epsilon.$$

(ii) Let $\epsilon > 0$ be given. By definition of q there exists a μ-net such that $\mu < q(\Omega) + \epsilon$. If we let $\delta = \|T\|\mu$, $N_\delta = T(N_\mu)$ and $P_\delta = R_{TP_\mu}$ it is clear that N_δ is a δ-net for $T(\Omega)$. Hence $q(T(\Omega)) \le \|T\|(q(\Omega) + \epsilon)$. Since $\epsilon > 0$ has been chosen arbitrarily, the result follows. □

3.12 Theorem

$T \in \Phi^+$ iff there exists a constant $c > 0$ such that $q(T(\Omega)) \geq cq(\Omega)$ for all bounded subsets $\Omega \subseteq H$.

Proof For $T \in \Phi^+$ let $T = U|T|$ be the polar decomposition. Then, from Proposition 1.6 we have that $|T| \in \Phi$ and hence there exists an $S \in \mathcal{A}$ such that $S|T| - I := R \in \mathcal{I}$. By using Lemma 3.11 it follows that

$$\begin{aligned} q(\Omega) &= q((S|T| - R)(\Omega)) \\ &\leq q(S|T|(\Omega)) \\ &= q(SU^*T(\Omega)) \\ &\leq \|S\| q(T(\Omega)) \end{aligned}$$

for all bounded subsets $\Omega \subseteq H$.

Conversely, suppose the condition holds and $T \notin \Phi^+$. Choose ϵ such that $0 < \epsilon < c$. By Proposition 1.6, Lemma 1.2 and [17], Proposition 4.2 there exists a projection $Q_\epsilon \notin \mathcal{I}$ such that $\|TQ_\epsilon\| < \epsilon$. For $\Omega = Q_\epsilon(U)$ it follows that
$q(T(\Omega)) \leq \|TQ_\epsilon\| < \epsilon$. From the condition we however get
$q(T(\Omega)) \geq c > \epsilon$: a contradiction, hence $T \in \Phi^+$. □

Recall (cf. Remark 3.6) that if $\mathcal{I} = \mathcal{K}$ we have that $q(\Omega) = 0$ if and only if Ω is $\mathcal{A}$-relatively compact. Hence as a direct corollary from the theorem above we obtain:

3.13 Corollary

Let $T \in \Phi^+(\mathcal{A},\mathcal{K})$. The only bounded sets which T maps onto $\mathcal{A}$-relatively compact sets are those which are $\mathcal{A}$-relatively compact.

3.14 **Remark**

Suppose X and Y are Banach spaces, then Yood (cf. [29]) showed that operators in $\Phi^+(X,Y)$ are characterized by the fact that the only bounded sets which they map into compact sets are those whose closures are compact. It is not known whether 3.13 is a characterization for $\Phi^+(\mathcal{A},\mathcal{K})$ in general. However, with certain conditions on $\mathcal{A}$ we shall see that 3.13 characterizes left Fredholm operators relative to $\mathcal{K}$.

3.15 **Proposition**

Let $\mathcal{A}$ be a semifinite von Neumann algebra with a non-large center (i.e. any sequence of mutually orthogonal infinite central projections is finite). Then $T \in \Phi^+(\mathcal{A},\mathcal{K})$ if and only if the only bounded sets which T maps onto $\mathcal{A}$-relatively compact sets are those which are $\mathcal{A}$-relatively compact.

Proof From Corollary 3.13 it is sufficient to prove the converse implication. It was proved by Kaftal that if $\mathcal{A}$ is semifinite with non-large center, then $\Phi^+(\mathcal{A},\mathcal{K}) = \{T \in \mathcal{A} :$ if $P \in \mathcal{A}$ is a projection such that $TP \in \mathcal{K}$ then $P \in \mathcal{K}\}$ (cf. [15], Proposition 1.7). Now, let $T \in \mathcal{A}$ be such that the only bounded sets mapped onto sets that are $\mathcal{A}$-relatively compact, are already $\mathcal{A}$-relatively compact. If $T \notin \Phi^+(\mathcal{A},\mathcal{K})$, there exists an infinite projection $P \in \mathcal{A}$ such that $TP \in \mathcal{K}$. If we let $\Omega = P(U)$ we get $q(T(\Omega)) = 0$, but $q(\Omega) = 1$: a contradiction. □

It was shown by Pfaffenberger [18] that under certain conditions on a Banach space X the left and right Fredholm operators can be characterized respectively in terms of the algebraic left and right zero-divisors of the Calkin algebra.

Subsequently, again under additional conditions on X, it was shown by Lebow and Schechter [16] (cf. also [2] and [3]) that the left and right topological zero-divisors coincide with the algebraic left and right zero-divisors, respectively. After some preliminaries we show that similar results hold in a semi-finite von Neumann algebra with non-large center.

Let $\mathcal{I}$ be any closed two-sided ideal in $\mathcal{A}$. We shall denote by S_ℓ the set of all topological left divisors of zero in the quotient algebra $\mathcal{A}/\mathcal{I}$, i.e. $\pi_\mathcal{I}(T) \in S_\ell$ if there exists a normalized sequence $(\pi_\mathcal{I}(S_n))$ such that $\lim_{n\to\infty} \pi_\mathcal{I}(T)\pi_\mathcal{I}(S_n) = 0$. The set of right topological divisors of zero is defined in an obvious similar way. We shall also denote by Z_ℓ (resp. Z_r) the set of left (resp. right) algebraic divisors of zero in $\mathcal{A}/\mathcal{I}$.

Consider the classes

$$\Phi^+(\mathcal{A},\mathcal{I}) = \{T \in \mathcal{A} : \text{if } P \in \mathcal{P}(\mathcal{A}) \text{ and } TP \in \mathcal{I} \text{ then } P \in \mathcal{I}\}$$

$$\Phi^-(\mathcal{A},\mathcal{I}) = \{T \in \mathcal{A} : \text{if } P \in \mathcal{P}(\mathcal{A}) \text{ and } PT \in \mathcal{I} \text{ then } P \in \mathcal{I}\}.$$

Kaftal studied these classes in the case where $\mathcal{I} = \mathcal{K}$ (cf. [15]).

3.16 **Proposition**

(i) $\Phi^+(\mathcal{A},\mathcal{I}) = \pi_\mathcal{I}^{-1}((Z_\ell)^c)$

(ii) $\Phi^-(\mathcal{A},\mathcal{I}) = \pi_\mathcal{I}^{-1}((Z_r)^c)$

(($)^c$ denotes set complementation)

Proof (i) Let $T \in \Phi^+(\mathcal{A},\mathcal{I})$, then $\pi_\mathcal{I}(T) \in (Z_\ell)^c$. For if not, there exists an $S \notin \mathcal{I}$ such that $TS \in \mathcal{I}$. Since $S \notin \mathcal{I}$ it follows from Proposition 2.4(c) that there exists a projection $P \notin \mathcal{I}$ such that $P(H) \subset S(H)$. Then $TP \in \mathcal{I}$, for if $Q \in \mathcal{P}(\mathcal{A})$ such that $Q(H) \subset TP(H)$ it follows that $Q(H) \subset TS(H)$. Hence $Q \in \mathcal{I}$ (by 2.4(c)) and by applying 2.4(c) again, we have that $TP \in \mathcal{I}$, so since $T \in \tilde{\Phi}^+(\mathcal{A},\mathcal{I})$, $P \in \mathcal{I}$, a contradiction.

Conversely, let $T \in \pi_{\mathcal{I}}^{-1}((Z_\ell)^c)$, then $T \in \tilde{\Phi}^+(\mathcal{A},\mathcal{I})$. For if not, there exists a projection $P \notin \mathcal{I}$ such that $TP \in \mathcal{I}$. Hence $\pi_{\mathcal{I}}(T) \in Z_\ell$, a contradiction.

(ii) It is trivial to show that $T \in \tilde{\Phi}^+$ iff $T^* \in \tilde{\Phi}^-$ and $T \in \pi_{\mathcal{I}}^{-1}((Z_\ell)^c)$ iff $T^* \in \pi_{\mathcal{I}}^{-1}((Z_r)^c)$. Hence (ii) follows directly from (i). □

3.17 **Theorem**

Let $\mathcal{A}$ be a semi-finite von Neumann algebra with non-large center. Then

(i) $\Phi^+(\mathcal{A},\mathcal{K}) = \pi_{\mathcal{K}}^{-1}((S_\ell)^c)$ *and* $S_\ell = Z_\ell$

(ii) $\Phi^-(\mathcal{A},\mathcal{K}) = \pi_{\mathcal{K}}^{-1}((S_r)^c)$ *and* $S_r = Z_r$.

Proof (i) Recall that $\alpha(T) = \inf_{K\in\mathcal{K}}\|T - K\|$. If $\pi_{\mathcal{K}}(T) \in S_\ell$ then by definition there exists a sequence $(S_n) \subset \mathcal{A}$ with $\alpha(S_n) = 1$ and $\lim_{n\to\infty} \alpha(TS_n) = 0$. Thus $T \notin \Phi^+(\mathcal{A},\mathcal{K})$, for if $T \in \Phi^+(\mathcal{A},\mathcal{K})$ one can choose a left Fredholm inverse $R \in \mathcal{A}$ such that $\alpha(TS_n) \geq 1/\|R\|$ for every $n \in \mathbb{N}$. We hence have the inclusions (the second one being trivial) :

$$\Phi^+(\mathcal{A},\mathcal{K}) \subseteq \pi_{\mathcal{K}}^{-1}((S_\ell)^c) \subseteq \pi_{\mathcal{K}}^{-1}((Z_\ell)^c).$$

To conclude (i), it suffices to show the inclusions

$$\pi_{\mathcal{K}}^{-1}((Z_\ell)^c) \subseteq \Phi^+(\mathcal{A},\mathcal{K}) \quad \text{and} \quad S_\ell \subseteq Z_\ell.$$

The first inclusion follows directly from Proposition 3.16 and [15], Proposition 1.7. For the second inclusion, suppose $\pi_{\mathcal{K}}(T) \in (Z_\ell)^c$. From the first inclusion it then follows that

$$T \in \Phi^+(\mathcal{A},\mathcal{K}) \subseteq \pi_{\mathcal{K}}^{-1}((S_\ell)^c), \text{ and hence } \pi_{\mathcal{K}}(T) \in (S_\ell)^c.$$

(ii) follows trivially by noting that $T \in \Phi^-(\mathcal{A},\mathcal{K})$ iff $T^* \in \Phi^+(\mathcal{A},\mathcal{K})$ and

$$T \in \pi_{\mathcal{K}}^{-1}((S_r)^c) \text{ iff } T^* \in \pi_{\mathcal{K}}^{-1}((S_\ell)^c). \qquad \square$$

3.18 **Remark**

It is clear from the proof of Theorem 4.19 that the inclusions,

$$\Phi^+(\mathcal{A},\mathcal{K}) \subseteq \pi_{\mathcal{K}}^{-1}((S_\ell)^c) \text{ and } \Phi^-(\mathcal{A},\mathcal{K}) \subseteq \pi_{\mathcal{K}}^{-1}((S_r)^c)$$

hold for any von Neumann algebra $\mathcal{A}$. We shall illustrate with an example that von Neumann algebras do exist for which the equalities of the theorem do not hold.

3.19 **Example** (cf. [15], Example 1.8)

Let $\mathcal{A}$ be a purely infinite von Neumann algebra and let (P_n) be an infinite sequence of mutually orthogonal non-zero projec tions such that $I = \sum_{n=1}^{\infty} P_n$.

Let $T = \sum_{n=1}^{\infty} 1/n\ P_n$. Then $T \geq 0$ and $E[0,\epsilon) = \sum_{1/n<\epsilon} P_n$ is infinite for every $\epsilon > 0$. So $T \notin \Phi^+(\mathcal{A},\mathcal{K})$ (cf. Proposition 1.6, Lemma 1.2 and [17] Proposition 4.2). If there exists an $S \in \mathcal{A}$ such that $\pi_{\mathcal{K}}(T)\pi_{\mathcal{K}}(S) = 0$, then

$$TS = \sum_{n=1}^{\infty} 1/n\ P_nS \in \mathcal{K} = \{0\}.$$

Thus for each $n \in \mathbb{N}$, $P_nS = 0$, from which it follows that $S = 0$ and therefore $\pi_{\mathcal{K}}(S) = 0$.

Hence $\pi_{\mathcal{K}}(T) \in (Z_\ell)^c$. $\square$

We conclude this section by defining a few other quantities related to semi-Fredholm operators. Let α be the $\mathcal{I}$-essential seminorm on $\mathcal{A}$ and let $Q \notin \mathcal{I}$ be any projection in $\mathcal{A}$.

We define

$$\lambda_Q(T) := \inf\{\alpha(TP) : P \notin \mathcal{I} \text{ and } P \leq Q\}$$

3.20 **Proposition**

Let $\mathcal{I}$ *be any closed ideal in* $\mathcal{A}$. *Then*

$$m_{\mathcal{I}}(T) = \lambda_I(T)$$

Proof Let E be the spectral measure for $|T|$ and for any $\epsilon > 0$ take $Q = E[m_{\mathcal{I}}(T) - \epsilon, m_{\mathcal{I}}(T) + \epsilon]$. Then $Q \notin \mathcal{I}$ and $\alpha(TQ) \leq m_{\mathcal{I}}(T) + \epsilon$. Hence $\lambda_I(T) \leq m_{\mathcal{I}}(T)$. For the other direction let $\epsilon > 0$ be given and choose $S = |T|E[m_{\mathcal{I}}(T) - \epsilon, \infty)$. Since

$$\alpha(TR) = \alpha(RSS^*R)^{1/2} \geq (m_{\mathcal{I}}(T) - \epsilon)\alpha(R) \quad \text{for any } R \in \mathcal{A},$$

it follows that $\lambda_I(T) \geq m_{\mathcal{I}}(T)$. □

3.21 **Proposition**

Let $S,T \in \mathcal{A}$ *be such that* $\alpha(S) < m_{\mathcal{I}}(T)$. *Then* T *and* $T + S$ *are left Fredholm relative to* $\mathcal{I}$ *and index* (T) = *index* $(T+S)$.

Proof For all $S,T \in \mathcal{A}$ (thus also for S and T chosen as above), it follows that

$$m_{\mathcal{I}}(T+S) \leq m_{\mathcal{I}}(T) + \alpha(S) \tag{1}$$

To see this, let $\epsilon > 0$ be given and let $Q \notin \mathcal{I}$ be any projection in $\mathcal{A}$. Then there exists a projection $P_\epsilon \notin \mathcal{I}$ such that $P_\epsilon \leq Q$ and

$$\alpha(SP_\epsilon) < \lambda_Q(S) + \epsilon.$$

Hence,

$$m_{\mathcal{I}}(T+S) \leq \alpha(TP_\epsilon) + \alpha(SP_\epsilon) \leq \alpha(TQ) + \lambda_Q(S) + \epsilon \leq \alpha(TQ) + \alpha(S) + \epsilon$$

Since $\epsilon > 0$ was chosen arbitrarily, it follows that

$$m_{\mathcal{I}}(T+S) \leq \alpha(TQ) + \alpha(S).$$

Thus, by taking the infimum over all projections $Q \notin \mathcal{I}$, relation (1) follows.

Since $\alpha(T) < m_{\mathcal{I}}(T)$, it follows from (1) that

$$0 < m_{\mathcal{I}}(T) - r\alpha(S) \leq m_{\mathcal{I}}(T+rS) \quad \text{for any} \quad r \in [0,1].$$

Hence, by Proposition 1.6 it is clear that $T+rS \in \Phi^+$. Since the index map is locally constant on Φ^+, the result follows. □

3.22 **Corollary**

Let $T \in \Phi$ *and* $S \in \mathcal{A}$ *be such that* $\alpha(S) < m_{\mathcal{I}}(T)$. *Then* $T+S \in \Phi$.

Proof From Proposition 3.21 and [17], Theorem 4.7(vi) it suffices to show that $N_{(T+S)^*} \in \mathcal{I}$. From [17], Proposition 9.3 there exists a central projection P such that $P\mathcal{I}$ is completely noncompact in $P\mathcal{A}$, and $(I-P)\mathcal{I}$ is an ideal contained in the ideal of relatively compact elements of $(I-P)\mathcal{A}$ (cf. [17], 9.2 for the definition of a completely noncompact ideal). Let i be the index map relative to $(I-P)\mathcal{I}$ and let $\bar{i}$ be the index map relative to $P\mathcal{I}$. Then if we use the notation in [17], 11.1 we denote by $\tilde{i} := i \oplus \bar{i}$ the index map relative to $\mathcal{I}$, where $\mathcal{I} = P\mathcal{I} \oplus (I-P)\mathcal{I}$. From Proposition 3.21 we have seen that $\tilde{i}(T) = \tilde{i}(T+S)$. Hence

$$i(TP) = i((T+S)P) \quad \text{and} \quad \bar{i}(T(I-P)) = \bar{i}((T+S)(I-P)).$$

Since $TP \in \Phi(P\mathcal{A},P\mathcal{I})$ and $T(I-P) \in \Phi((I-P)\mathcal{A},(I-P)\mathcal{I})$ it follows from [17], 5.17 and 10.1 that

$$(T+S)P \in \Phi(P\mathcal{A},P\mathcal{I}) \quad \text{and} \quad (T+S)(I-P) \in \Phi((I-P)\mathcal{A},(I-P)\mathcal{I}).$$

Hence

$$N_{(T+S)^*P} \in P\mathcal{I} \quad \text{and} \quad N_{(T+S)^*(I-P)} \in (I-P)\mathcal{I}.$$

Since

$$PN_{(T+S)^*} \leq N_{(T+S)^*}P \quad \text{and} \quad (I-P)N_{(T+S)^*} \leq N_{(T+S)^*}(I-P)$$

it follows that

$$PN_{(T+S)^*} \in P\mathcal{I} \quad \text{and} \quad (I-P)N_{(T+S)^*} \in (I-P)\mathcal{I}.$$

Thus $N_{(T+S)^*} \in \mathcal{I}$. □

4. RIESZ OPERATORS RELATIVE TO A CLOSED TWO-SIDED IDEAL IN A VON NEUMANN ALGEBRA

In this section we define Riesz operators in a natural way via the Fredholm operators relative to any closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$. The results are similar to those known for the classical case and they will be used in the sequel to prove characterization theorems of Riesz operators as well as a Riesz decomposition theorem (cf. [8] and [11] for the classical theory of Riesz operators).

Let $\pi_{\mathcal{I}}:\mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the quotient map. Recall that in section 3 we called the spectrum of $\pi_{\mathcal{I}}(T)$ the $\mathcal{I}$-essential spectrum and denoted it by $\sigma_{\mathcal{I}}(T)$ and the $\mathcal{I}$-essential spectral radius was denoted by $r_{\mathcal{I}}(T)$.

An operator $T \in \mathcal{A}$ will be called a Riesz operator (relative to $\mathcal{I}$) if $\lambda I - T \in \Phi$ for every $\lambda \neq 0$. Since $\sigma_{\mathcal{I}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi\}$ it follows that T is Riesz iff $\sigma_{\mathcal{I}}(T) = \{0\}$. We shall denote the set of all Riesz operators by $\mathcal{R}$ and if the reference to $\mathcal{A}$ and $\mathcal{I}$ is necessary we denote this set by $\mathcal{R}(\mathcal{A},\mathcal{I})$.

4.1 Remarks

1. It is clear that $T \in \mathcal{R}(\mathcal{A},\mathcal{I})$ iff $\lim_{n\to\infty} (\inf_{S\in\mathcal{I}} \|T^n - S\|)^{1/n} = 0$. Let $\mathcal{I}_0 = \{T \in \mathcal{A} : R_T \in \mathcal{I}\}$. Then by Lemma 1.5 we may replace $\mathcal{I}$ with $\mathcal{I}_0$ in this characterization of Riesz operators.

2. For any $T \in \mathcal{I}$ one has $r_{\mathcal{I}}(T) = 0$. Thus from the remark above it follows that $\mathcal{I} \subseteq \mathcal{R}$. There are many cases where this inclusion is strict.

3. If $\mathcal{I} = \{0\}$, the Riesz operators coincide with the quasi-nilpotent operators in $\mathcal{A}$, and if $\mathcal{I} = \mathcal{A}$ it is clear that $\mathcal{R} = \mathcal{A}$. The theory of Riesz operators in both these cases is trivial.

From the remark above, we have seen that $\mathcal{I} \subseteq \mathcal{R}$. In the following proposition we show that if $\mathcal{J}$ is any other two-sided ideal such that $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{R}(\mathcal{A},\mathcal{I})$, then $\mathcal{I} = \mathcal{J}$.

4.2 **Proposition**

$\mathcal{I}$ is the largest two-sided ideal contained in $\mathcal{R}(\mathcal{A},\mathcal{I})$.

Proof We first show that every Riesz projection is an element of $\mathcal{I}$. Thus, let $E \in \mathcal{P}(\mathcal{A}) \cap \mathcal{R}(\mathcal{A},\mathcal{I})$. Then, since $E^n = E$ for each $n \in \mathbb{N}$ it follows from

$$\lim_{n\to\infty} (\inf_{S\in\mathcal{I}} \|E^n - S\|)^{1/n} = 0$$

that $\inf_{S\in\mathcal{I}} \|E - S\| = 0$, which implies that $E \in \mathcal{I} \cap \mathcal{P}(\mathcal{A})$.
Now let $\mathcal{J}$ be any two-sided ideal such that $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{R}(\mathcal{A},\mathcal{I})$. Then the argument above implies that $\mathcal{J} \cap \mathcal{P}(\mathcal{A}) \subseteq \mathcal{I} \cap \mathcal{P}(\mathcal{A})$. It follows from Lemma 1.4 that $\mathcal{J} \subseteq \mathcal{I}$, hence $\mathcal{J} = \mathcal{I}$. □

We list a few properties of the class $\mathcal{R}(\mathcal{A},\mathcal{I})$ of relatively Riesz operators.

4.3 **Proposition**

(a) *For* $T \in \mathcal{A}$ *we have that* $T \in \mathcal{R}$ *if and only if* $T^* \in \mathcal{R}$.

(b) *$\mathcal{R}$ is stable under perturbations of elements of $\mathcal{I}$ (i.e. if* $T \in \mathcal{R}$ *and* $S \in \mathcal{I}$, *then* $T+S \in \mathcal{R}$).

Proof From the fact that $\mathcal{I}$ is a self-adjoint ideal in $\mathcal{A}$ we have for any $T \in \mathcal{A}$ and $S \in \mathcal{I}$ that

$$r_{\mathcal{I}}(T) = r_{\mathcal{I}}(T^*) \quad \text{and} \quad r_{\mathcal{I}}(T + S) = r_{\mathcal{I}}(T).$$

Hence (a) and (b) follow directly. □

Actually we can show that the class of Riesz operators is stable only under perturbations of elements of $\mathcal{I}$.

For any subset $B \subseteq \mathcal{A}$ we define the perturbation class of B by

$$P(B) = \{T \in \mathcal{A} : T+S \in B \text{ for all } S \in B\}.$$

4.4 **Proposition**

The perturbation class of $\mathcal{R}(\mathcal{A},\mathcal{I})$ is the ideal $\mathcal{I}$.

Proof Let $Q(\mathcal{A}/\mathcal{I})$ be the class of quasinilpotent elements of $\mathcal{A}/\mathcal{I}$. From a theorem due to Zemanek (cf. [4], BA 2.8) we have

$$\begin{aligned} \mathrm{rad}(\mathcal{A}/\mathcal{I}) &= \{\pi_{\mathcal{I}}(T) : T \in \mathcal{A} \text{ and } \pi_{\mathcal{I}}(T) + Q(\mathcal{A}/\mathcal{I}) \subset Q(\mathcal{A}/\mathcal{I})\} \\ &= \{\pi_{\mathcal{I}}(T) : T \in \mathcal{A} \text{ and } T+S \in \mathcal{R} \text{ for all } S \in \mathcal{R}\} \end{aligned}$$

Since $\mathcal{A}/\mathcal{I}$ is a C^*-algebra $\mathrm{rad}(\mathcal{A}/\mathcal{I}) = \{0\}$, it then follows that

$$\begin{aligned} \mathcal{I} = \pi_{\mathcal{I}}^{-1}(\mathrm{rad}(\mathcal{A}/\mathcal{I})) &= \{T \in \mathcal{A} : T+S \in \mathcal{R} \text{ for all } S \in \mathcal{R}\} \\ &= P(\mathcal{R}) \end{aligned}$$

□

The set $\mathcal{R}(\mathcal{A},\mathcal{I})$ is not necessarily an ideal in $\mathcal{A}$. In fact, for the case where $\mathcal{A} = \mathcal{L}(H)$ (H a separable Hilbert space) and $\mathcal{I} = \mathcal{K}$, examples to show this can be found in [11] (cf. [11], Example 3.6). We can, however, show the following result which also holds for the classical case. We denote by $[S,T]$ the commutator of S and T, i.e. $[S,T] = ST - TS$.

By using the well-known property that in any Banach algebra the relations $r(xy) \leq r(x)r(y)$ and $r(x+y) \leq r(x) + r(y)$ hold for any two commuting x and y, one easily obtains the following.

4.5 Proposition

(a) *If* $S \in \mathcal{R}$, $T \in \mathcal{A}$ *and* $[S,T] \in \mathcal{I}$ *then* $ST, TS \in \mathcal{R}$.

(b) *If* $S,T \in \mathcal{R}$ *and* $[S,T] \in \mathcal{I}$ *then* $T+\alpha S \in \mathcal{R}$ *for any* $\alpha \in \mathbb{C}$.

(c) *If a sequence* (T_n) *of Riesz operators is uniformly convergent to* $T \in \mathcal{A}$ *and if* $[T_n,T] \in \mathcal{I}$ *for all* $n \in \mathbb{N}$ *then* $T \in \mathcal{R}$.

4.6 Corollary

Let $T \in \mathcal{R}$. *Then the closed algebra generated by* T *is contained in* $\mathcal{R}$.

We now prove a generalization of the classical Weyl theorem which states that if $T \in \mathcal{L}(H)$ and $K \in \mathcal{K}$, then $\sigma_{\mathcal{K}}(T) = \sigma_{\mathcal{K}}(T+K)$.

4.7 Proposition

If $T \in \mathcal{A}$, $S \in \mathcal{R}$ *and* $[S,T] \in \mathcal{I}$ *then* $\sigma_{\mathcal{I}}(T+S) = \sigma_{\mathcal{I}}(T)$.

Proof For any two commuting elements x, y in a Banach algebra it is well-known that $\sigma(x+y) \subseteq \sigma(x) + \sigma(y)$. In particular we have that

$$\sigma_{\mathcal{I}}(T+S) \subseteq \sigma_{\mathcal{I}}(T) + \sigma_{\mathcal{I}}(S).$$

By assumption $\sigma_{\mathcal{I}}(S) = \{0\}$. Hence $\sigma_{\mathcal{I}}(T+S) \subset \sigma_{\mathcal{I}}(T)$.

Similarly $\sigma_{\mathcal{I}}(T) = \sigma_{\mathcal{I}}(T+S-S) \subseteq \sigma_{\mathcal{I}}(T+S)$. □

The following result gives a converse for the proposition mentioned above.

4.8 Proposition

Let $S \in \mathcal{A}$. *If* $\sigma_{\mathcal{I}}(T+S) = \sigma_{\mathcal{I}}(T)$ *for every* $T \in \mathcal{A}$ *with* $[S,T] \in \mathcal{I}$, *then* $S \in \mathcal{R}$.

Proof Choose any $T \in \Phi$ such that $[S,T] \in \mathcal{I}$. Since $\sigma_{\mathcal{I}}(T+S) = \sigma_{\mathcal{I}}(T)$ we clearly have that

$$0 \in \{\lambda : T-\lambda I \in \Phi\} = \{\lambda : T+S-\lambda I \in \Phi\}.$$

Hence $T+S \in \Phi$. By choosing $T = \lambda I$ $(\lambda \neq 0)$ the conditions of the proposition are satisfied. Thus it follows that $S+\lambda I \in \Phi$ for every $\lambda \neq 0$, which proves the proposition. □

These two propositions mentioned above can be used to prove a characterization of relatively Riesz operators in a von Neumann algebra $\mathcal{A}$.

4.9 Corollary

$S \in \mathcal{R}$ *iff* $T+S \in \Phi$ *for all* $T \in \Phi$ *for which* $[S,T] \in \mathcal{I}$.

Proof Let $S \in \mathcal{R}$ and $T \in \Phi$ with the property that $[S,T] \in \mathcal{I}$. Then we know that $0 \notin \sigma_{\mathcal{I}}(T) = \sigma_{\mathcal{I}}(T+S)$, thus $T+S \in \Phi$.
Since $[\lambda I, S] = 0$ for every λ, the converse is trivial. □

By application of this theorem we obtain the following result which will be used in the sequel.

4.10 Proposition

For $T \in \mathcal{A}$ *we have that* $T \in \mathcal{R}$ *iff* $T^n \in \mathcal{R}$ *for any (and hence for all)* $n \in \mathbb{N}$.

Proof If $T \in \mathcal{R}$ and $n \in \mathbb{N}$, then $T^n \in \mathcal{R}$ follows trivially from Proposition 4.5.

Conversely, if $T^n \in \mathcal{R}$ it follows by definition that

$$\lim_{k\to\infty} \alpha(T^{nk})^{1/nk} = 0.$$

Since $r_{\mathcal{I}}(T) = \lim_{k\to\infty} \alpha(T^k)^{1/k}$ is finite one clearly has that

$$r_{\mathcal{I}}(T) = 0.$$ □

For the Riesz operators we obtain a functional calculus similar to the classical case. The proof of this result is simply a transposition to von Neumann algebras of Theorem R.1.3 in [4].

4.11 **Proposition**

Let f *be a holomorphic function on an open set* U *containing* $\sigma(T)$ *with* $f(0) = 0$. *Then*

(a) *If* $T \in \mathcal{R}$ *then* $f(T) \in \mathcal{R}$.

(b) *If* $f(T) \in \mathcal{R}$ *and* f *does not vanish on* $\sigma(T)\backslash\{0\}$ *it follows that* $T \in \mathcal{R}$.

Proof (a) From our assumptions it follows that $f(T) = Tg(T)$ where g is holomorphic on U and $[T,g(T)] = 0$. Then (a) follows directly from Proposition 4.5(a).

(b) Since $\sigma_{\mathcal{I}}(T) \subseteq \sigma(T)$ (cf. [13], Proposition 3.1 for the case $\mathcal{I} = \mathcal{K}$, for the general case the proof is similar), the functional calculus in $\mathcal{A}/\mathcal{I}$ shows that $\pi_{\mathcal{I}}(f(T)) = f(\pi_{\mathcal{I}}(T))$, and by the spectral mapping theorem $f(\sigma_{\mathcal{I}}(T)) = \sigma_{\mathcal{I}}(f(T)) = \{0\}$. By hypothesis f does not vanish on $\sigma(T)\backslash\{0\}$, leaving $\sigma_{\mathcal{I}}(T) = \{0\}$ as the only possibility. □

In any unital $\mathcal{C}^*$-algebra $\mathcal{B}$ it is known that $\sigma(xy)\backslash\{0\} = \sigma(yx)\backslash\{0\}$ and $\sigma(uxu^*)\backslash\{0\} = \sigma(x)\backslash\{0\}$ hold for $x,y \in \mathcal{B}$ and $u \in \mathcal{B}$ unitary. The following proposition therefore follows:

4.12 **Proposition**

(a) $TS \in \mathcal{R}$ *if and only if* $ST \in \mathcal{R}$.

(b) *If* S *and* T *are unitary equivalent, then* $S \in \mathcal{R}$ *if and only if* $T \in \mathcal{R}$.

One can easily see from the next proposition that if a von Neumann algebra contains quasinilpotent operators which are not contained in $\mathcal{I}$, then $\mathcal{I}$ is properly contained in $\mathcal{R}$.

4.13 **Proposition**

If $S \in \mathcal{I}$ *and* $T \in \mathcal{A}$ *is quasinilpotent, then* $S+T \in \mathcal{R}$.

Proof This clearly follows from

$$\|\pi_{\mathcal{I}}(T+S)^n\|_{\mathcal{A}/\mathcal{I}}^{1/n} = \|\pi_{\mathcal{I}}(T)^n\|_{\mathcal{A}/\mathcal{I}}^{1/n} \leq \|T^n\|^{1/n} \quad \text{for all } n \in \mathbb{N}. \qquad \square$$

4.14 **Remark**

By the well-known West decomposition theorem (cf. [4], $\mathcal{C}^*$2.1) the converse of this proposition holds in the case where $\mathcal{A} = \mathcal{L}(\mathbb{H})$ and $\mathcal{I} = \mathcal{K}$. It is an open problem whether this is true in general von Neumann algebras. A partial converse can be obtained by using a result of Akemann and Pedersen [1]: If $T \in \mathcal{A}$ with $T^n \in \mathcal{I}$ for some $n \in \mathbb{N}$ (note that in this case $T \in \mathcal{R}$ by 4.10), then $T = S + Q$ where $S \in \mathcal{I}$ and Q is nilpotent. This follows from the fact that [1], 4.3 implies that there exists an $S \in \mathcal{I}$ such that $(T-S)^n = 0$.

In [5], Breuer proved a generalized Fredholm alternative which states that $I - T$ is Fredholm of index zero if $T \in \mathcal{K}$. Using the index theory of Olsen (cf. [17]), we prove a similar result for the Riesz operators relative to any closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$.

4.15 **Proposition** (Generalized Fredholm alternative)

If $T \in \mathcal{A}$ *is Riesz relative to* $\mathcal{I}$, *then* $I - T$ *is Fredholm relative to* $\mathcal{I}$ *with index zero.*

Proof By definition $I-\lambda T \in \Phi$ for all $\lambda \neq 0$. Since the index map on Φ is locally constant (cf. [17], Theorems 11.10 and 11.12), it is clear that the index is constant on $\{I-\lambda T | \lambda \in [0,1]\}$ and the result follows. □

Let $\Phi_S = \Phi^+ \cup \Phi^-$. Since the set of elements Φ_S is the class of all operators which are left or right invertible modulo $\mathcal{I}$, one trivially notes that an element of $\mathcal{I}$ cannot be contained in Φ_S. Actually one can show that not even a Riesz operator can be contained in Φ_S.

4.16 **Proposition**

If $T \in \Phi_S$ *then* $T \notin \mathcal{R}$.

Proof If $T \in \Phi^+$, it is clear that $r_{\mathcal{I}}(T) > 0$. Hence $T \notin \mathcal{R}$. If $T \in \Phi^-$ then $T^* \in \Phi^+$ and similarly $T^* \notin \mathcal{R}$. Thus by Proposition 4.3 $T \notin \mathcal{R}$. □

The following theorem characterizes the elements of an arbitrary closed ideal $\mathcal{I}$ in $\mathcal{A}$.

4.17 **Theorem**

Let $\mathcal{I}$ *be any closed two-sided ideal in* $\mathcal{A}$. *Then* $T \in \mathcal{I}$ *iff* $ST \in \mathcal{R}$ *for each* $S \in \Phi$.

Proof Let $\mathcal{J} = \{T \in \mathcal{A}: ST \in \mathcal{R}$ for each $S \in \Phi\}$. It is clear that $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{R}$. Hence the theorem will follow from Proposition 4.2 if we can show that $\mathcal{J}$ is a two-sided ideal in $\mathcal{A}$. Since Φ is a self-adjoint subset of $\mathcal{A}$ and $TS \in \mathcal{R}$ iff $ST \in \mathcal{R}$ (cf. Proposition 4.12), one has that $\mathcal{J}$ is self-adjoint. Thus it suffices to show that $\mathcal{J}$ is a left ideal.

Clearly $\alpha T \in \mathcal{J}$ for $\alpha \in \mathbb{C}$ and $T \in \mathcal{J}$.

Let T_1 and T_2 be elements of $\mathcal{J}$. We show that $T_1+T_2 \in \mathcal{J}$:

Take any $S \in \Phi$, then there exists an $R \in \Phi$ such that

$$\pi_{\mathcal{I}}(SR) = \pi_{\mathcal{I}}(RS) = \pi_{\mathcal{I}}(I).$$

Since $T_2 \in \mathcal{J}$, $ST_2 \in \mathcal{R}$ and hence $ST_2 - \lambda I \in \Phi$ for every $\lambda \neq 0$.

Let $R_\lambda = T_2 - \lambda R$. Then, since

$$\pi_{\mathcal{I}}(R_\lambda) = \pi_{\mathcal{I}}(R(ST_2 - \lambda I))$$

it is clear that $R_\lambda \in \Phi$ for every $\lambda \neq 0$. Hence, for every $\lambda \neq 0$, there exists an $S_\lambda \in \Phi$ such that $\pi_{\mathcal{I}}(R_\lambda S_\lambda) = \pi_{\mathcal{I}}(S_\lambda R_\lambda) = \pi_{\mathcal{I}}(I)$ and so

$$\pi_{\mathcal{I}}(S_\lambda(T_1 + R_\lambda)) = \pi_{\mathcal{I}}(S_\lambda T_1 + I);$$

also $\qquad S_\lambda T_1 + I \in \Phi$, hence $T_1 + R_\lambda \in \Phi$.

Since

$$T_1+R_\lambda = T_1 + T_2 - \lambda R \text{ and } \pi_{\mathcal{I}}(S(T_1+T_2) - \lambda I) = \pi_{\mathcal{I}}(S(T_1+R_\lambda))$$

for every $\lambda \neq 0$, it follows that $S(T_1 + T_2) \in \mathcal{R}$.

Hence $T_1+T_2 \in \mathcal{J}$.

Let S be a non-zero element of $\mathcal{A}$ and $T \in \mathcal{J}$. Then S can be written as a sum of two regular elements of $\mathcal{A}$, hence as a sum of two elements of Φ, say $S = S_1 + S_2$. Then clearly S_1T and S_2T are elements of $\mathcal{J}$. Thus $ST = S_1T+S_2T \in \mathcal{J}$. □

In the last two results of this section we show that the class of Riesz operators relative to $\mathcal{K}$ behaves well under reduction with respect to central projections as well as under decompositions of the von Neumann algebra.

Similar results for other classes were obtained by Kaftal (cf. [15], 2.1 and 2.2). Let E be a central projection in $\mathcal{A}$. For $T \in \mathcal{A}$, let us identify TE with its restriction $TE_{|E(H)}$ to E(H), i.e. let us identify the algebra $\mathcal{A}E \subset \mathcal{A}$ with the reduced algebra $\mathcal{A}_E$.

4.18 **Lemma**

With the notation above one has that $\mathcal{R}(\mathcal{A},\mathcal{K})E = \mathcal{R}(\mathcal{A}_E,\mathcal{K}(\mathcal{A}_E))$.

Proof Let $T \in \mathcal{R}(\mathcal{A},\mathcal{K})E$ and $\lambda \neq 0$ be given. There exists an $S \in \mathcal{R}(\mathcal{A},\mathcal{K})$ such that $T = SE$. Then $S_\lambda := \lambda I - S$ is invertible modulo $\mathcal{K}$, i.e. there exists an S'_λ such that $S_\lambda S'_\lambda \in I+\mathcal{K}$ and $S'_\lambda S_\lambda \in I+\mathcal{K}$. Hence

$$ES_\lambda ES'_\lambda \in E+\mathcal{K}E \quad \text{and} \quad ES'_\lambda ES_\lambda \in E+\mathcal{K}E.$$

By [15], Lemma 2.1 we know that $\mathcal{K}E = \mathcal{K}(\mathcal{A}_E)$ and therefore $\lambda E-T$ $(= ES_\lambda)$ is invertible modulo $\mathcal{K}(\mathcal{A}_E)$. Hence $T \in \mathcal{R}(\mathcal{A}_E,\mathcal{K}(\mathcal{A}_E))$.
Conversely, suppose $T \in \mathcal{R}(\mathcal{A}_E,\mathcal{K}(\mathcal{A}_E))$ and $\lambda \neq 0$. Then

$$S_\lambda := E - \frac{1}{\lambda}T \in \Phi(\mathcal{A}_E,\mathcal{K}(\mathcal{A}_E)).$$

Thus there exists an $S'_\lambda \in \mathcal{A}_E$ such that $S_\lambda S'_\lambda \in E+\mathcal{K}E$ and $S'_\lambda S_\lambda \in E+\mathcal{K}E$.
Let $A_\lambda = S_\lambda+I-E$ and $B_\lambda = S'_\lambda+I-E$. Then $A_\lambda,B_\lambda \in \mathcal{A}$ such that

$$A_\lambda B_\lambda \in I+\mathcal{K} \quad \text{and} \quad B_\lambda A_\lambda \in I+\mathcal{K}.$$

Thus $I - \frac{1}{\lambda}T = A_\lambda \in \Phi(\mathcal{A},\mathcal{K})$ for all $\lambda \neq 0$. Hence $T \in \mathcal{R}(\mathcal{A},\mathcal{K})$ and since $T = TE$, we have that $T \in \mathcal{R}(\mathcal{A},\mathcal{K})E$. □

Let I be an index set and let $\mathcal{A} = \sum_{i\in I} \oplus \mathcal{A}_i$ be the direct sum of von Neumann algebras $\mathcal{A}_i$ (cf. [10], Part I, Chapter 2, Section 2). We may identify the identity of $\mathcal{A}_i$ with a central projection $E_i \in \mathcal{A}$ and $\mathcal{A}_i$ with $\mathcal{A}E_i$. Denote by $\sum_{i\in I} \oplus \mathcal{R}(\mathcal{A}_i,\mathcal{K}_i)$ the set

$$\{T \in \mathcal{A}: TE_i \in \mathcal{R}(\mathcal{A}_i,\mathcal{K}_i)\}.$$

For $L \subseteq I$ we may identify $\sum_{i\in L} \oplus \mathcal{A}_i$ with a closed subalgebra of $\sum_{i\in I} \oplus \mathcal{A}_i$ in an obvious way.

4.19 **Proposition**

$\mathcal{R}(\mathcal{A},\mathcal{K}) \subseteq \sum_{i\in I} \oplus \mathcal{R}(\mathcal{A}_i,\mathcal{K}_i)$ and equality holds if at most finitely many E_i are infinite.

Proof The inclusion follows directly by application of Lemma 4.18. Suppose then that E_i is finite for all $i \notin J$, where J is some finite subset of I. Let $T \in \sum_{i\in I} \oplus \mathcal{R}(\mathcal{A}_i,\mathcal{K}_i)$ and $\lambda \neq 0$.
Then if $T = \sum_{i\in I} \oplus T_i$ one has that $S_{i,\lambda} := E_i - \frac{1}{\lambda}T_i \in \Phi(\mathcal{A}_i,\mathcal{K}_i)$ from which it follows that there exist $S'_{i,\lambda}$ and $K_{i,\lambda}, K'_{i,\lambda} \in \mathcal{K}_i$ such that

$$S_{i,\lambda}S'_{i,\lambda} = E_i+K_{i,\lambda} \quad\text{and}\quad S'_{i,\lambda}S_{i,\lambda} = E_i + K'_{i,\lambda}.$$

For $i \notin J$ we may choose $S'_{i,\lambda} = 0$, and $K'_{i,\lambda} = K_{i,\lambda} = -E_i$. Let

$$S_\lambda = \sum_{i\in I} \oplus S_{i,\lambda} \quad\text{and}\quad S'_\lambda = \sum_{i\in I} \oplus S'_{i,\lambda}.$$

S'_λ is an element of $\mathcal{A}$ since it actually reduces to a finite sum by our choices of $S'_{i,\lambda}$. Then clearly, $S_\lambda S'_\lambda \in I + \sum_{i\in I} \oplus \mathcal{K}_i = I + \mathcal{K}$ (cf. [15], 2.2). Similarly $S'_\lambda S_\lambda \in I + \mathcal{K}$.
Since $S_\lambda = I - \frac{1}{\lambda}T$, it follows that $T \in \mathcal{R}(\mathcal{A},\mathcal{K})$. □

5. GEOMETRICAL CHARACTERIZATIONS OF RIESZ OPERATORS RELATIVE TO A CLOSED TWO-SIDED IDEAL IN A VON NEUMANN ALGEBRA

In section 4 we proved some characterizations of Riesz operators relative to any closed ideal $\mathcal{I}$ in $\mathcal{A}$. In order to obtain a Riesz decomposition theorem we need a geometrical characterization of Riesz operators which is similar to a result of Smyth where Riesz operators on a general Banach space are characterized:
An operator $T \in \mathcal{L}(X)$ is Riesz if and only if for every $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $T^n(U)$ has a finite ϵ^n-net, where U is the unit ball of X (cf. [4], 0.3.5). In proving this result a somewhat laborious machinery of vector sequence spaces was needed. The proof of our result for general von Neumann algebras also gives an elegant proof of Smyth's result for the $\mathcal{L}(H)$-case.

For an operator T in a von Neumann algebra $\mathcal{A}$ the following property (referred to as property A) will be used to characterize the Riesz operators relative to $\mathcal{I}$.

A. For every $\epsilon > 0$ there exist an $n \in \mathbb{N}$, a projection $P_\epsilon \in \mathcal{I}$ and a bounded set $N_\epsilon \subseteq P_\epsilon(H)$ such that for each $x \in U$ there exists a $y \in N_\epsilon$ with $\|T^n x - y\| < \epsilon^n$. (Here and in the following U will denote the unit ball of H.)

5.1 **Lemma**

If $T \in \mathcal{A}$ *has property* A *then* T^m *also has property* A *for all* $m \in \mathbb{N}$.

Proof Without loss of generality we may assume that $T^m \neq 0$. For $\epsilon > 0$, put $\delta = \epsilon/\|T^{m-1}\|$. By assumption there exist an $n \in \mathbb{N}$, a

projection $P_\delta \in \mathcal{I}$ and a bounded set $N_\delta \subseteq P_\delta(H)$ such that for each $w \in U$ there exists a $z \in N_\delta$ with $\|T^n w - z\| < \delta^n$.
Let $N_\epsilon = \|T^{m-1}\|^n N_\delta$ and $P_\epsilon = P_\delta$. Then for $x \in U$ there exists a $y \in N_\epsilon$ such that $\|T^{mn}x - y\| < \epsilon^n$. □

5.2 Theorem

Let $T \in \mathcal{A}$. *Then* $T \in \mathcal{R}$ *if and only if* T *has property* A.

Proof Let $T \in \mathcal{R}$ and $\epsilon > 0$. Then by Remark 4.1(1) we have that $\lim_{n\to\infty} (\inf_{S\in\mathcal{I}_0} \|T^n - S\|^{1/n}) = 0$, where $\mathcal{I}_0 = \{S \in \mathcal{A}: R_S \in \mathcal{I}\}$. Hence there exist an $n \in \mathbb{N}$ and an $S_\epsilon \in \mathcal{I}_0$ such that

$$\|T^n - S_\epsilon\| < \epsilon^n \tag{1}$$

Let $P_\epsilon = R_{S_\epsilon}$ and $N_\epsilon = S_\epsilon(U)$, then $P_\epsilon \in \mathcal{I}$ and N_ϵ is a bounded subset of $P_\epsilon(H)$. By (1) $\|T^n x - S_\epsilon x\| < \epsilon^n$ for all $x \in U$. This proves property A.
Conversely, suppose T has property A. We are going to show that there exists a subsequence of $\{\inf_{S\in\mathcal{I}}\|T^n - S\|^{1/n}\}_n$ which converges to zero, implying that the $\mathcal{I}$-essential spectral radius of T vanishes.
Let $\epsilon > 0$ be given. Then there exist an $n \in \mathbb{N}$, a projection $P_\epsilon \in \mathcal{I}$ and a bounded set $N_\epsilon \subseteq P_\epsilon(H)$ such that for every $x \in U$ there is a $y \in N_\epsilon$ with $\|T^n x - y\| < \epsilon^n$.
Thus $\|T^n x - P_\epsilon T^n x\| = \inf_{w\in P_\epsilon(H)} \|T^n x - w\| < \epsilon^n$. This holds for every $x \in U$, hence $\|T^n - P_\epsilon T^n\| \le \epsilon^n$.
Since $P_\epsilon \in \mathcal{I}$ and therefore $P_\epsilon T^n \in \mathcal{I}$, it follows that for any $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $(\inf_{S\in\mathcal{I}} \|T^n - S\|)^{1/n} \le \epsilon$. We

now find the zero converging subsequence recursively: There exists an $n_1 \in \mathbb{N}$ such that $(\inf_{S\in\mathcal{I}} \|T^{n_1} - S\|)^{1/n_1} < 1$. Since T^{n_1+1} has property A by Lemma 5.1, there exists an $m_1 \in \mathbb{N}$ such that

$$(\inf_{S\in\mathcal{I}} \|(T^{n_1+1})^{m_1} - S\|)^{1/m_1} < (1/2)^{n_1+1} \tag{2}$$

Let $n_2 = (n_1 + 1)m_1$. Then clearly $n_1 < n_2$ and from (2) it follows that

$$(\inf_{S\in\mathcal{I}} \|T^{n_2} - S\|)^{1/n_2} < 1/2.$$

Repeating this argument one finds a monotone increasing sequence of positive integers $n_1 < n_2 < n_3 < \dots n_k < \dots$ such that

$$\inf_{S\in\mathcal{I}} \|T^{n_k} - S\|^{1/n_k} < 1/k \quad \text{for every } k \in \mathbb{N}. \qquad \square$$

5.3 **Remarks**

1. It should be noted that in the case where $\mathcal{A} = \mathcal{L}(H)$ and $\mathcal{I} = \mathcal{K}(H)$ property A coincides with the notion of a finite ϵ^n-net for $T^n(U)$ (cf. [4], §0.3 for the definition of an ϵ-net). Hence $T \in \mathcal{R}$ iff for every $\epsilon > 0$ there exist an $n \in \mathbb{N}$ and vectors $\{u_1,\dots,u_k\} \subset H$ such that $T^n(U) \subset \bigcup_{i=1}^{k} \{u_i + \epsilon^n U\}$.
2. If $\mathcal{A}$ is a von Neumann algebra such that $\mathcal{A} \cap \mathcal{K}(H) \neq \{0\}$ and if we let $\mathcal{I} = \mathcal{A} \cap \mathcal{K}(H)$ then $T \in \mathcal{R}(\mathcal{A},\mathcal{I})$ clearly implies that for every $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $T^n(U)$ has a finite ϵ^n-net. However, for the converse it seems that one should require more. For instance, if for every $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $T^n(U)$ has

a finite ϵ^n-net $\{u_1,\ldots,u_k\}$ and for this net it is true that the projection onto $\mathrm{span}\{u_1,\ldots,u_k\}$ is an element of $\mathcal{A}$, then $T \in \mathcal{R}(\mathcal{A},\mathcal{I})$.

From the proof of Theorem 5.2 we have:

5.4 **Corollary**

$T \in \mathcal{R}$ if and only if for every $\epsilon > 0$ there exist an $n \in \mathbb{N}$ and a projection $Q \in \mathcal{P}(\mathcal{A})$ such that $\|QT^n\| \le \epsilon^n$ and $I-Q \in \mathcal{I}$.

Proof If $T \in \mathcal{R}$ it has property A. Now if we put $Q = I - P_\epsilon$ in the converse part of the proof of Theorem 5.2 the condition holds. Clearly the condition implies property A and the result follows.

□

With Theorem 5.2 in hand, we are now able to give the following result on the class of Riesz operators.

5.5 **Proposition**

Let $S,T \in \mathcal{A}$ be commuting. If $T \in \mathcal{R}$ and $S(H) \subseteq T(H)$ then $S \in \mathcal{R}$.

Proof Let $\epsilon > 0$ be given. Under the conditions of the theorem there exists an $\alpha > 0$ such that for any $n \in \mathbb{N}$ one has

$$S^n(U) \subseteq \alpha^n\overline{T^n(U)} \qquad (1)$$

(cf. [4], 0.4.1 and 0.4.3).

Since $T \in \mathcal{R}$ there exist an $n \in \mathbb{N}$, a projection $P_\epsilon \in \mathcal{I}$ and a bounded set $N_\epsilon \subseteq P_\epsilon(H)$ such that for each $x \in U$ there is a $y \in N_\epsilon$ with

$$\|T^n x - y\| < (\frac{\epsilon}{2\alpha})^n \tag{2}$$

Let $x \in U$, then it follows from (1) that there exists a $z \in U$ such that

$$\|S^n x - \alpha^n T^n z\| < \frac{\epsilon^n}{2} .$$

By (2) there exists a $w \in N_\epsilon$ such that $\|\alpha^n T^n z - \alpha^n w\| < (\frac{\epsilon}{2})^n$. Thus $\|S^n x - \alpha^n w\| < \epsilon^n$. By noting that the set $\alpha^n N_\epsilon \subset P_\epsilon(H)$ is bounded the result follows. □

6. A DECOMPOSITION THEOREM FOR RIESZ OPERATORS RELATIVE TO SPECIFIC CLOSED IDEALS IN A VON NEUMANN ALGEBRA

In [5] and [7], the sequences of the null and range projections of the elements $(I-T)^n$, $n=1,2,3,\ldots$, where $T \in \mathcal{K}$ were studied. The well-known decomposition theorem of F. Riesz for compact operators was generalized to von Neumann algebras. With our geometrical characterization 5.4 of Riesz operators in hand, we can use the techniques of [5] and [7] to obtain a Riesz type of decomposition for Riesz operators relative to specific closed ideals in a semi-finite von Neumann algebra $\mathcal{A}$. For $T \in \mathcal{A}$ let

$$N_n := N_{(I-T)^n} \;;\; F_n := N_{n+1} - N_n \,,\; n=0,1,2,\ldots$$

$$R_n := R_{(I-T)^n} \;;\; G_n := R_n - R_{n+1} \,,\; n=0,1,2,\ldots$$

Note that (N_n) is non-decreasing (i.e. $N_n \leq N_{n+1}$ for all $n=0,1,2,\ldots$) and (R_n) is non-increasing (i.e. $R_{n+1} \leq R_n$ for all $n=0,1,2,\ldots$).

The range projection R_T will be called (relatively) cofinite if $I - R_T$ is finite.

6.1 Remark

If $\mathcal{A} = \mathcal{L}(H)$, this definition of cofiniteness coincides with the classical definition (i.e. the range of the projection is cofinite).

The following lemma will be crucial in the proof of the Riesz decomposition theorem.

6.2 **Lemma**

With the notation above

(a) $N_{n+r}T^kN_n = T^kN_n$

(b) $F_nT^kF_n = F_n$

(c) $R_nT^kR_{n+r} = T^kR_{n+r}$

(d) $G_nT^kG_n = G_n$

for $n=0,1,2,\ldots$; $r=0,1,2,\ldots$; $k=1,2,\ldots$

Proof (a) Since $(I-T)^{n+1}N_{n+1} = 0$, for any $n=0,1,2,3,\ldots,$ it is obvious that

$$N_n(I-T)N_{n+1} = (I-T)N_{n+1} \tag{1}$$

By multiplying (1) from the right with N_n it follows that $N_nTN_n = TN_n$. For any $r \in \mathbb{N}$ we have that

$$N_{n+r}TN_n = N_{n+r}TN_{n+r}N_n = TN_{n+r}N_n = TN_n.$$

This proves (a) for $k = 1$. In general the relation (a) follows by induction on k: Suppose $N_{n+r}T^kN_n = T^kN_n$. Then

$$N_{n+r}T^{k+1}N_n = N_{n+r}T^kN_nTN_n = T^kN_nTN_n = T^{k+1}N_n.$$

(b) From (1) it follows that $(I-N_n)TN_{n+1} = F_n$. By multiplying from both sides with F_n it follows immediately that $F_nTF_n = F_n$. If $k \in \mathbb{N}$, then $F_nT^kF_n = (N_{n+1} - N_n)T^k(N_{n+1} - N_n)$

$$= (I - N_n)T^kN_{n+1}$$

$$= (I - N_n)TN_{n+1}T^{k-1}N_{n+1}$$

$$= (N_{n+1} - N_n)T^{k-1}N_{n+1}$$

$$= (I - N_n)T^{k-1}N_{n+1}$$
$$\vdots$$
$$= F_n$$

(c) and (d) follow similarly by using the relation $R_{n+1}(I-T)R_n = (I-T)R_n$. This relation follows from the fact that

$$R_{n+1}(I - T)^{n+1} = (I - T)^{n+1}. \qquad \square$$

6.3 **Remark**

From the relations in Lemma 6.2 it follows that for each $k \in \mathbb{N}$, T^kN_n is invertible in the reduced algebra $\mathcal{A}_{N_n} = N_n\mathcal{A}N_n$: Since $T^kN_n = N_nT^kN_n$, it is clear that $T^kN_n \in \mathcal{A}_{N_n}$. By using the binomial formula and Lemma 6.2(a) one can write the relation $(I - T)^nN_n = 0$ in the form $N_n - TS_n = 0$ with $S_n \in \mathcal{A}_{N_n}$. Hence T has a right inverse in $\mathcal{A}_{N_n}$. The existence of a left inverse can be proved similarly. Now since

$$T^kN_n = T^{k-1}N_nTN_n = T^{k-2}N_nTN_nTN_n = \dots = (TN_n)^k$$

and TN_n is invertible in $\mathcal{A}_{N_n}$, it follows that T^kN_n is invertible in $\mathcal{A}_{N_n}$ for any $k \in \mathbb{N}$.

6.4 **Theorem**

Let $T \in \mathcal{R}(\mathcal{A},\mathcal{K})$. *Then the following hold:*

(a) N_n *is relatively finite and* R_n *relatively cofinite*

(b) *If* $N_\infty = \sup_{n\in\mathbb{N}} N_n$ *and* $R_\infty = \inf_{n\in\mathbb{N}} R_n$ *then* $N_\infty(H)$ *and* $R_\infty(H)$ *are invariant under* T^k *for any* $k \in \mathbb{N}$

(c) N_∞ *is relatively finite and* R_∞ *relatively cofinite, with* $N_\infty \sim I-R_\infty$

(d) $\inf(N_\infty,R_\infty) = 0$ *and* $\sup(N_\infty,R_\infty) = I$.

Proof

(a) Since $T \in \mathcal{R}(\mathcal{A},\mathcal{K})$, it is clear that $I - T \in \Phi(\mathcal{A},\mathcal{K})$. From [13], Theorem 2.2 N_1 is relatively finite and R_1 relatively cofinite. For $n \in \mathbb{N}$, $n > 1$ it follows by using the binomial formula and Proposition 4.5 that

$$(I - T)^n = I - T_0 \quad \text{with} \quad T_0 \in \mathcal{R}(\mathcal{A},\mathcal{K}).$$

The argument above implies that N_n is relatively finite and R_n relatively cofinite.

(b) If we let $r = 0$ in Lemma 6.2(a) and (c), and take the strong-operator limit on both sides we get

$$N_\infty T^k N_\infty = T^k N_\infty \quad \text{and} \quad R_\infty T^k R_\infty = T^k R_\infty \qquad (k \in \mathbb{N})$$

(c) By Corollary 5.4 there exist a projection $E \in \mathcal{K}$ and a $k \in \mathbb{N}$ such that

$$\|T^k - ET^k\| < 1/3 \tag{1}$$

Since $F_n = F_nT^kF_n = F_nT^{*k}F_n$, and

$$F_n - F_nET^kF_nT^{*k}EF_n = (F_n - F_nET^kF_n) + (F_n - F_nT^{*k}EF_n) - (F_n - F_nET^kF_n)(F_n - F_nT^{*k}EF_n)$$

it follows from (1) that

$$\|F_n - F_nET^kF_nT^{*k}EF_n\| < 7/9, \quad n=0,1,2,3,\ldots \tag{2}$$

Hence $S_n := F_nET^kF_nT^{*k}EF_n$ is invertible in the reduced algebra

$\mathcal{A}_{F_n} = F_n \mathcal{A} F_n$ for each $n=0,1,2,\ldots$. Let S_n' be the inverse of S_n in $\mathcal{A}_{F_n}$. Now for any $x \in H$ it follows that

$$\sum_{n=1}^{\infty} \|S_n x\|^2 = \sum_{n=1}^{\infty} \|S_n x - F_n x + F_n x\|^2$$
$$\leq [(\tfrac{7}{9})^2 + 1] \sum_{n=1}^{\infty} \|F_n x\|^2$$
$$\leq k\|N_\infty x\|^2$$
$$\leq k\|x\|^2 \text{ where } k = (\tfrac{7}{9})^2 + 1$$

Since the set $\{S_n x : n \in \mathbb{N}\}$ is an orthogonal set it follows that $S = \Sigma S_n$ is strong-operator convergent. Similarly one shows that $S' = \Sigma S_n'$ is strong-operator convergent.

Moreover, it is clear that S, S' are contained in $\mathcal{A}_{N_\infty} := N_\infty \mathcal{A} N_\infty$ and

$$N_\infty = SS' = S'S \qquad (3)$$

Since $\mathcal{A}$ is semi-finite and $E \in \mathcal{K}$, there exists a semi-finite normal trace τ on $\mathcal{A}$ such that $\tau(E) < \infty$ (cf. [20], Lemma 2.5.3). Since

$\mathcal{N}_\tau = \{T \in \mathcal{A} : \tau(T^* T) < \infty\}$ is a two-sided ideal it is clear that

$$\tau(N_\infty E T^k T^{*k} E N_\infty) < \infty .$$

Thus, since τ is normal and $0 < S_n < F_n E T^k T^{*k} E F_n$ it follows that

$$\tau(S) = \Sigma\tau(S_n) < \Sigma\tau(F_n E T^k T^{*k} E F_n)$$
$$= \tau(N_\infty E T^k T^{*k} E N_\infty)$$

(the last equality follows from [10], Part I, Chapter 6, Section 1, Proposition 2). Hence, $\tau(S) < \infty$ and relation (3) imply that

$$\tau(N_\infty) \leq \|S'\|\tau(S) < \infty \qquad (4)$$

Since $\mathcal{A}$ is semi-finite there exists a faithful family of semi-finite normal traces $\{\tau_i\}_{i \in I}$ such that $\tau_i(E) < \infty$ (cf. [20], Lemma 2.5.3). Then, (4) implies that $\tau_i(N_\infty) < \infty$ for all $i \in I$. Hence, by using [20], 2.5.3 again, it follows that N_∞ is finite relative to $\mathcal{A}$.

By Proposition 4.15 we have that $I - T$ is Fredholm (relative to $\mathcal{K}$) with index zero. Using the binomial formula and Proposition 4.5 it follows that $(I - T)^n$ is Fredholm with index zero for each $n \in \mathbb{N}$. Hence

$$N_n \sim I - R_n \quad \text{for all} \quad n=0,1,2,\ldots .$$

Since all the projections N_n are finite relative to $\mathcal{A}$, Proposition 2.4.2 in [20] implies that

$$F_n \sim G_n \quad \text{for all} \quad n=0,1,2,\ldots .$$

Hence

$$N_\infty = \sum_{n=0}^{\infty} F_n \sim \sum_{n=0}^{\infty} G_n = I - R_\infty \tag{5}$$

(d) Let $T_{(k)} = I - (I - T)^k$, then $N_k = N_{I-T_{(k)}}$ and $R_k = R_{I-T_{(k)}}$. Define $R_n^{(k)} = \inf(R_{nk}, N_k)$. By using Lemma 1 in [6] we obtain

$$N_{(n+1)k} - N_{nk} \sim R_n^{(k)} \tag{6}$$

Since N_∞ is finite, the reduced algebra $\mathcal{A}_{N_\infty}$ is finite. For any finite normal trace ϕ on $\mathcal{A}_{N_\infty}$ one has

$$\lim_{n\to\infty} \phi(N_{(n+1)k} - N_{nk}) = 0.$$

By the normality of ϕ, relation (6) and [5], Lemma 9 it follows that

$$\inf(R_\infty, N_k) = 0 \quad \text{for all} \quad k.$$

Hence by Lemma 1.1 we have that

$$\inf(N_\infty, R_\infty) = 0.$$

Using (5) and [25] (Chapter V, 1.6) we obtain

$$I - R_\infty \sim \sup(R_\infty, N_\infty) - R_\infty \leq I - R_\infty \tag{7}$$

Since $I - R_\infty$ is finite relative to $\mathcal{A}$ relation (7) implies that

$$\sup(R_\infty, N_\infty) = I. \qquad \square$$

It is well-known that the sequences (N_n) and (R_n) eventually become stationary in the classical case (i.e. the $\mathcal{L}(H)$ case). The following example shows that this is not always the case in general von Neumann algebras.

6.5 Example

Let $\mathcal{A} = \sum_{n=1}^{\infty} \oplus \mathcal{L}(H_n)$ where $H_n = H$ is a separable Hilbert space. Let $T_k \in \mathcal{L}(H)$ be defined by

$$T_k\left(\sum_{i=1}^{\infty} x_i\phi_i\right) = x_1\phi_1 + \sum_{i=2}^{k+1} (x_i - x_{i-1})\phi_i ,$$

where $\{\phi_i | i \in \mathbb{N}\}$ is any orthonormal basis for H. It is easy to see that

$$N_{I-T_k} \neq N_{(I-T_k)^2} \neq \cdots \neq N_{(I-T_k)^{k+1}} = N_{(I-T_k)^{k+r}}$$

for all $k,r \in \mathbb{N}$.

Let $\mathrm{I} := \sum_{n=1}^{\infty} \oplus I_n$ where $I_n = I$ for all $n \in \mathbb{N}$ and $T := \sum_{n=1}^{\infty} \oplus T_n$. Then $T \in \mathcal{K}(\mathcal{A}) \subseteq \mathcal{R}(\mathcal{A},\mathcal{K})$. However

$$N_{(\mathrm{I}-T)^k} \neq N_{(\mathrm{I}-T)^{k+r}} \quad \text{for all } k,r \in \mathbb{N}.$$

6.6 Remarks

1. Similar results such as [7], Theorem 3 (iii) and Theorem 4 also hold for the class of relatively Riesz operators. Since the proofs are exactly the same, we omit them.
2. An interesting question to ask is whether a similar result such as Theorem 6.4 can be proved for the class of Riesz operators relative to any closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$, and if not, to characterize those ideals for which such a theorem holds.

We give a partial answer to the question raised in the remark above.

6.7 Proposition

Let $\mathcal{I}$ *be a strong-operator-closed two-sided ideal contained in* $\mathcal{K}$. *Then, if* $T \in \mathcal{R}(\mathcal{A},\mathcal{I})$ *the following hold:*

(a) $N_n \in \mathcal{I}$ *and* $I-R_n \in \mathcal{I}$

(b) $N_\infty(H)$ *and* $R_\infty(H)$ *are invariant under* T^k *for any* $k \in \mathbb{N}$

(c) $N_\infty \in \mathcal{I}$, $I-R_\infty \in \mathcal{I}$ *with* $N_\infty \sim I-R_\infty$

(d) $\inf(N_\infty,R_\infty) = 0$ *and* $\sup(N_\infty,R_\infty) = I$.

Proof

(a) and (b) follow by a similar proof as Theorem 6.4(a), by using [17], Theorem 4.7 and Proposition 4.5.

(c) By using Corollary 5.4 we can, as in the proof of Theorem 6.4(c), obtain elements S and S' such that $N_\infty = SS' = S'S$ (cf. Theorem 6.4 relation(3)). Since $\mathcal{I}$ is strong-operator-closed and S is a strong-operator limit of elements in $\mathcal{I}$, it follows that $N_\infty \in \mathcal{I}$. Since $(I-T)^n$ is Fredholm relative to $\mathcal{I}$ with index zero for each $n \in \mathbb{N}$, it follows from [17], 5.1(ii) and 4.12 that

$$N_n \sim I - R_n \quad \text{for all} \quad n=0,1,2,\ldots$$

Since $\mathcal{I} \subseteq \mathcal{K}$, all the N_n are finite relative to $\mathcal{A}$ and we obtain as in the proof of 6.4(c), that

$$I - R_\infty \sim N_\infty.$$

(d) Since $\mathcal{I} \subseteq \mathcal{K}$, $\Phi(\mathcal{A},\mathcal{I}) \subseteq \Phi(\mathcal{A},\mathcal{K})$. Thus for $R_n^{(k)} = \inf(R_{nk},N_k)$ we can apply Lemma 1 in [6] to obtain

$$N_{(n+1)k} - N_{nk} \sim R_n^{(k)}.$$

Consider the reduced algebra $\mathcal{A}_{N_\infty}$. This algebra is finite, hence we can proceed as in the proof of 6.4(d) to obtain (d). □

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Closed two-sided ideals in a von Neumann algebra and applications

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SUMMARY

The aim of this thesis is to study closed two-sided ideals in a von Neumann algebra $\mathcal{A}$, not only by looking into the structure of these ideals, but by using them in several applications on the theory of von Neumann algebras. For example, one of the main objects of this thesis is to develop a Riesz theory relative to any closed ideal in a von Neumann algebra by proving some characterization theorems of relatively Riesz operators and then to use this to prove a Riesz decomposition theorem.

Section 1 contains the definitions of some basic facts concerning von Neumann algebras used throughout this work. The main issue of section 2 is to consider three specific examples of closed two-sided ideals in a semifinite algebra with a non-zero type I direct summand, namely the ideals of operators compact relative to the von Neumann algebra, the ideal of compact operators contained in $\mathcal{A}$ and the ideal of the so called Rosenthal operators relative to $\mathcal{A}$. These ideals are used to obtain factorization results as well as a duality theorem.

In the third section we deduce geometrical characterizations as well as a spectral characterization for the quotient norm on $\mathcal{A}/\mathcal{I}$, where $\mathcal{I}$ is any closed ideal in $\mathcal{A}$. We then prove some characterization theorems on the semi-Fredholm elements relative to $\mathcal{I}$. In section 4 Riesz operators relative to a closed two-sided ideal are defined. The results in this section are similar to those known for the classical case and they are used in the sequel to prove characterization theorems for relatively Riesz operators as well as a Riesz decomposition theorem. In section 5 a geometrical characterization of Riesz operators relative to any closed ideal is proved. This geometrical characterization is used in section 6 to obtain a Riesz decomposition theorem for Riesz operators relative to specific closed ideals in a semifinite von Neumann algebra.

Geslote tweesydige ideale in 'n von Neumann algebra en toepassings

deur

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OPSOMMING

Die doel van hierdie proefskrif is om geslote tweesydige ideale in 'n von Neumann algebra $\mathcal{A}$ te ondersoek. Die struktuur van sulke ideale word bestudeer deur byvoorbeeld karakteriseringstellings vir die elemente van sulke ideale te bewys. Hierdie stellings lei dan tot verskeie toepassings in die teorie van von Neumann algebras. Ons toon onder andere aan dat Riesz operatore op 'n natuurlike wyse relatief tot enige geslote ideaal gedefinieer kan word en bewys dan karakteriseringstellings van sulke operatore wat lei tot 'n Riesz-ontbindingstelling.

Afdeling 1 bevat die nodige definisies asook die basiese feite ten opsigte van von Neumann algebras wat deurgaans gebruik word. In afdeling 2 word die struktuur van drie spesifieke voorbeelde van geslote tweesydige ideale in 'n semi-eindige algebra met 'n nie-nul tipe I direkte sommand ondersoek, nl die ideaal van kompakte operatore relatief tot $\mathcal{A}$, die ideaal van kompakte operatore bevat in $\mathcal{A}$ en die sogenaamde Rosenthal operatore relatief tot $\mathcal{A}$. Hierdie ideale word

gebruik om faktoriseringstellings sowel as 'n dualiteitstelling te bewys.

In die derde afdeling bewys ons geometriese karakteriserings sowel as 'n spektraalkarakterisering vir die kwosientnorm op $\mathcal{A}/\mathcal{I}$. Hierdie resultate word dan gebruik om semi-Fredholm operatore relatief tot enige geslote ideaal in $\mathcal{A}$ te karakteriseer.

Riesz operatore relatief tot enige geslote ideaal word in die vierde afdeling bestudeer. Die resultate word dan in afdelings 5 en 6 gebruik om geometriese karakteriserings van Riesz operatore sowel as 'n Riesz-ontbindingstelling te bewys.