Declaration

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

Signature

Name

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Derivations on operator algebras

by
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Preface

This work primarily provides some detail of results on domain properties of closed (unbounded) derivations on $C^\star$- algebras. The focus is on Section 4: *Domain Properties* where a combination of topological and algebraic conditions for certain results are illustrated. Various earlier results are incorporated into the proofs of Section 4.

Section 1: *Basics* lists some basic functional analysis results, operator algebra theory (of particular importance is the continuous functional calculus and certain results on the state and pure state space) and a special section on operator closedness. Some Hahn-Banach results are also listed. The results of this section were obtained from various sources (Zhu, K. [24], Kadison, R.V. and Ringrose, J.R [8], Goldberg, S. [6], Rudin, W. [20], Sakai, S. [22], Labuschagne, L.E. [10] and others). The development of the representation theory presented in Section 1.1.7 was compiled from Bratteli, O. and Robinson, D.W. [3], Section 2.3.

Section 2: *Derivations* provides some background to the roots of derivations in quantum mechanics. The results of Section 2.2 (Commutators) are due to various authors, mainly obtained from Sakai, S. [22]. A detailed proof of Theorem 45 is given. Section 2.3 (Differentiability) contains some Singer-Wermer results mainly obtained from Mathieu, M. and Murphy, G.J. [13] and Theorem 50 is proved in detail. Section 2.4 deals with conditions for bounded derivations (Sakai, S. [22]) and (Johnson-Sinclair, cf. (Sakai, S. [22])), and Theorem 51 is proved in detail. Section 2.5 deals with the well published derivation theorem (Sakai, S. [22], Section 2.5 and Bratteli, O. and Robinson, D.W. [3], Corollary 3.2.47) and a slightly weaker version of the $\mathcal{W}^*$- algebra derivation theorem as published in Bratteli, O. and Robinson, D.W. [3], Corollary 3.2.47, is proved here.

Section 3: *Derivations as generators* first introduces some basic semi-group theory (obtained from Pazy, A. [16], Section 1.1 and 1.2) after which the well-behavedness property is introduced in Section 3.2. Some general results mainly obtained from Sakai, S. [22], Section 3.2, is detailed. The proofs of Theorems 61 and 62 makes use of various previous results and were conducted in detail. Section 3.3 (Well-behavedness and generators) draws a link between the well-behavedness property and conditions for a derivation to be a semi-group generator. The results are obtained from Pazy, A. [16], Section 1.4, and Bratteli, O. and Robinson, D.W. [3], Section 3.2.4. Special care was taken in the outlined proof of Theorem 68. A proof of a domain characterization theorem (due to Bratteli, O. and Robinson, D.W. [3], Proposition 3.2.55) is provided (Theorem 69) and used in the construction of the counter example of Section 4.6.

Section 4: *Domain properties* is occupied with un-bounded derivations on C^*- algebras and their domain properties. Some initial complex function theory is developed after which four important domain preserving theorems are proved in full detail: the inverse function (Section 4.2), the exponential function (Section 4.3), Fourier analysis on the domain (Section 4.4) and C^2- functions on the domain (Section 4.5). The non domain preserving C^1 function counter example is presented in Section 4.6.

The results of Section 4 appear in Bratteli, O. and Robinson, D.W. [3], Section 3.2.2, and Sakai, S. [22], Section 3.3, and the counter example is due to McIntosh, A. [11]. All the results in Section 4 are presented in full detail not available in this format from any of the sources used. Some Toeplitz operator theory is used with reference to Brown, A. and Halmos, P.R. [4], 94, and the Fourier coefficients of a required function is calculated. Some results on direct sum spaces and the core of a linear operator were used from Kadison, R.V. and Ringrose, J.R. [8], Section 2.6 and page 160, as well as Zhu, K. [24], Section 14.2.

Contents

Notation

Unless otherwise specified, the following notation is adopted. Where symbols have more than one definition, the relevant one will be clearly specified in the context of the application.

ϕ	linear functional, state, pure state, multiplicative linear functional
ρ	multiplicative linear functional
$\mathcal{H}$	Hilbert space
φ	element of a Hilbert space
$B(\mathcal{H})$	bounded linear operators on a space $\mathcal{H}$
$\mathcal{A}$, $\mathcal{B}$	$\mathcal{C}^*$- algebra or a Banach algebra
π	a *-morphism a *-automorphism a *-isomorphism a *-homomorphism
$\mathcal{C}$	continuous functions
$\mathcal{X}$, $\mathcal{Y}$, $\mathcal{W}$, $\mathcal{K}$	normed linear spaces
x, y	elements of a Banach algebra elements of a $\mathcal{C}^*$- algebra elements of a normed linear space elements of $\mathbb{R}$
$\mathcal{D}(T)$	the domain of a linear operator T
p, n, m, i	indexing numbers
$\mathcal{R}(T)$	range of a linear operator T
$\mathcal{N}(T)$	kernel of a linear operator T
$\mathcal{G}(T)$	graph of a linear operator T
$\mathcal{C}(T)$	core for a linear operator T
λ, α, β	elements of $\mathbb{C}$
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathcal{C}^1(\mathbb{R})$	once continuously differentiable functions on $\mathbb{R}$
$\mathcal{C}^2(\mathbb{R})$	twice continuously differentiable functions on $\mathbb{R}$
$\mathcal{C}_0(\mathbb{R})$	continuous functions on $\mathbb{R}$ vanishing at $\pm$ infinity
$L_1(\mathbb{R})$	integrable functions on $\mathbb{R}$ ($\int_{\mathbb{R}} \|f\| < \infty$)
$L_2(\mathbb{R})$	square integrable functions on $\mathbb{R}$ ($\int_{\mathbb{R}} \|f\|^2 < \infty$)
$\hat{f}$	Fourier transform of f
$\check{f}$	inverse Fourier transform

1 Basics

1.1 Operator algebra

This section lists some of the standard operator algebra results used later in this work. The most important results are contained in the subsections on states and representations and the continuous functional calculus which is used in a number of proofs throughout.

1.1.1 Banach algebras

Definition 1 (Banach algebra) *A Banach algebra is a unital algebra $\mathcal{A}$ with unit $\mathbb{1}$ and with a complete norm $||\cdot||$ satisfying the following conditions:*

$$||\mathbb{1}|| = 1$$

$$||xz|| \leq ||x||||y|| \qquad \forall x, y \in \mathcal{A}$$

The complex field $\mathbb{C}$ with $||z|| = |z|$ is the simplest example of a Banach algebra. $B(\mathcal{H})$, the space of bounded linear operators on a Hilbert space $\mathcal{H}$ together with the operator norm is another example of a Banach algebra.
The spectrum (denoted as $\sigma(x)$) of an element x in a Banach algebra is defined as the set of all complex numbers λ such that $\mathbb{1}\lambda - x$ is not invertible in $\mathcal{A}$. The compliment of $\sigma(x)$ in $\mathbb{C}$ is called the resolvent of x. The spectrum of any $x \in \mathcal{A}$ is non-empty (Zhu, K. [24], 17), is a compact subset of the complex field $\mathbb{C}$ and is contained in the closed disk $\{z \in C \big| |z| \leq ||x||\}$ (Zhu, K. [24], 18). The spectral radius $r(x)$ for $x \in \mathcal{A}$ is defined as

$$r(x) = \sup\{|\lambda| \big| \lambda \in \sigma(x)\}$$

1.1.2 Multiplicative linear functionals

Definition 2 (Multiplicative linear functional) *A linear functional ϕ on a Banach algebra $\mathcal{A}$ is called a multiplicative linear functional if ϕ is non-trivial and $\phi(xy) = \phi(x)\phi(y)$ for every $x, y \in \mathcal{A}$. Denote*

$$\mathcal{M}_{\mathcal{A}} = \{\phi \in \mathcal{A}^* | \phi(xy) = \phi(x)\phi(y) \, for \, x, y \in \mathcal{A}\}$$

It will be shown later that in the case of $\mathcal{A}$ commutative, $\mathcal{M}_A$ corresponds to the maximal ideal space of the algebra $\mathcal{A}$.

Theorem 1 *For a multiplicative linear functional ϕ on a Banach algebra $\mathcal{A}$ we always have that $||\phi|| = 1$.*

Proof: From $||\phi|| \geq |\phi(1\!\!1)| = |\phi(1\!\!1 1\!\!1)| = |\phi(1\!\!1)|^2$ we have $|\phi(1\!\!1)| = 1$ and $||\phi|| \geq 1$. Now assume $||\phi|| > 1$. Then there exists an element $x \in \mathcal{A}$ with $||x|| = 1$ with $|\phi(x)| > 1$. Put $x_0 = x - \phi(x)1$. Then $\phi(x_0) = \phi(x) - \phi(x) = 0$. Also $||1 + \frac{x_0}{\phi(x)}|| = ||\frac{x}{\phi(x)}|| = \frac{||x||}{|\phi(x)|} < 1$ which implies that $\frac{x_0}{\phi(x)}$ is invertible. Thus $\phi(x_0)\phi(x_0^{-1}) = 1$ which contradicts $\phi(x_0) = 0$ earlier.

Remark 1 *From Theorem 1, it is clear that $\mathcal{M}_\mathcal{A}$ is contained in the closed unit ball of the continuous dual space $\mathcal{A}^*$. Naturally, $\mathcal{M}_\mathcal{A}$ is topologized by the weak* topology inherited from the continuous dual $\mathcal{A}^*$. The following result follows from Alaoglu's theorem.*

Theorem 2 *$\mathcal{M}_\mathcal{A}$ is a weak*- closed and compact Hausdorff space contained in the closed unit ball of $\mathcal{A}^*$.*

Proof: It is easy to verify that $\mathcal{M}_\mathcal{A}$ is closed in the *weak** topology of $\mathcal{A}^*$. From Alaoglu's theorem, it follows that $\mathcal{B}_{\mathcal{A}^*}$ is *weak**- compact. Since $\mathcal{M}_\mathcal{A}$ is a closed set contained in a compact Hausdorff set $\mathcal{B}_{\mathcal{A}^*}$, it follows that $\mathcal{M}_\mathcal{A}$ is *weak**- compact and Hausdorff as well.

1.1.3 Maximal ideal space

Proposition 1 *If $\mathcal{I}$ is a proper maximal ideal in a unital Banach algebra $\mathcal{A}$, then $\mathcal{I}$ is norm closed and the quotient algebra $\mathcal{A}/\mathcal{I}$ is a Banach algebra. If $\mathcal{A}$ is commutative then $\mathcal{A}/\mathcal{I}$ is a division algebra isomorphic to $\mathbb{C}$.*

Proof: If $\mathcal{I}$ is a proper maximal ideal in $\mathcal{A}$, then no element in $\mathcal{I}$ is invertible. From (Zhu, K. [24], 10) we have $||1\!\!1 - x|| \geq 1$ for every $x \in \mathcal{I}$ which shows that $1\!\!1$ is not in the closure of $\mathcal{I}$ so that the closure of $\mathcal{I}$ is a proper ideal in $\mathcal{A}$. From the maximality of $\mathcal{I}$ it follows that $\mathcal{I} = \overline{\mathcal{I}}$ which implies that $\mathcal{I}$ is closed. The quotient algebra $\mathcal{A}/\mathcal{I}$ is a Banach algebra with norm $||[x]|| = \inf\{||x - y|| | y \in \mathcal{I}\}$ where $[x] \in \mathcal{A}/\mathcal{I}$.
If $\mathcal{A}$ is commutative and $\mathcal{I}$ a maximal ideal in $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is a division algebra so that from the Gelfand - Mazur theorem (Zhu, K. [24], 19) we know that $\mathcal{A}/\mathcal{I}$ is isometrically isomorphic to $\mathbb{C}$.

Proposition 2 *If $\mathcal{A}$ is an arbitrary Banach algebra and ρ is a multiplicative linear functional on $\mathcal{A}$, then the kernel $\mathcal{M}$ $(= \mathcal{N}(\rho))$ of ρ is a proper maximal ideal in $\mathcal{A}$.*

Proof: It is clear that $\mathcal{N}(\rho)$ is a proper ideal for all $\rho \in \mathcal{M}_\mathcal{A}$. For maximality, let $x \in \mathcal{A} - \mathcal{N}(\rho)$. Then

$$1\!\!1 = (1\!\!1 - \frac{x}{\rho(x)}) + \frac{x}{\rho(x)}$$

The vector in parentheses is in $\mathcal{N}(\rho)$ so that the linear span of x and $\mathcal{N}(\rho)$ contains $\mathbb{1}$ so that any ideal containing both $\mathcal{N}(\rho)$ and x must be the whole algebra. Thus $\mathcal{N}(\rho)$ is maximal.

Theorem 3 (Commutative Banach algebras) *If $\mathcal{A}$ is a commutative Banach algebra then the set of maximal ideals is in one-to-one correspondence with $\mathcal{M}_{\mathcal{A}}$, the multiplicative linear functionals on $\mathcal{A}$.*

Proof: From Proposition 2 it follows that we can always associate a multiplicative linear functional of $\mathcal{A}$ with a maximal ideal in $\mathcal{A}$.
Assume that $\mathcal{I}$ is a proper maximal ideal in $\mathcal{A}$. From Proposition 1 it follows that $\mathcal{I}$ is closed. Since $\mathcal{A}$ is commutative and $\mathcal{I}$ is closed it follows from Proposition 1 that $\mathcal{A}/\mathcal{I}$ is a division algebra. Again by Proposition 1 there exists an isometric isomorphism $\phi : \mathcal{A}/\mathcal{I} \to \mathcal{C}$. Let π be the quotient mapping from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}$. Then the composition $\phi \circ \pi$ is a multiplicative linear functional on $\mathcal{A}$ with kernel $\mathcal{I}$. This correspondence is one-to-one: Let ρ_1 and ρ_2 be multiplicative linear functionals on $\mathcal{A}$ with common kernel $\mathcal{I}$. For any $x \in \mathcal{A}$ we can write

$$(\rho_1(x) - \rho_2(x))\mathbb{1} = (x - \rho_2(x)\mathbb{1}) - (x - \rho_{\mathbb{1}}(x)\mathbb{1})$$

Both terms on the right are in $\mathcal{I}$ and therefore $(\rho_1(x) - \rho_2(x))\mathbb{1}$ is in $\mathcal{I}$ so that $\rho_1(\rho_1(x) - \rho_2(x))\mathbb{1}) = 0$. Therefore $\rho_1(x) = \rho_2(x)$ and $\rho_1 = \rho_2$.

Remark 2 *The commutativity of the algebra $\mathcal{A}$ is important here: If $\mathcal{A}$ is not commutative and $\mathcal{I}$ is a maximal ideal in $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ may not be a division algebra so that there might be no multiplicative linear functional associated with $\mathcal{I}$.*

The following result gives some more connections between the multiplicative linear functionals and the spectrum of an element $x \in \mathcal{A}$:

Theorem 4 (Multiplicative linear functionals) *Let x be an element of a commutative Banach algebra $\mathcal{A}$, $\sigma(x)$ the spectrum of x in $\mathcal{A}$, $r(x)$ the spectral radius of x, $\phi \in \mathcal{M}_{\mathcal{A}}$ a multiplicative linear functional on $\mathcal{A}$ and $\mathcal{G}(\mathcal{A})$ the invertible elements in $\mathcal{A}$. Then:*

$\lambda \in \sigma(x) \Leftrightarrow \phi(x) = \lambda$ *for some* $\phi \in \mathcal{M}_{\mathcal{A}}$

$x \in \mathcal{G}(\mathcal{A}) \Leftrightarrow \phi(x) \neq 0 \qquad \forall \phi \in \mathcal{M}_{\mathcal{A}}$

$\phi(x) \in \sigma(x) \qquad \forall x \in \mathcal{A}, \phi \in \mathcal{M}_{\mathcal{A}}$

$|\phi(x)| \leq r(x) \leq ||x|| \qquad \forall x \in \mathcal{A}, \phi \in \mathcal{M}_{\mathcal{A}}$

Proof: Rudin, W. [20], 364

1.1.4 The Gelfand Transform

Definition 3 (Gelfand transform) *Let $\mathcal{A}$ be a Banach algebra. The mapping Γ : $\mathcal{A} \longrightarrow \mathcal{C}(\mathcal{M}_{\mathcal{A}})$, where $\mathcal{M}_{\mathcal{A}}$ is topologized with the weak**- *topology inherited from* $\mathcal{A}^*$, *defined by*

$$\Gamma(x)(\phi) = \phi(x) == \langle \phi, x \rangle$$

mapping the Banach algebra $\mathcal{A}$ into the continuous complex valued functions on $\mathcal{M}_{\mathcal{A}}$, is called the Gelfand transform.

Remark 3 *From $\Gamma(xy)(\phi) = \phi(xy) = \phi(x)\phi(y)$ it is clear that the Gelfand transform is a homomorphism. From $||\Gamma(x)(\phi)|| = ||x(\phi)|| \leq ||\phi|| ||x|| = ||x||$ (ϕ is multiplicative) it also follows that Γ is contractive.*

Theorem 5 *If $\mathcal{A}$ is a commutative Banach algebra, then $x \in \mathcal{A}$ is invertible in $\mathcal{A}$ if and only if $\Gamma(x)$ is invertible in $\mathcal{C}(\mathcal{M}_{\mathcal{A}})$. It then follows that*

$$\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{C}(\mathcal{M}_{\mathcal{A}})}(\Gamma(x)) = \mathcal{R}(\Gamma(x))$$

where $\mathcal{R}(\Gamma(x))$ denotes the range of $\Gamma(x)$, and

$$r(x) = ||\Gamma(x)||_{\infty} = \sup\{|\phi(x)| \,|\, \phi \in \mathcal{M}_{\mathcal{A}}\}$$

Proof: If x is invertible in $\mathcal{A}$, then $\Gamma(x^{-1})(\phi) = \phi(x^{-1})$ and $\phi(x^{-1})\phi(x) = \phi(1\!\!1) = 1$ $\forall \phi \in \mathcal{M}_{\mathcal{A}}$ so that $\Gamma(x)$ is invertible in $\mathcal{C}(\mathcal{M}_{\mathcal{A}})$.
If x is not invertible then x is in a proper maximal ideal $\mathcal{I}$ in $\mathcal{A}$. By Theorem 3 there exists a multiplicative linear functional $\phi_0 \in \mathcal{M}_{\mathcal{A}}$ with $\mathcal{I} = \mathcal{N}(\phi_0)$. Now

$$\Gamma(x)(\phi_0) = \phi_0(x) = 0$$

so that $\Gamma(x)$ is not invertible in $\mathcal{C}(\mathcal{M}_{\mathcal{A}})$.
Any z in the range of $\Gamma(x)$ will be in $\sigma_{\mathcal{A}}(x)$ because if $\Gamma(x)(\phi) = \phi(x) = z$, we have that $\phi(z1\!\!1 - x) = 0$.
Conversely take any $z \in \sigma(x)$. Then the element $z1\!\!1 - x$ must be in a maximal ideal $\mathcal{I}$ of $\mathcal{A}$ so that by Theorem 3 there exists $\phi \in \mathcal{M}_{\mathcal{A}}$ with $\phi(z1\!\!1 - x) = 0$. Then $\phi(x) = z$ so that z is in the range of Γ.

Theorem 6 (Spectral mapping theorem) *If x is any element in a Banach algebra $\mathcal{A}$ and f is an analytic function in $|z| \leq ||x||$, then*

$$\sigma(f(x)) = \{f(z) | z \in \sigma(x)\}$$

Proof: Zhu, K. [24], 29.

Theorem 7 (Spectral radius theorem) *For every element x in a Banach algebra $\mathcal{A}$ we have*

$$r(x) \quad = \quad \lim_{n \to \infty} ||x^n||^{\frac{1}{n}}$$

Proof: Zhu, K. [24], 31.

1.1.5 $\mathcal{C}^*$- algebras

Definition 4 (Involution mapping) *A mapping $x \to x^*$ on a Banach algebra $\mathcal{A}$ is called an involution on the algebra $\mathcal{A}$ if it satisfies the following:*

(i) $(x^*)^* = x$ *for all* $x \in \mathcal{A}$

(ii) $(ax + by)^* = \overline{a}x^* + \overline{b}y^*$ *for all* $x, y \in \mathcal{A}$ *and* $a, b \in \mathcal{C}$

(iii) $(xy)^* = y^*x^*$ *for all* $x, y \in \mathcal{A}$

Definition 5 ($\mathcal{C}^*$- algebra) *A $\mathcal{C}^*$- algebra is defined as a Banach algebra $\mathcal{A}$ with an involution mapping $x \to x^*$ on $\mathcal{A}$ satisfying $||x^*x|| = ||x||^2$ for all $x \in \mathcal{A}$. The involution preserves the norm and the mapping $x \to x^*$ is therefore a continuous mapping on $\mathcal{A}$.*

Definition 6 (Special elements of a $\mathcal{C}^*$- algebra) *For x in a $\mathcal{C}^*$- algebra $\mathcal{A}$,*

x *is* self-adjoint *if* $x = x^*$

x *is* normal *if* $xx^* = x^*x$

x *is* positive *if* $x = y^*y$ *for some* $y \in \mathcal{A}$

For a $\mathcal{C}^*$- algebra $\mathcal{A}$, we have the following inclusions:

$$\{x \in \mathcal{A} | \, x = y^*y, \, y \in \mathcal{A}\} \subset \{x \in \mathcal{A} | \, x = x^*\} \subseteq \{x \in \mathcal{A} | \, x^*x = xx^*\}$$

Remark 4 (General decomposition) *Any general element x of a $\mathcal{C}^*$- algebra $\mathcal{A}$ has a unique decomposition in terms of self-adjoint elements $x_1, x_2 \in \mathcal{A}$ such that*

$$x = x_1 + ix_2$$

The real part of x is given by $x_1 = (x + x^*)/2$ and the imaginary part is given by $x_2 = (x - x^*)/2i$. Clearly x_1 and x_2 are self-adjoint.

The square root $\sqrt{x}$ (or $x^{\frac{1}{2}}$) of a positive element $x \in \mathcal{A}$ can be defined as the unique positive element $y \in \mathcal{A}$ such that $y^2 = x$. More specifically, for self-adjoint $x \in \mathcal{A}$, the modulus (or absolute value) $|x|$ can be defined as $\sqrt{x^2}$. This leads to the following orthogonal decomposition theorem:

Theorem 8 *Let $x = x^*$ be a self-adjoint element of a C^*- algebra $\mathcal{A}$. Define $x_\pm = (|x| \pm x)/2$ where $|x|$ is defined as above. Then $x_\pm$ is a unique positive map under the conditions listed below:*

$$x = x_+ - x_-$$

$$x_+ x_- = 0$$

$$||x|| = \max\left(||x_+||,\, ||x_-||\right)$$

Proof: Bratteli, O. and Robinson, D.W. [1], 35, or Zhu, K. [24], 37.

Theorem 9 *Let $x \in \mathcal{A}$ be a normal element in a C^*- algebra $\mathcal{A}$. Then $r(x) = ||x||$.*

Proof: Zhu, K. [24], 52.

Theorem 10 *Let x be a self-adjoint element in a C^*- algebra $\mathcal{A}$. Then $\sigma(x) \subset \mathbb{R}$.*

Proof: Zhu, K. [24], 53.

From Theorem 9 and 10 and Remark 4, it follows that for any self-adjoint x in a C^*- algebra $\mathcal{A}$,

$$\begin{aligned} \sigma(x) &\subseteq [-||x||, ||x||] \qquad \textit{and} \\ \sigma(x^2) &\subseteq [0, ||x||^2] \end{aligned}$$

1.1.6 States and Pure states

The states of a C^*- algebra $\mathcal{A}$ is a special class of linear functionals that takes positive values on the positive elements of $\mathcal{A}$ and one on the unit of $\mathcal{A}$. The states and pure states on $\mathcal{A}$ play an important role in representations of C^*- algebras. First some basic functional analysis:

Theorem 11 (Hahn-Banach) *If f is a bounded linear functional on a subspace $\mathcal{M}$ of a normed linear space $\mathcal{X}$, then f can be extended to a bounded linear functional F on $\mathcal{X}$ so that $||f|| = ||F||$.*

Proof: Rudin, W. [20], 104.

The following corollary to the Hahn-Banach theorem is used in the proof of Theorems 62 and 63. Here $d = d(x, \mathcal{M}) = \inf_{y \in \mathcal{M}} ||x - y||$.

Corollary 1 (Hahn-Banach) *Let $\mathcal{M}$ be a subspace of a normed linear space $\mathcal{X}$. For every $x \in \mathcal{X}$ with $d(x, \mathcal{M}) > 0$, there exists a bounded linear functional f on $\mathcal{X}$ such that*

$$||f|| = 1$$

$$f(\mathcal{M}) = 0$$

$$f(x) = d(x, \mathcal{M})$$

Proof: Goldberg, S. [6], 20.

Definition 7 (Positive linear functionals and states) *Let ϕ be a linear functional on a $\mathcal{C}^*$-algebra $\mathcal{A}$. Then:*

ϕ *is* positive *if* $\phi(x) \geq 0$ *for all* $x \geq 0$ *(positive)*

ϕ *is a* state *if* ϕ *is positive and* $\phi(\mathbb{1}) = 1$

Positivity of a linear functional ϕ can also be defined by the requirement $\phi(xx^*) \geq 0$ for all x in a $\mathcal{C}^*$- algebra $\mathcal{A}$. The Schwartz inequality holds for all positive linear functionals:

$$|\phi(y^*x)|^2 \leq \phi(x^*x)\phi(y^*y)$$

The following result for positive linear functionals is required in the proof of Theorem 62:

Theorem 12 *If ϕ is a bounded linear functional on a $\mathcal{C}^*$- algebra $\mathcal{A}$ with $\phi(x) = ||\phi||||x||$ for some positive $x \in \mathcal{A}$, then ϕ is a positive linear functional.*

Proof: Sakai, S. [21], 9.

Theorem 13 *A linear functional ϕ on a $\mathcal{C}^*$- algebra $\mathcal{A}$ is positive if and only if ϕ is bounded with $||\phi|| = \phi(\mathbb{1})$. Therefore, ϕ is a state on $\mathcal{A}$ if and only if $||\phi|| = \phi(\mathbb{1}) = 1$.*

Proof: Zhu, K. [24], 80.

Let $\mathcal{S}(\mathcal{A})$ denote the space of all states on a $\mathcal{C}^*$- algebra $\mathcal{A}$. Then $\mathcal{S}(\mathcal{A})$ is contained in the closed unit ball of the dual space (linear functionals) of $\mathcal{A}$. $\mathcal{S}(\mathcal{A})$ is topologized with the weak star topology inherited from the dual space of $\mathcal{A}$. $\mathcal{S}(\mathcal{A})$ is always non-empty:

Theorem 14 (State space) *If x is any element in a C^*- algebra $\mathcal{A}$, then for each $\lambda \in \sigma(x)$ there exists a state ϕ on $\mathcal{A}$ with $\phi(x) = \lambda$.*

Proof: Zhu, K. [24], 80.

This correspondence between the spectrum of an element $x \in \mathcal{A}$ and the states allows the following useful characterization of positive, self-adjoint and normal elements in terms of $\mathcal{S}(\mathcal{A})$:

Theorem 15 ($\mathcal{S}(\mathcal{A})$) *Let $\mathcal{A}$ be a C^*- algebra with $\mathcal{S}(\mathcal{A})$ the state space of $\mathcal{A}$. Then for $x \in \mathcal{A}$,*

$$x = 0 \Leftrightarrow \phi(x) = 0 \qquad \forall \phi \in \mathcal{S}(\mathcal{A})$$

$$x = x^* \Leftrightarrow \phi(x) \in \mathbb{R} \qquad \forall \phi \in \mathcal{S}(\mathcal{A})$$

$$x \geq 0 \Leftrightarrow \phi(x) \geq 0 \qquad \forall \phi \in \mathcal{S}(\mathcal{A})$$

$$xx^* = x^*x \Rightarrow \exists \phi \in \mathcal{S}(\mathcal{A}) \big| ||x|| = |\phi(x)|$$

Proof: Zhu, K. [24], 81.

The state space $\mathcal{S}(\mathcal{A})$ is a convex, weak-star compact and Hausdorff subspace of $\mathcal{A}^*$ - the dual space of $\mathcal{A}$ (Zhu, K. [24], 81). Therefore, from the Krein-Milman theorem (Zhu, K. [24], 6), $\mathcal{S}(\mathcal{A})$ is the weak-star closed convex hull of the set of extreme points of $\mathcal{S}(\mathcal{A})$. The set of extreme points of $\mathcal{S}(\mathcal{A})$ is denoted by $\mathcal{P}(\mathcal{A})$ and elements in $\mathcal{P}(\mathcal{A})$ are called *pure states* of $\mathcal{A}$. Every $\phi \in \mathcal{S}(\mathcal{A})$ can therefore be approximated in the weak-star topology by elements of the form $t_1\phi_1 + t_2\phi_2 + ... + t_n\phi_n$ with $\phi_i \in \mathcal{P}(\mathcal{A})$ and $t_i \in (0, 1)$ with $t_1 + t_2 + ... + t_n = 1$. In other words, a pure state is a state that cannot be written as a convex combination of other states.

Like the state space, the pure state space is sufficiently large to distinguish certain properties of elements of $\mathcal{A}$. Theorem 15 can be re-stated with $\mathcal{S}(\mathcal{A})$ replaced by $\mathcal{P}(\mathcal{A})$.

Theorem 16 *A non-trivial linear functional ϕ on a commutative C^*-algebra $\mathcal{A}$ is a pure state if and only if ϕ is multiplicative.*

Remark 5 *From Theorem 16 it can be seen that (for a commutative C^*-algebra $\mathcal{A}$) the pure states (multiplicative linear functionals / maximal ideals) is a weak-star closed subspace of the state space of $\mathcal{A}$*

1.1.7 Representations

This section lists a short summary of basic representation theory. The main result is the Gelfand, Neumark, Segal representation construction which states that every $\mathcal{C}^*$ algebra $\mathcal{A}$ is $\mathcal{C}^*$- isomorphic to a $\mathcal{C}^*$- subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. First some definitions: ($B(\mathcal{H})$ denotes the bounded linear operators on a Hilbert space $\mathcal{H}$)

Definition 8 (∗- morphism) *A ∗-morphism between two ∗- algebras $\mathcal{A}$ and $\mathcal{B}$ is a mapping π defined for all $x \in \mathcal{A} \mapsto \pi(x) \in \mathcal{B}$ such that:*

(i) $\pi(\alpha x + \beta y) = \alpha\pi(x) + \beta\pi(y)$

(ii) $\pi(xy) = \pi(x)\pi(y)$

(iii) $\pi(x^*) = \pi(x)^*$

for $x, y \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$.

For $\mathcal{C}^*$- algebras, ∗ - morphisms are continuous:

Theorem 17 (∗ - morphisms) *For $\mathcal{C}^*$- algebras $\mathcal{A}$ and $\mathcal{B}$, and a ∗ - morphism π of $\mathcal{A}$ into $\mathcal{B}$ we have*

$x \geq 0 \Rightarrow \pi(x) \geq 0$ *(positivity preserving)*

π *is continuous with* $||\pi(x)|| \leq ||x|| \qquad \forall x \in \mathcal{A}$

Proof: Bratteli, O. and Robinson, D.W. [3], 42.

The range $\{\pi(x)|x \in \mathcal{A}\}$ is a closed $\mathcal{C}^*$- subalgebra of $\mathcal{B}$ (in the setting of Theorem 17). If $\{\pi(x)|x \in \mathcal{A}\} = \mathcal{B}$ and each element of $\mathcal{B}$ is the image of a unique element in $\mathcal{A}$ (onto and one-to-one), then π is called a ∗- isomorphism. A ∗ - morphism π of a $\mathcal{C}^*$- algebra $\mathcal{A}$ onto a $\mathcal{C}^*$- algebra $\mathcal{B}$ is a ∗- isomorphism if $\mathcal{N}(\pi) = \{x \in \mathcal{A}|\pi(x) = 0\} = \{0\}$.

The kernel $\mathcal{N}(\pi) = \{x \in \mathcal{A}|\pi(x) = 0\}$ is a two-sided ideal of $\mathcal{A}$: For $x, y \in \mathcal{A}$ and $y \in \mathcal{N}(\pi)$ we have $\pi(xy) = \pi(x)\pi(y) = 0$ and $\pi(yx) = \pi(y)\pi(x) = 0$. Also, from the inequality $||\pi(x)|| \leq ||x||$ it follows that $\mathcal{N}(\pi)$ is closed. This leads to the construction of the quotient $\mathcal{C}^*$- algebra $\mathcal{A}_\pi = \mathcal{A}/\mathcal{N}(\pi)$ with the equivalence classes defined as $\hat{x} = \{x + k|k \in \mathcal{N}(\pi)\}$. The morphism π induces a morphism $\hat{\pi}$ from $\mathcal{A}_\pi$ to $\mathcal{B}$ by $\hat{\pi}(\hat{x}) = \pi(x)$ and since $\mathcal{N}(\hat{\pi}) = \{0\}$, $\hat{\pi}$ is a isomorphism between $\mathcal{A}_\pi$ and $\hat{\pi}(\mathcal{A}_\pi) \subseteq \mathcal{B}$.

Definition 9 (Basic representation) *A representation of a $\mathcal{C}^*$- algebra $\mathcal{A}$ is defined as a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and π a ∗ - morphism of $\mathcal{A}$ into $B(\mathcal{H})$, the bounded linear operators on the Hilbert space $\mathcal{H}$.*

A representation $(\mathcal{H}, \pi)$ is *faithful* if and only if π is a $*$ - isomorphism between $\mathcal{A}$ and $\pi(\mathcal{A})$ (if and only if $\mathcal{N}(\pi) = \{0\}$). The terminology π *is a representation of* $\mathcal{A}$ *on* $\mathcal{H}$ is also often used. Every representation $(\mathcal{H}, \pi)$ of a C^*- algebra $\mathcal{A}$ defines a faithful representation of the quotient algebra $\mathcal{A}_\pi$.

A $*$- automorphism π of a C^*- algebra $\mathcal{A}$ is a $*$ - isomorphism of $\mathcal{A}$ into itself. π is a $*$ - automorphism if it is a $*$ - morphism of $\mathcal{A}$ with range equal to $\mathcal{A}$ and kernel equal to $\{0\}$. Each $*$ - automorphism is norm-preserving - $\|\pi(x)\| = \|x\| \, \forall x \in \mathcal{A}$.

A trivial representation of a C^*- algebra $\mathcal{A}$ is given by the trivial $*$ - morphism $\pi = 0$ with $\pi(x) = 0 \, \forall x \in \mathcal{A}$. A representation may be non-trivial (in general) but still have trivial (invariant) parts. If $\mathcal{H}_0 \subset \mathcal{H}$ is defined as $\mathcal{H}_0 = \{\varphi \in \mathcal{H} | \pi(x)\varphi = 0 \, \forall x \in \mathcal{A}\}$ then $\mathcal{H}_0$ is invariant under π and the corresponding representation $\pi_0 = P_{\mathcal{H}_0} \pi P_{\mathcal{H}_0}$ is trivial ($P_{\mathcal{H}_0}$ is the orthogonal projector with range $\mathcal{H}_0$). A representation π is *non-degenerate* if $\mathcal{H}_0 = \{0\}$. In general, a set of bounded linear operators B acts non-degenerately on $\mathcal{H}$ if $\{\varphi \in \mathcal{H} | x(\varphi) = 0 \, \forall x \in B\} = \{0\}$.

An element $\varphi \in \mathcal{H}$ is *cyclic* for a set B of bounded linear operators on $\mathcal{H}$ if the set $\{x(\varphi) | x \in B\}$ is dense in $\mathcal{H}$.

Definition 10 (Cyclic representation) *A cyclic representation of a C^*- algebra $\mathcal{A}$ is a triple $(\mathcal{H}, \pi, \varphi)$ where $(\mathcal{H}, \pi)$ is a representation of $\mathcal{A}$ and $\varphi \in \mathcal{H}$ is cyclic in the Hilbert space $\mathcal{H}$ for the set $\{\pi(x) | x \in \mathcal{A}\}$.*

Theorem 18 (Cyclic representation) *Every non-degenerate representation $(\mathcal{H}, \pi)$ of a C^*- algebra $\mathcal{A}$ is the direct sum of a family $(\mathcal{H}_\alpha, \pi_\alpha)_{\alpha \in I}$ of cyclic representations of $\mathcal{A}$.*

Proof: Bratteli, O. and Robinson, D.W. [3], 46.

Theorem 18 allows a reduction from general representations to cyclic representation. This is useful in the construction of representations.

Definition 11 (Irreducible representation) *A set B of bounded linear operators on a Hilbert space $\mathcal{H}$ is* irreducible *if the only closed subspaces of $\mathcal{H}$ invariant under B is $\mathcal{H}$ and $\{0\}$. A representation $(\mathcal{H}, \pi)$ of a C^*- algebra $\mathcal{A}$ is irreducible if the set $\pi(\mathcal{A})$ is irreducible on $\mathcal{H}$.*

The following theorem establishes some identification of irreducible sets of operators. The *commutant* $\mathcal{M}'$ of a set of bounded linear operators $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ is defined as $\mathcal{M}' = \{y \in B(\mathcal{H}) | xy = yx \quad \forall x \in \mathcal{M}\}$.

Theorem 19 (Irreducible sets) *Let B be a self adjoint set of bounded linear operators on a Hilbert space $\mathcal{H}$. Then B is irreducible if and only if B' consists of multiples of* $\mathbb{1}$.

Proof: Bratteli, O. and Robinson, D.W. [3], 47.

Positive linear functionals, and in particular the states and pure states play an important role in the existence proof and construction of representations. Let $(\mathcal{H}, \pi)$ be a representation of a $\mathcal{C}^*$- algebra $\mathcal{A}$ and $\varphi \in \mathcal{H}$ a non-zero unit vector. Define

$$\omega_\varphi(x) \quad = \quad \langle \varphi, \pi(x)\varphi \rangle$$

for all $x \in \mathcal{A}$. Then ω_φ is a *vector state* on $\pi(\mathcal{A})$. Every representation therefore has a (vector) state associated with it. The following theorem shows that the converse is also true: every state ϕ over a $\mathcal{C}^*$- algebra is a vector state in a suitable representation.

Theorem 20 (State representation) *If ϕ is a state on a $\mathcal{C}^*$ - algebra $\mathcal{A}$, then there exists a cyclic representation $(\mathcal{H}_\phi, \pi_\phi, \varphi_\phi)$ of $\mathcal{A}$ such that*

$$\phi(x) \;=\; \langle \varphi_\phi, \pi_\phi(x)\varphi_\varphi \rangle$$

for all $x = \mathcal{A}$, so that $||\varphi_\phi||^2 \;=\; ||\phi|| \;=\; 1$.

Proof: Bratteli, O. and Robinson, D.W. [3], 56.

This cyclic representation $(\mathcal{H}_\phi, \pi_\phi, \varphi_\phi)$ constructed from the state ϕ on $\mathcal{A}$ is defined as the *canonical representation* of $\mathcal{A}$ associated with ϕ.

The nature of pure states and their canonical representation is given in the following theorem:

Theorem 21 (Pure state representation) *Let ϕ be a state over a $\mathcal{C}^*$ - algebra $\mathcal{A}$ with $(\mathcal{H}_\phi, \pi_\phi, \varphi_\phi)$ the associated cyclic representation. Then $(\mathcal{H}_\phi, \pi_\phi)$ is irreducible if and only if ϕ is a pure state.*

Proof: Bratteli, O. and Robinson, D.W. [3], 57.

1.1.8 Commutative $\mathcal{C}^*$- algebras

Theorem 22 *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{C}^*$- algebras and $\pi : \mathcal{A} \to \mathcal{B}$ a $\mathcal{C}^*$- homomorphism. Then for every $x \in \mathcal{A}$, $\sigma(\pi(x)) \subset \sigma(x)$ and $||\pi(x)|| \leq ||x||$.*

Proof: Zhu, K. [24], 55.

Theorem 23 *Let π be a $\mathcal{C}^*$- isomorphism of $\mathcal{C}^*$- algebras $\mathcal{A}$ onto $\mathcal{B}$. Then $\sigma(\pi(x)) = \sigma(x)$ and $||\pi(x)|| = ||x||$ for every $x \in \mathcal{A}$.*

Proof: Zhu, K. [24], 56.

Theorem 24 (Gelfand transform) *Every commutative $\mathcal{C}^*$- algebra is $*$ - isomorphic to $\mathcal{C}(\mathcal{K})$ for some compact Hausdorff space $\mathcal{K}$. In particular, for a commutative $\mathcal{C}^*$-algebra $\mathcal{A}$, the Gelfand transform is a $\mathcal{C}^*$- isomorphism from $\mathcal{A}$ onto $\mathcal{C}(\mathcal{M}_{\mathcal{A}})$.*

Proof: Let $\Gamma : \mathcal{A} \rightarrow \mathcal{M}_{\mathcal{A}}$ be the Gelfand transform: $\Gamma(x)(\phi) = < x, \phi > = \phi(x)$ for every $\phi \in \mathcal{M}_{\mathcal{A}}$ and $x \in \mathcal{A}$. From Remark 3 we know that Γ is an algebraic homomorphism into $\mathcal{C}(\mathcal{M}_{\mathcal{A}})$. It needs to be shown that Γ is one-to-one, onto and involution preserving.

Involution preserving: (To prove: $\Gamma(x^*) = \overline{\Gamma(x)}$. $\overline{z}$ denotes the complex conjugate for $z \in \mathbb{C}$.) Define for every $x \in \mathcal{A}$

$$\begin{aligned} x_1 &= \frac{x + x^*}{2} \qquad \text{and} \\ x_2 &= \frac{x - x^*}{2i} \end{aligned}$$

Now $x = x_1 + ix_2$ and $x^* = x_1 - ix_2$ and both x_1 and x_2 are self adjoint. Since $\mathcal{A}$ is commutative it follows from Theorem 5 that the range of $\Gamma(x_i)$ is $\sigma(x_i)$ which is (by Theorem 10) contained in $\mathbb{R}$ for $i = 1, 2$. Now $\Gamma(x^*) = \Gamma(x_1) - i\Gamma(x_2) = \overline{\Gamma(x_1) + i\Gamma(x_2)} = \overline{\Gamma(x)}$.

One-to-one: (To prove: $||\Gamma(x)||_\infty = ||x||$) From Theorem 5 we have $r(x) = ||\Gamma(x)||_\infty$. Now $||\Gamma(x)||^2_\infty = ||\Gamma(x)^*\Gamma(x)||_\infty = ||\Gamma(xx^*)||_\infty = r(x^*x)$. But clearly $(x^*x) = (xx^*)$ so that (x^*x) is self adjoint and normal. From Theorem 9 it follows that $r(x^*x) = ||x^*x|| = ||x||^2$ so that Γ is norm preserving and hence one-to-one.

Onto: (To Prove: Image Γ equal to $\mathcal{C}(\mathcal{M}_{\mathcal{A}[x]})$). Since $\mathcal{A}$ is a Banach algebra and Γ is an isometry, we know that the image of Γ in $\mathcal{C}(\mathcal{M}_{\mathcal{A}})$ is a closed $*$- subalgebra of $\mathcal{C}(\mathcal{M}_{\mathcal{A}})$ which contains the constant functions. Furthermore, for any two distinct elements ϕ_1 and ϕ_2 of $\mathcal{M}_{\mathcal{A}}$, we can choose an $x \in \mathcal{A}$ not in the kernel of $\phi_1 - \phi_2$ so that $\Gamma(\phi_1 - \phi_2)(x) = (\phi_1 - \phi_2)(x) \neq 0$ so that $\phi_1(x) \neq \phi_2(x)$ which shows that the image of Γ separates points of $\mathcal{M}_{\mathcal{A}}$. By the Stone-Weierstrass theorem, Γ is onto.

Remark 6 *From Theorem 16 we know that $\mathcal{M}_{\mathcal{A}} = \mathcal{P}(\mathcal{A})$ for a commutative $\mathcal{C}^*$- algebra $\mathcal{A}$ where $\mathcal{P}(\mathcal{A})$ is topologized with the weak*- topology. Therefore, Theorem 24 can be restated with $\mathcal{M}_{\mathcal{A}}$ replaced by $\mathcal{P}(\mathcal{A})$.*

Theorem 25 *Let $\mathcal{A}$ be a C^*- subalgebra of a C^*- algebra $\mathcal{B}$ with $x \in \mathcal{A}$. Then x is invertible in $\mathcal{A}$ if and only if x is invertible in $\mathcal{B}$. From this it follows that $\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{B}}(x)$.*

Proof: Zhu, K. [24], 59.

1.1.9 Continuous functional calculus

Let $\mathcal{A}[x]$ denote the C^*- subalgebra generated by $x \in \mathcal{A}$. If x is a normal element of a C^*- algebra $\mathcal{A}$, then $\mathcal{A}[x]$ is commutative. The next theorem will make it clear that an element $y \in \mathcal{A}$ is in $\mathcal{A}[x]$ if and only if y can be approximated in norm by polynomials in $\mathbb{1}$, x and x^*.

Theorem 26 (Spectral theorem) *Let x be a normal element of a C^*- algebra $\mathcal{A}$. Then $\mathcal{M}_{\mathcal{A}[x]}$ is homeomorphic to $\sigma(x)$. If we identify $\mathcal{M}_{\mathcal{A}[x]}$ with $\sigma(x)$, then the Gelfand transform Γ on $\mathcal{A}[x]$ has the property that $\Gamma(p(x, x^*)) = p(z, \bar{z})$ for every polynomial p of two variables ($z \in \mathcal{C}$).*

Proof: From Theorem 24 and Remark 3 at the beginning of this section if follows that $\Gamma : \mathcal{A}[x] \to \mathcal{C}(\mathcal{M}_{\mathcal{A}})$ is a surjective C^*- isomorphism. From Theorem 5 and Theorem 25 we can define the mapping

$$\Psi : \mathcal{M}_{\mathcal{A}[x]} \to \sigma(x)$$

by

$$\Psi(\phi) = \Gamma(x)(\phi) = \phi(x)$$

with $\phi \in \mathcal{M}_{\mathcal{A}[x]}$ so that Ψ is well defined and onto. To show that Ψ is one-to-one, suppose $\Psi(\phi_1) = \Psi(\phi_2)$ or $\phi_1(x) = \phi_2(x)$. From the proof of Theorem 24 we have that $\phi_1(x^*) = \phi_2(x^*)$ so that

$$\phi_1(p(x, x^*)) = \phi_2(p(x, x^*))$$

for every polynomial p of two variables. Such polynomials are dense in $\mathcal{A}[x]$ and hence $\phi_1 = \phi_2$ so that Ψ is one-to-one.
For Ψ to be a homeomorphism, we need to show that Ψ preserves convergence (the topology). Therefore, take any net ϕ_α in $\mathcal{M}_{\mathcal{A}[x]}$ converging to ϕ with respect to the *weak**- topology. Then clearly $\phi_\alpha(y) \to \phi(y)$ for every $y \in \mathcal{A}[x]$. Also $\phi_\alpha(x) \to \phi(x)$ since $x \in \mathcal{A}[x]$, so that $\Psi(\phi_\alpha) \to \Psi(\phi)$ which shows that Ψ is continuous. Now Ψ is a one-to-one continuous function from one compact Hausdorff space $\mathcal{M}_{\mathcal{A}[x]}$ to another

compact Hausdorff space $\sigma(x)$ and therefore Ψ is a homeomorphism.
By the above we can identify $\mathcal{M}_{\mathcal{A}[x]}$ with $\sigma(x)$ and then the Gelfand transform

$$\Gamma : \mathcal{A}[x] \quad \rightarrow \quad \mathcal{C}(\sigma(x))$$

satisfies $\Gamma(x)(z) = z$ for all $z \in \sigma(x)$. (Note that $z \in \sigma(x)$ is identified with $\phi \in \mathcal{M}_{\mathcal{A}}$ such that $\phi(x) = z$ (Theorem 14)). Since Γ is a $\mathcal{C}^*$- homomorphism, it follows that $\Gamma(p(x, x^*))(z) = p(z, \overline{z})$ for every polynomial p of two variables.

Theorem 27 *Let $x \in \mathcal{A}$ be normal in the $\mathcal{C}^*$-algebra $\mathcal{A}$ and $\mathcal{C}(\sigma(x))$ the continuous complex valued functions on $\sigma(x)$. Define $\imath \in \mathcal{C}(\sigma(x))$ by $\imath(t) = t$ for all $t \in \sigma(x)$. Then there exists a unique $*$-isomorphism $\pi : \mathcal{C}(\sigma(x)) \rightarrow \mathcal{A}$ such that $\pi(\imath) = x$. For each $f \in \mathcal{C}(\sigma(x))$, $\pi(f)$ is normal in $\mathcal{A}$ and $\pi(f)$ is the limit of a sequence of polynomials in* $\mathbb{1}$, *x and x^*. The set*

$$\mathcal{A}[x] \quad = \quad \{\pi(f) \big| f \in \mathcal{C}(\sigma(x))\}$$

is the smallest commutative $\mathcal{C}^*$-subalgebra of $\mathcal{A}$ that contains x.

Proof: Existence of $\mathcal{C}^*$- isomorphism:
Let $\mathcal{B}$ be any commutative $\mathcal{C}^*$-subalgebra of $\mathcal{A}$ that contains x. Then (by Theorem 24) there exists a $*$- isomorphism π from $\mathcal{B}$ onto $\mathcal{C}(\mathcal{X})$ where $\mathcal{X}$ is compact Hausdorff. With $u = \pi(x)$ in $\mathcal{C}(\mathcal{X})$, it follows from Theorem 24 that

$$\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{C}(\mathcal{X})}(\pi(x)) \quad = \quad \sigma_{\mathcal{C}(\mathcal{X})}(u) = \{u(x) \big| x \in \mathcal{X}\}$$

For every $f \in \mathcal{C}(\sigma_{\mathcal{B}}(x)) = \mathcal{C}(\{u(x) \big| x \in \mathcal{X}\})$, the composite function $f \circ u$ is continuous on $\mathcal{X}$. Thus the mapping $f \rightarrow f \circ u$ is a $*$- isomorphism from $\mathcal{C}(\sigma(x))$ into $\mathcal{C}(\mathcal{X})$. From the $*$- isomorphisms $f \rightarrow f \circ u$ and $\pi^{-1} : \mathcal{C}(\mathcal{X}) \rightarrow \mathcal{B}$, the composition $\psi : f \rightarrow \pi^{-1}(f \circ u)$ is a $*$- isomorphism from $\mathcal{C}(\sigma(x))$ into $\mathcal{A}$. Also

$$\psi(\imath) = \pi^{-1}(\imath \circ u) \quad = \quad \pi^{-1}(u) = x$$

Uniqueness of the $\mathcal{C}^*$- isomorphism π follows from the fact that $\pi : \imath \rightarrow x$ implies that polynomials in $\imath$ are mapped to polynomials in x. By the Stone-Weierstrass theorem, this determines the action of π on all of $\mathcal{C}(\sigma(x))$.

1.2 Closedness

Some very interesting parts of the study of derivations relates to unbounded, closed derivations. A short introduction to closed operator theory with definitions, properties and some results are given here. Theorem 28 is used often throughout this work.

1.2.1 Open mapping theorem

Assume throughout the rest of this section that $\mathcal{D}(T) \subset \mathcal{X}$ and $\mathcal{R}(T) \subset \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are normed linear spaces. Also, let T denote a linear operator mapping the domain $\mathcal{D}(T)$ into $\mathcal{Y}$.

Definition 12 (Closed operator) *Let $\mathcal{X} \times \mathcal{Y}$ be the normed linear space of all ordered pairs (x, y) with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ and norm given by $||(x, y)|| = max\{||x||, ||y||\}$. Let T be a linear operator mapping from $\mathcal{X}$ into $\mathcal{Y}$ with $\mathcal{D}(T) \subset \mathcal{X}$. Define the graph of T as $\mathcal{G}(T) = \{(x, Tx)|x \in \mathcal{D}(T)\}$. If $\mathcal{G}(T)$ is closed in $\mathcal{X} \times \mathcal{Y}$ then the operator T is said to be closed.*

Theorem 28 (Closedness) *An operator T is closed if and only if: $x_n \in \mathcal{D}(T)$; $x_n \to x$ and $Tx_n \to y$ imply $x \in \mathcal{D}(T)$ and $Tx = y$.*

Proof: For every $x \in \mathcal{D}(T)$, $z = (x, y)$ is in the graph $\mathcal{G}(T)$ of T if and only if $Tx = y$. For every $x_n \in \mathcal{D}(T)$, $(x_n, Tx_n) \to (x, y) \in \overline{\mathcal{G}(T)}$ if and only if $x_n \to x$ and $Tx_n \to y$. Therefore, if T is closed, then $x_n \to x$ and $Tx_n \to y$ imply $(x_n, Tx_n) \to (x, y) \in \overline{\mathcal{G}(T)} = \mathcal{G}(T)$ which implies $Tx = y$. Conversely, if $x_n \in \mathcal{D}(T)$, $x_n \to x$ and $Tx_n \to y$ imply $x \in \mathcal{D}(T)$ and $Tx = y$, then $(x, y) \in \mathcal{G}(T)$ and $(x_n, Tx_n) \to (x, y) \in \overline{\mathcal{G}(T)}$ so that $\overline{\mathcal{G}(T)} = \mathcal{G}(T)$.

Definition 13 (Closable operator) *Let T be a linear operator mapping from $\mathcal{X}$ into $\mathcal{Y}$ with $\mathcal{D}(T) \subset \mathcal{X}$. T is closable if there exists a linear extension of T which is closed in $\mathcal{X}$.*

Theorem 29 (Closability) *An operator T is closable if and only if for any $y \neq 0$ in $\mathcal{Y}$, $(0, y) \notin \overline{\mathcal{G}(T)}$.*

Proof: Goldberg, S. [6], 54.

This result is often used to prove closability by assuming $(0, y) \in \overline{\mathcal{G}(T)}$ and showing $y = 0$. The assumption $(0, y) \in \overline{\mathcal{G}(T)}$ implies the existence of $x_n \in \mathcal{D}(T)$ with $x_n \to 0$ and $T(x_n) \to y \neq 0$. Therefore, Theorem 29 can be restated as:

A linear operator T mapping from a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ with $\mathcal{D}(T) \subset \mathcal{X}$, is closable (pre-closed) if and only if the existence of a sequence $\{x_n\} \in \mathcal{D}(T)$ with $x_n \to 0$ and $Tx_n \to z$ implies $z = 0$.

Remark 7 (Partially defined inverse) *In the following theorem the notation T^{-1} refers to a 'partial inverse' of T whereby T is only required to be 1-1. T is neither onto nor defined on the whole of $\mathcal{X}$. This inverse injectively maps $\mathcal{R}(T)$ onto $\mathcal{D}(T)$ and will exist if T is 1-1.*

Theorem 30 (T 1-1 and closed) *If T mapping $\mathcal{D}(T)$ into $\mathcal{Y}$ is 1-1 and closed, then T^{-1} is closed.*

Proof: Suppose T is 1-1 and closed. Clearly T^{-1} exists. Also $T : \mathcal{D}(T) \to \mathcal{R}(T)$ is onto and $T^{-1} : \mathcal{D}(T^{-1}) \to \mathcal{R}(T^{-1})$ is onto. Consider $\mathcal{G}(T^{-1}) = \{(Tx, T^{-1}Tx) | x \in \mathcal{D}(T)\} = \{(Tx, x) | x \in \mathcal{D}(T)\}$. Since $\mathcal{G}(T) = \{(x, Tx) | x \in \mathcal{D}(T)\}$ is closed in $\mathcal{X} \times \mathcal{Y}$ we have that $\mathcal{G}(T^{-1})$ is closed in $\mathcal{Y} \times \mathcal{X}$ so that T^{-1} is closed.

Definition 14 (Open mapping) *An operator mapping from X into Y is called open if it maps open sets in X onto open sets in Y. An operator is called relatively open if it maps open sets in X onto sets open in $\mathcal{R}(T)$.*

Remark 8 (Open mapping) *The notion of openness (resp. relatively openness) of an operator T mapping from $\mathcal{X}$ into $\mathcal{Y}$ can also be characterized as an operator that maps interior points of an arbitrary set $\mathcal{W} \subset \mathcal{X}$ onto interior points of $T\mathcal{W} \subset \mathcal{Y}$ (resp. interior points of $T\mathcal{W} \subset \mathcal{R}(T)$). For if this is the case, then clearly all open sets in $\mathcal{X}$ will be mapped onto open sets in $\mathcal{Y}$ (resp. $\mathcal{R}(T)$).*
Adopt the following notation:

$$\begin{aligned} S_{\mathcal{X}}(r) &= \{z | z \in \mathcal{X}, \|z\| \leq r\} \text{ and} \\ S^0_{\mathcal{X}}(r) &= \{z | z \in \mathcal{X}, \|z\| < r\} \end{aligned}$$

with $r \in \mathbb{R}$, $r > 0$. Note that $S^0_{\mathcal{X}}(r)$ denotes the set of interior point of $S_{\mathcal{X}}(r)$ with $S_{\mathcal{X}}(r)$ a neighborhood of 0 in $\mathcal{X}$.

Remark 9 (Interior points) *The following theorem makes use of the inclusion $S^0_{\mathcal{Y}}(r) \subset TS_{\mathcal{X}}(1)$ with $r \in \mathbb{R}$, $r > 0$. This inclusion implies that $0 \in \mathcal{X}$, an interior point of $S_{\mathcal{X}}(1)$, is mapped onto the interior point $0 \in \mathcal{Y}$ of $T\mathcal{S}_{\mathcal{X}}(1)$, because clearly $S^0_{\mathcal{Y}}(r)$ is a neighborhood of 0 in $\mathcal{Y}$ that is contained in $TS_{\mathcal{X}}(1)$. Observe that for any open set $\mathcal{V} \subset \mathcal{X}$ and any $x \in \mathcal{X}$, $x + \mathcal{V}$ is also open. By linearity of T, it follows for any point x of $\mathcal{D}(T)$ that $T(x + S_{\mathcal{X}}(1)) \supset Tx + S^0_{\mathcal{Y}}(r)$ so that Tx is an interior point of $T(x + \mathcal{S}_{\mathcal{X}}(1))$. Remark 8 and 9 together thus yield:*

$$T \text{ open} \Leftrightarrow \mathcal{S}^0_{\mathcal{Y}}(r) \subset T\mathcal{S}_{\mathcal{X}}(1)$$

for some $r \in \mathbb{R}$, $r > 0$. T is called nearly open *when $\mathcal{S}^0_{\mathcal{Y}}(r) \subset \overline{T\mathcal{S}_{\mathcal{X}}(1)}$ for some $r \in \mathbb{R}$ and $r > 0$.*

Theorem 31 (Nearly open implies open) *Let $\mathcal{X}$ be complete and $T : \mathcal{X} \to \mathcal{Y}$ be closed. If for some $r > 0$, $S^0_{\mathcal{Y}}(r) \subset \overline{TS_{\mathcal{X}}(1)}$ then $S^0_{\mathcal{Y}}(r) \subset TS_{\mathcal{X}}(1)$.*

Proof: Assume first that for every $0 < \epsilon < 1$ we have $S^0_{\mathcal{Y}}(r) \subset TS_{\mathcal{X}}(\frac{1}{1-\epsilon})$. Since $S^0_{\mathcal{Y}}(r)$ is open, $y \in S^0_{\mathcal{Y}}(r)$ implies that $\frac{y}{1-\epsilon} \in S^0_{\mathcal{Y}}(r)$ for ϵ small enough. Hence there exists $x \in S_{\mathcal{X}}(\frac{1}{1-\epsilon})$ such that $Tx = \frac{y}{1-\epsilon}$ or $T((1-\epsilon)x) = y$. Since $||(1-\epsilon)x|| \leq 1$, we have $y \in TS_{\mathcal{X}}(1)$. Therefore $S^0_{\mathcal{Y}}(r) \subset TS_{\mathcal{X}}(\frac{1}{1-\epsilon})$ $\forall 0 < \epsilon < 1$ implies $S^0_{\mathcal{Y}}(r) \subset TS_{\mathcal{X}}(1)$. It needs to be shown that $S^0_{\mathcal{Y}}(r) \subset \overline{TS_{\mathcal{X}}(1)}$ implies $S^0_{\mathcal{Y}}(r) \subset TS_{\mathcal{X}}(\frac{1}{1-\epsilon})$ for every $0 < \epsilon < 1$.
Assume $S^0_{\mathcal{Y}}(r) \subset \overline{TS_{\mathcal{X}}(1)}$ and T is closed and let $y \in S^0_{\mathcal{Y}}(r)$ be given. Since for every set $\mathcal{K} \subset \mathcal{X}$, $a\overline{\mathcal{K}} = \overline{a\mathcal{K}}$ for $a > 0$ a scalar and $S^0_{\mathcal{Y}}(r) \subset \overline{TS_{\mathcal{X}}(1)}$ given, it follows that $S^0_{\mathcal{Y}}(r\epsilon^n) \subset \overline{TS_{\mathcal{X}}(\epsilon^n)}$. Taking $n = 0$, there exists $x_0 \in S_{\mathcal{X}}(1)$ such that $||y - Tx_0|| < r\epsilon$ or $y - Tx_0 \in S^0_{\mathcal{Y}}(r\epsilon)$. Taking $n = 1$, there exists $x_1 \in S_{\mathcal{X}}(\epsilon)$ such that $||y - Tx_0 - Tx_1|| < r\epsilon^2$ or $y - Tx_0 - Tx_1 \in S^0_{\mathcal{Y}}(r\epsilon^2)$. Proceeding in this manner, there exists a sequence $x_i \in S_{\mathcal{X}}(\epsilon^i)$ with $||y - \sum_{i=0}^n Tx_i|| < r\epsilon^{n+1}$. Now define the sequence $z_n = \sum_{i=0}^n x_i$ where $||x_i|| \leq \epsilon^i$. Then $||x_0|| \leq 1$, $||x_1|| \leq \epsilon$, $||x_2|| \leq \epsilon^2$ and so on, so that $\sum_{i=0}^\infty ||x_i|| \leq \frac{1}{1-\epsilon} < \infty$ (geometric series). Thus, $\{z_n\}$ is Cauchy and since $\mathcal{X}$ is complete, it follows that there exists $x \in \mathcal{X}$ such that $z_n \to x$ with $||x|| \leq \frac{1}{1-\epsilon}$. Clearly $Tz_n = T(\sum_{i=1}^n (x_i)) \to y$, and since T is closed, $x \in \mathcal{D}(T) \cap S_{\mathcal{X}}(\frac{1}{1-\epsilon})$ and $Tx = y$. Thus $y \in TS_{\mathcal{X}}(\frac{1}{1-\epsilon})$ and hence $S^0_{\mathcal{Y}}(r) \subset TS_{\mathcal{X}}(\frac{1}{1-\epsilon})$.

Lemma 1 (T nearly open) *Let $\mathcal{Y}$ be of the second category and T onto. Then T is nearly open.*

Proof: It needs to be shown that there exists $r > 0$ such that $S^0_{\mathcal{Y}}(r) \subset \overline{TS_{\mathcal{X}}(\epsilon)}$. Since $\mathcal{R}(T) = \mathcal{Y}$, we may write $\mathcal{Y} = \cup_{n=1}^\infty \overline{nTS_{\mathcal{X}}(1)}$, and since $\mathcal{Y}$ is of second category, at least one of the sets $\overline{pTS_{\mathcal{X}}(1)}$ has a non-empty interior. The map g_p defined by $g_p(x) = px$ is homeomorphic. Therefore $\overline{TS_{\mathcal{X}}(1)}$ must also contain an open set (non-empty). Also $g_{\frac{\epsilon}{2}}(\overline{TS_{\mathcal{X}}(1)}) = \overline{TS_{\mathcal{X}}(\frac{\epsilon}{2})}$ and thus $\overline{TS_{\mathcal{X}}(\frac{\epsilon}{2})}$ also contains a non-empty open set $\mathcal{V}$. Now

$$0 \in \mathcal{V} - \mathcal{V} \subset \overline{TS_{\mathcal{X}}(\frac{\epsilon}{2})} - \overline{TS_{\mathcal{X}}(\frac{\epsilon}{2})} \subset \overline{TS_{\mathcal{X}}(\epsilon)}$$

The second inclusion above is justified by noting that $TS_{\mathcal{X}}(\frac{\epsilon}{2}) - TS_{\mathcal{X}}(\frac{\epsilon}{2}) = T(S_{\mathcal{X}}(\frac{\epsilon}{2}) - S_{\mathcal{X}}(\frac{\epsilon}{2})) \subset TS_{\mathcal{X}}(\epsilon) \subset \overline{TS_{\mathcal{X}}(\epsilon)}$.
Now $\mathcal{V} - \mathcal{V}$ is an open set around 0, which indicates that there exists $r > 0$ such that $S^0_{\mathcal{Y}}(r) \subset \mathcal{V} - \mathcal{V} \subset \overline{TS_{\mathcal{X}}(\epsilon)}$.

With Baire's category theorem in mind, the open mapping theorem now follows:

Theorem 32 (Open mapping theorem) *Let $\mathcal{X}$ be complete and $\mathcal{Y}$ of second category. If T is closed and onto, then T is an open mapping.*

Proof: Goldberg, S. [6], 45.

1.2.2 Closed graph theorem

Theorem 33 (Closed graph theorem) *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and T a closed operator mapping $\mathcal{X}$ into $\mathcal{Y}$. Then T is continuous.*

Proof: Since $\mathcal{G}(T)$ is closed in the Banach space $\mathcal{X} \times \mathcal{Y}$, $\mathcal{G}(T)$ is complete. Define the mapping $f : \mathcal{G}(T) \to \mathcal{X}$ by $f((x, Tx)) = x$. f is bijective and since $\|f((x, Tx))\| = \|x\| \leq \|(x, Tx)\|$, f is continuous. (The norm on $\mathcal{X} \times \mathcal{Y}$ is defined as $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}$). From Theorem 32 we know that $f^{-1} : \mathcal{X} \to \mathcal{G}(T)$ is continuous. Now

$$\|Tx\| \leq \|(x, Tx)\| = \|f^{-1}(x)\| \leq \|f^{-1}\|\|x\|$$

so that T is bounded (continuous).

Theorem 34 (Extension of closed graph theorem) *Let $\mathcal{X}$ and $\mathcal{Y}$ be complete. Any two of the following imply the third:*

1. *$\mathcal{D}(T)$ is closed*
2. *T is closed*
3. *T is continuous*

Proof:(1,2 $\Rightarrow$ 3) Let $\mathcal{D}(T)$ be closed and T mapping $\mathcal{D}(T)$ into $\mathcal{Y}$ be closed. Since $\mathcal{D}(T)$ is closed, we have that $\mathcal{D}(T)$ is complete and by the closed graph theorem, T is continuous.
(1,3 $\Rightarrow$ 2) Suppose $\mathcal{D}(T)$ is closed and T continuous. Now for $x_n \in \mathcal{D}(T)$ and $x_n \to x$ we have $x \in \mathcal{D}(T)$, since $\mathcal{D}(T)$ is closed. Furthermore, if also $Tx_n \to y$, then since by continuity, $Tx_n \to Tx$, we get $Tx = y$. Thus by Theorem 28 we have that T is closed.
(2,3 $\Rightarrow$ 1) Let T be closed and continuous. Now for $x \in \overline{\mathcal{D}(T)}$, there exists $x_n \in \mathcal{D}(T)$ such that $x_n \to x$. Consider

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\|\|x_n - x_m\|.$$

Therefore Tx_n is Cauchy and $Tx_n \to y\,(\in \mathcal{Y})$ since $\mathcal{Y}$ is complete. Since T is closed, it follows from Theorem 33 that $x \in \mathcal{D}(T)$ (and $Tx = y$). Therefore $\overline{\mathcal{D}(T)} = \mathcal{D}(T)$ because $x \in \overline{\mathcal{D}(T)}$ was arbitrary.

Theorem 35 (Kernel of a closed operator) *The kernel $\mathcal{N}(T)$ of an operator T is closed if T is closed.*

Proof: Suppose T is closed. Take any $x \in \overline{\mathcal{N}(T)}$. Then there exists $\{x_n\} \in \mathcal{N}(T) \subset \mathcal{D}(T)$ such that $x_n \to x$. Clearly $Tx_n = 0$ for all n and therefore $Tx_n \to 0$. Since T is closed, it follows from Theorem 28 that $x \in \mathcal{N}(T)$ (since $Tx = 0$) so that $\mathcal{N}(T) = \overline{\mathcal{N}(T)}$ and $\mathcal{N}(T)$ is closed.

Theorem 36 (Induced operator) *Let $\mathcal{N}(T)$ be closed and $\hat{T} : \mathcal{D}(T)/\mathcal{N}(T) \to \mathcal{Y}$ the 1-1 operator induced by T defined as $\hat{T}([x]) = Tx$ where $[x] \in \mathcal{D}(T)/\mathcal{N}(T)$. Then T is closed if and only if $\hat{T}$ is closed.*

Proof:($\Rightarrow$) If T is closed, suppose $[x_n] \to [x]$, with $[x_n] \in \mathcal{D}(T)/\mathcal{N}(T)$, and $\hat{T}([x_n]) \to y$. Then there exists $\{v_n\} \in \mathcal{N}(T)$ such that $x_n - v_n \to x$. Now $T(x_n) = T(x_n - v_n) = \hat{T}([x_n]) \to y$. Since T is closed, $x \in \mathcal{D}(T)$ and $T(x) = y$ by Theorem 28. Thus $[x] \in \mathcal{D}(\hat{T})$ and $\hat{T}([x]) = y$. Hence $\hat{T}$ is closed.
($\Leftarrow$) Let $\hat{T}$ be closed. Suppose $x_n \to x$ and $Tx_n \to y$. Then $[x_n] \to [x]$ and $\hat{T}([x_n]) = Tx_n \to y$. By the closedness of $\hat{T}$, it follows that $[x] \in \mathcal{D}(\hat{T})$ and $\hat{T}([x]) = y$. Thus $x \in \mathcal{D}(T)$ and $Tx = y$ and hence T is closed by Theorem 28.

Theorem 37 (Existence of the inverse - 1) *Let $\mathcal{D}(T) = \mathcal{X}$. T^{-1} exists and is continuous if and only if there exists $m > 0$ such that $||Tx|| \geq m||x||$ for every $x \in \mathcal{X}$.*

Proof: ($\Leftarrow$) Suppose $||Tx|| \geq m||x||$. Then $x \neq 0$ implies $Tx \neq 0$ so that T is injective. Hence T^{-1} exists. Now $||T^{-1}Tx|| = ||x|| \leq m^{-1}||Tx||$. Therefore T^{-1} is bounded and continuous.
($\Rightarrow$) Let T^{-1} be continuous. Then $||x|| = ||T^{-1}Tx|| \leq ||T^{-1}||||Tx||$ for every $x \in \mathcal{X}$. Choosing $m = \frac{1}{||T^{-1}||}$, the desired property follows.

Theorem 38 (Existence of the inverse - 2) *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and T closed. Then T^{-1} exists (as a map from $\mathcal{R}(T)$ to $\mathcal{D}(T)$) and is bounded if and only if T is 1-1 and $\mathcal{R}(T)$ is closed.*

Proof:($\Rightarrow$) Suppose T has a bounded inverse. Suppose $Tx_n \to y \in \mathcal{Y}$. From the existence of the bounded inverse it follows from Theorem 37 that there exists $m > 0$ such that $||Tx_n - Tx_m|| \geq m||x_n - x_m||$ which shows that $\{x_n\}$ is Cauchy in $\mathcal{X}$ and hence converges to some $x \in \mathcal{X}$. Since T is closed, $x \in \mathcal{D}(T)$ and $Tx = y$ by Theorem 28. Thus $\mathcal{R}(T)$ is closed.

($\Leftarrow$) Suppose T is 1-1 and $\mathcal{R}(T)$ is closed. Then T^{-1} is closed (by Theorem 30) as an operator from $\mathcal{R}(T)$ into $\mathcal{X}$. Since $\mathcal{R}(T)$ is closed and hence complete in the Banach space $\mathcal{Y}$, it follows from the closed graph theorem that T^{-1} is continuous.

1.2.3 Minimum modulus

The following consideration leads to the definition of a number $\gamma(T)$ associated to each operator T having a closed kernel. Let T be a closed operator and $\mathcal{X}$ and $\mathcal{Y}$ complete. Let $\hat{T}$ be the 1-1 operator induced by T. Since T is closed, we have from Theorem 35 that $\mathcal{N}(T)$ is closed and hence $\mathcal{D}(T)/\mathcal{N}(T)$ is a normed linear space with norm $||[x]|| = d(x, \mathcal{N}(T))$. Now by Theorem 38, $\mathcal{R}(T) = \mathcal{R}(\hat{T})$ is closed if and only if $\hat{T}$ has a bounded inverse ($\hat{T}$ is 1-1). By Theorem 37, this means that there exists an $m > 0$ such that for every $[x] \in \mathcal{D}(\hat{T})$, $m||[x]|| \leq ||\hat{T}[x]||$. Equivalently $0 < \inf\{\frac{||\hat{T}[x]||}{||[x]||} | [x] \in \mathcal{D}(\hat{T}), x \notin \mathcal{N}(T)\}$. Since $||T[x]|| = ||Tx||$ and $||[x]|| = d(x, \mathcal{N}(T))$ it follows that $0 < \inf\{\frac{||Tx||}{d(x,\mathcal{N}(T))} | x \in \mathcal{D}(T),\, x \notin \mathcal{N}(T)\}$. This infimum is called the minimum modulus of the operator T.

The following observation will be used in Theorem 39. Consider the normed linear space $\mathcal{D}(T)/\mathcal{N}(T)$ with elements $[x]$. Then $y \in [x]$ if and only if $\lambda y \in [\lambda x]$. This can be seen by noting that $y \in [x]$ implies $x - y = m$ for some m in $\mathcal{N}(T)$. Then $\lambda x - \lambda y = \lambda m$ and since $\lambda m \in \mathcal{N}(T)$ if and only if $m \in \mathcal{N}(T)$ (from linearity), it follows that $\lambda y \in [\lambda x]$.

Definition 15 (Minimum modulus) *Let $\mathcal{N}(T)$ be closed. The minimum modulus $\gamma(T)$ of T is defined as $\gamma(T) = \inf_{x\in\mathcal{D}(T)} \frac{||Tx||}{d(x,\mathcal{N}(T))}$ where $\frac{0}{0}$ is defined to be ∞.*

Theorem 39 (Minimum modulus) *T is relatively open if and only if $\gamma(T) > 0$.*

Proof:($\Leftarrow$) Let $\gamma(T) > 0$ and assume $T \neq 0$ to avoid $\gamma(T) = \infty$. By definition of $\gamma(T) = \inf_{n\in\mathcal{N}(T)} \frac{||Tx||}{d(x,\mathcal{N}(T))}$ ($= \lambda > 0$) we have $\frac{||Tx||}{d(x,\mathcal{N}(T))} \geq \lambda \,\forall x \notin \mathcal{N}(T)$ which implies $||Tx|| \geq \lambda d(x, \mathcal{N}(T))$. Now if $||Tx|| < \lambda = \inf_{y\in\mathcal{D}(T)} \frac{||Ty||}{d(y,\mathcal{N}(T))}$ then since $||Tx|| < \lambda \leq \frac{||Tx||}{d(x,\mathcal{N}(T))}$ it follows that $||Tx|| < \frac{||Tx||}{d(x,\mathcal{N}(T))}$ from which it follows that $d(x, \mathcal{N}(T)) < 1$. From $||Tx|| < \lambda$ we have that $Tx \in \lambda\mathcal{S}^0_{\mathcal{R}(T)}(1)$. Now let $n \in \mathcal{N}(T)$ be such that $||x - n|| < 1$ (n exists, since $d(n, \mathcal{N}(T)) < 1$). Then $x - n \in \mathcal{S}_{\mathcal{D}(T)}(1)$. Call $z = x - n$ ($||z|| \leq 1$). Now $T(z) = T(x - n) = Tx - Tn$, but $Tn = 0$. Thus $Tz = Tx$ where $z \in \mathcal{S}_{\mathcal{D}(T)}(1)$, so that $\lambda\mathcal{S}_{\mathcal{R}(T)}(1) \subset T\mathcal{S}_{\mathcal{D}(T)}(1)$. This shows that the interior point 0 of $\mathcal{S}_{\mathcal{D}(T)}(1)$, is mapped onto the interior point 0 of $\mathcal{S}_{\mathcal{R}(T)}(1)$. It follows that T is relatively open.

($\Rightarrow$) If T is relatively open then there exists a $\lambda > 0$ such that

$$\mathcal{S}^0_{\mathcal{R}(T)}(\lambda) \subset T(\mathcal{S}^0_{\mathcal{D}(T)}(1)) \tag{1}$$

Choose $y = Tx \in \mathcal{S}^0_{\mathcal{R}(T)}(\lambda)$ $(x \in \mathcal{D}(T))$ so that $\lambda - \epsilon \leq \|Tx\| < \lambda$ $(\lambda > \epsilon > 0)$. Then (from equation (1)) there exists $z \in \mathcal{S}^0_{\mathcal{D}(T)}(1)$ $(\|z\| < 1)$ with $Tz = Tx$. But $Tz - Tx = 0$ so that $n = z - x \in \mathcal{N}(T)$. Now $d(x, \mathcal{N}(T)) = \inf_{n \in \mathcal{N}(T)} \|x - n\| \leq \|x - z - x\| = \|z\| < 1$. Then

$$0 < \lambda - \epsilon < \frac{\|Tx\|}{d(x, \mathcal{N}(T))}$$

Theorem 40 (Extension of open mapping theorem) *Let $\mathcal{X}$ and $\mathcal{Y}$ be complete. Any two of the following three imply the third:*

1. *$\mathcal{R}(T)$ is closed*
2. *T is closed*
3. *$\gamma(T) > 0$*

Proof: (1,2 $\Rightarrow$ 3) Let $\mathcal{R}(T)$ be closed in $\mathcal{Y}$ and T closed. Since $\mathcal{Y}$ is complete, $\mathcal{R}(T)$ is complete and by the open mapping theorem, T is relatively open so that $\gamma(T) > 0$.
(2,3 $\Rightarrow$ 1) Let T be closed and $\gamma(T) > 0$. By Theorem 39 we have, since $\gamma(T) > 0$, T is relatively open. Now define the 1-1 operator $\hat{T}$ induced by T where $\mathcal{D}(\hat{T}) = \mathcal{D}(T)/\mathcal{N}(T)$ and $\hat{T} : \mathcal{D}(\hat{T})$ into $\mathcal{Y}$, by $\hat{T}[x] = Tx$. Since $\mathcal{N}(T)$ is closed by Theorem 12, $\mathcal{D}(T)/\mathcal{N}(T)$ is a normed linear space with norm $\|[x]\| = d(x, \mathcal{N}(T))$. Clearly $\mathcal{R}(T) = \mathcal{R}(\hat{T})$. $\mathcal{R}(\hat{T})$ will be closed if (and only if) $\hat{T}$ has a bounded inverse, by Theorem 38. Now

$$\begin{aligned}
\gamma(\hat{T}) &= \inf\left\{\frac{\|\hat{T}[x]\|}{\|[x]\|} \,\middle|\, [x] \in \mathcal{D}(\hat{T}),\, x \notin \mathcal{N}(T)\right\} \\
&= \inf\left\{\frac{\|Tx\|}{d(x, \mathcal{N}(T))} \,\middle|\, x \in \mathcal{D}(T),\, x \notin \mathcal{N}(T)\right\} \\
&= \gamma(T) > 0
\end{aligned}$$

Therefore there exists $\gamma > 0$ such that $\|\hat{T}[x]\| \geq \gamma\|[x]\|$ for all $[x] \in \mathcal{D}(T)/\mathcal{N}(T)$. Since $\hat{T}$ is 1-1, we have that $\hat{T}^{-1}$ exists. Clearly $\|\hat{T}^{-1}\hat{T}[x]\| = \|[x]\| \leq \frac{1}{\gamma}\|\hat{T}[x]\|$ which shows (by Theorem 38) that $\hat{T}^{-1}$ is bounded. Therefore $\mathcal{R}(T) = \mathcal{R}(\hat{T})$ is closed.
(3,1 $\Rightarrow$ 2) Let $\gamma(T) > 0$ and $\mathcal{R}(T)$ closed. Then T is relatively open. As before, let $\hat{T}$ be the 1-1 operator induced by T. Since $\mathcal{R}(\hat{T})$ is closed and $\hat{T}$ is 1-1, $\hat{T}^{-1}$ exists and is bounded (by Theorem 38), and hence continuous. Furthermore $\mathcal{D}(\hat{T}^{-1}) = \mathcal{R}(T)$ is closed in $\mathcal{Y}$ and therefore $\hat{T}^{-1}$ is closed by Theorem 34. Since $\hat{T}^{-1}$ is 1-1 and closed, $(\hat{T}^{-1})^{-1} = \hat{T}$ is also closed (by Theorem 30). Hence by Theorem 36, T is closed.

2 Derivations

2.1 Introduction

Definition 16 *Let $\mathcal{A}$ be a Banach algebra and δ a linear mapping in $\mathcal{A}$. Then δ is a derivation if:*

1. *$\mathcal{D}(\delta)$ is a dense sub-algebra in $\mathcal{A}$*
2. $\delta(ab) = \delta(a)b + a\delta(b) \qquad \forall a, b \in \mathcal{A}$

   If $\mathcal{A}$ is a $*$- Banach algebra, then δ is a $*$- derivation if
3. $\delta(a^*) = \delta(a)^* \qquad \forall a \in \mathcal{A}$

Derivations with the property $\delta(x^*) = \delta(x)^*$ are also referred to as *symmetric* derivations. Derivations that can be expressed as a commutator ($\delta(b) = ab - ba$) for all $b \in \mathcal{A}$ and for $a \in \mathcal{A}$, are called *inner* derivations.

Derivations in operator algebras originate from quantum mechanics where the coordinates of particle momentum and position are identified with operators p and q respectively, satisfying the commutation relations

$$\begin{aligned} p_i p_j - p_j p_i &= 0 \\ q_i q_j - q_j q_i &= 0 \\ &\textit{and} \\ p_i q_j - q_j p_i &= -ih\delta_{ij}\mathbb{1} \end{aligned}$$

where h is Planck's constant. These operators p and q were tentatively proposed (by Heisenberg) in terms of matrix operators. It will be shown shortly that the relation $pq - qp = -ih\mathbb{1}$ cannot hold for both p and q bounded. Thus operators p and q were assumed to act on an infinite dimensional Hilbert space $\mathcal{H}$. Physically each vector $\psi \in \mathcal{H}$ corresponds to a state of the system and for ψ normalized, the inner-product $\langle \psi, A_t\psi \rangle$ corresponds to the expected value of the observable A at time t. The equation determining the change of any such observable A with the time t was specified as

$$\frac{dA}{dt} = i\frac{(HA_t - A_tH)}{h} \tag{2}$$

where

$$H = \frac{p^2}{2m} + V(q) \tag{3}$$

is the Hamiltonian, combining kinetic and potential energy as functions of p and q respectively.

The theory of commutators in quantum mechanics stimulates the study of bounded derivations on operator algebras in the following way: Let $x, y \in \mathcal{B}(A)$ where A is a Banach space. ($\mathcal{B}(A)$) denotes the algebra of bounded linear operators on A). Define the derivation δ_x by

$$\delta_x(y) = xy - yx$$

That δ_x is a derivation, follows from the associativity of the algebra $\mathcal{B}(A)$ and the observation that if $\delta_x(y) = xy - yx$ and $\delta_x(z) = xz - zx$ then

$$\begin{aligned}\delta_x(yz) &= x(yz) - (yz)x \\ &= (xy)z - y(zx) \\ &= (\delta_x(y) + yx)z - y(xz - \delta_x(z)) \\ &= \delta_x(y)z + yxz - yxz + y\delta_x(z)\end{aligned}$$

The observation $||\delta_x(y)|| \leq 2||xy|| \leq 2||x||||y||$ shows that δ_x is bounded.
Consider the Schrödinger operators p and q on the Hilbert space $\mathcal{L}^2[-\infty, +\infty]$ defined by

$$\begin{aligned}p(f) &= -i\frac{df}{dx} \\ q(f)(x) &= xf(x)\end{aligned}$$

Then the commutation relation for p and q yields

$$\begin{aligned}(pq - qp)f &= (pq)f - (qp)f \\ &= p(xf) - q(-i\frac{df}{dx}) \\ &= -i\frac{d}{dx}(xf) + ix\frac{d}{dx}f \\ &= -if - ix\frac{d}{dx}f + ix\frac{d}{dx}f \\ &= -if\end{aligned}$$

This study of derivations begins with some general results on derivations as commutators.

2.2 Commutators

The following result is used in the proof of some of the commutator results as well as elsewhere in this work:

Theorem 41 (Kleinecke-Sirokov, cf. (Sakai, S. [22], Theorem 2.2.1)) *Let $\mathcal{A}$ be a Banach algebra and δ a bounded derivation on $\mathcal{A}$ ($\mathcal{D}(\delta) = \mathcal{A}$). Suppose that $\delta^2(x) = 0$ for some $x \in \mathcal{A}$. Then $\delta(x)$ is a generalized nilpotent - i.e. $(\|\delta(x)^n\|)^{\frac{1}{n}} \to 0$ as $n \to \infty$.*

Proof: The proof that $\delta^n(x^n) = n!\delta(x)^n$ makes use of induction of $\delta^n(x^n)$. For $n = 1$ we have $\delta^1(x^1) = 1!\delta(x)^1$. Now suppose that $\delta^n(x^n) = n!\,\delta(x)^n$. Consider Leibniz's formula:

$$\begin{aligned}\delta^n(vu) &= \sum_{k=1}^{n}\binom{n}{k}\delta^k(v)\delta^{n-k}(u) \qquad \textit{where}\\ \binom{n}{k} &= \frac{n(n-1)(n-2)...(n-k+1)}{k!}\end{aligned}$$

Then, for the case $n+1$ we have that

$$\begin{aligned}\delta^{n+1}(x^{n+1}) &= \delta^{n+1}(x^n x)\\ &= \sum_{k=1}^{n+1}\binom{n+1}{k}\delta^k(x^n)\delta^{n+1-k}(x)\\ &= \binom{n+1}{1}\delta(x^n)\delta^n(x) + \binom{n+1}{2}\delta^2(x^n)\delta^{n-1}(x) + ...\\ &+ \binom{n+1}{n-1}\delta^{n-1}(x^n)\delta^2(x) + \binom{n+1}{n}\delta^n(x^n)\delta(x) + \binom{n+1}{n+1}\delta^{n+1}(x)x\\ &= \delta^{n+1}(x^n)x + (n+1)\delta^n(x^n)\delta(x)\\ &= \delta(\delta^n(x^n))x + (n+1)n!\,\delta(x)^{n+1}\\ &= \delta(n!\,\underbrace{\delta(x)\cdots\delta(x)}_{n\ times})x + (n+1)!\,\delta(x)^{n+1}\\ &= n!\,\underbrace{[\delta^2(x)\delta(x)^{n-1} + \delta(x)\delta^2(x)\delta(x)^{n-2} + ... + \delta(x)^{n-1}\delta^2(x)]}_{n-1\ times} + (n+1)!\,\delta(x)^n\\ &= (n+1)!\,\delta(x)^{n+1}\end{aligned}$$

Thus

$$\delta^n(x^n) = n!\,\delta(x)^n \ for\ n = 1, 2, 3\cdots.$$

Therefore

$$\begin{aligned} \|\delta(x)^n\|^{\frac{1}{n}} &= \|\delta^n(x^n)\frac{1}{n!}\|^{\frac{1}{n}} \\ &\leq \|\delta\|\|x\|(\frac{1}{n!})^{\frac{1}{n}} \to 0 \qquad\qquad n \to \infty \end{aligned}$$

Theorem 42 (Wielandt-Wintner, cf. (Sakai, S. [22], Corollary 2.2.2)) *Let $\mathcal{A}$ be a Banach algebra. Then there exist no two elements $a,\, b \in \mathcal{A}$ such that $ab - ba = \mathbb{1}$.*

Proof: For $x \in \mathcal{A}$ let $\delta_a(x) = ax - xa = [a,\, x]$. Then $\delta_a(x)$ is continuous since for every $\delta < 0$ there exists $\epsilon = 2\|a\|\delta < 0$ such that if $\|x - x_0\| < \delta$ then $\|\delta_a(x) - \delta_a(x_0)\| \leq 2\|a\|\|x - x_0\| = 2\|a\|\delta = \epsilon$. Hence $\delta_a(x)$ is continuous at x_0 and linear; and thus bounded on $\mathcal{A}$.

Now if $ab - ba = \mathbb{1}$, then

$$\begin{aligned} \delta_a^2(b) &= \delta_a(ab - ba) \\ &= \delta_a(\mathbb{1}) \\ &= a - a \\ &= 0 \end{aligned}$$

But then $\delta_a(b)$ is a generalized nilpotent (as in theorem 41) which contradicts the fact that $\|\delta_a(b)^n\|^{\frac{1}{n}} = \|(ab - ba)^n\|^{\frac{1}{n}} = 1$ for all n.

As noted in the introduction, the physical significance of this result is that the relation $pq - qp = -ih\mathbb{1}$ only holds for at least one of p or q unbounded, which leads to the unbounded-operators-on-Hilbert-space formalism proposed by Heisenberg.

In terms of derivations as operators on normed algebras, this result reveals that the identity element of $\mathcal{A}$ can not be expressed as the commutator of two elements in the algebra. Also, if we define $\delta_a(b) = ab - ba$, then no such derivation can map onto the identity operator. We extend the trivial fact that if $\mathcal{A}$ is commutative, we have that $ab - ba = 0$ for all $a,\, b \in \mathcal{A}$ and therefore all derivations defined as $\delta_a(b) = ab - ba$ will equal the trivial operator. This result is formalized in Theorem 46 and following are some required results.

Theorem 43 (Rosenblum, cf. (Sakai, S. [22], Theorem 2.2.5)) *Let $\mathcal{A}$ be a C^*-algebra and δ a bounded, everywhere defined derivation on $\mathcal{A}$. Suppose $\delta(x) = 0$ for some normal x ($x^*x = xx^*$) of $\mathcal{A}$. Then $\delta(x^*) = 0$.*

Proof: Consider the representation $x \mapsto e^{i\lambda x}$. Then

$$\begin{aligned} \delta(e^{i\lambda x^*}) &= \delta(e^{i\lambda x^*} e^{i\bar{\lambda} x} e^{-i\bar{\lambda} x}) \\ &= \delta(e^{i\lambda x^*} e^{i\bar{\lambda} x}) e^{-i\bar{\lambda} x} + e^{i\lambda x^*} e^{i\bar{\lambda} x} \delta(e^{-i\bar{\lambda} x}) \qquad \lambda \in \mathcal{C} \qquad (4) \end{aligned}$$

It is given that $\delta(x) = 0$. Now suppose $\delta(x^n) = 0$. Then $\delta(x^{n+1}) = \delta(x^n)x + x^n\delta(x) = 0$. Thus by induction in n we have $\delta(x^n) = 0$ for $(n = 1, 2, 3 \cdots)$. We have $\delta(\mathbb{1}) = 0$ and since δ is bounded,

$$\begin{aligned}\delta(e^{-i\bar{\lambda}x}) &= \delta(\mathbb{1}) + \delta(-i\bar{\lambda}x) + \delta(\frac{1}{2!}(-i\bar{\lambda}x)^2) + \delta(\frac{1}{3!}(-i\bar{\lambda}x)^3) + \cdots \\ &= 0\end{aligned}$$

Therefore (4) reduces to

$$\delta(e^{i\lambda x^*}) = \delta(e^{i\lambda x^*}e^{i\bar{\lambda}x})e^{-i\bar{\lambda}x}$$

But

$$\begin{aligned}\delta(e^{i\lambda x^*})e^{-i\lambda x^*} &= \delta(e^{i\lambda x^*}e^{i\bar{\lambda}x})e^{-i\bar{\lambda}x}e^{-i\lambda x^*} \\ &= \delta(e^{i(\lambda x^* + \bar{\lambda}x)})e^{-i(\bar{\lambda}x + \lambda x^*)}\end{aligned}$$

Normality of x is used to show that $e^{i\lambda x^*}e^{i\overline{\lambda x}} = e^{i(\lambda x^* + \overline{\lambda}x)}$. Now put

$$\begin{aligned}f(\lambda) &= \delta(e^{i(\lambda x^* + \bar{\lambda}x)})e^{-i(\bar{\lambda}x + \lambda x^*)} \\ &= \delta(e^{i\lambda x^*})e^{-i\lambda x^*}\end{aligned}$$

Then $f(\lambda)$ is differentiable on the whole plane $\mathbb{C}$ and

$$\begin{aligned}||f(\lambda)|| &= ||\delta(e^{i\lambda x^*})e^{-i\lambda x^*}|| \\ &= ||\delta(e^{i(\lambda x^* + \bar{\lambda}x)})e^{-i(\bar{\lambda}x + \lambda x^*)}|| \\ &\leq ||\delta|| ||e^{i(\lambda x^* + \bar{\lambda}x)}|| ||e^{-i(\bar{\lambda}x + \lambda x^*)}|| \\ &= ||\delta||\end{aligned}$$

so that $f(\lambda)$ is a constant by Liouville's theorem. But $f(0) = \delta(e^{i(0)x^*})e^{-i(0)x^*} = 0$ since $\delta(\mathbb{1}) = 0$. Hence $\delta(e^{i\lambda x^*}) = 0$ and so $0 = \frac{d}{d\lambda}(\delta(e^{i\lambda x^*}))|_{\lambda=0} = \delta(\frac{d}{d\lambda}e^{i\lambda x^*})|_{\lambda=0} = \delta(ix^*)$.

Theorem 44 (Fuglede, cf. (Sakai, S. [22], Corollary 2.2.6)) *Let T be a bounded normal operator on a Hilbert space $\mathcal{H}$ and S a bounded operator on $\mathcal{H}$. If $[S, T] = ST - TS = 0$ then $[S, T^*] = ST^* - T^*S = 0$ for all $x \in \mathcal{H}$.*

Proof: Consider the C^*- algebra $\mathcal{B}(\mathcal{H})$ and define the derivation δ_S on $\mathcal{B}(\mathcal{H})$ by $\delta_S(T) = ST - TS$ for T and $S \in \mathcal{B}(\mathcal{H})$. We have that $\delta_S(T)$ is bounded. Now by Theorem 43, since δ_S is bounded on $\mathcal{B}(\mathcal{H})$, we have that given $\delta_S(T) = 0$ it follows that $\delta_S(T^*) = ST^* - T^*S = 0$.

Theorem 45 (Sakai, S. [22], Theorem 2.2.7) *Let $\mathcal{A}$ be a $\mathcal{C}^*$ algebra and δ a derivation on $\mathcal{A}$. If $[\delta(x), x] = 0$ for some normal $x \in \mathcal{A}$, then $\delta(x) = 0$.*

Proof: From Theorem 44 it follows that if $[\delta(x), x] = 0$ for normal $x \in \mathcal{A}$, then $[\delta(x), x^*] = 0$. Then

$$\begin{aligned} \delta(x^*)x + x^*\delta(x) &= \delta(x^*x) \\ &= \delta(xx^*) \\ &= \delta(x)x^* + x\delta(x^*) \end{aligned}$$

From this we get

$$\begin{aligned} \delta(x^*)x - x\delta(x^*) &= \delta(x)x^* - x^*\delta(x) \\ [\delta(x^*), x] &= [\delta(x), x^*] = 0 \end{aligned}$$

so that again by Theorem 44 $[\delta(x^*), x] = 0$ implies $[\delta(x^*), x^*] = 0$. Now x is in the center of the sub-algebra $\mathcal{B}$ of $\mathcal{A}$ generated by $\{1\!\!1, x, x^*, \delta(x), \delta(x^*)\}$. The idea is to show that $\phi(\delta(x)) = 0$ for every $\phi \in \mathcal{P}(\mathcal{A})$, the pure states on $\mathcal{A}$, so that $\delta(x) = 0$.

Since $\mathcal{B}$ is a $\mathcal{C}^*$- algebra, we can write $x = x_1 + i\,x_2$ where x_1 and x_2 are both self-adjoint (Theorem 8). From Subsection 1.1.7 there exists a pure state $\phi \in \mathcal{P}(\mathcal{B})$ and an irreducible representation $(\mathcal{H}, \pi_\phi)$ of $\mathcal{B}$ generated by the GNS construction, which implies that $\pi_\phi(\mathcal{B})'$ (the commutant of $\pi_\phi(\mathcal{B})$) consists of multiples of $1\!\!1$. Therefore, the center $Z(\pi_\phi(\mathcal{B})) = \pi_\phi(\mathcal{B}) \cap \pi_\phi(\mathcal{B})'$ consists only of multiples of $1\!\!1$. (Note that $\pi_\phi(Z(\mathcal{B})) = Z(\pi_\phi(\mathcal{B}))$).

Since $x = x_1 + i\,x_2$ is in the center of $\mathcal{B}$, then (because by normality $x_1, x_2 \in Z(\mathcal{B})$) $\pi_\phi(x_1) \in Z(\pi_\phi(\mathcal{B}))$ which implies $\pi_\phi(x_1) = \lambda 1\!\!1$ for some λ. But then $y = x_1 - \lambda 1\!\!1 \in \mathcal{N}(\pi_\phi)$, the kernel of π_ϕ which is a closed ideal in $\mathcal{B}$. Now $y = x_1 - \lambda 1\!\!1 = y_+ - y_-$ and by letting $\sqrt{y_+} = a_1$ and $\sqrt{y_-} = a_2$, we obtain $x_1 - \lambda 1\!\!1 = a_1^2 - a_2^2$ with $a_1, a_2 \in \mathcal{N}(\pi_\phi)$ (because $\sqrt{y_\pm} \in \mathcal{N}(\pi_\phi)$). Then

$$\delta(x_1) = \delta(x_1 - \lambda 1\!\!1) = \delta(a_1)a_1 + a_1\delta(a_1) - \delta(a_2)a_2 - a_2\delta(a_2)$$

and $\delta(a_1), \delta(a_2) \in \mathcal{B}$. From the Cauchy-Schwartz inequality we get

$$\begin{aligned} |\phi(\delta(x_1))| &\leq |\phi(\delta(a_1)a_1)| + |\phi(a_1\delta(a_1))| + |\phi(\delta(a_2)a_2)| + |\phi(a_2\delta(a_2))| \\ &\leq \phi(\delta(a_1)^*\delta(a_1))^{\frac{1}{2}}\phi({a_1}^2)^{\frac{1}{2}} + \phi(\delta(a_2)^*\delta(a_2))^{\frac{1}{2}}\phi({a_2}^2)^{\frac{1}{2}} \\ &\quad + \phi(\delta(a_1)\delta(a_1)^*)^{\frac{1}{2}}\phi({a_1}^2)^{\frac{1}{2}} + \phi(\delta(a_2)\delta(a_2)^*)^{\frac{1}{2}}\phi({a_2}^2)^{\frac{1}{2}} \\ &= 0 \end{aligned}$$

so that $\delta(x_1) \in \mathcal{N}(\phi)$. Since this holds for arbitrary ϕ, we have $\delta(x_1) = 0$. Similarly $\delta(x_2) = 0$ so that $\delta(x) = 0$.

Theorem 46 (Singer, cf. (Sakai, S. [22], Corollary 2.2.8)) *Let $\mathcal{A}$ be a commutative $\mathcal{C}^*$-algebra and let δ be a bounded everywhere defined derivation on $\mathcal{A}$. Then $\delta = 0$.*

Proof: Since $\mathcal{A}$ is commutative, $xx^* = x^*x$ and

$$[\delta(x), x] = \delta(x)x - x\delta(x) = 0$$

for every $x \in \mathcal{A}$. Then by Theorem 45 we have that $\delta(x) = 0$ for every $x \in \mathcal{A}$.

Theorem 47 (Putnam, cf. (Sakai, S. [22], Corollary 2.2.9)) *Let T be a bounded normal operator on a Hilbert space $\mathcal{H}$ and let S be a bounded operator on $\mathcal{H}$. If $[T, [T, S]] = 0$ then $[T, S] = 0$.*

Note that (by expansion) $[T, [T, S]] = [[S, T], T]$.

Proof: Let $B(\mathcal{H})$ be the $\mathcal{C}^*$- algebra of all bounded operators on $\mathcal{H}$. Then $T, S \in B(\mathcal{H})$. Put $\delta_S(X) = [S, X]$ where $X \in B(\mathcal{H})$. Now

$$\begin{aligned} \delta_S(T) &= [S, T] \\ &= ST - TS \end{aligned}$$

and therefore

$$\begin{aligned} [\delta_S(T), T] &= \delta_S(T)T - T\delta_S(T) \\ &= [S, T]T - T[S, T] \\ &= [[S, T], T] \\ &= -[T, [S, T]] \end{aligned}$$

Now if $[[S, T], T] = 0$ then $[\delta_S(T), T] = 0$ and by Theorem 45, since T is a normal element of the $\mathcal{C}^*$- algebra $B(\mathcal{H})$, we have that $\delta_S(T) = [S, T] = 0$.

Corollary 2 (Putnam, cf. (Sakai, S. [22], Corollary 2.2.10)) *Let $A \in B(\mathcal{H})$ and suppose $[A, [A^*, A]] = 0$. Then $[A^*, A] = 0$ -i.e. A is normal.*

Proof: Put $\delta_A(X) = [A, X]$ with $X \in \mathcal{A}$. Now consider

$$\begin{aligned}
\delta_A^2(A^*) &= \delta_A([A, A^*]) \\
&= [A, [A, A^*]] \\
&= [A, (AA^* - A^*A)] \\
&= [A, -(A^*A - AA^*)] \\
&= [A, -[A^*, A]] \\
&= -[A[A^*, A] - [A^*, A]A] \\
&= -[A, [A^*, A]] \\
&= 0
\end{aligned}$$

Then by Theorem 41, $\delta_A(A^*)$ is a generalized nilpotent. Thus

$$r(AA^* - A^*A) = \lim_{n\to\infty} ||(AA^* - A^*A)^n||^{\frac{1}{n}} = 0$$

Now since $(AA^* - A^*A)^* = (AA^*)^* - (A^*A)^* = AA^* - A^*A$, it follows that $AA^* - A^*A$ is self adjoint and hence normal. Therefore

$$||AA^* - A^*A|| = r(AA^* - A^*A) = 0$$

which leads to $AA^* - A^*A = 0$.

2.3 Differentiability

After this brief look at derivations as commutators, we now investigate the differentiation properties inherent in derivations. The conclusion from this section is that if a process of differentiation behaves as intuitively expected and is everywhere defined and bounded, then it is trivial ($\delta = 0$). Therefore, any reasonable non-trivial process of differentiation must admit elements which are not differentiable. The *Singer-Wermer* theorem is the main result.

The commutation relation $[a, \delta(a)] = a\delta(a) - \delta(a)a = 0$ for every a, holds if and only if $\delta(p(a)) = p'(a)\delta(a)$ for every a and p a polynomial in a. This intuitive *chain rule* can therefore be translated into the requirement $[a, \delta(a)] = 0$.

First some commutative Banach algebra basics:

- Every maximal ideal $\mathcal{M}$ of a commutative Banach algebra $\mathcal{A}$ is the kernel of some homomorphism of $\mathcal{A}$ onto the complex plane (multiplicative linear functionals).

- For $x \in \mathcal{A}$ we have $\lambda \in \sigma(x)$ if and only if $h(x) = \lambda$ for some $h \in \mathcal{M}_{\mathcal{A}}$
- From these two results and the spectral radius formula, $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = 0$ implies that x is in the radical of $\mathcal{A}$ (the intersection of all maximal ideals of $\mathcal{A}$).

The notation $rad\mathcal{A}$ refers to the radical of a Banach algebra $\mathcal{A}$. A Banach algebra is said to be semi-simple if $rad\mathcal{A} = \{0\}$.

The Singer-Wermer theorem states that bounded derivations on semi-simple commutative algebras are trivial:

Theorem 48 (Singer-Wermer, cf. (Sakai, S. [22], Corollary 2.2.3)) *If $\mathcal{A}$ is a commutative Banach algebra and δ a bounded derivation on $\mathcal{A}$, then $\delta(\mathcal{A}) \subset rad\mathcal{A}$.*

Proof: Sakai, S. [22], 20.

The following two results derive a local version of the Singer-Wermer theorem for non-commutative Banach algebras. The notation $Q(\mathcal{A})$ denotes the set of all quasi-nilpotent elements of $\mathcal{A}$: $Q(\mathcal{A}) = \{x \in \mathcal{A} | \lim_n \|x^n\|^{\frac{1}{n}} = 0\}$.

Theorem 49 (Mathieu, M. and Murphy, G.J. [13], Theorem 2.2) *Let δ be a bounded derivation on a Banach algebra $\mathcal{A}$. If $[a, \delta(a)] = 0 \,\forall\, a \in \mathcal{A}$ then $\delta(\mathcal{A}) \subset Q(\mathcal{A})$.*

Proof: Let $\mathcal{B} = B(\mathcal{A})$ be the Banach algebra of all bounded linear operators on $\mathcal{A}$. If $\delta \in \mathcal{B}$ is a derivation, then it can be extended uniquely to a bounded derivation on the unitization of $\mathcal{A}$ by setting $\delta(\mathbb{1}) = 0$. Now $\mathcal{A}$ may be considered as a closed subalgebra of $\mathcal{B}$ by means of the representation $a \mapsto L_a$ defined by $L_a(b) = ab$. Under this identification δ becomes an inner derivation as $L_{\delta(a)} = -[L_a, \delta]$.
Consider

$$\begin{aligned} \delta^2_{L_a}(\delta) = [L_a, [L_a, \delta]] &= -[L_a, L_{\delta(a)}] \\ &= -L_{[a, \delta(a)]} \\ &= 0 \end{aligned}$$

because $[a, \delta(a)] = 0$ is given. By the spectral mapping theorem we have $r(\delta(a)) = r(L_{\delta(a)})$. It follows from Theorem 41 that $r(L_{\delta(a)}) = r(\delta(a)) = 0$, or equivalently $(\|\delta(a)^n\|)^{\frac{1}{n}} \to 0$ as $n \to \infty$.

This result will now be used to somewhat relax the restrictions in the statement of the Singer-Wermer theorem.

Theorem 50 (Mathieu, M. and Murphy, G.J. [13], Theorem 3.6) *Let $\mathcal{A}$ be a Banach algebra and δ a derivation on $\mathcal{A}$. Suppose $\delta(\mathcal{A}) \subset Q(\mathcal{A})$. Then $\delta(\mathcal{A})$ is contained in the radical of $\mathcal{A}$.*

Proof: The following facts are used to reduce the proof to the invertible elements of $\mathcal{A}$: Obviously $\delta(\mathbb{1}) = 0$ and therefore for $\lambda > ||a||$ we have $\delta(a + \lambda\mathbb{1}) = \delta(a)$ for every $a \in \mathcal{A}$. Thus $\delta(\mathcal{A}) = \delta(\mathcal{A} \cap \mathcal{A}^{-1})$ ($(a + \lambda\mathbb{1})$ is invertible in $\mathcal{A}$ since $\lambda \notin \sigma(a)$).
For $\mathcal{X}$ a Banach space, consider the homomorphism $\pi : \mathcal{A} \to B(\mathcal{X})$ defined by $\pi(a)(x) = a(x)$ for every $a \in \mathcal{A}$, $x \in \mathcal{X}$, which is an irreducible representation of $\mathcal{A}$ in $B(\mathcal{X})$. By composition of π and δ, we may define the map $\tilde{\delta} := \pi \circ \delta : \mathcal{A} \to B(\mathcal{X})$ by $\pi(\delta(a))(x) = \delta(a)(x)$. Now, for $a, b \in \mathcal{A}$

$$\begin{aligned} \tilde{\delta}(ab) &= \pi(\delta)(ab) \\ &= \pi(\delta(ab)) \\ &= \pi\big(\delta(a)b + a\delta(b)\big) \\ &= \pi(\delta(a))\pi(b) + \pi(a)\pi(\delta(b)) \\ &= \tilde{\delta}(a)\pi(b) + \pi(a)\tilde{\delta}(b) \end{aligned}$$

so that $\tilde{\delta}$ is a 'derivation-like' operator from $\mathcal{A}$ to $\pi(\mathcal{A})$.
It is given that $\delta(\mathcal{A}) \subset Q(\mathcal{A})$. Since π is a continuous homomorphism, it follows that

$$\begin{aligned} ||\pi(\delta(a))^n||^{\frac{1}{2}} &= ||\pi(\delta(a)^n)||^{\frac{1}{n}} \\ &\leq ||\pi||^{\frac{1}{n}}||\delta(a)^n||^{\frac{1}{n}} \\ &\to_n^\infty 0 \end{aligned}$$

which shows that $\tilde{\delta}(\mathcal{A}) = \pi(\delta(\mathcal{A})) \subseteq Q(\pi(\mathcal{A}))$ for every irreducible $\pi : \mathcal{A} \to B(\mathcal{X})$. Therefore, showing $\tilde{\delta}(\mathcal{A}) = \pi(\delta(\mathcal{A})) = \{0\}$ for every irreducible representation π of $\mathcal{A}$, is the same as showing $\delta(\mathcal{A}) \subset rad(\mathcal{A})$. The intersection of the kernels of all irreducible representations π is contained in the radical of $\mathcal{A}$. We may therefore assume that $\mathcal{A}$ is unital and that it acts irreducibly on a Banach space $\mathcal{X}$.
The proof is conducted in two steps: Step one shows that for any $x \in \mathcal{X}$, ax and $\delta(a)x$ are linearly dependent; and step two uses this fact to show that $\pi(\delta(a)) = 0$ for the invertible elements a in $\mathcal{A}$, and for every irreducible π.

Step one: We want to show that $\pi(a)(x)$ and $\pi(\delta(a))(x)$ are linearly dependent. Take $a \in \mathcal{A}$ and $x \in \mathcal{X}$ such that $\pi(a)(x) = 0$. Then $\pi(\delta(a))(x) = 0$, because if not, then since π is irreducible, there exist $b \in \mathcal{A}$ such that $\pi(b(\delta(a)))(x) = x$. Hence $\pi(\delta(ba))(x) = \pi(b\delta(a))(x) + \pi(\delta(b)a)(x) = x$ so that 1 is an eigenvalue of $\delta(ba)$, which contradicts $r(\delta(ba)) = 0$. Thus if $\pi(a)(x) = 0$ then $\pi(\delta(a))(x) = 0$.

Consider for any $x \in \mathcal{X}$ and $a \in \mathcal{A}$ the vectors $a(x)$ and $\delta(a)(x)$. We claim that there exists $\lambda(x) \in \mathbb{C}$ such that $\delta(a)(x) = \lambda(x)a(x)$. For if not, then by the Jacobson density theorem (Palmer, T.W. [15], 465), there exists $b \in \mathcal{A}$ such that $ba(x) = 0$ and $b\delta(a)(x) = -a(x)$. Then $(\delta(b)a)(x) = (\delta(ba) - b\delta(a))(x)$. But from the previous paragraph we have that $\delta(ba)(x) = 0$. Thus $(\delta(b)a)(x) = a(x)$ so that 1 is an eigenvalue of $\delta(b)$, which contradicts nil-potency.

Step two: Take $a \in \mathcal{A}$ invertible, then for every $x \in \mathcal{X}$ we showed that $\pi(\delta(a))(x) = \lambda(x)\pi(a)(x)$ for some $\lambda(x) \in \mathbb{C}$. We need to show that λ is independent of x.
Choose any linearly independent $x, y \in \mathcal{X}$ - i.e if $\alpha x + \beta y = 0$ then $\alpha = \beta = 0$ for $\alpha, \beta \in \mathbb{C}$. Now

$$\begin{aligned} \lambda(x+y)(x+y) &= \pi(a)^{-1}\pi(\delta(a))(x+y) \\ &= \pi(a)^{-1}\pi(\delta(a))(x) + \pi(a)^{-1}\pi(\delta(a))(y) \\ &= \lambda(x)x + \lambda(y)y \end{aligned}$$

so that

$$(\lambda(x+y) - \lambda(x))x = (\lambda(y) - \lambda(x+y))y$$

From the linear independence of x and y it follows that $\lambda(x) = \lambda(x+y) = \lambda(y)$. Therefore

$$\pi(a)^{-1}\pi(\delta(a)) = \lambda \mathbb{1} \tag{5}$$

Now if $\lambda \neq 0$, then

$$\begin{aligned} r(\pi(a)^{-1}\pi(\delta(a))) &= \lim_n \|(\pi(a)^{-1}\pi(\delta(a)))^n\|^{\frac{1}{n}} \\ &\geq |\lambda| > 0 \end{aligned}$$

It is given that $\delta(\mathcal{A}) \subset Q(\mathcal{A})$ (for all $a \in \mathcal{A}$, $\|\delta(a)^n\|^{\frac{1}{n}} \to 0$). It follows from the commutativity of $\pi(a)^{-1}$ and $\pi(\delta(a))$ (equation (5)) and the spectral radius theorem that

$$\begin{aligned} r(\pi(a)^{-1}\pi(\delta(a))) &\leq r(\pi(a)^{-1})r(\pi(\delta(a))) \\ &= \lim_{n\to\infty} \|\pi(a)^{-n}\|^{\frac{1}{n}} \lim_{n\to\infty} \|\pi(\delta(a))^n\|^{\frac{1}{n}} \\ &= 0 \end{aligned}$$

which contradicts $\lambda(x) \neq 0$. Therefore $\pi(\delta(a))(x) = 0$ (for every x) whenever a is invertible.

Remark 10 *Let $\mathcal{A}$ be a Banach algebra and suppose that for every $a \in \mathcal{A}$, $[a, \delta(a)] \in rad(\mathcal{A})$ would imply $\delta(\mathcal{A}) \subset Q(\mathcal{A})$ without requiring δ to be bounded. Then, from the previous result, an unbounded version of the Singer-Wermer theorem would be proved.*

2.4 Boundedness

This study of bounded derivations is limited (as will become evident later) and the focus of this work is on unbounded derivations. This section briefly states when derivations can be expected to be bounded.

Theorem 51 (Sakai, S. [22], Theorem 2.3.1) *Let $\mathcal{A}$ be a C^* algebra and δ a derivation on $\mathcal{A}$. Then δ is bounded.*

Proof: It can be assumed that $\mathcal{A}$ has an identity because if not, then an identity can be added to $\mathcal{A}$ to form the new algebra $\tilde{\mathcal{A}}$. Define on $\tilde{\mathcal{A}}$ the derivation $\tilde{\delta}$ with $\tilde{\delta}(1) = 0$ and let $\delta^*(x) = \delta(x^*)^*$ where $x \in \mathcal{A}$. Clearly δ^* is also a derivation. Then $\delta = \frac{\delta+\delta^*}{2} + i\frac{i\delta^*-i\delta}{2}$, which shows that every derivation is a unique combination of *-derivations. Henceforth we may assume δ to be a $*$- derivation.
Take $x \in \mathcal{A}_{sa}$ - i.e $x = x^*$. Let ϕ be a state on $\mathcal{A}$ such that $|\phi(x)| = ||x||$. We shall see that $\phi(\delta(x)) = 0$ for $x \in \mathcal{A}_{sa}$. (From $|\phi(x)| = ||x||$, we may assume that $\phi(x) = ||x||$, otherwise consider $-x$ instead of x.)

Put $||x||\mathbb{1} - x = h^2$ with ($h \geq 0$, $h \in \mathcal{A}$). Then

$$\begin{aligned} \phi(h^2) &= \phi(||x||\mathbb{1} - x) \\ &= ||x||\phi(\mathbb{1}) - \phi(x) \\ &= ||x|| - ||x|| = 0 \end{aligned}$$

From

$$\begin{aligned} \delta(||x||\mathbb{1} - x) &= \delta(||x||\mathbb{1}) - \delta(x) \\ &= ||x||\delta(\mathbb{1}) - \delta(x) \\ &= -\delta(x) \end{aligned}$$

we have

$$
\begin{aligned}
|-\phi(\delta(x))| &= |\phi(\delta(\|x\|1\!\!1 - x))| \\
&= |\phi(\delta(h^2))| \\
&= |\phi(h\delta(h)) + \phi(\delta(h)h)| \\
&\leq |\phi(h\delta(h))| + |\phi(\delta(h)h)| \\
&= \phi(h^2)^{\frac{1}{2}}\phi(\delta(h)^2)^{\frac{1}{2}} + \phi(\delta(h)^2)^{\frac{1}{2}}\phi(h^2)^{\frac{1}{2}} = 0 \qquad (6)
\end{aligned}
$$

by the Cauchy-Schwarz inequality. Hence $\phi(\delta(x)) = 0$.

We want to show that δ is a closed (or closable) linear operator. Then since $\mathcal{D}(\delta)$ $(= \mathcal{A})$ is complete, it will follow from the closed graph theorem (Theorem 33) that δ is bounded.

Therefore suppose that $x_n = x_n^* \to 0$ and $\delta(x_n) \to y$. y is self adjoint since δ is a $*$-derivation. From the discussion following Theorem 29 we need to show that $y = 0$.
Let $\phi_n \in \mathcal{S}(\mathcal{A})$ such that $|\phi_n(y+x_n)| = \|y+x_n\|$. Then by equation (6), $|\phi_n(\delta(y+x_n))| = 0$. Let ϕ_o be an accumulation point of (ϕ_n) in the *weak**- topology of $\mathcal{S}(\mathcal{A})$ (compact). Now

$$
\begin{aligned}
|\phi_{n_j}(y + x_{n_j}) - \phi_0(y)| &= |\phi_{n_j}(y + x_{n_j}) - \phi_{n_j}(y) + \phi_{n_j}(y) - \phi_0(y)| \\
&\leq |\phi_{n_j}(y + x_{n_j}) - \phi_{n_j}(y)| + |\phi_{n_j}(y) - \phi_0(y))| \\
&= |\phi_{n_j}(x_{n_j})| + |\phi_{n_j}(y) - \phi_0(y))| \\
&\leq \|x_{n_j}\| + |\phi_{n_j}(y) - \phi_0(y)| \to 0
\end{aligned}
$$

for some subsequence (n_j) of (n).

Therefore $|\phi_0(y)| = \|y\|$ and from equation (6) we have $\phi_0(\delta(y)) = 0$. But $0 = \phi_{n_j}(\delta(y + x_{n_j})) = \phi_{n_j}((\delta(y) + \delta(x_{n_j})) \to \phi_0(\delta(y) + y) = \phi_0(\delta(y)) + \phi_0(y)$ which leads to $\|y\| = \phi_0(y) = 0$ so that $y = 0$. Therefore δ is a closed linear operator and since $\mathcal{D}(\delta) = \mathcal{A}$, by the closed graph theorem (Theorem 33) we have that δ is bounded.

Theorem 52 (Johnson-Sinclair, cf. (Sakai, S. [22], Theorem 2.3.2)) *Let $\mathcal{A}$ be a semi-simple Banach algebra and δ a derivation on $\mathcal{A}$. Then δ is continuous.*

Proof: Sakai, S. [22], 23.

2.5 Derivation Theorem

The derivation theorem is well documented in Sakai, S. [22], Section 2.5 and Bratteli, O and Robinson, D.W. [3], Corollary 3.2.47. The version proved here was obtained from Bratteli, O and Robinson, D.W. [3], Corollary 3.2.47, slightly relaxed by not requiring the result to include $||h|| \leq ||\delta||/2$, where h is the element in $\mathcal{M}$ given below.

Theorem 53 (Bratteli, O. and Robinson, D.W. [3], Corollary 3.2.47) *Let δ be an everywhere defined, bounded, symmetric derivation of a Von Neumann algebra $\mathcal{M}$. Then there exists a self-adjoint $h = h^* \in \mathcal{M}$ such that $\delta(x) = i[h, x]$ for every $x \in \mathcal{D}(\delta) = \mathcal{M}$.*

Proof: The proof of the derivation theorem depends on results in the next section and is therefore deferred to after Corollary 3 in Section 3.

3 Derivations as generators

Derivations arise as generators for various operator groups. This section provides some basic semi-group theory required for the understanding of derivations as generators, and the well known *Lumer-Phillips* theorems for derivations are only arrived at after the introduction of *well-behavedness.*

3.1 Semi groups

Definition 17 *A one parameter family $\Gamma(t)$ of bounded linear operators from a Banach space $\mathcal{A}$ into $\mathcal{A}$ is a semi-group of bounded linear operators on $\mathcal{A}$ if:*

$$\Gamma(0) = I$$

$$\Gamma(t + s) = \Gamma(t)\Gamma(s) \text{ for every } t,\, s \geq 0$$

The linear operator A defined by

$$\begin{aligned} \mathcal{D}(A) &= \{x \in \mathcal{A} |\lim_{t \downarrow 0} \frac{\Gamma(t)x - x}{t} \text{ exists}\}, \text{ and} \\ Ax = \lim_{t \downarrow 0} \frac{\Gamma(t)x - x}{t} &= \frac{d^+\Gamma(t)x}{dt}\Big|_{t=0} \quad \text{for } x \in \mathcal{D}(A) \end{aligned}$$

is the infinitesimal generator of the semi-group $\Gamma(t)$ and $\mathcal{D}(A)$ the domain of A.

3.1.1 Uniform continuity

A semi-group of bounded linear operators is *uniformly continuous* (continuity at I) if

$$\lim_{t \downarrow 0} ||\Gamma(t) - I|| = 0$$

Characterization of the generators and conditions that will ensure a linear operator to be a generator is important and in the case of uniform continuity, the answer is quite simple:

Theorem 54 (Pazy, A. [16], Theorem 1.2) *A linear operator A is the infinitesimal generator of a uniformly continuous semi-group if and only if A is a bounded linear operator.*

Proof: Pazy, A. [16], 2.

Thus, everywhere defined derivations will always be a generator of a uniformly continuous semi-group. It is easy to verify uniqueness of the generator, and uniqueness of the semi-group can be shown. By uniqueness of the semi-group we mean that if two semi-groups have the same infinitesimal generator, then they must agree.

Theorem 55 (Pazy, A. [16], Theorem 1.4) *If $\Gamma(t)$ is a uniformly continuous semi-group of bounded linear operators, then there exists a unique bounded linear operator A such that $\Gamma(t) = e^{tA}$ and A is the infinitesimal generator of $\Gamma(t)$.*

Proof: From Theorem 54 we know that the generator A of $\Gamma(t)$ is bounded, and from the (norm converging) series expansion

$$e^{tA} = \sum_{n=0}^{n=\infty} \frac{(tA)^n}{n!}$$

it follows that A is also the generator of e^{tA} so that by the uniqueness of the semi-group we have $\Gamma(t) = e^{tA}$.

From Theorem 51 (Bounded derivations) and Theorem 54, we have the following corollary for derivations:

Corollary 3 (Uniform continuous generators) *Let $\mathcal{A}$ be a C^*- algebra. Then a linear operator δ defined on $\mathcal{A}$ is an everywhere defined symmetric derivation of $\mathcal{A}$ if and only if δ is the generator of a norm-continuous one-parameter group of $\star$- automorphisms of $\mathcal{A}$.*

Proof: The proof of this result is a simplification of the proof of Theorem 68 and will therefore not be expounded in any detail.

The derivation theorem (Theorem 53) from the previous section can now be proved:

Proof of Theorem 53 : Since δ is a bounded $*$ - derivation, it follows from Corollary 3 that δ is the infinitesimal generator of a uniformly continuous group of $*$ - automorphisms on $\mathcal{M}$. Let α_t, $t\,(\geq 0) \in \mathbb{R}$ be the group of $*$ - automorphisms on $\mathcal{M}$. Then, by Theorem 55

$$\begin{aligned} \alpha_t &= e^{t\delta} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!}\delta^k \end{aligned}$$

From (Pederson, G.K. [17], 324), the uniform continuity of α_t assures the existence of a uniformly continuous unitary group $\{u_t\} \subseteq \mathcal{M}$ such that

$$\alpha_t(x) = u_t x u_t^*$$

for every $x \in \mathcal{M}$. Let A be the infinitesimal generator of the unitary group u_t. Then from Stone's theorem (Pazy, A. [16], 41), it follows that $h = h^*$ where $A = ih$ (since A is skew adjoint) so that we can write

$$u_t = e^{iht} \qquad \forall t$$

Uniform continuity of u_t implies (from Theorem 54) that h is bounded and

$$\begin{aligned} \frac{1}{t}(u_t - 1\!\!1) &= \frac{1}{t}(e^{ith} - 1\!\!1) \\ &\to ih \end{aligned}$$

in the strong operator topology on $\mathcal{M}$. Since $u_t - 1\!\!1 \in \mathcal{M}\ \forall t$ and $\mathcal{M}$ is strong-operator closed, it follows that $h \in \mathcal{M}$. We now have

$$\begin{aligned} e^{t\delta}(x) &= u_t x u_t^* \\ &= e^{ith} x e^{-ith} \end{aligned}$$

and with differentiation at $t = 0$:

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} e^{t\delta}(x) &= \delta e^{t\delta}(x)\Big|_{t=0} \\ &= \delta(x) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} e^{ith} x e^{-ith} &= (ihe^{ith} x e^{-ith} - ihe^{ith} x e^{-ith})\Big|_{t=0} \\ &= ihx - ixh \\ &= i[h, x] \end{aligned}$$

so that $\delta(x) = i[h, x]$.

Since we are more interested in unbounded (closed and densely defined) derivations, uniform continuity is too general and we therefore need to look at strongly continuous semi-groups.

3.1.2 Strong continuity

A semi-group of bounded linear operators on a Banach space $\mathcal{A}$ is *strongly continuous* ($\mathcal{C}_0$) if

$$\lim_{t \downarrow 0} \Gamma(t)x = x \qquad \forall x \in \mathcal{A}$$

Theorem 56 (Pazy, A. [16], Theorem 2.2) *Let $\Gamma(t)$, $0 \le t < \infty$, be a $\mathcal{C}_0$ semi-group of bounded linear operators on a Banach space $\mathcal{A}$. There exists constants $\omega \ge 0$ and $M \ge 1$ such that*

$$||\Gamma(t)|| \le Me^{\omega t} \qquad \text{for } 0 \le t < \infty \tag{7}$$

Proof: Assume that there exists a $\eta > 0$ such that $||\Gamma(t)||$ is bounded for $0 \le t \le \eta$. If not, there is a sequence t_n with $t_n \ge 0$, $\lim_{n\to\infty} t_n = 0$ and $||\Gamma(t_n)|| \ge n$. From uniform boundedness it follows that for some $x \in \mathcal{A}$, $||\Gamma(t_n)x||$ is unbounded, which contradicts the definition of $\mathcal{C}_0$- continuity. Therefore, $||\Gamma(t)|| \le M$ for $0 \le t \le \eta$. Since $||\Gamma(0)|| = 1$, $M \ge 1$. Let $\omega = \frac{1}{\eta}\ln(M) \ge 0$. Given $t \ge 0$ we have $t = n\eta + \delta$ where $0 \le \delta < \eta$ and therefore by the semi-group property,

$$||\Gamma(t)|| = ||\Gamma(\delta)\Gamma(\eta)^n|| \le M^{n+1} < Me^{\omega t}$$

Characterization of the generator of a $\mathcal{C}_0$- semi-group is not so obvious and the Hille-Yosida or Lumer-Philips theories needs to be developed to present a full account of the generator character. The following 'one-way' result is however useful and sufficient for the construction of the counter example in Section 4.

Theorem 57 (Pazy, A. [16], Corollary 2.5) *If A is the infinitesimal generator of a $\mathcal{C}_0$- semi-group $\Gamma(t)$ then the domain $\mathcal{D}(A)$ is dense in $\mathcal{A}$ and A is a closed linear operator.*

Proof: Pazy, A. [16], 5.

The 'closed, densely defined' character allows for unbounded derivations to be infinitesimal generators of $\mathcal{C}_0$- semi-groups.

The proof of Theorem 57 requires the following standard semi-group result. $\mathcal{A}$ is a Banach space.

Theorem 58 (Pazy, A. [16], Theorem 2.4) *Let $\Gamma(t)$ be a $\mathcal{C}_0$- semi-group on $\mathcal{A}$ and A its infinitesimal generator. Then:*

(i) For $x \in \mathcal{A}$,

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} \Gamma(s)x ds = \Gamma(t)x$$

with the convergence in norm $\forall x$.

(ii) For $x \in \mathcal{A}$, $\frac{1}{t}\int_0^t \Gamma(s)xds \in \mathcal{D}(A)$ *and*

$$A(\int_0^t \Gamma(s)xds) = \Gamma(t)x - x$$

(iii) For $x \in \mathcal{D}(A)$, $\Gamma(t)x \in \mathcal{D}(A)$ *and*

$$\frac{d}{dt}\Gamma(t)x = A\Gamma(t)x = \Gamma(t)Ax$$

(iv) For $x \in \mathcal{D}(A)$,

$$\Gamma(t)x - \Gamma(s)x = \int_s^t \Gamma(\gamma)Axd\gamma = \int_s^t A\Gamma(\gamma)xd\gamma$$

Proof of Theorem 57:

- (Denseness) Set for every $x \in \mathcal{A}$, $x_t = \frac{1}{t}\int_0^t \Gamma(s)xds$. Since $x_t \in \mathcal{D}(A)$ and $x_t \to x$ as $t \downarrow 0$, $\mathcal{D}(A)$ is dense in $\mathcal{A}$.

- (Closedness) Let $x_n \in \mathcal{D}(A)$, $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. From (iv) in Theorem 58

$$\begin{aligned} \Gamma(t)x_n - \Gamma(0)x_n &= \int_0^t \Gamma(\gamma)Ax_nd\gamma. \\ \textit{Let } n \to \infty: \qquad \Gamma(t)x - x &= \int_0^t \Gamma(\gamma)yd\gamma \end{aligned}$$

  Since $Ax_n \to y$ in norm and $\Gamma(\gamma)$ is continuous for every $\gamma \geq 0$, we may write $\lim_n \int_0^t \Gamma(\gamma)Ax_nd\gamma = \int_0^t \Gamma(\gamma)\lim_n Ax_nd\gamma$ to obtain the above. Then

$$\begin{aligned} \frac{\Gamma(t)x - x}{t} &= \frac{1}{t}\int_0^t \Gamma(\gamma)yd\gamma. \\ \textit{Let } t \downarrow 0: \qquad Ax &= y. \end{aligned}$$

  So $x \in \mathcal{D}(A)$ from (i) in Theorem 58 and the definition of the semi-group generator.

As mentioned previously, more semi-group theory is required for proper generator characterization. More applicable to derivations would be the development of the well-behavedness of derivations.

3.2 Well-behavedness

Before we can continue with the study of derivations as generators, the notion of *well-behavedness* needs to be introduced. Well-behavedness properties are also relevant in minima-maxima problems.

3.2.1 Definitions

Definition 18 (Well-behaved element) *Let δ be a $\star$- derivation in a $\mathcal{C}^\star$- algebra $\mathcal{A}$. Then $x = x^\star \in \mathcal{D}(\delta)$ is* well-behaved *with respect to δ if there is a state ϕ_x on $\mathcal{A}$ such that $|\phi_x(x)| = ||x||$ and $\phi_x(\delta(x)) = 0$.*

The existence of $\phi_x \in \mathcal{S}(\mathcal{A})$ with $|\phi_x(x)| = ||x||$ is guaranteed by $x = x^\star$ (or $||x|| = r(x)$) and the fact that the compactness of $\sigma(x)$ implies the existence of a $\lambda \in \sigma(x)$ such that $||x|| = |\lambda|$ and the fact that for every $\lambda \in \sigma(x)$ there is a $\phi \in \mathcal{S}(\mathcal{A})$ such that $\phi(x) = \lambda$. Well-behavedness therefore intuitively refers to the behavior of the derivation at a minimum / maximum.

Definition 19 (Well-behaved derivation) *A $\star$- derivation δ on a $\mathcal{C}^*$- algebra $\mathcal{A}$ is* well-behaved *if every $x = x^\star \in \mathcal{D}(\delta)$ is well-behaved with respect to δ. (δ is then also called* conservative *and both $-\delta$ and $+\delta$ are dissipative).*

Let $\mathcal{C}_0(\mathbb{R})$ be the continuous functions on $\mathbb{R}$ (complex valued) which are vanishing at $\pm$ infinity, and let $\mathcal{C}_0^1(\mathbb{R})$ be the subset of elements in $\mathcal{C}_0(\mathbb{R})$ that are once continuously differentiable. If $\mathcal{A} = \mathcal{C}_0(\mathbb{R})$ and $\delta = \frac{d}{dt}$ with $\mathcal{D}(\delta) = \mathcal{C}_0^1(\mathbb{R})$, then δ is well-behaved.

Definition 20 (Quasi well-behaved derivation) *A $\star$- derivation is* quasi well-behaved *if the self-adjoint portion in $\mathcal{D}(\delta)$ has a dense open subset of well-behaved elements.*

If $\mathcal{A} = \mathcal{C}([0, 1])$ and $\delta = \frac{d}{dt}$ with $\mathcal{D}(\delta) = \mathcal{C}^1([0, 1])$, then δ is quasi well-behaved. This follows from the observation that, for $f \in \mathcal{C}^1[0, 1]$ to be well-behaved, it must attain its maximum or minimum on the interval $(0, 1)$. Such functions are dense in $\mathcal{C}^1[0, 1]$.

3.2.2 General results

The first result stated connects closability of a derivation to the well-behavedness property. The proof of Theorem 51 can be simplified using this result:

Theorem 59 (Sakai, S. [22], Theorem 3.2.9) *If a *-derivation δ in a C^*-algebra $\mathcal{A}$ is quasi-well-behaved, then δ is closable and its closure $\overline{\delta}$ is again quasi-well-behaved.*

Proof: Sakai, S. [22], 60.

Theorem 60 (Sakai, S. [22], Corollary 3.2.10) *If δ is well-behaved, then $\overline{\delta}$ is also well-behaved.*

Proof: Sakai, S. [22], 61.

Theorem 61 (Sakai, S. [22], Proposition 3.2.12) *Let δ be a *-derivation in a C^*-algebra $\mathcal{A}$ with identity. Then δ is well-behaved if and only if for every $x \in \mathcal{D}(\delta)^+$ (positive) there is a state $\phi_x \in \mathcal{S}(\mathcal{A})$ such that $\phi_x(x) = ||x||$ and $\phi_x(\delta(x)) = 0$.*

Proof:($\Rightarrow$) Positive elements $x \in \mathcal{D}(\delta)^+$ are self-adjoint and the result follows from the definition of well-behavedness.
($\Leftarrow$) Let y be a self adjoint element of $\mathcal{D}(\delta)$ and $y = y_+ - y_-$ be the orthogonal decomposition of y. Then y_+ and y_- are positive (and self-adjoint), $y_+y_- = y_-y_+ = 0$ and $||y|| = \max(||y_+||, ||y_-||)$. From the self-adjointness of y it follows that $\sigma(||y||\mathbb{1} \pm y) \in \mathbb{R}^+$ so that the elements $||y||\mathbb{1} \pm y$ are positive. By assumption, there exist states $\phi_{||y||\mathbb{1}\pm y}$ with $\phi_{||y||\mathbb{1}\pm y}(||y||\mathbb{1} \pm y) = ||||y||\mathbb{1} \pm y||$ and $\phi_{||y||\mathbb{1}\pm y}(\delta(||y||\mathbb{1} \pm y)) = \phi_{||y||\mathbb{1}\pm y}(\delta(y)) = 0$.

If $||y|| = ||y_+||$ then consider

$$\phi_{||y||\mathbb{1}+y}(||y||\mathbb{1} + y) = ||||y||\mathbb{1} + y_+ - y_-||$$

By the continuous functional calculus we can assume that y_+ and y_- are real valued functions, so that the right hand side can be reduced to

$$\begin{aligned} \sup_{t \in supp(y_+)} |(||y_+|| + y_+ - y_-)(t)| &= 2||y_+|| \\ &= 2||y|| \end{aligned}$$

But,

$$\phi_{||y||\mathbb{1}+y}(||y||\mathbb{1} + y) = \phi_{||y||\mathbb{1}+y}(||y||\mathbb{1}) + \phi_{||y||\mathbb{1}+y}(y)$$

and

$$\phi_{||y||\mathbb{1}+y}(||y||\mathbb{1}) = ||y||$$

so that we get $\phi_{||y||\mathbb{1}+y}(y) = ||y||$. A similar argument shows that if $||y|| = ||y_-||$ then $\phi_{||y||\mathbb{1}-y}(y) = ||y||$.

The following three theorems establish the connection between well-behavedness and the requirements of a linear operator to be a generator of a strongly continuous group of automorphisms, to be used in the Lumer-Phillips results (see Theorem 68).

Theorem 62 (Sakai, S. [22], Proposition 3.2.17) *Let δ be a *-derivation in $\mathcal{A}$. An element $x \in \mathcal{D}(\delta)^+$ is well-behaved if and only if $||(\mathbb{1} + \lambda\delta)(x)|| \geq ||x||$ for all $\lambda \in \mathbb{R}$.*

Proof: ($\Rightarrow$) Suppose $x (= x^*)$ is well-behaved. Then (from Theorem 61)

$$||(\mathbb{1} + \lambda\delta)(x)|| \geq |\phi_x((\mathbb{1} + \lambda\delta)(x))| = |\phi_x(x)| = ||x||$$

($\Leftarrow$) Given $x > 0$ and $||(\mathbb{1} + \lambda\delta)(x)|| \geq ||x||$ for $\lambda \in \mathbb{R}$, define $\mathcal{B}$ as the subspace of $\mathcal{A}$ spanned by $\delta(x)$. Then, since $Re((\mathbb{1} + \lambda\delta)(x) + i\mu\delta(x)) = (\mathbb{1} + \lambda\delta)(x)$,

$$\begin{aligned} ||(\mathbb{1} + (\lambda + i\mu)\delta)(x)|| &= ||(\mathbb{1} + \lambda\delta)(x) + i\mu\delta(x)|| \\ &\geq ||(\mathbb{1} + \lambda\delta)(x)|| \\ &\geq ||x|| \end{aligned}$$

for all λ, $\mu \in \mathbb{R}$ so that $d(x, \mathcal{B}) \geq ||x|| > 0$. From Corollary 1 we can choose a linear functional ϕ on $\mathcal{A}$ with $||\phi|| = 1$, $\phi(x) = d(x, \mathcal{B})$ and $\phi(\mathcal{B}) = 0$. But $\phi(x) = d(x, \mathcal{B}) \leq ||x||$ so that $\phi(x) = ||x||$. Since $x \geq 0$ and $\phi(x) = ||\phi||||x||$, it follows from Theorem 12 that ϕ is bounded and positive. By the Hahn-Banach theorem (Theorem 11), ϕ can be extended to a bounded, positive linear functional on $\mathcal{A}$. ϕ is a state since $||\phi|| = 1$.

Theorem 63 (Sakai, S. [22], Proposition 3.2.18) *If a *-derivation δ in $\mathcal{A}$ with identity is well-behaved, then there is a $\phi \in \mathcal{S}(\mathcal{A})$ such that $\phi(\delta(x)) = 0$ for all $x \in \mathcal{D}(\delta)$.*

Proof: Since δ is well-behaved, we have that for every self-adjoint $x(= x^*) \in \mathcal{D}(\delta)$, there exists $\phi_x \in \mathcal{S}(\mathcal{A})$ such that $|\phi_x(x)| = ||x||$ and $\phi_x(\delta(x)) = 0$. Assume that for $x = x^* \in \mathcal{D}(\delta)$, $||\mathbb{1} - \delta(x)|| < 1$. Then

$$\begin{aligned} |\phi_x(\mathbb{1})| &= |\phi_x(\mathbb{1}) - \phi_x(\delta(x))| \\ &= |\phi_x(\mathbb{1} - \delta(x))| \\ &\leq ||\mathbb{1} - \delta(x)|| \\ &< 1 \end{aligned}$$

so that $\phi_x(\mathbb{1}) < 1$ which contradicts the fact that $\phi_x \in \mathcal{S}(\mathcal{A})$. Therefore $d(\mathbb{1}, \delta(\mathcal{A})) = \inf\{||\mathbb{1} - \delta(x)|||x = x^* \in \mathcal{D}(\delta)\} \geq 1$. From the proof of Theorem 62, we can choose a bounded linear functional ϕ on $\mathcal{A}$ with $||\phi|| = 1$, $\phi(\mathbb{1}) = 1$ and $\phi(\delta(a)) = 0$ for every $a \in \mathcal{D}(\delta)$.

Theorem 64 (Sakai, S. [22], Proposition 3.2.19) *Let δ be a *-derivation in $\mathcal{A}$ and suppose that δ is well-behaved; then $||(\mathbb{1} + \lambda\delta)(x)|| \geq ||x||$ for all $x \in \mathcal{D}(\delta)$ and $\lambda \in \mathbb{R}$.*

Proof: Take the state ϕ_{x^*x} as in the definition of well-behavedness. Then

$$\begin{aligned} \phi_{x^*x}((\mathbb{1} + \lambda\delta)(x^*)(\mathbb{1} + \lambda\delta)(x)) &= \phi_{x^*x}((x^* + \lambda\delta(x^*))(x + \lambda\delta(x))) \\ &= \phi_{x^*x}(x^*x + \lambda\delta(x^*)x + \lambda x^*\delta(x) + \lambda^2\delta(x^*)\delta(x)) \end{aligned}$$

But $\delta(x^*)x + x^*\delta(x) = \delta(x^*x)$, thus

$$\phi_{x^*x}((\mathbb{1} + \lambda\delta)(x^*)(\mathbb{1} + \lambda\delta)(x)) = \phi_{x^*x}(x^*x + \lambda\delta(x^*x) + \lambda^2\delta(x^*)\delta(x))$$

and since $\phi_{x^*x}(\delta(x^*)\delta(x)) = \phi_{x^*x}(|\delta(x)|^2) \geq 0$ we have

$$\begin{aligned}\phi_{x^*x}((\mathbb{1} + \lambda\delta)(x^*)(\mathbb{1} + \lambda\delta)(x)) &\geq \phi_{x^*x}(x^*x + \lambda\delta(x^*x)) \\ &= \phi_{x^*x}(x^*x) + \lambda\phi_{x^*x}(\delta(x^*x))\end{aligned}$$

But $\phi_{x^*x}(\delta(x^*x)) = 0$ since $x^*x \in \mathcal{D}(\delta)$ and δ is well-behaved, so that

$$\begin{aligned}\phi_{x^*x}((\mathbb{1} + \lambda\delta)(x^*)(\mathbb{1} + \lambda\delta)(x)) &\geq \phi_{x^*x}(x^*x) \\ &= ||x^*x|| = ||x||^2\end{aligned}$$

Therefore $||(\mathbb{1} + \lambda\delta)(x)||^2 \geq ||x||^2$.

3.3 Well-behavedness and generators

We now return to the classification of derivations as generators for strongly continuous one-parameter semi-groups. The well-behavedness property and results from the previous subsection is utilized in the Lumer-Phillips theorems.

First some definitions:

Let $\Gamma(t)$ be a $\mathcal{C}_0$ semi-group. From Theorem 56 we know that there exists constants $\omega \geq 0$ and $M \geq 1$ such that $||\Gamma(t)|| \leq Me^{\omega t}$. When $M = 1$ and $\omega = 0$, then $\Gamma(t)$ is called a $\mathcal{C}_0$ *semi-group of contractions.*

If $\mathcal{A}$ is a Banach space with dual $\mathcal{A}^\star$, we denote the value of $x^\star \in \mathcal{A}^\star$ at $x \in \mathcal{A}$ by the inner-product $\langle x^\star, x\rangle$ or $\langle x, x^\star\rangle$. For every $x \in \mathcal{A}$ the *duality set* $F(x) \subseteq \mathcal{A}^\star$ is defined as

$$F(x) = \{x^\star | x^\star \in \mathcal{A}^\star; \langle x^\star, x\rangle = ||x||^2 = ||x^\star||^2\}$$

From Hahn-Banach, $F(x) \neq \emptyset$ for every $x \in \mathcal{A}$.

Definition 21 (Dissipativeness) *A linear operator A is dissipative if for every $x \in \mathcal{D}(A)$ there exists a $x^\star \in F(x)$ such that $Re\langle Ax, x^\star\rangle \leq 0$.*

The following standard semi-group theory result is a useful characterization of dissipative operators:

Theorem 65 (Pazy, A. [16], Theorem 4.2) *A linear operator A is dissipative if and only if*

$$||(\lambda \mathbb{1} - A)x|| \geq \lambda ||x|| \qquad \text{for all} \qquad x \in \mathcal{D}(A),\ \lambda > 0$$

Proof: Pazy, A. [16], 14.

The following three versions of the Lumer-Phillips theorem were compiled from Pazy, A. [16], Theorem 4.3, and Bratteli, O. and Robinson, D.W. [3], Theorem 3.2.50. The result for Banach space contractions is as follows:

Theorem 66 (Lumer-Phillips for contractions) *Let A be a linear operator with a dense domain $\mathcal{D}(A)$ in a Banach space $\mathcal{A}$.*

(i) If A is dissipative and there is a λ_0 such that $R(\lambda_0 \mathbb{1} - A) = \mathcal{A}$, then A is the infinitesimal generator of a $\mathcal{C}_0$ semi-group of contractions.

(ii) If A is the infinitesimal generator of a $\mathcal{C}_0$ semi-group of contractions on $\mathcal{A}$, then $R(\lambda \mathbb{1} - A) = \mathcal{A}$ for all $\lambda > 0$ and A is dissipative.

Proof: Pazy, A. [16], 14.

The result for Banach space isometries is as follows:

Theorem 67 (Lumer-Phillips for isometries) *If δ is an operator on a Banach space $\mathcal{A}$, then δ is the infinitesimal generator of a strongly continuous group of isometries Γ_t if and only if δ is closed, $\mathcal{D}(\delta)$ is dense, and*

$$\begin{aligned} ||(\mathbb{1} - \alpha\delta)x|| &\geq ||x|| \\ \mathcal{R}(\mathbb{1} - \alpha\delta) &= \mathcal{A} \end{aligned}$$

for all $\alpha \in \mathbb{R}$ and $x \in \mathcal{D}(\delta)$.

Proof: Pazy, A. [16], 14.

The result for closed operators on $\mathcal{C}^{\star}$- algebras and automorphisms is as follows:

Theorem 68 (Lumer-Phillips for $\star$- automorphisms) *Let δ be a norm-closed operator on a $C^\star$ algebra $\mathcal{A}$ with dense domain $\mathcal{D}(\delta)$. δ is the generator of a strongly continuous one-parameter group of $\star$- automorphisms of $\mathcal{A}$ if and only if:*

(i) $\mathcal{D}(\delta)$ *is a $\star$- algebra and δ is a symmetric derivation*

(ii) $(\mathbb{1} + \lambda\delta)(\mathcal{D}(\delta)) = \mathcal{A}$ *for every* $\lambda \in \mathbb{R}\backslash\{0\}$

(iii) $\|(\mathbb{1} + \lambda\delta)A\| \geq \|A\|$ *for every* $\lambda \in \mathbb{R}$ *and* $A \in \mathcal{D}(\delta)$

Proof outline: ($\Rightarrow$) Assume first that δ is the generator of a strongly continuous one-parameter group $t \rightarrow \Gamma_t$ of $\star$- automorphisms of $\mathcal{A}$. Property (i) (the derivation property) is arrived at by differentiation of the $\star$- automorphism properties at $t = 0$. Let $A, B \in \mathcal{D}(\delta)$.
Since

$$\begin{aligned} \Gamma_t(AB) &= \Gamma_t(A)\Gamma_t(B) \qquad \textit{then} \\ \frac{d}{dt}\Gamma_t(AB)\big|_{t=0} &= \frac{d}{dt}[\Gamma_t(A)\Gamma_t(B)]\big|_{t=0} \\ \Rightarrow \delta(AB) &= \frac{d}{dt}\Gamma_t(A)\big|_{t=0}\Gamma_0(B) + \Gamma_0(A)\frac{d}{dt}\Gamma_t(B)\big|_{t=0} \\ \Rightarrow \delta(AB) &= \delta(A)B + A\delta(B) \end{aligned} \tag{8}$$

and since

$$\begin{aligned} \Gamma_t(A^\star) &= \Gamma_t(A)^\star \qquad \textit{then} \\ \frac{d}{dt}\Gamma_t(A^\star)\big|_{t=0} &= \delta\Gamma_0(A^\star) = \delta(A^\star) \\ \frac{d}{dt}\Gamma_t(A)^\star\big|_{t=0} &= \delta\Gamma_0(A)^\star = \delta(A)^\star = \delta(A^\star) \end{aligned} \tag{9}$$

Properties (ii) and (iii) of the theorem follows from the Lumer-Phillips theorem by noting that Γ_t and Γ_{-t} are inverses of each other and hence isometric by Theorem 67 since both are contractive.

($\Leftarrow$) Next assume conditions (i), (ii) and (iii) hold. Again from the Lumer-Phillips theorem it follows that conditions (ii) and (iii) suffices for δ to be a generator of a strongly continuous one-parameter group Γ_t of isometries. It remains to be shown that condition (i) implies the $\star$- automorphism properties 8 and 9. This is done by proving the result on the dense set of analytic elements for δ, from first principles. An element $A \in \mathcal{D}^\infty(\delta) = \cap_{n=1}^\infty \mathcal{D}(\delta^n)$ is *analytic* with respect to δ if there is a positive number t

(depending on A) such that $\sum_{n=0}^{\infty} \frac{t^n}{n!} \|\delta^n(A)\| < \infty$.

$$\begin{aligned}
\Gamma_t(A^*) &= \sum_{n}^{\infty} \frac{t^n}{n!} \delta^n(A^*) \\
&= A^* + t\delta(A^*) + \frac{t^2}{2!}\delta^2(A^*) + \frac{t^3}{3!}\delta^3(A^*) ... \\
&= (A)^* + t\delta(A)^* + \frac{t^2}{2!}\delta^2(A)^* + \frac{t^3}{3!}\delta^3(A)^* ... \\
&= \sum_{n}^{\infty} \frac{t^n}{n!} \delta^n(A)^* \\
&= \Gamma_t(A)^*
\end{aligned}$$

The second property (9) requires the use of Leibniz's formula:

$$\begin{aligned}
\Gamma_t(AB) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \delta^n(AB) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} [\sum_{k=0}^{n} \binom{n}{k} \delta^{(n-k)}(A)\delta^{(k)}(B)] \\
&= \underbrace{\frac{t^0}{0!0!}AB}_{n=0} + \underbrace{\frac{t^1}{0!1!}\delta(A)B + \frac{t^1}{1!0!}A\delta(B)}_{n=1} + \underbrace{\frac{t^2}{0!2!}\delta^2(A)B + \frac{t^2}{1!1!}\delta(A)\delta(B) + \frac{t^2}{2!0!}A\delta^2(B)}_{n=2} + ... \\
&= \underbrace{[\frac{t^0}{0!0!}AB + \frac{t^1}{0!1!}\delta(A)B + \frac{t^2}{0!2!}\delta^2(A)B + ...]}_{k=0} \\
&+ \underbrace{[\frac{t^1}{1!0!}A\delta(B) + \frac{t^2}{1!1!}\delta(A)\delta(B) + \frac{t^3}{1!2!}\delta^2(A)\delta(B) + ...]}_{k=1} \\
&+ \underbrace{[\frac{t^2}{2!0!}A\delta^2(B) + \frac{t^3}{2!1!}\delta(A)\delta^2(B) + \frac{t^4}{2!2!}\delta^2(A)\delta^2(B) + ...]}_{k=2} + ... \\
&= [\sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^{(n)}(A)]\frac{t^0}{0!}B + [\sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^{(n)}(A)]\frac{t^1}{1!}\delta(B) + [\sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^{(n)}(A)]\frac{t^2}{2!}\delta^{(2)}(B) + ... \\
&= [\sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^{(n)}(A)][\sum_{k=0}^{\infty} \frac{t^k}{k!}\delta^{(k)}(B)] \\
&= \Gamma_t(A)\Gamma_t(B)
\end{aligned}$$

The un-conditional convergence (to allow inter changing the arrangement of n and k terms above) follows from the absolute convergence of the series

$$
\begin{aligned}
\lim_{m\to\infty} ||S_m|| &= \lim_{m\to\infty} ||\sum_{n=0}^{m} \frac{t^n}{n!}[\sum_{k=0}^{n} \frac{n!}{n!(n-k)!}\delta^{(n-k)}(A)\delta^{(k)}(B)]|| \\
&\leq \lim_{m\to\infty} \sum_{n=0}^{m}[\sum_{k=0}^{n} \frac{t^n}{n!(n-k)!}||\delta^{(n-k)}(A)||||\delta^{(k)}(B)||] \\
&< \infty
\end{aligned}
$$

The last step follows from the assumption that A and B are analytic. Note that condition (iii) is equivalent to well-behavedness (from Theorem 64).

The following theorem characterizes the domain of a generator and is required in the construction of the domain property counter example of Section 4:

Theorem 69 (Bratteli, O. and Robinson, D.W. [3], Proposition 3.2.55) *Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and let*

$$\alpha_t(A) = e^{itH}Ae^{-itH} \qquad\qquad A \in B(\mathcal{H})$$

be the corresponding one-parameter group of automorphisms of $B(\mathcal{H})$. If δ is the infinitesimal generator of α_t, then $A \in \mathcal{D}(\delta)$ if and only if $A(\mathcal{D}(H)) \subseteq \mathcal{D}(H)$ and $\phi \to \delta(A)\phi = i[H, A]\phi$ ($\phi \in \mathcal{D}(H)$) is bounded; i.e $||HA - AH|| < \infty$.

Proof: Assume $A \in \mathcal{D}(\delta)$ and $\varphi, \phi \in \mathcal{D}(H)$. From Definition 17 we have

$$
\begin{aligned}
\langle\varphi, \delta(A)\phi\rangle &= \lim_{t\to 0}\frac{1}{t}\{\langle\varphi, e^{itH}Ae^{-itH}\phi\rangle - \langle\varphi, A\phi\rangle\} \\
&= \lim_{t\to 0}\frac{1}{t}\{\langle e^{-itH}\varphi, Ae^{-itH}\phi\rangle - \langle\varphi, Ae^{-itH}\phi\rangle + \langle\varphi, Ae^{-itH}\phi\rangle - \langle\varphi, A\phi\rangle\} \\
&= \lim_{t\to 0}\frac{1}{t}\{\langle(e^{-itH} - I)\varphi, Ae^{-itH}\phi\rangle + \langle\varphi, A(e^{-itH} - I)\phi\rangle\} \\
&= \langle -iH\varphi, A\phi\rangle + \langle\varphi, A(-iH)\phi\rangle
\end{aligned}
$$

which implies that the sesqui-linear form $|\sigma(\varphi, \phi)| = |\langle\varphi, \delta(A)\phi\rangle|$ is bounded, since

$$
\begin{aligned}
|\sigma(\varphi, \phi)| &= |\langle\varphi, \delta(A)\phi\rangle| = |i\langle H\varphi, A\phi\rangle - i\langle\varphi, AH\phi\rangle| \\
&\leq ||\delta(A)||||\varphi||||\phi||
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sigma(\varphi, \phi) &= \langle\varphi, \delta(A)\phi\rangle \\
&= -\langle iH\varphi, A\phi\rangle - i\langle\varphi, AH\phi\rangle \qquad (10)
\end{aligned}
$$

This can be re-written as $\langle H\varphi,\, A\phi\rangle = \langle\varphi,\, AH\phi\rangle - i\langle\varphi,\, \delta(A)\phi\rangle$ for $\varphi,\, \phi \in \mathcal{D}(H)$. Then

$$\begin{aligned} |\langle H(\varphi_1 - \varphi_2),\, A\phi\rangle| &\le |\langle\varphi_1 - \varphi_2,\, AH\phi\rangle| + |i\langle\varphi_1 - \varphi_2,\, \delta(A)\phi\rangle| \\ &\le ||\varphi_1 - \varphi_2||||AH\phi|| + ||\varphi_1 - \varphi_2||||\delta(A)\phi|| \\ &= ||\varphi_1 - \varphi_2||(||AH\phi|| + ||\delta(A)\phi||) \end{aligned}$$

so that the mapping $\varphi \to \langle H\varphi,\, A\phi\rangle$ is continuous for fixed $\phi \in \mathcal{D}(H)$. Further, we then have that $A\phi \in \mathcal{D}(H^*) = \mathcal{D}(H)$, that $\varphi \to \langle\varphi,\, H^* A\phi\rangle$ is continuous, and $\langle H\varphi,\, A\phi\rangle = \langle\varphi, HA\phi\rangle$. Equation (10) can now be written as $\langle\varphi,\, \delta(A)\phi\rangle = i\langle\varphi,\, [H,\, A]\phi\rangle$. Therefore $\phi \to \delta(A)\phi = i[H,\, A]\phi$ is bounded.

Conversely, if $A(\mathcal{D}(H)) \subseteq \mathcal{D}(H)$ and $||HA - AH|| < \infty$, then (to prove $A \in \mathcal{D}(\delta)$) we need to show that

$$\lim_{s\downarrow 0} \frac{\alpha_s(A) - A}{s}$$

exists.
If B is the bounded extension of $HA - AH$ to all of $\mathcal{H}$, then with a similar argument as before one gets

$$\frac{d}{dt}\langle\varphi,\, \alpha_t(A)\phi\rangle = \langle\varphi,\, \alpha_t(B)\phi\rangle$$

for all $\varphi,\, \phi \in \mathcal{D}(H)$. So on integrating $(\alpha_0(A) = \mathbb{1}(A))$

$$\langle\varphi,\, [\alpha_s(A) - A]\phi\rangle = \langle\varphi,\, \int_0^s \alpha_t(B)\phi dt\rangle$$

By the density of $\mathcal{D}(H)$ we get

$$\alpha_s(A) - A = \int_0^s \alpha_t(B)dt$$

Then

$$\begin{aligned} \lim_{s\downarrow 0} \frac{\alpha_s(A) - A}{s} &= \lim_{s\downarrow 0} \frac{1}{s} \int_0^s \alpha_t(B)dt \\ &= B \end{aligned}$$

by Theorem 58(i) so that $A \in \mathcal{D}(\delta)$ with $\delta(A) = B$.

4 Domain properties

This section investigates the domain properties, and specifically the so-called 'chain-rule', on the domain $\mathcal{D}(\delta)$ of a closed $*$-derivation δ defined in a general $\mathcal{C}^*$ - algebra $\mathcal{A}$. Consider first the case where $\mathcal{A}$ is an Abelian algebra and let $A = A^* \in \mathcal{D}(\delta)$. (Unless otherwise stated, in this section A, B will denote elements of a $\mathcal{C}^*$- algebra and x, y elements in $\mathbb{R}$).

By commutativity we have

$$\delta(A^n) = nA^{(n-1)}\delta(A)$$

and thus for any polynomial $P(A)$ with $A \in \mathcal{D}(\delta)$ we have

$$\begin{aligned} \delta(P(A)) &= \delta(a_0) + a_1\delta(A) + a_2\delta(A^2) + a_3\delta(A^3) + \ldots + a_n\delta(A^n) \\ &= a_1\delta(A) + a_2 2A\delta(A) + a_3 3A^2\delta(a) + a_4 4A^3\delta(A) + \ldots + a_n nA^{n-1}\delta(A) \\ &= \delta(A)[a_1 + a_2 2A + a_3 3A^2 + a_4 4A^3 + \ldots + a_n nA^{n-1}] \\ &= \delta(A)\sum_{j=1}^{j=n}(j)a_jA^{j-1} \end{aligned} \tag{11}$$

so that $P(A) \in \mathcal{D}(\delta)$ and $\delta(P(A)) = \delta(A)P'(A)$. From $A = A^*$ we know that $\mathcal{A}[A]$ (the smallest $\mathcal{C}^*$- algebra containing A) is commutative and $\mathcal{M}_{\mathcal{A}[A]}$, the maximal ideal space of $\mathcal{A}[A]$, can be identified with $\sigma(A) \subset [-||A||, ||A||]$. The continuous functional calculus mapping each $f \in \mathcal{C}(\sigma(A))$ to an element in $\mathcal{A}[A]$, is defined by $f(A) = \Gamma^{-1}(f)$ (Γ the Gelfand transform). For example, the function $f(t) = t^2$ is in $\mathcal{C}(\sigma(A))$ and therefore there exists a (unique) $B \in \mathcal{A}[A]$ with $B = f(A)$ and $B = A^2$.

Because $A = A^*$, we know that $\sigma(A) \subset \mathbb{R}$ is closed, bounded and hence compact and (given $f \in \mathcal{C}^1(\sigma(A))$) from the Stone-Weierstrass theorem we can select polynomials P_n with $P_n \to f$ and $P_n' \to f'$ on the spectrum of A. Replacing P_n into P earlier, it follows from the closedness of δ that $f(A) \in \mathcal{D}(\delta)$ and

$$\delta(f(A)) = \delta(A)f'(A)$$

This result relies heavily on two assumptions: the commutativity of $\mathcal{D}(\delta) \subset \mathcal{A}$ and the function $f \in \mathcal{C}^1$. Firstly, if we drop the commutativity assumption, then equation (11) above fails to hold and the argument is no longer valid. Section 4.6 provides an example of an $f \in \mathcal{C}^1$ on a non-commutative algebra with $f(A) \notin \mathcal{D}(\delta)$.

Less obvious, however, is the effect of the choice of functions f for which the result is still valid. If we restrict the functions to $f \in \mathcal{C}^2(\sigma(A))$ and maintain the commutativity of

$\mathcal{A}$ then of course the property still holds, but relaxing commutativity immediately raises the following questions:

1. Is the domain property valid for non-commutative $\mathcal{A}$ and $f \in \mathcal{C}^2$?

2. If 1 holds, by how much can the restrictions on f be relaxed without ruining the result for non-commutative $\mathcal{A}$?

4.1 Background

Consider the complex function $f_\lambda(z) = \frac{z}{\lambda - z}$ where λ, $z \in \mathbb{C}$ - the complex plane. By Taylor series expansion there exists a unique power series expansion of f_λ around $a \in \mathbb{C}$ of the form

$$\sum_{k=0}^{\infty} a_k(z - a)^k$$

where

$$a_k = \frac{f_\lambda^{(k)}(a)}{k!}$$

If we consider $a = 0$, then $a_0 = 0$ and $a_k = \frac{\lambda}{\lambda^{k+1}}$ for $k \geq 1$ so that

$$f_\lambda(z) = \sum_{k=0}^{\infty}(\frac{z}{\lambda})^k \tag{12}$$

The limit $\lim_{k\to\infty}|\frac{a_{k+1}}{a_k}| = \frac{1}{\lambda}$ and the series converges on $|z| < \lambda$ and diverges on $|z| > \lambda$.

Let $A = A^\star \in \mathcal{A}$ (a $\mathcal{C}^\star$-algebra) so that $\sigma(A) \subset [-||A||, ||A||]$ ($\subset \mathbb{R}$) and consider the element $A(\lambda 1\!\!1 - A)^{-1}$. By the continuous functional calculus for $A = A^\star$, this element corresponds to the element $f : z \to \frac{z}{\lambda - z}$ in $\mathcal{C}(\sigma(A))$. Therefore, from equation (12) we can write

$$f_\lambda(A) = \sum_{k=0}^{\infty}\frac{1}{\lambda^k}A^k \tag{13}$$

where $\lambda \in \mathbb{C}$, $A \in \mathcal{A}$. This series converges for all $|\lambda| > ||A||$ (outside the spectrum of A). (Equation (13) is a so called Neumann series and its convergence can also be established as a convergent geometric series for all $|\lambda| > ||A||$)

We now restrict the focus to the subalgebra $\mathcal{D}(\delta)$ - domain of a closed $\star$-derivation in a $\mathcal{C}^\star$- algebra $\mathcal{A}$ - and we wish to investigate the behavior of functions on $\mathcal{D}(\delta)$. Both the inverse and the exponential function are fundamental to functional analysis and the following two sections deal with them.

4.2 The inverse function on $\mathcal{D}(\delta)$

Theorem 70 (Bratteli, O. and Robinson, D.W. [3], Proposition 3.2.29) *Let δ be a norm closed derivation on a $C^\star$-algebra $\mathcal{A}$ with identity $1\!\!1$. If $A = A^\star \in \mathcal{D}(\delta)$ and $\lambda \notin \sigma(A)$, then $A(\lambda 1\!\!1 - A)^{-1} \in \mathcal{D}(\delta)$ and*

$$\delta(A(\lambda 1\!\!1 - A)^{-1}) = \lambda(\lambda 1\!\!1 - A)^{-1}\delta(A)(\lambda 1\!\!1 - A)^{-1}$$

Proof: This theorem is proved in three steps.
Step 1 proves the result for all $|\lambda| > ||A||$ -i.e $|\lambda|$ larger than the spectral radius of $A = A^\star$.
Step 2 proves the result for λ in a small neighborhood of each λ_0 with $|\lambda_0| > ||A||$.
Step 3 consists of an analytic continuation argument proving the result for general $\lambda \notin \sigma(A)$.

First consider the function $f_\lambda(A) = A(\lambda 1\!\!1 - A)^{-1} = A_\lambda$ (as before). By the continuous functional calculus on $\sigma(A)$ for $A = A^\star$ we know that $A(\lambda 1\!\!1 - A)^{-1}$ corresponds to the element in $C[\sigma(A)]$ defined by $f : x \to \frac{x}{\lambda - x}$ for $\lambda \notin \sigma(A)$. We then have

$$||A_\lambda|| = ||A(\lambda 1\!\!1 - A)^{-1}|| = ||f||_\infty = \sup_{\gamma \in \sigma(A)} |\frac{\gamma}{\lambda - \gamma}|$$

This representation always reduces the complexity of problems in that we only need to investigate functions on $\mathbb{C}$ (the complex plane).

Step 1: Set $A_\lambda = A(\lambda 1\!\!1 - A)^{-1}$ ($= f_\lambda(A)$) and let $|\lambda| > ||A||$. As shown before, the Neumann series (geometric series)

$$S_m = \sum_{n=1}^{m} (\frac{A}{\lambda})^n$$

converges to $A_\lambda = A(\lambda 1\!\!1 - A)^{-1}$ and

$$\begin{aligned} ||A_\lambda|| = ||\sum_{n=1}^{\infty}(\frac{A}{\lambda})^n|| &= ||\frac{A}{\lambda} + (\frac{A}{\lambda})^2 + (\frac{A}{\lambda})^3 + ... + (\frac{A}{\lambda})^n + ...|| \\ &\leq \frac{||A||}{|\lambda|} + (\frac{||A||}{|\lambda|})^2 + (\frac{||A||}{|\lambda|})^2 + ... \\ &= |\lambda|(|\lambda| - ||A||)^{-1} \end{aligned}$$

because $|\lambda| > ||A_\lambda||$. Furthermore, $A^{n+1} \in \mathcal{D}(\delta)\,\forall n$ so that $S_m \in \mathcal{D}(\delta)$. Finally we need to look at $\delta(S_m)$. From $\delta(A^{n+1}) = \sum_{p=0}^{n} A^p\delta(A)A^{n-p}$ and the Neumann series $S_m = \sum_{n\geq 0}^{m}(\frac{A}{\lambda})^{n+1}$ we construct the double series (sum)

$$(\delta(S_m)) := \frac{1}{\lambda}\sum_{n\geq 0}^{m}\sum_{p=0}^{n}(\frac{A}{\lambda})^p\delta(A)(\frac{A}{\lambda})^{n-p}$$

Now

$$
\begin{aligned}
||\delta(S_m)|| &= ||\frac{1}{\lambda}\sum_{n\geq 0}^{m}\sum_{p=0}^{n}(\frac{A}{\lambda})^p\delta(A)(\frac{A}{\lambda})^{n-p}|| \\
&\leq \frac{1}{|\lambda|}\sum_{n\geq 0}^{m}\sum_{p=0}^{n}||\frac{A}{\lambda}||^p||\delta(A)||||\frac{A}{\lambda}||^{n-p} \\
&= \frac{1}{|\lambda|}||\delta(A)||\sum_{n\geq 0}^{m}\sum_{p=0}^{n}||\frac{A}{\lambda}||^n \\
&= \frac{1}{|\lambda|}\sum_{n\geq 0}^{m}(n+1)||\frac{A}{\lambda}||^n
\end{aligned}
$$

and with the ratio test

$$
\begin{aligned}
\lim_{m\to\infty}\frac{s_{m+1}}{s_m} &= \lim_{m\to\infty}\frac{(m+2)||\frac{A}{\lambda}||^{m+1}}{(m+1)||\frac{A}{\lambda}||^m} \\
&= ||\frac{A}{\lambda}|| < 1
\end{aligned}
$$

this double series also converges. An expansion of $\delta(A_\lambda)$ yields

$$
\delta(A_\lambda) = \frac{1}{\lambda}\Bigg[\underbrace{[\overbrace{\delta(A)}^{p=0}]}_{n=0} + \underbrace{[\overbrace{\delta(A)(\frac{A}{\lambda})}^{p=0} + \overbrace{(\frac{A}{\lambda})\delta(A)}^{p=1}]}_{n=1} + \underbrace{[\overbrace{\delta(A)(\frac{A}{\lambda})^2}^{p=0} + \overbrace{\frac{A}{\lambda}\delta(A)\frac{A}{\lambda}}^{p=1} + \overbrace{(\frac{A}{\lambda})^2\delta(A)}^{p=2}]}_{n=2} + ...\Bigg]
$$

and by grouping terms with equal p- indexes together, we get

$$
= \frac{1}{\lambda}\Bigg[[\delta(A) + \delta(A)(\frac{A}{\lambda}) + \delta(A)(\frac{A}{\lambda})^2 ...] + [(\frac{A}{\lambda})\delta(A) + (\frac{A}{\lambda})\delta(A)(\frac{A}{\lambda}) + (\frac{A}{\lambda})\delta(A)(\frac{A}{\lambda})^2 ...] + ...\Bigg]
$$

so that for each fixed value of p (say $p = 2$), n can assume p, $p+1$, $p+2$ .. (2,3,4,...). The following matrix indexation helps to visualize the fact that no terms are gained or lost in the re-arrangement. For any pair (p, n), there is only one term:

			p		
	0	1	2	3	...
	1	2	3	4	...
n	2	3	4	5	...
	3	4	5	6	...
	...	...	...	...	...

The absolute convergence of $\delta(S_m)$ implies its un-conditional convergence, which assures the same limit irrespective of the arrangement. It follows from the closedness of δ that

$A_\lambda \in \mathcal{D}(\delta)$ and

$$\begin{aligned}\delta(A_\lambda) &= \frac{1}{\lambda}\sum_{p\geq 0}(\frac{A}{\lambda})^p\delta(A)\sum_{n\geq 0}(\frac{A}{\lambda})^n \\ &= \lambda(\lambda 1\!\!1 - A)^{-1}\delta(A)(\lambda 1\!\!1 - A)^{-1}\end{aligned}$$

Step 2: Assume now that $0 \neq \lambda_0 > ||A||$ so that λ_0 is outside the disk in $\mathbb{C}$ with radius equal to the spectral radius of A. From step 1 we know that $A_{\lambda_0} \in \mathcal{D}(\delta)$ and

$$\delta(A_{\lambda_0}) = \lambda_0(\lambda_0 1\!\!1 - A)^{-1}\delta(A)(\lambda_0 1\!\!1 - A)^{-1}$$

We will now prove the result of step 1 for all λ with $|\frac{\lambda_0 - \lambda}{\lambda}| < ||A_{\lambda_0}||^{-1}$, i.e we prove the result for all λ inside a (small) neighborhood of $|\lambda_0| > ||A||$. Since $|\frac{\lambda}{\lambda_0 - \lambda}| > ||A_{\lambda_0}||$, we may substitute $\frac{\lambda}{\lambda_0 - \lambda}$ for λ into the Neumann series discussed in step 1 to get the series

$$S_m = \sum_{n\geq 0}^{m}(\frac{A_{\lambda_0}}{\lambda(\lambda_0 - \lambda)^{-1}})^{n+1}$$

This can be expanded and manipulated as follows:

$$\begin{aligned}[\frac{\lambda_0}{\lambda_0 - \lambda}][\frac{\lambda_0 - \lambda}{\lambda}A_{\lambda_0} &+ (\frac{\lambda_0 - \lambda}{\lambda})^2A_{\lambda_0}^2 + (\frac{\lambda_0 - \lambda}{\lambda})^3A_{\lambda_0}^3 + ...] \\ &= \frac{\lambda_0}{\lambda}A_{\lambda_0} + \frac{\lambda_0}{\lambda}(\frac{\lambda_0 - \lambda}{\lambda})A_{\lambda_0}^2 + \frac{\lambda_0}{\lambda}(\frac{\lambda_0 - \lambda}{\lambda})^2A_{\lambda_0}^3 + ... \\ &= \frac{\lambda_0}{\lambda}A_{\lambda_0}[1\!\!1 + (\frac{\lambda_0 - \lambda}{\lambda})A_{\lambda_0} + (\frac{\lambda_0 - \lambda}{\lambda})^2A_{\lambda_0}^2 + (\frac{\lambda_0 - \lambda}{\lambda})^2A_{\lambda_0}^3 + ...] \\ &= \frac{\lambda_0}{\lambda}A_{\lambda_0}[1\!\!1 + \frac{A_{\lambda_0}}{(\frac{\lambda}{\lambda_0 - \lambda})} + \frac{A_{\lambda_0}^2}{(\frac{\lambda}{\lambda_0 - \lambda_0})^2} + \frac{A_{\lambda_0}^3}{(\frac{\lambda}{\lambda_0 - \lambda})^3} + ...] \\ &= \frac{\lambda_0}{\lambda}A_{\lambda_0}(1\!\!1 - \frac{A_{\lambda_0}}{(\frac{\lambda}{\lambda_0 - \lambda})})^{-1} \qquad \textit{since} \qquad |\frac{\lambda_0 - \lambda}{\lambda}| < ||A_{\lambda_0}||^{-1} \\ &= \frac{\lambda_0}{\lambda}A_{\lambda_0}(1\!\!1 - \frac{(\lambda_0 - \lambda)A_{\lambda_0}}{\lambda})^{-1} \\ &= \frac{\lambda_0}{\lambda}A_{\lambda_0}(\frac{\lambda 1\!\!1 - (\lambda_0 - \lambda)A_{\lambda_0}}{\lambda})^{-1} \\ &= \lambda_0 A_{\lambda_0}(\lambda 1\!\!1 - (\lambda_0 - \lambda)A_{\lambda_0})^{-1} \\ &= \lambda_0 A(\lambda_0 1\!\!1 - A)^{-1}\big(\lambda 1\!\!1 - \lambda_0 A(\lambda_0 1\!\!1 - A)^{-1} + \lambda A(\lambda_0 1\!\!1 - A)^{-1}\big)^{-1} \\ &= \lambda_0 A(\lambda(\lambda_0 1\!\!1 - A) - \lambda_0 A + \lambda A)^{-1} \\ &= A(\lambda 1\!\!1 - A)^{-1}\end{aligned} \tag{14}$$

so that we have

$$A_\lambda = \frac{\lambda_0}{\lambda_0 - \lambda}\sum_{n\geq 0}(\frac{\lambda_0 - \lambda}{\lambda}A_{\lambda_0})^{n+1}$$

Again by the closedness of δ it follows that $A_\lambda \in \mathcal{D}(\delta)$ and

$$\begin{aligned}
\delta(A_\lambda) &= (\frac{\lambda_0}{\lambda}) \sum_{p \geq 0} [(\frac{\lambda_0 - \lambda}{\lambda}) A_{\lambda_0}]^p \delta(A_{\lambda_0}) \sum_{n \geq 0} [(\frac{\lambda_0 - \lambda}{\lambda}) A_{\lambda_0}]^n \\
&= \frac{\lambda_0}{\lambda} [\frac{\lambda}{\lambda} 1\!\!1 - \frac{(\lambda_0 - \lambda) A_{\lambda_0}}{\lambda}]^{-1} \delta(A_{\lambda_0}) [\frac{\lambda}{\lambda} 1\!\!1 - \frac{(\lambda_0 - \lambda) A_{\lambda_0}}{\lambda}]^{-1} \\
&= \frac{\lambda_0}{\lambda} [((\lambda_0 1\!\!1 - A) - (\frac{\lambda_0}{\lambda} - 1) A)(\lambda_0 1\!\!1 - A)^{-1}]^{-1} \delta(A_{\lambda_0}) \\
&\quad [((\lambda_0 1\!\!1 - A) - (\frac{\lambda_0}{\lambda} - 1) A)(\lambda_0 1\!\!1 - A)^{-1}]^{-1} \\
&= \frac{\lambda_0}{\lambda} \frac{\lambda}{\lambda_0} [(\lambda_0 1\!\!1 - A)(\lambda 1\!\!1 - A)^{-1}] \delta(A_{\lambda_0}) \frac{\lambda}{\lambda_0} [(\lambda_0 1\!\!1 - A)(\lambda 1\!\!1 - A)^{-1}]
\end{aligned}$$

From step 1, $\delta(A_{\lambda_0}) = \lambda_0 (\lambda_0 1\!\!1 - A)^{-1} \delta(A)(\lambda_0 1\!\!1 - A)^{-1}$ so that

$$\delta(A_\lambda) = \lambda(\lambda 1\!\!1 - A)^{-1} \delta(A)(\lambda 1\!\!1 - A)^{-1}$$

Step 3: Recall $|\lambda_0| > ||A||$ and the region $|\frac{\lambda}{\lambda_0 - \lambda}| > ||A_{\lambda_0}||$ from step 2. By the continuous functional calculus for $A = A^\star$, $A(\lambda_0 1\!\!1 - A)$ corresponds to an element $f : \lambda \to \lambda(\lambda_0 - \lambda)^{-1}$ for all $\lambda \in \mathcal{C}(\sigma(A))$. This allows us to investigate the region in $\mathbb{C}$ for which

$$\begin{aligned}
\frac{|\lambda - \lambda_0|}{|\lambda|} &< ||A_{\lambda_0}||^{-1} \\
&= ||f||_\infty^{-1} \\
&= (\sup_{\gamma \in \sigma(A)} |\frac{\gamma}{\lambda_0 - \gamma}|)^{-1}
\end{aligned}$$

To continue the proof for general $\lambda \notin \sigma(A_\lambda)$, we need to be able to select a sequence $\lambda_n \notin \sigma(A)$ with $\lambda_n \to \lambda$ such that $A_{\lambda_n} \in \mathcal{D}(\delta)$ and $\delta(A_{\lambda_n}) = \lambda_n(\lambda_n 1\!\!1 - A)^{-1}\delta(A)(\lambda_n 1\!\!1 - A)^{-1}$, so that the result for general $\lambda \notin \sigma(A_\lambda)$ then follows from an analytic continuation argument.

Therefore, we have to understand the region introduced in step 2 a little better. To simplify matters we set $||f||_\infty^{-1} = \alpha$. Let $\lambda_0 = (x_0, y_0)$ and $\lambda = (x, y)$ with x, y, x_0, y_0 all real numbers. Then

$$\begin{aligned}
|\frac{(x_0, y_0) - (x, y)}{(x, y)}| &= \alpha \\
|(x_0 - x), (y_0 - y)| &= |x, y|\alpha \\
x_0^2 - 2xx_0 + x^2 + y_0^2 - 2yy_0 + y^2 &= (x^2 + y^2)\alpha^2 \qquad (15)
\end{aligned}$$

Now investigate the following three cases:

Case 1: $\alpha = 1$

$$\begin{aligned}
x_0^2 - 2xx_0 + y_0^2 - 2y_0 y &= 0 \\
y &= -\frac{x_0}{y_0} x + (\frac{x_0^2 + y_0^2}{2y_0})
\end{aligned}$$

which describes a line perpendicular to the line joining (x_0, y_0) and $(0, 0)$ and intersecting it in $(\frac{1}{2}x_0, \frac{1}{2}y_0)$.

Cases 2 and 3 require completion of the squares ($\alpha \neq 1$) in equation (14):

$$x^2\alpha^2 - x^2 + y^2\alpha^2 - y^2 + 2xx_0 + 2yy_0 = x_0^2 + y_0^2$$

becomes

$$\begin{aligned}
(\alpha^2 - 1)[x + \frac{x_0}{\alpha^2 - 1}]^2 - \frac{x_0^2}{\alpha^2 - 1} + (\alpha^2 - 1)[y + \frac{y_0}{\alpha^2 - 1}]^2 - & \\
\frac{y_0^2}{\alpha^2 - 1} &= x_0^2 + y_0^2 \\
[x + \frac{x_0}{(\alpha^2 - 1)}]^2 + [y + \frac{y_0}{(\alpha^2 - 1)}]^2 &= \frac{x_0^2}{\alpha^2 - 1} + \frac{y_0^2}{\alpha^2 - 1} + \frac{x_0^2}{(\alpha^2 - 1)^2} + \frac{y_0^2}{(\alpha^2 - 1)^2} \\
&= \frac{\alpha^2(x_0^2 + y_0^2)}{(\alpha^2 - 1)^2}
\end{aligned}$$

This is a circular region with center on the line joining (x_0, y_0) and $(0, 0)$ and radius $\frac{\alpha\sqrt{x_0^2 + y_0^2}}{|\alpha^2 - 1|}$.

Case 2: $\alpha > 1$

Select λ_0 on the positive imaginary axis. Then $(\alpha^2 - 1) > 0$ so that $\frac{y_0}{\alpha^2 - 1} > 0$ and $\frac{y_0}{\alpha^2 - 1} \to \infty$ as $\alpha \downarrow 1$ so that the center of the circle approaches $-\infty\, i$ as $\alpha \downarrow 1$. Also, the radius $\frac{\alpha\sqrt{x_0^2 + y_0^2}}{|\alpha^2 - 1|} \to \infty$ as $\alpha \downarrow 1$.

Case 3: $\alpha < 1$

Here $\alpha^2 - 1 < 0$ so that $\frac{y_0}{\alpha^2 - 1} < 0$ and $\frac{y_0}{\alpha^2 - 1} \to -\infty$ as $\alpha \uparrow 1$ so that the center of the circle approaches $+\infty\, i$ as $\alpha \uparrow 1$.

Recall $\alpha = (\sup_{\gamma \in \sigma(A)} |\frac{\gamma}{\lambda_0 - \gamma}|)^{-1} = \inf_{\gamma \in \sigma(A)} |\frac{\lambda_0 - \gamma}{\gamma}| = \inf_{\gamma \in \sigma(A)} |\frac{\lambda_0}{\gamma} - 1|$ and restrict λ_0 to the positive imaginary axis. Then (since $\sigma(A) \subset \mathbb{R}$ is compact) it follows that $|\frac{\lambda_0}{\gamma} - 1| = \sqrt{1 + (\frac{y_0}{\gamma})^2} > 1$ for all $\gamma \in \sigma(A)$ so that $\alpha > 1$ for λ_0 on the imaginary axis. This is case 2 above and we see that the circle will intersect the imaginary axis at

$$\begin{aligned}
[y + \frac{y_0}{\alpha^2 - 1}]^2 &= \frac{\alpha^2 y_0^2}{(\alpha^2 - 1)^2} \\
y + \frac{y_0}{\alpha^2 - 1} &= \pm\frac{\alpha y_0}{\alpha^2 - 1} \\
y &= -\frac{y_0}{\alpha^2 - 1} \pm \frac{\alpha y_0}{\alpha^2 - 1}
\end{aligned}$$

For λ_0 on the positive imaginary axis, we have that $\alpha > 1$ and therefore $\frac{|\frac{\lambda_0}{2} - \lambda_0|}{\frac{\lambda_0}{2}} = 1 < \alpha$ shows that $\frac{\lambda_0}{2}$ is also in the region $|\frac{\lambda_0 - \lambda}{\lambda}| < \alpha$. A similar argument shows that if λ_0 is on the negative imaginary axis, then $\frac{\lambda_0}{2}$ is also in this region. Furthermore, for any λ in the half plane given by $sgnIm\lambda = sgnIm\lambda_0$ and $|Im\lambda| > |Im\lambda_0|$, we have

$$\begin{aligned} |\frac{\lambda_0 - \lambda}{\lambda}| &= |\frac{(0, y_0) - (x, y)}{(x, y)}| \\ &= \sqrt{\frac{x^2 + (y_0 - y)^2}{x^2 + y^2}} \\ &\leq 1 < \alpha \end{aligned}$$

because $|Im\lambda| > |Im\lambda_0|$ and $(y_0 - y)^2 < y^2$.

To conclude, we now have the following: Given any $\lambda \notin \sigma(A)$, there exists a sequence $\lambda_n \notin \sigma(A)$ with $\lambda_n \to \lambda$ such that $A_{\lambda_n} \in \mathcal{D}(\delta)$ and $\delta(A_{\lambda_n}) = \lambda_n(\lambda_n 1\!\!1 - A)^{-1}\delta(A)(\lambda_n 1\!\!1 - A)^{-1}$ for all n. Thus

$$\begin{aligned} A_{\lambda_n} &\in \mathcal{D}(\delta) \\ A_{\lambda_n} &= A(\lambda_n 1\!\!1 - A)^{-1} \\ &\to A(\lambda 1\!\!1 - A)^{-1} = A_\lambda \end{aligned}$$

in norm as $\lambda_n \to \lambda$ (since $(\lambda_n 1\!\!1 - A)^{-1} \to (\lambda 1\!\!1 - A)^{-1}$).
Also

$$\begin{aligned} \delta(A_{\lambda_n}) &= \lambda_n(\lambda_n 1\!\!1 - A)^{-1}\delta(A)(\lambda_n 1\!\!1 - A)^{-1} \\ &\to \lambda(\lambda 1\!\!1 - A)^{-1}\delta(A)(\lambda 1\!\!1 - A)^{-1} \end{aligned}$$

so that by the closedness of δ (Theorem 28), it follows that

$$\begin{aligned} A_\lambda &\in \mathcal{D}(\delta) \\ \delta(A_\lambda) &= \lambda(\lambda 1\!\!1 - A)^{-1}\delta(A)(\lambda 1\!\!1 - A)^{-1} \end{aligned}$$

for all $\lambda \notin \sigma(A)$ and $A = A^\star \in \mathcal{D}(\delta)$.

Remark 11 *If $1\!\!1 \in \mathcal{D}(\delta)$, then the result of Theorem 70 can be restated with the element $A(\lambda 1\!\!1 - A)^{-1}$ replaced by $(\lambda 1\!\!1 - A)^{-1}$. We write*

$$\begin{aligned} 1\!\!1 &= (\lambda 1\!\!1 - A)(\lambda 1\!\!1 - A)^{-1} \\ &= \lambda(\lambda 1\!\!1 - A)^{-1} - A(\lambda 1\!\!1 - A)^{-1} \end{aligned}$$

so that $A(\lambda 1\!\!1 - A)^{-1} = \lambda(\lambda 1\!\!1 - A)^{-1} - 1\!\!1$. Since $1\!\!1 \in \mathcal{D}(\delta)$, we have $(\lambda 1\!\!1 - A)^{-1} \in \mathcal{D}(\delta)$. Also, $\delta(A(\lambda 1\!\!1 - A)^{-1}) = \lambda\delta((\lambda 1\!\!1 - A)^{-1})$ since $\delta(1\!\!1) = 0$.

4.3 The exponential function on $\mathcal{D}(\delta)$

Theorem 71 (Bratteli, O. and Robinson, D.W. [3], Lemma 3.2.31) *Let δ be a norm closed derivation of the $C^\star$- algebra $\mathcal{A}$ and let $A = A^\star \in \mathcal{D}(\delta)$. Define $U_z = e^{zA}$ for $z \in C$. Then $U_z \in \mathcal{D}(\delta)$ and $\delta(U_z) = z\int_0^1 U_{tz}\delta(A)U_{(1-t)z}dt$.*

Proof: The proof follows the following steps:
Step 1 uses the binomial theorem for general $r \in \mathbb{R}$ to show that $\lim_{n\to\infty} ||U_z - (1 - \frac{zA}{n})^{-n}|| = 0$.
Step 2 sets $f(t) = U_{tz}\delta(A)U_{(1-t)z}$, defines $f_n(t) = (\mathbb{1} - \frac{tzA}{n})^{-n}\delta(A)(\mathbb{1} - \frac{(1-t)zA}{n})^{-n}$ and shows that $f_n(t) \to f(t)$ uniformly on $[0, 1]$.
Step 3 defines a partition P_k of $[0, 1]$ and shows that for all Riemann-integrable $g(t)$ we have $S(P_k, g(t)) \to \int_0^1 g(t)dt$ where $S(P_k, g(t)) = \sum_{m=0}^k (p_m - p_{m-1})g(p_m)$. This step involves looking at the adjusted left sums over the interval $[0, 1]$.
Step 4 combines steps 2 and 3 and implements the double limit (iterated limit) theorem to show that $\lim_k[S(P_k, \lim_n f_n(t))] = \lim_k[\lim_n S(P_k, f_n(t))] = \lim_k[S(P_k, f(t))] = z\int_0^1 f(t)dt$.

Step 1: Consider the binomial series expansion for powers of general $r \in \mathbb{R}$: If $|x| < 1$, then

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k \tag{15}$$

where $\binom{r}{k}$ is defined as

$$\binom{r}{k} = \frac{r(r-1)(r-2)...(r-k+1)}{k!}$$

For n large enough we have $||(\frac{-zA}{n})|| < 1$ so that we can replace $x := (\frac{-zA}{n})$ and $r := -n$ in 15. Then

$$\begin{aligned}(1 + (\frac{-zA}{n}))^{-n} &= \sum_{k}^{\infty} \binom{-n}{k} (\frac{-zA}{n})^k \\ &= \sum_{k}^{\infty} \binom{-n}{k} (\frac{-1}{n})^k (zA)^k\end{aligned}$$

The first two coefficients of the right hand side is 1, and

For $k = 2$:

$$\binom{-n}{2}(\frac{-1}{n})^2 = \frac{-n(-n-1)}{2!n^2}$$

$$
\begin{aligned}
&= \frac{n^2+n}{2!n^2} \\
&\to_n^\infty \frac{1}{2!}
\end{aligned}
$$

For $k = 3$:

$$
\begin{aligned}
\binom{-n}{3}\left(\frac{-1}{n}\right)^3 &= \frac{-n(-n-1)(-n-2)}{3!}\left(\frac{-1}{n}\right)^3 \\
&= \frac{n^3+3n^2+2n}{3!n^3} \\
&\to_n^\infty \frac{1}{3!}
\end{aligned}
$$

In general,

$$
\begin{aligned}
\binom{-n}{k}\left(\frac{-1}{n}\right)^k &= \frac{(-n)(-n-1)...(-n-k+1)}{k!n^k}(-1)^k \\
&= \frac{1}{k!}\frac{n}{n}\frac{(n+1)}{n}...\frac{(n+k-1)}{n} \\
&\to_n^\infty \frac{1}{k!}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|e^{zA} - (\mathbb{1} - \frac{zA}{n})^{-n}\| &\leq \sum_{k=0}^{\infty}\left(\frac{1}{k!}\frac{n}{n}\frac{n+1}{n}...\frac{n+k-1}{n} - \frac{1}{k!}\right)\|zA\|^k \\
&= \left(1 - \frac{\|zA\|}{n}\right)^{-n} - e^{\|zA\|} \\
&\to_n^\infty 0
\end{aligned}
$$

so that $\lim_{n\to\infty}\|U_z - (\mathbb{1} - \frac{zA}{n})^{-n}\| = 0$.

Step 2:

$$
\begin{aligned}
&\lim_n^\infty \|U_{tz}\delta(A)U_{(1-t)z} - (\mathbb{1} - \frac{tzA}{n})^{-n}\delta(A)(\mathbb{1} - \frac{(1-t)zA}{n})^{-n}\| = \\
&\lim_n^\infty \|U_{tz}\delta(A)U_{(1-t)z} - (\mathbb{1} - \frac{tzA}{n})^{-n}\delta(A)(\mathbb{1} - \frac{(1-t)zA}{n})^{-n} - \\
&(\mathbb{1} - \frac{tzA}{n})^{-n}\delta(A)U_{(1-t)z} + (\mathbb{1} - \frac{tzA}{n})^{-n}\delta(A)U_{(1-t)z}\| = \\
&\lim_n^\infty \|\big[U_{tz} - (\mathbb{1} - \frac{tzA}{n})^{-n}\big]\delta(A)U_{(1-t)z} + (\mathbb{1} - \frac{tzA}{n})^{-n}\delta(A)\big[U_{(1-t)z} - (\mathbb{1} - \frac{(1-t)zA}{n})^{-n}\big]\| \leq \\
&\lim_n^\infty \|U_{tz} - (\mathbb{1} - \frac{tzA}{n})^{-n}\|\|\delta(A)U_{(1-t)z}\| + K\|\delta(A)\|\lim_n^\infty \|U_{(1-t)z} - (\mathbb{1} - \frac{(1-t)zA}{n})^{-n}\| \\
= \; & 0
\end{aligned}
$$

follows from step 1, the continuity of multiplication of the Banach algebra and the boundedness of $\{(\mathbb{1} - \frac{tzA}{n})^{-n}\}$. K (above) is chosen as an upper bound for $\|(\mathbb{1} - \frac{tzA}{n})^n\|$.

Step 3: Define a partition P_k of $[0, 1]$ by $t_0 = 0$, $t_1 = \frac{1}{k}$, ..., $t_k = 1$ so that $t_m = \frac{m}{k}$. Now construct the lower (left) sum as follows:

$$
\begin{aligned}
z\int_0^1 U_{tz}\delta(A)U_{(1-t)z}dt &= \lim_k^{\infty} \frac{z}{k}\sum_{m=0}^{k-1} U_{\frac{m}{k}z}\delta(A)U_{(1-\frac{m}{k})z} \\
&= \lim_k^{\infty} \frac{z}{k}\sum_{m=0}^{k-1} U_{m\frac{z}{k}}\delta(A)U_{(k-m)(\frac{z}{k})} \\
&= \lim_k^{\infty} e^{\frac{1}{k}zA}\frac{z}{k}\sum_{m=0}^{k-1} U_{(m)\frac{z}{k}}\delta(A)U_{(k-m)(\frac{z}{k})} \\
&= \lim_k^{\infty} \frac{z}{k}\sum_{m=0}^{k-1} U_{(m+1)\frac{z}{k}}\delta(A)U_{(k-m)(\frac{z}{k})} \qquad (16)
\end{aligned}
$$

The last step above is achieved by multiplying (from the left) by $U_{\frac{z}{k}} = e^{\frac{1}{k}zA}$ without affecting the limit in k, since $\lim_{k\to\infty} e^{\frac{1}{k}zA} \to \mathbb{1}$.

Step 4: From the formula (Leibnitz) $\delta(A^n) = \sum_{k=0}^{n-1} A^k\delta(A)A^{n-1-k}$ and an application of Theorem 70 (Remark 11) it follows that

$$
\begin{aligned}
\delta\big((\mathbb{1} - \frac{zA}{n})^{-n}\big) &= \sum_{m=0}^{n-1}(\mathbb{1} - \frac{zA}{n})^{-m}\delta\big((\mathbb{1} - \frac{zA}{n})^{-1}\big)(\mathbb{1} - \frac{zA}{n})^{-n+m+1} \\
&= (\frac{z}{n})\sum_{m=0}^{n-1}(\mathbb{1} - \frac{zA}{n})^{-m-1}\delta(A)(\mathbb{1} - \frac{zA}{n})^{-n+m} \\
&= (\frac{z}{n})\sum_{m=0}^{n-1}(\mathbb{1} - \frac{zA}{n})^{-(m+1)}\delta(A)(\mathbb{1} - \frac{zA}{n})^{-(n-m)} \qquad (17)
\end{aligned}
$$

We now construct a double limit from equations (16) and (17) and in each term of the Darboux sum, we have (from step 2) $(\mathbb{1} - \frac{zA}{n})^{-(m+1)} = [(\mathbb{1} - \frac{zA}{n})^{-\frac{n}{z}}]^{(m+1)\frac{z}{n}}$ with $(\mathbb{1} - \frac{zA}{n})^{-\frac{n}{z}} \to e^A$ as $n \to \infty$. Also $(\mathbb{1} - \frac{zA}{n})^{-(n-m)} = [(\mathbb{1} - \frac{zA}{n})^{-\frac{n}{z}}]^{(n-m)\frac{z}{n}}$. Consider the sequence

$$
x_{kn} = (\frac{z}{k})\sum_{m=0}^{k-1}[(\mathbb{1} - \frac{zA}{n})^{-n}]^{(m+1)\frac{1}{k}}\delta(A)[(\mathbb{1} - \frac{zA}{n})^{-n}]^{(k-m)\frac{1}{k}}
$$

In order for the iterated limit theorem to be applicable, we need to show that both $\lim_k x_{kn} = z_n$ and $\lim_n x_{kn} = y_k$ exist, and that for at least one set of limits the convergence is uniform.

For the limit in k, we use the partition as defined in step 3 and construct an adjusted left sum:

$$\begin{aligned}
& z\int_0^1 (\mathbb{1}-\frac{zA}{n})^{-nt}\delta(A)(\mathbb{1}-\frac{zA}{n})^{-n(1-t)}dt \\
= \;& \lim_k(\frac{z}{k})\sum_{m=0}^{k-1}[(\mathbb{1}-\frac{zA}{n})^{-n}]^{\frac{m}{k}}\delta(A)[(\mathbb{1}-\frac{zA}{n})^{-n}]^{\frac{k-m}{k}} \\
= \;& \lim_k(\frac{z}{k})[(\mathbb{1}-\frac{zA}{n})^{-n}]^{\frac{1}{k}}\sum_{m=0}^{k-1}[(\mathbb{1}-\frac{zA}{n})^{-n}]^{\frac{m}{k}}\delta(A)[(\mathbb{1}-\frac{zA}{n})^{-n}]^{\frac{k-m}{k}} \\
= \;& \lim_k(\frac{z}{k})\sum_{m=0}^{k-1}[(\mathbb{1}-\frac{zA}{n})^{-n}]^{\frac{1+m}{k}}\delta(A)[(\mathbb{1}-\frac{zA}{n})^{-n}]^{\frac{k-m}{k}}
\end{aligned}$$

so that

$$\lim_k x_{kn} = z\int_0^1 (\mathbb{1}-\frac{zA}{n})^{-nt}\delta(A)(\mathbb{1}-\frac{zA}{n})^{-n(1-t)}dt = z_n$$

with

$$\lim_n x_{kn} = \frac{z}{k}\sum_{m=0}^{k-1} e^{(m+1)\frac{zA}{k}}\delta(A)e^{(k-m)\frac{zA}{k}} = y_k$$

The uniform convergence of $\lim_n f_n(t) = \lim_n(\mathbb{1}-\frac{zA}{n})^{-nt} = e^{tzA}$ for $t \in [0, 1]$ assures the uniform convergence in n of the series $x_{nk} \to y_k$ above, so that the iterated limit theorem applies.
Now from step 3 and equation (17) it follows that

$$\begin{aligned}
\lim_n \delta\big((\mathbb{1}-\frac{zA}{n})^{-n}\big) &= \lim_n(\frac{z}{n})\sum_{m=0}^{n-1}(\mathbb{1}-\frac{zA}{n})^{-(m+1)}\delta(A)(\mathbb{1}-\frac{zA}{n})^{-(n-m)} \\
&= \lim_k\lim_n\frac{z}{k}\sum_{m=0}^{k-1}[(\mathbb{1}-\frac{(zA)}{n})^{-n}]^{(m+1)\frac{1}{k}}\delta(A)[(\mathbb{1}-\frac{(zA)}{n})^{-n}]^{(k-m)\frac{1}{k}} \\
&= \lim_k\frac{z}{k}\sum_{m=0}^{k-1}e^{(m+1)\frac{zA}{k}}\delta(A)e^{(k-m)\frac{zA}{k}} \\
&= z\int_0^1 U_{tz}\delta(A)U_{(1-t)z}dt
\end{aligned}$$

From the closedness of δ (Theorem 28), we now have the following:

$(\mathbb{1}-\frac{zA}{n})^{-n} \in \mathcal{D}(\delta)$

$\lim_n(\mathbb{1} - \frac{zA}{n})^{-n} = U_z$

$\delta[(\mathbb{1} - \frac{zA}{n})^{-n}] \to z\int_0^1 U_{tz}\delta(A)U_{(1-t)z}dt$

which implies

$U_z \in \mathcal{D}(\delta)$

$\delta(U_z) = z\int_0^1 U_{tz}\delta(A)U_{(1-t)z}dt$

4.4 Fourier analysis on $\mathcal{D}(\delta)$

Theorem 72 (Bratteli, O. and Robinson, D.W. [3], Theorem 3.2.32) *Let δ be a norm-closed derivation of a C^* -algebra $\mathcal{A}$ with identity $\mathbb{1}$ and assume $\mathbb{1} \in \mathcal{D}(\delta)$. Let $f \in \mathcal{L}^1(\mathbb{R})$ be a function of one real variable so that $\hat{f}$ exists, and assume further the condition*

$$\int |\hat{f}(x)||x|dx < \infty$$

is satisfied. If $A = A^* \in \mathcal{D}(\delta)$, then $f(A) \in \mathcal{D}(\delta)$ and

$$\begin{aligned} f(A) &= (2\pi)^{-\frac{1}{2}} \int \hat{f}(x)e^{ixA}dx \\ \delta(f(A)) &= i(2\pi)^{-\frac{1}{2}} \int \hat{f}(x)x\{\int_0^1 e^{itxA}\delta(A)e^{i(1-t)xA}dt\}dx \end{aligned}$$

Proof: The proof follows the following steps:
Step 1 proves $\hat{f} \in \mathcal{C}_0(\mathbb{R})$ by means of the dominated convergence theorem.
Step 2 proves that $\hat{f} \in \mathcal{L}^1(\mathbb{R})$.
Step 3 proves $\check{\hat{f}}$ exists, $\check{\hat{f}} = f$ a.e and $\check{\hat{f}} \in \mathcal{C}_0(\mathbb{R})$.
Step 4 involves a Riemann sum construction of $f = \check{\hat{f}}$, shows that $\sum_N(f(A) \in \mathcal{D}(\delta)$ and then shows that $\delta(\sum_N f(A))$ converges (the previous exponential function theorem is used here). The result follows from the closedness of δ.

Step 1: By definition, $\hat{f}(t) = \int f(x)e^{-ixt}dx$. If $t_n \to t \in \mathbb{R}$, then

$$\begin{aligned} |\hat{f}(t_n) - \hat{f}(t)| &= |\int_{\mathbb{R}} [f(x)e^{-ixt_n} - f(t)e^{-ixt}]dx| \\ &\leq \int_{\mathbb{R}} |f(x)||e^{-ixt_n} - e^{-ixt}|dx \end{aligned}$$

Now $|e^{-ixt_n} - e^{-ixt}| \leq 2$ implies the integrand is bounded by $2|f(x)|$ and tends to 0 for every x as $n \to \infty$. From the dominated convergence theorem (Rudin, W. [20], 29) we have

$$\begin{aligned}\lim_n |\hat{f}(t_n) - \hat{f}(t)| &\leq \lim_n \int_{\mathbb{R}} |f(x)||e^{-ixt_n} - e^{-ixt}|dx \\ &= \int_{\mathbb{R}} \lim_n \left[|f(x)||e^{-ixt_n} - e^{-ixt}|\right]dx \\ &= 0 \ \forall \ x \in \mathbb{R}\end{aligned}$$

so that $\hat{f}(t_n) \to \hat{f}(t)$ which implies $\hat{f}$ is continuous. To prove vanishing at infinity, note that $e^{\pi i} = -1$ so that

$$\begin{aligned}\hat{f}(t) &= \int_{-\infty}^{\infty} f(x)e^{i\pi}e^{-i\pi}e^{-ixt}dx \\ &= -\int_{-\infty}^{\infty} f(x)e^{-it(x+\frac{\pi}{t})}dx \\ &= -\int_{-\infty}^{\infty} f(x-\frac{\pi}{t})e^{-itx}dx\end{aligned}$$

(Note the substitution $x := x + \frac{\pi}{t}$). Hence

$$2\hat{f}(t) = \int_{-\infty}^{\infty} \{f(x) - f(x-\frac{\pi}{t})\}e^{-itx}dx$$

The map $\frac{\pi}{t} \to f_{\frac{\pi}{t}} = f(x - \frac{\pi}{t})$ is uniformly continuous (Rudin, W. [20], 182), so that

$$2|\hat{f}(t)| \leq \|f - f_{\frac{\pi}{t}}\| \to_t^{\pm\infty} = 0$$

Step 2: From step 1, $\hat{f}$ is continuous on the compact subset $[-1, 1]$ so that $\hat{f}$ is bounded. Now

$$\int_{\mathbb{R}} |\hat{f}| = \int_{-\infty}^{-1} |\hat{f}| + \int_{-1}^{+1} |\hat{f}| + \int_{1}^{\infty} |\hat{f}|$$

and clearly $|\hat{f}(x)| \leq |x||\hat{f}(x)|$ on $\mathbb{R}\backslash[-1, +1]$ so that

$$\int_{-\infty}^{-1} |\hat{f}(x)| \leq \int_{-\infty}^{-1} |x||\hat{f}(x)| \leq \int_{-\infty}^{\infty} |x||\hat{f}(x)| < \infty$$

by assumption. $\int_1^{\infty} |\hat{f}(x)| < \infty$ follows from a similar argument.

Step 3: Since $f, \hat{f} \in \mathcal{L}_1(\mathbb{R})$ (steps 1 and 2) it follows from the inversion theorem (Rudin, W. [20], 185) that $g(x) = \int \hat{f}(t)e^{ixt}dt$, $g \in \mathcal{C}_0$ and $f = g = \check{\hat{f}}$ a.e. Since $\check{\hat{f}}$ is continuous on $\mathbb{R}$, we can replace f by $\check{\hat{f}}$ if necessary and assume f to be continuous.

Step 4: Since $\hat{f} \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{C}_0(\mathbb{R})$, we can construct a Riemann sum

$$\sum_N (f) = \sum_{j=1}^{N} (x_j - x_{j-1}) \hat{f}(x_j) e^{ix_j A}$$

From $\hat{f} \in \mathcal{L}_1(\mathbb{R})$ it follows that $\sum_N (f)$ is convergent. Looking at $\delta(\sum_N (f))$, it follows from the exponential function theorem that

$$\delta(e^{ix_j A}) = ix_j \int_0^1 e^{itx_j A} \delta(A) e^{i(1-t)x_j A} dt$$

so that

$$\delta(\sum_N (f)) = \sum_{j=1}^{N} (x_j - x_{j-1}) \hat{f}(x_j) ix_j \int_0^1 e^{itx_j A} \delta(A) e^{i(1-t)x_j A} dt$$

From the closedness of δ, we now have that the following (Theorem 28):

$\sum_N (f(A)) \in \mathcal{D}(\delta)$

$\sum_N (f(A)) \to \mathcal{X}$

$\delta[\sum_N (f(A))] \to \mathcal{Y}$

imply

$\mathcal{X} = f(A) = \check{\hat{f}}(A) = \int_{-\infty}^{\infty} \hat{f}(x) e^{ixA} dx \in \mathcal{D}(\delta)$

$\delta(\mathcal{X}) = \delta(f(A)) = i \int_{-\infty}^{\infty} \hat{f}(x) x \int_0^1 e^{itxA} \delta(A) e^{i(1-t)xA} dx$

which concludes the theorem.

Note that from first principles and step 3

$$\begin{aligned} \check{\hat{f}}'(x) &= \lim_{h \to 0} \frac{\check{\hat{f}}(x+h) - \check{\hat{f}}(x)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \int_{-\infty}^{\infty} \left[e^{i(x+h)t} - e^{ixt} \right] \hat{f}(t) dt \end{aligned}$$

The integrand is bounded by $2|\hat{f}(t)|$ ($\hat{f} \in \mathcal{L}_1(\mathbb{R})$) so that the dominated convergence theorem implies

$$\begin{aligned} \check{\hat{f}}'(x) &= \int_{-\infty}^{\infty} \lim_{h \to 0} \frac{1}{h} \left[e^{i(x+h)t} - e^{ixt} \right] \hat{f}(t) dt \\ &= \int_{-\infty}^{\infty} it \hat{f}(t) e^{ixt} dt \end{aligned}$$

Also $\left| \int_{-\infty}^{\infty} it\hat{f}(t) e^{ixt} dt \right| \leq \int_{\infty}^{\infty} |t||\hat{f}(t)| dt < \infty$ by assumption, implying $\delta(\sum_N (f))$ converges. This shows that the conditions for f in the hypothesis is stronger than $\mathcal{C}^1$.

4.5 $\mathcal{C}^2$ functions on $\mathcal{D}(\delta)$

The following theorem illustrates the domain preserving nature of $\mathcal{C}^2$ functions on $\mathcal{D}(\delta)$:

Theorem 73 (Sakai, S. [22], Theorem 3.3.7) *Let $A = A^\star \in \mathcal{D}(\delta)$ (δ is norm-closed) and $f \in \mathcal{C}^2([-||A||, ||A||])$ a twice continuously differentiable function. Then $f(A) \in \mathcal{D}(\delta)$.*

Proof: The theorem can be proved by showing that the conditions for f given, implies $\int_{-\infty}^{\infty} |x||\hat{f}(x)| < \infty$, which will imply the result from the previous theorem.

Define $g(x) = |x+i|^{-1}$ and $h(x) = |x^2+ix||\hat{f}(x)|$. Then

$$\begin{aligned} < g, h > &= \int_{-\infty}^{\infty} \frac{|x^2+ix|}{|x+i|}|\hat{f}(x)|dx \\ &= \int_{-\infty}^{\infty} \frac{\sqrt{x^4+x^2}}{\sqrt{x^2+1}}|\hat{f}(x)|dx \\ &= \int_{-\infty}^{\infty} \sqrt{x^2(\frac{x^2+1}{x^2+1})}|\hat{f}(x)|dx \\ &= \int_{-\infty}^{\infty} |x||\hat{f}(x)|dx \end{aligned}$$

Looking at $||g||||h||$;

$$\begin{aligned} ||g||||h|| &= (\int_{-\infty}^{\infty} |g(x)|^2 dx)^{\frac{1}{2}} (\int_{-\infty}^{\infty} |h(x)|^2 dx)^{\frac{1}{2}} \\ &= (\int_{-\infty}^{\infty} \left||x+i|^{-1}\right|^2 dx)^{\frac{1}{2}} (\int_{-\infty}^{\infty} \left||x^2+ix||\hat{f}(x)|\right|^2 dx)^{\frac{1}{2}} \end{aligned}$$

The first integral can be reduced to $\int_{-\infty}^{\infty} |x+i|^{-2}dx = \int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \pi$.

For the second integral, note that the function $f \in \mathcal{C}^2([-||A||, ||A||])$ can be extended to $f \in \mathcal{C}^2(\mathbb{R})$ with the support $\overline{f^{-1}((0, \infty))}$ compact, by adding polynomial pieces to the end points (at $x = -||A||$ and $x = +||A||$) so that for some $r > ||A||$, the polynomial extensions are $\mathcal{C}^2$ on the interval $[-r, r]$ with $f^{(i)}(\pm r) = 0$ for $i, = 0, 1, 2$. If we set $f = 0$ on $\mathbb{R}\backslash[-r, r]$, then f extends to a $\mathcal{C}^2$ function on $\mathbb{R}$ with compact support.
Now

$$\begin{aligned} (\widehat{\frac{d^2 f}{dt^2}} - \widehat{\frac{df}{dt}})(t) &= (it)^2\hat{f}(t) - it\hat{f}(t) \\ &= -(t^2+it)\hat{f}(t) \end{aligned}$$

Applying Parseval's identity (Rudin, W. [20], 187), the second integral therefore reduces to

$$\begin{aligned}\int_{\infty}^{\infty} |x^2 + ix|^2 |\hat{f}(x)|^2 dx &= \int_{-r}^{r} |\frac{d^2 f}{dx^2} - \frac{df}{dx}|^2 dx \\ &< \infty\end{aligned}$$

where the support $\overline{\{x|f(x) \neq 0\}} \subseteq [-r, r]$.

By the Cauchy-Schwarz in-equality ($| < g, h > | \leq ||g||||h||$) it follows that $f \in \mathcal{C}^2[-||A||, ||A||]$ implies $\int_{-\infty}^{\infty} |x||\hat{f}(x)|dx < \infty$, and hence $f(A) \in \mathcal{D}(\delta)$.

4.6 A counter example

In this section it is shown that the results obtained in Section 4, can not be extended for functions $f \in \mathcal{C}^1$. The counter example uses a function of the form $f(x) = |x|$ (suitably smoothed at $x = 0$) defined on a closed interval in $\mathbb{R}$ containing the spectrum of $A = A^\star$ and a derivation arrived at as the infinitesimal generator of a strongly continuous one-parameter group of $\star$- automorphisms of a $\mathcal{C}^\star$- algebra $\mathcal{A}$. A boundedness property for the elements in the domain $\mathcal{D}(\delta)$ of the derivation is then violated by the element $f(A)$, proving the results of Section 4 false for general $f \in \mathcal{C}^1$.

The function used is defined for all $\alpha \geq 0$ on $[-e^{-1}, e^{-1}]$ as

$$\begin{aligned}f_\alpha(x) &= \frac{|x|}{(\ln|\ln|e^{-1}|x|||)^\alpha} && \forall\, 0 < |x| \leq e^{-1} \\ &= 0 && x = 0\end{aligned}$$

$f_0 = |x|$ and $f_\alpha \in \mathcal{C}^1$ when $\alpha > 0$.

The following three results are required in the proof of the counter example. We want to construct two operators A and B in a Hilbert space with A bounded, $A(\mathcal{D}(B)) \subset \mathcal{D}(B)$ and $AB - BA$ bounded, but $f_\alpha(A)B - Bf_\alpha(A)$ un-bounded.

The proof of Theorem 74 requires some background on Toeplitz matrices and their corresponding functions. A brief summary of (Brown, A. and Halmos, P.R [4], 89-94) is listed here. Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$ and $\mathbb{D}$ the disk in $\mathbb{C}$ with radius 1 and consider the orthogonal basis $\{e_n\}$ for $\mathfrak{L}^2(\mathbb{T})$ defined by the bounded functions $e_n(z) = z_n$ for all $|z| = 1$ and $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. A function $f \in \mathfrak{L}^2(\mathbb{T})$ is analytic if the negatively indexed Fourier coefficients vanish - i.e if $\int f\overline{e_n} = 0$ for $n = -1, -2, -3 \ldots$. The analytic functions in $\mathfrak{L}^2$ is denoted by $\mathfrak{H}^2$. A bounded function ϕ on $\mathbb{T}$ induces two operators in the following ways:

For every $f \in \mathfrak{L}^2$, define the *Laurent operator* $L = L_\phi : \mathfrak{L}^2 \to \mathfrak{L}^2$ by $Lf = \phi f$

For every $f \in \mathfrak{L}^2$, define the *Toeplitz operator* $T = T : \mathfrak{H}^2 \to \mathfrak{H}^2$ as the compression of L to $\mathfrak{H}^2$ by $Tf = PL_\phi f = P(\phi f)$, where P is the orthogonal projection from $\mathfrak{L}^2$ onto $\mathfrak{H}^2$.

Clearly $||Lf||_2 \leq ||\phi||_\infty ||f||_2$. Thus $||T_\phi|| \leq ||L_\phi|| \leq ||\phi||_\infty$

Laurent operators

The special Laurent operator L_{e_1} is called the bilateral shift - denoted by W. The description 'shift' is evident from the fact the $We_n = e_{n+1}$ for n, ± 1, ± 2, $\pm 3 \ldots$

Result 1: An operator L on $\mathfrak{L}^2$ is Laurent if and only if L commutes with the bilateral shift operator.

Every operator A on $\mathfrak{L}^2$ has an infinite matrix $[a_{ij}]$ with respect to the basis $\{e_n\}$ given by $[a_{ij}] = \langle Ae_j, e_i\rangle$. Now if $L_\phi = A$ for some bounded function ϕ, then the matrix $[a_{ij}]$ of A has an expansion in terms of the Fourier coefficients of ϕ:

$$\phi = \sum_i \alpha_i e_i$$

where

$$\alpha_i = \langle \phi, e_i \rangle$$

For a Laurent operator L we have

$$\begin{aligned} \alpha_{ij} &= \langle Ae_j, e_i\rangle = \langle \phi e_j, e_i\rangle = \langle e_j \phi, e_i \rangle \\ &= \langle W^j \phi, e_i \rangle = \langle \phi, W^{*j} e_i \rangle = \langle \phi, e_{i-j}\rangle = \alpha_{i-j} \end{aligned}$$

which motivates the definition of a Laurent matrix:

Define a Laurent matrix as a two way infinite matrix $[a_{ij}]$ $(i, j = 0, \pm 1, \pm 2, \pm 3, \ldots)$ with $[a_{i+1, j+1}] = [a_{ij}]$

Result 2: An operator L on $\mathfrak{L}^2$ is Laurent if and only if the matrix $[a_{ij}]$ of L with respect to the basis $\{e_n | n = 0, \pm 1, \pm 2, \pm 3, \ldots\}$ is a Laurent matrix.

How do we recapture ϕ from the matrix of A? Since $\phi = Ae_0$, it follows that the Fourier coefficients of ϕ are the matrix $\langle Ae_0, e_i \rangle$ which is the 0- column of the matrix of A.

Toeplitz operators

The following results for Toeplitz operators follows from the preceding:

If $\phi = 1$, then T_ϕ is the identity.

If $\phi = \alpha\varphi + \beta\gamma$ then $T_\phi = \alpha T_\varphi + \beta T_\gamma$.

If $\phi = \overline{\phi}$ then $T_\phi = T_\phi^*$.

The mapping $\phi \to T_\phi$ is one-to-one.

T_ϕ is positive if and only if ϕ is positive.

If ϕ is a bounded function, then $T_\phi f = PL_\phi f$ for every $f \in \mathfrak{H}^2$. Therefore, for $i, j \geq 0$ we have (as before)

$$\begin{aligned} \langle T_\phi e_j, e_i \rangle &= \langle PL_\phi e_j, e_i \rangle = \langle L_\phi e_{j+1}, e_{i+1} \rangle \\ &= \langle PL_\phi e_{j+1}, e_{i+1} \rangle = \langle T_\phi e_{j+1}, e_{i+1} \rangle \end{aligned}$$

which leads to the definition of a Toeplitz matrix:

Define a Toeplitz matrix as a one way infinite matrix $[a_{ij}]$ $(i, j = 0, 1, 2, ...)$ with $[a_{i+1,j+1}] = [a_{ij}]$

Result 3: An operator T on $\mathfrak{H}^2$ is a Toeplitz operator if and only if the matrix $[a_{ij}]$ of T with respect to the basis $\{e_n | n = 0, 1, 2, ...\}$ is a Toeplitz matrix.

The proof is listed in (Brown, A. and Halmos, P.R [4], 93), and the proof of sufficiency is briefly as follows:

Assume A is an operator on $\mathfrak{H}^2$ such that $\langle Ae_{j+1}, e_{i+1} \rangle = \langle Ae_j, e_i \rangle$ for $i, j = 0, 1, 2, ...$ It needs to be shown that A is a Toeplitz operator. Consider the operator on $\mathfrak{L}^2$ given by

$$A_n = W^{*n} A P W^n \qquad \forall n \geq 0 \tag{18}$$

where W is the bilateral shift operator. It can be shown (Brown, A. and Halmos, P.R [4], 93) that the sequence $\{A_n\}$ of operators on $\mathfrak{L}^2$ is weakly convergent to a bounded operators A_∞ on $\mathfrak{L}^2$, that A_∞ has a Laurent matrix which implies that A_∞ is a Laurent operator, and that

$$PA_\infty f = Af \qquad \forall f \in \mathfrak{H}^2$$

Therefore, A is the compression to $\mathfrak{H}^2$ of the Laurent operator A_∞ and is hence a Toeplitz operator.

How do we recapture ϕ from the matrix of A? If $A = T_\phi$, then $A_\infty = L_\phi$ (as in the proof of Result 3) so that the Fourier coefficients of ϕ are the entries in the 0- column of the matrix of A_∞. An expression in terms of A is evident from the following: If $i, j \geq 0$, then

$$\langle Ae_j, e_i \rangle = \langle A_\infty e_j, e_i \rangle = \langle \phi, e_{i-j} \rangle$$

so that

$$\langle \phi, e_i \rangle = \langle Ae_0, e_i \rangle \qquad \forall i \geq 0$$

and

$$\langle \phi, e_{-j} \rangle = \langle Ae_j, e_0 \rangle \qquad \forall j \geq 0$$

Therefore, ϕ is the function who's positively indexed Fourier coefficients are the entries of the 0- column of the matrix of A, and who's negatively indexed Fourier coefficients are the entries of the 0- row of the matrix of A.

In particular, given the (Toeplitz) matrix W with

$$W = \begin{bmatrix} 0 & 1 & \frac{1}{2} & \cdots \\ -1 & 0 & 1 & \cdots \\ -\frac{1}{2} & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

it needs to be shown (Brown, A. and Halmos, P.R [4], 94) that the function $f(x) = i(\pi - x) \forall x \in (0, \pi)$ has its forward Fourier coefficients (positive index) equal to the terms in the 0 - column of the matrix of W and its backward Fourier coefficients (negative index) equal to the 0 - row of the matrix of W.
The general Fourier series expansion of $f(x)$ is as follows:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad where$$
$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$$

Therefore, for $f(x) = i(\pi - x) \forall x \in (0, 2\pi)$ we have

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} i(\pi - x)dx = 0$$

and

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_0^{2\pi} i\pi e^{-inx}dx - \frac{1}{2\pi}\int_0^{2\pi} ixe^{-inx}dx \\ &= \underbrace{\frac{i}{2}\int_0^{2\pi} e^{-inx}dx}_{1} - \underbrace{\frac{i}{2\pi}\int_0^{2\pi} xe^{-inx}dx}_{2} \end{aligned} \tag{19}$$

Partial integration (factor integration) yields the following:

$$\begin{aligned} Integral\,1: &= \frac{i}{2}(\frac{-1}{in})e^{-inx}\Big|_0^{2\pi} \\ &= -\frac{1}{2n}[\cos(nx) - i\sin(nx)\Big|_0^{2\pi}] \\ &= -\frac{1}{2n}[(1-0)-(1-0)] \\ &= 0 \qquad\qquad n \neq 0 \end{aligned}$$

and

$$\begin{aligned} Integral\,2: &= \frac{i}{2\pi}x\int e^{-inx}dx - \frac{i}{2\pi}\int[\int e^{-inx}dx]1dx\Big|_0^{2\pi} \\ &= \frac{ix}{2\pi}(\frac{-1}{in})e^{-inx} - \frac{i}{2\pi}\int(\frac{-1}{in})e^{-inx}dx\Big|_0^{2\pi} \\ &= \frac{-x}{2n\pi}e^{-inx} + \frac{i}{2n^2\pi}e^{-inx}\Big|_0^{2\pi} \\ &= [\frac{-x}{2n\pi} + \frac{i}{2n^2\pi}](\cos(nx) - i\sin(nx))\Big|_0^{2\pi} \\ &= [\frac{-2\pi}{2n\pi} + \frac{i}{2n^2\pi}]\cos(2n\pi) - [0 + \frac{i}{2n^2\pi}](1) \\ &= -\frac{1}{n} + \frac{i}{2n^2\pi} - \frac{i}{2n^2\pi} \\ &= -\frac{1}{n} \end{aligned}$$

Replacing 1 and 2 back into equation (19) yields the required result

$$\begin{aligned} c_n &= 0 - \frac{-1}{n} \\ &= \frac{1}{n} \qquad\qquad n \neq 0 \end{aligned}$$

Toeplitz matrix operators and their corresponding functions are useful in deriving the norm of the matrix operator. The following theorem makes use of this connection:

Theorem 74 (McIntosh, A. [11], Theorem 1) *For every integer $m \geq 3$ there is a self-adjoint operator $U = U^{\star}$ and a skew-adjoint operator V in the Hilbert space $\mathbb{C}^m$ satisfying $e^{-m}\mathbb{1} \leq U \leq e^{-1}\mathbb{1}$, $\|UV + VU\| \leq \pi$, and $\|f_\alpha(U)V - Vf_\alpha(U)\| > \frac{1}{8}(\log(\frac{1}{2}m))^{1-\alpha}$ for $0 \leq \alpha < 1$.*

Proof: Define a skew-adjoint operator W on $\mathbb{C}^m$ by $W_{ij} = (j-i)^{-1}$ when $i \neq j$ and $W_{ii} = [0]$ when $i = j$:

$$W = \begin{bmatrix} 0 & 1 & \frac{1}{2} & \cdots & \frac{1}{m-1} \\ -1 & 0 & 1 & \cdots & \frac{1}{m-2} \\ -\frac{1}{2} & -1 & 0 & \cdots & \frac{1}{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{m-1} & \frac{-1}{m-2} & \frac{-1}{m-3} & \cdots & 0 \end{bmatrix}$$

W is a $m \times m$ Toeplitz matrix with corresponding function $g(\theta) = i(\pi - \theta)$ on $0 < \theta < 2\pi$ so that $\|W\| \leq \pi$.
Define U by the diagonal matrix $U_{ii} = u_i = e^{-i}$:

$$U = \begin{bmatrix} e^{-1} & 0 & 0 & \cdots \\ 0 & e^{-2} & 0 & \cdots \\ 0 & 0 & e^{-3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Define the skew-adjoint operator V by the matrix $V_{ij} = W_{ij}(u_i + u_j)^{-1} = W_{ij}(e^{-i} + e^{-j})^{-1}$:

$$V = \begin{bmatrix} 0 & 1(e^{-1}+e^{-2})^{-1} & \frac{1}{2}(e^{-1}+e^{-3})^{-1} & \frac{1}{3}(e^{-1}+e^{-4})^{-1} & \cdots \\ -1(e^{-2}+e^{-1})^{-1} & 0 & 1(e^{-2}+e^{-3})^{-1} & \frac{1}{2}(e^{-2}+e^{-4})^{-1} & \cdots \\ -\frac{1}{2}(e^{-3}+e^{-1})^{-1} & -1(e^{-3}+e^{-2})^{-1} & 0 & 1(e^{-3}+e^{-4})^{-1} & \cdots \\ -\frac{1}{3}(e^{-4}+e^{-1})^{-1} & -\frac{1}{2}(e^{-4}+e^{-2})^{-1} & -1(e^{-4}+e^{-3})^{-1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then

$$(UV+VU)=\begin{bmatrix} 0 & [\frac{e^{-1}+e^{-2}}{e^{-1}+e^{-2}}] & \frac{1}{2}[\frac{e^{-1}+e^{-3}}{e^{-1}+e^{-3}}] & \cdots \\ -[\frac{e^{-1}+e^{-2}}{e^{-1}+e^{-2}}] & 0 & [\frac{e^{-2}+e^{-3}}{e^{-2}+e^{-3}}] & \cdots \\ -\frac{1}{2}[\frac{e^{-1}+e^{-3}}{e^{-1}+e^{-3}}] & -[\frac{e^{-2}+e^{-3}}{e^{-2}+e^{-3}}] & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}=W$$

so that $||UV+VU|| \le \pi$. Now put $S = f_\alpha(U)V - Vf_\alpha(U)$. Then $S_{ij} = (f_\alpha(e^{-i}) - f_\alpha(e^{-j}))W_{ij}(e^{-i}+e^{-j})$. We need to show that S is unbounded as $m \to \infty$.

First, observe that $f_\alpha(e^{-i}) = \frac{|e^{-i}|}{\ln|\ln|e^{-1}e^{-i}||^\alpha} = \frac{e^{-i}}{(\ln(i+1))^\alpha}$ so that

$$S_{ij} = \frac{\frac{e^{-i}}{(\ln(i+1))^\alpha} - \frac{e^{-j}}{(\ln(j+1))^\alpha}}{(j-i)(e^{-i}+e^{-j})}$$

so that $S_{ij} \ge 0 \forall i, j = 1, 2, ..., m$. Looking at the upper triangle only $(i < j)$, then $\ln(i+1) < \ln(j+1)$ and $e^{-i} > e^{-j}$ so that

$$\begin{aligned} S_{ij} &\ge \frac{e^{-i} - e^{-j}}{(j-i)2e^{-i}[\ln(j+1)]^\alpha} \\ &= \frac{1-e^{-j+i}}{2(j-i)[\ln(j+1)]^\alpha} \\ &\ge \frac{1-e^{-1}}{2(j-i)[\ln(j+1)]^\alpha} \end{aligned}$$

Hence for $2 \le j \le m$

$$\begin{aligned} \sum_{i=1}^{m} S_{ij} &\ge \sum_{i=1}^{j-1} S_{ij} \ge \frac{1}{2}(1-e^{-1})[\ln(j+1)]^{-\alpha}\{1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{j-2}+\frac{1}{j-1}\} \\ &> \frac{1}{2}(1-e^{-1})[\ln(j+1)]^{-\alpha}\int_1^j \frac{1}{t}dt \\ &= \frac{1}{2}(1-e^{-1})[\ln(j+1)]^{-\alpha}\ln(j) \\ &> 26^{-\frac{1}{2}}(\ln(j+1))^{1-\alpha} \end{aligned}$$

The last step follows from the fact that $\frac{\ln(j)}{\ln(j+1)} \ge \frac{\ln(2)}{\ln(3)}$ for $2 \le j \le m$ which is motivated by noting that $\log_2(j) = \frac{\ln(j)}{\ln(2)}$ and $\log_3(j+1) = \frac{\ln(j+1)}{\ln(3)}$ and setting $x = \log_2(j)$. Then for $(j \geq 2,\ x \geq 1)$

$$\begin{aligned} j &= 2^x \\ j+1 &= 2^x + 1 \\ &\leq 3^x \end{aligned}$$

which implies $x \geq \log_3(j+1)$.

Let $L = [1, 1, 1, ..., 1]$ be a vector in $\mathbb{C}^m$ so that $||L||^2 = m$ and

$$\begin{aligned} 26m||S||^2 = 26||L||^2||S||^2 &\geq 26||S(L)||^2 = \sum_{j=1}^{m}\{\sum_{i=1}^{m} S_{ij}\}^2 \\ &> \sum_{j=2}^{m}(\ln(j+1))^{2-2\alpha} \\ &> \sum_{j=\lfloor(m+1)/2\rfloor}^{m}(\ln(\frac{1}{2}m))^{2-2\alpha} \\ &> (\frac{1}{2}m)(\ln(\frac{1}{2}m))^{2-2\alpha} \end{aligned}$$

It follows that $||f_\alpha(U)V - Vf_\alpha(U)|| = ||S|| > \frac{1}{\sqrt{52}}(\ln(\frac{1}{2}m))^{1-\alpha} > \frac{1}{8}(\ln(\frac{1}{2}m))^{1-\alpha}$.

Theorem 75 (McIntosh, A. [11], Theorem 2) *There exists for every integer $m \geq 3$ self-adjoint operators A_m and B_m in the Hilbert space $\mathbb{C}^{2m}$ satisfying $e^{-m}\mathbb{1} \leq |A_m| \leq e^{-1}\mathbb{1}$, $||A_mB_m - B_mA_m|| \leq \pi$ but $||f_\alpha(A_m)B_m - B_mf_\alpha(A_m)|| > \frac{1}{8}(\ln(\frac{m}{2}))^{1-\alpha}$ for $0 \leq \alpha < 1$.*

Proof: Define A_m and B_m as

$$A_m = \begin{bmatrix} U & 0 \\ 0 & -U \end{bmatrix} \qquad \textit{and} \qquad B_m = \begin{bmatrix} 0 & V \\ V^\star & 0 \end{bmatrix}$$

with U and V as defined in Theorem 74. Then

$$A_mB_m = \begin{bmatrix} 0 & UV \\ -UV^\star & 0 \end{bmatrix} \qquad \textit{and} \qquad B_mA_m = \begin{bmatrix} 0 & -VU \\ V^\star U & 0 \end{bmatrix}$$

so that (since $U = U^\star$) it follows that

$$A_m B_m - B_m A_m = \begin{bmatrix} 0 & UV + VU \\ -(UV + VU)^\star & 0 \end{bmatrix}$$

Again from Theorem 74 we have

$||A_m B_m - B_m A_m|| = ||UV + VU|| \leq \pi$. Also (since f_α is an *even function*) we have

$$f_\alpha(A_m) = \begin{bmatrix} f_\alpha(U) & 0 \\ 0 & f_\alpha(-U) \end{bmatrix} = \begin{bmatrix} f_\alpha(U) & 0 \\ 0 & f_\alpha(U) \end{bmatrix}$$

so that

$$f_\alpha(A_m)B_m - B_m f_\alpha(A_m) = \begin{bmatrix} 0 & f_\alpha(U)V - V f_\alpha(U) \\ -(f_\alpha(U)V - V f_\alpha(U))^\star & 0 \end{bmatrix}$$

It then follows from Theorem 74 that

$||f_\alpha(A_m)B_m - B_m f_\alpha(A_m)|| > \frac{1}{8}(\ln(\frac{m}{2}))^{1-\alpha}$

We now want to construct an infinite dimensional case from the cases $\mathbb{C}^{2m}$, A_m and B_m and some *direct sum* techniques for Hilbert spaces are required.

Suppose $\{\mathcal{H}_m\}_{m\in\mathbb{N}}$ is a family of Hilbert spaces. The direct sum $\mathcal{H} = \bigoplus_{m\in\mathbb{N}} \mathcal{H}_m$ consists of elements $x = \{x_m\}_{m\in\mathbb{N}}$ with $x_m \in \mathcal{H}_m$ such that

$$||x||^2 = \sum_{m\in\mathbb{N}} ||x_m||^2 < \infty$$

$\mathcal{H} = \bigoplus_{m\in\mathbb{N}} \mathcal{H}_m$ is a Hilbert space with the inner product defined as

$$\langle x, y\rangle = \sum_{m\in\mathbb{N}} \langle x_m, y_m\rangle \qquad \textit{with} \qquad x, y \in \mathcal{H}$$

If A_m is a bounded linear operator defined on $\mathcal{H}_m$ for every $m \in \mathbb{N}$ with $\sup_m \|A_m\| < \infty$, define

$$\bigoplus_{m\in\mathbb{N}} A_m : \bigoplus_{m\in\mathbb{N}} \mathcal{H}_m \to \bigoplus_{m\in\mathbb{N}} \mathcal{H}_m$$

as

$$\bigoplus_{m\in\mathbb{N}} A_m(\{x_m\}) = \{A_m x_m\} = \{A_1 x_1,\, A_2 x_2,\, ...,\, A_m x_m,\, ...\}$$

Then $\bigoplus_{m\in\mathbb{N}} A_m$ is linear on $\bigoplus_{m\in\mathbb{N}} \mathcal{H}_m$ with norm defined as

$$\|\bigoplus_{m\in\mathbb{N}} A_m\|_{\mathcal{H}} = \sup_m\{\|A_m\|_{\mathcal{H}_m} | m \in \mathbb{N}\} = \sup_m \big\{ \sup_{\|x_m\|=1} \{\|A_m x_m\|\}_{\mathcal{H}_m} \big| m \in \mathbb{N}\big\}$$

The direct sum of bounded linear operators have the following properties:

$$\begin{aligned}
\bigoplus_{m\in\mathbb{N}} (\alpha A_m + \beta A_m) &= \alpha \bigoplus_{m\in\mathbb{N}} A_m + \beta \bigoplus_{m\in\mathbb{N}} A_m \\
\big(\bigoplus_{m\in\mathbb{N}} A_m\big)^* &= \bigoplus_{m\in\mathbb{N}} A_m^* \\
\big(\bigoplus_{m\in\mathbb{N}} A_m\big)\big(\bigoplus_{m\in\mathbb{N}} B_m\big) &= \bigoplus_{m\in\mathbb{N}} A_m B_m
\end{aligned}$$

The notion of a *core* $\mathcal{C}(B)$ of a linear operator B is required in the proof of the following theorem:

Definition 22 (Core) *Let $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator. A subset $\mathcal{C}(B)$ of $\mathcal{D}(B)$ is a* core *of B if it is graph norm-dense $\big(\|\cdot\|_B = \|\cdot\| + \|B(\cdot)\|\big)$ in $\mathcal{D}(B)$.*

Remark 12 (Core) *The definition of a core can be re-stated as follows:*
A subset $\mathcal{C}(B)$ is a core for a linear operator B if for every $x \in \mathcal{D}(B)$ there exists a sequence $\{x_n\} \in \mathcal{C}(B)$ such that

$$\begin{aligned}
x_n &\to x \qquad\qquad \textit{and} \\
B(x_n) &\to B(x)
\end{aligned}$$

Definition 23 (Symmetric operator) *Let $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator with dense domain in the Hilbert space $\mathcal{H}$. If $\langle Bx, y\rangle = \langle x, By\rangle\ \forall\, x, y, \in \mathcal{D}(B)$, then B is called symmetric.*

Remark 13 (Symmetric operator) *Since B^* (the adjoint of B) is always closed (Remark 2.7.6 Kadison, R.V. and Ringrose, J.R. [8], 157) and $\mathcal{G}(B) \subseteq \mathcal{G}(B^*)$, it follows that B is closable if it is symmetric. If $B = B^*$ (self-adjoint), then B is symmetric and closed. Conditions for a closed (closable) operator B to be self-adjoint is given in Theorem 2.7.10 of Kadison, R.V. and Ringrose, J.R. [8], 160.*

Theorem 76 (McIntosh, A. [11], Theorem 3) *There exist self-adjoint operators A and B in a Hilbert space $\mathcal{H}$ such that $||A|| \leq e^{-1}$, $A(\mathcal{D}(B)) \subset \mathcal{D}(B)$, $||AB - BA|| \leq \pi$ but $f_\alpha(A)B - Bf_\alpha(A)$ is unbounded for $0 \leq \alpha < 1$.*

Proof: Define $H = \bigoplus_{m\geq 3} \mathbb{C}^{2m}$, $A = \bigoplus_{m\geq 3} A_m$ and $B = \bigoplus_{m\geq 3} B_m$ with A_m and B_m as defined in Theorem 75. The proof is conducted along the following steps:

Step 1 Show that $||A|| \leq e^{-1}$
Step 2 Show that there exists a core $\mathcal{C}(B)$ for B with $A(\mathcal{C}(B)) \subset \mathcal{C}(B)$.
Step 3 Show that $||AB - BA|| \leq \pi$
Step 4 Show that A and B are self-adjoint and hence closed.
Step 5 Show that $A\mathcal{D}(B) \subset \mathcal{D}(B)$ from $A(\mathcal{C}(B)) \subset \mathcal{C}(B)$, closedness of B and the boundedness of A and $AB - BA$ on $\mathcal{C}(B)$.
Step 6 Show that $f_\alpha(A)B - Bf_\alpha(A)$ is un-bounded.

Step 1: From Theorem 75

$$\begin{aligned} ||A||_{\mathcal{H}} &= \sup_m\{||A_m||_{\mathbb{C}^{2m}}\,|\, m \geq 3\} \\ &\leq e^{-1} \end{aligned}$$

Step 2: Consider the set $\mathcal{C}(B) = \{x = \{x_m\} | all\, but\, finitely\, many\, of\, the\, x_m\, are\, zero\}$. According to Definition 22, this set is a core of B if for every $x = \{x_m\} \in \mathcal{D}(B) = \mathcal{D}(\bigoplus_{m\geq 3} B_m)$, there exists a sequence $z_n \in \mathcal{C}(B)$ such that $z_n \to x = \{x_m\}$ $(m \geq 3)$ and $B(z_n) \to B(x) = \bigoplus_{m\geq 3} B_m(\{x_m\})$. Choose $z_n = \{x_3, x_4, x_5, ..., x_n, 0, 0, ...\}$. Then clearly $z_n \in \mathcal{C}(B)$, $z_n \to x = \{x_m\} \in \mathcal{D}(B)$ and

$$Bz_n = \{B_3x_3, B_4x_4, B_5x_5, ..., B_nx_n, B_{n+1}0, B_{n+2}0, ...\} \to Bx$$

as $n \to \infty$. From the boundedness of A it is clear that $Az_n \to Ax$ and $B(Az_n) \to BAx$.

Step 3: To show that $||AB - BA|| \leq \pi$, note that since $AB - BA$ is densely defined and bounded on $\mathcal{C}(B)$, $\bigoplus_{m\geq 3}(A_m B_m - B_m A_m)$ extends $AB - BA$ to $\mathcal{H}$ and we have

$$\begin{aligned} || \bigoplus_{m\geq 3}(A_m B_m - B_m A_m)||_{\mathcal{H}} &= \sup_m \{||A_m B_m - B_m A_m||_{\mathbb{C}^{2m}} | m \geq 3\} \\ &\leq \pi \end{aligned}$$

from Theorem 75.

Step 4: The self-adjointness of A follows from the self-adjointness and boundedness of A_m $\forall\, m$ and Theorem 75:

$$A = \bigoplus_m A_m = \bigoplus_m A_m^* = \big(\bigoplus_m A_m\big)^* = A^*$$

Since $||B_m||$ $\forall\, m$ is un-bounded, the same argument cannot be used to show self-adjointness for B. It needs to be shown that B is symmetric ($\langle Bx,\, y\rangle = \langle x,\, By\rangle$, $B \subseteq B^*$, B is closable) and the operator $B \pm i\mathbb{1}$ has a dense range, because then from Theorem 2.7.10 (Kadison, R.V. and Ringrose, J.R. [8], 160)) B can be extended to a self-adjoint (closed) linear operator on $\mathcal{D}(B)$.

To show that B is symmetric (and closable), we know from Theorem 75 that B_m is self-adjoint and hence (closed and) symmetric (Kadison, R.V. and Ringrose, J.R [8], 160). By definition of the inner product on $\mathcal{H} = \bigoplus_{m\geq 3}\mathcal{H}_m$ it follows that ($\forall\, x,\, y \in \mathcal{C}(B)$)

$$\langle Bx,\, y\rangle = \sum_m \langle B_m x_m,\, y_m\rangle = \sum_m \langle x_m,\, B_m y_m\rangle = \langle x,\, By\rangle$$

so that B is symmetric.

To show that $(B \pm i\mathbb{1})$ has a dense range, note that with B_m self-adjoint (symmetric and closed), it follows from Theorem 2.7.10 (Kadison, R.V. and Ringrose, J.R. [8], 160) that $\forall\, m$,

$$(B_m \pm i\mathbb{1}_m)(\mathcal{H}_m) = \mathcal{H}_m$$

Then $\forall\, N \geq 3$

$$\begin{aligned} (B \pm i\mathbb{1})\Big(\bigoplus_{m=3}^{N}\mathcal{H}_m\Big) \oplus \Big(\bigoplus_{m\geq N+1}\{0\}\Big) &= \Big(\bigoplus_{m=3}^{N}(B_m \pm i\mathbb{1})\mathcal{H}_m\Big) \oplus \Big(\bigoplus_{m\geq N+1}\{0\}\Big) \\ &= \Big(\bigoplus_{m=1}^{N}\mathcal{H}_m\Big) \oplus \Big(\bigoplus_{m\geq N+1}\{0\}\Big) \end{aligned}$$

so that the range of $(B \pm i\mathbb{1})$ is dense in $\bigoplus_{m \geq 3} \mathcal{H}_m = \mathcal{H}$. The closedness of B can now be used in the following step:

Step 5: The following is true at this point: For any $x \in \mathcal{D}(B)$, there exists a sequence $x_n \in \mathcal{C}(B)$ with $x_n \to x$ and $Bx_n \to Bx$ and from the boundedness of A, we also have $ABx_n \to ABx$ and $Ax_n \to Ax$.

If we can show that $B(Ax_n) \to y$ (for some y), then by the closedness of B (from Step 3) it follows that $Ax \in \mathcal{D}(B)$ so that we have the result $A(\mathcal{D}(B)) \subset \mathcal{D}(B)$.

To show that $B(Ax_n)$ converges, consider (for $x_n \in \mathcal{C}(B)$)

$$\begin{aligned} BAx_n &= BAx_n - ABx_n + ABx_n \\ &= -(AB - BA)x_n + ABx_n \end{aligned}$$

Now $ABx_n \to ABx$ and $(AB - BA)$ is bounded (step 3) so that BAx_n is convergent.

Step 6:

$$\begin{aligned} \|f_\alpha(A)B - Bf_\alpha(A)\| &= \|f_\alpha(\bigoplus_m A_m) \bigoplus_m B_m - \bigoplus_m B_m f_\alpha(\bigoplus_m A_m)\| \\ &= \|\bigoplus_m (f_\alpha(A_m)B_m - B_m f_\alpha(A_m))\| \\ &= \sup_m\{\|f_\alpha(A_m)B_m - B_m f_\alpha(A_m)\| | m \geq 3\} \end{aligned}$$

and un-boundedness follows again from Theorem 75.

The counter example is given in the following theorem:

Theorem 77 (McIntosh, A. [11], Theorem 4) *There is a closed symmetric derivation δ of a $\mathcal{C}^\star$- algebra $\mathcal{A}$ with dense domain $\mathcal{D}(\delta)$, an element $A = A^\star \in \mathcal{D}(\delta)$, and a function $f \in \mathcal{C}^1$ on a closed interval containing the spectrum of A, such that $f(A) \notin \mathcal{D}(\delta)$.*

Proof: Choose A, B and $\mathcal{H}$ as above and define the one parameter family $\Gamma_t : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ by $\Gamma_t(C) = e^{itB}Ce^{-itB}$ for each $t \in \mathbb{R}$, $C \in \mathcal{L}(\mathcal{H})$. Let $\mathcal{A}$ be the $\mathcal{C}^\star$- algebra generated by $\{\Gamma_t(A) | t \in \mathbb{R}\}$. The construction of the counter example follows the following steps:

Step 1 shows that Γ_t is a group of $\star$- automorphisms of $\mathcal{A}$.
Step 2 shows that Γ_t is strongly continuous.

Step 3 shows that the infinitesimal generator δ for Γ_t is a symmetric derivation with the required domain properties $\mathcal{D}(\delta) = \{C \in \mathcal{A} | C(\mathcal{D}(B)) \subset \mathcal{D}(B)\, and \, ||BC - CB|| < \infty\}$. *step 4* shows that δ is closed and $\mathcal{D}(\delta)$ is dense in $\mathcal{A}$.

By taking f as one of the functions f_α defined earlier with $\alpha \in (0, 1)$, it then follows that $A \in \mathcal{D}(\delta)$, but $f(A)B - Bf(A)$ is unbounded (from Theorem 76) so that $f(A) \notin \mathcal{D}(\delta)$.

Step 1: To show that for any given t, Γ_t is $\star$- automorphic on the larger algebra $\mathcal{L}(H)$ is straight forward: For example for $C, D \in \mathcal{L}(H)$, $\Gamma_t(CD) = e^{itB}CDe^{-itB} = e^{itB}C[e^{-itB}e^{itB}]De^{-itB} = \Gamma_t(C)\Gamma_t(D)$ and $\Gamma_t(C)^\star = [e^{itB}Ce^{-itB}]^\star = (e^{-itB})^\star(e^{itB}C)^\star = e^{itB}C^\star e^{-itB} = \Gamma_t(C^\star)$. From $\Gamma_t(e^{it'B}Ae^{-it'B}) = e^{i(t+t')B}Ae^{-i(t+t')B}$, it is clear that the generators of $\mathcal{A}$ is mapped into $\mathcal{A}$. Also, $\Gamma_{-t}(e^{itB}Ae^{-itB}) = A$, so that the mapping is one-one and surjective. The semi-group properties $\Gamma_0(A) = A$ and $\Gamma_{t+s}(A) = e^{(t+s)iB}Ae^{-(t+s)iB} = \Gamma_t(e^{isB}Ae^{-isB}) = \Gamma_t(\Gamma_s(A))$ are easily verified.

Step 2: Strong continuity follows directly from the continuity of e^{itB} on the Hilbert space $\mathcal{H}$: $\lim_{t\to 0} < e^{itB}Ae^{-itB}x, y > = \lim_{t\to 0} < Ae^{-itB}x, e^{-itB}y > = < Ax, y >$.

Step 3: The infinitesimal generator δ of Γ_t is obtained by differentiating Γ_t at $t = 0$:

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}\Gamma_t(A) &= \frac{d}{dt}\Big|_{t=0} e^{itB}Ae^{-itB} \\ &= iB(e^{itB}Ae^{-itB}) + e^{itB}A(-iB)e^{-itB}\Big|_{t=0} \\ &= i(BA - AB) \end{aligned}$$

for $A \in \mathcal{D}(\delta)$. Then from Theorem 69, δ is a symmetric derivation with domain $\mathcal{D}(\delta) = \{C \in \mathcal{A} | C(\mathcal{D}(B)) \subset \mathcal{D}(B) \; and \; ||BC - CB|| < \infty\}$.

Step 4: It follows from Theorem 57 section 3.1.2 that $\mathcal{D}(\delta)$ is dense in $\mathcal{A}$ and δ is closed.
Now select A, B and $\mathcal{H}$ as in the previous theorem and a function $f \in \mathcal{C}_1$ from the family f_α with $\alpha \in (0, 1)$. Clearly $A \in \mathcal{D}(\delta)$, but $f(A)$ will result in the unbounded element $f(A)B - Bf(A)$ (from Theorem 76), violating the domain restrictions so that $f(A) \notin \mathcal{D}(\delta)$.

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