# Comparative performance of time spectral methods for solving hyperchaotic finance and cryptocurrency systems

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**Abstract**

A comparative performance analysis of two spectral methods on the hyperchaotic finance system (HCFS) and the cryptocurrency pricing problem (CPP) is proposed. The first approach uses a differentiation matrix, the second method considers an integration matrix on a single and multiple domains when the time intervals are large. Numerical simulations are performed against the well established numerical method, Chebfun. It turns out that the spectral method using integration matrix is more efficient than the other methods on both problems.

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## 1. Introduction

Nonlinear chaotic systems have attracted many research works in the area of physics, control theory, telecommunication artificial neural networks, biological networks, chemical reactors, etc. They can describe the evolution of more complex systems in a reasonable manner.

In the area of economics, the phenomenon of chaos was first discovered in 1985 and later it was found in finance by Ma and Chen [12, 11]. Financial system dynamics have a significant role in micro-economics. They become more and more complicated with economic growth and contain many complex factors such as interest rate, the price of goods, investment demand and stock. A typical case of an appearance of chaos in the financial system took place during the world economic crisis where in 2007, the U.S subprime mortgage crisis triggered the global economic crisis showing thus the existence of chaos in the finance system.

Over the past few years, many more hyperchaotic systems have been discovered in the high-dimensional social economical systems. A hyperchaotic system is usually defined as a chaotic system with at least two positive Lyapunov exponents. Jahanshahi et al. [13, 14, 15] gives a good analysis of the dynamics of behaviour in varying several parameters and initial conditions. Furthermore, a synchronization via sliding mode control and fuzzy method is

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provided. See also [21, 22, 26] for more details.

In addition to hyperchaotic system, we look at cryptocurrency pricing problem, which comes with the world digital revolution. As the concepts stands new, very few literature is found around cryptocurrency, however one can mention that a cryptocurrency is essentially a type of digital asset used as money in a sense of the Austrian school of economics, i.e. money emerges from a competition of medium of exchanges [3, 23]. A cryptocurrency value is dictated by the supply and demand in a free economy where the intervention of any entity or government is not possible in the issuance of its new units as to opposed fiat currencies. From an asset flow perspective the pricing of cryptocurrency involves the resolution of nonlinear systems of ODEs [4].

In general, analytical solutions for the nonlinear chaotic systems are almost unachievable. Therefore, we rely on numerical methods for computation of solutions since the HCFS is actually a dynamical system that is described by a set of nonlinear ODEs. However most methods used to solve hyperchaotic systems based on finite difference methods [20, 19], finite elements [16], homotopy analysis or perturbation methods (HAM/HPM) together with Adomian decomposition [7], etc. are known to suffer from the curse of dimensionality [1, 18] and sometimes not capable of handling stiffness issues [9] that may arise in financial systems.

In this paper, we perform a comparative analysis of robust spectral methods to numerically solve hyperchaotic finance and cryptocurrency systems. Spectral methods, however, have the advantage of being fast converging methods. Their truncation error decays as fast as the global smoothness of the underlying solution permits, their definite integrals are calculated once by the quadrature rule [8].

Various modified and quadrature rules can be found in the literature of spectral methods, including quadratures based on Chebyshev polynomials. The shifted Chebyshev-Tau method is used to solve the Klein-Gordon equation [10]. An extension of this method is applied in the case of fractional differential equations [5]. Bhrawy [2] introduces an operational matrix to the shifted Chebyshev method to generate an even faster algorithm for fractional integration in the sense that only a small number of shifted Chebyshev polynomials is needed to obtain satisfactory results. Driscoll [6] presents a fast practical algorithm based on operational matrices in which the matrices have a lower density. Trefethen [24] proposed a Matlab package known as Chebfun, that exploits results from approximation theory, spectral methods, and object-oriented software design. Trif [25] introduced the chebpack package that is based on the Chebyshev-Tau method where the focus is more on the spectral space of coefficients rather than the physical space. This approach takes advantage of the spectral properties of Chebyshev polynomials resulting in sparse upper triangular matrices that allow to achieve a tremendous gain in the computation.

In this paper, we use the technique proposed in Trif [25] to construct spectral methods coupled with a domain decomposition to solve the HCFS and the cryptocurrency pricing problem (CPP) using the integral and differential approaches. The methods are not entirely new, but the combinations are very novel.

The rest of the paper is organised as follows, Section 2 presents the spectral methods based on Chebyshev polynomials. In Section 3, we apply these numerical methods to the HCFS and CPP, and compare the results with solutions obtained from Chebfun. The last section is allocated to the conclusion.

## 2. Chebyshev polynomials

The Chebyshev polynomial $T_n(x)$ of 1st kind is a polynomial in $x \in [-1, 1]$ of degree $n > 0$ defined by the relation:

$$\begin{aligned} T_n(x) &= \cos n\theta, \text{ for } x = \cos\theta \\ ie. \quad T_n(x) &= \cos(n \arccos(x)) \end{aligned}$$

Note that the definition of the Chebyshev polynomials can easily be extended to any interval $[a, b]$ by just applying a shift mapping $s : x \to s(x) = \frac{2}{b-a}x - \frac{b+a}{b-a}$. For this reason we shall work on the interval $[-1, 1]$ then apply the inverse shift mapping we can always get back to any interval $[a, b]$.
From the trigonometric relation.

$$\cos(n\theta) + \cos(n-2)\theta = 2\cos\theta\cos(n-1)\theta \tag{2.1}$$

we get

$$\begin{aligned} T_0(x) &= 1, && (2.2) \\ T_1(x) &= x, && (2.3) \\ T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, ... && (2.4) \end{aligned}$$

Which in turn can be expressed in a matrix form as:

$$\begin{bmatrix} 1 & & & & \\ -2x & 1 & & & \\ 1 & -2x & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2x & 1 \end{bmatrix} \begin{bmatrix} T_0(x) \\ T_1(x) \\ T_2(x) \\ \vdots \\ T_n(x) \end{bmatrix} = \begin{bmatrix} 1 \\ -x \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.5}$$

The zeros of $T_n$ are the points

$$x_k = -\cos\frac{(k - \frac{1}{2})\pi}{n}, \qquad k = 1, 2, \dots, n. \tag{2.6}$$

The set $\{x_k\}_k$ is termed as collocation points, also called Chebyshev points of first kind. For any point $x$, the set $\{T_0(x), T_1(x), \dots\}$ is an orthogonal basis according to the weighted inner product defined by:

$$< f, g >= \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\, dx \tag{2.7}$$

for any continuous function $f, g$ defined on $[-1, 1]$. This means that for any polynomial of degree $n > 0$, there exists a unique set of coefficients $\{c_1, c_2, ..., c_n\}$ such that

$$p_n(x) = \sum_{k=0}^{n} c_k T_k(x). \tag{2.8}$$

Considering that polynomials are dense in $C([-1,1])$, and since the set of Chebyshev polynomials is complete, therefore we have the following theorem

**Theorem 2.1.** *Let f be a Lipschitz continuous function on the interval [-1,1]. Then f admits a unique representation as a series of the form:*

$$u(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k T_k(x). \tag{2.9}$$

*where $T_k(x)$ are Chebyshev polynomials,*

$$c_k = \frac{2}{\pi}\int_{-1}^{1} \frac{u(x)T_k(x)}{\sqrt{1-x^2}}dx, \qquad k = 0, 1, 2, 3, ... \tag{2.10}$$

*This series converges uniformly and absolutely.*

A Chebyshev approximation of order $n > 0$ of a function $u$ continuous on an interval $[-1, 1]$ is defined as

$$\begin{aligned} u_n(x) &= \sum_{k=0}^{n} c_k T_k(x) && (2.11)\\ &= \underline{c} \cdot T(x) && (2.12) \end{aligned}$$

where $\underline{c} = (c_0, c_1, \ldots, c_n)$ is the coefficient vector associated with the approximation $u_n$. It is usually termed as the spectral representation of $f_n$. The set of Chebyshev coefficient vectors $\{\underline{c}\}$ of continuous functions on $[-1, 1]$ is referred to as the frequency space.

For simplicity of notation, we shall write $u(x)$ in place of $u_n(x)$ to denote the Chebyshev approximation of order $n$ of $u$ at $x$. Another discrete representation of the function $u$ is to directly interpolate $u$ at the collocation points $x_k's$. This means $u$ can be represented by a vector $\underline{v}$ of its values on the grid $\underline{x} = (x_0, x_1, ..., x_n)$, that is $\underline{v} = (u(x_0), u(x_1), ..., u(x_n))$. We shall call $\underline{v}$ the physical representation of $u$.
On the collocation point, one writes

$$\begin{aligned} \underline{v}(x) &= T(x).\underline{c} && (2.13)\\ (v(x_0), ..., v(x_n)) &= \left(\sum_{k=0}^{n} c_k T_k(x_0), ..., \sum_{k=0}^{n} c_k T_k(x_n)\right), && (2.14) \end{aligned}$$

where $T$ is the matrix defined as follows

$$T = \begin{bmatrix} T_0(x_0) & T_1(x_0) & \ldots & T_n(x_0) \\ T_0(x_1) & T_1(x_1) & \ldots & T_n(x_1) \\ \vdots & & \ddots & \vdots \\ T_0(x_n) & T_1(x_n) & \ldots & T_n(x_n) \end{bmatrix}.$$

Since

$$\begin{aligned} \underline{v} &= T\underline{c} \\ \text{implying} \qquad \underline{c} &= T^{-1}\underline{v}. \end{aligned}$$

From the nature of $T_k's$, the matrix $T$ is sparse and FFT enables to get $T^{-1}$.

*2.1. Some useful properties*

Consider two functions $a$ and $u$ of a variable $x$, with spectral representation $\underline{a}$ and $\underline{u}$ respectively. Then the product $a(x) \cdot u(x)$ admits also a spectral representation, denoted as $\underline{\phi}$ which is defined as:

$$\underline{\phi} = \mathbf{a}.\underline{c} \tag{2.15}$$

where $\mathbf{a}$ is termed as the matrix representation of the function $a(x)$ and $\underline{c}$ is the spectral representation of function $u$, see [6].

An efficient way of getting the matrix $\mathbf{a}$ is to write the product in its discrete form.

$$\text{Since} \qquad a(x)f(x) = \left[\sum_{k=0}^{n} a_k T_k(x)\right]\left[\sum_{k=0}^{n} c_k T_k(x)\right], \tag{2.16}$$

$$\text{then} \qquad \sum_{k=0}^{n} \phi_k T_k(x) = \sum_{k=0}^{n}\sum_{l=0}^{n} \alpha_{kl} a_k c_l T_k T_l \tag{2.17}$$

for some coefficients $\alpha_{kl}$, $0 \leq k, l \leq n$. In addition, Given the following relation

$$T_k(x)T_l(x) = \frac{1}{2}\left[T_{k+l}(x) + T_{|k-l|}(x)\right], \qquad \text{for all } k, l = 0, 1, ...n \tag{2.18}$$

and in rearranging terms properly, it brings to existence a matrix $\mathbf{a}$ such that

$$\sum_{k=0}^{n} \phi_k T_k(x) = \sum_{k=0}^{n}\left[\sum_{l=0}^{n} \mathbf{a}_{kl} c_l\right] T_k(x).$$

In the frequency space, this will write as:

$$\underline{\phi} = \mathbf{a}\underline{c}. \tag{2.19}$$

*2.2. Differentiation and integration*

Recall again equation (2.11) and differentiate it

$$u'(x) = \sum_{k=0}^{n} c_k T_k'(x). \tag{2.20}$$

The differentiation of relation (2.4) and relation (2.3) gives

$$T_0 = T_1' \tag{2.21}$$

$$T_1 = \frac{T_2'}{2} \tag{2.22}$$

$$T_{n+1}(x) = nT_{n-1}'(x) - 2(1-x^2)T_n'(x) \tag{2.23}$$

$$\text{ie.} \quad T_n = \frac{T_{n+1}'}{2(n+1)} - \frac{T_{n-1}'}{2(n-1)}, \quad n = 2, 3, ... \tag{2.24}$$

Inserting this back into (2.20) shows the existence of a matrix $D = (d_{kl})_{0<k,l<n}$ such that

$$\sum_{k=0}^{n} \underline{c}_k' T_k(x) = \sum_{k=0}^{n}\sum_{l=0}^{n} d_{kl} c_l T_k(x) \tag{2.25}$$

$$\textit{ie.} \quad \underline{c}' = D\underline{c} \tag{2.26}$$

where $\underline{c}'$ stands for the spectral representation of the derivative function $u'$. Moreover, $D$ is a sparse upper triangular matrix, with the following properties

$$\begin{cases} d_{kl} = 0, & \text{for } k \leq l, \\ d_{kl} = 0, & \text{if } l-k \text{ is even}, \\ d_{kl} = 2k, & \text{if } l-k \text{ is odd}. \end{cases} \tag{2.27}$$

Applying the above result recursively we get the spectral representation $c^{(p)}$ of the derivative at order $p$ of $u$ as:

$$\underline{c}^{(p)} = D^p \underline{c}. \tag{2.28}$$

For the case of integration, recall again relation

$$\begin{aligned} T_{n+1}(x) &= nT_{n-1}(x) - 2(1-x^2)T_n(x) && (2.29)\\ \int T_n(x)dx &= \frac{1}{2}\Big[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\Big], n = 2,3, && (2.30)\\ \int T_1(x)dx &= \frac{1}{4}T_2(x), && (2.31)\\ \int T_0(x)dx &= \frac{1}{2}T_1(x). && (2.32) \end{aligned}$$

As a linear operator, the integral of $u$ will also be a continuous Lipschitz function in $[-1,1]$, which will in turn have a unique expansion series of the form

$$\int u(x)dx = \sum_{k=0}^{n} I_k T_k(x), \qquad x \in [a,b]$$

where $I_k$'s are coefficients of the integral of $u$, and similarly as with differentiation there exists a $n \times n$-matrix $J$ such that

$$I_k = \sum_{l=0}^{n} J_{kl} c_l \tag{2.33}$$

or simply

$$\underline{I} = J \cdot \underline{c} \tag{2.34}$$

where $\underline{I}$ is the spectral representation of the integral of $u$. In fact,

$$\begin{aligned} \int u(x)dx &= \int \sum_{k=0}^{n-1} c_k T_k(x) \\ i.e \qquad \sum_{k=0}^{n-1} I_k T_k(x) &= \int \sum_{k=0}^{n-1} c_k T_k(x)\, dx \\ &= \sum_{k=0}^{n-1} c_k \int T_k(x)\, dx \\ \sum_{k=0}^{n-1}\sum_{j=2}^{n-1} J_{kj} c_j T_k(x) &= \sum_{k=2}^{n} c_k \frac{1}{2}\Big[\frac{T_{k+1}}{k+1} - \frac{T_{k-1}}{k-1}\Big]. \end{aligned}$$

Performing a smart multiplication and rearranging terms we get the coefficients of $J$ recursively as follows:

$$J_{kk} = 0, \quad J_{01} = \frac{1}{2}, \quad J_{k,k-1} = -J_{kk+1} = \frac{1}{k}.$$

So then, the spectral representation of the integral of $u$ is the vector $\underline{d} = J.\underline{c}$, and for any continuous function $a(x)$, the corresponding spectral representation for the integral of the product $a(x)\,u(x)$ is $J\,\mathbf{a}\,\underline{c}$ where $\mathbf{a}$ is the matrix representation of $a$. We write

$$\int a(x)u(x)dx \to J\,\mathbf{a}\,\underline{c}. \tag{2.35}$$

Consequently it can be seen with the help of elementary technique of integration by parts that

$$\begin{aligned}
\int a_1(x)u'(x)dx &\to (\mathcal{I} - JD)\mathbf{a}_1\,\underline{c} \\
\int\int a_2(x)u"(x)dx &\to (\mathcal{I} - JD)^2\mathbf{a}_2\,\underline{c}. \\
\int\int\int a_3(x)\frac{d^3u}{dx^3}(x)dx &\to (\mathcal{I} - JD)^3\mathbf{a}_3\,\underline{c} \\
&\vdots \\
\int...\int a_m(x)\frac{d^mu}{dx^m}(x)dx...dx &\to (\mathcal{I} - JD)^m\mathbf{a}_m\,\underline{c}
\end{aligned}$$

where $\mathcal{I}$ stands for the identity matrix. Thus, for a general linear differential operator $L$

$$L\,u(x) = \sum_{i=0}^{m} a_i(x)\frac{d^iu}{dx^i}(x) \tag{2.36}$$

we have

$$\int...\int L\,u(x)dx^m \to \sum_{i=0}^{m} J^{m-i}(\mathcal{I} - JD)^i\mathbf{a}_i\,\underline{c}. \tag{2.37}$$

The matrix

$$A = \sum_{i=0}^{m} J^{m-i}(\mathcal{I} - JD)^i\mathbf{a}_i \tag{2.38}$$

is the spectral representation of the integral operator of $L$.

Consider a general differential equation $\mathcal{A}u = f$ of order $m$ for which the differential operator can be written as $\mathcal{A} = L + N$, where $L$ and $N$ are respectively the linear part and the nonlinear part. The differential equation then writes as

$$Lu(t) + Nu(t) = f(t), \tag{2.39}$$

$$Lu(t) = -Nu(t) + f(t) \tag{2.40}$$

Using the differential approach the equation (2.40) is directly taken into its spectral representation

$$\begin{aligned}
\sum_{k=0}^{m} D^k \underline{c} &= -\mathbf{n} + \tilde{\mathbf{f}}, && (2.41)\\
A\,\underline{c} &= \mathbf{f}, && (2.42)\\
\text{implying} \qquad \underline{c} &= A^{-1}(\mathbf{f}), && (2.43)
\end{aligned}$$

for some $A = \sum_{k=0}^{m} D^k$, where $\mathbf{n}, \tilde{\mathbf{f}}$ are the spectral representation of $\mathcal{N}u$ and $f$ respectively, and $\mathbf{f} = -\mathbf{n} + \tilde{\mathbf{f}}$.

Using the integral approach, we first take the integral of the equation (2.40) then write the resulting equation into its spectral form. That is,

$$\begin{aligned}
\int ... \int Lu(t) \to A\underline{c} &= -\mathbf{n} + J^m \underline{f}, && (2.44)\\
A\underline{c} &= \mathbf{f}, && (2.45)\\
\text{implying} \qquad \underline{c} &= A^{-1}\mathbf{f} && (2.46)
\end{aligned}$$

where $\mathbf{n}$ is the spectral representation of the integral of $Nu$ at order $m$, and $\mathbf{f} = -\mathbf{n} + J^m \underline{f}$ is the spectral representation of $-Nu + f(t)$. We use the following algorithm 1 for both differentiation and integration method:

**Algorithm 1** Pseudo code

```
1: u_0 ← initial solution
2: INITIALIZE L
3: Evaluate N, and f at u_0
4: u := L^{-1} * (N + f)
5: while ||u − u_0|| > ϵ do
6:     u_0 ← u
7:     Evaluate N, and f at u_0
8:     u = L^{-1} * (N + f)
9: RETURN u
```

*2.3. The multistep Spectral Method*

For this section we consider $I_h$ to be a mesh on the interval $[0, T]$ and $N$ be the number of subintervals and

$$I_h := \{t_k : 0 = t_0 < t_1 < \cdots < t_N = T\}.$$

We denote by $\Lambda_k = [t_{k-1}, t_k]$, $h_k = t_k - t_{k-1}$ and $u^k(t)$ the solution of (2.39) on the $k$-th element, namely

$$u^k(t) = u(t), \qquad \forall t \in \Lambda_k, \qquad 1 \leq k \leq N.$$

Let $M_k > 0$ be an integer and consider $\mathcal{P}_{M_k}$ to be the space of polynomials of order at most $M_k$ built on $\Lambda_k$. We apply the spectral method as described in algorithm 1 to obtain a numerical solution $U_{M_k} \in \mathcal{P}_{M_k}$ on $\Lambda_k$. The spectral method using integral or differentiation approach, on the interval $[0, T]$ consists of a successive application of the spectral method on each $\Lambda_k$ to obtain a global numerical solution $U_M(t)$ of (2.39) defined such that:

$$U_M(t) = U_{M_k}(t), \quad t \in \Lambda_k, \quad 1 \leq k \leq N.$$

where $M$ is taken to be the smallest of the $M_k$s: that is, $M = \inf_{0<k\leq N} M_k$.

For each subinterval $[t_i, t_{i+1}]$, the equation (2.45) is applied.

$$A^{(i)}\mathbf{c}^{(i)} = \mathbf{f}^{(i)}, \qquad i = 0, ..., m-1. \tag{2.47}$$

The overall matrix $A$ of the entire problem is then a diagonal matrix of the block matrices $A^{(i)}$.

$$\begin{pmatrix} A^{(1)} & 0 & & \\ 0 & A^{(2)} & 0 & \\ & \ddots & \ddots & \\ & & 0 & A^{(m)} \end{pmatrix} \begin{pmatrix} \mathbf{c}^{(1)} \\ \mathbf{c}^{(2)} \\ \vdots \\ \mathbf{c}^{(m)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \\ \vdots \\ \mathbf{f}^{(m)} \end{pmatrix} \tag{2.48}$$

By inversion of the matrix $A^{(i)}$ on each domain $\Lambda_i$, we obtain $c^{(i)}$ and therefore $u_{M_i}$ which is $U_M$ on $\Lambda_i$.

In this case a global error can arise and jeopardise the convergence. However the following theorem 2.2 still guarantees an exponential convergence even after discretization.

**Theorem 2.2.** *Assume that $u$ belongs to the broken Sobolev space: $u \in H^1(0, T)$ and $u_{|\Lambda_k} \in H^{r_k}(\Lambda_k)$, $1 \leq k \leq N$ with integers $2 \leq r_k \leq M_k + 1$, and there exists a constant $L \geq 0$ such that for any $z_1$ and $z_2$,*

$$|f(z_1, t) - f(z_2, t)| \leq L|z_1 - z_2|. \tag{2.49}$$

*Then for*

$$2\sqrt{2\pi}h_{\max}L \leq \beta < 1, \tag{2.50}$$

*we have*

$$\|u - U_M\|^2_{H^1(0,T)} \leq c_\beta T \exp(c_\beta T) \sum_{i=1}^{N} h_i^{2r_i-2} M_i^{2-2r_i} |u|^2_{H^{r_i}(\Lambda)}, \tag{2.51}$$

*where $c_\beta$ is a positive constant depending only on $\beta$.*

Where $\Lambda_k = [t_{k-1}, t_k]$, $h_k = t_k - t_{k-1}$ and the constant $M_k$ is of the order of the Chebyshev polynomial of approximation $u_k$ defined on $\Lambda_k$. The proof can be found in [12, 27].

## 3. Applications and numerical results on the hyperchaotic finance system

In this section, we apply our methods to HCFS and CPP then test the convergence, and efficiency of the proposed methods. Since the exact solution is not available we choose the ODE15s with relative and absolute tolerance $10^{-14}$ to serve as the benchmark solution. The error $E$ we consider is the maximal error given by:

$$||E|| = ||Sol_{Benchmark} - Sol_{Numerical}||_\infty. \tag{3.52}$$

Let us now apply the above technique described in Section 2 to the nonlinear hyperchaotic problems and cryptocurrency pricing problem.

*3.1. The hyperchaotic finance system*

It has been shown (see Zhao et al. [29]) that four sub-blocks actually drive the dynamics of the finance model: production, money, stocks and labor force. Their interaction is reported by three nonlinear differential equations defining what is termed as the chaotic finance system. Technically and more explicitly, the finance system describes the time variation of three main state variable: the interest rate $x$, the investment demand $y$ and the price index of stock $z$. The interest rate is an amount expressed as the percentage of the principal by lender to a borrower for an asset. The investment demand can be defined as the desired capital and inventories by firms. The chaotics finance system is expressed as follows:

$$\begin{cases} \dot{x} = z + (y - a)\,x, \\ \dot{y} = 1 - by - x^2, \\ \dot{z} = -x - cz, \end{cases} \tag{3.53}$$

where the parameters $a, b, c$ are respectively the saving, the per-investment cost and the elasticity of the demand [17]. These parameters are all considered to be non-negative and constant. From the chaotic finance system (3.53) Yu et al. [28] found that the factors affecting the interest rates are related not only to investment demand and price index but also to the average profit margin. Moreover, the average profit margin and interest rate are proportional. Hence an improved chaotic finance system is constructed by including an additional state variable $w$ that will stand for the average profit margin. The system is now of four dimensional differential equations:

$$\begin{cases} \dot{x} = z + (y - a)x + w, \\ \dot{y} = 1 - by - x^2, \\ \dot{z} = -x - cz, \\ \dot{w} = -d\,xy - ew. \end{cases} \tag{3.54}$$

In oder to apply our numerical methods based on spectral Chebyshev methods, we first write the system in the framework of Equation (2.40). That is,

$$\begin{cases} \dot{x} + ax - z - w & = xy, \\ \dot{y} + by & = 1 - x^2, \\ \dot{z} + x + cz & = 0, \\ \dot{w} + ew & = -d\,xy, \end{cases} \tag{3.55}$$

which can also be written as:

$$\underline{a}\,u'(t) + B\,u(t) = f(t), \quad t \in [0, T] \tag{3.56}$$

where $\underline{a} = (1, 1, 1, 1)$, $u(t) = [x(t), y(t), z(t), w(t)]$, $B = \begin{bmatrix} a & 0 & -1 & -1 \\ 0 & b & 0 & 0 \\ 1 & 0 & c & 0 \\ 0 & 0 & 0 & e \end{bmatrix}$ and $f(t) = [x(t)y(t), 1 - x^2(t), 0, -d\,x(t)y(t)]$.

The spectral representation of the Equation (3.56) is

$$\begin{bmatrix} D+a\mathcal{I} & 0 & -\mathcal{I} & -\mathcal{I} \\ 0 & D+b\mathcal{I} & 0 & 0 \\ Id & 0 & D+c\mathcal{I} & 0 \\ 0 & 0 & 0 & D+e\mathcal{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \\ \mathbf{f}^4 \end{bmatrix} \tag{3.57}$$

where $\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w}$ are the spectral representation of the unknown functions $[x(t), y(t), z(t), w(t)]$ respectively, and similarly $[\mathbf{f}^1, \mathbf{f}^2, \mathbf{f}^3, \mathbf{f}^4]$ which represent the coefficient vectors of the nonlinear part $[xy, 1-x^2, 0, dxy]$ respectively.

On the other hand we can approach the hyperchaotic problem by integration first then apply Chebychev approximation to the resulting integral problem.

It is not difficult to see that the Chebyshev approximation of problem (3.54) at order $n$ in the space of Chebyshev polynomials after integration yields the following simultaneous equations

$$\begin{cases} (\mathcal{I}+aJ)\mathbf{x} - J\mathbf{z} - J\mathbf{w} & = \mathbf{f}^1 \\ (\mathcal{I}+bJ)\mathbf{y} & = \mathbf{f}^2 \\ (\mathcal{I}+cJ)\mathbf{z} + J\mathbf{x} & = \mathbf{f}^3 \\ (\mathcal{I}+eJ)\mathbf{w} & = \mathbf{f}^4 \end{cases} \tag{3.58}$$

where $\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w}$ are defined as above, and similarly $[\mathbf{f}^1, \mathbf{f}^2, \mathbf{f}^3, \mathbf{f}^4]$ which represent the coefficient vectors of the integral of the nonlinear part $[xy, 1-x^2, 0, dxy]$ respectively. In other words,

$$\begin{bmatrix} \mathcal{I}+aJ & 0 & -J & -J \\ 0 & \mathcal{I}+bJ & 0 & 0 \\ J & 0 & \mathcal{I}+cJ & 0 \\ 0 & 0 & 0 & \mathcal{I}+eJ \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \\ \mathbf{f}^4 \end{bmatrix} \tag{3.59}$$

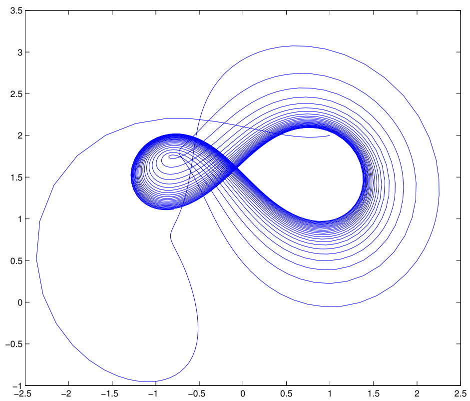
The two approaches generates nonlinear problem, we will apply an iterative method to equations (3.59) and (3.57). The aim is to get the coefficient vector $\underline{c}$ of $u(t) = [x(t), y(t), z(t)]$. Lets then consider the fix point problem

$$A\underline{c} = \mathbf{f}. \tag{3.60}$$

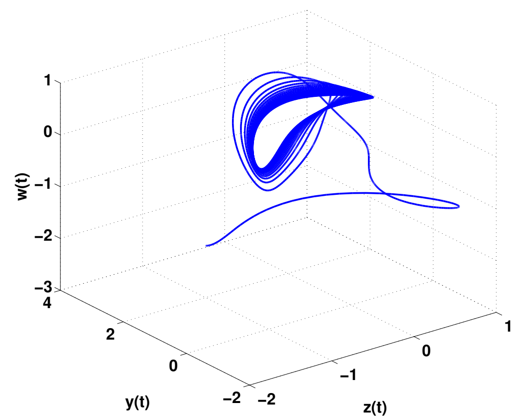
We shall start with an initial guess coming out of the initial condition then get the new $\underline{c}$ by $\mathbf{c} = A^{-1}\mathbf{f}$ where the old $\underline{c}$ is used to compute $\mathbf{f}$ in the iterations. Keeping in mind that the chaotic finance (3.54) is also highly nonlinear on some interval, and in order to speed up convergence we suggest the use of a splitting method on the interval $[0, T]$ into $N$-domains $0 = t_0 < t_1 < ... < t_N = T$ and apply the spectral methods.

The results are implemented for $a = 0.9, b = 0.2, c = 1.2, d = 0.2, e = 0.17$. Figure 1 shows the phase portraits between variables for a long time $T = 200$. They both exhibit chaos as expected.

The solution functions $x(t), y(t), z(t), w(t)$ are plotted in Figure 2 where a 3-domains decomposition has been used with 16 collocation points per domain and $T = 5$. It is clear, the numerical solutions from both spectral approaches match the benchmark solution from ODE15s.




(a) phase portrait $xy$



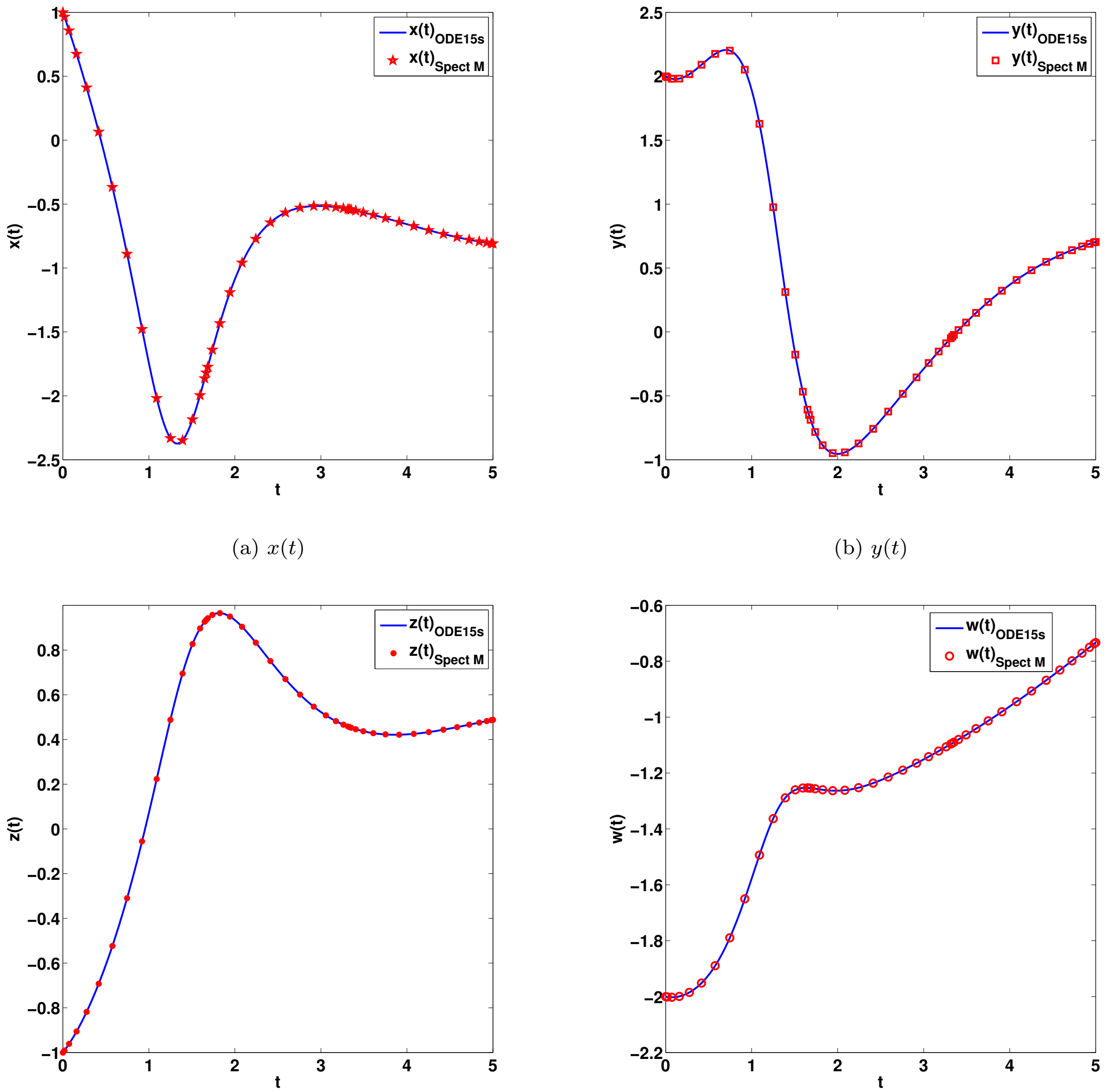

(b) phase portrait $yzw$

Figure 1: phase potraits 2D and 3D

We go on into investigating the effect of $n$ and $N$ on the decay error. Figure 3a shows that employing more collocation points on a domain enhances the precision of the numerical solution both from integration as well as differentiation method. It is also remarkable to see that while it takes close 80 points for Chebfun to reach an accuracy of $10^{-4}$, the integral and differentiation spectral methods only require 20 points to achieve same accuracy. However the later methods tend to slightly lose this quality as the number of points gets larger (here $n = 120$) as compare to Chebfun. This makes us consider a domain decomposition of the interval $[0, T]$. Introducing decompositions (2 and 4 sub-intervals) the spectral decay is recovered, see Figure 3b and 3c and better accuracy is obtained.

However, better accuracy can also bring along a cost in time. Figure 3e presents the efficiency of the three methods. From the reading of that plot one can see that the spectral integral method outperforms better than the other two methods on the hyperchaotic finance system for time $T = 5$. In fact, it takes 0.01s for the integral method to achieve a precision error of order $10^{-4}$ while the differentiation method and Chebfun would take respectively 1.25 and 3.2 times more. The reason for such is mainly due to the level of sparsity of the matrices generated by the schemes. Figure 5 shows the sparsity structures of matrices generated for each method. The integral method has a matrix structure more porous than others, giving it the advantage to be invertible faster than the matrix from differentiation method which is upper triangular, and also matrix from chebfun which is full.

Another point to consider is the factor $T$. It is worth observing that during the course of our simulations, the differentiation method starts malperforming when the size of the domain gets greater than 2 and same remark would apply to the integral method but for $T > 4$. For this additional reason, as $T$ gets larger, we shall again require to decompose our interval $[0, T]$ into multiple domains. The same remark also holds for Chebfun, it degenerates whenever $T > 11$ as shown in Figure 3d. The remedy is the same, one has to consider a domain decomposition into sub-intervals. Doing so turns out also to produce improvement in the CPU running time. For instance Figure 4 shows that using 8-domains decomposition,




(a) $x(t)$

(b) $y(t)$

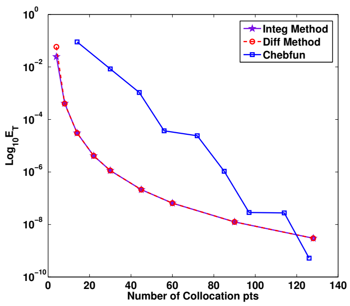
(c) $z(t)$

(d) $w(t)$

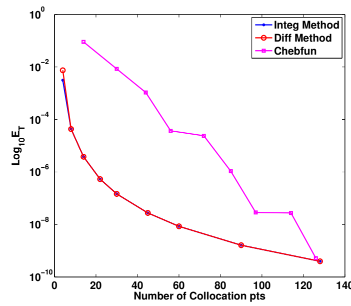
Figure 2: plot of $x, y, z, w$ variables using 3 domains and 16 collocation pts and $T = 5$

the integral method takes 0.1s to achieve an accuracy of order $10^{-8}$, while it would take 1s reach the same level of precision when using a singe domain only. The same observation appears in the differentiation method (see Figure 4b) where it takes 1320s ie. 23min to obtain an error of order $10^{-12}$ and will require 2048 points using one domain but with 8-domains decomposition the algorithm will take less that 2min using differentiation method and less than 1min using the integral method and obtain nearly same accuracy with the very same total number of points 2048. In all cases, still the integral method performs faster than the differentiation method.

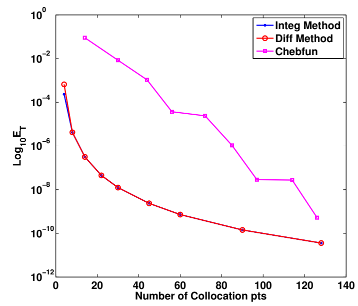
As we vary the total number of collocation points from $n = 32, 64, 128, 256, 512, 1024, 2048$ and 4096, we record the error on each variable $x, y, z$ and $w$. Figure 4c shows that when



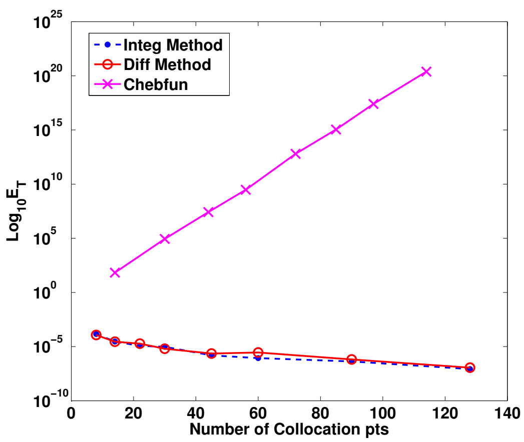

(a) Error versus $n$ for $T = 2$ $N = 1$



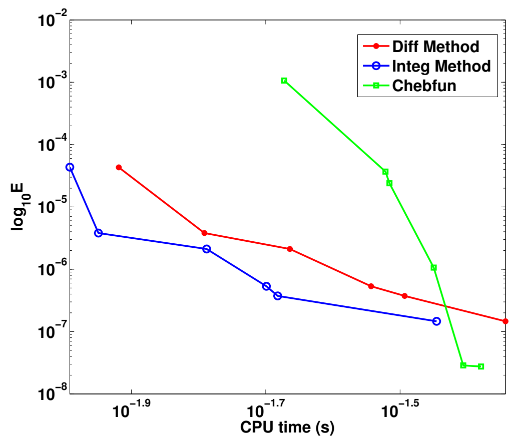

(b) Error versus $n$ for $T = 2$ $N = 2$




(c) Error versus $n$ for $T = 2$ $N = 4$

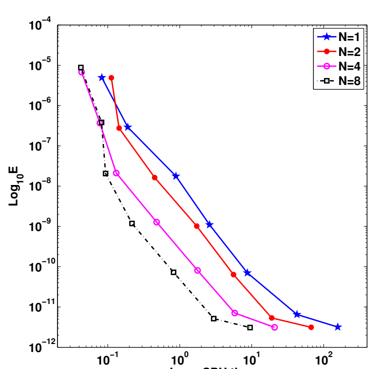



(d) Error versus $n$ for $T = 11$ $N = 4$

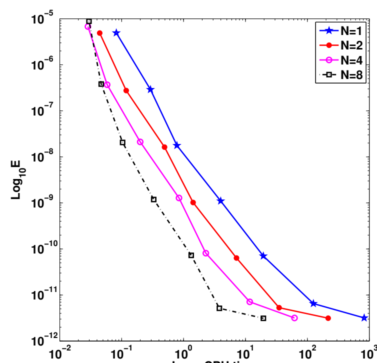



(e) Efficiency

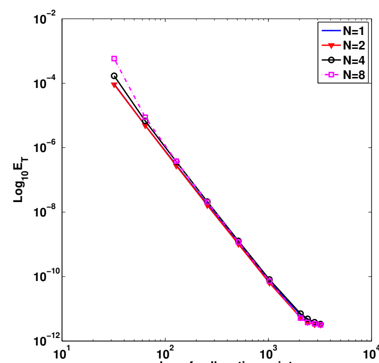
Figure 3: Convergence and efficiency of the three methods a we vary the number of collocation points




(a) Efficiency with integration method



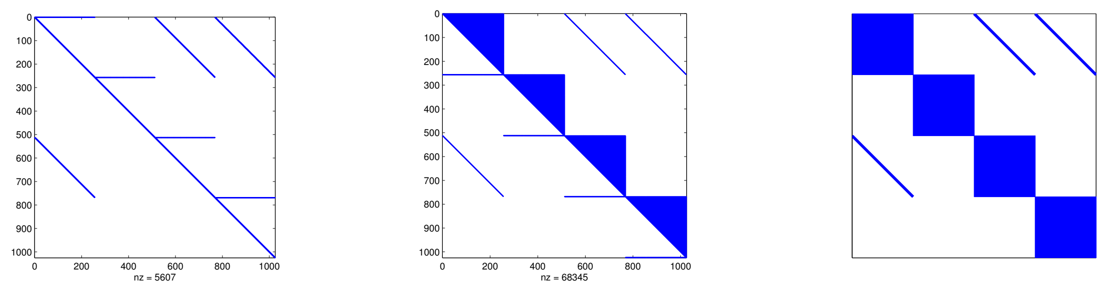

(b) Efficiency with differentiation method




(c) Convergence of integration/differention method

Figure 4: Efficiency and convergence of integral and differentiation method as we vary the number of domains

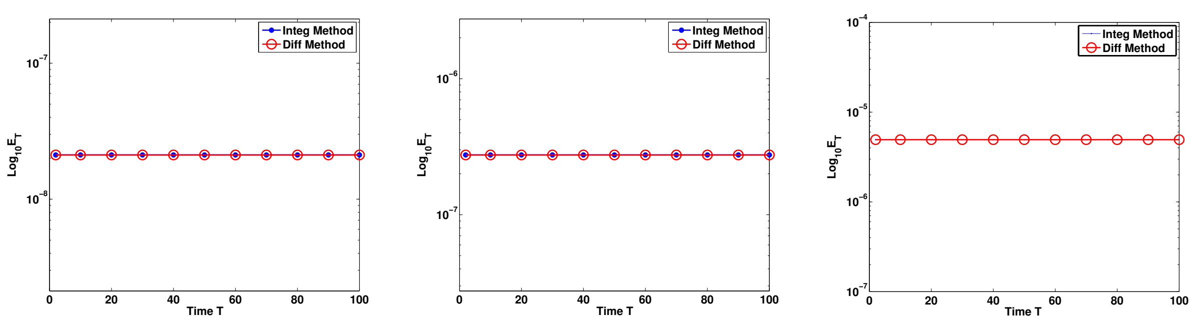
the number of collocation points get larger (here $n > 2000$) on each subinterval, the spectral methods tend to converge with same rate no matter the number of subintervals. This agrees with the observation we mentionned earlier regarding the number of points being enough to allow convergence. In this condition, even when we vary the number of subinterval, We may only experience a slight variation on the precision.




(a) 1-Domain Integration matrix (b) 1-Domain Differentiation matrix (c) Chebfun matrix

Figure 5: Plots of the underlying matrix $A$ of all three methods.

We also look at the influence of the number of domains $N$ and the time $T$ as it increases, and maintaining the number $n$ of collocation points constant equal 64. Figure 6 shows that the error remains the same no matter how big is $T$, as long as the domain sizes are enough to allow for the spectral method to run on each domain. The number of domains $N$ does not influence the error but helps in speeding up the algorithm.




(a) $0 \leq T \leq 100, N = 200$ ie. h=0.5 (b) $0 \leq T \leq 100, N = 100$ ie. $h = 1$ (c) $0 \leq T \leq 100, N = 50$ i.e h=2

Figure 6: Plots of error as $T$ increases to 100 and $n = 64$, for different number of domains

### *3.2. The cryptocurrency pricing model*

Our next problem to investigate is cryptocurrency pricing problem. As we mentioned earlier, From an asset flow perspective, Caginalp [4] proposed a model which discribes the interaction between the market price of cryptocurrency $P(t)$, the liquidity price $L(t)$at time $t$ and the trend-based component of investor preference at time $t$ denoted as $\zeta_1(t)$. This interaction is described by the following system:

$$\begin{cases} \tau_0 \frac{dP}{dt} &= (1+2\zeta_1)L - P \\ \\ c_0 \frac{dL}{dt} &= 1 - L + q(1+2\zeta_1)L - qP \\ \\ c_1 \frac{d\zeta_1}{dt} &= q_1(1+2\zeta_1)\frac{L}{P} - q_1 - \zeta_1 \end{cases} \tag{3.61}$$

The system admits only one equilibrium point obtained for $L = P$ and $\zeta_1 = 0$.

The differentiation approach applied in problem (3.61) produces the following discrete system:

$$\begin{bmatrix} \tau_0 D + \mathcal{I} & \mathcal{I} & 0 \\ q\mathcal{I} & c_0 D + (1-q)\mathcal{I} & 0 \\ 0 & 0 & c_1 D + \mathcal{I} \end{bmatrix} \begin{bmatrix} \mathbf{P} \\ \mathbf{L} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix}. \tag{3.62}$$

where **P, L, Z** are the coefficient vectors the variables $P, L, \zeta_1$; similarly $[\mathbf{f}^1, \mathbf{f}^2, \mathbf{f}^3]$ represent the coefficient vectors of the nonlinear part $[2\zeta_1 L, 1 + 2\zeta_1 L, q_1(1+2\zeta_1)\frac{L}{P}]$ respectively.

As for the integral approach, we get the following system:

$$A\underline{c} = \mathbf{f}. \tag{3.63}$$

where

$$\mathbf{A} = \begin{bmatrix} \tau_0\mathcal{I} + J & -J & 0 \\ qJ & c_0\mathcal{I} + (1-q)J & 0 \\ 0 & 0 & c_1\mathcal{I} + J \end{bmatrix}$$

and $\mathbf{f}$ is the coefficient vector of the integral of the nonlinear part $N$,

$$N = \begin{bmatrix} 2J\zeta_1 L \\ J(1 + 2q\zeta_1 L) \\ J\left(q_1(1+2\zeta_1)\frac{L}{P} - q_1\right) \end{bmatrix}.$$

We run our numerical methods for the following constant parameters $\tau_0 = 1, c_0 = -1, q = 0.5, q_1 = 1.2, c_1 = 1, T = 1$ and compare the performance of the three methods as in section 3.1. The graph of efficiencies of the methods reported in Figure 7a shows again that the integral method performs better than the differentiation method and better than the Chebfun as it takes less than 0.01s to achieve an accuracy of order $10^{-6}$ while it takes more than 0.04s for Chebfun to achieve the same order of precision.

Let us now focus on the spectral method using integration matrix, lets test its robustness in varying the time $T$. For $T = 4$, we see already in Figure 7b that using a single domain make the method to degenerate when the number of points is not large enough. This weakness is covered when the domain is split into subintervals. The exponential convergence is recovered with a slight loss in the accuracy as we increase the number of subintervals. This observation is confirmed in Figure 7d where $T$ is now equal to 8 and starting with 4-domain decomposition, ie., convergence remains exponential. As for the efficiency, Figure 7c and 7e show that increasing the number of subintervals improves the the method efficiency, that is, the more we increase the number of subintervals, the faster and more precise the spectral method gets. This comment is again in line with what was already pointed out in the hyperchaotic problem in section 3.1.

## 4. conclusion

In this article, a Chebyshev spectral method has been applied on time multiple domain using differentiation matrix and also using integration approach. The methods prove to be robust with the integral approach showing to be more efficient for hyperchaotic finance problem and cryptocurrency pricing problem, than the method from differentiation approach. The results are also compared with solutions obtained from other numerical methods in the literature to confirm reliability of the solutions. The spectral methods presented here are simple, fast and accurate for handling even more complicated ODEs. For future investigation we intend to extend the spectral method designed though to the fractional case of hyperchaotic systems.

## 5. Conflict of Interest

The authors have no affiliation with any organization with a direct or indirect financial interest in the subject matter discussed in the manuscript.

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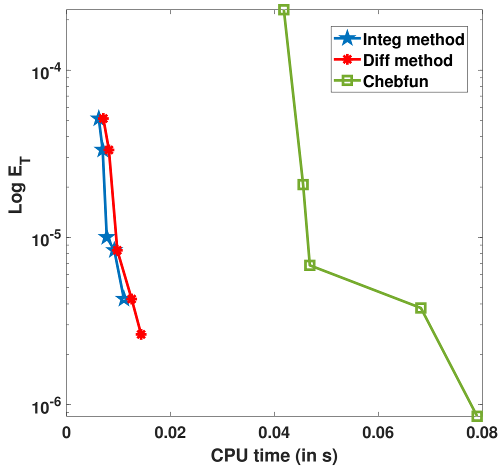
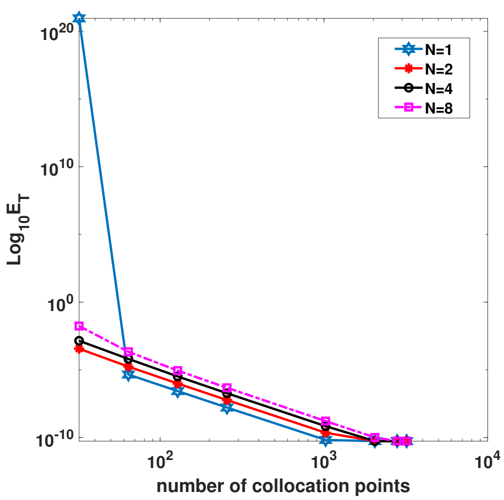
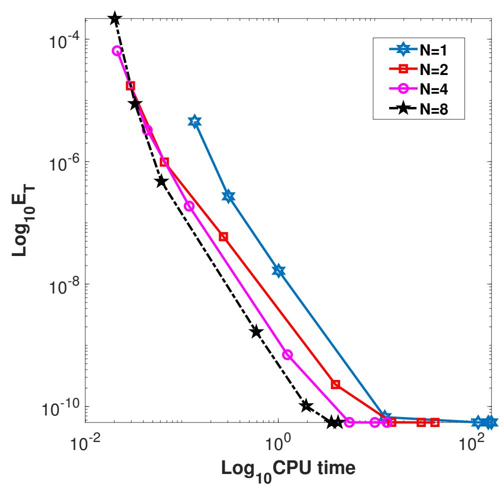


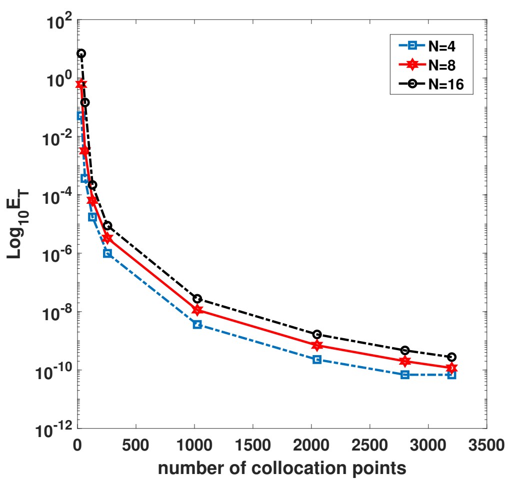

(a) Efficiency of the three methods



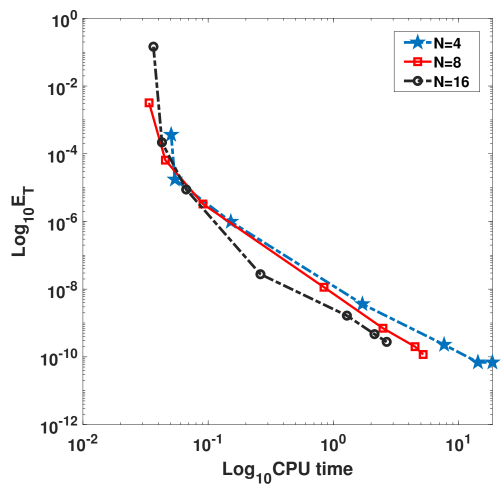

(b) Convergence for $T = 4$




(c) Efficiency for $T = 4$




(d) Convergence for $T = 8$




(e) Efficiency for $T = 8$

Figure 7: Convergence and efficiency of the three methods on the cryptocurrency problem