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The cap set problem and standard diagrams

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Abstract

In n-dimensional affine space over the field $\mathbb{Z}/3\mathbb{Z}$, a *cap* is given by a set of points no three of which are in a line, and the cap set problem asks for the largest possible size of an arbitrary cap. The solution to the cap set problem is known for n at most 6.

In this paper, we define and apply standard diagrams. These pictures interpret a well-known technique for solving the cap set problem in a new way, allowing conclusions to be derived more easily and intuitively than before. We use standard diagrams to find caps in dimensions up to and including 4 systematically. We prove the apparently new result that in dimension 4, up to isomorphism there are exactly 20 size-18 caps, which we give explicitly.

This article is the first of a series. In later articles, we plan to use the methods and results of this paper to investigate dimensions 5 and higher. The eventual goal is to solve the cap set problem in dimension 7, the first unsolved case.

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Contents

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1. Introduction and setup

1.1. Introduction

An *s-cap n-flat C* (or an *n-dimensional cap C of size s*) is given by a collection S_C of exactly s points, no three of which are collinear, in an n-dimensional affine space F_C over the field $\mathbb{Z}/3\mathbb{Z}$. The cap set problem asks: for each positive integer n, what is the largest possible size of an arbitrary n-dimensional cap?

As mentioned by Davis and Maclagan [1], the special case $n = 4$ is equivalent to asking: what is the largest possible number of cards from the card game SET[1] without a so-called "Set"? In general (see Grochow [2]), the question is a simplified version of other questions in number theory and combinatorics dealing with arithmetic progressions.

A recent breakthrough (Ellenberg and Gijswijt [3], adapting a paper of Croot, Lev, and Pach [4]) showed that there is a constant c such that c is strictly less than 3 and for all positive integers n, each n-dimensional cap has at most c^n points. The cap set problem is solved for dimensions n at most 6 (see Davis and Maclagan [1], Edel et al. [5], and Potechin [6]): in dimensions 1, 2, 3, 4, 5, and 6 respectively, the largest possible cap sizes are 2, 4, 9, 20, 45, and 112 respectively.

This paper is the first in a planned series of articles; the aim of the series is to solve the cap set problem for the first unsolved case, namely $n = 7$. In this article, we define and apply standard diagrams to give a new and intuitive interpretation of a well-known method for solving the cap set problem. We show how standard diagrams can be applied to find caps in dimensions up to and including 4 systematically. The climax is an apparently new result: we determine all possible caps of size 18 in dimension 4 up to isomorphism. In later papers, we plan to use the methods and results of the current paper to look at higher dimensions.

1.2. Reference list: notation and terms

For clarity and rigour, this subsection carefully sets notation and defines terms.

Two or more $\pm$ signs in the same expression are independent of one another: "$(\pm 1, \pm 1)$" refers to the four points $(-1, -1)$, $(-1, 1)$, $(1, -1)$, and $(1, 1)$;

[1]Registered trademark of Cannei, LLC.

"$(\pm 1, \pm 1, \pm 1)$" refers to eight points; "$\pm(1,1)$" refers to the two points $(-1,-1)$ and $(1,1)$.

For nonnegative integers n and k, the binomial coefficient $\binom{n}{k}$ is defined to be $(\prod_{i=0}^{k-1}(n-i))/k!$. In particular, if $0 \leq n \leq k-1$, then $\binom{n}{k} = 0$.

For the rest of this section, the integers n and m satisfy $0 \leq m \leq n$.

Let $\mathbb{R}^2$ have the usual pair (x, y) of co-ordinates. We write $\mathbb{F}_3$ for the finite field $\mathbb{Z}/3\mathbb{Z}$. A vector v in $\mathbb{F}_3^n$ can be written as $(v_1, \ldots, v_n)$ or as $(v_i)_{i=1}^n$.

Throughout this section, F is an arbitrary *n-flat*, or *n-dimensional affine space*, over $\mathbb{F}_3$; that is, F is a set with a simply transitive action

$$F \times V_F \to F \colon (v_0, v) \mapsto v_0 + v$$

by (the additive group of) an n-dimensional $\mathbb{F}_3$-vector space V_F. For v_1 and v_2 in F, we write $v_2 - v_1$ for the vector v such that v is in V_F and $v_1 + v = v_2$. In the fundamental example of an n-flat, F and V_F are both $\mathbb{F}_3^n$, and the expression $v_0 + v$ equals the vector sum of v_0 and v in $\mathbb{F}_3^n$.

An *m-flat* in F (or an *m-dimensional affine subspace* of F) is a set F' equal to $\{v_0 + v \colon v \in V_{F'}\}$ for some v_0 in F and some m-dimensional $\mathbb{F}_3$-vector subspace $V_{F'}$ of V_F, with the associated simply transitive action of $V_{F'}$ on F' obtained by restricting the action of V_F on F. A *line* and a *hyperplane* in F are a 1-flat and an $(n-1)$-flat in F respectively.

For each n'-flat F' (not necessarily such that $F' \subseteq F$), an *affine transformation from F to F'* is a map $T \colon F \to F'$ such that for v_1 and v_2 in F, the map

$$T_L \colon V_F \to V_{F'} \colon v_2 - v_1 \mapsto T(v_2) - T(v_1)$$

is well defined and $\mathbb{F}_3$-linear; T is *invertible* if it is bijective (in which case $n = n'$). For $F' = F$, the map T is an *affine transformation of F*. An affine transformation T of F is a *translation of F* if T_L is the identity. The image of a subset S of F under some translation is a *translation of S*. The composition $T_2 T_1$ (applying T_1 first and T_2 second) is always affine for affine transformations T_1 and T_2 of F.

For this paragraph, take integers m_1 and m_2 such that $0 \leq m_1 < m_2 \leq n$, and take an arbitrary m_2-flat F_2 in F. An *m_1-flat direction* of F_2 is a partition of F_2 into m_1-flats that are translations of one another. For each m_1-flat F_1 such that $F_1 \subseteq F_2$, the *m_1-flat direction* of F_1 in F_2 is the partition of F_2 into translations of F_1. If D is an m_1-flat direction of F such that F_2 is the union of some of the m_1-flats in D, then the *restriction* of D to F_2 is the m_1-flat direction of F_2 consisting of the m_1-flats in D that are in F_2. If $\widetilde{F}_2$ is a translation of F_2 and the directions D and $\widetilde{D}$ are m_1-flat directions of F_2 and $\widetilde{F}_2$ respectively, then D and $\widetilde{D}$ are *parallel* if they are translations of each other, that is, if some m_1-flat in $\widetilde{D}$ is a translation (in F) of some m_1-flat in D, that is, if D and $\widetilde{D}$ are restrictions of the same m_1-flat direction of F to F_2 and $\widetilde{F}_2$ respectively. A *line direction* is a 1-flat direction; a *hyperplane direction* of F_2 is an (m_2-1)-flat direction of F_2.

A *co-ordinate* of F is an affine transformation $x \colon F \to \mathbb{F}_3$; for example, $\mathbb{F}_3^n$ has the standard co-ordinates $x_i \colon \mathbb{F}_3^n \to \mathbb{F}_3 \colon (v_j)_{j=1}^n \mapsto v_i$ for i in $\{1, \ldots, n\}$.

A *dual vector* of F is an $\mathbb{F}_3$-linear map from V_F to $\mathbb{F}_3$; the *dual vector* of a co-ordinate x of F is x_L, where we recall that for each affine transformation $T : F \to F'$, there is a well-defined $\mathbb{F}_3$-linear map $T_L : V_F \to V_{F'}$ by our earlier definition of an affine transformation. A *projective dual vector* of F is a set $\{f, -f\}$, where f is a nonzero dual vector of F; if a co-ordinate x of F is not a constant map, then the *projective dual vector* of x is $\{x_L, -x_L\}$.

A *dual m-flat*, *dual line*, or *dual hyperplane* of F is a vector subspace, with dimension m, 1, or $n-1$ respectively, of the space of dual vectors of F. A *projective dual* $(m-1)$*-flat* (*projective dual line* respectively) of F is the partition of a dual m-flat (a dual 2-flat respectively) of F minus the zero dual vector into projective dual vectors; the projective dual (-1)-flat is $\emptyset$.

A list $(x_1, \ldots, x_k)$ of co-ordinates of F is *independent* if the list $((x_1)_L, \ldots, (x_k)_L)$ is $\mathbb{F}_3$-linearly independent[2].

For this paragraph, let x be a nonconstant co-ordinate of F. The three *x-hyperplanes* in F are the hyperplanes in F each given by one of the equations $x = 1$, $x = 0$, and $x = -1$. The *x-hyperplane direction* D of F is the hyperplane direction that consists of the x-hyperplanes in F; in that situation, x is a *co-ordinate for* D, and the *projective dual vector* of D is $\{\pm x_L\}$. The map $D \mapsto$ (projective dual vector of D) is a well-defined bijection from the set of hyperplane directions of F to the set of projective dual vectors of F. For integers m_2 such that $1 \le m_2 \le n$, if an m_2-flat F_2 in F is such that the restriction $x|F_2 \colon F_2 \to \mathbb{F}_3$ of x to F_2 is nonconstant, then the *x-hyperplane direction* D_2 of F_2 is the $(x|F_2)$-hyperplane direction of F_2; in that situation, D_2 consists of the three *x-hyperplanes* in F_2, and x is a *co-ordinate for* D_2.

For each independent list $(x_1, \ldots, x_k)$ of co-ordinates of F, an $(n-k)$-flat direction D of F has *co-ordinates* x_1*, …,* x_k if each $(n-k)$-flat in D is defined by k equations $x_i = a_i$ with constants a_i in $\mathbb{F}_3$ for i in $\{1, \ldots, k\}$. For brevity, given $(a_1, \ldots, a_k)$ in $\mathbb{F}_3^k$, we write "the $(n-k)$-flat $(x_1, \ldots, x_k) = (a_1, \ldots, a_k)$" to mean the $(n-k)$-flat in which every point P satisfies $(x_1(P), \ldots, x_k(P)) = (a_1, \ldots, a_k)$. The expression "$(n-1)$-flats $x_1 = \pm 1$" refers to the two $(n-1)$-flats each given by one of the equations $x_1 = -1$ and $x_1 = 1$. The expression "$(n-2)$-flats $(x_1, x_2) = (\pm 1, \pm 1)$" refers to the four $(n-2)$-flats each given by one of the equations $(x_1, x_2) = (-1, -1)$, $(x_1, x_2) = (-1, 1)$, $(x_1, x_2) = (1, -1)$, and $(x_1, x_2) = (1, 1)$. Other such expressions are interpreted similarly.

For each independent list $(x_1, \ldots, x_n)$ of co-ordinates of F, and for i in $\{1, \ldots, n\}$, the *x_i-axis* is the line in F given by the equations $x_j = 0$ for $j \ne i$, and the *x_i-axis direction* of F is the line direction of the x_i-axis in F.

A *line segment* is a set consisting of two different points P_1 and P_2 in F; the *midpoint* of the line segment is the point other than P_1 and P_2 in the line containing P_1 and P_2. A *triangle* is a set consisting of three different noncollinear points in F, any two of which form a *side* of the triangle.

[2] As in the textbook *Linear Algebra Done Right* by Axler [7, chapter 2, section "Span and Linear Independence"], and for the reason described there, we use linearly independent lists instead of linearly independent sets.

The *line direction of a line segment* in F consists of the translations of the line through both points of the line segment. The *side directions of a triangle* in F are the line directions of the sides of the triangle.

Take a subset S of F. For a hyperplane direction D of F with co-ordinate x, if the hyperplanes $x = -1$, $x = 0$, and $x = 1$ respectively pass through exactly a, b, and c points of S respectively, then the *($(n-1)$-flat) point count* of S (or of (S, F)) for D, for x, or for the projective dual vector $\{\pm x_L\}$ is the multiset $\{a, b, c\}$. In that situation, D is an $\{a, b, c\}$ *hyperplane direction* of S (or of (S, F)), and $\{\pm x_L\}$ is an $\{a, b, c\}$ *projective dual vector* of S (or of (S, F)). We make statements like "the hyperplane direction is $\{a, b, c\}$", using the multiset $\{a, b, c\}$ as an adjective to describe hyperplane directions and projective dual vectors.

The *($(n-2)$-flat) point count* of S (or of (S, F)) for an independent pair (x_1, x_2) of co-ordinates of F is the 3-by-3 matrix

$$\begin{pmatrix} n_{-1,1} & n_{0,1} & n_{1,1} \\ n_{-1,0} & n_{0,0} & n_{1,0} \\ n_{-1,-1} & n_{0,-1} & n_{1,-1} \end{pmatrix}, \tag{1}$$

where $n_{a,b}$ is the number of points in the intersection of S with the $(n-2)$-flat $(x_1, x_2) = (a, b)$, for a and b in $\mathbb{F}_3$. The *($(n-3)$-flat) point count* of S (or of (S, F)) for an independent triple (x_1, x_2, x_3) of co-ordinates of F is a 3-by-3-by-3 tensor in which each of the 27 entries is the number of points in the intersection of S with a certain $(n-3)$-flat $(x_1, x_2, x_3) = (a_1, a_2, a_3)$, and so on.

A *cap*, an *s-cap*, or an *s-cap n-flat* C is a pair (S_C, F_C) such that F_C is an n-flat and S_C is a set of exactly s different points, no three of which are collinear, in F_C. Such a C is *n-dimensional* or *in dimension* n, and it is *in* F_C. The *size* of the cap is s, which is also denoted $|C|$. An *m-flat* of C is an m-flat of F_C; we similarly define *m-flat directions*, *co-ordinates*, *dual vectors*, *projective dual vectors*, *dual m-flats*, and *projective dual $(m-1)$-flats* for C. For a given cap C, a *cap line segment* and *cap triangle* are respectively a line segment and triangle consisting of elements of S_C, and a *cap point* is an element of S_C.

If F_C is clear from the context, then S_C is treated as a synonym for C. If C is clear from the context and F is some m-flat of F_C, then F is treated as a synonym for the cap $(S_C \cap F, F)$, which is *in* the m-flat direction of F in C. For example, the phrase "the 9-cap 3-flat $x_1 = 1$" refers to a size-9 cap of the form $(S_C \cap F, F)$, where F is the 3-flat $x_1 = 1$ in C.

For each two (not necessarily different) s-cap n-flats C_1 and C_2, an *isomorphism from C_1 to C_2* is an invertible affine transformation from F_{C_1} to F_{C_2} under which the image of S_{C_1} is S_{C_2}; if such an isomorphism exists, then C_1 and C_2 are *isomorphic* or *affinely equivalent.* A *symmetry* of a cap is an isomorphism from the cap to itself.

2. Standard diagrams and projective dual lines

2.1. Standard diagrams

In this subsection, we describe a tool that we apply repeatedly in this paper.

The idea behind this tool is well established; a paper of Davis and Maclagan [1] calls it "counting marked hyperplanes". However, when it is applied in the literature, a linear combination of certain equations is taken, and the coefficients in that linear combination can seem to come out of the blue, without explanation. We describe the idea in a visual and intuitive way.

Let an s-cap n-flat C be given. For nonnegative integers a, b, and c with $a \geq b \geq c$ and $a+b+c=s$, let $t_{abc}(C)$ be the number of hyperplane directions of C with point count $\{a,b,c\}$.

Counting the hyperplane directions D of C yields

$$\sum_{a \geq b \geq c} t_{abc}(C) = \frac{3^n - 1}{2} \tag{2}$$

for $n \geq 1$.

Counting the pairs (L, D) such that D is a hyperplane direction of C and L is a cap line segment in some hyperplane in D yields

$$\sum_{a \geq b \geq c} \left(\binom{a}{2} + \binom{b}{2} + \binom{c}{2} \right) t_{abc}(C) = \binom{s}{2} \cdot \frac{3^{n-1} - 1}{2} \tag{3}$$

for $n \geq 2$.

Counting the pairs (T, D) such that D is a hyperplane direction of C and T is a cap triangle in some hyperplane in D yields

$$\sum_{a \geq b \geq c} \left(\binom{a}{3} + \binom{b}{3} + \binom{c}{3} \right) t_{abc}(C) = \binom{s}{3} \cdot \frac{3^{n-2} - 1}{2} \tag{4}$$

for $n \geq 3$.

Equations (2), (3), and (4) quickly imply the following fundamental result, which we use often in the arguments to come.

Definition-Proposition 2.1 (Standard diagram)**.** *Let C be some s-cap n-flat with $n \geq 3$.*

For nonnegative integers a, b, and c with $a \geq b \geq c$, if $\{a, b, c\}$ is potentially a hyperplane point count of some s-cap n-flat, then define P_{abc} to be the point

$$\left(\binom{a}{2} + \binom{b}{2} + \binom{c}{2}, \binom{a}{3} + \binom{b}{3} + \binom{c}{3} \right)$$

in $\mathbb{R}^2$. Define the critical point *P_{Cr} to be the point*

$$\left(\binom{s}{2} \cdot \frac{3^{n-1} - 1}{3^n - 1}, \binom{s}{3} \cdot \frac{3^{n-2} - 1}{3^n - 1} \right)$$

in $\mathbb{R}^2$. The collection of points P_{abc} together with P_{Cr} is the standard diagram*.*

For each hyperplane direction D of C, let the point P_D be P_{abc}, where the point count of C for D is $\{a, b, c\}$ with $a \geq b \geq c$.

If we take the centre of mass of the $(3^n - 1)/2$ points P_D (counting multiplicity if $P_{D_1} = P_{D_2}$ for some two different hyperplane directions D_1 and D_2), then we obtain P_{Cr}. Therefore, the following statements hold for each line L in $\mathbb{R}^2$.

- *If P_{Cr} is off the line L, then some P_D off L is on the same side of L as P_{Cr}.*
- *Suppose that one side of L contains no points P_D that are off L. If P_{Cr} is on L, then so is every P_D.*

Proof. The centre of mass of the points P_{abc} with multiplicities $t_{abc}(C)$ is

$$\frac{\left(\sum_{a\geq b\geq c}\left(\binom{a}{2}+\binom{b}{2}+\binom{c}{2}\right)t_{abc}(C), \sum_{a\geq b\geq c}\left(\binom{a}{3}+\binom{b}{3}+\binom{c}{3}\right)t_{abc}(C)\right)}{\sum_{a\geq b\geq c}t_{abc}(C)},$$

which is P_{Cr} by equations (2), (3), and (4). □

The point P_{Cr} is sometimes very close to but not on a line L; in such cases, our sketches of standard diagrams shift P_{Cr} noticeably away from L to make it clear that P_{Cr} is off L, on a particular side.

We now refine this idea by grouping most points P_D. (We use this refinement when we consider 18-cap 4-flats.) In C, let some $\{a,b,c\}$ hyperplane direction D_1 have co-ordinate x_1. Projective dual vectors correspond to hyperplane directions, so each of the $(3^{n-1}-1)/2$ projective dual lines through the projective dual vector $\{\pm(x_1)_L\}$ corresponds to $T\cup\{D_1\}$ for some triple T of hyperplane directions, and these $(3^{n-1}-1)/2$ triples T form a partition of the set of hyperplane directions of C other than D_1. For each T, let P_T be the sum in $\mathbb{R}^2$ of the three points P_D such that D is in T.

Definition-Proposition 2.2 (Standard diagram for triples)**.** *Let C be some s-cap n-flat with $n\geq 3$. Let D_1, x_1, and $\{a,b,c\}$ be as above.*

For each matrix of the form (1), if that matrix is potentially the $(n-2)$-flat point count of some s-cap n-flat for some independent pair (y_1,y_2) of co-ordinates with y_1 giving an $\{a,b,c\}$ hyperplane direction, then, letting $n_{3k+i,3\ell+j}$ be equal to $n_{i,j}$ for i and j both in $\{-1,0,1\}$ and k and ℓ both in $\mathbb{Z}$, define

$$P_{n_{-1,-1},n_{-1,0},n_{-1,1};n_{0,-1},n_{0,0},n_{0,1};n_{1,-1},n_{1,0},n_{1,1}}$$

to be the point

$$\left(\sum_{i=-1}^{1}\sum_{j=-1}^{1}\binom{n_{-1,j-i}+n_{0,j}+n_{1,j+i}}{2}, \sum_{i=-1}^{1}\sum_{j=-1}^{1}\binom{n_{-1,j-i}+n_{0,j}+n_{1,j+i}}{3}\right).$$

Define the critical point $\widetilde{P}_{Cr}$ *to be the point*

$$\left(\binom{s}{2}-\frac{\binom{a}{2}+\binom{b}{2}+\binom{c}{2}}{\frac{3^{n-1}-1}{2}}, \binom{s}{3}\cdot\frac{3^{n-2}-1}{3^{n-1}-1}-\frac{\binom{a}{3}+\binom{b}{3}+\binom{c}{3}}{\frac{3^{n-1}-1}{2}}\right).$$

The collection of points $P_{n_{-1,-1},\ldots,n_{1,1}}$ together with $\widetilde{P}_{Cr}$ is the standard diagram for triples*.*

For each triple T as above, (i) the cap C has a co-ordinate x_2 such that the three hyperplane directions in T have respective co-ordinates $x_2 + cx_1$, where c ranges over $\mathbb{F}_3$, and (ii) for such an x_2, if the point count of C for (x_1, x_2) is (1)*, then $P_T = P_{n_{-1,-1}, \ldots, n_{1,1}}$. The centre of mass of the $(3^{n-1}-1)/2$ points P_T is $\widetilde{P}_{Cr}$.*

Proof. The last statement holds because, by Definition-Proposition 2.1,

$$P_{abc} + \sum_T P_T = \left(\binom{s}{2} \cdot \frac{3^{n-1}-1}{2}, \binom{s}{3} \cdot \frac{3^{n-2}-1}{2} \right). \qquad \square$$

*2.2. The congruent-mod-*3 *projective dual flat*

In the coming arguments, we often explore the link between m-flat point counts and $(m-1)$-flat point counts of a cap. The following result is a useful example of this idea in action.

In this subsection, let C be some s-cap n-flat, where s is a multiple of 3. The *congruent-mod-*3 *projective dual flat* F_0 of C is the set of the projective dual vectors of C with point count $\{a_0, b_0, c_0\}$ such that $a_0 \equiv b_0 \equiv c_0 \bmod 3$.

Lemma 2.3 (Congruent-mod-3 projective dual flat)**.** *The congruent-mod-*3 *projective dual flat F_0 of C is, in fact, a projective dual $(n-1)$- or $(n-2)$-flat.*

If the four projective dual vectors $\{\pm(x_i)_L\}$ on some projective dual line have point counts congruent to $\{k_i, k_i, k_i\}$ respectively modulo 3 *for i in $\{1, 2, 3, 4\}$, then $\sum_{i=1}^{4} k_i \equiv 0 \bmod 3$.*

Proof. Let L be some projective dual line of C. Take two different elements $\{\pm(x_1)_L\}$ and $\{\pm(x_2)_L\}$ of L, where x_1 and x_2 are associated co-ordinates of C. The two other elements of L must be $\{\pm(x_1 + x_2)_L\}$ and $\{\pm(x_1 - x_2)_L\}$. Let the hyperplanes $x_1 = j$ and $x_2 = j$ respectively have exactly q_j and r_j cap points respectively, for each j in $\mathbb{F}_3$. Now the point count of C for (x_1, x_2) is

$$\begin{pmatrix} a & b & r_1 - a - b \\ d & e & r_0 - d - e \\ q_{-1} - a - d & q_0 - b - e & q_1 + r_{-1} - s + a + b + d + e \end{pmatrix}$$

for some nonnegative integers a, b, d, and e. Therefore, letting u be $-a+b+d-e$, the point count of C for x_1+x_2 modulo 3 is $\{q_0+r_1+u, q_1+r_{-1}+u, q_{-1}+r_0+u\}$, and the point count of C for $x_1 - x_2$ modulo 3 is $\{q_0 + r_0 - u, q_1 + r_{-1} - u, q_{-1} + r_1 - u\}$. It is a simple exercise to confirm that (i) the set F_0 includes exactly one or all four of the projective dual vectors on L, and (ii) if all four projective dual vectors on L are in F_0 with point counts $\{k_i, k_i, k_i\}$ modulo 3, then $\sum_{i=1}^{4} k_i \equiv 0 \bmod 3$.

Let $\widetilde{F_0}$ be the set of dual vectors of C consisting of 0 and all dual vectors f with $\{\pm f\}$ in F_0. Now $\widetilde{F_0}$ is an $\mathbb{F}_3$-vector subspace of the space of dual vectors of C, since if two of the four projective dual vectors on L are in F_0, then so are the other two. Therefore, F_0 is a projective dual m-flat for some integer m.

If some projective dual vector $\{\pm f\}$ is not in F_0, then each of the $(3^{n-1}-1)/2$ projective dual lines through $\{\pm f\}$ has exactly one projective dual vector in F_0, so F_0 is a projective dual $(n-2)$-flat. $\square$

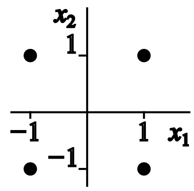


Figure 1: The 4-cap 2-flat $(\{\pm 1\}^2, \mathbb{F}_3^2)$.

3. Dimensions up to and including 3

As a brief review of some basic cases and a simple application of the standard diagram, we find, up to isomorphism, every cap in dimensions up to and including 3. The results of this section are useful for dimensions 4 and (in later papers) higher.

For completeness, we record the obvious

Proposition 3.1 (Dimension 1)**.** *In dimension* 1*, there are exactly three caps up to invertible affine transformations: the* 0*-cap,* 1*-cap, and* 2*-cap (the empty set, singleton set, and line segment respectively).*

The first nontrivial case is dimension 2.

Proposition 3.2 (Dimension 2: large caps)**.** *In dimension* 2*, the largest possible size of an arbitrary cap is* 4*, and every* 4*-cap is isomorphic to* $(\{\pm 1\}^2, \mathbb{F}_3^2)$*.*

Proof. Equations (2) and (3) give $4 = t_{221}(C) = 5$ for each 5-cap 2-flat C.

Now let C be an arbitrary 4-cap 2-flat. The system consisting of equations (2) and (3) is

$$\left\{ \begin{array}{rcl} t_{220}(C) + t_{211}(C) & = & 4 \\ 2t_{220}(C) + t_{211}(C) & = & 6 \end{array} \right\}$$

and it has the unique solution $(t_{220}(C), t_{211}(C)) = (2, 2)$. The two $\{2, 2, 0\}$ line directions have respective co-ordinates x_1 and x_2 such that no cap points are on the lines $x_i = 0$ for i in $\{1, 2\}$, so the four points in F_C off both of those lines are the cap points. Therefore, $(x_1, x_2)\colon C \to (\{\pm 1\}^2, \mathbb{F}_3^2)$ is an isomorphism (and $(\{\pm 1\}^2, \mathbb{F}_3^2)$ is easily checked to be a cap). □

Figure 1 suggests the following useful definitions. A *square* is a 4-cap 2-flat. In each square C, let a *diagonal* (respectively, a *side*) of C be a line L such that L is in one of the $\{2, 1, 1\}$ (respectively, $\{2, 2, 0\}$) line directions of C and there are two cap points on L; the two diagonals of C intersect in the *centre* of C. Let the line directions of the diagonals (respectively, sides) of C in F_C be the *diagonal directions* (respectively, *side directions*) of C. Every square determines its diagonals, sides, and centre independently of co-ordinates. In $(\{\pm 1\}^2, \mathbb{F}_3^2)$, the diagonals (respectively, sides) are the lines $x_1 \pm x_2 = 0$ (respectively, the lines $x_1 = \pm 1$ and $x_2 = \pm 1$).

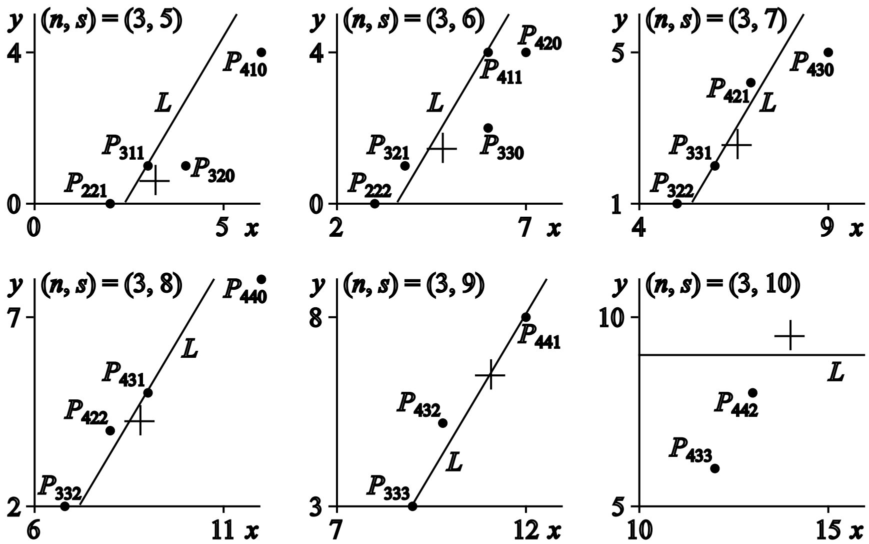


Figure 2: Standard diagrams for $n = 3$ and $5 \leq s \leq 10$.

Corollary 3.3 (Dimension 2: all caps)**.** *In dimension* 2*, there are exactly five caps up to invertible affine transformations: the* 0*-cap,* 1*-cap,* 2*-cap,* 3*-cap, and* 4*-cap (the empty set, singleton set, line segment, triangle, and square respectively).*

We now find every cap in dimension 3 up to isomorphism. We begin by showing, via standard diagrams, that each 3-dimensional cap has at most nine cap points and at least one 2-flat with at most one cap point.

Proposition 3.4 (Dimension 3: standard diagrams)**.** *In dimension* 3*, the following statements hold.*

- *In each cap of size at most* 8*, some* 2*-flat has no cap points.*
- *In each* 9*-cap, there are exactly nine* $\{4, 4, 1\}$ *hyperplane directions, exactly four* $\{3, 3, 3\}$ *hyperplane directions, and no other hyperplane directions.*
- *There is no cap of size* 10*.*

Proof. We deal with small caps first. In each cap of size at most 3, some 2-flat has no cap points. Each 4-cap is a square or has four cap points not all on the same 2-flat; in both cases, some 2-flat has no cap points.

For caps C of size s at least 5, apply the standard diagrams for $n = 3$ and $5 \leq s \leq 10$ in Figure 2, where the + signs denote the critical points P_{Cr}. The line L has the equation $3y = 5x + 18 - 6s$ (respectively, the equation $y = 9$) for $5 \leq s \leq 9$ (respectively, $s = 10$).

Suppose s is in $\{5, 6, 7, 8\}$. The point P_{Cr} is below L, so some point P_D is below L.

Suppose $s = 9$. The line L goes through the points P_{Cr}, P_{441}, and P_{333} (but not P_{432}), so the standard diagram implies $t_{432}(C) = 0$, so the system consisting of equations (2), (3), and (4) is

$$\left\{ \begin{array}{rcr} t_{441}(C) + t_{333}(C) & = & 13 \\ 12t_{441}(C) + 9t_{333}(C) & = & 144 \\ 8t_{441}(C) + 3t_{333}(C) & = & 84 \end{array} \right\}$$

and it has the unique solution $(t_{441}(C), t_{333}(C)) = (9, 4)$.

In the case $s = 10$, the points P_{442} and P_{433} are on the opposite side of L from P_{Cr}. □

Now, we list the 3-dimensional caps up to isomorphism.

Theorem 3.5 (Dimension 3: all caps)**.** *In dimension* 3*, every cap is isomorphic to exactly one of the following* 17 representative caps *in* $\mathbb{F}_3^3$ *(shown in the figures to come for sizes* 4 *and higher). In the following list, the name immediately before each representative cap* C *refers to an arbitrary cap isomorphic to* C*; for example,* a square pyramid *is a* 5*-cap* 3*-flat that is isomorphic to the representative cap* $(\{(\pm 1, \pm 1, -1), (0, 0, 1)\}, \mathbb{F}_3^3)$*.*

- *The* 0*-cap: empty set:* $\emptyset$*.*
- *The* 1*-cap: singleton set:* $\{(0, 0, 0)\}$*.*
- *The* 2*-cap: line segment:* $\{(0, 0, 0), (1, 0, 0)\}$*.*
- *The* 3*-cap: triangle:* $\{(0, 0, 0), (0, 1, 0), (1, 0, 0)\}$*.*
- *The* 4*-caps:*
  - *square:* $\{(\pm 1, \pm 1, 0)\}$*.*
  - *tetrahedron:* T_4*, which we define to be* $\{(-1, -1, 1), (-1, 1, -1), (1, -1, -1), (1, 1, 1)\}$*.*
- *The* 5*-caps:*
  - *square pyramid:* $\{(\pm 1, \pm 1, -1), (0, 0, 1)\}$*.*
  - *tetrahedron plus centre:* $T_4 \cup \{(0, 0, 0)\}$*.*
- *The* 6*-caps:*
  - *cube minus edge:* $\{\pm 1\}^3 - \{(-1, 1, 1), (1, 1, 1)\}$*.*
  - *cube minus face diagonal:* $\{\pm 1\}^3 - \{(-1, -1, 1), (1, 1, 1)\}$*.*
  - *cube minus long diagonal:* $\{\pm 1\}^3 - \{(-1, -1, -1), (1, 1, 1)\}$*.*
- *The* 7*-caps:*
  - *cube minus point:* $\{\pm 1\}^3 - \{(1, 1, 1)\}$*.*
  - *cube with contracted edge:* $\{\pm 1\}^3 \cup \{(0, 1, 1)\} - \{(\pm 1, 1, 1)\}$*.*

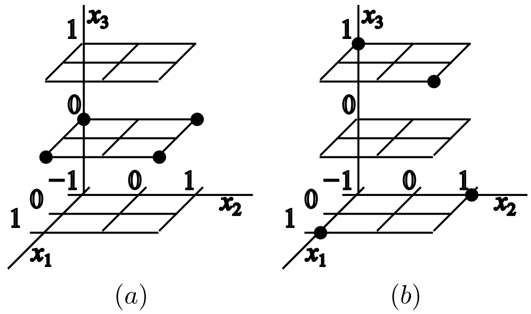


Figure 3: The representative 4-cap 3-flats. (a) The square. (b) The tetrahedron.

- *The* 8*-caps:*
  - *cube:* $\{\pm 1\}^3$.
  - *saddled cube:* $T_4 \cup \{(\pm 1, 0, \pm 1)\}$.
  - *square antiprism:* $\{(\pm 1, 0, -1),\ (0, \pm 1, -1),\ (\pm 1, \pm 1, 1)\}$.
- *The* 9*-cap: square antiprism plus centre:* $\{(\pm 1, 0, -1),\ (0, \pm 1, -1),\ (0, 0, 0),\ (\pm 1, \pm 1, 1)\}$.

For each 8*-cap* 3*-flat (respectively,* 9*-cap* 3*-flat)* C_0 *with some given* $\{4, 4, 0\}$ *(respectively,* $\{4, 4, 1\}$*) hyperplane direction* D*, there is some isomorphism from* C_0 *to some representative* 8*-cap* 3*-flat (respectively, to the representative* 9*-cap* 3*-flat)* C *that sends* D *to the* x_3*-hyperplane direction of* C*.*

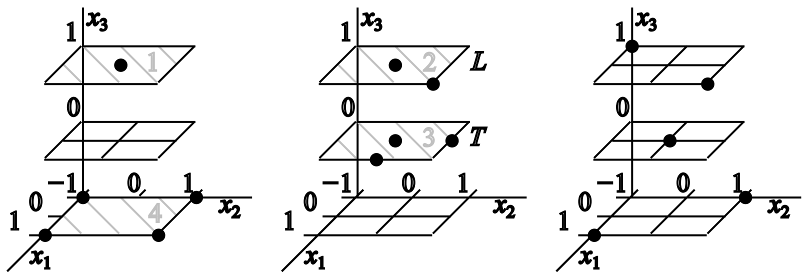
Proof. The result is clear for caps of size at most 4. For example, see Figure 3 for the representative 4-cap 3-flats.

We now consider larger caps. In each case, we obtain the point count of some 2-flat direction from Proposition 3.4, and we consider what the cap points are in each 2-flat in that 2-flat direction.

The 5**-cap 3-flats.** By Proposition 3.4, each 5-cap 3-flat C has at least one $\{4, 1, 0\}$ or $\{3, 2, 0\}$ hyperplane direction.

Suppose C has at least one $\{4, 1, 0\}$ hyperplane direction. For some co-ordinate x_3 of C, the 2-flats $x_3 = -1$ and $x_3 = 1$ have four cap points and one cap point respectively. Without loss of generality, C has co-ordinates x_1 and x_2 such that (i) the square in the 2-flat $x_3 = -1$ is as in Figure 4(a), and (ii) the cap point in the 2-flat $x_3 = 1$ is also as in that figure. Now C is the representative square pyramid in that figure.

Suppose C has no $\{4, 1, 0\}$ hyperplane directions. For some co-ordinate x_3 of C, the 2-flats $x_3 = 0$ and $x_3 = 1$ contain a cap triangle T and a cap line segment L respectively. Without loss of generality, T is as in Figure 4(b) for some co-ordinates x_1 and x_2 of C. Since no side direction of T in the 2-flat $x_3 = 0$ is parallel to the line direction of L in the 2-flat $x_3 = 1$ (or C has a



(a) (b) (c)

Figure 4: The 5-cap 3-flats. (a) The representative square pyramid. (b) A tetrahedron plus centre. (c) The representative tetrahedron plus centre.

$\{4,1,0\}$ hyperplane direction), without loss of generality L is also as in Figure 4(b). Now the affine transformation

$$(x_1, x_2, x_3) \mapsto (x_2 + x_3 + 1, x_1 + x_3 + 1, x_1 + x_2 + 1)$$

turns C into $T_4 \cup \{(0,0,0)\}$ in (x_1, x_2, x_3) co-ordinates, as in Figure 4(c).

The representative square pyramid has a $\{4,1,0\}$ hyperplane direction, and the representative tetrahedron plus centre does not, so these two caps are non-isomorphic.

The 6-cap 3-flats. By Proposition 3.4, each 6-cap 3-flat C has at least one $\{4,2,0\}$ or $\{3,3,0\}$ hyperplane direction.

Suppose that C has at least one $\{4,2,0\}$ hyperplane direction. A square S and a cap line segment L are in the 2-flats $x_3 = -1$ and $x_3 = 1$ respectively, for some co-ordinate x_3 of C. Without loss of generality, S is as in Figures 5(a)-(b) for some co-ordinates x_1 and x_2 of C. Up to isomorphism, we obtain the representative cube minus edge in Figure 5(a) or the representative cube minus face diagonal in Figure 5(b). (The line direction of L in the 2-flat $x_3 = 1$ is parallel to, respectively, a side direction or a diagonal direction of S.)

Suppose that C has no $\{4,2,0\}$ hyperplane directions. Cap triangles T_{-1} and T_1 are in the 2-flats $x_3 = -1$ and $x_3 = 1$ respectively, for some co-ordinate x_3 of C. Now T_a has three side directions in the 2-flat $x_3 = a$ for each a in $\{\pm 1\}$, and every 2-flat has four line directions, so there are at least two pairs $\{D_{-1}, D_1\}$ such that D_{-1} and D_1 are parallel side directions of T_{-1} and T_1 respectively in the 2-flats $x_3 = -1$ and $x_3 = 1$ respectively. Choose two such pairs $\{D_{-1}, D_1\}$; they consist of the restrictions of the x_1- and x_2-axis directions of C to the 2-flats $x_3 = \pm 1$, for some co-ordinates x_1 and x_2 of C. Without loss of generality, T_{-1} is as in Figure 5(c). Without loss of generality, T_1 is as in that figure (if T_1 is not a translation of what is shown in the figure, then C has a $\{4,2,0\}$ hyperplane direction). Now C is the representative cube minus long diagonal in that figure.

The numbers of $\{4,2,0\}$ hyperplane directions in the representative cube

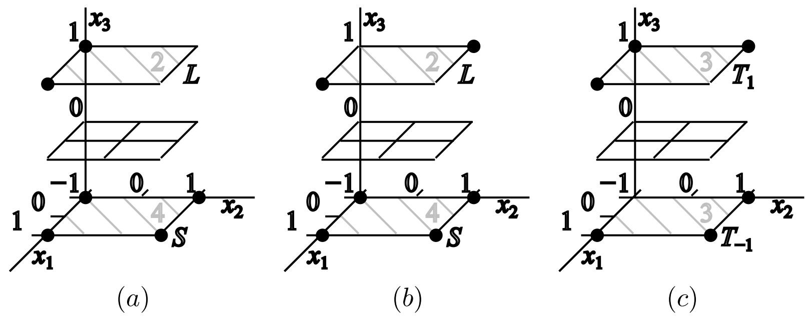


Figure 5: The representative 6-cap 3-flats. (a) Cube minus edge. (b) Cube minus face diagonal. (c) Cube minus long diagonal.

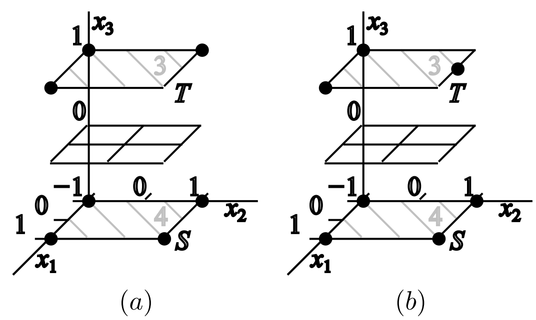
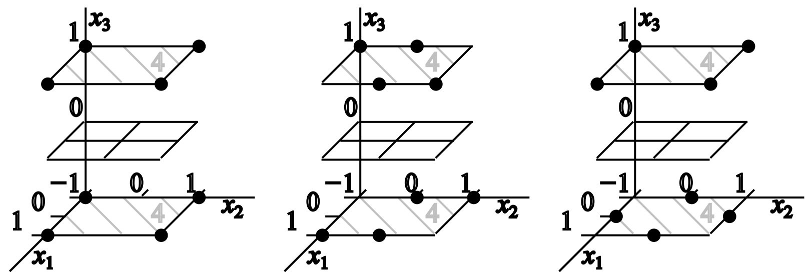


Figure 6: The representative 7-cap 3-flats. (a) Cube minus point. (b) Cube with contracted edge.

minus edge, cube minus face diagonal, and cube minus long diagonal are 3, 1, and 0 respectively, so no two of these caps are isomorphic.

The 7-cap 3-flats. For each 7-cap 3-flat, there is some co-ordinate x_3 for which a square S and a cap triangle T are in the 2-flats $x_3 = -1$ and $x_3 = 1$ respectively. Without loss of generality, S is as in Figures 6(a)-(b) for some co-ordinates x_1 and x_2. The number of side directions of T in the 2-flat $x_3 = 1$ that are parallel to side directions of S is 2 or 1. Up to isomorphism, we obtain, respectively, the representative cube minus point in Figure 6(a) and the representative cube with contracted edge in Figure 6(b). Their numbers of $\{4, 3, 0\}$ hyperplane directions are 3 and 2 respectively, so the caps are nonisomorphic.

The 8-cap 3-flats. Each 8-cap 3-flat has at least one $\{4, 4, 0\}$ hyperplane direction. Take a $\{4, 4, 0\}$ hyperplane direction D with co-ordinate x_3 such that there is a square in each of the two 2-flats $x_3 = \pm 1$. These two squares have two pairs, one pair, or no pairs of parallel diagonal directions; up to isomorphism, these cases yield the representative cube, saddled cube, and square antiprism in Figures 7(a), 7(b), and 7(c) respectively. Their numbers of $\{4, 4, 0\}$ hyperplane



(a) (b) (c)

Figure 7: The representative 8-cap 3-flats. (a) Cube. (b) Saddled cube. (c) Square antiprism.

directions are 3, 2, and 1 respectively, so no two of these caps are isomorphic.

The 9-cap 3-flat. Each 9-cap 3-flat C has at least one $\{4, 4, 1\}$ hyperplane direction. Let such a direction D be given with co-ordinate x_3 such that the numbers of cap points in the 2-flats $x_3 = -1$, $x_3 = 0$, and $x_3 = 1$ are 4, 1, and 4 respectively.

Removing the cap point in the 2-flat $x_3 = 0$ yields an 8-cap $\widetilde{C}$ such that x_3 is a co-ordinate for a $\{4, 4, 0\}$ hyperplane direction, so for some co-ordinates x_1 and x_2, after negating x_3 if necessary, the cap $\widetilde{C}$ is one of the three representative 8-cap 3-flats in (x_1, x_2, x_3) co-ordinates.

Now $\widetilde{C}$ cannot be the representative cube or the representative saddled cube, since neither of these options would allow C to have a ninth cap point in the 2-flat $x_3 = 0$: the open circles in Figures 8(a) and 8(b) – as well as in other figures – indicate midpoints of cap line segments, and such midpoints cannot be cap points. Therefore, $\widetilde{C}$ is the representative square antiprism. As Figure 8(c) illustrates, no possible ninth cap point exists except $(x_1, x_2, x_3) = (0, 0, 0)$, so adding that point to $\widetilde{C}$ yields the unique 9-cap 3-flat up to isomorphism – even up to only the isomorphisms preserving D. □

In an arbitrary square antiprism plus centre, let the *axis line direction* be the unique line direction such that the following statements hold. (Each bullet point by itself fixes the axis line direction, which is the x_3-axis direction in Figure 9. The following list of facts about the structure of an arbitrary 9-cap 3-flat is used often in the arguments to come.)

- Each of the nine lines in the axis line direction has exactly one cap point on it. (In Figure 9, those nine lines are the vertical lines that are shown in light grey.)
- If D is one of the 12 line directions other than the axis line direction, then some three lines in D contain two cap points each.
- The axis line direction is "the intersection of the four $\{3, 3, 3\}$ hyperplane directions": if D is some hyperplane direction, then D has point count

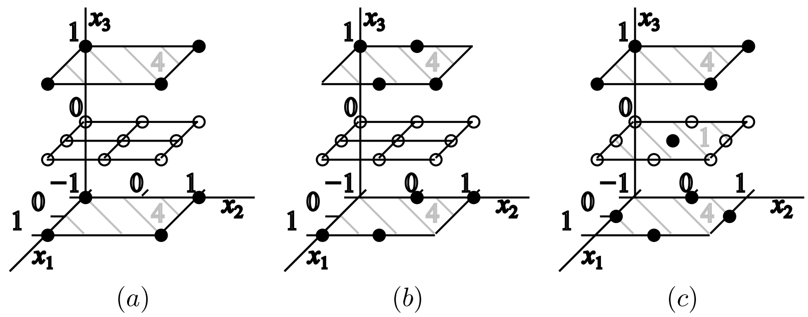


Figure 8: (a) Trying to add a ninth cap point to a cube. (b) Trying to add a ninth cap point to a saddled cube. (c) The representative square antiprism plus centre.

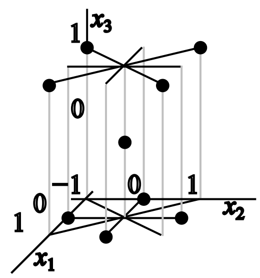


Figure 9: The representative square antiprism plus centre.

$\{3,3,3\}$ iff each of its 2-flats contains at least one line in the axis line direction. (Figure 9 shows a 2-flat in each of the four $\{3,3,3\}$ hyperplane directions, and those four 2-flats intersect in the line $(x_1, x_2) = (0,0)$, which is in the axis line direction.)

- Suppose D is some $\{3,3,3\}$ hyperplane direction. In this case, each of the three 2-flats in D contains one of three disjoint cap triangles T that are translations of one another, and the axis line direction is the only line direction in which each line is contained in some 2-flat in D but is not in a side direction of the triangle T in that 2-flat. (In the representative 9-cap 3-flat, consider the x_1- or x_2-hyperplanes.)

- Suppose D is some $\{4,4,1\}$ hyperplane direction. In this case, the single cap point in the 1-cap 2-flat in D and the centres of the two 4-cap 2-flats in D are collinear, and the line on which they lie is in the axis line direction. (In the representative 9-cap 3-flat, consider the x_3-hyperplanes.)

It is a straightforward exercise to prove that (i) the representative square antiprism plus centre has 144 symmetries, and (ii) if (A_1, A_2) and (B_1, B_2) are ordered pairs of cap points in the representative square antiprism plus centre, and $A_1 \neq A_2$ and $B_1 \neq B_2$, then exactly two symmetries of the cap send A_1 and A_2 to B_1 and B_2 respectively. (An outline of a proof is as follows. The axis line direction is intrinsically determined, so each symmetry induces an action on the set of nine lines $(x_1, x_2) = (a_1, a_2)$ with a_1 and a_2 in $\mathbb{F}_3$. No two different symmetries induce the same action, since each line in the axis line direction has only one cap point on it. Now determine the group of such induced actions arising from symmetries.)

4. Dimension 4

We now proceed to dimension 4. We show that in this situation, the 19-cap is unique up to isomorphism. (The 20-cap is too; it has maximum size, as is well known.) Further, we show that there are 20 isomorphism classes of 18-cap 4-flats, and we give representatives of the classes.

4.1. Excluded configurations

To prepare for later arguments, we show the following useful lemmas, which say that certain types of configurations cannot occur. Throughout this subsection, C is an arbitrary cap in dimension 4.

A *triangle of* 414 *edges* is a 2-flat point count of C of the form

$$\begin{pmatrix} 4 & 1 & 4 \\ 1 & 1 & * \\ 4 & * & * \end{pmatrix}.$$

Lemma 4.1 (Dimension 4: no triangles of 414 edges)**.** *The cap C does not have a triangle of* 414 *edges.*

Proof. Suppose to the contrary that the 2-flat point count of C for some independent pair (x_1, x_2) of co-ordinates is a triangle of 414 edges.

If some $\{4, 4, 1\}$ hyperplane direction D of an arbitrary 9-cap 3-flat is given, then by Theorem 3.5, the diagonal directions of each 4-cap 2-flat in D are parallel to the side directions of the other 4-cap 2-flat in D.

Consider the 9-caps in the three 3-flats $x_1 = -1$, $x_2 = x_1$, and $x_2 = 1$, that is, in the three 3-flats corresponding to the matrix entries

$$\begin{pmatrix} 4 & \_ & \_ \\ 1 & \_ & \_ \\ 4 & \_ & \_ \end{pmatrix}, \begin{pmatrix} \_ & \_ & 4 \\ \_ & 1 & \_ \\ 4 & \_ & \_ \end{pmatrix}, \text{and } \begin{pmatrix} 4 & 1 & 4 \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix}.$$

The diagonal directions of the square in the 2-flat $(x_1, x_2) = (-1, 1)$ are parallel to the side directions of the square in $(x_1, x_2) = (-1, -1)$, which are parallel to the diagonal directions in $(x_1, x_2) = (1, 1)$, which are parallel to the side directions in $(x_1, x_2) = (-1, 1)$. Therefore, in $(x_1, x_2) = (-1, 1)$, the diagonal directions are the side directions, which is impossible. □

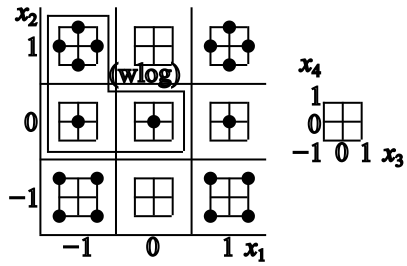


Figure 10: Trying to construct an H of 414 edges.

An *H of* 414 *edges* is a 2-flat point count of C of the form

$$\begin{pmatrix} 4 & * & 4 \\ 1 & 1 & 1 \\ 4 & * & 4 \end{pmatrix}.$$

Lemma 4.2 (Dimension 4: no H of 414 edges)**.** *The cap C does not have an H of* 414 *edges.*

Proof. Suppose otherwise; let the relevant pair of co-ordinates be (x_1, x_2). Without loss of generality, let co-ordinates x_3 and x_4 be such that the 2-flats $(x_1, x_2) = (-1, 1)$, $(x_1, x_2) = (-1, 0)$, and $(x_1, x_2) = (0, 0)$ are as in Figure 10.

By the structure of the 9-cap 3-flat $x_1 = -1$, the 2-flat $(x_1, x_2) = (-1, -1)$ is as in the figure: the cap point in $(x_1, x_2) = (-1, 0)$ and the centres of the squares $(x_1, x_2) = (-1, \pm 1)$ are collinear, and the diagonal directions of $(x_1, x_2) = (-1, -1)$ are parallel to the side directions of $(x_1, x_2) = (-1, 1)$.

Similarly, the 9-cap 3-flats $x_2 = \pm x_1$ force the 4-cap 2-flats $(x_1, x_2) = (1, \pm 1)$ to be as in the figure. Next, the 9-cap 3-flat $x_1 = 1$ forces the 2-flat $(x_1, x_2) = (1, 0)$ to be as in the figure. Now the 3-flat $x_2 = 0$ has a line of three cap points. □

An *H of* 431 *edges* is a 2-flat point count of C of the form

$$\begin{pmatrix} 4 & * & 4 \\ 3 & 3 & 3 \\ 1 & * & 1 \end{pmatrix}.$$

Lemma 4.3 (Dimension 4: no H of 431 edges)**.** *The cap C does not have an H of* 431 *edges.*

Proof. Suppose otherwise; let the relevant pair of co-ordinates be (x_1, x_2). Without loss of generality, let x_3 and x_4 be such that the 2-flats $(x_1, x_2) = (-1, -1)$, $(x_1, x_2) = (-1, 0)$, and $(x_1, x_2) = (0, 0)$ are as in Figure 11. (We may do that by the structure of the 9-cap 3-flat $x_2 = 0$ and its $\{3, 3, 3\}$ hyperplane direction with co-ordinate x_1.) To avoid lines of three cap points in the 3-flat $x_2 = 0$, the 2-flat $(x_1, x_2) = (1, 0)$ is also as in the figure.

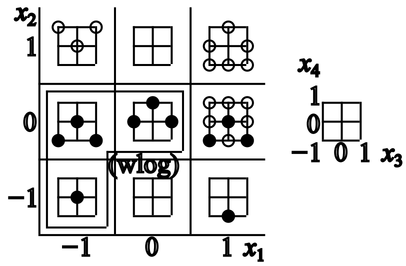


Figure 11: Trying to construct an H of 431 edges.

The cap point in the 2-flat $(x_1, x_2) = (1, -1)$ must be $(x_1, \ldots, x_4) = (1, -1, 0, -1)$: if it is at some other position, then all but three or four points in the 4-cap 2-flat $(x_1, x_2) = (-1, 1)$ are excluded as potential cap points (avoid lines of three cap points in the 3-flats $x_1 = -1$ and $x_1 + x_2 = 0$), and if four nonexcluded points remain, then three of them are in a line.

Now all but three points in the 2-flat $(x_1, x_2) = (1, 1)$ cannot be cap points (avoid lines of three cap points in the 3-flats $x_1 = 1$ and $x_1 - x_2 = 0$), but that 2-flat has four cap points. □

The last preparatory lemma looks at how 3-flat point counts interact via 2-flat point counts.

Lemma 4.4 (Dimension 4: projective dual lines)**.** *Let (x_1, x_2) be some independent pair of co-ordinates of C. Consider the point counts of C for the four 3-flat directions with respective co-ordinates x_1, x_2, $x_2 + x_1$, and $x_2 - x_1$.*

(a) No three of the four point counts are all $\{8, 8, 3\}$.

(b) If three of the four point counts are $\{8, 7, 3\}$, then the fourth point count is neither $\{8, 7, 3\}$ nor $\{6, 6, 6\}$.

Proof. Let the point count of C for (x_1, x_2) be

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

(a) Without loss of generality, let $\{8, 8, 3\}$ be the point count of C for each of x_1, x_2, and $x_2 - x_1$, with two 3-cap 3-flats $x_1 = 0$ and $x_2 = 0$. Now the point count of C for (x_1, x_2) is

$$\begin{pmatrix} a & b & 8-a-b \\ d & e & 3-d-e \\ 8-a-d & 3-b-e & -3+a+b+d+e \end{pmatrix}$$

and, considering remainders modulo 3, the statement

$$a + f + h = 6 + a - b - d - 2e \equiv -3 + a + 2b + 2d + e = b + d + i \bmod 3$$

holds, so $a+f+h=b+d+i=8$ and $c+e+g=3$. Therefore,

$$b+c+d+e+f+g+h \leq (b+e+h)+(c+e+g)+(d+e+f) = 3(3) = 9,$$

which implies $a+i \geq 10$ (there are 19 cap points in total), so at least one of a and i is at least 5. However, every 2-dimensional cap has at most four cap points.

(b) By Lemma 2.3, not all four point counts are $\{8,7,3\}$. Suppose that the point count for $x_2 - x_1$ is $\{6,6,6\}$ and that the other three point counts are $\{8,7,3\}$.

Let the numbers of cap points in the 3-flats $x_1 = -1$, $x_1 = 0$, $x_1 = 1$, $x_2 = -1$, $x_2 = 0$, and $x_2 = 1$ be 3, 7, 8, $11-k$, 7, and k respectively, where k is in $\{3,8\}$. Now the point count of C for (x_1, x_2) is

$$\begin{pmatrix} a & b & k-a-b \\ d & e & 7-d-e \\ 3-a-d & 7-b-e & 1-k+a+b+d+e \end{pmatrix}.$$

Now $6 = a+f+h = 14+a-b-d-2e$ and $6 = b+d+i = 1-k+a+2b+2d+e$. Comparing these two equations modulo 3 gives $k \equiv 2 \bmod 3$, so $k=8$. Solving the equations for d and e in terms of a and b, we see that the point count is

$$\begin{pmatrix} a & b & 8-a-b \\ 6-a-b & 1+a & b \\ -3+b & 6-a-b & a \end{pmatrix}.$$

Now the equations

$$\begin{aligned} a+e+i &= 1+3a, \\ b+f+g &= -3+3b, \text{ and} \\ c+d+h &= 20-3a-3b \end{aligned}$$

hold. Since the numbers $a+e+i$, $b+f+g$, and $c+d+h$ are 8, 7, and 3 in some order, their remainders modulo 3 imply $a=b=2$. Now the number of cap points in the 2-flat $(x_1, x_2) = (-1,-1)$ is $b-3$, which is -1. □

4.2. Caps of sizes 19*,* 20*, and* 21

We start by finding the unique 19-cap 4-flat up to isomorphism and proving uniqueness (without reference to the 20-cap 4-flat). We show how to tweak that argument to prove that the 20-cap 4-flat is unique up to isomorphism and no 21-cap 4-flats exist.

The 19-cap 4-flat argument consists of three steps. First, we use the standard diagram to obtain at least one 3-flat point count of the cap. Second, we obtain a 2-flat point count of the cap. Third, we use that result to determine the 19-cap 4-flat.

We begin with the standard diagram.

Proposition 4.5 (Dimension 4: some 3-flats in 19-caps)**.** *Each* 19*-cap* 4*-flat has at least one* $\{9,9,1\}$ *or* $\{9,8,2\}$ *hyperplane direction.*

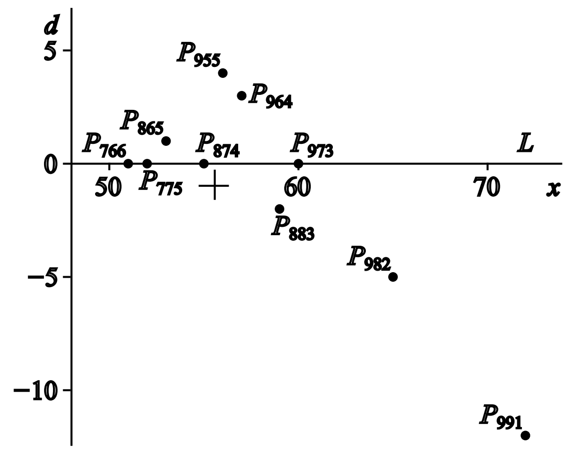


Figure 12: Standard diagram for $(n, s) = (4, 19)$.

Proof. Figure 12 shows the standard diagram for $(n, s) = (4, 19)$. Instead of the usual pair (x, y) of co-ordinates for $\mathbb{R}^2$, the figure uses the pair (x, d) of co-ordinates, where the value of the new co-ordinate d at each point is, in the original (x, y) diagram, the signed vertical distance of that point above the line L passing through the points P_{766}, P_{775}, P_{874}, and P_{973}. (At points below L, the co-ordinate d has negative values.) The line L has the equation $y = 5x - 180$, and the equality $d = y - 5x + 180$ holds.

The diagram implies that each 19-cap 4-flat has at least one $\{9, 9, 1\}$, $\{9, 8, 2\}$, or $\{8, 8, 3\}$ hyperplane direction.

Suppose that some 19-cap 4-flat C has no $\{9, 9, 1\}$ and no $\{9, 8, 2\}$ hyperplane directions. Now $d(P_{Cr}) = -39/40$ and $d(P_{883}) = -2$, and the inequality $d(P_D) \geq 0$ holds for all other P_D. The critical point P_{Cr} is the centre of mass of the 40 points P_D, so $40d(P_{Cr}) = \sum_D d(P_D)$, so

$$-39 = (-2)t_{883}(C) + (\text{nonnegative number}) \geq -2t_{883}(C),$$

so $t_{883}(C) \geq \lceil 39/2 \rceil = 20$, which is impossible by the following lemma. □

Lemma 4.6 (Dimension 4: condition on $\{8, 8, 3\}$ hyperplane directions)**.** *In each* 19*-cap* 4*-flat, the number of* $\{8, 8, 3\}$ *hyperplane directions is at most* 14.

Proof. Suppose that some $\{8, 8, 3\}$ hyperplane direction D_1 of some 19-cap 4-flat C has co-ordinate x_1. As in the standard diagram for triples, the 13 projective dual lines of C through $\{\pm(x_1)_L\}$ yield a partition of the set of 39 hyperplane directions other than D_1 into 13 triples T. Each triple T has at most one $\{8, 8, 3\}$ hyperplane direction: otherwise, for some triple T_0, the set $\{D_1\} \cup T_0$ has at least three $\{8, 8, 3\}$ hyperplane directions, contradicting Lemma 4.4.[3] □

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[3]Many thanks to an anonymous referee for suggesting Lemma 4.6 as stated (a previous version gave $t_{883}(C) \leq 19$) and the idea of its proof via adapting the proof of Lemma 4.12.

We now obtain a 2-flat point count.

Proposition 4.7 (Dimension 4: some 2-flats in 19-caps)**.** *Let C be some 19-cap 4-flat. For some independent pair (x_1, x_2) of co-ordinates of C, the 2-flat point count of C for (x_1, x_2) is*

$$\begin{pmatrix} 4 & 1 & 4 \\ 1 & 0 & 1 \\ 4 & 0 & 4 \end{pmatrix}.$$

Proof. By Proposition 4.5, C has at least one $\{9, 9, 1\}$ or $\{9, 8, 2\}$ hyperplane direction.

Suppose that C has at least one $\{9, 9, 1\}$ hyperplane direction. For some co-ordinate x_1, the two 3-flats $x_1 = \pm 1$ have nine cap points each. Every 9-cap 3-flat has exactly four $\{3, 3, 3\}$ hyperplane directions, and the number of hyperplane directions of every 3-flat is 13, which is more than 2×4, so for some co-ordinate x_2 with (x_1, x_2) independent, the point count of C for (x_1, x_2) is

$$\begin{pmatrix} 4 & * & 4 \\ 1 & * & 1 \\ 4 & * & 4 \end{pmatrix}. \tag{5}$$

Each $*$ in that point count is at most 1 (avoid 10-cap 3-flats), and C has no H of 414 edges by Lemma 4.2, so without loss of generality, the point count is the required matrix.

Now suppose that C has at least one $\{9, 8, 2\}$ hyperplane direction. For some co-ordinate x_1, the 3-flats $x_1 = -1$, $x_1 = 0$, and $x_1 = 1$ have nine, two, and eight cap points respectively.

The 8-cap 3-flat $x_1 = 1$ has at least one $\{4, 4, 0\}$ hyperplane direction; let some such 2-flat direction have a co-ordinate that is the restriction to $x_1 = 1$ of some co-ordinate x_2 of C. That 2-flat direction is parallel to a $\{3, 3, 3\}$ or $\{4, 4, 1\}$ hyperplane direction D of the 3-flat $x_1 = -1$ (Theorem 3.5).

If D has point count $\{3, 3, 3\}$, then without loss of generality, the point count of C for (x_1, x_2) is

$$\begin{pmatrix} 3 & * & 4 \\ 3 & * & 0 \\ 3 & * & 4 \end{pmatrix}.$$

In this case, to avoid $\{3, 2, 4\}$ hyperplane directions of 9-cap 3-flats (Proposition 3.4), two of the entries $*$ are 1 and the third $*$ is 0, so without loss of generality, the point count is

$$\begin{pmatrix} 3 & 1 & 4 \\ 3 & 0 & 0 \\ 3 & 1 & 4 \end{pmatrix},$$

which yields an H of 431 edges, contradicting Lemma 4.3.

Therefore, D has point count $\{4, 4, 1\}$; without loss of generality, the point count of C for (x_1, x_2) is

$$\begin{pmatrix} 4 & * & 4 \\ 1 & * & 0 \\ 4 & * & 4 \end{pmatrix}.$$

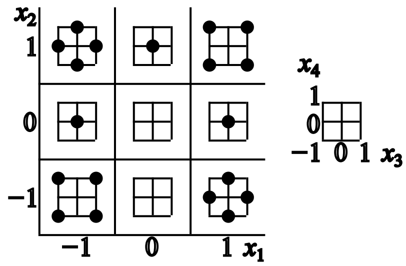

Figure 13: The representative 19-cap 4-flat.

Again, each $*$ is at most 1. By Lemma 4.1, no triangle of 414 edges is possible. Therefore, the point count is

$$\begin{pmatrix} 4 & 1 & 4 \\ 1 & 0 & 0 \\ 4 & 1 & 4 \end{pmatrix},$$

which is isomorphic to the required matrix. □

Finally, we find the unique 19-cap 4-flat up to isomorphism and we prove uniqueness.

Theorem 4.8 (Dimension 4: the 19-cap)**.** *Every* 19*-cap* 4*-flat is isomorphic to the representative cap in* $\mathbb{F}_3^4$ *shown in Figure* 13*. In that cap, the numbers of* 3*-flat directions with point counts* $\{9,9,1\}$*,* $\{9,8,2\}$*,* $\{8,6,5\}$*, and* $\{7,6,6\}$ *are* 1*,* 9*,* 18*, and* 12 *respectively.*

Proof. Let C be some 19-cap 4-flat; choose co-ordinates x_1 and x_2 as in Proposition 4.7. Now the point count of C for (x_1, x_2) is

$$\begin{pmatrix} 4 & 1 & 4 \\ 1 & 0 & 1 \\ 4 & 0 & 4 \end{pmatrix}.$$

Without loss of generality, let co-ordinates x_3 and x_4 be such that the 2-flats $(x_1, x_2) = (-1, 1)$, $(x_1, x_2) = (-1, 0)$, and $(x_1, x_2) = (0, 1)$ are as in Figure 14(a). By the structure of the 9-cap 3-flats $x_1 = -1$ and $x_2 = 1$, the 2-flats $(x_1, x_2) = \pm(1, 1)$ are also as in that figure. Without loss of generality, the 2-flat $(x_1, x_2) = (1, 0)$ is as in Figure 14(b) or as in Figure 14(c).

Suppose that it is as in Figure 14(b). The structure of the 9-cap 3-flat $x_1 = 1$ implies that the square $(x_1, x_2) = (1, -1)$ is also as in that figure. That is impossible: the line $(x_1 - x_2, x_3, x_4) = (-1, 0, 0)$ has three cap points.

Therefore, the 2-flat $(x_1, x_2) = (1, 0)$ is as in Figure 14(c). The structure of the 9-cap 3-flat $x_1 = 1$ forces the square $(x_1, x_2) = (1, -1)$, so all of C is as in Figure 14(c). This figure does indeed show a cap, so it is the unique 19-cap 4-flat up to isomorphism. Its 3-flat point counts can be verified directly. □

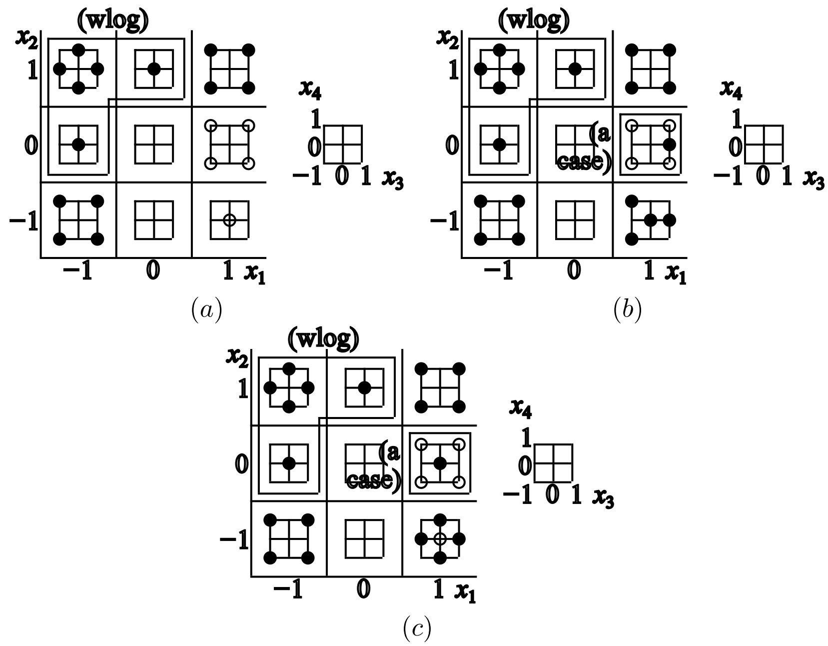

Figure 14: Constructing the 19-cap 4-flat. (a) Top row and left column constructed. (b) Trying to construct an impossible case. (c) The final 19-cap 4-flat.

The previous argument easily adapts to 20- and 21-cap 4-flats.

Theorem 4.9 (Dimension 4: the 20-cap)**.** *Every* 20*-cap* 4*-flat is isomorphic to the representative cap in* $\mathbb{F}_3^4$ *shown in Figure* 15*. In that cap, the numbers of* 3*-flat directions with point counts* $\{9, 9, 2\}$ *and* $\{8, 6, 6\}$ *are* 10 *and* 30 *respectively.*

For each 20*-cap* 4*-flat with some given* $\{9, 9, 2\}$ *hyperplane direction* D*, there is some isomorphism from that cap to Figure* 15 *that sends* D *to the* x_1*-hyperplane direction.*

For each 20*-cap* 4*-flat with some given independent pair* (x_1, x_2) *of co-ordinates for which the point count of the cap is*

$$\begin{pmatrix} 4 & 1 & 4 \\ 1 & 0 & 1 \\ 4 & 1 & 4 \end{pmatrix}, \tag{6}$$

there is some isomorphism from that cap to Figure 15 *that sends each* 2*-flat* $(x_1, x_2) = (a_1, a_2)$ *to the* 2*-flat* $(x_1, x_2) = (a_1, a_2)$*.*

Proof. The standard diagram in Figure 16 uses (x, d) co-ordinates, where d is, in the (x, y) diagram, the signed vertical distance above the line L passing through the points P_{776} and P_{983}; the line L has the equation $10y = 51x - 2007$. By

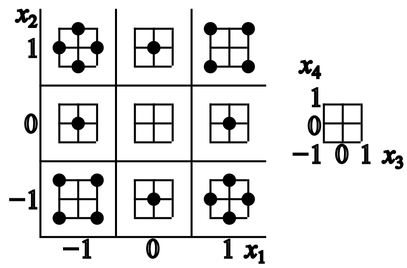

Figure 15: The representative 20-cap 4-flat.

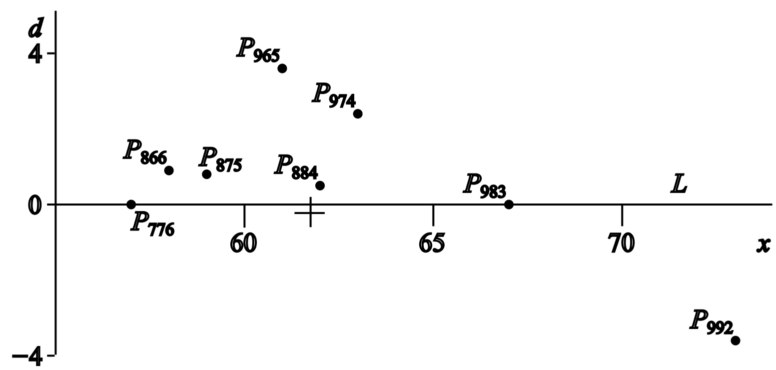

Figure 16: Standard diagram for $(n, s) = (4, 20)$.

that standard diagram, each 20-cap 4-flat has at least one $\{9, 9, 2\}$ hyperplane direction.

If some such $\{9, 9, 2\}$ hyperplane direction is given, then, as before, we obtain a 2-flat point count of the form (5), where each $*$ is at most 1; to avoid triangles of 414 edges, the point count matrix is (6).

From that point count, by the argument for 19-cap 4-flats, without loss of generality the cap minus its cap point in the 2-flat $(x_1, x_2) = (0, -1)$ is as in Figure 13. Now the structure of the 9-cap 3-flat $x_2 = -1$ forces the cap to be as in Figure 15. This figure does indeed show a cap, so it is the unique 20-cap 4-flat up to isomorphism. Its 3-flat point counts can be verified directly. □

Theorem 4.10 (Dimension 4: no 21-caps)**.** *No* 4*-dimensional cap has size* 21*.*

Proof 1. The standard diagram in Figure 17 uses (x, d) co-ordinates, where d is, in the (x, y) diagram, the signed vertical distance above the line L_1 passing through the points P_{777} and P_{984}; the line L_1 has the equation $7y = 39x - 1722$. By that standard diagram, each 21-cap 4-flat has at least one $\{9, 9, 3\}$ hyperplane direction; we obtain a 2-flat point count of the form (5) where each $*$ is 1, so we obtain a triangle of 414 edges, contradicting Lemma 4.1. □

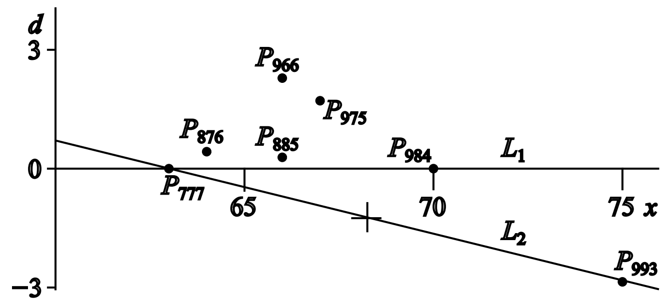

Figure 17: Standard diagram for $(n, s) = (4, 21)$.

Proof 2. (Using the standard diagram, this proof rephrases an argument from Davis and Maclagan [1].) All points P_D that are not in $\{P_{993}, P_{777}\}$ are on the same side of the line L_2 passing through the points P_{993}, P_{777}, and P_{Cr}; the line L_2 has the equation $3y = 16x - 693$. Therefore, in each 21-cap 4-flat C, all 3-flat directions have point count $\{9, 9, 3\}$ or $\{7, 7, 7\}$. The equations (2), (3), and (4) form the system

$$\left\{ \begin{array}{rcr} t_{993}(C) + t_{777}(C) & = & 40 \\ 75t_{993}(C) + 63t_{777}(C) & = & 2730 \\ 169t_{993}(C) + 105t_{777}(C) & = & 5320 \end{array} \right\},$$

which has the unique solution $(t_{993}(C), t_{777}(C)) = (35/2, 45/2)$, for which $t_{993}(C)$ and $t_{777}(C)$ are not in $\mathbb{Z}$. □

4.3. Caps of size 18

The argument to find all 18-cap 4-flats has a structure similar to the argument for 19-cap 4-flats: start by considering 3-flat point counts, and after that, refine further to determine the caps exactly.

Proposition 4.11 (Dimension 4: some 3-flats in 18-caps)**.** *Each* 18*-cap* 4*-flat has at least one* $\{9, 9, 0\}$*,* $\{9, 8, 1\}$*,* $\{9, 7, 2\}$*, or* $\{8, 8, 2\}$ *hyperplane direction.*

Proof. Suppose otherwise. Apply the standard diagram in Figure 18 with the line L passing through the points P_{Cr}, P_{873}, and P_{666}. (The figure uses the line L, which has the equation $7y = 32x - 1020$, to obtain the co-ordinate d as before.) If C is some 18-cap 4-flat, then the equations (2), (3), and (4) imply $(t_{873}(C), t_{666}(C)) = (27, 13)$, which is impossible by the following lemma. □

Lemma 4.12 (Dimension 4: condition on $\{8, 7, 3\}$ and $\{6, 6, 6\}$ hyperplane directions)**.** *In each* 18*-cap* 4*-flat with at least one* $\{8, 7, 3\}$ *hyperplane direction, at least* 13 *hyperplane directions are neither* $\{8, 7, 3\}$ *nor* $\{6, 6, 6\}$*.*

Proof. Suppose that some 18-cap 4-flat C does not satisfy the result. Let some $\{8, 7, 3\}$ hyperplane direction D_1 of C have co-ordinate x_1. As in the standard

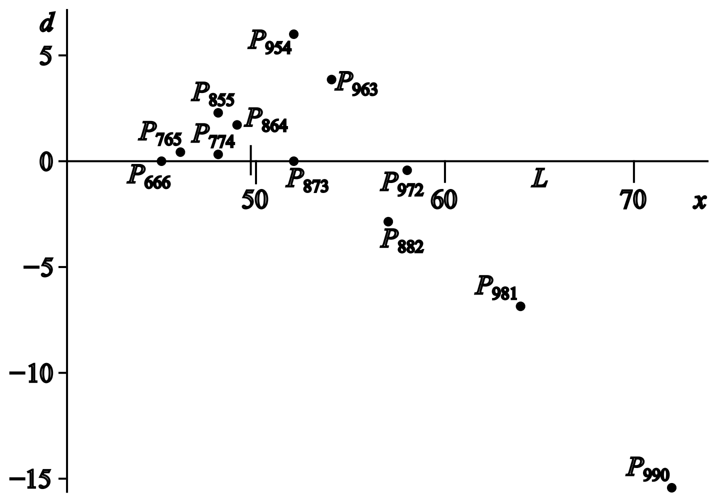

Figure 18: Standard diagram for $(n, s) = (4, 18)$.

diagram for triples, the 13 projective dual lines of C through $\{\pm(x_1)_L\}$ yield a partition of the set of 39 hyperplane directions other than D_1 into 13 triples T.

At most 12 hyperplane directions are neither $\{8,7,3\}$ nor $\{6,6,6\}$, so some triple T_0 has only $\{8,7,3\}$ and/or $\{6,6,6\}$ hyperplane directions, so $\{D_1\} \cup T_0$ has (i) at least two $\{6,6,6\}$ hyperplane directions and at least one $\{8,7,3\}$ hyperplane direction, contradicting Lemma 2.3, or (ii) at least three $\{8,7,3\}$ hyperplane directions, in which case the fourth hyperplane direction has point count $\{8,7,3\}$ or $\{6,6,6\}$, contradicting Lemma 4.4. □

We now find all possible 18-cap 4-flats. In each of several cases, we refine to a 3-flat point count plus extra information or to a 2-flat point count, and then we determine the caps.

Theorem 4.13 (Dimension 4: the 18-caps)**.** *Each* 18*-cap* 4*-flat* C *is isomorphic to exactly one of the* 20 *representative* 18*-caps in* $\mathbb{F}_3^4$ *shown in the figures to come. Each of the* 20 *names*

$$990A_1,\ 990A_2,\ 990A_3,\ 990B,$$
$$981A,\ \ldots,\ 981J,$$
$$972A,\ 963A,\ 963B,\ 954A,\ 882A_1,\ \text{and}\ 882A_2$$

means an arbitrary image of its representative under an isomorphism.

Remarks. In the name of each type of cap, the first part *abc* refers to the "largest 3-flat point count" of the cap, that is, the 3-flat point count $\{a, b, c\}$ such that (i) the statement $a \geq b \geq c$ holds, (ii) the number a is as large as possible, and (iii) if there are two or more possibilities with the same a, then b is as large as possible. The letter following the "largest 3-flat point count" distinguishes different tuples t of the form $(t_{abc}(C))_{a \geq b \geq c}$. In cases where t is the same for

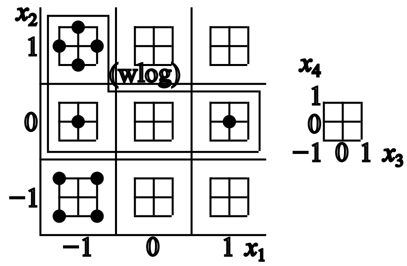

Figure 19: Part of the representative caps $990A_1$, $990A_2$, and $990A_3$.

two or more representatives, a subscript following the letter distinguishes them. For instance, an $882A_2$ is an 18-cap 4-flat that has no 9-cap 3-flat, has at least one $\{8, 8, 2\}$ hyperplane direction, and is not isomorphic to any $882A_1$.

Proof. By Proposition 4.11, exactly one of the following four statements holds: (i) C has at least one $\{9, 9, 0\}$ hyperplane direction, (ii) C has at least one $\{9, 8, 1\}$ hyperplane direction but no $\{9, 9, 0\}$ hyperplane directions, (iii) C has at least one $\{9, 7, 2\}$ hyperplane direction but no $\{9, 9, 0\}$ or $\{9, 8, 1\}$ hyperplane directions, or (iv) C has at least one $\{8, 8, 2\}$ hyperplane direction but no $\{9, 9, 0\}$, $\{9, 8, 1\}$, or $\{9, 7, 2\}$ hyperplane directions. We consider those four cases one by one.

Case 1: the cap C has at least one $\{9, 9, 0\}$ hyperplane direction. Let the co-ordinate x_1 of C be such that the 3-flats $x_1 = \pm 1$ have nine cap points each. The associated 9-cap 3-flats have parallel or nonparallel axis line directions. Each subcase is considered in turn.

Subcase 1(a): the 9-cap 3-flats $x_1 = \pm 1$ have parallel axis line directions. Without loss of generality, let x_2, x_3, and x_4 be such that $(x_1, \ldots, x_4)$ is independent and the following conditions hold: the x_2-axis direction of C restricts to the axis line direction of each of the 9-cap 3-flats $x_1 = \pm 1$, the point count of C for (x_1, x_2) is the matrix $M_{1(a)}$ that is equal to

$$\begin{pmatrix} 4 & 0 & 4 \\ 1 & 0 & 1 \\ 4 & 0 & 4 \end{pmatrix},$$

and both the 3-flat $x_2 = 0$ and the 2-flat $(x_1, x_2) = (-1, 1)$ are as in Figure 19. By the structure of the 9-cap 3-flat $x_1 = -1$, the 2-flat $(x_1, x_2) = (-1, -1)$ is also as in that figure.

The axis line directions of the 9-cap 3-flats $x_1 = \pm 1$ are parallel, so by the structure of the 9-cap 3-flat $x_1 = 1$, the squares $(x_1, x_2) = (1, \pm 1)$ have their centres at $(x_3, x_4) = (0, 0)$. We obtain three caps up to isomorphism: the representative caps $990A_1$, $990A_2$, and $990A_3$ in Figures 20, 21, and 22 respectively. (The diagonal directions of the square $(x_1, x_2) = (1, 1)$ are parallel to, respectively, the following line directions of the square $(x_1, x_2) = (-1, 1)$:

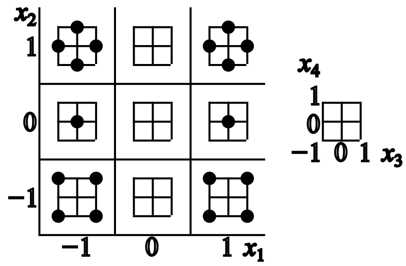

Figure 20: The representative $990A_1$.

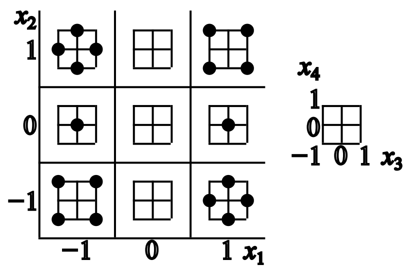

Figure 21: The representative $990A_2$.

the two diagonal directions, the two side directions, or one diagonal direction and one side direction.)

Subcase 1(b): the 9-cap 3-flats $x_1 = \pm 1$ have nonparallel axis line directions. Without loss of generality, let x_2, x_3, and x_4 be such that $(x_1, \ldots, x_4)$ is independent and the x_3- and x_4-axis directions of C restrict to the axis line directions of the 3-flats $x_1 = -1$ and $x_1 = 1$ respectively. The point count of C for (x_1, x_2) is forced to be the matrix $M_{1(b)}$ that is equal to

$$\begin{pmatrix} 3 & 0 & 3 \\ 3 & 0 & 3 \\ 3 & 0 & 3 \end{pmatrix}.$$

In the 3-flat $x_1 = -1$ (respectively, $x_1 = 1$), the x_3-hyperplane direction

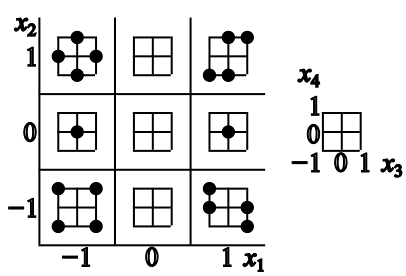

Figure 22: The representative $990A_3$.

(respectively, the x_4-hyperplane direction) has point count $\{4,4,1\}$. Without loss of generality, each of the 2-flats $(x_1,x_3)=(-1,1)$ and $(x_1,x_4)=(1,1)$ is the underlying space of a 1-cap 2-flat that is as in Figure 23(a).

It is also true, without loss of generality, that the triangles in the 2-flats $(x_1,x_2)=(\pm1,0)$ are as in Figure 23(b). (Apply one or both of the maps

$$(x_1,\ldots,x_4)\mapsto(x_1,x_2,1-x_1-x_3,x_4) \text{ and}$$
$$(x_1,\ldots,x_4)\mapsto(x_1,x_2,x_3,1+x_1-x_4)$$

if necessary.)

It is also true, without loss of generality, that the 3-flat $x_1=-1$ is as in Figure 23(c). (Recall the structure of each $\{3,3,3\}$ hyperplane direction D of every 9-cap 3-flat: each of the three 2-flats in D contains one of three disjoint cap triangles that are translations of one another. To obtain Figure 23(c), do the following. Firstly, apply a power of

$$(x_1,\ldots,x_4)\mapsto(x_1,x_2,x_3,x_4+x_2)$$

to obtain the 3-flat $x_1=-1$ in the figure. Secondly, restore the 2-flat $(x_1,x_4)=(1,1)$ via a shear that (i) preserves each point in $x_1=-1$, and (ii) sends the 1-cap 2-flat in the current x_4-hyperplane direction of the 3-flat $x_1=1$ to the 2-flat $(x_1,x_4)=(1,1)$.)

Therefore, without loss of generality, C is the representative $990B$ in Figure 23(d). (Apply a power of

$$(x_1,\ldots,x_4)\mapsto(x_1,x_1+x_2-1,-x_1+x_2+x_3+1,x_4)$$

to obtain the desired 3-flat $x_1=1$.)

Case 2: the cap C has at least one $\{9,8,1\}$ hyperplane direction but no $\{9,9,0\}$ hyperplane directions. For some co-ordinate x_1 of C, let the numbers of cap points in the 3-flats $x_1=-1$, $x_1=0$, and $x_1=1$ be 9, 1, and 8 respectively. For some co-ordinate x_2 of C with (x_1,x_2) independent, the x_2-hyperplane direction of the 8-cap 3-flat $x_1=1$ has point count $\{4,4,0\}$. In the 9-cap 3-flat $x_1=-1$, the x_2-hyperplane direction has point count $\{4,4,1\}$ or $\{3,3,3\}$. Therefore, without loss of generality, the point count of C for (x_1,x_2) is among the matrices $M_{2(a)}$, $M_{2(b)}$, and $M_{2(c)}$ that are equal to

$$\begin{pmatrix}4&0&4\\1&1&0\\4&0&4\end{pmatrix},\ \begin{pmatrix}4&1&4\\1&0&0\\4&0&4\end{pmatrix},\text{ and }\begin{pmatrix}3&0&4\\3&1&0\\3&0&4\end{pmatrix}$$

respectively. Each subcase is considered in turn.

Subcase 2(a): the point count is $M_{2(a)}$. Without loss of generality, let x_3 and x_4 be such that the 2-flats $(x_1,x_2)=(-1,1)$, $(x_1,x_2)=(-1,0)$, and $(x_1,x_2)=(0,0)$ are as in Figure 24. By the structure of the 9-cap 3-flat $x_1=-1$, the 2-flat $(x_1,x_2)=(-1,-1)$ is also as in that figure. Similarly, the 3-flats $x_2=\pm x_1$ now force the rest of the figure, yielding the representative $981A$.

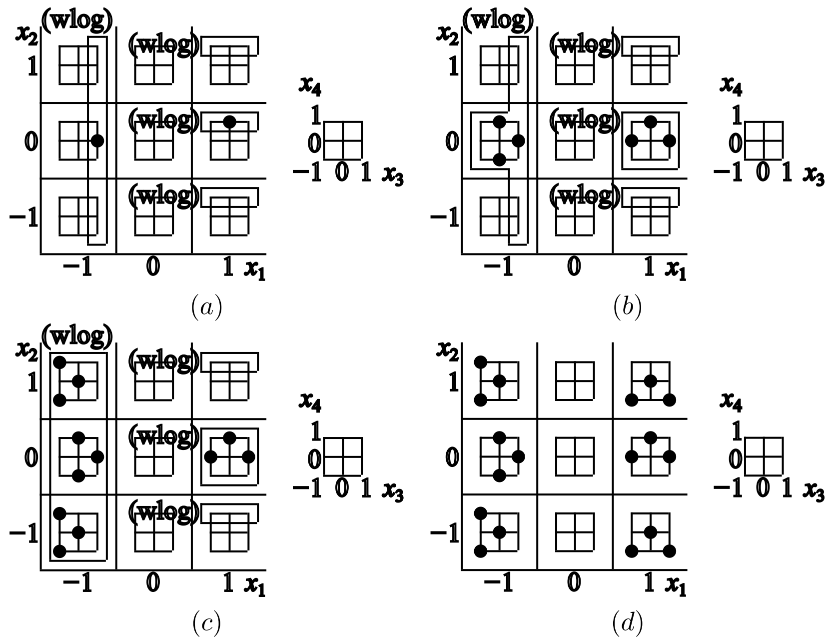

Figure 23: Constructing the representative $990B$. (a) Fixing the 2-flats $(x_1, x_3) = (-1, 1)$ and $(x_1, x_4) = (1, 1)$, without loss of generality. (b) Fixing the 2-flats $(x_1, x_2) = (\pm 1, 0)$, without loss of generality. (c) Fixing the 3-flat $x_1 = -1$, without loss of generality. (d) The representative $990B$.

Subcase 2(b): the point count is $M_{2(b)}$. Without loss of generality, let x_3 and x_4 be such that the 2-flats $(x_1, x_2) = (-1, 1)$, $(x_1, x_2) = (-1, 0)$, and $(x_1, x_2) = (0, 1)$ are as in Figure 25. The 9-cap 3-flats $x_1 = -1$ and $x_2 = 1$ must, by their structure, also be as in that figure.

Without loss of generality, the 2-flat $(x_1, x_2) = (1, -1)$ is among the seven options in Figure 26, yielding the representative caps $981B$ to $981H$. (In $(x_1, x_2) = (1, -1)$, we may assume that the centre of the square has (x_3, x_4) co-ordinates among $(0, 0)$, $(1, 0)$, and $(1, 1)$, without loss of generality. In the case of $(0, 0)$, there are three choices for the diagonal directions of the square up to isomorphism, giving $981B$, $981C$, and $981D$. Each other case has two choices for the diagonal directions of the square up to isomorphism, because the point $(x_1, \ldots, x_4) = (1, -1, 0, 0)$ is the midpoint of a cap line segment; we obtain $981E$ to $981H$.)

Subcase 2(c): the point count is $M_{2(c)}$. Without loss of generality, let x_3 and x_4 be such that the 2-flats $(x_1, x_2) = (-1, -1)$, $(x_1, x_2) = (-1, 0)$, and $(x_1, x_2) = (0, 0)$ are as in Figure 27. (The cap triangles in the first two of those three 2-flats are translations of each other, by the structure of the 9-cap 3-flat $x_1 = -1$.) To avoid lines of three cap points in the 3-flat $x_1 = -1$, the 2-flat

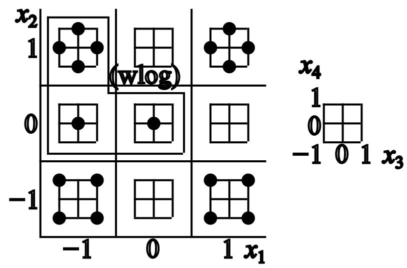

Figure 24: The representative 981A.

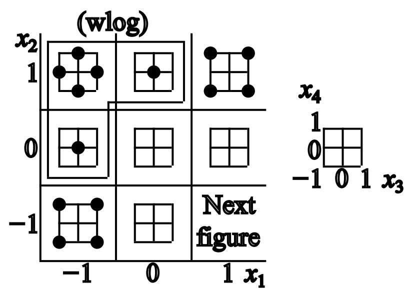

Figure 25: The representative caps 981B to 981H.

$(x_1, x_2) = (-1, 1)$ is also as in that figure. The squares $(x_1, x_2) = (1, \pm 1)$ are still to be considered.

Suppose both of the points $(x_1, \ldots, x_4) = (1, \pm 1, 1, -1)$ are not cap points. In each of the 2-flats $(x_1, x_2) = (1, \pm 1)$, the line $x_3 + x_4 = 0$ has no cap points, so each of the lines $x_3 + x_4 = \pm 1$ has two cap points. Therefore, the point count of C for $(x_1, 1 - x_1 - x_3 - x_4)$ is $M_{2(b)}$ and we are in Subcase 2(b).

Similarly, we are in Subcase 2(b) if both of the points $(x_1, \ldots, x_4) = (1, \pm 1, -1, -1)$, or both of the points $(x_1, \ldots, x_4) = (1, \pm 1, 0, 1)$, are not cap points: apply

$$(x_1, \ldots, x_4) \mapsto (x_1, x_2, -x_3, x_4) \text{ or}$$
$$(x_1, \ldots, x_4) \mapsto (x_1, x_2, x_3 + x_1, x_4 - x_3 + x_1)$$

respectively, and use the previous argument.

Therefore, we may assume that in $(x_1, \ldots, x_4)$ co-ordinates, each of the pairs $\{(1, \pm 1, 1, -1)\}$, $\{(1, \pm 1, -1, -1)\}$, and $\{(1, \pm 1, 0, 1)\}$ has at least one cap point.

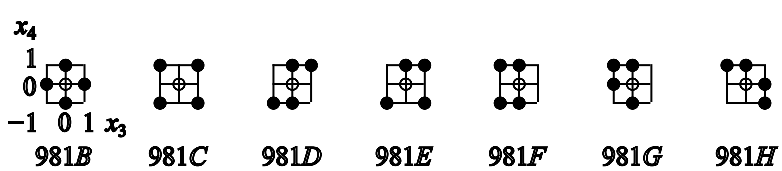

Figure 26: The 2-flat $(x_1, x_2) = (1, -1)$ for the representative caps 981B to 981H.

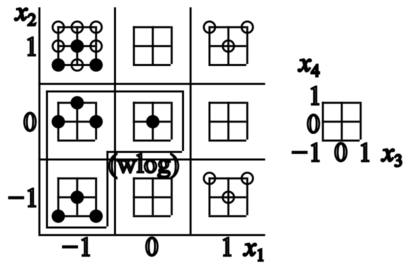

Figure 27: Part of the representative caps $981I$ and $981J$.

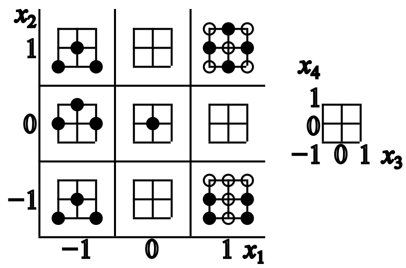

Figure 28: The representative $981I$.

In each of the 2-flats $(x_1, x_2) = (1, \pm 1)$, among the three points with (x_3, x_4) co-ordinates $(1, -1)$, $(-1, -1)$, and $(0, 1)$, the number of cap points is 1 or 2. (If all three points are cap points, then each position for the fourth cap point in the 2-flat yields a line of three cap points.) We obtain the following two options.

- **Each of the pairs $\{(1, \pm 1, 1, -1)\}$, $\{(1, \pm 1, -1, -1)\}$, and $\{(1, \pm 1, 0, 1)\}$ has exactly one cap point.** Without loss of generality (negate x_2 and/or apply a power of

$$(x_1, \ldots, x_4) \mapsto (x_1, x_2, x_3 + x_1, x_4 - x_3 + x_1)$$

as needed), the points $(1, -1, 1, -1)$, $(1, -1, -1, -1)$, and $(1, 1, 0, 1)$ are cap points, so $(1, -1, 0, -1)$ is not a cap point (avoid three cap points in a line), so C is the representative $981I$ in Figure 28.

- **One pair has two cap points, and each of the other two pairs has one cap point.** Without loss of generality (use the same transformations as before), the points $(1, -1, 1, -1)$, $(1, 1, -1, -1)$, and $(1, \pm 1, 0, 1)$ are cap points, so the points $(1, 1, 1, 0)$ and $(1, -1, -1, 0)$ are not cap points (avoid three cap points in a line), so C is the representative $981J$ in Figure 29.

Case 3: the cap C has at least one $\{9, 7, 2\}$ hyperplane direction but no $\{9, 9, 0\}$ or $\{9, 8, 1\}$ hyperplane directions.[4] For some co-ordinate x_1 of

[4]Many thanks to an anonymous referee for suggesting the outline of the argument that

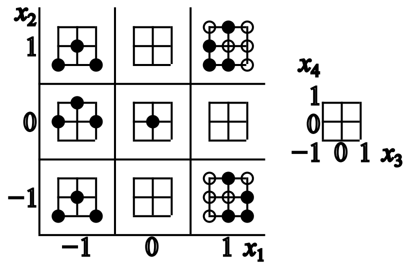

Figure 29: The representative $981J$.

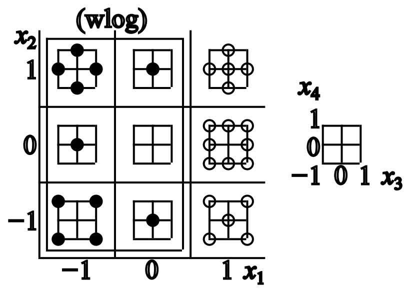

Figure 30: Trying to construct a cap in Subcase 3(a).

C, let the numbers of cap points in the 3-flats $x_1 = -1$, $x_1 = 0$, and $x_1 = 1$ be 9, 2, and 7 respectively. The axis line direction D_{-1} of the 9-cap 3-flat $x_1 = -1$ is parallel or not parallel to the line direction D_0 of the cap line segment in the 3-flat $x_1 = 0$. Each subcase is considered in turn.

Subcase 3(a): the line directions D_{-1} and D_0 are parallel. Without loss of generality, the 3-flats $x_1 = -1$ and $x_1 = 0$ are as in Figure 30 for some x_2, x_3, and x_4. To avoid lines of three cap points, there are now exactly nine options for cap points in the 3-flat $x_1 = 1$, as Figure 30 shows. Those nine points together with the cap points in the 3-flats $x_1 = -1$ and $x_1 = 0$ form the representative 20-cap 4-flat $\widetilde{C}$ in Figure 15. Therefore, C is just $\widetilde{C}$ minus two cap points.

In $\widetilde{C}$, each cap point is in some 2-cap 3-flat. In particular, if P is one of the two cap points that is removed from $\widetilde{C}$ to obtain C, then P is in a 2-cap 3-flat in some $\{9, 9, 2\}$ hyperplane direction of $\widetilde{C}$. Therefore, C has a $\{9, 9, 0\}$ or $\{9, 8, 1\}$ hyperplane direction, which is impossible.

Subcase 3(b): the line directions D_{-1} and D_0 are not parallel. Without loss of generality, the 3-flats $x_1 = -1$ and $x_1 = 0$ are as in Figure 31 for some x_2, x_3, and x_4. The points at the open circles in that figure are not cap points, since they are midpoints of known cap line segments. We now prove

appears here for Case 3. Because of that suggestion, the overall argument for 18-cap 4-flats was simplified significantly.

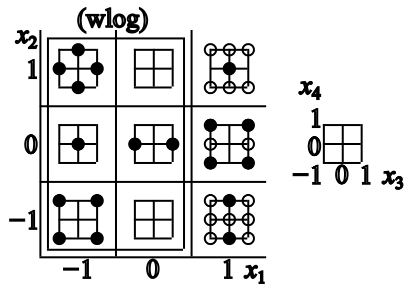

Figure 31: The representative $972A$.

that, up to an invertible affine transformation, all of C is as in that figure.

For the rest of this subcase, we use $(x_1,\ldots,x_4)$ co-ordinates to describe points. Consider the 3-flats F such that (i) the intersection of F with the 3-flat $x_1=-1$ is a 2-flat with exactly one cap point on it, and (ii) the intersection of F with the 3-flat $x_1=0$ has no cap points on it. There are 12 such 3-flats F, and their equations are

$$x_1+x_2+x_4=1,\ x_1-x_2=-1,\ x_1+x_2-x_4=1,$$
$$x_1+x_2-x_3+x_4=0,\ x_1+x_2+x_3+x_4=0,\ x_1-x_2-x_3=0,$$
$$x_1+x_2+x_3-x_4=0,\ x_1+x_2-x_3-x_4=0,\ x_1-x_2+x_3=0,$$
$$x_2+x_4=-1,\ x_2-x_4=-1,\text{ and } x_1+x_2=-1.$$

If there is only one cap point on some such 3-flat F, then the 3-flat direction of F in C is a $\{9,8,1\}$ hyperplane direction, which is impossible. Therefore, the intersection of each F with the 3-flat $x_1=1$ has at least one cap point on it.

The first three 3-flats F given above imply that out of each pair of points among $(1,-1,0,1)$, $(1,-1,0,-1)$, and $(1,0,0,0)$, at least one is a cap point. It follows that at least two of those three points are cap points. Without loss of generality, we may assume that $(1,-1,0,1)$ and $(1,-1,0,-1)$ are cap points (apply a power of the map

$$(x_1,\ldots,x_4)\mapsto(x_1,x_2-x_4-x_1,x_3,x_4-x_1)$$

to permute $(1,-1,0,1)$, $(1,-1,0,-1)$, and $(1,0,0,0)$ while maintaining the configurations in the 3-flats $x_1=-1$ and $x_1=0$).

The other nine options for F, in the order above, yield the conclusion that at least one point in each of the following sets is a cap point:

$$\{(1,0,-1,1),(1,0,0,-1),(1,1,-1,0)\},\ \{(1,0,0,-1),(1,0,1,1),(1,1,1,0)\},$$
$$\{(1,0,1,1),(1,0,1,-1),(1,1,0,0)\},\ \{(1,0,0,1),(1,0,1,-1),(1,1,1,0)\},$$
$$\{(1,0,0,1),(1,0,-1,-1),(1,1,-1,0)\},\ \{(1,0,-1,1),(1,0,-1,-1),(1,1,0,0)\},$$
$$\{(1,0,-1,-1),(1,0,0,-1),(1,0,1,-1)\},\ \{(1,0,-1,1),(1,0,0,1),(1,0,1,1)\},$$
$$\text{and } \{(1,1,-1,0),(1,1,0,0),(1,1,1,0)\}.$$

Also, to avoid a line of three cap points passing through at least one of $(1,-1,0,\pm1)$, the following statements hold for each a in $\mathbb{F}_3$: the pair $\{(1,0,a,-1),(1,1,-a,0)\}$

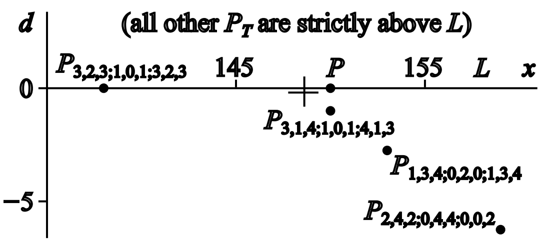

Figure 32: Standard diagram for triples for $(n, s) = (4, 18)$ and $\{a, b, c\} = \{8, 8, 2\}$.

has at most one cap point, and the pair $\{(1, 0, a, 1), (1, 1, -a, 0)\}$ has at most one cap point.

Taking the 3-flat $x_1+x_2 = -1$ as F, we see that the line $(x_1, x_2, x_4) = (1, 1, 0)$ has at least one cap point, so it has one or two cap points.

Suppose that it has two cap points. Since each of the 2-flats $(x_1, x_2 \pm x_4) = (1, 1)$ has at most four cap points, at most one point in each of the lines $(x_1, x_2, x_4) = (1, 0, \pm 1)$ is a cap point. The 3-flat $x_1 = 1$ has seven cap points, so each of the lines $(x_1, x_2, x_4) = (1, 0, \pm 1)$ has exactly one cap point, and $(1, 0, 0, 0)$ is a cap point. There are three options for the pair of cap points in the line $(x_1, x_2, x_4) = (1, 1, 0)$; each option forces the cap points in the lines $(x_1, x_2, x_4) = (1, 0, \pm 1)$, yielding a line of three cap points or a 3-flat F such that F is of the form $x_1 + x_2 \pm x_3 \pm x_4 = 0$ and the intersection of F with the 3-flat $x_1 = 1$ has no cap points.

It follows that the line $(x_1, x_2, x_4) = (1, 1, 0)$ has exactly one cap point. If that point is $(1, 1, \pm 1, 0)$, then to avoid lines of three cap points, the line $(x_1, x_2, x_3) = (1, 0, \mp 1)$ has no cap points, so some 3-flat F of the form $x_1 - x_2 \pm x_3 = 0$ is such that its intersection with the 3-flat $x_1 = 1$ has no cap points. Therefore, $(1, 1, 0, 0)$ is a cap point.

The last 2-flat is $(x_1, x_2) = (1, 0)$, which now has four cap points. To avoid lines of three cap points, it is forced to be as in Figure 31. We obtain the representative $972A$ in that figure.

Case 4: the cap C has at least one $\{8, 8, 2\}$ hyperplane direction D_1, but no $\{9, 9, 0\}$, $\{9, 8, 1\}$, or $\{9, 7, 2\}$ hyperplane directions. Use the standard diagram for triples in Figure 32; the co-ordinate d is, in the (x, y) diagram, the signed vertical distance above the line L passing through the points $P_{3,2,3;1,0,1;3,2,3}$ and P, where $P = P_{3,1,4;1,0,1;3,1,4} = P_{3,1,4;0,2,0;2,4,2}$; the line L has the equation $12y = 55x - 5262$.

Since $\widetilde{P}_{Cr}$ (indicated by $+$) is strictly below L, so is some P_T. Therefore, without loss of generality, the point count of C for some (x_1, x_2) is among the matrices $M_{4(a)}$, $M_{4(b)}$, and $M_{4(c)}$ that are equal to

$$\begin{pmatrix} 2 & 4 & 2 \\ 4 & 4 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 4 \\ 3 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 4 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

respectively. Each subcase is considered in turn. (After $M_{4(a)}$ is dealt with, no

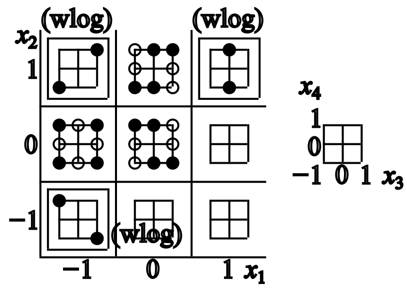

Figure 33: Another $990A_3$.

more types of 18-cap 4-flats remain to be discovered. The arguments for $M_{4(b)}$ and $M_{4(c)}$ are described briefly.)

Subcase 4(a): the point count of C for (x_1, x_2) is $M_{4(a)}$ (a "triangle of 242 edges"). Without loss of generality, let x_3 and x_4 be such that in each of the 2-cap 2-flats $(x_1, x_2) = (-1, -1)$, $(x_1, x_2) = (-1, 1)$, and $(x_1, x_2) = (1, 1)$, the midpoint of the cap line segment is $(x_3, x_4) = (0, 0)$. These three cap line segments are in three different line directions of C, or all three cap line segments are in the same line direction of C, or two but not all three cap line segments are in the same line direction of C. Each subsubcase is considered in turn.

Subsubcase 4(a)(i): the three cap line segments are in three different line directions of C. Without loss of generality, let x_3 and x_4 be such that those cap line segments are as in Figure 33. Now all of C is as in the figure (the squares $(x_1, x_2) = (-1, 0)$, $(x_1, x_2) = (0, 0)$, and $(x_1, x_2) = (0, 1)$ are forced), so the x_4-hyperplane direction of C has point count $\{9, 9, 0\}$ (in fact, C is a $990A_3$), which yields a contradiction.

Subsubcase 4(a)(ii): the three cap line segments are in the same line direction of C. Without loss of generality, let x_3 and x_4 be such that those cap line segments are as in Figure 34. The three 2-flats $(x_1, x_3) = (-1, 0)$, $(x_1 - x_2, x_3) = (0, 0)$, and $(x_2, x_3) = (1, 0)$ have no more cap points, so the six lines $(x_1, x_2, x_3) = (-1, 0, -1)$, $(x_1, x_2, x_3) = (-1, 0, 1)$, $(x_1, x_2, x_3) = (0, 0, -1)$, $(x_1, x_2, x_3) = (0, 0, 1)$, $(x_1, x_2, x_3) = (0, 1, -1)$, and $(x_1, x_2, x_3) = (0, 1, 1)$ have two cap points each; let the products x_3x_4 for the points other than cap points in those lines be a, b, c, d, e, and f respectively.

The triple of pairs $((a, b), (c, d), (e, f))$ is affected as follows by affine transformations preserving the cap line segments in the three 2-cap 2-flats $(x_1, x_2) = (-1, -1)$, $(x_1, x_2) = (-1, 1)$, and $(x_1, x_2) = (1, 1)$.

- The shear $(x_1, \ldots, x_4) \mapsto (x_1, x_2, x_3, x_4 + kx_3)$ increases all of a to f by the constant k in $\mathbb{F}_3$.
- Negating x_4 negates all of a to f.
- Negating both x_3 and x_4 takes $((a, b), (c, d), (e, f))$ to $((b, a), (d, c), (f, e))$.

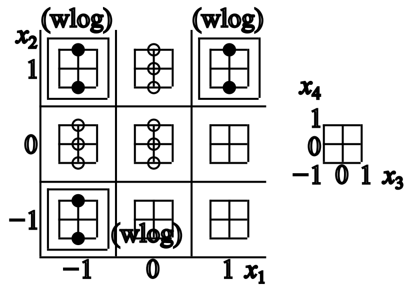

Figure 34: Part of some caps with triangles of 242 edges.

- The pairs (a,b), (c,d), and (e,f) can be re-arranged, using the two swaps

$$\begin{array}{rcl} ((a,b),(c,d),(e,f)) & \mapsto & ((c,d),(a,b),(e,f)) \\ (x_1,\ldots,x_4) & \mapsto & (x_2-x_1-1,x_2,x_3,x_4) \end{array}$$

and

$$\begin{array}{rcl} ((a,b),(c,d),(e,f)) & \mapsto & ((a,b),(e,f),(c,d)) \\ (x_1,\ldots,x_4) & \mapsto & (x_1,x_1-x_2+1,x_3,x_4). \end{array}$$

Therefore, the 22 values for $((a,b),(c,d),(e,f))$ in Table 1, with their associated caps, cover all options up to isomorphism. (Proof: Add some k in $\mathbb{F}_3$ to all of a to f so that of the options 0, 1, and -1, the value 0 occurs most often among a to f, so at least twice. Among (a,b), (c,d), and (e,f), the number of pairs that are $(0,0)$ is 3, 2, 1, or 0. If there is one $(0,0)$, then in both other pairs combined, the number of components that are 0 is 2, 1, or 0. Suppose that there is no $(0,0)$. Now in all three pairs combined, the number of components that are 0 is 3 or 2; if it is 2, then without loss of generality no $(1,1)$ or $(-1,-1)$ pairs exist, because if some pair $(-k,-k)$ exists, then it can be converted to $(0,0)$ by adding k to all of a to f, which yields a previous case.)

The new types of caps (that is, the types of caps with neither $\{9,9,0\}$ nor $\{9,8,1\}$ nor $\{9,7,2\}$ hyperplane directions) are a $963A$, a $963B$, a $954A$, and an $882A_1$; the corresponding representatives are in Figures 35, 36, 37, and 38 respectively, using the first $((a,b),(c,d),(e,f))$ value in Table 1 for each cap. The $963B$ in Figure 39 is the image of Figure 36 under

$$(x_1,\ldots,x_4)\mapsto(1+x_1+x_2-x_3,1-x_1-x_4,1-x_1-x_2-x_3,-x_1+x_2+x_3)$$

and the $882A_1$ in Figure 40 is the image of Figure 38 under

$$(x_1,\ldots,x_4)\mapsto(-x_2,-1+x_1+x_2+x_3,-1+x_1+x_2-x_3,1+x_2+x_4).$$

In Figure 40, the nine $\{8,8,2\}$ (respectively, nine $\{8,5,5\}$) hyperplane directions of the $882A_1$ have co-ordinates $x_1+ax_2+bx_3$ (respectively, $x_4+ax_2+bx_3$), where a and b are in $\mathbb{F}_3$.

Subsubcase 4(a)(iii): two, but not all three, of the cap line segments are in the same line direction of C. Without loss of generality, let x_3

$((a,b),(c,d),(e,f))$	Cap type	$((a,b),(c,d),(e,f))$	Cap type
$((0,0),(0,0),(0,0))$	$990A_1$	$((0,0),(1,1),(-1,-1))$	$963B$
$((0,0),(0,0),(0,1))$	$981A$	$((0,0),(1,-1),(1,-1))$	$981A$
$((0,0),(0,0),(1,-1))$	$972A$	$((0,0),(1,-1),(-1,1))$	$963A$
$((0,0),(0,0),(1,1))$	$963A$	$((0,1),(0,1),(0,1))$	$963B$
$((0,0),(0,1),(0,1))$	$972A$	$((0,1),(0,1),(0,-1))$	$972A$
$((0,0),(0,1),(0,-1))$	$963A$	$((0,1),(0,1),(1,0))$	$972A$
$((0,0),(0,1),(1,0))$	$963A$	$((0,1),(0,1),(-1,0))$	$963A$
$((0,0),(0,1),(-1,0))$	$972A$	$((0,1),(0,-1),(1,0))$	$954A$
$((0,0),(0,1),(1,1))$	$954A$	$((0,1),(0,-1),(1,-1))$	$954A$
$((0,0),(0,1),(1,-1))$	$954A$	$((0,1),(-1,0),(1,-1))$	$882A_1$
$((0,0),(0,1),(-1,1))$	$963A$		
$((0,0),(0,1),(-1,-1))$	$972A$		

Table 1: Caps with triangles of 242 edges.

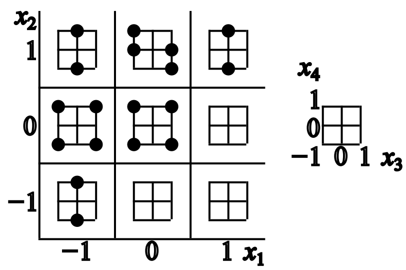

Figure 35: The representative $963A$.

and x_4 be such that the three 2-flats $(x_1, x_2) = (-1, -1)$ and $(x_1, x_2) = (\pm 1, 1)$ are as in Figure 41. Now the 3-flat $x_2 = 0$ must be as in that figure (avoid lines of three cap points in the 3-flats $x_1 = -1$ and $x_1 - x_2 = 0$). Each of the 3-flats $x_3 + x_4 = \pm 1$ and $x_3 - x_4 = \pm 1$ has seven known cap points so far.

The last 2-flat remaining is $(x_1, x_2) = (0, 1)$. In that 2-flat, if at least three of the four cap points are outside the line $x_3 + x_4 = 0$ (respectively, the line $x_3 - x_4 = 0$), then the $(x_3 + x_4)$-hyperplane direction (respectively, the $(x_3 - x_4)$-hyperplane direction) of C has point count $\{9, 9, 0\}$ or $\{9, 8, 1\}$, which is impossible. Therefore, the lines $(x_1, x_2, x_3 \pm x_4) = (0, 1, 0)$ have two cap points each.

The line $(x_1, x_2, x_3) = (0, 1, 0)$ has no cap points (look at the 2-flat $(x_2, x_3) = (1, 0)$). We obtain the representative $882A_2$ in Figure 41, of which the image under

$$(x_1, \ldots, x_4) \mapsto (x_3, x_2 + 1, x_1 + x_2 - 1, x_4)$$

is the $882A_2$ in Figure 42.

Subcase 4(b): the point count of C for (x_1, x_2) is $M_{4(b)}$. A tedious search yields no new caps: up to isomorphism, the 3-flat $x_1 = -1$ has two options, each with three options for $(x_1, x_2) = (0, 0)$, each with a list of options

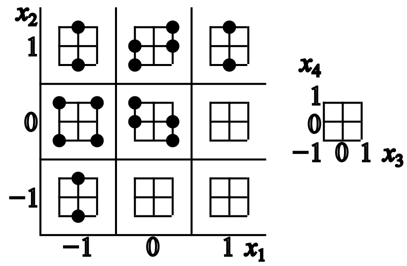

Figure 36: The representative $963B$.

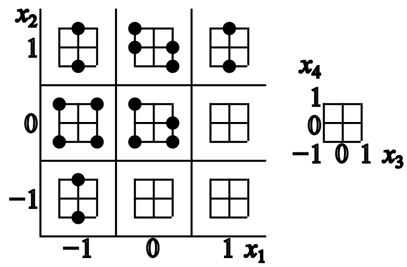

Figure 37: The representative $954A$.

for $x_1 = 1$; in all cases, C has a $\{9,9,0\}$ or $\{9,8,1\}$ hyperplane direction or a 2-flat direction with point count $M_{4(a)}$.

Subcase 4(c): the point count of C for (x_1, x_2) is $M_{4(c)}$. Again, no new caps come from a case-by-case search: set the 2-flats $(x_1, x_2) = (-1, 1)$, $(x_1, x_2) = (-1, 0)$, and $(x_1, x_2) = (0, 1)$ without loss of generality; up to isomorphism, there are eight options for the pair of triangles in the 2-flats $(x_1, x_2) = \pm(1, 1)$, each with a list of options for the rest of C; in all cases, C has a $\{9,9,0\}$ or $\{9,8,1\}$ hyperplane direction or a 2-flat direction with point count $M_{4(a)}$.

No duplicates. Table 2 shows $t_{972}(C)$, $t_{963}(C)$, $t_{882}(C)$, and $u(C)$ for each representative cap C, where $u(C)$ is the number of 2-flat directions of C such that, for some independent pair of co-ordinates of the 2-flat direction, the point

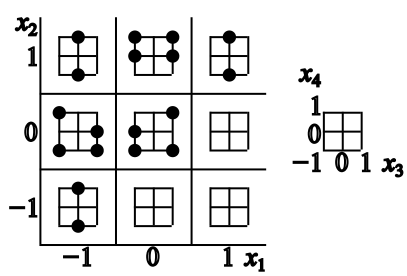

Figure 38: The representative $882A_1$.

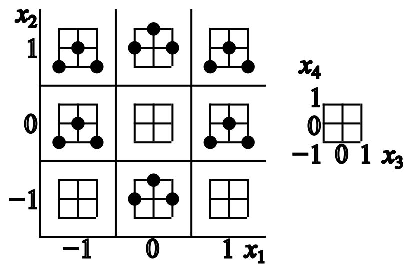

Figure 39: Another $963B$.

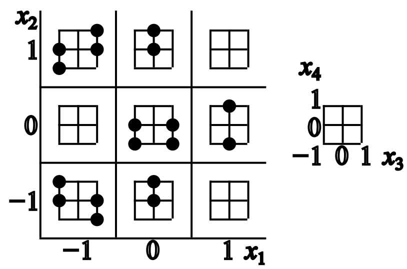

Figure 40: Another $882A_1$.

count of C is $M_{4(a)}$. Computer calculations found Table 2 and the "largest 3-flat point counts". No two representatives C have the same values for all of $t_{972}(C)$, $t_{963}(C)$, $t_{882}(C)$, $u(C)$, and the "largest 3-flat point count" of C combined, so no two representatives are isomorphic. The proof is done. □

4.4. Chance that a random subset of $\mathbb{F}_3^4$ *of size* 18, 19, *or* 20 *forms a cap*

As a final flourish, we now calculate the probability that a random collection of 18, 19, or 20 different points in $\mathbb{F}_3^4$ forms a cap (that is, the chance that among a random collection of 18, 19, or 20 different cards in the card game SET, there is no "Set").

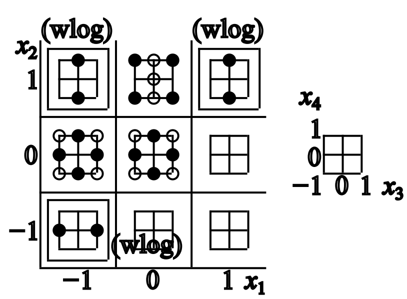

Figure 41: The representative $882A_2$.

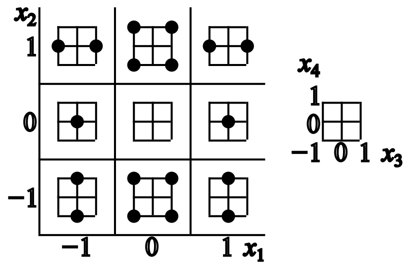

Figure 42: Another $882A_2$.

C	$990A_1$	$990A_2$	$990A_3$	$990B$	$981A$	$981B$	$981C$
$t_{972}(C)$	0	0	0	0	0	4	0
$t_{963}(C)$	0	0	0	0	0	0	0
$t_{882}(C)$	9	9	9	6	9	4	0
$u(C)$	12	0	6	1	12	0	0
Symmetries	288	288	72	4	16	16	16
C	$981D$	$981E$	$981F$	$981G$	$981H$	$981I$	$981J$
$t_{972}(C)$	2	0	0	0	0	1	0
$t_{963}(C)$	0	0	1	1	0	0	0
$t_{882}(C)$	2	2	0	3	5	5	4
$u(C)$	0	0	0	0	1	1	0
Symmetries	4	2	12	6	4	2	4
C	$972A$	$963A$	$963B$	$954A$	$882A_1$	$882A_2$	
$t_{972}(C)$	1	0	0	0	0	0	
$t_{963}(C)$	0	2	6	0	0	0	
$t_{882}(C)$	9	10	9	9	9	9	
$u(C)$	12	12	12	12	12	2	
Symmetries	4	4	72	4	72	16	

Table 2: Data for the representative 18-cap 4-flats.

Table 2 shows the number of symmetries that each representative 18-cap 4-flat has. Those values were verified by a computer search of all $3^4 \prod_{i=0}^{3}(3^4 - 3^i)$ affine transformations of $\mathbb{F}_3^4$ for each of the 20 types of 18-cap 4-flats.

Each affine transformation of $\mathbb{F}_3^4$ sends the representative $990A_1$ to some $990A_1$. Since each $990A_1$ has exactly 288 symmetries, it follows that in $\mathbb{F}_3^4$, there are exactly $(1/288) \cdot 3^4 \prod_{i=0}^{3}(3^4 - 3^i)$ caps C such that C is a $990A_1$. Similar reasoning holds for the other types of 18-cap 4-flats, so

$$\text{(the number of 18-cap 4-flats in } \mathbb{F}_3^4\text{)} = \left(\frac{2}{288} + \frac{3}{72} + \frac{4}{16} + \frac{1}{12} + \frac{1}{6} + \frac{7}{4} + \frac{2}{2}\right) \cdot 3^4 \prod_{i=0}^{3}(3^4 - 3^i) = 6\ 482\ 268\ 000.$$

Therefore, if a subset S of $\mathbb{F}_3^4$ of size 18 is chosen randomly under a uniform probability distribution (meaning that each possible subset has the same chance as each other possible subset of being chosen), then the chance that $(S, \mathbb{F}_3^4)$ is a cap is $(6\ 482\ 268\ 000)/\binom{81}{18}$, which is 1 in approximately 70 454 350, where we round the reciprocal of the probability to the nearest integer.

Similarly, it was verified by computer that each 19-cap 4-flat has 144 symmetries and each 20-cap 4-flat has 2 880 symmetries, so

$$(\text{the number of 19-cap 4-flats in } \mathbb{F}_3^4) = \frac{1}{144} \cdot 3^4 \prod_{i=0}^{3} (3^4 - 3^i) = 13\ 646\ 880$$

and

$$(\text{the number of 20-cap 4-flats in } \mathbb{F}_3^4) = \frac{1}{2\ 880} \cdot 3^4 \prod_{i=0}^{3} (3^4 - 3^i) = 682\ 344.$$

Therefore, if a subset S of $\mathbb{F}_3^4$ of size 19 (respectively, size 20) is chosen randomly under a uniform probability distribution, then the chance that $(S, \mathbb{F}_3^4)$ is a cap is $(13\ 646\ 880)/\binom{81}{19}$ (respectively, $(682\ 344)/\binom{81}{20}$), which is 1 in approximately 110 965 601 988 (respectively, 1 in approximately 6 879 867 323 284), where we round the reciprocal of each probability to the nearest integer.

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References

[1] B. L. Davis, D. Maclagan, The card game Set, The Mathematical Intelligencer 25 (3) (2003) 33–40. doi:https://doi.org/10.1007/BF02984846.

[2] J. A. Grochow, New applications of the polynomial method: The cap set conjecture and beyond, Bulletin of the American Mathematical Society 56 (1) (2019) 29–64. doi:https://doi.org/10.1090/bull/1648.

[3] J. S. Ellenberg, D. Gijswijt, On large subsets of $\mathbb{F}_q^n$ with no three-term arithmetic progression, Annals of Mathematics 2nd Series 185 (1) (2017) 339–343. doi:https://doi.org/10.4007/annals.2017.185.1.8.

[4] E. Croot, V. F. Lev, P. P. Pach, Progression-free sets in $\mathbb{Z}_4^n$ are exponentially small, Annals of Mathematics 2nd Series 185 (1) (2017) 331–337. doi:https://doi.org/10.4007/annals.2017.185.1.7.

[5] Y. Edel, S. Ferret, I. Landjev, L. Storme, The classification of the largest caps in AG(5,3), Journal of Combinatorial Theory Series A 99 (1) (2002) 95–110. doi:https://doi.org/10.1006/jcta.2002.3261.

[6] A. Potechin, Maximal caps in AG(6,3), Designs, Codes and Cryptography 46 (3) (2008) 243–259. doi:https://doi.org/10.1007/s10623-007-9132-z.

[7] S. Axler, Linear algebra done right, Corrected Printing of 2nd Edition, Undergraduate Texts in Mathematics, Springer-Verlag New York, New York, NY, 2004.