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ON CERTAIN PRODUCTS OF PERMUTABLE SUBGROUPS

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Dedicated to the memory of Alexander Grant Robertson Stewart

Abstract

In this paper, we study the structure of finite groups $G = AB$ which are a weakly mutually *sn*-permutable product of the subgroups A and B, that is, A permutes with every subnormal subgroup of B containing $A \cap B$ and B permutes with every subnormal subgroup of A containing $A \cap B$. We obtain generalisations of known results on mutually *sn*-permutable products.

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1. Introduction

All groups considered here will be finite.

Mutually permutable products, that is, products $G = AB$ such that A permutes with every subgroup of B and B permutes with every subgroup of A, have been extensively studied by many authors [3]. In recent years, some other permutability connections between the factors have also been considered. In particular, the rich normal structure of a mutually permutable product of two nilpotent groups [3, Ch. 5] has motivated interest in the study of mutually *sn*-permutable products.

DEFINITION 1.1. We say that a group $G = AB$ is the mutually *sn*-permutable product of the subgroups A and B if A permutes with every subnormal subgroup of B and B permutes with every subnormal subgroup of A.

Carocca [5] showed that a mutually *sn*-permutable product of two soluble groups is also soluble. In [1], the authors analyse the structure of mutually *sn*-permutable products and prove the following extension of a classical result of Asaad and Shaalan [2].

THEOREM 1.2 [1, Theorem B]. *Let $G = AB$ be the mutually sn-permutable product of the subgroups A and B, where A is supersoluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is supersoluble.*

Following [8], we say that a subgroup H of a group G is $\mathbb{P}$-subnormal in G whenever either $H = G$ or there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for every $i = 1, \ldots, n$. It turns out that supersoluble groups are exactly those groups in which every subgroup is $\mathbb{P}$-subnormal. Having in mind this result and the influence of the embedding of Sylow subgroups on the structure of a group, the following extension of the class of supersoluble groups introduced in [8] seems to be natural.

DEFINITION 1.3. A group G is called widely supersoluble, w-supersoluble for short, if every Sylow subgroup of G is $\mathbb{P}$-subnormal in G.

The class of all finite w-supersoluble groups, denoted by $w\mathcal{U}$, is a saturated formation of soluble groups containing $\mathcal{U}$, the class of all supersoluble groups, which is locally defined by a formation function f, such that for every prime p, $f(p)$ is composed of all soluble groups G whose Sylow subgroups are abelian of exponent dividing $p - 1$ [8, Theorems 2.3 and 2.7]. Not every group in $w\mathcal{U}$ is supersoluble [8, Example 1]. However, every group in $w\mathcal{U}$ has an ordered Sylow tower of supersoluble type [8, Proposition 2.8].

In [4], mutually sn-permutable products in which the factors are w-supersoluble are analysed. The following extension of Theorem 1.2 holds.

THEOREM 1.4 [4, Theorem 4]. *Let $G = AB$ be the mutually sn-permutable product of the subgroups A and B, where A is w-supersoluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is w-supersoluble.*

Assume that $G = AB$ is the mutually sn-permutable product of the subgroups A and B. Then, by [3, Proposition 4.1.16 and Corollary 4.1.17], $A \cap B$ is subnormal in G and permutes with every subnormal subgroup of A and B. Assume now that $G = AB$ and $A \cap B$ satisfy this condition. Then G is the mutually sn-permutable product of A and B if and only if A permutes with every subnormal subgroup V of B such that $A \cap B \leqslant V$ and B permutes with every subnormal subgroup U of A such that $A \cap B \leqslant U$. This motivates the following definition.

DEFINITION 1.5. Let A and B be two subgroups of a group G such that $G = AB$. We say that G is the weakly mutually sn-permutable product of A and B if A permutes with every subnormal subgroup V of B such that $A \cap B \leqslant V$ and B permutes with every subnormal subgroup U of A such that $A \cap B \leqslant U$.

Obviously, mutually sn-permutable products are weakly mutually sn-permutable, but the converse is not true in general, as the following example shows.

EXAMPLE 1.6. Let $G = \Sigma_4$ be the symmetric group of degree 4. Consider a maximal subgroup A of G which is isomorphic to Σ_3, and $B = A_4$, the alternating group of degree 4. Then $G = AB$ is the weakly mutually *sn*-permutable product of the subgroups A and B. However, the product is not mutually *sn*-permutable because A does not permute with a subnormal subgroup of order 2 of B.

The first goal of this paper to prove weakly mutually *sn*-permutable versions of the aforesaid theorems. We show that Theorem 1.4 holds for weakly mutually *sn*-permutable products.

THEOREM A. *Let $G = AB$ be the weakly mutually sn-permutable product of the subgroups A and B, where A is w-supersoluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is w-supersoluble.*

The next corollary follows from the proof of Theorem A and generalises Theorem 1.2.

COROLLARY B. *Let $G = AB$ be the weakly mutually sn-permutable product of the subgroups A and B, where A is supersoluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is supersoluble.*

The second part of the paper is concerned with weakly mutually *sn*-permutable products with nilpotent derived subgroups. Our starting point is the following extension of a classical result of Asaad and Shaalan [2].

THEOREM 1.7 [1, Theorem C]. *Let $G = AB$ be the mutually sn-permutable product of the supersoluble subgroups A and B. If the derived subgroup G' of G is nilpotent, then G is supersoluble.*

A natural question is whether this result is true for weakly mutually *sn*-permutable products under the same conditions. The following example answers this question negatively.

EXAMPLE 1.8. Let $G = \langle a, b, c : a^5 = b^5 = c^6 = 1, ab = ba, a^c = a^2b^3, b^c = a^{-1}b^{-1}\rangle \simeq [C_5 \times C_5]C_6$. Then $G = AB$ is the weakly mutually *sn*-permutable product of $A = \langle c\rangle$ and $B = [\langle a\rangle \times \langle b\rangle]\langle c^3\rangle$. Note that B is a normal subgroup of G; therefore, it permutes with every subgroup of A. Moreover, $A \cap B = \langle c^3\rangle$ and the unique subnormal subgroup of B containing $A \cap B$ is the whole of B. It is not difficult to see that B is supersoluble. Therefore, A and B are supersoluble and G' is nilpotent. Moreover, A is nilpotent and B is a normal subgroup of G. Thus, in particular, it permutes with every Sylow subgroup of A.

However, an additional assumption allows us to get supersolubility.

THEOREM C. *Let $G = AB$ be the weakly mutually sn-permutable product of the supersoluble subgroups A and B. If B permutes with each Sylow subgroup of A, A permutes with every Sylow subgroup of B and the derived subgroup G' of G is nilpotent, then G is supersoluble.*

By [7, Theorem 2.6], a group G is w-supersoluble if and only if every metanilpotent subgroup of G is supersoluble. In particular, if we have a group with G' nilpotent, every w-supersoluble subgroup is supersoluble. Therefore, the following result is clear.

COROLLARY D. *Let $G = AB$ be the weakly mutually sn-permutable product of the w-supersoluble subgroups A and B. If B permutes with each Sylow subgroup of A, A permutes with every Sylow subgroup of B and the derived subgroup G' of G is nilpotent, then G is w-supersoluble.*

2. Preliminary results

In this section we will prove some results needed for the proofs of our main results. We start by showing that factor groups of weakly mutually sn-permutable products are also weakly mutually sn-permutable products.

LEMMA 2.1. *Let $G = AB$ be the weakly mutually sn-permutable product of A and B, and let N be a normal subgroup of G. Then $G/N = (AN/N)(BN/N)$ is the weakly mutually sn-permutable product of AN/N and BN/N.*

PROOF. Let us consider $G/N = (AN/N)(BN/N)$. Suppose that HN/N is a subnormal subgroup of AN/N such that $AN/N \cap BN/N \leqslant HN/N$. Note that $U = HN \cap A$ is a subnormal subgroup of A such that $UN = HN$ and $A \cap B \leq U$. Since U permutes with B, it follows that $HN = UN$ permutes with BN.

Interchanging A and B and arguing in the same manner proves the result. □

LEMMA 2.2. *Let $G = AB$ be the weakly mutually sn-permutable product of A and B.*

(a) *If H is a subnormal subgroup of A such that $A \cap B \leqslant H$, then HB is a weakly mutually sn-permutable product of H and B.*
(b) *If $A \cap B = 1$, then $G = AB$ is a totally sn-permutable product of A and B.*

PROOF. Since every subnormal subgroup of H is a subnormal subgroup of A, it follows that B permutes with every subnormal subgroup L of H such that $A \cap B \leqslant L$. On the other hand, let M be a subnormal subgroup of B such that $A \cap B \leqslant M$. Then we have $HM = H(A \cap B)M = (A \cap HB)M = AM \cap HB = MA \cap BH = M(A \cap BH) = M(A \cap B)H = MH$. Hence HB is a weakly mutually sn-permutable product of H and B.

For (b), every subnormal subgroup of A permutes with B by (a) and every subnormal subgroup of B permutes with A. So $G = AB$ is the mutually sn-permutable product of A and B. Hence $G = AB$ is the totally sn-permutable product of A and B since $A \cap B = 1$. □

Observe that Lemma 2.2 implies that if $G = AB$ is the weakly mutually sn-permutable product of A and B, H is a subnormal subgroup of A such that $A \cap B \leqslant H$ and K is a subnormal subgroup of B such that $A \cap B \leqslant K$, then HK is a weakly mutually sn-permutable product of H and K. In the next result we analyse the behaviour of minimal normal subgroups of weakly mutually *sn*-permutable products containing the intersection of the factors.

LEMMA 2.3. *Let $G = AB$ be the weakly mutually sn-permutable product of A and B. If N is a minimal normal subgroup of G such that $A \cap B \leqslant N$, then either $A \cap N = B \cap N = 1$ or $N = (N \cap A)(N \cap B)$.*

PROOF. Observe that $A \cap N$ is a normal subgroup of A such that $A \cap B \leqslant A \cap N$ and consequently $H = (A \cap N)B$ is a subgroup of G. Note that $N \cap H = N \cap (A \cap N)B = (A \cap N)(B \cap N)$. Since $N \cap H$ is a normal subgroup of H, it follows that B normalises $N \cap H = (A \cap N)(B \cap N)$.

By the same argument, $K = A(B \cap N)$ is a subgroup of G such that $K \cap N = A(B \cap N) \cap N = (A \cap N)(B \cap N)$. Moreover, A normalises $K \cap N = (A \cap N)(B \cap N)$. It follows that $(A \cap N)(B \cap N)$ is a normal subgroup of G. By the minimality of N, $A \cap N = B \cap N = 1$ or $N = (N \cap A)(N \cap B)$ as required. □

LEMMA 2.4. *Let $G = AB$ be the weakly mutually sn-permutable product of the subgroups A and B, where B is nilpotent. If B permutes with each Sylow subgroup of A, then $A \cap B$ is a nilpotent subnormal subgroup of G.*

PROOF. It is clear that $A \cap B$ is nilpotent. The Sylow subgroups of B are normal in B, so $A \cap B$ permutes with every Sylow subgroup of B. Let A_q be a Sylow subgroup of A, with q a prime dividing $|A|$. Since B permutes with every Sylow subgroup of A, it follows that BA_q is a subgroup of G. Hence $BA_q \cap A = A_q(A \cap B)$. Therefore $A \cap B$ permutes with every Sylow subgroup of A. Applying [3, Theorem 1.2.14(3)], $A \cap B$ is a subnormal subgroup of both A and B. By [3, Theorem 1.1.7], $A \cap B$ is a subnormal subgroup of G. □

LEMMA 2.5. *Let $G = AB$ be the weakly mutually sn-permutable product of the subgroups A and B, where A is soluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is soluble.*

PROOF. Suppose that the theorem is false, and let G be a minimal counterexample. If N is a minimal normal subgroup of G, then $G/N = (AN/N)(BN/N)$ is the weakly mutually *sn*-permutable product of the subgroups AN/N and BN/N by Lemma 2.1. Since BN/N permutes with each Sylow subgroup of AN/N, it follows that G/N is soluble by the minimality of G. Let N_1 and N_2 be two minimal subgroups of G. Then $G \cong G/(N_1 \cap N_2)$ is soluble, a contradiction. Hence G has a unique minimal normal subgroup N of G and we may assume that N is nonabelian. This means that $\mathbf{F}(G) = 1$.

On the other hand, $A \cap B \leqslant \mathbf{F}(G)$ using Lemma 2.4. Therefore $A \cap B = 1$ and then $G = AB$ is the totally *sn*-permutable product of A and B. The result then follows by applying [5, Theorem 6]. □

LEMMA 2.6 [1, Lemma 3]. *Let G be a primitive group and let N be its unique minimal normal subgroup. Assume that G/N is supersoluble. If N is a p-group, where p is the largest prime dividing $|G|$, then $N = \mathbf{F}(G) = \mathbf{O}_p(G)$ is a Sylow p-subgroup of G.*

3. Main results

We are ready to prove our main results.

PROOF OF THEOREM A. Suppose the theorem is not true and let G be a minimal counterexample. We shall prove our theorem in five steps.

(a) *G is a primitive soluble group with a unique minimal normal subgroup N and $N = \mathbf{C}_G(N) = \mathbf{F}(G) = \mathbf{O}_p(G)$ for a prime p.* Let N be a minimal normal subgroup of G. By Lemma 2.1, $G/N = (AN/N)(BN/N)$ is a weakly mutually sn-permutable product of AN/N and BN/N and it is clear that BN/N permutes with every Sylow subgroup of AN/N. Moreover, AN/N is w-supersoluble and BN/N is nilpotent. By the minimality of G, it follows that G/N is w-supersoluble. Note that $w\mathcal{U}$, the class of finite w-supersoluble groups, is a saturated formation of soluble groups by [8, Theorems 2.3 and 2.7]. This implies that G is a primitive soluble group and so G has a unique minimal normal subgroup N with $N = \mathbf{C}_G(N) = \mathbf{F}(G) = \mathbf{O}_p(G)$ for some prime p as required.

(b) *The subgroup BN is w -supersoluble and* $1 \neq A \cap B \leqslant N$. If $A \cap B = 1$, then the result follows by Lemma 2.2 and Theorem 1.4. Applying Lemma 2.4, it follows that $A \cap B$ is a nilpotent subnormal subgroup of G. Therefore $1 \neq A \cap B \leqslant \mathbf{F}(G) = N$ and so $N = (N \cap A)(N \cap B)$ by Lemma 2.3. Hence $NB = (N \cap A)(N \cap B)B = (N \cap A)B$ is a weakly mutually sn-permutable product of $N \cap A$ and B. Also note that B permutes with every Sylow subgroup of $N \cap A$ (there is only one Sylow subgroup of $N \cap A$, which is $N \cap A$). If $NB < G$, then NB is w-supersoluble by the choice of G. So we may assume that $G = NB$. In this case, consider a subgroup $N_1 \leqslant A \cap B \leqslant N$. Note that N_1 is normal in N since N is abelian. Hence $N = N_1^G = N_1^{NB} = N_1^B \leqslant B$ and $G = B$, a contradiction. Hence the result follows.

(c) *N is the Sylow p-subgroup of G and p is the largest prime dividing* $|G|$. Let q be the largest prime dividing $|G|$ and suppose that $q \neq p$. Suppose first that q divides $|BN|$. Since BN has a Sylow tower of supersoluble type, BN has a unique Sylow q-subgroup, say $(BN)_q$. This means that $(BN)_q$ centralises N. Thus $(BN)_q = 1$, since $\mathbf{C}_G(N) = N$, a contradiction.

We may assume that q divides $|A|$ but does not divide $|BN|$. Since A has a Sylow tower of supersoluble type, A has a unique Sylow q-subgroup, A_q say. This means that A_q is normalised by $N \cap A$. Consider $A_q(N \cap B) = A_q(A \cap B)(N \cap B)$, a weakly mutually permutable product of $A_q(A \cap B)$ and $N \cap B$ by Lemma 2.2. Also $N \cap B$ permutes with each Sylow subgroup of $A_q(A \cap B)$. Suppose that $A_q(N \cap B) < G$. Then $A_q(N \cap B)$ is w-supersoluble by the choice of G. It follows that $A_q(N \cap B)$ has a unique Sylow q-subgroup since it has a Sylow tower of supersoluble type. In other words, A_q is normalised by $N \cap B$. Hence A_q is normalised by $(N \cap A)(N \cap B) = N$. This means that A_q centralises N, a contradiction. We may assume that $A_q(N \cap B) = G$. Then $N \cap B = B$ and so B is an elementary abelian p-group. Moreover, $A = A_q(A \cap B)$. Since $A \cap B$ is a Sylow p-subgroup of A which is subnormal in A, it is normal in A. Hence $A \cap B$ is normal in G because $A \cap B$ is normal in the abelian group B. By the minimality of N, it follows that $N = A \cap B$, that is, $G = A_q(N \cap B) = A_q(A \cap B) = A$, a contradiction. Therefore p is the largest prime dividing $|G|$.

We now prove that N is the Sylow p-subgroup of G. Since G is a primitive soluble group, $G = NM$, where M is a maximal subgroup of G and $N \cap M = 1$. Then $M \cong G/N$ is w-supersoluble. By [6, Theorem A.15.6], $\mathbf{O}_p(M) = 1$. If p divides $|M|$, then since M has a Sylow tower of supersoluble type, $\mathbf{O}_p(M) \neq 1$, a contradiction. Hence p does not divide $|M|$ and therefore N is the unique Sylow p-subgroup of G.

(d) *N is contained in A and N is not contained in B.* Suppose that B is a p-group. Then $G = AN$. Let $N_1 \leqslant A \cap B$. Since B is abelian, $N \leq N_1^G = N_1^{AN} = N_1^A \leqslant A$ and so $G = AN = A$, a contradiction. So we may assume that B is not a p-group. If N is contained in B, then since B is nilpotent and $N = \mathbf{C}_G(N)$, it follows that B is a p-group, a contradiction. Therefore N is not contained in B. Hence B has a nontrivial Hall p'-subgroup, $B_{p'}$, which is normal in B. Consequently, $AB_{p'} = A(A \cap B)B_{p'}$ is a subgroup of G. Then $1 \neq B_{p'}^G \leqslant AB_{p'}$ and so $N \leqslant AB_{p'}$. Hence $N \leqslant A$ as required.

(e) *Final contradiction.* Let $A_{p'}$ be a Hall p'-subgroup of A. If $A_{p'} = 1$, then $G = BN$ is w-supersoluble by (b), a contradiction. Hence $A_{p'} \neq 1$. Since B permutes with every Sylow subgroup of A, it follows that $A_{p'}B$ is a subgroup of G. But N is not contained in B, so $A_{p'}B$ is a proper subgroup of G. Since $NA_{p'}B = G$, it follows that $N \cap A_{p'}B = N \cap B$ is normal in G. The minimality of N implies that $N = N \cap B$ or $N \cap B = 1$. If $N = N \cap B$, we get a contradiction with (d). Therefore $N \cap B = 1$, and then $A \cap B \leqslant N \cap B = 1$, contradicting (b). □

PROOF OF THEOREM C. Assume the result is not true and let G be a minimal counterexample. It is clear that $G' \neq 1$, A and B are proper subgroups of G, and G is a primitive soluble group. Hence there exists a unique minimal normal subgroup N of G, such that $N = F(G) = C_G(N)$. Moreover, $G' = N$. We may assume that $A' \neq 1$ and $B' \neq 1$, otherwise A or B is nilpotent and the result follows from Corollary B. If $A \cap B = 1$, then G is the mutually sn-permutable product of A and B. By [1, Theorem C], the group is supersoluble, a contradiction. Thus we may assume $A \cap B \neq 1$. Since A permutes with every Sylow subgroup of B and B permutes with every Sylow subgroup of A, it follows that $A \cap B$ permutes with every Sylow subgroup of A and every Sylow subgroup of B. Hence $A \cap B$ is subnormal in A and it is a subnormal subgroup of B. Let N_1 denote a minimal normal subgroup of A such that $N_1 \leq A'$. Since A is supersoluble, it is clear that $|N_1| = p$. Note that $N_1(A \cap B)$ is a subnormal subgroup of A. Therefore $BN_1(A \cap B) = BN_1$ is a subgroup of G. Now $1 \neq N_1^G = N_1^B \leq BN_1$. Hence $N \leq BN_1$ and then $N = N_1(N \cap B)$. Consequently, either $N_1 \leq N \cap B$ or $N_1 \leq N \cap B$. Denote $T = BN$. If $N_1 \leq N \cap B$, then $T = B$ is a supersoluble normal subgroup of G. Assume $N_1 \cap (N \cap B) = 1$. Then $N \cap B$ is a maximal subgroup of N and so T is the weakly mutually sn-permutable product of B and N. Consequently, T satisfies the hypotheses of the theorem. If T is a proper subgroup of G, then $T = BN$ is supersoluble. Assume that $G = BN$. Then B is a maximal subgroup of G such that $B \cap N = 1$, $B' \leq N \cap B = 1$ and B is nilpotent. By Corollary B, G is supersoluble, contrary to assumption. Hence either B is a normal subgroup of G or BN is a supersoluble normal subgroup of G.

Arguing in an analogous manner with A shows that if AN is a proper subgroup of G, then it is supersoluble. Consequently if BN and AN are both proper subgroups of G, then G is the product of two supersoluble normal subgroups with G' nilpotent. Then G is supersoluble, a contradiction. Therefore we may assume that $G = BN$ or $G = AN$. Suppose without loss of generality that $G = BN$. Then $N \cap B$ is a normal subgroup of G. If $N \cap B = N$, then $G = B$, a contradiction. Hence $N \cap B = 1$. Now $B' \leq N \cap B = 1$ and B is nilpotent, the final contradiction. □

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