To appear in *Quantitative Finance*, Vol. 00, No. 00, Month 20XX, 1–20

Why has the equal weight portfolio underperformed and what can we do about it?

B. H. TALJAARD*† and E. MARÉ‡

†Department of Actuarial Science, University of Pretoria, Private Bag X20, Hatfield, South Africa, 0028
‡Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20, Hatfield, South Africa, 0028

(*31 May 2020*)

It is widely noted that market capitalization weighted portfolios are inefficient and underperform an equal weighted portfolio over the long-term. However, at least since 2016, an equal weighted portfolio of stocks in the S&P500 has significantly underperformed the market capitalization weighted portfolio. In this paper we analyse this underperformance using stochastic portfolio theory. We show that the equal weighted portfolio does appear to outperform the market capitalization weighted portfolio over the long-term but with periods of significant short-term underperformance. In addition, we find that concentration in the market capitalization weighted portfolio has increased in recent years and has contributed to the recent underperformance together with a significantly lower level of diversification benefits. Furthermore, we highlight an approach to improve the performance of a portfolio by dynamically selecting a market cap or an equal weighting using a rudimentary linear regression model.

Keywords: Stochastic portfolio theory; Equities; Equal weight portfolio; Portfolio optimization; Diversification;

JEL Classification: G11, G15, G17

*Corresponding author. Email: byran.taljaard@gmail.com

1. Introduction

It is widely published that an equal weighted equity portfolio outperforms a market capitalization weighted equity portfolio (cap weighted) over the long term (Malladi and Fabozzi 2017, Bolognesi *et al.* 2013, Plyakha *et al.* 2012). Furthermore, DeMiguel *et al.* (2009) have shown that, not only is the equal weighted portfolio more efficient than the cap weighted portfolio, but that it outperforms mean-variance based portfolio strategies out of sample. However, while long-term performance is of course important, in the short-term, the equal weighted portfolio can suffer significant underperformance relative to the cap weighted portfolio. It may be difficult for investors to hold equal weighted portfolios during these periods.

There can be various reasons for this short-term underperformance. The act of rebalancing an equal weighted portfolio essentially involves selling stocks that have outperformed the average stock and buying those that have underperformed. In the short-term this would require some level of mean reversion in stocks to outperform the cap weighted portfolio. However, in momentum driven markets this is unlikely to be the case and, in such instances, the equal weighted portfolio is bound to underperform the cap weighted portfolio in the short-term.

Another source of underperformance could be the diversification benefits of the equal weighted portfolio. The act of rebalancing an equal weighted portfolio is also sometimes referred to as volatility harvesting or volatility return, for example, in Bouchey *et al.* (2012, 2015), and Hallerbach (2014), while Booth and Fama (1992) refer to the additional return obtained from rebalancing as the diversification return. It is possible that the diversification return is too low, either because individual stock volatilities are low or correlation is high, and does not warrant rebalancing of the portfolio back to equal weights. This is analysed further in Taljaard and Maré (2019) and Cuthbertson *et al.* (2016).

We attempt to understand and improve the performance of an equal weighted portfolio of S&P500 stocks using an approach from stochastic portfolio theory. Stochastic portfolio theory was first introduced by Robert Fernholz (Fernholz and Shay 1982, Fernholz 1999, Fernholz and Square 1998), which culminated in a book (Fernholz 2002). Our use of stochastic portfolio theory is on the basis that it requires relatively few assumptions, is based on a widely used model of stocks, and provides a direct approach to modelling and understanding the drivers of returns of the equal weighted portfolio relative to the cap weighted portfolio.

Making use of the tools derived in stochastic portfolio theory, we analyse the relative performance of the equal weighted portfolio, decomposing the relative performance into two main parts, namely the change in the concentration of the cap weighted portfolio and the excess return generated by a diversification benefit. We also analyse a third component, called leakage, which measures the impact of changes in the constituents of an index, or subset of the equity market, on a portfolio.

We focus mainly on the S&P500 in this article given its status as the largest, most liquid equity market in the world and that much of the commentary on equal weight and market cap in recent months has been in relation to the US equity markets. However, this approach of analysing the equal weight and cap weighted portfolios is applicable to other markets and, although the matter deserves a dedicated consideration, we touch very briefly on a few other equity markets.

As it pertains to the S&P500, although an equal weighted portfolio seems to have outperformed the cap weighted portfolio since 1995, we show that there have been periods of significant underperformance over shorter time horizons. These periods have, primarily, either coincided with higher levels of concentration in the cap weighted portfolio and/or lower benefits of diversification due to lower average volatilities and higher correlations among stocks. As a trading strategy, we attempt to dynamically switch from the equal weighted portfolio to the cap weighted portfolio to improve short-term, relative, risk-adjusted performance.

The article is structured as follows: in Section 2 we derive some key formulae and ideas from stochastic portfolio theory, in Section 3 we show the long-term and short-term outperformance of the equal weighted portfolio of S&P500 stocks from 1995 and compare the performance with the theoretical constructs in Section 2. In Section 4 we attempt to optimise the performance of our portfolio by alternating between the cap weighted and equal weighted portfolios using a rudimentary linear model.

2. Stochastic portfolio theory

In this section we give a brief overview of the necessary expressions from stochastic portfolio theory, as found in Fernholz (2002). We also suggest Fernholz and Karatzas (2009) and Karatzas (2006) for a complete overview of stochastic portfolio theory. We cover two main parts of stochastic portfolio theory in the following subsections, namely the portfolio price process and portfolio generating functions. Portfolio generating functions allow us to consider the relative performance of the equal weighted portfolio in relation to the cap weighted portfolio. We leave other topics of stochastic portfolio theory, such as the capital distribution curve (Fernholz 2001a) and stock selection by rank (Fernholz 2001b) for further study.

2.1. *Portfolio price process*

Consider a stock that follows the popular logarithmic model for the continuous-time stock process:

$$d\log X_i(t) = \gamma_i(t)dt + \sum_{\nu=1}^{n} \xi_{i\nu} dW_\nu(t) \qquad t \in [0, \infty), \tag{1}$$

where $\xi_{i\nu}$ is the sensitivity of stock $X_i(t)$ to the ν-th source of randomness for $\nu = 1, 2, \ldots, n$ and $dW_\nu(t)$ represent Brownian motions. $\gamma_i(t)$ is the geometric growth rate of stock $X_i(t)$ related to the arithmetic growth rate, $\alpha_i(t)$, by

$$\gamma_i(t) = \alpha_i(t) - \frac{1}{2}\sum_{\nu=1}^{n} \xi_{i\nu}^2(t).$$

If we consider a set of stocks $X_1, \ldots, X_n$ that each follow the price process as in Equation (1) and construct a portfolio with weights given by $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))$, such that

$$\sum_{i=1}^{n} \pi_i(t) = 1,$$

and

$$\pi_i(t) \geq 0 \quad \forall i = 1, \ldots, n,$$

then the portfolio price process, $Z_\pi(t)$, follows:

$$d\log Z_\pi(t) = \gamma_\pi(t) + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu} dW_\nu(t), \tag{2}$$

where

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right),$$

and the cross-variance processes for $\log X_i$ and $\log X_j$ are given by

$$\sigma_{ij}(t)dt = d\left\langle \log X_i, \log X_j \right\rangle_t = \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t),$$

with the covariance process of X_i given by

$$\sigma_{ii}(t) = d\left\langle \log X_i \right\rangle_t.$$

The portfolio's growth rate, $\gamma_\pi(t)$, consists of two distinct parts: the weighted growth rates of the individual stocks and a second term involving the stocks' weighted volatilities and covariances. This second term is referred to as the excess growth rate and is given by

$$\gamma_\pi^*(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right). \tag{3}$$

We can regard $\gamma_\pi^*(t)$ as the benefit of diversification. If the correlation between stocks is very small, then $\gamma_\pi^*(t)$ would be larger leading to a higher contribution to the overall portfolio growth rate. In a similar vein, should individual stock volatilities be low, then $\gamma_\pi^*(t)$ would also be low as there would be less scope for a reduction in portfolio volatility.

Combining Equations (1), (2), and (3) we obtain

$$d\log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t) d\log X_i(t) + \gamma_\pi^*(t)dt.$$

That is, the price process of the portfolio $Z_\pi(t)$ is a function of two parts: the weighted price processes of the individual stocks and a term representing the diversification benefit within the portfolio.

The above formulae are easily extended to include dividends. Given a dividend process for stock $X_i(t)$, represented as $\delta_i(t)$, then the total return for stock $X_i(t)$ is

$$\hat{X}_i(t) = X_i(t)\exp\left(\int_0^t \delta_i(s)ds\right), \qquad t \in [0, \infty).$$

Therefore, Equation (1) can be extended for the total return process,

$$d\log \hat{X}_i(t) = d\log X_i(t) + \delta_i(t)dt,$$

and the total return process for the portfolio given by weights $\pi(t)$ becomes,

$$d\log \hat{Z}_\pi(t) = d\log Z_\pi(t) + \delta_\pi(t)dt, \tag{4}$$

where

$$\delta_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\delta_i(t).$$

2.2. *Portfolio generating functions and relative performance*

We are, of course, not necessarily interested in the equal weighted portfolio's price process on its own, but rather its price process relative to the cap weighted portfolio.

Consider the cap weighted portfolio price process given by $Z_\mu(t)$, where the portfolio weight of stock i at time t, $\mu_i(t)$, is given by

$$\mu_i(t) = \frac{X_i(t)}{\sum_{i=1}^n X_i(t)},$$

where $X_i(t)$ is stock i's market capitalization at time t. $Z_\mu(t)$ is just like any other portfolio price process and, therefore, the expressions in Section 2.1 apply to $Z_\mu(t)$, with the only difference being the weights themselves.

To derive expressions for the relative performance of a portfolio, we first need to define portfolio generating functions. These are functions that generate various portfolio weights using, as an input, the market capitalization weights, $\mu(t)$. They are defined as follows (see Fernholz (2002), Definition 3.1.1, reproduced below for convenience):

Definition 1 Let $\mathbf{S}$ be a positive continuous function defined on Δ^n (the unit n-simplex) and let π be a portfolio. Then $\mathbf{S}$ generates π if there exists a measurable process of bounded variation Θ such that

$$\log\left(Z_\pi(t)/Z_\mu(t)\right) = \log\mathbf{S}(\mu(t)) + \Theta(t), \qquad t \in [0,T], \quad \text{a.s.} \tag{5}$$

The process Θ is called the drift process corresponding to $\mathbf{S}$.

We can also express Equation (5) in differential form,

$$d\log\left(Z_\pi(t)/Z_\mu(t)\right) = d\log\mathbf{S}(\mu(t)) + d\Theta(t), \qquad t \in [0,T], \quad \text{a.s.} \tag{6}$$

If we include dividends then, using Equation (4), the total relative return process becomes

$$d\log\left(\hat{Z}_\pi(t)/\hat{Z}_\mu(t)\right) = d\log\mathbf{S}(\mu(t)) + \int_0^t (\delta_\pi(s) - \delta_\mu(s))ds + d\Theta(t), \qquad t \in [0,T], \quad \text{a.s.} \tag{7}$$

In other words, the relative performance of a portfolio with weights $\pi(t)$ is a function of the portfolio generating function $\mathbf{S}$, the difference in dividend rates and the drift process.

The expressions for the weights, $\pi_i(t)$, as well as the drift process, $\Theta(t)$, can be derived using Theorem 3.1.5 in Fernholz (2002). We reproduce the relevant theorem below.

Theorem 2.1 *Let $\mathbf{S}$ be a positive C^2 function defined on a neighborhood U of Δ^n such that for all i, $x_i D_i \log\mathbf{S}(x)$ is bounded on Δ^n. Then $\mathbf{S}$ generates the portfolio π with weights*

$$\pi_i(t) = \left(D_i\log\mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^n \mu_j(t) D_j \log\mathbf{S}(\mu(t)) \right) \mu_i(t),$$

for $t \in [0,T]$ and $i = 1, \ldots, n$ and with a drift process Θ such that a.s., for $t \in [0,T]$,

$$d\Theta(t) = \frac{-1}{2\mathbf{S}(\mu(t))} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt.$$

The notation D_i represents the partial derivative with respect to the ith variable and $\tau_{ij}(t)$ represents the relative covariance of stock i and j. That is,

$$\tau_{ij}(t) = \langle \log\left(X_i/Z_\mu\right), \log\left(X_j/Z_\mu\right) \rangle .$$

We can show that the function given by

$$\mathbf{S}(\mu) = \left(\prod_{i=1}^{n} \mu_i \right)^{\frac{1}{n}}, \tag{8}$$

generates the equal weighted portfolio with a drift process given by

$$d\Theta(t) = \gamma_\pi^*(t)dt, \tag{9}$$

where $\gamma_\pi^*(t)$ is defined as the excess growth rate in Equation (3).

Considering this in the context of Equation (6), the relative performance of the equal weighted portfolio over some time period is given by the addition of the change in $\frac{1}{n}\log\left(\mu_1(t)\cdots\mu_n(t)\right)$ and $\gamma_\pi^*(t)dt$.

We know that $\gamma_\pi^*(t)$ is always positive, however, the change in $\log\mathbf{S}(\mu)$ is dependent on the change of the distribution of weights, $\mu_i(t)$ in the cap weighted portfolio. If the concentration increases, the change in $\log\mathbf{S}(\mu)$ is negative and detracts from the equal weight portfolio's performance relative to the cap weighted portfolio and vice versa.

In a scenario where the cap weighted portfolio becomes increasingly concentrated, the change in $\log\mathbf{S}(\mu)$ becomes a short-term, but consistent, drag on relative performance through Equation (6). As a result, the excess growth rate will have to offset this term in order for the equal weighted portfolio to outperform the cap weighted portfolio. Although always positive, it is not a given that in every period the excess growth rate will be high enough to offset the growing concentration of market capitalization weights and this can, therefore, lead to short-term underperformance.

2.3. *Portfolios on subsets of the whole market*

In the previous section, portfolios formed on the entire equity market were considered. However, in practice, an index formed on the top stocks is used as a benchmark and its constituents are used to form active portfolios. In our examples we consider the S&P500, which would be a subset of the entire US equity market. As a result, we are implicitly selecting stocks by rank in both the cap weighted and equal weighted portfolios. This requires us to modify the drift process, $d\Theta(t)$, in Theorem 2.1 by introducing a new term $dL_\pi(t)$.

$$d\Theta(t) = \frac{-1}{2\mathbf{S}(\mu(t))} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt + dL_\pi(t),$$

where

$$dL_{\pi}(t) = \frac{1}{2}\sum_{k=1}^{n=1}\left(\pi_{(k+1)}(t) - \pi_{(k)}(t)\right) d\Lambda_{\log\mu_{(k)} - \log\mu_{(k+1)}}(t), \qquad t \in [0, T]. \tag{10}$$

The terms $\pi_{(k)}(t)$ and $\mu_{(k)}(t)$ represent the portfolio weight and the market cap weight of the k-th largest stock, respectively, where the stocks are ranked by market cap weights from largest to smallest over the entire market.

Λ is the semi-martingale local time process defined as,

$$\Lambda_X(t) = \frac{1}{2}\left(|X(t)| - |X(0)| - \int_0^t sgn(X(s))dX(s)\right), \qquad t \in [0, T].$$

We can think of this new term dL_{π} as the shift in weights within the market, and specifically how that affects portfolio selection both in terms of the equal weighted portfolio and the benchmark, which is a subset of the whole market. In our case here, with the S&P500 or any other index, the changes that would affect portfolio selection occurs at the edges. That is, the 500th stock, for example, becoming the 501st stock and falling out of the index and therefore out of the individual portfolios. This term is, therefore, defined as leakage, given that its size is determined by stocks "leaking" out of the portfolio.

As a result, and in the specific case of the equal weighted portfolio, this leakage term, over small time periods, usually only involves the last position in the index. That is, in the case of the S&P500, we are only interested in changes in the 500th position over a small time period. This change (the last stock being replaced) is likely to affect the equal weighted portfolio much more than a cap weighted portfolio, given that the final few weights are likely to already be small in the case of the cap weight. The equal weight on the other hand, typically, has larger weights in those last few stocks and is, therefore, more affected by exclusions.

The leakage term defined in Equation (10) is of a portfolio relative to the entire market. In the case of the equal weighted portfolio with weights $\pi_{(k)}(t)$ relative to the cap weighted portfolio with weights $\mu_{(k)}(t)$, where both portfolios are a subset of the entire market, the leakage term corresponding to the relative return of the equal weighted against this cap weighted portfolio can be expressed as

$$dL_{\pi/\mu}(t) = \frac{1}{2}\sum_{k=1}^{n=1}\left([\mu_{(k+1)}(t) - \mu_{(k)}(t)] - [\pi_{(k+1)}(t) - \pi_{(k)}(t)]\right) d\Lambda_{\log\mu_{(k)} - \log\mu_{(k+1)}}(t). \tag{11}$$

This is the difference in impact of the leakage term on the cap weighted subset portfolio and the equal weighted portfolio in our case. As explained above, the equal weighted portfolio is likely to hold more weight in the stocks exiting the index and, therefore, Equation (11) is a negative drag on the equal weighted portfolio's performance relative to the cap weighted portfolio in most cases.

Therefore, rewriting Equation (7), the relative return of the equal weighted portfolio (with weights π) to a cap weighted portfolio (with weights μ) formed on a subset of the market can be given by

$$d\log\left(\hat{Z}_{\pi}(t)/\hat{Z}_{\mu}(t)\right) = d\log\mathbf{S}(\mu(t)) + d\Theta(t) + \int_0^t (\delta_{\pi}(s) - \delta_{\mu}(s))ds + dL_{\pi/\mu}(t), \qquad t \in [0, T], \quad \text{a.s.} \tag{12}$$

3. Empirical performance of the equal weight portfolio

In this section we compare the performance of monthly rebalanced equal and cap weighted portfolios of the S&P500 from 1996 to March 2020. We focus on the S&P500, given, its status as the most liquid equity market in the world and highlight that the S&P500 equal weight index was only launched in 2003. We use data obtained from Bloomberg, including S&P500 constituents and daily price and total return data. Portfolios are rebalanced monthly, with dividends and capital adjustments treated as cash in the portfolio before being reinvested at the next rebalance date. We do not include transaction costs at this stage.

Figure 1 shows the cumulative return (log scale) since 1996 of both the equal and cap weighted portfolios with monthly rebalancing. Notwithstanding the underperformance at the beginning of our sample period, the equal weighted portfolio does indeed outperform the cap weighted portfolio over the entire period.

Table 1 shows the risk adjusted returns over the entire period and, although the equal weighted portfolio has a higher volatility, its higher annualised growth rate leads to higher Sharpe and Sortino ratios than the cap weighted portfolio.

Figure 1. Log cumulative returns for the equal weight and cap weighted portfolios with monthly rebalancing.

Table 1. Risk adjusted performance of equal and cap weighted portfolios, monthly rebalanced.

Portfolio	CAGR[a]	Volatility	Sharpe ratio[b]	Sortino ratio
Equal weighted	9.6%	20.3%	0.449	0.559
Cap weighted	8.2%	19.4%	0.394	0.500

[a] Compound annual growth rate.

[b] One-month US Treasury bill used as risk free rate for both Sharpe and Sortino ratios.

In Figure 2 we show the relative cumulative return over time of the equal weighted portfolio, which visually highlights periods of under- or out-performance over the cap weighted portfolio. It confirms that the equal weighted portfolio outperformed the cap weighted portfolio between 1996 and 2020. However, the majority of this outperformance is generated during the period 2000 to 2008, with the relative performance being largely sideways from 2008 onwards and somewhat negative since 2016.

The relative returns of the equal weighted portfolio on a rolling one-year basis (Figure 3) also confirms that the majority of the outperformance generated by the equal weighted portfolio is due to the earlier parts of our sample period. Relative returns in the latter part of our sample period

Figure 2. Equal weighted portfolio cumulative return relative to cap weighted portfolio with monthly rebalancing.

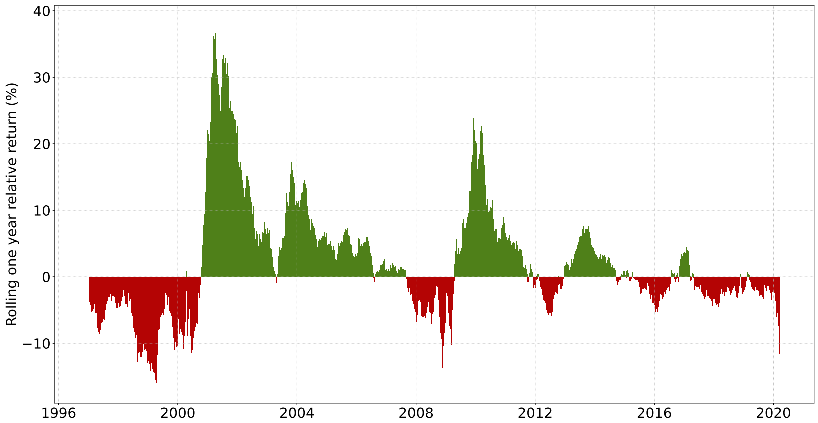

Figure 3. Equal weighted portfolio rolling one-year return relative to cap weighted portfolio with monthly rebalancing.

Table 2. Total return over selected five year periods.

Period	Equal Weight	Cap Weight	Compounded Difference
2000 - 2005	53.4%	-10.0%	71.9%
2005 - 2010	13.6%	3.4%	13.3%
2010 - 2015	125.3%	104.9%	11.1%
2015 - 2020	60.7%	74.5%	-7.7%

have been rather lacklustre with the equal weighted portfolio underperforming for the most part since 2016.

Although the equal weighted portfolio appears to outperform the cap weighted portfolio over the whole period, most of the outperformance has been due to the 2000 to 2012 period. This is also evident from Table 2, which shows five-year total returns for the equal and cap weighted portfolios since 2000. Relative returns are the highest in the early 2000s and degrade slightly for later five-year periods. In particular, the last five years have been especially tough for the equal weighted portfolio, which has underperformed by 14% from 2015 to 2020.

Following the theoretical results in the previous section, and in particular given Equation (12), the main components of the relative performance consist of changes in the portfolio generating function, the equal weighted portfolio's drift process and the net impact of leakage. The drift process, in the case of the equal weighted portfolio, is the excess growth rate given by Equation

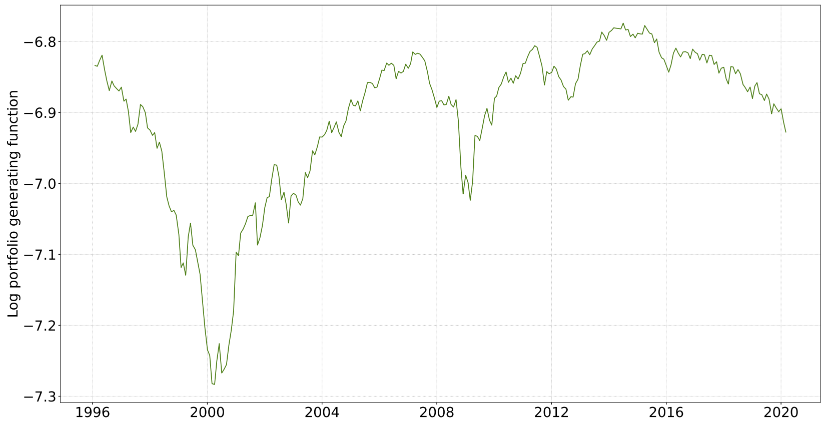

Figure 4. Log portfolio generating function for the S&P500 equal weighted portfolio as per Equation (8).

(3) and is intuitively a measure of the benefits of diversification.

On the other hand, per Equation (12), decreases in the portfolio generating function detracts from the relative performance of the equal weighted portfolio and vice versa. In the case of the equal weighted portfolio, the portfolio generating function, Equation (8), is also a measure of the concentration of the cap weighted portfolio. Therefore, as the cap weighted portfolio becomes more concentrated, the equal weighted portfolio is bound to underperform unless the drift process, namely the excess growth rate per Equation (9), can offset this.

Therefore, in order to understand the relative performance of the equal weight portfolio, we analyse the excess growth rate and portfolio generating function over the period 1996 to 2020.

Figure 4 shows how the log portfolio generating function for the equal weighted portfolio increases during the period 2000 to 2008, in the same period the equal weighted portfolio generates the majority of its outperformance, and how, more recently, the portfolio generating function has been declining. That is, the cap weighted portfolio has increasingly become more concentrated. This has a negative impact on the equal weighted portfolio that relies in part on some level of mean reversion.

We show the excess growth rate over the period 1996 to 2020 in Figure 5. We estimate the covariance matrix required to calculate the excess growth rate using the historical three-month covariance matrix at each point in time.

Figure 5 highlights another reason for the underperformance in recent years. The excess growth rate has declined substantially from an average of between 5% and 10% in the first half of our sample period to between 2% and 3% in recent years. In fact, while the portfolio generating function has declined substantially since 2014, the excess growth rate has remained low. In terms of Equation (6), this would lead to underperformance of the equal weighted portfolio relative to the cap weighted portfolio.

The decline in - and low levels of - the excess growth rate in recent years is also interesting from a diversification point of view. Given Equation (3), the excess growth rate would be low when either average stock volatility is low (so that diversification is less beneficial) or correlations are high (so that there is no way to achieve higher diversification). Figure 6 highlights that average volatilities have remained reasonably low in recent years, at least compared to the periods 2000 to 2003 and 2008 to 2012. Similarly, average correlations seem to have been higher on average for the period 2009 to 2020 as compared to the earlier part of our sample period (1996 to 2004). The impact of both of these findings would lead to lower excess growth rates for the equal weighted portfolio.

Finally, as we discussed in Section 2.3, given that we are operating on a subset of the entire US equity market, we also have to consider the impact of leakage. In other words, the impact of stocks exiting the S&P500. Figure 7 highlights the cumulative impact of leakage on the equal weighted portfolio's relative performance. As expected, this term is a consistent drag on the equal weighted

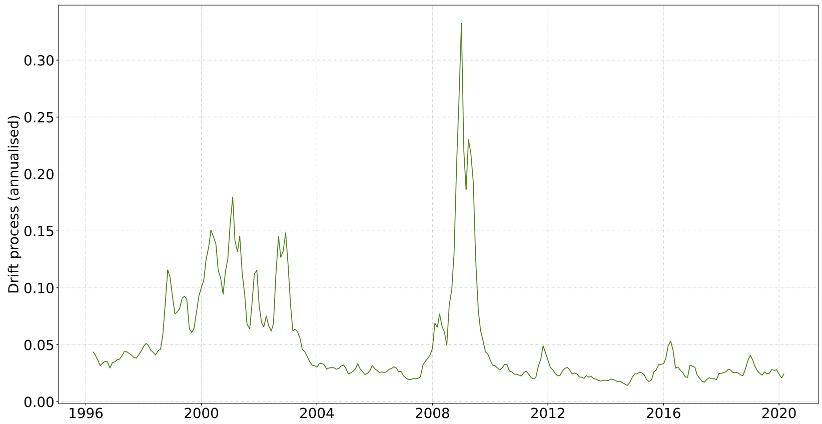

Figure 5. Excess growth for the S&P500 equal weighted portfolio as per Equation (3), which also represents the drift process of the equal weight portfolio per Equation (9).

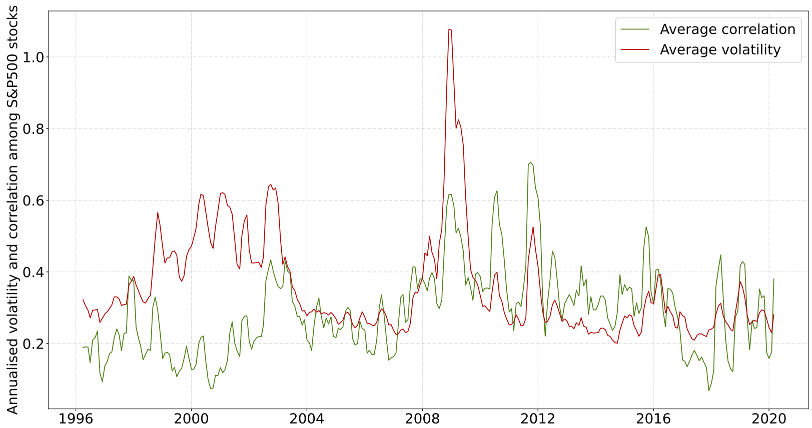

Figure 6. Average volatilities and correlations for the stocks in the S&P500 index. These components make up the excess growth rate in Figure 5 and Equation (3).

portfolio, of approximately 1.8% per annum since 1996. The size of the impact is proportionate to the number of changes in constituents, which we depict in Figure 8 on a rolling one-year basis. In Figure 7, for example, we can see that the impact of leakage on the equal weighted portfolio slowed between 2003 and 2006, which corresponds to an unusually low number of index changes over the same period. More recently, the number of index changes has been more consistent, between 25 and 35, on an annual basis.

In Figure 9 we combine all the theoretical components addressed above into a single view of the decomposition of the rolling one-year relative returns of the equal weighted portfolio. Table 3 highlights the impact of these factors over five-year windows.

Most of the annual variability in returns is attributable to changes in the portfolio generating function highlighting the importance of monitoring this term to improve short-term performance. The figures in Table 3 show how the portfolio generating function contributed significantly to the outperformance of the equal weighted portfolio in the early 2000s but has been a drag on performance in the recent years.

The leakage impact is also a small drag on performance and oscillates as the number of stocks entering and exiting the S&P500 changes. It appears from Table 3 that the leakage impact has grown. However, Figure 8, which shows the changes in the index constituents over a rolling one-year period, highlights how in 2000 to 2005 the number of changes in the index was relatively low. This

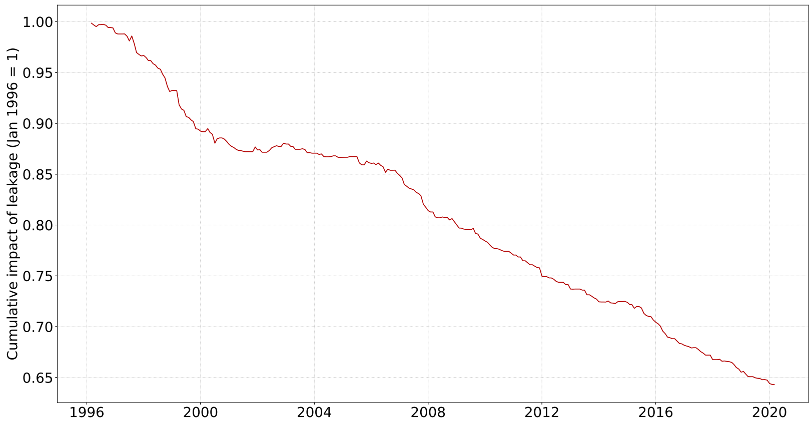

Figure 7. Cumulative impact of leakage on the equal weight portfolio relative to the cap weighted portfolio per Equation (11).

Figure 8. Rolling one year changes in index constituents for the S&P500

Table 3. Decomposition of equal weighted portfolio relative return over selected periods

Period	Excess growth rate	Port. gen. function	Leakage	Dividends	Costs	Residual	Total
2000 - 2005	48.3%	40.2%	-2.9%	-0.1%	-0.3%	-16.8%	68.0%
2005 - 2010	36.3%	-0.8%	-9.5%	-0.9%	-0.3%	-10.3%	8.8%
2010 - 2015	12.9%	9.3%	-7.7%	-0.8%	-0.2%	-2.9%	9.8%
2015 - 2020	14.4%	-10.4%	-11.0%	0.4%	-0.2%	0.4%	-8.1%

led to a smaller than usual leakage. This is similar to the period 2010 to 2015 where the number of changes was smaller than that of the period 2015 to 2020. As a result leakage for 2015 to 2020 is higher at -11% in contrast to -7.7% for the period 2010 to 2015.

Both Figure 8 and Table 3 highlight the decline in the contribution of the excess growth rate from the early 2000s to more recently.

We also note a residual effect when the theoretical decomposition is compared to the empirical returns. This residual tends to be correlated to the excess growth rate, increasing when the contribution of the excess growth rate is large and vice versa. This leads us to believe that the residual is a consequence of errors and noise in our covariance matrix estimation. Improving the covariance matrix estimation will likely lead to an improved estimate of the excess growth rate and a smaller residual effect.

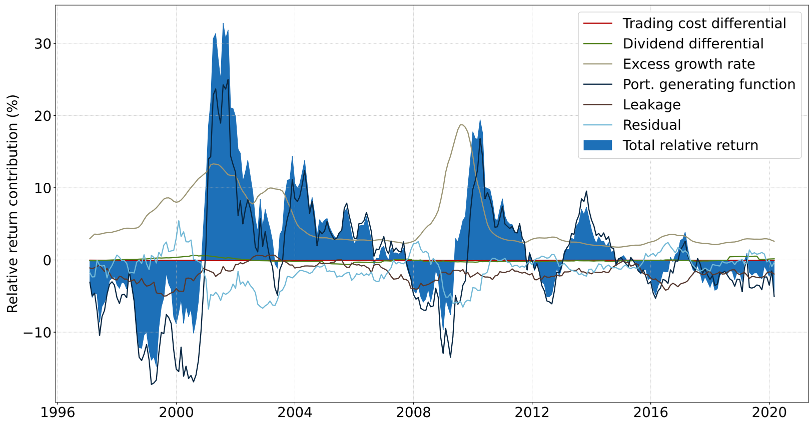

Figure 9. Annual decomposition of equal weighted portfolio relative return including dividends, Equation (12), and transaction cost differentials.

3.1. *Impact of technology stocks in the S&P500 and its impact on concentration*

In an aside, we take a look at how removing the major technology stocks would impact the portfolio generating function of the equal weighted portfolio. We focus on the stocks commonly referred to as the FAANG stocks or more recently expanded version FAAANM: Facebook, Alphabet (Google), Amazon, Apple, Netflix, and Microsoft.

To gauge the impact on the equal weighted portfolio's relative return, we remove these stocks and redistribute their cumulative weight proportionate to the remaining market capitalisation weights. We then recalculate the portfolio generating function as per Equation (8) and compare this to the actual portfolio generating function. Any difference in the two would correspond to a difference in the relative performance of the equal weighted portfolio through Equation (6).

Figure 10 shows the impact of removing these stocks on a rolling five-year basis. The improvement in the portfolio generating function becomes more significant from 2010 onwards, moderating between 2013 and 2017. More recently, the improvement in the portfolio generating function is approximately 8% over the past five years. If we compare this to Table 3, we see that this effect has accounted for a large portion of the negative contribution from the portfolio generating function over the last five years (-10.4%). To be clear, we are not arguing that technology stocks should be removed from the index, but rather highlighting an interesting application of stochastic portfolio theory to understand their impact on the equal weighted portfolio in recent years.

4. Optimising the equal weighted portfolio

As mentioned in Section 2, Equation (8), the portfolio generating function of the equal weighted portfolio, implies that $\log \mathbf{S}(\mu(t))$ is bounded, assuming that the stock market does not become concentrated in a single stock. Furthermore, the drift process of the equal weight portfolio, represented by the excess growth rate in Equation (9), is always greater than, or equal to, zero. As a result, Equation (6), the relative return process of the equal weighted portfolio, implies that the equal weighted portfolio will almost surely outperform the cap weighted portfolio in the long-run.

This does appear to be the case, given our results in Section 3. However, the stock market can, in theory and practice, continue to become increasingly concentrated over the short-term. In Figure 4, these periods of increasing concentration seem to span years. As a result, Equation (8) can become a consistent drag on the relative performance and impact an investor's ability and willingness to hold the equal weight portfolio in favour of a cap weighted index.

In this section we look to negate some of these periods of steep drawdowns, relative to the cap weighted portfolio. Our aim is to create as simple a model as possible and in turn highlight that

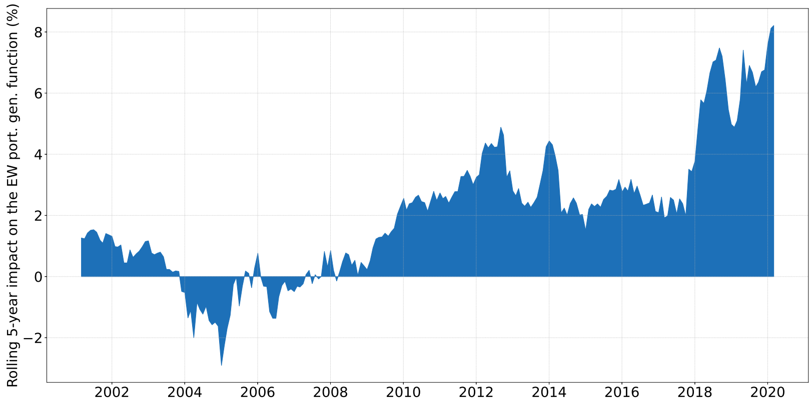

Figure 10. Improvement in the portfolio generating function of the equal weighted portfolio (over a rolling five year basis) when removing the stocks: Facebook, Alphabet (Google), Amazon, Apple, Netflix, and Microsoft and redistributing their weight proportionately among the remaining stocks.

monitoring the portfolio generating function, as given by Equation (8), the excess growth rate, and the impact of leakage, Equation (11), are important when improving the short-term performance of the equal weighted portfolio relative to a cap weighted index. While this rudimentary approach appears to perform adequately in reducing relative drawdown, we highlight some areas of improvement at the end of the section.

4.1. *A rudimentary linear regression approach*

Our rudimentary approach is to make use of the portfolio generating function, the excess growth rate and leakage to forecast what the relative performance of the equal weight will be in the following month and switching between the cap weight and equal weighted portfolios. We do this since, in the previous sections we showed how, theoretically and empirically, the relative return of an equal weighted portfolio is a combination of the changes in concentration of the cap weighted portfolio weights (the portfolio generating function), the excess growth rate (heuristically the benefits of diversification of the equal weighted portfolio) and the rate of leakage as stocks move out of the S&P500. Therefore, we focus our attempts here to optimise the equal weighted portfolio on these three components.

We make use of a rudimentary linear regression model that looks to forecast the next month's relative performance using the average monthly change in the log portfolio generating function, Equation (8), over the past three months, the most recent estimate of the drift process, and the average of the last three month's leakage as defined in Equation (11). In this case the drift process is the excess growth rate of the equal weighted portfolio, Equation (3), where we estimate volatilities and correlations over the prior three months. We fit this model on the prior three years' of data and attempt to forecast the next month's relative return.

Mathematically this would be expressed as

$$\hat{y}_{t+1} = \beta_0 + \beta_1 G_t, \qquad t = 3, \ldots, n$$

where $\hat{y}_{t+1}$ is the next month's relative return of the equal weighted portfolio and G_t is given by

$$G_t = \frac{1}{12}\gamma^*_\pi(t) + \frac{1}{3}\sum_{i=t-2}^{t} d\log S(\mu(i))) + \frac{1}{3}\sum_{i=t-2}^{t} dL_{\pi/\mu}(i),$$

where $d\log\mathbf{S}(\mu(i))$ is the one-month change in the log of the portfolio generating function and $dL_{\pi/\mu}(i)$ represents the net leakage effect, Equation (11), for the equal weighted portfolio at time $t = i$ with μ representing the cap weights of the S&P500 constituents.

The intercept β_0 and coefficient β_1 are fit using an ordinary least squares approach. The model is fit over the preceding three years' monthly data at the beginning of each month prior to any monthly rebalancing in the portfolios.

Since the dynamic portfolio will be shifting between equal and cap weighted, turnover is likely to increase and we, therefore, include trading costs of 15 basis points in our analysis on all portfolios. This is roughly in line with analysis done by Frazzini *et al.* (2018) for stocks over the period 1998 to 2016.

As the analysis in this section includes transaction costs of 15 basis points for all portfolios, we only switch from equal weights to cap weights (or vice versa) if the relative return is predicted to, at least, offset these transaction costs.

Practically, to generate the equal and cap weights for the dynamic portfolio, we make use of the diversity weighted portfolio (see Fernholz 2002). This portfolio has the following generating function

$$D_p(\mu) = \left(\sum_{i=1}^{n} \mu_i^p\right)^{\frac{1}{p}}. \tag{13}$$

This generates weights given by

$$\pi_i(t) = \frac{\mu_i^p(t)}{\sum_{i=1}^{n} \mu_i^p(t)}. \tag{14}$$

As $p \to 0$ the portfolio's weights tend to the equal weight portfolio and as $p \to 1$, the portfolio's weights tend to the cap weighted portfolio. There have been attempts to directly optimise this value p (see, for example, Samo and Vervuurt 2016); although, this tends to result in portfolios with $p < 0$. This leads to an inverse cap weighted portfolio where smaller stocks have much larger weights than the largest stocks in the market.

In our analysis we restrict p to either very close to 0, or equal to 1, to select directly between either the equal weighted or cap weighted portfolios.

4.2. *Results*

We highlight the main results in Table 4. The optimised portfolio seems to outperform the equal weighted portfolio by approximately 60 basis points per annum, on average, with lower volatility and higher Sharpe and Sortino ratios than the equal weighted portfolio. There also appears to be a marked improvement in the information ratio of 0.517 for the optimal portfolio over the equal weighted portfolio's 0.265.

We perform a statistical test for the significance of the difference in the Sharpe ratios of the equal weighted and optimal portfolios relative to the cap weighted portfolio in Table 4. The method used is that of Ledoit and Wolf (2008), which makes use of a studentised circular bootstrap approach to construct a confidence interval at a given significance level. This test can be altered to provide a p-value for the null hypothesis. The hypothesis test is a two-sided test with the null hypothesis as H_0 : the difference in Sharpe ratios is zero. We perform the same test for the information ratios.

We note that in both cases the p-values are large, however, it is worth bearing in mind that the optimal portfolio is a combination of the equal weight and cap weighted portfolio. Therefore, the bootstrapping approach would cover many areas where both the optimal portfolio and cap weighted portfolio have the same return series. We show the proportion of time spent in each

Table 4. Risk adjusted performance of optimised, equal, and cap weighted portfolios, monthly rebalanced with 15bps of costs.

Portfolio	CAGR[a]	Volatility	Sharpe ratio[b]	Information ratio[c]	Sortino ratio
Optimised	10.1%	19.8%	0.476 (0.18)	0.517 (0.15)	0.606
Equal weighted	9.5%	20.3%	0.442 (0.89)	0.265 (–)	0.551
Cap weighted	8.2%	19.4%	0.392 (–)	–	0.498

[a] Compound annual growth rate.

[b]One-month US Treasury bill used as risk free rate for both Sharpe and Sortino ratios. P-value for two sided hypothesis test as per Ledoit and Wolf (2008) shown in brackets. The null hypothesis is that the Sharpe ratio is equal to that of the cap weighted portfolio.

[c]Information ratio relative to the cap weighted portfolio. P-value for two sided hypothesis test as per Ledoit and Wolf (2008) shown in brackets. The null hypothesis is that the IR is equal to that of the equal weighted portfolio.

specific portfolio weighting scheme in Figure 11. The optimal portfolio spends approximately two-thirds of the time within an equal weighting and reverts to cap weighting the remainder of the time. In this context we would argue the p-values are not too extreme considering the proportion of time the optimal portfolio spends in each weighting methodology.

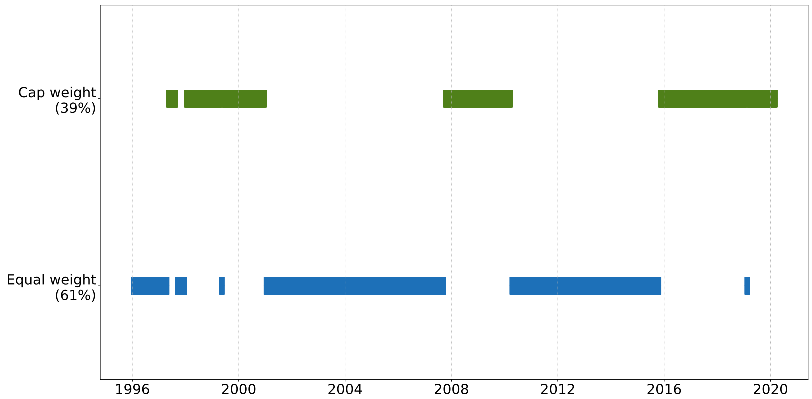

Figure 11. Allocation of optimal portfolio between cap weighted and equal weighted portfolios. Proportion of overall time allocated to a specific portfolio is shown in brackets.

The overall outperformance of the optimised portfolio is largely as a result of avoiding some of the steep drawdowns relative to the cap weighted portfolio. In Table 5 we show the maximum relative return drawdowns of the optimised and equal weighted portfolios (relative to the cap weighted portfolio). This is reduced from a maximum underperformance of 30.7% for the equal weighted portfolio to 11.3% for the optimised portfolio. Furthermore, we show the alpha (over the cap weighted portfolio) per unit of average relative drawdown. The optimised portfolio's alpha per unit of drawdown is over two times higher than that of the equal weighted portfolio, given its higher absolute return and lower relative drawdowns.

This improvement is most evident in a chart of the cumulative returns of the optimised and equal weighted portfolios (Figure 12). Here we see the optimised portfolio does well to switch into the cap weighted portfolio during the three main drawdown periods: around the 2000s, post-2008, and more recently post-2016. We have selected these three periods and shown the improved CAGR separately in Table 6. These periods appear to include the months just prior to a crisis as well as the crises themselves.

However, where the model does underperform is each period's recovery following a large relative drawdown. For example, the equal weighted portfolio outperforms the cap weighted portfolio sig-

Table 5. Relative return drawdowns of optimised and equal weighted portfolios.

Portfolio	Maximum relative drawdown[a]	Alpha / Avg DD[b]
Optimised	11.3%	10.85
Equal weighted	30.7%	5.04

[a] Maximum relative return drawdown.

[b]Alpha (relative to cap weighted portfolio) per unit of average relative return drawdown.

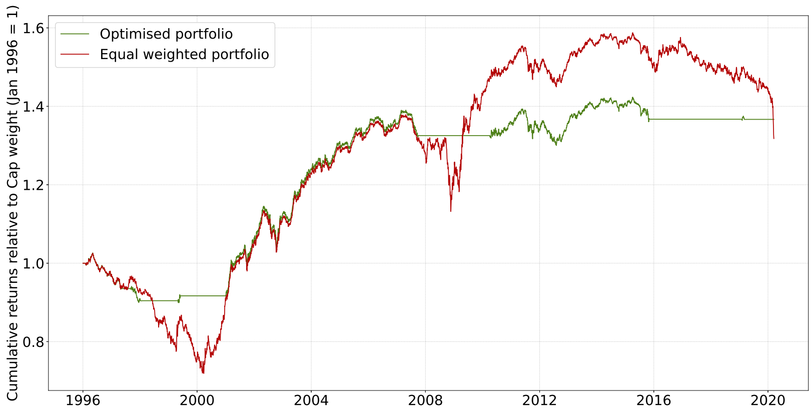

Figure 12. Optimised and equal weighted portfolio cumulative returns relative to cap weighted portfolio with monthly rebalancing.

Table 6. CAGR of portfolios over the three main periods of underperformance.[a]

Portfolio	Jan '96 - Jan '01	July '07 - Jan '09	May '05 - March '20
Optimised	17.8%	-33.4%	4.7%
Equal weighted	16.2%	-39.3%	1.7%
Cap weighted	18.5%	-31.4%	5.6%

[a] Rebalanced monthly, 15bps in transaction costs

nificantly starting in 2000, but the optimised portfolio only switches to equal weight much later, thereby missing out on at least a year's outperformance. That said, given the simplicity of the model, the optimised portfolio appears to do well to generate a higher return than the equal weighted portfolio while avoiding large relative drawdowns.

4.3. *Other countries*

Although beyond the scope of this article, we very briefly touch on performance in some other countries in Table 7. In all the countries highlighted in Table 7, the equal weighted portfolio outperforms the cap weighted portfolio. We note that while the equal weighted portfolio seems to have good performance in most countries, relative drawdowns to the cap weighted portfolio can still be high even in countries with excellent relative performance overall. In this sense, the optimised approach does improve the relative drawdown experience but it does so in some cases by giving up some return relative to the equal weighted portfolio. In the case of Japan, for example, where the equal weighted portfolio does very well relative to the cap weighted portfolio.

Given the simplicity of the model and different factors influencing various markets, a comprehensive treatment of the differences in the equal weighted and cap weighted portfolios within different

Table 7. Performance of equal weighted (EW), cap weighted (MC) and optimal portfolios in other countries over the last 20 years

Country	CAGR			Sharpe Ratio			Max. Relative Drawdowns	
	MC	EW	Optimal	MC	EW	Optimal	EW	Optimal
UK	3.1%	4.8%	4.4%	0.13	0.22	0.20	-26.7%	-17.1%
Germany	1.0%	3.7%	3.6%	0.11	0.22	0.21	-57.8%	-23.2%
France	2.6%	3.2%	3.2%	0.18	0.21	0.21	-24.5%	-10.9%
Canada	6.6%	7.1%	7.1%	0.35	0.37	0.37	-19.0%	-11.6%
Australia	9.1%	9.3%	7.9%	0.38	0.37	0.31	-39.8%	-35.0%
Japan	1.1%	7.1%	6.6%	0.16	0.44	0.41	-52.1%	-38.2%

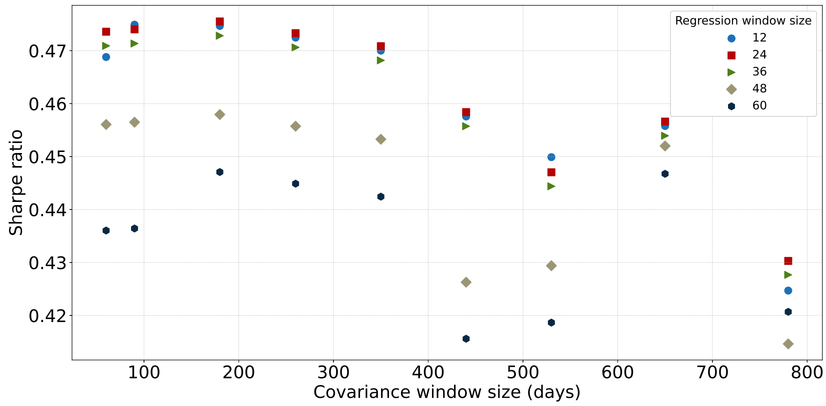

Figure 13. Sharpe ratio for optimal portfolio using different lags for linear regression and covariance estimate.

countries is likely to add significantly to the literature and we highlight this as one area of further research.

4.4. *Robustness and future areas of improvement*

Although the model is reasonably simple, there are a few moving parts. In particular, we are using historical estimates of the covariance matrix over the past three months and the trend of the portfolio generating function over the prior three months to estimate the next month's relative return. This is done using a linear regression fit over the prior three years' monthly data.

In Figure 13, we compare the Sharpe ratio of the optimal portfolio across various covariance estimation windows and various look-back periods on which the linear regression model was fit. There appears to be a clear trend along the covariance window size axis, with Sharpe ratios declining as the window size is increased regardless of the regression window size used. The optimal area appears to be less than 300 trading days of data used.

The results are also fairly consistent along the linear regression look-back periods with 12-, 24- and 36-month periods producing similar Sharpe ratios regardless of the covariance window size. Look-back periods of 48 and 60 months perform consistently worse.

A key area of improvement is the estimation of the covariance matrix, which is used in the estimate of the excess growth rate. In our model we have used a very basic approach and techniques such as Random Matrix Theory, the shrinkage approach and clustering techniques could improve the accuracy of the excess growth rate estimate. We point readers to Pantaleo *et al.* (2011) who compare the performance of nine covariance estimation techniques and compare them to the sample covariance matrix (the approach used here).

Secondly, an improved forecast of the portfolio generating function, essentially the level of concentration, would improve the model further. Although likely the more difficult of the two areas of improvement mentioned here, the level of concentration itself may be predictable (Figure 4) using various time series approaches or alternatively modelling the distribution of capital in a specific market using the ranked weights. See, for example, Fernholz (2001b) and Fernholz (2001a).

5. Conclusion

While the equal weighted portfolio appears to outperform the cap weighted portfolio over a long time horizon, it can suffer from significant periods of underperformance over the short-term. We have attempted to analyse this behaviour using findings from stochastic portfolio theory. In particular, we have shown more recently the equal weighted portfolio's underperformance seems to have coincided with the cap weighted portfolio's weights becoming increasingly concentrated, while at the same time the benefits of diversification from the equal weighted portfolio have been declining.

This concurs with the theoretical constructs of stochastic portfolio theory, specifically the theoretical drivers of the return of the equal weighted portfolio relative to the cap weighted portfolio. We have further attempted to improve the short-term underperformance of the equal weighted portfolio by using a rudimentary linear regression model with inputs given by the drivers identified in stochastic portfolio theory; the change in the log portfolio generating function; the excess growth rate of the equal weighted portfolio; and the impact of leakage as index constituents change. This rudimentary model appears to have significantly improved the risk return characteristics of the equal weighted portfolio by avoiding large relative return drawdowns while still maintaining longer-term relative outperformance over the cap weighted portfolio.

There are a number of possible future research areas. In the short-term, the change in the log portfolio generating function is likely to drive most of the performance. Forecasting this change more accurately for use in our rudimentary model or, even, by using either more complicated features or models to forecast the relative return itself, might be possible. Furthermore, modelling the capital distribution curve, another topic in stochastic portfolio theory that models the log weights relative to the log of the positions of stocks, might also be useful in predicting changes in the log portfolio generating function. Finally, it might be useful to attempt to forecast the covariance matrix to better estimate the future excess growth rate of the equal weighted portfolio.

Acknowledgements

The authors wish to express their appreciation for comments and suggestions received from two anonymous referees which aided considerably in improving the article.

References

Bolognesi, E., Torluccio, G. and Zuccheri, A., A comparison between capitalization-weighted and equally weighted indexes in the European equity market. *Journal of Asset Management*, 2013, **14**, 14–26.

Booth, D.G. and Fama, E.F., Diversification returns and asset contributions. *Financial Analysts Journal*, 1992, **48**, 26–32.

Bouchey, P., Nemtchinov, V., Paulsen, A. and Stein, D.M., Volatility harvesting: Why does diversifying and rebalancing create portfolio growth?. *The Journal of Wealth Management*, 2012, **15**, 26–35.

Bouchey, P., Nemtchinov, V. and Wong, T.K.L., Volatility harvesting in theory and practice. *The Journal of Wealth Management*, 2015, **18**, 89–100.

Cuthbertson, K., Hayley, S., Motson, N. and Nitzsche, D., What Does Rebalancing Really Achieve?. *International Journal of Finance & Economics*, 2016, **21**, 224–240.

DeMiguel, V., Garlappi, L. and Uppal, R., Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy?. *The review of Financial studies*, 2009, **22**, 1915–1953.

Fernholz, B.R., Stable models for the distribution of equity capital. , 2001a.
Fernholz, R., On the diversity of equity markets. *Journal of Mathematical Economics*, 1999, **31**, 393–417.
Fernholz, R., Equity portfolios generated by functions of ranked market weights. *Finance and Stochastics*, 2001b, **5**, 469–486.
Fernholz, R., *Stochastic portfolio theory*, 2002, Springer.
Fernholz, R. and Karatzas, I., Stochastic portfolio theory: an overview. *Handbook of numerical analysis*, 2009, **15**, 89–167.
Fernholz, R. and Shay, B., Stochastic portfolio theory and stock market equilibrium. *The Journal of Finance*, 1982, **37**, 615–624.
Fernholz, R. and Square, O.P., Arbitrage in equity markets. Technical report, Working Paper, Intech, Princeton, 1998.
Frazzini, A., Israel, R. and Moskowitz, T.J., Trading costs. *Available at SSRN 3229719*, 2018.
Hallerbach, W.G., Disentangling rebalancing return. *Journal of Asset Management*, 2014, **15**, 301–316.
Karatzas, I., A survey of stochastic portfolio theory. *The Eugene Lukacs Lectures, Bowling Green University*, 2006.
Ledoit, O. and Wolf, M., Robust performance hypothesis testing with the Sharpe ratio. *Journal of Empirical Finance*, 2008, **15**, 850–859.
Malladi, R. and Fabozzi, F.J., Equal-weighted strategy: Why it outperforms value-weighted strategies? Theory and evidence. *Journal of Asset Management*, 2017, **18**, 188–208.
Pantaleo, E., Tumminello, M., Lillo, F. and Mantegna, R.N., When do improved covariance matrix estimators enhance portfolio optimization? An empirical comparative study of nine estimators. *Quantitative Finance*, 2011, **11**, 1067–1080.
Plyakha, Y., Uppal, R. and Vilkov, G., Why does an equal-weighted portfolio outperform value-and price-weighted portfolios?. *Available at SSRN 2724535*, 2012.
Samo, Y.L.K. and Vervuurt, A., Stochastic portfolio theory: a machine learning perspective. *arXiv preprint arXiv:1605.02654*, 2016.
Taljaard, B. and Maré, E., Too Much Rebalancing Is Not a Good Thing. *Available at SSRN 3484337*, 2019.