A lot sizing model for a Multi-State System with deteriorating items, variable production rate and imperfect quality

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Abstract

The management and control of inventory has become a core part of management, which plays a significant role through achieving efficient and profitable operations of a business organization. Hence, considerable efforts have been made to develop models that can be implemented to optimize inventory systems without compromising customer needs. The classic Economic Production Quantity (EPQ) model is the most widely used of these models; however, this model presents certain limitations, leading researchers to extend some of the assumptions to increase its applicability to present-day organizations. In manufacturing, studies of the functional state of equipment have, for a long time, been based on binary modelling conditions where two states were considered: the operational state and the complete failure state. However, a growing literature takes into consideration the numerous scenarios that may occur during the lifetime of some equipment. Such systems are called Multi-State Systems (MSS). Thus, in this dissertation, a perishable replenishment policy is developed based on the MSS concept to optimize a EPQ model that operates in a degraded state, producing both perfect and imperfect products, under constant demand and backlog dependent-demand. The cycle was assumed to start with a particular production rate until a point when the inventory reached a certain level, and after which the failure mode was activated due to the deterioration of certain components, and the production rate was reduced to a lower rate to ensure the continuity of supply, until the maximum inventory level was reached. Production then stopped to restore the machine and the cycle started again. The model assumed that inventory was subject to deterioration, the demand rate was constant, and partial backlogging was allowed. The work done included an exploration of the modelling methods, analysis and evaluation of the performance of the multi-state system in which the level of service relies on the state of the equipment during the production cycle. An evaluation and optimisation of the system' performance indicators such as inventory levels, backorder level, cycle time and the total cost function were carried out. Due to model complexity, the Newton-Raphson approach was used to solve the model and numerical examples are provided to illustrate the solution procedure. Based on the results, the presence of imperfect quality outputs forced the system to produce more items to meet the needs for perfect quality items. As the proportion of imperfect quality items produced increased, the proportional increase in cost seems to have grown more quickly. As the production rate in the first production-consumption cycle increased, the total cost function increased; this was mainly due to higher production cost, holding and disposal costs incurred. However, as the inventory holding cost rate increased, the optimal inventory levels decreased, the cycle time decreased, but the shortage and the total cost increased. The decrease in production rate during the second production-consumption cycle was shown to have increased the cycle time and the inventory level in the first cycle, but decreased the inventory level in the second cycle and the total cost. Sensitivity analysis showed that working with low values of cost parameters provided better results in terms of optimizing the total cost.

The EPQ model presented in this research can be used by production managers, working in industries such as assembly lines, steel factory, hydrometallurgical plants under different operational scenarios, as a guideline when making production decisions.

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Acronyms

AC	Deteriorating cost
ADF	arbitrary probability distribution function
CT	Total cost
CB	Completely Backorder
DC	Disposal Cost
EOQ	Economic Order Quantity
EHC	Equivalent Holding Cost
EPQ	Economic Production Quantity
HJB	Hamilton-Jacobi-Bellman
H_M	Hessian Matrix
ICC	Inventory Carrying Cost
LC	Lost Sale Cost
MOGA	Multi-Objective Genetic Algorithm
MSS	Multi-State System
MPS	Multi Production System
NIKS	Number of Items Kept in Stock
PB	Partial Backorder
PC	Production Cost
PDFs	probability distribution functions
QB_1	Quantity Backordered

PS	production system
SUC	Setup Cost
SC	Shortage Cost
SCM	Supply Chain Management
WIP	Work-in-process

Chapter 1

Introduction

1.1 Background

Every organization holds inventory. According to Chase et al (2007), inventory represents the quantity of a given resource or item utilized by an organization. In other words, inventory is a quantity of material resources owned and stored by a business. For an organization to grow and survive, inventories play a vital role because excess inventory may arise as a result of poor management of inventory level. This in turn leads to an upsurge in the cost of inventory. In addition, inventory shortage may lead to poor service levels and customer dissatisfaction. Without inventories, most operations in an organization are impossible. It is thus critical to have the right levels of inventory because it allows operations to be more productive and efficient. Systems of managing inventories are essential in helping businesses with the minimization of costs incurred as well as maximization of profits. This is achieved by ensuring that customer demands are met mainly by delivering the correct quantity and quality, of goods at the right time and place.

Poor inventory management affects product availability, delivery lead time, organization's performance, customer satisfaction, service level and the perceived product value. They affect the operating cost, and consequently, the return on investment, profit and assets. Without proper inventory management, organizations may not function properly. It, therefore, becomes vital to manage the inventory that must be held. Inventory management affects multiple departments such as sales, marketing, procurement, production and finance within an organization and across the entire supply chain. Inventory management is a well-studied area that has been researched for over a century. Clodfelter (2010) argued that effective inventory management benefit businesses in: 1) delivering good services; 2) optimizing the investment level required for proper inventory planning and allocation; 3) earning discounts on trade procurements; 4) ensuring the procurement and storage of materials that meet the required product specifications; and 5) efficient management of production schedules. Nevertheless, effective inventory management presents some challenges for businesses. Many business organizations face conflicting objectives of improving customer service by preventing under-stocking, which may result in lost orders and lost sales, and minimizing the costs of manufacturing finished goods. Therefore, businesses should meticulously manage their inventories to balance these conflicting objectives and achieve the best trade-off option between them in terms of optimality.

Another issue in the control of inventory is the cost associated with the deteriorating nature of stocked items. Wee (1993) defines deterioration or spoilage as any process that inhibits the usage of an item for its intended original use. Most inventory models assume implicitly that stored items have infinite shelf life, meaning that stocked products remain unchanged and fully usable for future demand satisfaction. The effect of the deterioration rate can be ignored if it is low and negligible; however, in most cases the effect of deterioration rate plays a significant role and its impact must be considered explicitly.

Manufacturing organizations face significant challenges such as, availability of equipment and other resources, flexibility of manufacturing systems, reliability of the processes, the quality of the output products, and the integration of new products and services into the existing production process. The optimal policy of production to satisfy a certain demand is determined using Economic Production Quantity (EPQ) models. The condition of a fixed rate of production is used while developing a classical EPQ model. However, production rate has been determined to be dependent on the type of process. Production processes with small production runs are less complex and have a lower setup cost requirement compared to long runs with several units of operations (Mukhopadhyay and Goswami, 2013). This means deterioration is more likely to occur in production processes with long production runs. Observations of production processes deterioration have been made in many industrial organizations such as plastic industry, assembly lines, steel manufacturing and food processing. The problem that arises is determining the production plan to satisfy the demand while minimising the system's total cost in a manufacturing system characterised by breakdowns, and degradation of the machines. A fundamental characteristic of a production system is that it behaves like a chain, in which each link has an impact on the rest of the chain; in other words, machines in a manufacturing system can enhance the production policy of a given system and deliver services more promptly or have the opposite impact on the system. That is, while developing EPQ models for inventory systems, the state of equipment should not be neglected. For example, any shortage of raw materials or machine breakdown in production could have repercussions on the end product.

The dependency between production run and the type of production process is also effected by the quality of the production output. Significant studies have been conducted on inventory models, lifting the assumption that only perfect quality items are produced within an organization. Various research have extended the model to represent different and more realistic inventory systems. Rosenblatt and Lee (1986) presented the first work dealing with inventory management with imperfect quality items in a production system, and there still seems to be opportunities for further research in this area. The effects of integrating quality control into the Economic Order Quantity (EOQ) model were studied by Porteus (1986). Schwaller (1998) presented a model that assumes the presence of imperfect quality items in a known proportion. His study further considered the application of fixed and variable inspection costs when finding and removing the item. Inspection policy and joint lot sizing were considered by Zhang and Gerchak (1990) in an EOQ model with random yield. Chiu et al. (2011) proposed a numerical method for determination of the optimal Lot Size for a manufacturing system with discontinuous issuing policy and rework. Cheng (1991) proposed an EOQ model with demand-dependent unit production cost and imperfect production processes. Chiu (2003) generalized the model in Hayek and Salameh's (2001) model by considering a production process with random defective rate where the defective items are reworked and unsatisfied demand is backlogged. El-Kassar et al. (2007) considered continuous demand of perfect and imperfect items in production systems.

A good production policy is crucial for optimising any system. This requires reliable equipment, good quality of finished products, and a production planning which considers contingencies and adequate inventory management. This study will extend the basic EPQ model by relaxing three implicit assumptions made in Bhowmick and Samantha (2011). The first assumption is that production systems are deterioration free and that performance indicators such as production rates, inventories levels, cycle time, backorder level are independent of production process state. They therefore, assumed that the equipment of the manufacturing system could operate efficiently, without breakdown over a finite planning horizon.

The study of manufacturing systems has long been based on binary modelling where a system is either perfectly operational or in complete failure state (out of service). Although such modelling has, in practice, numerous practical applications, it is still not adequate to capture many situations that may arise in real-life and that may affect their performance. A growing literature considers numerous cases that may occur during the lifetime of some systems, Multi-State System (MSS) is the term associated with such systems. MSS's are usually subject to many different types of changes with different impacts on the systems' performances. Deteriorated or degraded state is considered as one of the states; it allows the MSS to continue to perform its service with a decrease in the performance caused by the breakdowns.

The second assumption relaxed is that all the finished products are of acceptable quality. Realistically, it is not always possible in any production process to produce all items of perfect quality. The production of imperfect quality items is a natural occurrence in production process. Hence, it is very unlikely that perfect quality will be achieved for all the items produced from a production process. The third assumption relaxed is that shortages are fully backlogged. Sometimes customers may not be willing to wait until the next replenishment before their order is fulfilled. As a result, some of the customer orders received might be lost, and this is the assumption made in this study, that is some proportion of customer demand not met immediately is lost.

1.2 Motivation

Nowadays, challenges like increasing difficulties in procurement, difficult production management conditions and the complexities of the operational techniques applied within a business organization (such as identifying suitable methods for managing the flow of goods and services, the number of variety of products, the complex nature of demand predictions, fast-changing customer preferences, the competitiveness of manufacturing firms, shortened product lifetimes, as well as the impact of the deterioration in inventory management), make the production and replenishment policies for deteriorating goods rather complex and attract the attention of both business managers and researchers.

For years, researchers in production-inventory system have been studying different mathematical models that have constant production rates. However, in reality, the production rate in a manufacturing environment is subject to many challenges. The problems faced by manufacturing companies can be strategic, tactical or operational. These problems may be linked to various variables such as rapid growth, the complexity of planning and scheduling processes, the difficulties of delivering on time due to lead time changes, material flow, labour shortage, complex inventory management policies and raw material purchasing decisions, inability to assess various scenarios and measure their impacts, the nature of the product, the requirements of consumers, low equipment performance and reliability, ageing manufacturing components, costly distribution methods, and competition with other organizations.

When planning for production, possibilities of manufacturing defects have to be taken into consideration. Typical EPQ models assume perfect production processes with reliable equipment and perfect product quality. Imperfect production processes and imperfect quality items are common in manufacturing. During the production process, some of the items produced are considered imperfect, and this defect in the product quality may be the result of many factors, such as human error, process deterioration, wide tolerance, equipment failure, mishandling, and incorrect specifications for raw materials (Al-Salamah, 2019). Machines are usually set up at the beginning of the production of a batch. Moreover, it is usually assumed that the production process is under control and that the items produced are of acceptable quality. In general, manufacturers are often confronted with the problem of measuring system capacity. So if one looks at a manufacturing system like a complex sequence with several unit processes, each with its characteristics, questions such as what is the system capacity, what the capacity of the plant is, how much to produce, how the production depends on the equipment used, the type of equipment the operator uses and the way they operate, amongst others, become central issues that manufacturers have to address in production and capacity planning. Thus, running a system without correctly addressing the issues mentioned above may, for instance, accelerate the production process deterioration leading to a complete shutdown of the production chain (Shib et al., 2007). Deterioration of processes is a source of malfunction in manufacturing, and it affects the system in various ways (Hall, 1983). It can lead to product defects, which in turn may affect the quality of the product. It may cause process stoppages and failures, which may affect the entire process' availability. Ben-Daya et al. (2008) demonstrated that process deterioration could also be the cause of minor stoppages and the reduction of productivity (speed losses), which can affect the efficiency of the process. Over the past few years, many studies addressed the effects or process deterioration on process availability (Ben-Daya and Rahim, 2001). However, the combined effect on process efficiency, quality of the items produced, equipment's reliability and process deterioration have not been adequately discussed in literature. This dissertation is intended to contribute in this direction.

In addition, EPQ models are widely used in industries and despite the successful application of EPQ models in management of inventory in the previous century, EPQ models still bear quite a number of unrealistic assumptions. One such assumption is that all the items are deterioration free; however, some items change while in storage. Hence, they become either entirely or partially unfit for use with time. Deterioration refers to a product's vaporization, spoilage, damage, or obsolescence (Wee, 1993). Generally, it may be assumed that products deteriorate over time, leading to a decline in either their utility or their price compared to the original (Hsu et al., 2007). An ever-increasing variety of items such as food, health care products, perfumes, vegetables, biotechnology products, pharmaceuticals, cosmetics, radioactive and many chemical products are classified as deteriorating products. As per the definition, there are two categories of deteriorating items. The primary category are items that lose some or all of their value over a period, such as phones and computer chips. The secondary category refers to items that evaporate, decompose or expire, such as chemical products, vegetables, meat, flowers, pharmaceutical products, and fruits. The current market trend is that customers demand not only more product's varieties but also items of good quality. This fact makes modelling the inventory systems more challenging. Moreover, a price markdown policy is often used as the product's best before date approaches to remove deteriorated items from those still in a condition to satisfy the demand. Integrating these approaches into inventory control makes models more complex to develop.

Nowadays, the effects of deterioration cannot be neglected. Deterioration can also lead to product imperfections. This and the costs associated with recycling/disposal processes for expired/deteriorated products complicate the modelling of the system's costs. Therefore, efficient inventory management has a significant effect on a company's competitive advantage and profitability.

1.3 Problem statement and objectives

1.3.1 Problem statement

Production systems are combinations of materials handling equipment and processing machines set up to deliver the expected products. This is accomplished by maintaining a smooth flow of components throughout the system to prevent production waste. Technological progress has provided several possibilities for production managers to exercise better control over a production plant's performance, both from production logistics and quality perspectives. However, the integrated analysis of both the productivity of unreliable systems and the quality of finished goods in manufacturing has received limited attention.

In the context of this dissertation, special attention has been given to a continuous production system with unreliable equipment subject to breakdowns, and also having both deteriorating inventory and imperfect production outputs. Several problems may arise in such production systems, of which imperfect processes and imperfect products are considered here. Production process deterioration, which is the degradation of equipment in a production environment, represents one of the obstacles to achieving high productivity with minimal rejects. Generally, degradation or deterioration of production processes and deteriorating inventory, are known to be detrimental, but they cannot be permanently eliminated. For this reason, it is vital to identify the causes of degradation of processes, defects in quality outputs and deteriorating inventory system, understand their relationship with the manufacturing system, and quantify its impact on the system's performance. Also, modelling systems with imperfect products make inventory control and replenishment policy challenging, which leads some researchers such as Tsou et al. (2012) to restrict the expressions that represent the perfect, imperfect or defective products to constant values and establish a relationship between their models and the classical EPQ model.

1.3.2 Objectives

Studying a system from a production perspective consists of identifying the relationship and interaction between people, technology and resources, to understand how a system works, what constraints limits its performances, how to evaluate its performance levels and implement strategies for optimising them and how to quantify decisions trade-offs so that a company can plan, allocate and utilise its resources effectively. The following objectives were determined based on the problem statement:

1. To characterise the modelling of deterioration processes based on the assumptions of published works in the literature.
2. To explore modelling methods, and optimization of manufacturing systems with equipment subject to breakdowns.
3. To propose the scheme of multi-state production systems, which incorporates the continuity of production in the failure state.
4. To model, analyse and evaluate the performance of a manufacturing system with unreliable machines and the impact of its parameters on the manufacturing objective.

To meet these objectives, this dissertation aims to answer the following research questions:

1. How can one develop a replenishment policy for deteriorating inventory in a manufacturing system for both perfect and imperfect production based on a multi-state system?
2. What is the appropriate method for solving this model to handle the multiple state nature of the manufacturing system, the imperfect production outputs and the deterioration of inventory?
3. Does any evidence exist to support the robustness of the developed model and whether the model can handle different real-life scenarios?

1.4 Scope of the study

The themes presented in this study are deteriorating items, imperfect quality products, performance, backlog, unreliable equipment and multi-state system. The first theme implies that the available inventory of items produced undergoes changes with time while in storage. The second theme implies a production system that is not defect-free; as a result, a portion of items produced is considered defective and unfit to meet the customer demand. The third theme implies the measure of the physical nature of the outputs of the manufacturing system. The performance can be structural, functional, behavioural, etc. Thus, productivity or capacity, reliability, availability and speed rate may represent the performance measure of a given system. Backlog implies all the unfulfilled demand. Unreliable equipment implies machines that are subject to breakdowns. A lot of elements cause equipment failure, and many failures happen randomly. Lastly, a multi-state system involves a flexible industrial system based on automatic internal activation of mechanisms for reconfiguring a system to ensure the continuity of an operation when a failure occurs. In such a situation, a failure results in a decrease in the system's performance, but not systematically in its total shutdown.

1.5 Research methodology

Bertrand and Fransoo (2002) suggested several methodologies for performing quantitative research in operations research (OR) and operations management (OM). The following methodology, which was previously adapted from their methodologies, served as the basis in conducting the research discussed in this dissertation:

- **Conceptual model of the process or problem**: The first step is the identification of a problem. The proposed inventory systems are described in depth. These descriptions are based on real situations in organizations that produce items, certain number of which are defective, and the remaining are subject to deterioration once stored.

- **Scientific model of the process or process**: Mathematical models are developed based on the problem statements. Since mathematical models are intended to reproduce real situations as closely, clearly and concisely as possible, several assumptions are made to bridge the gap between reality and the mathematical formulation of the problems.

- **Solution to the scientific model**: Solution procedures for solving the proposed mathematical models are presented. The solution algorithm techniques are then applied to numerical examples to solve the conceptual problem or process.

- **Proof of the solution**: Finding the solution and proving the correctness of the solution play an essential role in research methodology. The proof of solution demonstrates that solutions to the mathematical models describing the proposed production systems exist.

- **Insights related to the conceptual model**: As seen from Figure (1.1), a sensitivity analysis is conducted to examine the robustness of the developed model. The results found are then used to make suggestions and recommendations on the manufacturing systems for deteriorating items.

1.6 Framework of the dissertation

This dissertation is divided into six chapters. The research topic and particularly the research gaps were introduced in chapter one. These gaps are mainly in the area of inventory management as discussed, leading to the definition of the research problem. In addition, the main research objectives, aims, research questions and the methodology adopted, which is a combination of mathematical modelling, analytical approaches and sensitivity analysis, have also been defined.

The literature review is composed of three main areas which are divided into three chapters:

Chapter two provides a literature review focusing on the fundamental thinking and principles of inventory management as shown in Figure (1.1). A definition of terms used, description of features in inventory management are provided as well as the classical EOQ model. A description of some extensions that have been made in

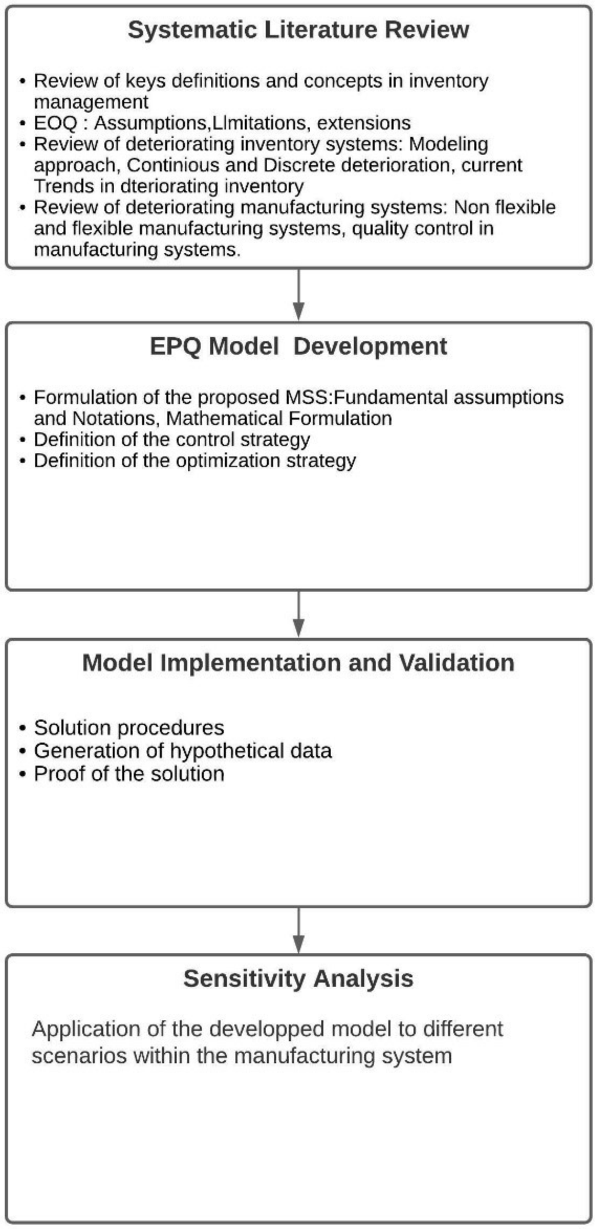

Figure 1.1: Proposed Research Methodology

recent years is also presented. This chapter provides a broad overview subsequently setting the scene before moving to other core chapters of the dissertation.

Chapter three: provides a comprehensive systematic literature review focusing on deteriorating inventory replenishment. Through this review, a description of the modelling philosophy of deteriorating replenishment policy as well as other categorizations based on demand functions and objective function's components are discussed. A classification of works on deteriorating inventory is also provided.

Chapter four: provides a review focusing on the deteriorating processes in manufacturing systems. A review of the literature on deterioration of production equipment, inflexible manufacturing systems, flexible manufacturing systems with unreliable equipment, as well as the integration of quality are examined. Finally, a concluding section and motivations for the model that will be proposed in the next chapter is also provided.

The remaining sections of this dissertation are divided as follows:
Chapter five: addresses the main objective of the study, which is the development of a deteriorating replenishment EPQ model for a MSS with imperfect quality and variable production rate. Firstly, the MSS model is described for a manufacturing system in which the deterioration of equipment leads to a change in the production rate i.e. switching from one rate to another. Prior to meeting the customer's demand, good items are separated from defective ones by screening with the defective products disposed of. The stock is depleted and a level of backlog is attained after which the production commences again, clearing the backlog, after which the cycle starts again. In developing the model, the relevant variables are first defined. The mathematical model is then formulated, as a set of differential equations representing the different time periods of the entire cycle. The relevant optimisation technique is used to solve the models and the feasibility conditions of the decision variables are introduced. To illustrate the use of the solution methodology, numerical examples are presented. Sensitivity analyses are also conducted to determine the situations under which each model parameter is most sensitive.

Chapter six: the conclusion, where a summary of possible practical applications and contributions are presented. Potential topics for future research work and research perspectives are also provided in this chapter.

Chapter 2

Review of keys definitions and concepts in inventory management

In modern manufacturing, monitoring the flow of materials, information and services from suppliers to end-users are critical issues for almost everyone in any sector of Supply Chain Management (SCM). In a typical SCM, raw materials are procured, shipped to one or more factories and then in factories items are produced and shipped for intermediate storage at warehouses before being shipped to retailers or end-users. The challenges in this area are well known by top management. As inventories are considered practically universal, it is worth starting with the fundamental question: "Why do companies keep inventories"? One of the main responses is that inventories are a way of dealing with variability and uncertainty in demand and supply. Inventories act as a buffer between suppliers and customers which helps in maintaining customer service when the supply chain has problems. However, this buffer comes at a price and organizations have to continuously devise ways of lowering the inventory costs while maintaining acceptable levels of customer service. Inventory does not exist in isolation, so consideration must be given to its possible implications on other areas of the business. Therefore, inventory management aims to balance opposing goals within an organization. One of these goals is to maintain inventory levels at a relatively low level to ensure that cash can be used for other functions, and the second objective is to carry enough inventory that provides a desired service level. This chapter starts with the fundamentals of inventory management as described in Section 2.1. Particularly, the need for inventory control, as well as definitions of basic concepts, types of inventory, and cost components involved are presented. The Economic Order Quantity (EOQ) model is introduced in section 2.2 including its assumptions, limitations and extensions.

2.1 Fundamental notions of inventory management

Most organizations carry inventories. These are the reserves of materials that are held until needed. When a business possesses resources that are not immediately needed, it stores these materials. For instance, there is stock of recorded programs for television companies; stock of goods purchased from wholesalers by various shops and kept until they sell them to customers; a bank carries cash on hand for its daily banking activities; hay held by a farmer to feed his cattle in winter; a factory carries a reserve of raw materials for its products; a research firm has a stock of information. (Donald, 2003). The following section provides a brief overview of basic concepts of inventory management that will be used in the next chapters; the description of the concepts mentioned below is based on Donald (2003):

- An item is a distinct product that is kept in stock: it is one entry in the inventory.
- A unit is the standard size or quantity of an item.
- An inventory is a list of the items held in stock.
- On-hand inventory: This is the physically available inventory; it determines if a given demand from a customer could be met directly from the stock.
- Lead-time: the amount of time that goes by, from start to finish of any given process. Shortages: represents the portion of the unmet demand. There are three types of shortages:
  - Partial backlog/back-order: some of the unfulfilled customer orders are lost, and the rest are backlogged.
  - Complete backlog demand: represents a state in which all unmet customer orders await fulfilment at the next replenishment.
  - Another possibility is loss of sale. Loss of sale represents a particular case in which unfulfilled demands are entirely lost.

2.1.1 Inventory Management

Inventory management refers to a systematic approach to the procurement, storage and sale of inventory, whether raw materials or finished goods.It's responsible for inventory-related decisions. These decisions are driven by critical factors such as the type of item, expected customer orders, and the inventory level. A primary aim of any inventory management system is to identify what item, when it should be ordered and how much should be ordered. Inventory management has received considerable attention from managers over the past few decades because managers became aware of the high costs that arise from holding unnecessary inventory in warehouses (Gourdin ,2001). Hence, many efforts have been made, and different approaches to managing inventory have been conducted to minimise excess inventory while maintaining customer service. An additional insight from Gourdin (2001) is the importance of carrying inventory in certain circumstances, such as meeting demands of global consumers, and therefore, business management aims to carry just what is needed to meet this objective. To this effect, Chase et al. (2007) described inventory as "the stock of any item or resource available for use in any given organization". From this perspective, to measure inventory levels regularly and make adjustments based on business needs, an appropriate set of controls and policies for the inventory system are required. In addition, replenishment procedures and inventory size are also considered important. Moreover, Pycraft et al. (2010) provided a more extensive definition of inventory as "the accumulated stock of material resources in a processing system.

2.1.2 Types of Inventory

Six main categories of inventory were identified by Stock and Lambert (2001). They discussed the role and use of each type within an organization as:

- **Fluctuation Inventory**: This inventory can be used in unplanned manufacturing situations where forecasts of the quantity of the finished goods required cannot be assured.

- **Anticipation inventory**: also known as Speculation inventory, is the accumulated inventory that a business intends to use for an anticipated or expected future peak in sales. Examples of speculation inventory are winter fashion goods or Christmas items.

- **Lot-size (batch-size) inventory**: This type of inventory does not consider individual units because the demand of the stock is considered in batch.

- **In-transit inventory**: In-transit inventory, also known as pipeline inventory, is the stock that has been ordered but has still not been delivered. Work-in-process (WIP) inventory is considered part of this category and is intended for the plant design and layout processes type.

- **Buffer Stock**: also called decoupling Inventory, buffer stocks are used to prevent a company from any potential failure, ensure the continuity of the production and give the company enough time to address or resolve the process.

- **Dead inventory**: also known as dead stock or obsolete inventory, refers to unwanted stocks that are not expected to be utilized for any long term or immediate purposes. Additional costs are therefore incurred to store and maintain dead inventory. In some cases, the stock may be stored to anticipate an eventual increase in demand or simply because disposal costs are higher than the storage costs. However, customer service is a primary reason that pushes businesses to stock dead inventory to enable occasional buyers to procure them at a salvage price in the future.

Depending on the nature of goods, the above types of inventory can be categorised into three major classes:

1. Unlimited lifetime inventory: refers to inventory, which has no deterioration.
2. Obsolete inventory: refers to inventory that has lost its worth over time as a result of new substitute items being introduced or due to quick changes in technology. Obsolete inventories are usually disposed of or sold at a salvage price after their season is over.
3. Deteriorating inventory: refers to damaged, decomposed, vaporized, spoiled, or degraded inventory.

2.1.3 Inventory Control

Wild (2017) defines inventory control as a term widely used to organize the inventory management procedure to ensure that customers obtain products when required. Procurement, production, storage, and delivery activities are primarily driven by the business's marketing and sales functions. Hence, inventory control is responsible for managing finished products, raw materials, defective goods and other necessary supplies. According to Jaber et al. (2009), production, logistics, and customer service functions depend significantly on effective inventory control. Businesses hardly fulfil customer demands with inefficient inventory management if production does not match procurement and sales needs. Research in operations management focuses extensively on inventory control and its applications in various industries. From this perspective, it is crucial for any firm to answer the three following questions regarding inventory management (Silver et al., 2017):

1. How frequently does the status of items in inventory need to be reviewed?
2. When does a replenishment order get placed?
3. How large should the order be?

Answering these questions can seem quite simple at first, however the properties or characteristics of the materials, resources being managed make inventory management difficult. For years, researchers and managers in inventory management have developed mathematical models to answer the three questions above while keeping in mind the characteristics of the items being managed. To this effect, a classification of these mathematical models is presented in Figure (2.1). This classification is based on Vrat's (2014) approach, which considered several factors such as probabilistic or deterministic demand, varying or constant replenishment lead time, single or multiple items productions, perishable or non-perishable items, and single or multi echelons supply models.

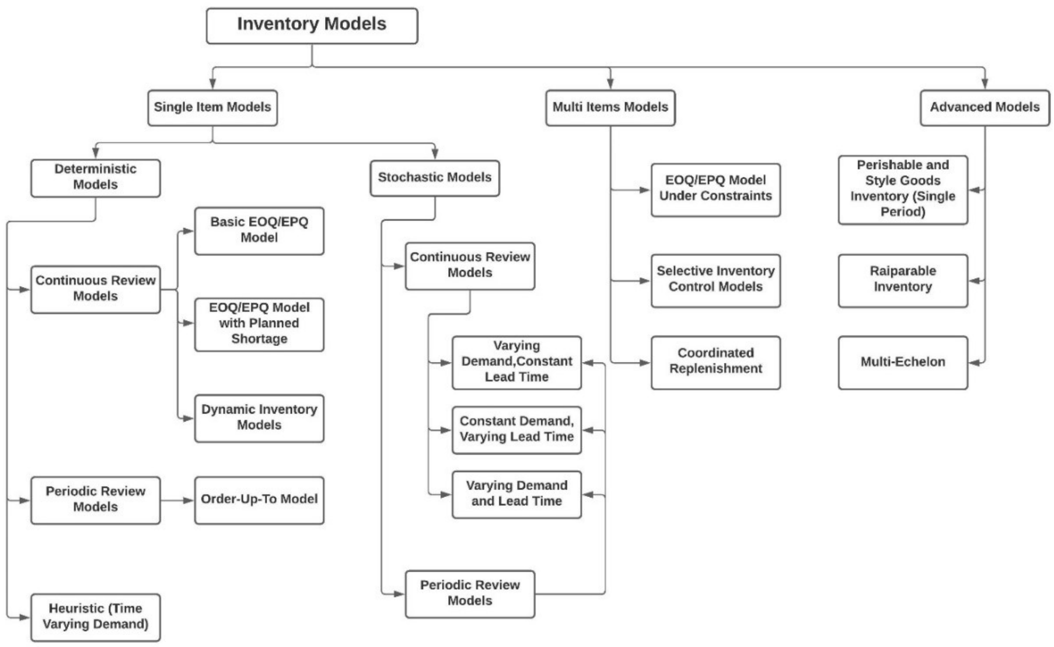

Figure 2.1: Types of inventory control systems

Silver et al. (1998) suggested that supply chain activities based on coordination are necessary to raise the efficiency levels of firms. Saxena (2003) suggested that the shortage cost or excess inventory cost incurred must be minimized to ensure the effectiveness of the inventory control in managing inventory turnover.

2.1.4 Costs components in inventory management

The performance criteria in inventory management is based on either minimizing the total cost or maximizing the profit. Costs in inventory management are generally divided into four major groups (Silver et al., 1998):

- **The carrying cost**: is the total of all the costs incurred due to the quantity of the physical inventory available at a particular time. The cost of carrying items in inventory includes counting and handling costs, the expenses incurred in running a warehouse operation cost, special storage requirements cost, deterioration of stock and obsolescence, and insurance.

- **The ordering or setup cost**: is the cost associated with ordering the inventory or replenishment. This cost has two components: the variable and the fixed costs. The variable cost typically includes the cost of both loading and unloading the truck, the costs of checking the orders, etc. (Chaaben, 2010). The fixed cost is irrespective of the replenishment size. In production, the fixed setup cost includes many components such as costs associated with interrupting production, costs associated with material handling and transportation, the cost of administration associated with the time and effort expended in preparing orders, etc.

- **The purchase cost**: which is the cost proportional to the quantity ordered. It may depend directly the replenishment size or there may be quantity discounts.

- **The Penalty (stock out, shortage or backorder) costs**: are costs to an organization when it is unable to satisfy a demand. This cost might include the cost of substitution of a less profitable item, the cost of emergency shipments or loss of sale cost.

2.2 Classic Economic Order Quantity (EOQ) model and Extensions

This section provides a review of the classic inventory control theory. It discusses the optimal size to order by balancing the various costs involved when conditions are rather stable with no uncertain demand. The EOQ model is the most basic of all inventory models that helps inventory managers in determining the optimal order quantity, and its applicability has been accepted by research throughout the century (Donald, 2003). Harris (1913) was the first to propose the formulation for determining the order quantity by balancing the setup cost, purchasing cost and inventory carrying cost; However, the actual calculation is mostly accredited to Wilson (1934), who marketed the results. Harris's (1913) model development involved several assumptions with some of the assumptions being pretty unrealistic and limit the model's application to many real-life inventory systems. In attempts to improve the application and practicality of classic EOQ model in real-life problems, several research studies have been conducted to determine which of these assumptions to relax, and how the results of such relaxation would impact the companies' costs (Andriolo et al., 2014).

2.2.1 The classic EOQ model

As mentioned earlier, Harris's EOQ model is the most straightforward inventory control model. This model considers an idealized inventory system and finds the fixed replenishment quantity that minimizes certain inventory-related costs. There are four fundamental categories of costs as discussed in section 2.1.4 namely the ordering and setup costs, the inventory carrying costs, the purchasing costs, and the penalty costs.

The model is restricted to an order quantity system. Therefore, the optimum quantity is independent of the control system costs. Since demand is assumed to be known, constant, and deterministic, the only relevant cost for the third category would be if the decision maker deliberately chooses to run out of stock before replenishing; however, no shortages are allowed. In addition, the purchasing cost doesn't affect the optimum quantity; therefore, this cost is not relevant. Finally, only the ordering and inventory carrying costs are relevant to the inventory model. In determining the appropriate order quantity, the criterion of minimization of total relevant costs that balances the ordering costs and holding is applied because as the order quantity increases, the ordering costs decrease, and the holding costs increase and vice versa (Sebatjane, 2018). These trade-offs are shown in Figures (2.2) and (2.3), demonstrating how the total cost change with order quantity and the changes to the inventory level with time, respectively.

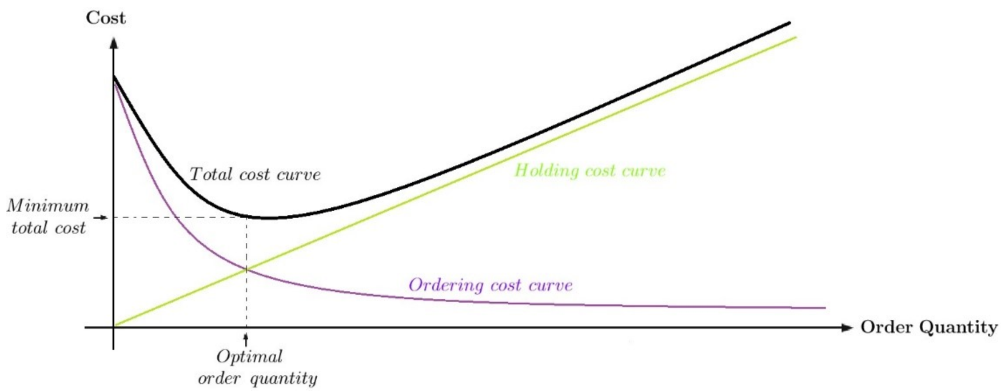

Figure 2.2: Ordering cost, carrying cost, and total cost as functions of order quantity

The inventory system shown in Figure (2.3 considers a model with only a single item in isolation, and it is assumed that the replenishment is instantaneous so that all the order quantity arrives in stock simultaneously and can be used immediately. Each time an order for Q items is placed, an ordering cost of C_ord is incurred. The items are consumed at a constant annual rate D; this means that the consumption is always at the same rate. Eventually, no stock remains at the end of cycle T.

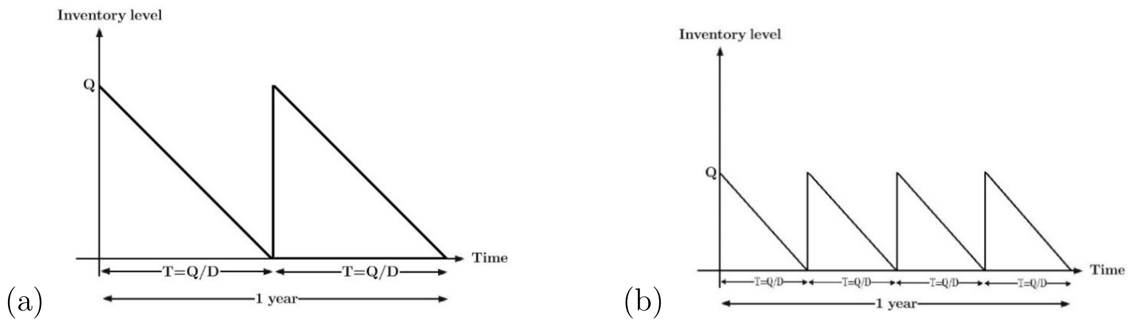

Figure 2.3: Typical inventory system behaviour for the classic EOQ model

A new order for quantity Q is received when the previous quantity Q runs out. A unit of the item is kept in stock at a holding cost h per unit time. The Total cost (CT) is the sum of ordering costs and holding costs

$$CT = h\left(\frac{Q}{2}\right) + C_{\text{ord}}\left(\frac{D}{Q}\right) \tag{2.1}$$

A convenient way to optimise the above equation is to differentiate CT with respect to Q and equate it to zero.

$$\frac{dCT(Q)}{dQ} = 0 \tag{2.2}$$

That is,

$$\frac{h}{2} - C_{\text{ord}}\frac{D}{Q^2} = 0 \tag{2.3}$$

$$Q_{\text{opt}} = \sqrt{\frac{2C_{\text{ord}}D}{h}} \tag{2.4}$$

Equation (2.4) represents the optimal order quantity. This expression shows, among other points that the optimal order quantity goes up with the square root of demand rather than being directly proportional. Substituting the value of Q from Equation (2.4) back into Equation (2.1), the following equation is obtained:

$$CT\left(EOQ_{opt}\right) = \sqrt{2C_{ord}Dh} \tag{2.5}$$

With:
C_{ord} : the inventory ordering cost
h : the inventory holding cost.

2.2.2 Extensions made to the classic EOQ model

Harris's (1913) EOQ model is based on some assumptions that make the model limited in real life. Woolsey (1988) argued that the assumptions made in the set of parameters of the EOQ model are too simplistic for practical implementation and showed the inapplicability of the model to some scenarios encountered in real life. Many possible constraints and some not explicitly incorporated in the model may prevent using the classic EOQ model. Therefore, development of many inventory models has been based on the publication of Harris's (1913) classic EOQ model (Andriolo et al., 2014). Some of the most significant extensions made throughout the century concerning Harris' model are represented in Figure (2.4), which also depicts the evolution in the area of inventory management.

The first significant extension in relation to Harris's classic EOQ model came in the form of what is now widely known as the EPQ model, which is utilized in the determination of the optimal lot size for a production system whereby the primary units of a lot can be utilized to meet customer demand while the remainder of the products is still in production. Two significant events occur simultaneously in developing the EPQ model, namely periodic production and continuous consumption. In the late 1950s, several extensions had been made in the area of lot sizing. Wagner and Whitin (1958) relaxed the assumption of constant deterministic to a periodic demand rate through developing the Dynamic Economic Lot (DEL) model using a recursive algorithm.

Harris (1913)'s model assumes that items can be stored for an indefinite period without changing the item's value, integrity or utility. However, this is not always true, especially for items such as vegetables, meat, flowers, and pharmaceutical products, to name a few. Such items are called perishable or deteriorating items. The customer's taste can be influenced by introducing a more attractive product compared to the product already present in the market. Researchers have also shown that there may be loss of consumers' confidence concerning an item's quality due to age, which has an undesirable effect on demand. Deterioration is an umbrella term, which encompasses any forms of damage, obsolescence, pilferage, evaporation or spoilage of goods. Whitin (1957) is considered as the pioneer of deteriorating inventory model, where for the first time, the deterioration of fashion items was taken into account when the prescribed storage period came to an end. Ghare et al. (1963) developed the first inventory model with an exponential deterioration rate. Hadley and Whitin (1963) considered quantity discounts when modelling inventory systems for the first time. These authors restricted the attention to types of discount structures, namely all-units and marginal quantity discounts. Subsequently, Hadley and Whitin (1963) studied inventory systems where shortages are permitted.

The year 1970 marked the start of exponential growth of extensions made to the original models over the following forty years. Consideration of multiples setup costs and stochastic environment are some of the extensions included in the EOQ. Lippmann (1971) proposed an EOQ policy with multiple set-up costs. Mohan (1978) developed a model under the working capital constraint. The stochastic Lead-Time is discussed in the study of Liberatore (1979). Another area of research that has emerged in this decade involves the relation between material requirements planning (MRP) and the EOQ. MRP was described as the "salvation of production and inventory control management" by Chamberlain (1977) due to the limitations of the EOQ and its inability to respond to plans aimed at reducing inventory.

During the 1990s, the classic EOQ model was still applicable in real business situations. Moreover, the EOQ remained the most studied in the area of inventory control to manage constraints in real life situations. Cheng (1990) discussed the effect of inventory investment constraints and storage space on the EOQ model with demand dependant unit production cost. Hill (1995) proposed the inventory model with ramp-type demand. Khouja and Mehrez (1994) extended the EPQ model by considering the production rate as a factor or decision variable. The research demonstrated that the deterioration was significant with the increase in the production rate. After that, Mandal et al. (1998) extended Hill's model to make the model more realistic by considering the effect of shortages while considering that the items were deteriorating.

The EOQ model was extended by research studies in the early 2000's to include the impact of rework, imperfect quality, quantity discounts, shortages, trade credits, capacity constraints, and probabilistic functions. However, two areas of research that increased in importance in recent years were the role of EOQ in relation to sustainability applications along with EOQ models and their influence on the performance and coordination of a supply chain system. Goyal and Giri (2000) presented a detailed review of deterioration models, providing further clarity on the literature regarding the deteriorating inventory models. Agrawal and Ferguson (2007) developed an EOQ model under stochastic demand. In this decade various extensions of the EOQ model have been developed in other areas including impact of discounts, inflation, the time value of money, imperfect quality, limited space area and the inclusion of Just-In-Time. Yen et al. (2012) proposed an optimal retailer's ordering policies with trade credit financing and limited storage capacity in the supply chain system. Abdul et al. (2012) discussed an EPQ model that includes the cost of raw materials required for production. In their model, the researchers considered a situation where the raw material purchased from the supplier contained a fraction of items of poor quality. A control process is put in place to detect poor quality items. The development of the mathematical model led the researchers to study two different scenarios in which the items of imperfect quality are sold at a discounted price; the alternative scenario implies that, after screening, the proportion of the imperfect raw material could be stored until the end of the inventory cycle and then returned to the supplier. Ghosh, et al. (2015) extended the classic EOQ to a space-dependent EOQ model of a multiple-item deteriorating inventory. Figure (2.4) below shows the century of evolution of the Wilson model written by Andriolo et al. (2014).

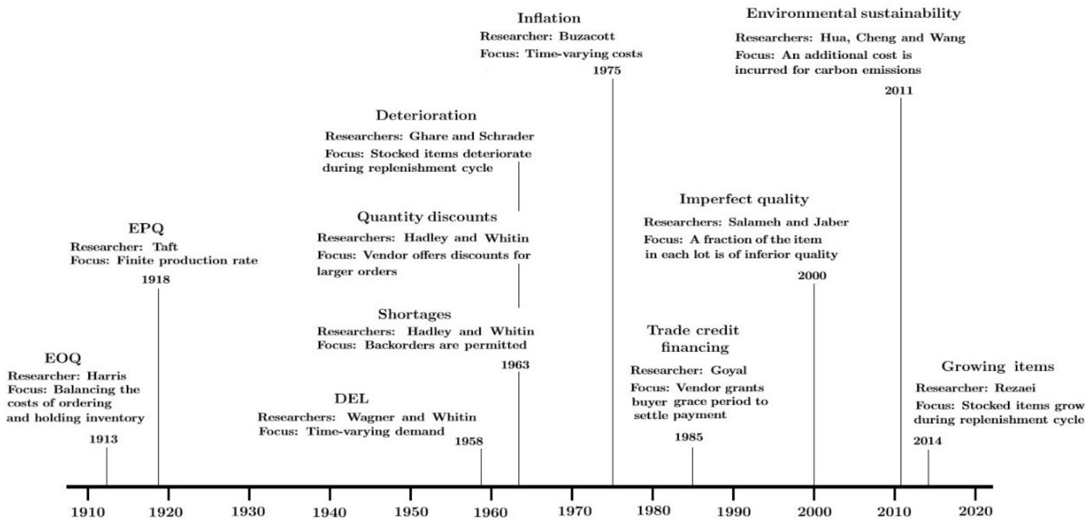

Figure 2.4: Timeline showing some of the major developments in inventory theory

Recently, Maiti, and Giri (2015) developed a closed loop supply chain under retail price and product quality dependent. Liu et al. (2015) extended Harris's (1913) model to a continuous replenishment policy model for perishable goods in which the demand depends on the product's quality. Gupta and Nisha (2015) extended the basic EPQ by relaxing the assumption that the production rate is constant. The assumption is that change in demand could be one of the critical factors that may change the production rate. Therefore, the research proposed a linear model that varies according to the demand with two production rates. It has been stated that the production could also fluctuate with the deterioration. Majumder et al. (2015) proposed a model considering an imperfect production process for breakable goods within a limited period. The formulation was based on two models, which included shortages. The research done in this area demonstrated that the cost of production is a function of raw materials, the production rate, the product reliability indicator, the workload and the cost of deterioration. In order to model specific and realistic inventory systems, some of the features of the classic EOQ model and its extensions are combined, either together or with new assumptions. This study is concerned with managing deteriorating inventory items under a set of realistic conditions, namely: discontinuous production, imperfect quality, time-dependent demand, rework and quantity discounts, which might arise in manufacturing plants. Gothi et al. (2017) discussed an inventory model in which the deterioration is represented through an exponential distribution.

This chapter is a discussion of the general evolution of the models of inventory management and covers foundational inventory control, the basic EOQ/EPQ models, their limitations and general extensions. The next chapter presents a review of deteriorating items inventory systems as an extension of interest to the problem studied in this research.

Chapter 3

Review of deteriorating inventory systems

The present chapter provides an overview of some existing primary studies on the replenishment policies of deteriorating inventory. A classification of the literature on deteriorating inventories is presented in Section 3.2. Additional classifications are also provided depending on the characteristics of the objective and demand functions. A summary of some of the reviewed studies is provided in Table (3.1).

3.1 Definitions of deterioration

Diverse definitions and classifications of deteriorating items with minor distinctions exist in the literature. These include the following:

- A process of degradation that prevents an item from performing its primary function, such as breakdown of equipment, degradation of radioactive substances, electronic parts and pharmaceutical drugs.
- Deterioration can also refer to damage, decay or obsolescence of an item resulting in a decrease in its utility such as spoilage of perishable goods (Wee, 1993).

There are a number of definitions of deteriorating and perishable inventories, including that provided by Goyal and Giri (2001), but the preferred definition in this study is that of Disney et al. (2012), who proposed the following distinction between perishable and deteriorating stocks. In the perishable inventory, items lose value but are not destroyed, as opposed to the deteriorating inventory, where items physically deteriorate and are destroyed over time. Different concepts of deterioration are involved when analysing the deteriorating of inventory. First, there are instances where all items in inventory become out-of-date, such as fashion items. Second, there are instances involving the deterioration of items throughout their planning horizon. Deteriorating items may also be categorized based on their utility or value over time. Constant value perishables undergo deterioration and face no substantial decay in value during their cycle, such as drugs and other pharmaceutical products. Decreasing-value perishable items lose their utility throughout the planning horizon, such as fresh fruits.

3.2 Modelling approaches of deteriorating inventory

This section provides a review on the modelling approaches in inventory control, giving special attention to three main classes within the context of modelling of deteriorating inventory. The purpose of the review is to gather insights from previous works and research gaps that may exist prior to developing a replenishment policy for a multi-state system and make a valuable contribution in inventory management. The literature on deterioration is extensive, and thus no single literature review can fully cover the field. Deteriorating inventory models can be classified intro three major groups based on the following modelling approach:

1. Non-linear inventory function;

2. Variable holding cost function;

3. Non-linear inventory function and variable holding cost function.

These three types are presented in Figure (3.1).

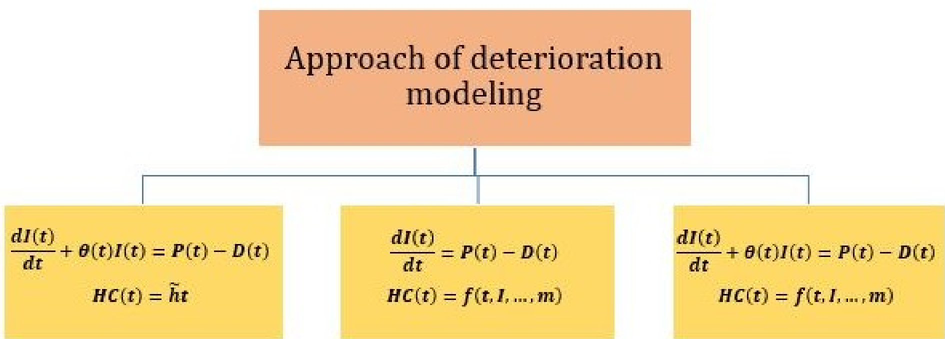

Figure 3.1: Categorization of deterioration modeling approaches

with:
$I(t)$: Instantaneous inventory per time unit

$\theta(t)$: the deterioration rate function
$D(t)$: Demand per time units
$P(t)$: Production rate per time unit
$HC(t)$: Inventory holding cost function
$\widetilde{h}$: Constant
$f(t, I, \ldots, m)$: a non-linear function of parameters such as on-hand inventory, time.

3.2.1 Non-linear inventory functions

Most researchers in inventory control assume that deterioration is time-dependent. In many real-life circumstances, this is because some stored goods can decrease in value as they stay longer in inventory. Therefore, a more extended storage period is expensive, as it involves sophisticated technology to conserve the value of the goods. This is the reason why the on-hand inventory function is essential in inventory studies. Thus, the change in inventory level can be determined by the following equation:

$$\frac{dI(t)}{dt} + \theta(t)I(t) = P(t) - D(t) \tag{3.1}$$

In this type of representation, the inventory holding cost function is assumed constant. This means that the inventory carrying cost is a linear function that depends on factors such as the inventory level, the storage time t, and can be expressed as follows:

$$HC(t) = \tilde{h}tI \tag{3.2}$$

Where:
$\tilde{h}$ is constant.

This type of modelling is best suited to decaying goods. Ghare and Schrader (1963) were pioneers of deteriorating inventory studies, who for the first time considered an inventory model with constant decaying rate and no-shortage. Since then, many researchers have pursued extensive studies on deteriorating inventories and have provided insights into how the stocks have changed over time. Manna and Chaudhuri (2001), Wang and Chen (2001), Gede and Hui (2012), fall into this category.

3.2.2 Variable holding cost

In addition to the common components of holding cost like expenses for running a warehouse, costs of special storage requirements, insurance, taxes, etc., in this class of models, the inventory holding cost also includes the cost of deterioration of items, and thus the modelling the carrying cost of inventory is directly affected. In this class, the function of the available inventory is similar in form to that of non-deteriorated products, and the differential equation can be obtained by the following:

$$\frac{dI(t)}{dt} = P(t) - D(t) \tag{3.3}$$

In this second category, the inventory holding cost, HC, is assumed to be a non-linear increasing function of parameters such as the storage time, t, or the available inventory, I. For this type of inventory systems, the deterioration rate function is considered as part of the on-hand inventory function. For items that deteriorate, particularly perishable goods such as food products like vegetables in which the value and quality of goods decrease with age, a non-linear time-dependent holding cost is assumed to be more appropriate. For products such as volatile liquids and radioactive substances, where more extensive security measures are required, a non-linear function may be appropriate to estimate the inventory holding cost. Weiss (1982) is considered to be among the pioneers to develop an inventory model for both stochastic and deterministic demands by treating the unit holding cost as a non-linear time dependent function of the stock duration and represented this cost as follows:

$$HC(t) = \tilde{h}t^{\gamma} \tag{3.4}$$

Where $\tilde{h} > 0$ and $\gamma 1$ are constant. He kept other conventional EOQ assumptions such as constant unit cost, zero supply lead-time, selling price and set-up cost, and developed two mathematical models for both deterministic and stochastic demand rate.

In 1994, Goh proposed two deterministic inventory models for a single item with infinite-horizon by considering stock-dependent demand. Both stock-dependent and time-dependent holding costs were considered in these two deterministic inventory models. He assumed that the variable holding cost is a polynomial function of the storage duration. Recently, Ferguson et al. (2007) extended Weiss's model by including extra delivery charges and price discounts for perishable items. Later that year, Alfares (2007) studied Goh's (1994) model by extending some of the assumptions made in Goh's (1994) to address the case of discrete holding costs dependent of storage time. A generalization of Alfares's model was made in 2008 by Urban with the objective of optimizing the profit. In addition, he relaxed the limitation that the inventory level is equal to zero at the end of each replenishment cycle by considering it as non-negative. Alfares (2012) presented an EPQ model with stock dependent demand and time dependent holding cost. Later, Alfares (2015) extended the EPQ model developed by Alfares (2014) by maximizing the profit while considering a variable unit purchase cost based on all-units quantity discounts. Recently, San-José et al. (2018) developed an EPQ model with time nonlinear dependent holding cost, in which demand is price dependent and shortages are partially backlogged.

3.2.3 Non-linear inventory function and variable holding cost

The approach for modelling this category of inventory system is considerably different from that of the first two classes presented in 3.2.1 and 3.2.2. Models in this third category are developed with consideration of the deterioration rate function $\theta(t)$ and the variable holding cost, as discussed in Giri and Chaudhuri (1998). They extended Goh's model by considering two models with non-linear stock and time-dependent holding costs, respectively, in addition to the constant deterioration function. Several researchers have studied this category intensively over the past twenty-five years. Chang (2004), Ferguson et al. (2007), Mahata and Goswami (2009) fall in this category.

3.3 Deterioration function $\theta(t)$

Generally, the deterioration function $\theta(t)$ is classified into two groups: continuous deterioration and discrete deterioration functions.

3.3.1 Continuous deterioration

The continuous deterioration $\theta(t)$can be classified into four major groups as follows:

Constant deterioration: Many studies consider models with constant deterioration rate. Such models are suitable for products such as perfumes, electronic equipment, pharmaceuticals or oil. Chung and Ting (1994) developed an inventory model with constant deterioration rate and time-dependent demand function. Soumendra (2010) developed a continuous order-level inventory model for deteriorating items under deterministic demand. In this research, Soumendra worked on maximizing total profit, including ordering cost, holding cost, shortage cost, purchase cost and loss of sales. The computational results indicated that the developed inventory model provided insightful information in helping inventory managers decide on the optimal quantity to keep in stock while maximizing the profit. Srivastava and Gupta (2007) proposed an infinite time-horizon inventory model for deteriorating items assuming

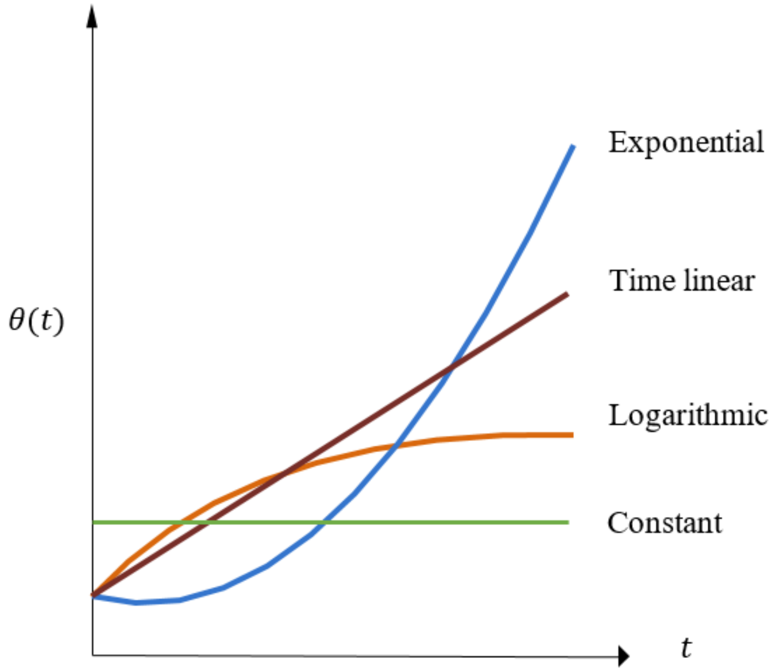

Figure 3.2: Graphical representation of the four major groups of continuous deterioration

the deterioration rate and the demand rate to be constant for a period of time and then as a linear function of time for another time period. Karthikeyan and Santhi (2015) proposed an EOQ inventory model for deteriorating items in which the deterioration rate is constant, and salvage value is incorporated into the deteriorated items.

Time linear deterioration: Generally, in deterministic inventory models, the deterioration is assumed to be constant and has nothing to do with the duration of the goods in the inventory. One of the models is Ghosh and Chaudhuri (2006), which considers a time-proportional deterioration function of the form:

$$\theta(t) = \theta_1 + \theta_2 t \tag{3.5}$$

Where: $0 \leq \theta(t) < 1$ and $0 < \theta_1, \theta_2 < 1$.

Trailokyanath and Hadibandhu (2013) developed an EOQ model for deteriorating items with linear demand patterns and linear deterioration rate function. The authors minimized the average total cost, including the inventory holding cost, shortage cost, ordering cost and the opportunity cost due to lost sales. Singh et al. (2016) proposed a replenishment policy with time-dependent deterioration and both constant and time-varying demand rate. Tripathia and Kaurb (2016) studied an optimum EOQ model for non- decreasing time dependent deterioration and decaying demand with shortages.

Logarithmic deterioration and multiple deterioration: Generally, the products are such that the deterioration increases dramatically in the initial stage and after certain time the rate of deterioration increases slowly. This category is often appropriate for items such as integrated circuits and other electronic equipment where the degradation during the initial stage changes drastically. Patel and Sheikh (2015) proposed a deteriorating items inventory model with different deterioration rates, linear trend in demand time varying holding cost and shortages. Patel (2018) developed an inventory model with different deterioration rates under inflation and permissible delay in payments for two level storage. Naik and Patel (2018) developed an inventory model for imperfect quality and repairable items with different deterioration rates under price and time dependent demand.

Exponential deterioration: Unlike the logarithmic form, items deteriorate at a slow rate in the initial phase and then shift rapidly after a certain period. Sivakumar (2009) introduced the notion continuous finite-source inventory system for a model with exponential lifetime for the items and exponential lead-time. Mahata (2011) discussed a replenishment policy with exponential distribution deterioration rate and time-dependent demand. The author proposed an exact formula of the optimal cost without carrying out any approximation over the deterioration rate. Lawrence et al (2013) considered an inventory system for deteriorating items with a quasi-random demand distribution and exponential deterioration rate. They assumed that both the lead-time and service time are independent phase type distributions. They determined optimal order quantity that minimizes the total cost rate. Recently, Gothi et al. (2017) proposed an inventory model for deteriorating and repairable items with exponential deterioration and linear demand rate.

Weibull deterioration: Covert and Phillip (1973) were the first to introduce the notion of Weibull deterioration for perishable inventory. Many researchers such as Chakrabarty et al. (1998), Kumar et al. (2012) and Sharmila and Uthayakumar (2016) have developed inventory models in which they assumed the instantaneous rate of deterioration (t) to be a two-parameter Weibull distribution of the form:

$$\theta(t) = \alpha\beta t^{\beta-1}, \alpha, \beta \geq 0 \tag{3.6}$$

The two-parameter Weibull distribution is adequate for goods with a decreasing rate of deterioration, provided that the initial rate of deterioration is considerably high.

Similarly, this type of distribution can also be used with goods with an increasing rate of deterioration, provided that the initial rate of deterioration is practically zero. The models of Covert and Philip (1973), Asoke and Ali (2014), etc., fall in this class.

3.3.2 Discrete deterioration function

Discrete deterioration also known as Semi-continuous deterioration are mostly used in inventory of non-instantaneous deterioration products such as foodstuffs like dry fruits, potatoes, and vegetables. Wu et al. (2006) were among the first to discuss the problem of determining the optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partially backlogged shortages. Further, Ouyang et al. (2006) presented an inventory model for non-instantaneous deteriorating items with permissible delay in payments. Valliathal and Uthayakumar (2011), proposed the optimal pricing and replenishment policies of an economic order quantity model for non-instantaneous deteriorating items with partial backlogging over an infinite time horizon.

3.4 Objective function

One of the characteristics that differentiates inventory management models is defining the objective function and the cost components that define that function. Many studies in deteriorating inventory aim to minimise the total cost (Uthayakumar and Tharani, 2017; Alfares, 2007; Girl and Chaudhuri, 1998; Weiss, 1982). Widyadana and Wee (2012) developed integrated EPQ model for deteriorating items with preventive maintenance, random machine breakdown. It is assumed that the corrective and preventive maintenance times are stochastic and unfilled demand is lost. Two models have been developed by considering corrective repair and maintenance times uniformly distributed for the first and exponentially distributed for the second. Other research focus on optimizing the total profit over an infinite planning horizon, while some consider optimizing the sum of the costs components over a finite planning horizon. Widyadana and Wee (2012) proposed a production system for deteriorating inventory under multiple production setups and rework to optimize the system's total cost function represented by:

$$TC\left(m, T_1\right) = \frac{mk_s + k_r + h_s\left[m\left(I_{t1} + I_{t2}\right) + I_{t3} + I_{t4}\right] + h_r TRI + DcDi}{m\left(T_1 + T_2\right) + T_3 + T_4} \qquad (3.7)$$

Where: TRI, k_s and k_s are the production setup cost and rework setup cost respectively. Similarly, m, h_s and h_r are the number of production setup in one cycle, the serviceable items holding cost and recoverable items holding cost. Respectively, $I_{t1}, I_{t2}, I_{t3}, I_{t4}, T_1, T_2, T_3,$ and T_4 are the total serviceable inventory in a production period, the total serviceable inventory in a non-production period, the total serviceable inventory in a rework production period, the total serviceable inventory in a rework non-production period, the production period, the nonproduction period, the rework process period, the non-rework process period. Di and Dc indicates the total number of deteriorating items and deteriorating cost respectively.

Sharmila and Uthayakumar (2016) considered an inventory model for deteriorating items with three different rates of production and stock dependent demand. They minimized the average total cost of the system, using a two-parameter Weibull distribution to represent the deterioration rate. They also developed a continuous inventory model with three rates of production under stock and time-dependent demand for time-varying deterioration rate with shortages. Other types of cost components such as transportation, greenhouse penalty costs or advertisement costs have been incorporated into some deteriorating inventory replenishment models, in addition to the usual cost components as discussed in section 3.4.2. Venkateswarlu and Mohan (2013) and Begum et al. (2009) developed replenishment policies for deteriorating items under profit maximization by treating the selling price as a decision variable. Cheng et al. (2012) proposed an inventory model considering the financial environment to find the optimal order quantity and payoff time for maximizing the retailer's total profit.

The following section describes in detail the shortage cost as an important cost component of the objective function of the replenishment policies for deteriorating inventory.

3.4.1 Shortage cost

To simplify both the modelling and the solution approaches, many replenishment policies for deteriorating items, particularly the basic models, have been developed with no shortage cost. Models without shortages are considered in Sana (2011), Ferguson et al. (2007), Rau et al. (2003), Girl and Chaudhuri (1998), Goh (1994), and Weiss (1982). Shortages are of great importance, especially in a system that takes into account delay in delivery or payment because shortages can affect a system in several ways; shortages can affect the profit due to an increase in loss of sales because of customers that are not willing to wait for the next replenishment; it can also affect the selling price since some items are sold at a price discount in order to not lose customers during the backlog cycle. Most researchers naturally address the question of how long the backlog cycle should be and whether the permissible delay's cycle is influenced by the batch size produced or the order quantity delivered. Jamal et al (1997) proposed an EOQ model for deteriorating items with allowable shortage and permissible delay payment. Kharde (2012) presented a replenishment policy for planned shortages using the concept of Equivalent Holding Cost (EHC) to optimize the EPQ system.

Some researchers dealt with shortages as partially backordered. Research under partial backorder can be divided into two categories, as shown in Figure(3.3), such as time-dependent and time-independent models. In time-dependent partial back-order models, it is assumed that the number of outstanding orders during the backlog cycle is time dependent demand. This implies that more demand can be fulfilled if the permissible period for shortage is short. However, in time-independent partial back-order models, it is assumed that the number of lost customers (or unfulfilled backordered) is independent of the waiting time for fulfilling the customer's needs. In 1995, Padmanabhan and Vrat dealt with profit maximization by considering an inventory-dependent demand function for deteriorating items with demand dependent shortage. Goyal and Giri (2003) and Lo et al. (2007) fall into this category. Pentico and Drake (2011) proposed the following Table (3.1) to classify time-dependent partial back-order models:

Table 3.1: Time-dependent partial Backorder categories

Form of stock-out	Equation	Range of τ
Linear 1	$\beta(\tau) = \beta_M - (\beta_M - \beta_0)\left(\frac{\tau}{T_1}\right)$	$0 \leq \tau \leq T_1$
Linear 2	$\beta(\tau) = \beta_M - \left(\frac{\beta_M}{\tau_M}\right)(\tau)$	$0 \leq \tau \leq \tau_M$
Rational	$\beta(\tau) = \left(\frac{\beta_M}{1 + a\tau}\right) a > 0$	$0 \leq \tau$
Step	$\beta(\tau) = 1$	$0 \leq \tau \leq \tau_M$
Exponential	$\beta(\tau) = \beta_M e^{-a\tau} a > 0$	$0 \leq \tau$
Mixed Exponential[1*]	$\beta(\tau_1, \tau_2) = \beta_1 e^{-a_1\tau_1} + \beta_2 e^{-a_2\tau_2}$ $\beta_1, \beta_2 \geq 0$ $\beta_1 + \beta_2 \leq 1$ $a_1, a_2 > 0$	$0 \leq \tau_1, \tau_2$

With:
τ : The remaining time until replenishment
$\beta(\tau)$: Stock out
β_0 : The initial stock out level

β_M : The maximum stock out level
T_1 : Permissible time for backordering τ_M : Maximum Customer's waiting time
^{1*} it is assumed that there are two kind of customers.

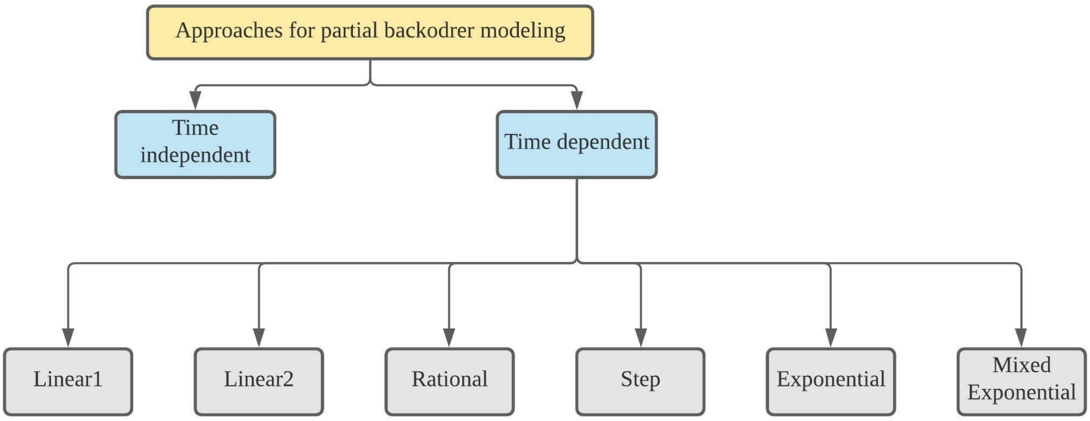

Figure 3.3: Partial backorder forms in the literature (Sazvar, 2013)

3.4.2 Other cost components

As previously mentioned, many researchers extended traditional models to consider certain aspects of reality by including costs such as the advertisement cost Luo (1998), inspection cost Chung and Wee (2008), the greenhouse gas penalty cost Wangsa (2016), price discounting Pandey and Vaish (2017) and loss of sales Gothi et al. (2017).

Pandey and Vaish (2017) developed an EOQ model considering the seasonal quadratic type of demand, which started with zero, reached its maximum and ended with zero. A fraction of the demand was backlogged until the next replenishment, and a price discount was given on the backorder quantity in order to reduce the lost sales. Wangsa (2016) developed a joint economic model for a buyer-vendor scenario in which penalty and incentive policies of the government to reduce emissions are considered. The proposed model involved the greenhouse gas emission from the industrial and transport sectors. He divided the emission into two types, namely: direct and indirect emissions for the buyer's demand that was normally distributed and partially backordered. The proposed model optimized the joint total cost incurred by a single manufacturer and a single buyer and involves the transportation costs of the freight forwarder, with Transportation costs dependent on the distance, shipping weight, fuel price and consumption. Finally, an algorithmic procedure was proposed to determine the optimal order quantity, the actual shipment weight, the total emission, the safety factor and the frequency of delivery. Chung and Wee (2008) dealt with a production inventory policy for items with negligible imperfect products. However, screening costs for defective items were considered as the items underwent further processing. Gothi et al (2017) proposed a production system for deteriorating items with time dependent demand and loss of sales. Their model assumes that defective items can be repaired and that shortages are partially backlogged.

3.5 Nature and function of the demand function

3.5.1 Nature of the demand

Problems in inventory management exist mainly due to demand. In general, the demand is not directly controllable, and in many circumstances, it cannot be controlled indirectly. The demand is mainly influenced by choices made by people outside the organization. However, properties such as the size, rate and patterns of the process may be evaluated. According to Giard (2003), analysing demand implies that three essential elements should be considered:

- The demand pattern
- The source of demand
- Factors influencing demand.

The demand pattern

The demand pattern is one of the critical elements contributing to the complexity of modelling replenishment for deteriorating items. The type of item and the quantity to fulfil often characterise demand patterns. Many researchers consider the demand size deterministic; however, the demand size is often a random variable that follows certain probability distributions in real-life situations.

Source of demand

Demand can be internal or external. The difference between internal and external demand is based on requirements and planning. For instance, for an internal demand the number of consumers can be limited, the requirements are dependent; however, for external demand, the number of clients can be unlimited and the requirements independent.

Factors influencing demand

The demand for many items has always been subject to variations. There are situations where the variations of certain factors are substantial and scenarios whereby it is essential to have specific items available. It is important to note that multiple variables such as quantity offered, quality, price, season, the discounts offered and budgetary constraints always impact demand.

- Effect of the income: A client may always buy several goods. However, it is impossible to buy everything due to limited income. Thus, the budget constraint becomes an essential element of the dynamic nature of demand. Income illustrates the combinations of goods that a particular consumer can purchase based on income and the prices of the goods. Therefore, income has a very considerable impact on the behaviour of demand components over time.
- Change in the price of goods: Change in the price of goods can affect the consumer's choice. To better understand this phenomenon, Mutombo (2015) analysed consumers' behaviours, when a change in the price of goods occurs. The author came to the conclusion that a decrease in price opens up new possibilities for consumers to purchase more. The change in the price of items can be broken down into two effects:
  - Since item x is cheaper, the income gives customers greater purchasing power. The clients are somehow wealthier. Thus, they may purchase more items x. This reaction describes the income effect.
  - Due to item x being cheaper, each item y sacrificed by the consumer gives the consumer more of item x. As the price of item y becomes relatively more expensive than item x, consumers are likely to purchase less of item y. This phenomenon describes the substitution effect. Lowering the price of an item improves the consumer's situation. The improvement in the consumer's purchasing power will have an impact on both goods. The income and substitution effects lead consumers to purchase more items compared to rival products.

Some goods escape the law of demand described by Mutombo (2015) and for which demand curve increases with the price. These relevant effects are described as follow:

- The Veblen effect: Anomalous market behaviour in which consumers buy higher-priced items while similar low-priced alternatives are available. This is caused either by the desire for conspicuous consumption or the belief that higher priced items are of a higher quality.

- Anticipation: In the event of a generalized price increase, the observed behaviour opposed the law of demand. Therefore, consumers may find themselves in a situation of increasing prices such that the best solution is to purchase goods in order to avoid being affected by the price increase.

- King effect: for agricultural products, demand is generally very inelastic to prices. As a result, good harvests may lead to a decrease in farmers' total income, while bad harvests will have the opposite effect. According to Peer (2013), King's Law or the King Effect refers to as a situation in which a deficit of an item on the market causes its price to rise to such an extent that the value of the product or crop increases disproportionately.

3.5.2 Demand function

The demand function is another crucial element in modelling deteriorating inventories. Studies in inventory management are subdivided into two major groups (Figure 3.4):

- Constant demand rate models
- Variable demand rate models.

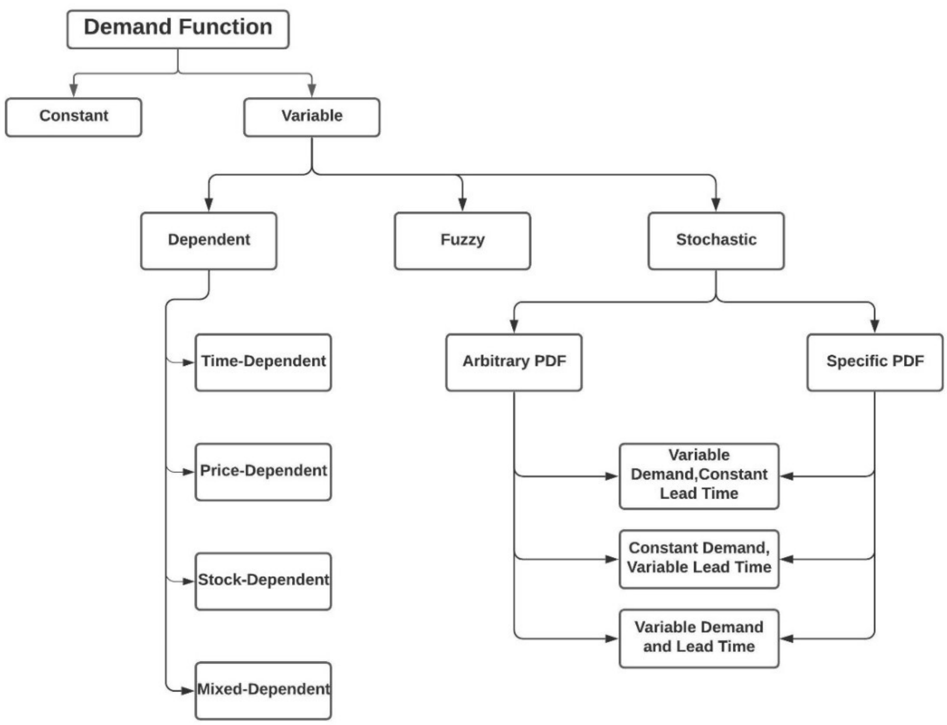

Figure 3.4: Classification of demand function

Constant deterministic demand rate

One of the most critical assumptions in inventory management is that demand is a constant deterministic function of stock, price or time over a finite or infinite planning horizon, as shown in Figure (3.5). This hypothesis and many others may look unrealistic; however, it is essential to keep in mind that all models are abstract versions of reality with the goal being to provide valuable insights rather than to be a precise representations of actual conditions. Secondly, this is a basic assumption that can be extended in several ways. Ghare and Schrader (1963) proposed an inventory

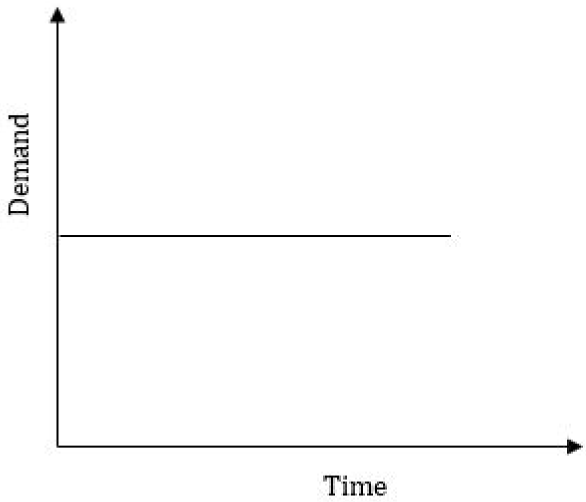

Figure 3.5: Constant demand function over time

model under constant demand. Heng et al. (1991) studied a replenishment policy with exponential decay and constant demand rate. Studies such as Muehlemann and Valtis-Spanopoulos (1980), Rau et al. (2003), Ferguson et al. (2007), Jhuma and Samanta (2011) and Wang et al. (2011) fall into this category.

Variable demand rate

Constant deterministic demand assumes that no change in the demand rate occurs, which is unrealistic because the demand for any given item cannot be subject to a fixed rate. In section 3.5.1, it has been shown that several factors influence demand. This observation prompted researchers to consider the demand function in several ways, such as time, seasonal, stock or price dependent. Many inventory researchers have paid their attention to variable demand functions, focusing on dependent, stochastic and fuzzy demands functions.

- **Dependent demand**: Some research in deteriorating inventory have been carried out with price-dependent, stock-dependent and time-dependent demand functions. Khanra et al. (2011) studied an EOQ model for a deteriorating item with time varying quadratic demand under permissible delay in payment. Tripathi et al. (2016) formulated an inventory model with time dependent demand. Venkateswarlu and Mohan (2013) developed a deterministic inventory for deteriorating items in which the demand rate is assumed to be quadratic price dependent together with a time dependent deterioration rate function. Kumar and Rajput (20.13) presented an EOQ model for deteriorating Weibull items with a price-dependent demand rate and a replenishment policy for profit maximization. Deteriorating Models in Sharmila and Uthayakumar (2016), and Krishna and Bani (2016) are considered mixed dependent demand.

- **Stochastic demand**: In section 3.5.2, it is assumed that the demand is deterministic. In reality, this is unlikely to be true. From an actual world perspective, a stochastic distribution of demand is more realistic. Stochastic demand functions can be categorized into two groups depending on the type of distribution Goyal and Giri (2001):

  - By considering specific types of probability distribution functions (PDFs), as in the work of Weiss (1982), in which the demand followed a Poisson distribution.
  - By considering the demand rate as an arbitrary probability distribution function (ADF). Tadashi et al. (1993) proposed a stochastic EOQ model with discounting. Berman and Perry (2006) formulated a stochastic EOQ type model with an arbitrary random demand.

- **Fuzzy demand**: models such as Yao and al. (2000), Yao and Lee (2003) consider a fuzzy function to represent the demand. Mahata and Dutta (2007) proposed an EPQ with fuzzy deterioration rate and fuzzy demand rate with a production loss incurred due to faulty or old machines, with manufacturing defect also being considered by assuming a fraction of production item deteriorates per unit time. The demand and deterioration were defuzzified through the signed distance method to find the fuzzy number's membership function through extension principles. Neha and Soni (2012) studied an economic production model with finite production rate and fuzzy deterioration rate. Jeyakumari et al.(2018) considered set up cost, inventory holding cost and demand as fuzzy parameters. They addressed the question of establishing the inventory model for time-degenerating items in the fuzzy sense using penalty cost under the conditions of infinite production.

Table (3.2) presents a summary of some significant studies that have been undertaken. For each research, the approach adopted to model the deteriorating inventory, the type of the supply chain system, the deterioration function, demand function, and shortage are presented together with the objective function. This leads to a review of models of deteriorating manufacturing processes, which is the subject of the next chapter.

Table 3.2: Summary of selected deteriorating inventories

Objective Function **R**: revenue **H**: holding cost **SH**: Shortage cost **P**: Procurement cost **S**: Setup cost **D**: Deterioration cost **LC**: the opportunity cost due to lost sales	$Max\{R - (S + H + SH + P + LC)\}$	$Min\left\{\sum_{R,D,P^{3*}} S + H + D\right\}$	$Max\left\{R - \left(S + H + D + R_{mc}^{3*} + S_{pc}^{4*}\right)_{Supplier\&retailer}\right\}$
Planning Horizon	Infinite	Infinite	Single period
Shortage ***PB***: *Partial backorder* ***CB***: *Completely backorder* ***CL***: *Completely lost sale*	A general non-increasing time-dependent function	NA	NA
Inventory System	EOQ	EOQ	EOQ
Lead-Time	0	0	0
Deterioration Function *CST: Constant*	Non-instantaneous	Time-Dependent	CST
Demand Function *CST: Constant*	$\{D(p,t) = g(p).f(t)\}^{1*}$	CST.	CST.
Solution Approach	Exact	Heuristic	Exact
Supplier	1	1	1
Retailer	1	1	1
Echelon	1	3	2
Class	1	1	1
Author	Valliathal & Uthayakumar	Wang et al.	Wee et al.
Year	2011	2011	2011

$g(p)^1$ is a non − negative, continuous, convex decreasing function of the selling price (p) and f(t) is a non − negative, continuous function of time.

R_{mc}^{2*}: Remanufacturing cost

R^3: retailer, D^3: distributor, P^3: producer.

R_{pc}^{4*}: Scrap processing cost

Objective Function **R**: revenue **H**: holding cost **SH**: Shortage cost **P**: Procurement cost **S**: Setup cost **D**: Deterioration cost **LC**: the opportunity cost due to lost sales	$Max\{R - (H + P + P_{ST}{}^{4*})\}$	$Max\{\sum_n R_n - (H_n + S_n(Minor\&Maj + (R - (H + S(Minor\&Majo$	$Min\{(S + H + D)_{FP} + (S + H + D)_{RM}\}$
Planning Horizon	Finite	Finite	Finite
Shortage ***PB***: *Partial backorder* ***CB***: *Completely backorder* ***CL***: *Completely lost sale*	NA	NA	NA
Inventory System	EOQ	EOQ	EOQ
Lead-Time	0	0	0
Deterioration Function *CST: Constant*	CST	CST	CST
Demand Function *CST: Constant*	$\{D_j(p_j) = \alpha - \beta p_j - \gamma p_j^2, \alpha, \beta, \gamma > 0\}^{1*}$ $\{D_j(p_j) = \alpha p_j^{-\varepsilon}, \alpha, \varepsilon > 0\}^{2*}$	$D(b, p, t) = (a_i - b_i p_{j,i}) exp(-\beta_i t$ $j = (I, NI)^{3*}$ $i = retailer$	A piecewise Function with time
Solution Approach	Exact	Exact	Exact
Supplier	0	1	1
Retailer	1	N	1
Echelon	1	2	1
Author	Sana	Chen & Chang	He et al.
Year	2011	2010	2010

1^* Quadratic price − dependent

2^* A negative power function of price

3^* (I, NI) → (Integrated, Non − Integrated),

4^* Pricing Setting Cost

5^* finished products

Objective Function **R**: revenue **H**: holding cost **SH**: Shortage cost **P**: Procurement cost **S**: Setup cost **D**:Deterioration cost	$Min\{(H + D + S)_{retailer} + (H + D + S + flexibility + Ins_{cd}^{3*})_{SCFM} + (H + RM^{4*} + repair + salvage)_{SCRM}\}^{1*}$	$Min\{S + H + D + A_{coc}{}^{3*}\}$	$Max\{R_{fresh\ and\ order\ product} - (H + P)\}$
Planning Horizon	Infinite	Infinite	Infinite
Shortage ***PB**: Partial backorder* ***CB**: Completely backorder* ***CL**:Completely lost sale*	NA	NA	NA
Inventory System	Periodic	Continuous	EOQ
Lead-Time	CST	0	0
Deterioration Function *CST: Constant*	CST	CST	Nonlinearly time-dependent holding cost
Demand Function *CST: Constant*	CST	CST	Inventory dependent $D(I) = D * I^{\beta}$ $D > 0, 0 < \beta < 1, I > 0$
Solution Approach	Exact	Exact	Heuristic
Supplier	1	0	-
Retailer	1	1	1
Echelon	2	1	1
Author	Wee & Chung	Liao	Urban
Year	2009	2008	2008

1* SCFM: supplier's cost of forward manufacturing, SCRM: supplier's cost of reverse manufacturing

2* the annual capital opportunity

3*inspection component_design_life

4*remanufacturing

Objective Function **R**: revenue **H**: holding cost **SH**: Shortage cost **P**: Procurement cost **S**: Setup cost **D**:Deterioration cost	$Min\{H + S\}$	$Min\{H + S\}$	$Min\{(S + H + P)_{sup\&ret}\}$	$Min\{H + S + D + interest\ payable - interest\ earned\}$
Planning Horizon	infinite	Infinite	finite	infinite
Shortage ***PB***: *Partial backorder* ***CB***: *Completely backorder* ***CL***:*Completely lost sale*	NA	NA	Only for retailer	NA
Inventory System	EOQ	EOQ	Continuous	periodic
Lead-Time	0	0	CST	0
Deterioration Function *CST: Constant*	$h * t^r, r > 0$	Nonlinearly time-dependent holding cost	CST	A constant $\theta + A$ fuzzy holding cost
Demand Function *CST: Constant*	CST	Inventory-dependent $D(I) = D * I^{\beta}$ $D > 0, 0 < \beta < 1, I > 0$	CST	A fuzzy number
Solution Approach	Regression	Exact	Exact	Heuristic
Supplier	-	-	1	-
Retailer	1	1	1	1
Echelon	1	1	2	1
Author	Ferguson et al.	Alfares	Lin &Lin	Mahata & Goswam
Year	2007	2007	2007	2007

Objective Function **R**: revenue **H**: holding cost **SH**: Shortage cost **P**: Procurement cost **S**: Setup cost **D**:Deterioration cost	$Min\{(H+S+D+delivery\ cos$ $+(reciving\ co$ $+H+D)_{producer(RM}$ $+(S+delivery\ cos$ $+H+D)_{producer(FC}$ $+(S+reciving\ cos$ $+H+D)_{retailer}\}$[1*]	$Min\{H+S\}$	$Min\{H+S+D\}$	$Min\{H+S\}$
Planning Horizon	infinite	Infinite/finit e	infinite	infinite
Shortage ***PB***: *Partial backorder* ***CB***: *Completely backorder* ***CL***:*Completely lost sale*	NA	PB: A constant fraction of no fulfilled demand	NA	NA
Inventory System	EOQ	periodic	EOQ	EOQ
Lead-Time	0	0	0	0
Deterioration Function *CST: Constant*	CST	A known function of time $\theta(t)$	A constant θ + 1. $ht^r, r \geq 1$ 2. $ht^r,\ r \geq 1$	$1-ht^r$, $r \geq 1$ $2-ht^r$, $r \geq 1$
Demand Function *CST: Constant*	CST	A known function of time $D(t)$	Inventory-dependent $D(I) = D * I^{\beta}$ $D > 0, 0 < \beta < 1, I > 0$	Inventory-dependent $D(I) = D * I^{\beta}$ $D > 0, 0 < \beta < 1, 0 < I < Q$
Solution Approach	Heuristic	Exact and Heuristic[2*]	Numerical	Exact
Supplier	1	-	-	-
Retailer	1	1	1	1
Echelon	3	1	1	1
Author	Rau et al.	Goyal & Giri	Girl & Chaudhuri	Goh
Year	2003	2003	1998	1994

1* RM – raw material, FC – finished cost.

Objective Function **R**: revenue **H**: holding cost **SH**: Shortage cost **P**: Procurement cost **S**: Setup cost **D**:Deterioration cost	$Min\{H + S\}$	$Min\{H + S\}$ *Multi-product
Planning Horizon	infinite	Infinite
Shortage ***PB***: *Partial backorder* ***CB***: *Completely backorder* ***CL***:*Completely lost sale*	NA	NA
Inventory System	EOQ	EOQ
Lead-Time	0	0
Deterioration Function *CST: Constant*	ht^r $r \geq 1$	Non-linearly dependent to the average value of capital invested in stock
Demand Function *CST: Constant*	1- CST 2- Poisson	CST
Solution Approach	Heuristic	Exact
Supplier	-	-
Retailer	1	1
Echelon	1	1
Author	Weiss	Muhlemann & Spanopoulos
Year	1982	1980

Chapter 4

Review of deteriorating manufacturing systems

4.1 introduction

Classical manufacturing systems are built on the idea that the process operates flawlessly with machines and equipment designed to last indefinitely, generating items of perfect quality at a constant production rate. This is not always true because facilities and equipment are subject to deterioration over time due to several causes such as age, deformation due to operation, wear and tear, and the effects of corrosive agents (chemicals, atmospheric agents, etc.). These damages can lead to a decrease in the production capacity, breakdowns which can result in the shutdown of the plants, the production of more rejects, or decrease in the market value of the products produced due to design or manufacturing failures; that is, due to the poor quality of the finished products. In any case, these damages are a source of additional direct or indirect costs. In some manufacturing systems, it is assumed that, stopping a machine for repair after a breakdown restores the damaged machine to its original state. However, if this were always the case, the systems could be operational for an almost infinite time, which is almost impossible.

This chapter is arranged as follows: a review of non-flexible manufacturing systems is presented in section 4.2; section 4.3 provides a comprehensive literature focusing on the flexibility and adaptability of systems subject to breakdowns, the most relevant topic in this research study. The integration of quality is presented in section 4.4; finally, a conclusion and motivations for the models we will propose in the next chapter is presented in section 4.5. Our objective is by no means to duplicate works already done, but rather to show the complexity of modelling manufacturing systems for perishable inventories with failure-states of machines while considering imperfect production

4.2 Non-flexible manufacturing systems

In today's competitive business environment, organizations are increasingly concerned with ensuring the efficiency of their production processes to optimize the total cost incurred, or profit earned. A production system (PS) being a set of interconnected sub-systems such as machines, assembly stations, conveyors, etc., used for the production of goods or services; their availability and reliability are the most crucial factors influencing the efficiency of the production system. While the production system was first proposed by Taft (1918) as an extension of the Harris model demonstrated in the previous chapter, it is only in the last 35 years that the field of manufacturing system optimization has been extended. The characterization of production systems provides a mechanism for identifying the conceptual and functional attributes that drive their multiple performance levels. The behaviour of a production process and its constituents can be modelled through two states with performance levels associated with each of these states (Figure 4.1), such as the functional state and the non-functional state (Zaitseva, 2003).

Figure 4.1: Functional diagram of a production system with two binary states

The problem of modelling production systems is often subject to assumptions, and the specificity of the cases studied. Thus, the choice of the methodology for evaluating the indicators of effectiveness is determined by the complexity of the systems being modelled. Kimemia and Gershwin (1983) are considered to be the first to introduce the concept of hedging point policies in manufacturing, a concept that consists of maintaining a certain level of stock called safety stock at an optimal level to prevent possible unforeseen circumstances that could occur such as machine breakdowns, unscheduled shutdowns in unreliable production systems. They proposed a control model using a heuristic approach to approximate the inventory level to minimise the total cost. Akella and Kumar (1986) developed an optimal control policy in a failure manufacturing system using Hamilton-Jacobi-Bellman (HJB) under linear assumptions for surplus and backlog cost of a production control problem dependent on random breakdowns and repairs. Their control policy was represented by the following equation:

$$u(x(t), \alpha(t)) = \begin{cases} \alpha(t)u_{\max}, \text{ if } x(t) < z^* \\ \alpha(t)d, \text{ if } x(t) = z^* \\ 0, \text{ if } x(t) > z^* \end{cases} \tag{4.1}$$

Their system controls the production rate $u(x(t), \alpha(t))$ in terms of stock $x(t)$ and the state of the machine at any given time $\alpha(t) \in \{0, 1\}$, with $\alpha(t) = 1$ if the equipment is operational, $\alpha(t) = 0$ if the equipment is non-operational, with z^* the hedging point. Bielecki and Kumar (1988) validated the control system proposed by Akella and Kumar (1986), considering a random state evolution modelled as a birth-death process to find real-time production rates in response to a stochastic disturbance and to optimise the long-term average cost. Sethi et al. (1991) proposed an asymptotic analysis of hierarchical production planning in a manufacturing system with two tandem machines that are subject to breakdown and repair. Srivatsan and Dallery (1996) extended Bielecki and Kumar (1988)'s model by considering a two-part-type manufacturing systems and minimized the total expected surplus and backlog costs.

Gharbi and Kenne (2003) proposed an optimal control policy for a system composed by several machines and producing several products using HJB equations. The authors combined analytical methods and experimental design simulation to find an estimation of the optimal policy because the HJB equations are complex to solve for a multi-product production system. Krasik et al. (2008) extended Gharbi and Kenne's (2003) model to a system with multiple machines in parallel, producing multiple products with non-zero setup costs. An algorithm based on dynamic programming was used to determine the optimal cost consisting of inventory holding cost and shortage costs. Gershwin et al. (2009) studied a manufacturing system by incorporating the fraction of customers who withdraw their orders when shortages have reached a certain level. Gupta and Arora (2010) studied a manufacturing system of perishable inventory intending to meet a linearly increasing demand. For the study and modelling of the functionality of such a system, a Multi Production System (MPS) for items that deteriorate according to a special form of Weibull density function is considered. Sharmila and Uthayakumar (2016) investigated the effect of stock dependent demand and time on the MPS. Their approach is based on a combination of demand flexibility and item shelf life expressed in terms of Weibull distribution to minimise the average total cost for operating the production system and keeping the stock in inventory. Swagatika et al. (2019) developed a MPS with three production rates in both crisp and fuzzy sense. Their study has been addressed by considering demand in the power form of frequency of advertisement and unit price of the item, particularly for the fuzzy model.

None of the proposed models considers an approach that simultaneously integrates production management and continuous operation of resources in a failure state. When a failure occurs, the production process is usually stopped for repair. However, in flexible manufacturing systems, after a random failure, the machines may continue to operate at a rate lower than the initial rate until a specific inventory is reached because stopping the production process, such as the case of the above-proposed models above, would incur high costs and customer losses. Therefore, it is necessary to address the concept of multi-state systems with unreliable equipment in continuous manufacturing planning and control.

4.3 Flexible manufacturing systems in inventory management

Interest in research dealing with manufacturing systems with machines subject to repairs and failures has been on the rise. An important manufacturing systems' class is the one where the system is continuous and flexible. Several systems and their components undergo different stages of deterioration during their shelf lives. Deterioration processes or systems degradation studied in literature are often due to environmental conditions such as corrosion, erosion, or physical phenomena such as vibration, wear, fatigue, shock, etc. (Lam and Yeh, 1994). Different states of deterioration are often used for the study and modelling of production systems and their components, each of which reflects the stage of the system or its components. If, from the production point of view, a system is conceived in a way that, at the occurrence of any failure, a reconfiguration is undertaken automatically, allowing the degraded machine or any other equipment to be functional, but with a decrease of the service delivered, we refer to this as a Multi-State System (MSS) or degraded system. Thus, a third state is added to the two previous states represented in Figure (4.1), referred to as the degraded state. A system is considered degraded if, after a failure, it remains operational but with a decreased level of performance. In other words, in a degraded state of a system, the process operates with a loss of performance due to a breakdown of one or more of its components. In contrast, a system is described as non-degradable if it is failure-free and continues to be functional by yielding the same level of performance at any point in time.

In the research based on non-flexible production systems illustrated in section 4.2, ensuring continuity of the production is an important issue. In those models developed, the occurrence of a failure would result in the complete shutdown of the production system (Figure 4.1). This type of system can be highly costly, disastrous, and lead to production line losses in manufacturing. Wu and Chan (2003) proposed a more robust system with unreliable machines while ensuring the continuity of production (Figure 4.2). This continuity of service depends mainly on

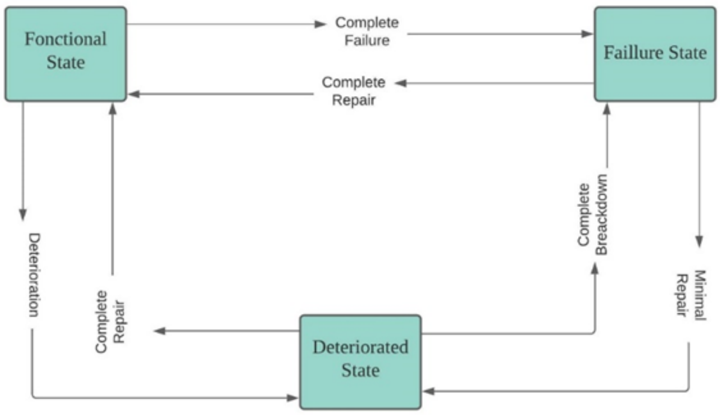

Figure 4.2: Functional diagram of a MSS with three binary states

the state of the partial degradation of the manufacturing system. For such systems, the breakdown of any component only minimally or at least partially disrupts their performance. In this way, the system can continue to provide service with an acceptable level of degradation. This approach differs from conventional methods implying the complete shutdown of the system, as it is a cost-effective approach.

To anticipate any severe failure of the MSS, proper analysis of the degraded mode is often carried out. This involves identifying the component that must be urgently restored. Such an evaluation aims to determine the most suitable compromise between the various possible options and tolerable losses. Having identified, assessed, and prioritised the risks, the procedures to govern the downgraded mode are implemented.
Generally, the approach adopted for operating a system in degraded mode is described as follows:

1. Identifying and classifying the potential failures in terms of degree of severity;

2. Determining the impact of these failures on the manufacturing system;

3. Determining alternative control strategies to ensure the robustness of the system once the failure mode is activated.

MSS with unreliable machines subject to failures and random repair have been widely investigated by several researchers. Boukas and Haurie (1990) studied a MSS composed of two machines to optimize the total expected cost in terms of production rates and maintenance rules over an infinite planning horizon. Boukas and Haurie (1990) extended Rishel's (1975) model by combining production planning of unreliable machines deteriorating with the concept of hedging point introduced by Kimemia and Gershwin (1983) and preventive maintenance. Koulamas (1993) considered a single-stage and serial production line systems with unreliable machines and general probability distributions by using a Markovian approach to characterize the states of the production process (busy, idle, undergoing repair).

Hu et al. (1994) proposed a single product, single- unreliable machine production system with production rate dependent on failure rate. The authors indicated that hedging point policies are optimal if the failure rate is linearly production-dependent and suggested that the machine's production rate be reduced as the hedging point is approached to account for its reliability. Martinelli (2007) validated the results provided by Hu et al. (1994) by considering an MSS with an unreliable machine subjected to two different failure rates. The author later extended Martinelli's (2007) problem by considering an unreliable machine with the failure rate as a piecewise increasing function of the production rate. Dehayem et al. (2011) considered a semi-Markov method for a deteriorating production system dependent on the age and the number of breakdowns. Moreover, the authors demonstrated that the process becomes more complex if the degradation of the system varies with the production rate. Figure (4.3) represents the behaviour an MSS with a failure state (degraded mode).
The MSS system represented in Figure 4.3 shows that at time t_1, the failure of a constituent of the system is identified. The system switches to a failure state. This state remains active until the repair of the damaged component is completed after which the system switches to full functionality at time t_2. Kouedeu et al. (2014) proposed a single product, two unreliable, non-identical machines model with random breakdowns and repairs. They indicated that the production speed should be reduced when approaching the hedging point of finished products to address machine reliability and reduce the total cost incurred. In addition, several methods have been developed to improve such a system, including redundancy, safe control-monitoring design and reconfiguration of functions. For systems with unreliable components in which failure rates are time-dependent, preventive replacement may be an option for enhancing the reliability of the system (Levitin and Lisnianski, 1999). However, the maintenance of systems having components of a system that are subject to breakdown prevents the risk of failure but also lead to high costs, particularly

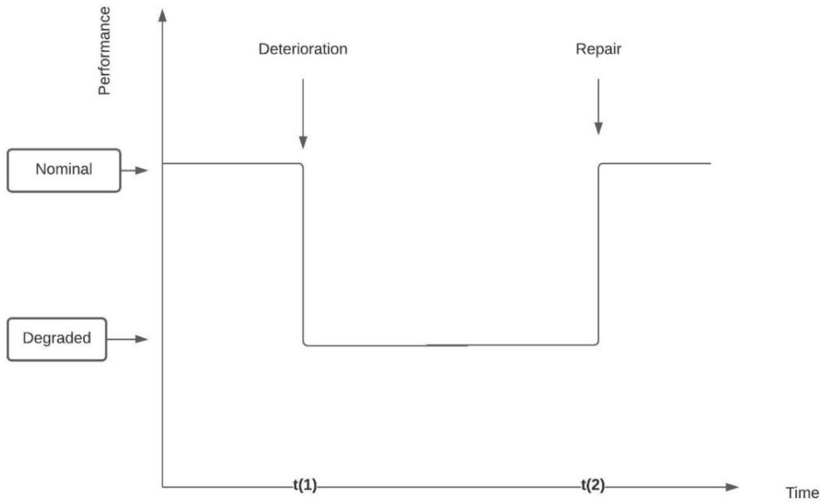

Figure 4.3: State of a MSS in terms of equipment deterioration

for systems with high sensitive components. Moreover, minimal repair can be considered the cheapest option that allows the system's constituents to resume their function after failure of the system due to cumulative breakdowns of components (Beihelt and Fisher, 1980). Several studies in the field of optimisation of multi-state systems' reliability aim to identify the optimal number of redundant components in each subsystem of a production system to obtain the optimal solution by using non-linear, integer or mixed programming approaches (Tillman et al., 1997). However, the processing time required to find an optimal solution is likely to increase exponentially with the size of the problem, which makes exact methods inefficient in large scale (Nahas, 2008). Heuristics have been developed for this purpose, and the most commonly used meta heuristics for multi-state systems are ant colonies, genetic algorithms, and neural networks. They do not guarantee an optimal solution; however, they provide an acceptable solution in the case of medium-sized combinatorial problems.

4.4 Integration of quality control in manufacturing systems

Conventional production systems are based on some simplistic assumptions that overlook many real-life factors. Several production systems assume that the outputs of a production system are always of perfect quality and ignored the process in which defective products are generated, but instead considered that the yield of the manufacturing process is perfect and equal to 100%. Production processes are often imperfect, and their outputs may contain some bad finished products. Such products can either be reworked, scrapped, sold at a salvage price, or undergo further processing. In all cases, this results in substantial costs. In the case where the items may need further processing, additional costs such as the cost of the additional resources needed to further process the items, and the costs associated with the increased lead-time resulting from the processing of the articles are incurred.

To ensure the quality expected by customers are met, manufacturers may need to inspect the finished products before being delivered to customers. At the end of this control, the products are classified as either being conforming or non-conforming based on their quality. Rosenblatt and Lee (1987) were among the first to introduce the concept of quality control in inventory control by modelling a process in which defectives articles are generated explicitly based on whether the production system is "in control" or "out of control". Porteus (1990) later extended Rosenblatt and Lee's (1987) work by considering a case where inspection can be performed during the production cycle to identify any defective during the production process. Shamsi et. al. (2009) integrated imperfect quality items, rework, backlogging and inspection error into a single EPQ model. Alimohamadi el. at. (2011) developed EPQ model by considering preventive maintenance work in process inventory with reworkable items and shortage. Rivera-Gomez et al (2013) discussed an unreliable production system in which the scrap rate varies with the age of the machines. Abdul et al (2012) and El-Kassar et al. (2012) have independently discussed a manufacturing system with imperfect raw materials purchased from the supplier and its reverse logistics. Krishnamoorthi and Panayappan (2012) developed an inventory system for a single stage production process with rework of imperfect and shortages. Li et al. (2015) examined a production model jointly considering product deterioration and unreliable machines with time increasing defective rates and rework.

Majumder et al (2016) proposed a model considering an imperfect production process for breakable goods within a limited time frame to optimize the profit as well as the reliability indicator (r) using the Euler-Lagrange equations. Patra (2018) presented a non-reliable production system for deteriorating inventory with advertisement and price dependent demand and no rework. A "Multi-Objective Genetic Algorithm (MOGA)" was used to maximize the profit and minimize the risk. Sakou (2019) investigated a non-reliable production system consisting of a single machine and a single product subject to random failures and repairs. The author then used control theory based on dynamic programming of the HJB type to optimize the production policy that minimized the total cost. Kumar (2020) proposed an imperfect production system with imperfect inspection processes and rework of the defectives produced and scrapping of the non-reworkable defectives under complete backlogging.

4.5 Conclusion

The quest for continuous improvement drives managers to develop strategies of seeking more significant levels of productivity, quality and safety in production systems. Faced with the globalization of markets and technological issues in production, manufacturing factories are confronted with issues related to optimizing their production systems. Production planning problems in a manufacturing system are often subject to constraints related to the equipment used, the quantity and the quality of the products produced. A company wins when it can meet its customers' needs without unnecessary costs. For this reason, every organization will seek to maximize its profit and minimize its costs. However, if one or more components of a production system are unavailable due to some failure, the system will not deliver the required service. An extensive literature has been dedicated to production systems with binary states in which a system is either operational or out of service. Strategies for improving or restoring the performance of these binary systems have often been based on the use of redundancy, preventive maintenance, etc.

Research on MSS is sparse and has only recently begun to receive attention, following the publication of Ben-Daya et al.'s (2009) work. Several extensions have been made, but none of them takes into account the MSS with unreliable machines and imperfect quality outputs, stock dependent demand, time exponential deterioration function of items, stochastic failure rate, green house policies, capacity limits, non-linear shortage costs, quantity discounts, salvage price, maintenance, rework and risk measures. Substitutability is rarely addressed in deteriorating inventories, whilst it is relevant to inventory managers. Owing to these gaps in the current body of knowledge, this study seeks to bridge some of the mentioned gaps in the next chapter. Therefore, when formulating the specific model, the assumption of deterioration of a machine, as used in the model developed by Ben-Daya et al. (2008), is adapted because, despite the occurrence of a failure, the system reconfiguration mechanisms with partial reduction of the nominal performance of the equipment ensures the continuation of service. Furthermore, in addition to Ben Daya et al.'s (2008) model, the use of modelling methods and implementation of optimisation strategies for performance measures proposed in Bhowmick and Samantha's (2011) model is also integrated. In addition, for market survival, the manufacturer also allows backorder. Obviously, the backordered demand depends on the portion of customers that are willing to stay until the next replenishment, as developed in Gothi et al. (2017). Numerical methods are proposed to evaluate the impact of failure state on performance measurements of the continuous production system.

Chapter 5

A lot sizing model for a Multi-State System with deteriorating items, variable production rate and imperfect quality

5.1 Introduction

The traditional economic production model made a number of simplifying assumptions that might be unrealistic in real-world situations. Ever since the Economic Production Quantity (EPQ) model was first introduced in the early decades of the 21st century, researchers have extended it in many ways through thee relaxation of key assumptions, including considerations of shortages, degradation of equipment, deterioration of goods, variable demand, imperfect quality of the outputs and some combinations of these relaxations. Many inventory models have been developed under the assumption that the lifetime of an item is infinite while it is in storage. In many real life situations, this assumption may not be true. The management of deteriorating inventories has received much attention by several researchers in recent years because deterioration of items is one of the important factors in inventory control problems. Chemicals, fruits, vegetables, fertilizers, perfumes, pharmaceutical products, radioactive substances, gasoline, and different types of oils are examples of deteriorating items. The classical production model of Taft (1918) assumes that the depletion of inventory is due only to the constant demand rate while in many inventory systems, the effect of deterioration cannot be ignored. Whitin (1957) was the first to consider the effect of deterioration in inventory. Ghare and Schrader (1963) proposed a replenishment policy for an exponentially decaying inventory. Covert et al. (1973) developed an EOQ model for deteriorating items by considering a two-parameter Weibull deterioration rate. Datta and Pal (1990) proposed a deterministic inventory system for deteriorating items with constant deterioration rate and demand rate that is a linear function of stock level. Giri et al. (1997) developed heuristic models for deteriorating items with shortages and time-changing request and expense. Panda et al (2008) developed an inventory model for perishable products with time varying demand. Pandey and Vaish (2017) developed an optimal inventory policy for deteriorating items with seasonal demand under the effect of price discounting on unit selling price for backordered quantity so as to enhance the demand and to reduce the lost sale.

Many researchers have also studied production systems where perfect quality items are always produced, but in actual situations, manufactured products may include a number of imperfect items. This defect in the product quality may be the result of many factors such as human errors, wide tolerance, equipment failure, mishandling, and incorrect specifications for raw materials (Muhammad, 2019) and is now being studied. Zang and Gerchak (1990) extended the classical EOQ model to systems with imperfect quality by studying an inspection policy with random yield on lot sizing and assuming that defective units are replaced by non-defective ones. Cheng (1991) presented an EOQ model with imperfect production processes and price dependent demand. Chang (2004) presented a model in which the proportion of items considered imperfect and the demand rate were assumed to be fuzzy variables. Ozdemir (2007) examined an EOQ model with defective items and backordering. Jaber et al. (2008) presented an EOQ model for imperfect quality items subject to learning effects.

Typical models of production systems do not consider processes with speed losses or breakdown of machines. However, considering the manufacturing system as a complex sequence with several unit processes, each with its characteristics, the issues of resource reliability has become an issue that producers have to address because the performance of a system depends on the availability of machinery. Process degradation is a natural phenomenon in a production process that runs for long time, hence, the problem of degradation of processes has been addressed by several authors. Hall (1983) studied the effect of malfunctions of equipment on the quality of products. Sana et al. (2007) extended Sethi and Sethi's (1990) model by considering a system with imperfect production (due to staff impatience), constant demand, loss of sale, and price discounting on the quality items. Ben-Daya et al. (2008) studied an EPQ model with a shifting production rate under stoppages due to speed losses. They demonstrated that process deterioration could be the result of minor stoppages and speed losses, which in practice may affect the efficiency of the process. Kenne and Nkeungoue (2008) proposed homogenous Markov Processes using the Hedging point policy with failures and repairs of machines. The authors assumed that the machine failures were age-dependent. Mehrgani et al. (2014) studied production systems with machines subjected to random breakdowns and repairs under preventive maintenance with human error.

The study of system's reliability has traditionally been based on binary modelling (using two states) namely the operational state and the complete failure state. However, growing literature now considers numerous situations that may occur during the lifetime of some production systems. Such systems may be Multi Production System (MPS) or Multi-State System (MSS). MPS usually starts with low production rates and then ramp up in order to lower the average holding cost as smaller stock level is held for longer time while large stocks are held for longer period. MSS however, may serve both the purpose of holding cost reduction and operation in a degraded state. MSS may be subject to multiple failure modes which may have different effects on their performance. Degradation, being one of these failure modes, allows a machine to continue to perform its function after a breakdown has occurred, but resulting in a partial reduction of its nominal performance. A manufacturing system's effectiveness is usually determined by combining the system's efficiency, the product's availability, and the product's quality. Over the past few years, many studies addressed the effects on process availability (Ben-Daya and Rahim, 2001). However, MSS and its effect on both the efficiency and the quality of the items produced has not been fully addressed in the literature. Some researchers have tried to address this problem by simply inflating the quantity produced in the lot size by an amount needed to take care of the quantity that needs to be discarded without considering the implications of degraded state of the machine and other factors such as the quality of items produced and the shelf life of items stored (Hu et al., 1994; Martinelli, 2007; Dehayem et al., 2011). While this assumption seems reasonable and may yield simpler and more direct mathematical solutions, the authors did not address the question of what happens when the machine continues to work at this higher rate of production, that may lead to more damage to the machine, and may increase the level of defective items produced beyond what would have happened by shifting to a lower production rate. Silver (1990) considered a case of a manufacturing equipment dedicated to the production of a family of items by deliberately slowing down the output to permit the individual item production rates to be treated as controllable variables without taking into account the degraded mode. Khouja and Mehrez (1994) developed an EPQ model where the production rate is a decision variable. Results of this model indicate that there are both weak and strong relationships between the rate of production and process quality. Eiamkanchanalai (1995) extended the work of Khouja and Mehrez (1994) to scenarios where the rate of production is a decision variable by including a linear penalty function due to unused capacity. Shib et al. (2007) considered a production system with adjustable rate with demand for both perfect and imperfect quality items. Ben-Daya et al. (2007) developed a two-state production system, demonstrating the effect of varying production rate on batch sizing due to speed loss. Bhowmick and Samanta (2011) developed a production model for deteriorating items with constant demand, increasing production rates and shortages. Uthayakumar and Sekar (2017) developed an EPQ model for deteriorating items with multi-production setups and rework of imperfect items and salvage value. We now discuss an identified gap in literature

Table 5.1: Gap analysis of related works in literature

References		Characteristics of the EPQ system					Shortage		Deterioration function
Year	Author	Single Production	MPS*	MSS*	Imperfect quality	rework	PB*	CB*	
1918	Taft	✓							
1994	Khouja and Mehrez	✓		✓	✓	✓			
2008	Ben-Daya et al.		✓	✓					
2012	Bhowmick and Samanta		✓					✓	✓
2017	Gothi et al.	✓					✓		✓
2019	Salamah	✓			✓	✓		✓	
This paper			✓	✓	✓		✓		✓

An analysis of the published EPQ model with MSS/MPS systems previously studied in the literature is provided in Table (5.1), which illustrates the various factors considered with the MSS/MPS models by different research articles in the extant literature, and what this paper adds to the research on production with machines in degraded mode. A review of current literature seems to suggest that there is no work published on inventory modelling for deteriorating items, which considered the assumptions of multi-state production systems with non-increasing production rate, imperfect quality and partial backlogging. This paper considers an MSS for a deteriorating item with imperfect items quality and degrading production rates, while also allowing partial backlogging of demand with lost sales. The model, extends the work of Khouja and Mehrez (1994) and Al-Salamah (2019). The goal is to examine deterioration of both products and processes and their impacts on the economic production quantity decisions. In particular, we assume that the degradation of equipment results in a change in the production rate at a time the operator may choose. This model may be applied in many industries in which machines can be subject to failures and where their production rates can be controlled, like in machinery and mechanical assemblies including automobile, milling, turning, drilling, aircraft engine and machine tools, and paper manufacturing plants. The rest of this chapter is structured in the following manner. In section 2, the development of the mathematical model is outlined. The total cost function and optimality conditions are presented in section 4. Numerical examples are presented in section 5. Lastly, the conclusion and future research suggestions are presented in Section 6.

5.2 Formulation of the proposed MSS

5.2.1 Notations and Fundamental assumptions

Assumptions

The following assumptions are made for development of the model:

- The production-inventory system produces a single item.
- Shortages are partially backlogged and partially lost.
- The changeover cost and time from k_1 to k_2 is assumed to be negligible
- All imperfect items are scrapped and disposed as a batch, and a disposal cost is incurred per item scrapped.
- The production shift occurs during the production run, and is consequent to the optimized inventory parameters.
- The deterioration of an item produced follows the exponential function $\theta e^{-\theta t}$ for $t \geq 0$, where θ is the deterioration rate; i.e a constant fraction $\theta (0 \leq \theta \ll 1)$ of the on-hand inventory deteriorates per unit time.

Notations

The following notations are adopted to develop the model:

Table 5.2: Notations used in the formulation of the mathematical model

Symbol	Description
C_a	The deterioration cost per item.
C_s	The shortage cost per item per time
C_b	The disposal cost per unit item
C_p	The penalty cost per unit lost sale
$D(t) = a$	The constant demand rate for item produced
d_1	Proportion of defective units produced during the time interval $[0, t_1]$
d_2	Proportion of defective units produced during the time interval $[0, t_2]$
G	The production setup cost
h	The inventory carrying cost per item per time
H_M	The Hessian Matrix
$I(t)$	The instantaneous state of the inventory level at any time t $(0 \leq t \leq T)$.
$\mathrm{k}_1, \mathrm{k}_2$	The Constant production rates during the time intervals $[0, \mathrm{t}_1]$ and $[\mathrm{t}_1, \mathrm{t}_2]$ respectively
I_1	The inventory level at the end of time $t = t_1)$
I_2	The inventory level at the end of time $t = t_2)$
r	The fraction of demand lost due to inventory stock $-$ out $(0 < \mathrm{r} < 1)$
p_{c1}, p_{c2}	The constant unit production costs when the rate of productions are k_1 and k_2 respectively
S	The maximum shortage, occuring at $\mathrm{t} = \mathrm{t}_4$
T	The total cycle time
CT	The average total cost for the time period $[0,\ \mathrm{T}]$.
$\theta(t)$	The deterioration rate per (units/unit time).
$\rho_1, \rho_2,,$ Ψ, β, τ B, C, E, F	Aggregation parameters for some known variables

5.2.2 Mathematical Formulation

In the following system, a company produces a certain item, which deteriorates over time. Figure (5.1) represents the behaviour of the MSS with a single unreliable machine that produces the item through a degradable process. The system is designed to start operating at a production rate of k_1 and an inventory of perfect quality items accumulates during the first part of the cycle at a rate $(1 - d_1)\, k_1 - a$ while the imperfect quality items accumulate at the rate $d_1 k_1$ and are disposed as a single batch at the end of the cycle. We assume the elapsed time until the production process switches to a 'degraded state' to be t_1, by which time the stock of good items had reached the level I_1. It is assumed that when breakdown occurs, the system is automatically reconfigured to continue to be operational but at a lower production rate, k_2. The same concept was used by Khouja (2005) and Ben-daya et al. (2009) dealing with shifts in the production rate. Therefore, the production rate switches over to k_2 and inventory of perfect items accumulates at the rate $(1 - d_2)\, k_2 - a$ until a level I_2 is reached. The quantity of imperfect quality item continues to accumulate at the rate $d_2 k_2$ in the second production consumption cycle which ends at time t_2 and is also disposed at the end of the cycle. We assume the 'failure state to be a static. This implies that no further deterioration occurs overtime after the system shifts to a "failure" state. Once the level I_2 is reached, production is then stopped for maintenance, and the stock in inventory decreases due to demand and deterioration until the stock level reaches zero. There are two unit costs of production associated with each of the two states of production. These costs are assumed constant in each state of the production, and increases in the second state of the machine due to the degraded state of the machine, which is assumed to need more power and probably labour. After the stock drops down to zero, the system goes into a state of backlog of demand up to S (the maximum backorder level) and thereafter production starts to clear the backlog.Gothi et al. (2017) assumed that demand during the time $[t_4, T]$ is satisfied as the production has already started at time $t = t_4$ and so loss of sales cost during this interval is not taken into account. But in reality some customers are not willing to wait as the company is still not able to meet all the outstanding demand instantaneously. Moreover, it is assumed that the production process is restored to the original production rate only at the beginning of the next cycle.

The differential equations that represent the problem statement of the EPQ model in the interval [0, T] are given by:

$$\frac{dI(t)}{dt} + \theta \times I(t) = (1 - d_1)\, k_1 - a \qquad 0 \le t \le t_1 \tag{5.1}$$

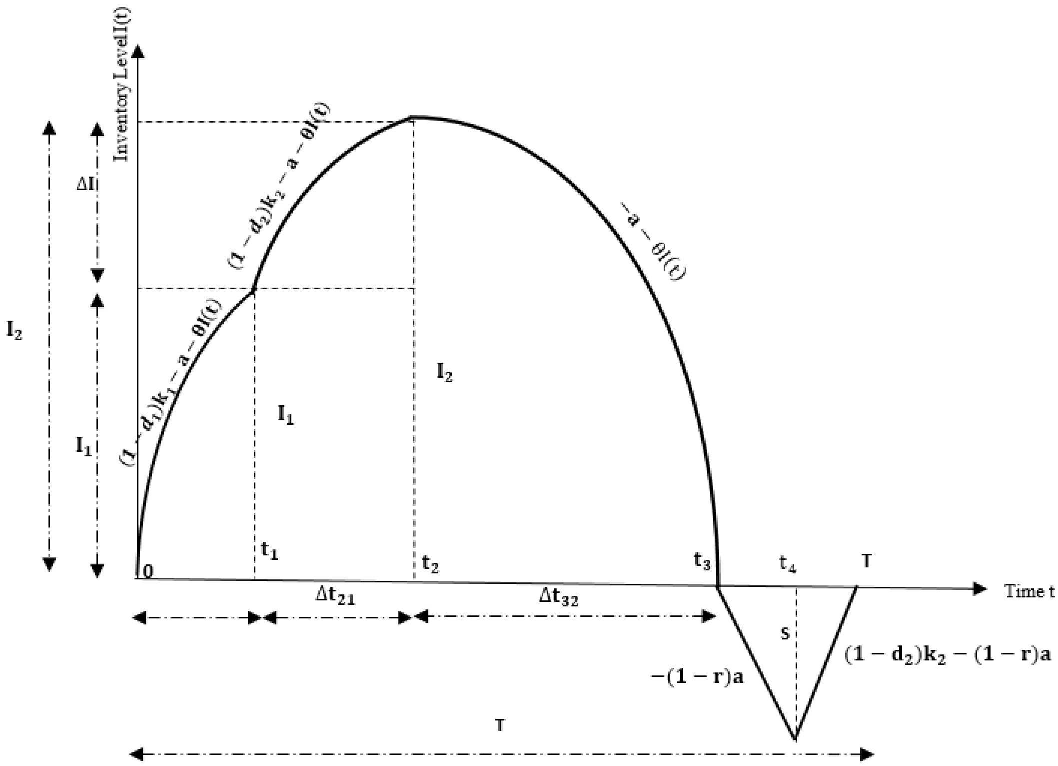

Figure 5.1: Inventory profile of the MSS with alternating production rates

$$\frac{dI(t)}{dt} + \theta \times I(t) = (1 - d_2)\, k_2 - a \qquad t_1 \leq t \leq t_2 \tag{5.2}$$

$$\frac{dI(t)}{dt} + \theta \times I(t) = -a \qquad t_2 \leq t \leq t_3 \tag{5.3}$$

$$\frac{dI(t)}{dt} = -(1 - r) \times a \qquad t_3 \leq t \leq t_4 \tag{5.4}$$

$$\frac{dI(t)}{dt} = [(1 - d_2)\, k_2 - (1 - r)a] \qquad t_4 \leq t \leq T \tag{5.5}$$

Next, the differential equations (5.1) -(5.5) of the manufacturing system are solved.

From equation (5.1)

$$\frac{dI(t)}{dt} + (\theta)I(t) = (1 - d_1)\, k_1 - a$$

$$I(t) = e^{-(\theta)t}\left[\int \frac{(1-d_1)\,k_1 - a}{e^{-(\theta)t}} \times dt + L_1\right]$$

$$I(t) = \left[\frac{(1-d_1))\,k_1}{\theta} - \frac{a}{\theta}\right] + L_1 e^{-\theta t} \tag{5.6}$$

By considering the boundary condition $I(0) = 0$, L_1 is obtained from equation (5.6)

$$\left[\frac{(1-d_1))\,k_1}{\theta} - \frac{a}{\theta}\right] + L_1 = 0$$

$$L_1 = \frac{a}{\theta} - \frac{(1-d_1))\,k_1}{\theta} \tag{5.7}$$

By substituting L_1 from Equation (5.7) into equation (5.6)

$$I(t) = \left[\frac{(1-d_1))\,k_1}{\theta} - \frac{a}{\theta}\right] + \left[-\frac{(1-d_1))\,k_1}{\theta} + \frac{a}{\theta}\right] e^{-(\theta)t}$$

$$I(t) = \left[\frac{(1-d_1))\,k_1}{\theta} - \frac{a}{\theta}\right]\left(1 - e^{-\theta t}\right) \qquad 0 \le t \le t_1 \tag{5.8}$$

By applying the same procedure to equations (5.2) and (5.3), the following are obtained:

$$I(t) = \left[\frac{(1-d_2)\,k_2}{\theta} - \frac{a}{\theta}\right] + L_2 e^{-\theta t} \tag{5.9}$$

From equation (5.9) under the boundary condition $I\,(t_1) = I_1$ we obtain:

$$L_2 = \left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta} - \frac{a}{\theta}\right]\right\} e^{\theta t_1} \tag{5.10}$$

Therefore, the general representation of the inventory level in the interval $[t_1, t_2]$ is:

$$I(t) = \left[\frac{(1-d_2)\,k_2}{\theta} - \frac{a}{\theta}\right] + \left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta} - \frac{a}{\theta}\right]\right\} e^{-\theta(t-t_1)} \qquad t_1 \le t \le t_2 \tag{5.11}$$

The solution of equation (5.3) is given by:

$$I(t) = -\frac{a}{\theta} + L_3 e^{-\theta t} \tag{5.12}$$

From equation (5.12) under the boundary condition, $I\left(t_2\right)=I_2$,we get L_3 in $\left[t_2, t_3\right]$:

$$-\frac{a}{\theta}+L_2 e^{-\theta t_2}=I_2$$

$$L_3=\left(I_2+\frac{a}{\theta}\right) e^{\theta t_2} \tag{5.13}$$

Therefore, the general representation of the inventory level in the interval $\left[t_2, t_3\right]$: becomes:

$$I(t)=-\frac{a}{\theta}+\left(I_2+\frac{a}{\theta}\right) e^{-\theta\left(t-t_2\right)} \qquad t_2 \leq t \leq t_3 \tag{5.14}$$

Since production stops at t_2, and can only resume at t_4, the items in inventory are consumed until the inventory reaches zero at t_3. However, not all customers are willing to wait until the next replenishment; therefore, a portion of demand is lost; the remaining order is backlogged until production begins. The quantity backlogged is represented by equation (5.4), and can be solved as follows:

$$\frac{d I(t)}{d t}=-(1-r) \times a \tag{5.15}$$

The solution of equation (5.15) is:

$$I(t)=-(1-r) a t+L_4 \tag{5.16}$$

From equation (5.16), under the boundary condition I $\left(t_3\right)=0$, we get:

$$L_4=(1-r) a t_3 \tag{5.17}$$

Substituting L_4 from (5.17) into (5.16), the inventory level during the backlog period is obtained as:

$$I(t)=-(1-r) a\left(t-t_3\right) \qquad t_3 \leq t \leq t_4 \tag{5.18}$$

Finally, from Equation (5.5):

$$I(t)=\int\left[\left(1-d_2\right) k_2-(1-r) \times a\right] \times d t \tag{5.19}$$

$$I(t)=\left(1-d_2\right) t k_2-(1-r) a t+L_5 \tag{5.20}$$

From equation (5.20), under the boundary condition $I\left(t_4\right)=-S$, we get:

$$L_5=-S-\left(1-d_2\right) t_4 k_2+(1-r) a t_4 \tag{5.21}$$

Substituting L_5 from equation (5.21) back into Equation (5.20), leads to the following:

$$I(t) = -S + (1-d_2)(t-t_4)k_2 - (1-r)a(t-t_4) \qquad t_4 \leq t \leq T \tag{5.22}$$

The model's objective is to determine the optimal cycle time T, the inventories I_1, I_2 and the optimal shortage that minimise the average total cost CT over the time horizon $[0, T]$. The evaluation of this manufacturing system is carried out using the analytical approach, which consists of modelling the system under study through mathematical equations (5.1) to (5.5), which are then solved through equations (5.8), (5.11), (5.14), (5.18), and (5.22) to determine the characteristics of the decisions variables that optimise this system. The planning horizon is divided into five cycles; the proportion of time the system reaches inventory I_1 during the first production-consumption cycle can be obtained by solving the following set of equations: From equation (5.8) we get:

$$I(t_1) = \left[\frac{(1-d_1)k_1}{\theta} - \frac{a}{\theta}\right]\left(1 - e^{-\theta t_1}\right) \tag{5.23}$$

From equation (5.11), we get:

$$I(t_1) = I_1 \tag{5.24}$$

extracting t_1 from Equation (5.23) and (5.24) leads to the following:

$$\begin{aligned} I_1 &= \left[\frac{(1-d_1)k_1}{\theta} - \frac{a}{\theta}\right]\left(1 - e^{-\theta t_1}\right) \\ e^{-\theta t_1} &= 1 - \frac{\theta I_1}{[(1-d)k_1 - a]} \\ \ln e^{-\theta t_1} &= \ln\left[1 - \frac{\theta I_1}{[(1-d_1)k_1 - a]}\right] \\ -\theta t_1 &= \ln\left[1 - \frac{\theta I_1}{(1-d_1)k_1 - a}\right] \end{aligned}$$

$$t_1 = -\frac{1}{\theta}\ln\left[1 - \frac{\theta I_1}{(1-d_1)k_1 - a}\right] \tag{5.25}$$

From the Taylor's series expansion, and under the assumption $\theta^2 \ll 1$ (neglecting higher powers of θ), the logarithmic function in equation (5.25) leads to

$$\begin{aligned} \ln\left[1 - \frac{\theta I_1}{[(1-d_1)k_1 - a]}\right] = &- \frac{\theta I_1}{(1-d_1)k_1 - a} - \frac{\theta^2 I_1^2}{2[(1-d_1)k_1 - a]^2} \\ &- \frac{2\theta^3}{6[(1-d_1)k_1 - a]^3} I_1^3 \end{aligned} \tag{5.26}$$

By substituting Equation (5.26) into (5.25), we get:

$$\begin{aligned} t_1 &= -\frac{1}{\theta}\left[-\frac{\theta I_1}{(1-d_1)\,k_1-a}-\frac{\theta^2 I_1^2}{2\left[(1-d_1)\,k_1-a\right]^2}-\frac{2\theta^3}{6\left[(1-d_1)\,k_1-a\right]^3}I_1^3\right] \\ t_1 &\approx \frac{I_1}{(1-d_1)\,k_1-a}+\frac{\theta I_1^2}{2\left[(1-d_1)\,k_1-a\right]^2} \end{aligned} \tag{5.27}$$

Thus, t_1 can be written in terms of I_1 and so, t_1 is not a decision variable.

The proportion of time t_2 during which the system reaches inventory I_2 can be obtained making t_2 the subject as follows From equation (5.11), we get:

$$I\left(t_2\right)=\left[\frac{(1-d_2)\,k_2}{\theta}-\frac{a}{\theta}\right]+\left\{I_1-\left[\frac{(1-d_2)\,k_2}{\theta}-\frac{a}{\theta}\right]\right\}e^{-\theta(t_2-t_1)} \tag{5.28}$$

In addition, from equation (5.14), we get:

$$I\left(t_2\right)=I_2 \tag{5.29}$$

Extracting t_2 using equations (5.28) and (5.29) leads

$$\frac{\frac{\theta I_2-(1-d_2)k_2+a}{\theta}}{\frac{\theta I_1-(1-d_2)k_2+a}{\theta}}=e^{-\theta(t_2-t_1)}$$

$$\frac{\frac{\theta I_1-(1-d_2)k_2+a}{-(1-d_2)k_2+a}}{\frac{\theta I_2-(1-d_2)k_2+a}{-(1-d_2)k_2+a}}=e^{\theta(t_2-t_1)}$$

$$e^{\theta(t_2-t_1)}=\left[1-\frac{\theta I_1}{(1-d_2)\,k_2-a}\right]\left[1-\frac{\theta I_2}{(1-d_2)\,k_2-a}\right]^{-1}$$

$$\ln e^{\theta(t_2-t_1)}=\ln\left[1-\frac{\theta I_1}{(1-d_2)\,k_2-a}\right]\left[1-\frac{\theta I_2}{(1-d_2)\,k_2-a}\right]^{-1}$$

$$t_2-t_1=\frac{1}{\theta}\ln\left[1-\frac{\theta I_1}{(1-d_2)\,k_2-a}\right]\left[1-\frac{\theta I_2}{(1-d_2)\,k_2-a}\right]^{-1}$$

$$t_2-t_1=\frac{1}{\theta}\ln\left[1-\frac{\theta I_1}{(1-d_2)\,k_2-a}\right]-\frac{1}{\theta}\ln\left[1-\frac{\theta I_2}{(1-d_2)\,k_2-a}\right] \tag{5.30}$$

From Taylor's series expansion, and under the assumption $\theta^2 \ll 1$ (neglecting higher powers of θ), the expansion of the logarithmic functions in (5.30) is represented by:

$$\begin{aligned}
&\frac{1}{\theta}\ln\left[1-\frac{\theta I_1}{(1-d_2)\,k_2-a}\right]-\frac{1}{\theta}\ln\left[1-\frac{\theta I_2}{(1-d_2)\,k_2-a}\right]\\
=&\frac{1}{\theta}\left[-\frac{\theta I_1}{(1-d_2)\,k_2-a}-\frac{\theta^2 I_1^2}{2\left[(1-d_2)\,k_2-a\right]^2}-\frac{2\theta^3 I_1^3}{6\left[(1-d_2)\,k_2-a\right]^3}\right]\\
&-\frac{1}{\theta}\left[-\frac{\theta I_2}{(1-d_2)\,k_2-a}-\frac{\theta^2 I_2^2}{2\left[(1-d_2)\,k_2-a\right]^2}-\frac{2\theta^3 I_2^3}{6\left[(1-d_2)\,k_2-a\right]^3}\right]
\end{aligned} \tag{5.31}$$

Substituting equation (5.31) into Equation (5.30), we get:

$$\begin{aligned}
t_2-t_1=&\frac{1}{\theta}\left[-\frac{\theta I_1}{(1-d_2)\,k_2-a}-\frac{\theta^2 I_1^2}{2\left[(1-d_2)\,k_2-a\right]^2}-\frac{2\theta^3 I_1^3}{6\left[(1-d_2)\,k_2-a\right]^3}\right]\\
&-\frac{1}{\theta}\left[-\frac{\theta I_2}{(1-d_2)\,k_2-a}-\frac{\theta^2 I_2^2}{2\left[(1-d_2)\,k_2-a\right]^2}-\frac{2\theta^3 I_2^3}{6\left[(1-d_2)\,k_2-a\right]^3}\right]
\end{aligned}$$

or:

$$t_2-t_1\approx\frac{I_2-I_1}{(1-d_2)\,k_2-a}+\frac{\theta\left(I_2^2-I_1^2\right)}{2\left[(1-d_2)\,k_2-a\right]^2} \tag{5.32}$$

That is,

$$\begin{aligned}
t_2=&\frac{I_2-I_1}{(1-d_2)\,k_2-a}+\frac{\theta\left(I_2^2-I_1^2\right)}{2\left[(1-d_2)\,k_2-a\right]^2}+\frac{I_1}{(1-d_1)\,k_1-a}\\
&+\frac{\theta I_1^2}{2\left[(1-d_1)\,k_1-a\right]^2}
\end{aligned} \tag{5.33}$$

Thus, t_2 can be written in terms of I_1 and I_2. Therefore, t_2 is not a decision variable.

By using equations (5.14) and (5.18), the time at which the system reaches the zero inventory is obtained. Therefore, from equation (5.14),

$$I\left(t_3\right)=-\frac{a}{\theta}+\left\{I_2+\frac{a}{\theta}\right\}e^{-\theta(t_3-t_2)} \tag{5.34}$$

And from equation (5.18), we get:

$$I\left(t_3\right)=0 \tag{5.35}$$

Again, eliminating $I\left(t_3\right)$ from equations (5.34) and (5.35), we get:

$$\begin{aligned}
&0=-\frac{a}{\theta}+\left\{I_2+\frac{a}{\theta}\right\}e^{-\theta(t_3-t_2)}\\
&\frac{\frac{a}{\theta}}{\left(\frac{a+\theta I_2}{\theta}\right)}=e^{-\theta(t_3-t_2)}\\
&e^{\theta(t_3-t_2)}=\frac{a+\theta I_2}{a}\\
&\theta\left(t_3-t_2\right)\ln e=\ln\left(\frac{a+\theta I_2}{a}\right)
\end{aligned}$$

$$t_3 - t_2 = \frac{1}{\theta} \ln \left(\frac{a + \theta I_2}{a} \right) \tag{5.36}$$

For small values of θ and using Taylor series approximation, we expand the logarithmic function in (5.36) as follow:

$$\ln \left(\frac{a + \theta I_2}{a} \right) = \frac{\theta I_2}{a} - \frac{\theta^2 I_2^2}{2a^2} + \frac{2\theta^3 I_2^3}{6a^3} \tag{5.37}$$

By substituting equation (5.37) into equation (5.36),we obtain:

$$t_3 - t_2 = \frac{1}{\theta} \left[\frac{\theta}{a} I_2 - \frac{\theta^2}{2a^2} I_2^2 + \frac{2\theta^3}{6a^3} I_2^3 \right]$$

$$= \frac{I_2}{a} - \frac{\theta I_2^2}{2a^2} + \frac{\theta^2 I_2^3}{3a^3}$$

$$t_3 \approx \frac{I_2}{a} - \frac{\theta I_2^2}{2a^2} + t_2$$

That is,

$$\begin{aligned} t_3 = \frac{I_2}{a} - \frac{\theta I_2^2}{2a^2} + \frac{I_2 - I_1}{(1-d_2)\,k_2 - a} + \frac{\theta\,(I_2^2 - I_1^2)}{2\left[(1-d_2)\,k_2 - a\right]^2} + \frac{I_1}{(1-d_1)\,k_1 - a} \\ + \frac{\theta I_1^2}{2\left[(1-d_1)\,k_1 - a\right]^2} \end{aligned} \tag{5.38}$$

Thus, t_3 can be written in terms of I_1 and I_2. Therefore, t_3 is not a decision variable.

From equation (5.18),

$$(t_4) = -(1-r)a\,(t_4 - t_3) \tag{5.39}$$

And from equation (5.22),

$$I\,(t_4) = -S \tag{5.40}$$

Eliminating $I\,(t_4)$ from equations (5.39) and (5.40) , we get:

$$\begin{aligned} -(1-r)a\,(t_4 - t_3) = -S \\ t_4 - t_3 = \frac{S}{(1-r)a} \end{aligned} \tag{5.41}$$

$$t_4 = \frac{S}{(1-r)a} + \frac{I_1}{(1-d_1)\,k_1 - a} + \frac{\theta I_1^2}{2\left[(1-d_1)\,k_1 - a\right]^2} + \frac{I_2 - I_1}{(1-d_2)\,k_2 - a} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\left[(1-d_2)\,k_2 - a\right]^2} + \frac{I_2}{a} - \frac{\theta I_2^2}{2a^2} \tag{5.42}$$

Thus, t_4 can be written in terms of I_1 and I_2, therefore, t_4 is not a decision variable, with S the maximum backlog of the given production system. The production-planning problem considered in this chapter also involves the determination of the optimal backlog level, which can be computed as follow

$$-S + (1-d_2)\,(T-t_4)\,k_2 - (1-r)a\,(T-t_4) = 0$$

$$-S + \left[(1-d_2)\,k_2 - (1-r)a\right](T-t_4) = 0$$

$$T - t_4 = \frac{S}{\left[(1-d_2)\,k_2 - (1-r)a\right]} \tag{5.43}$$

Substituting equation (5.43) into equation (5.41), leads to:

$$T - t_3 - \frac{S}{(1-r)a} = \frac{S}{\left[(1-d_2)\,k_2 - (1-r)a\right]}$$

$$\frac{(1-r)a\left[(1-d_2)\,k_2 - (1-r)a\right](T-t_3)}{(1-r)a\left[(1-d_2)\,k_2 - (1-r)a\right]} = \frac{S\left[(1-d_2)\,k_2 - (1-r)a\right]}{(1-r)a\left[(1-d_2)\,k_2 - (1-r)a\right]} + \frac{S(1-r)a}{(1-r)a\left[(1-d_2)\,k_2 - (1-r)a\right]}$$

$$S\,(1-d_2)\,k_2 = (1-r)a\left[(1-d_2)\,k_2 - (1-r)a\right](T-t_3)$$

$$S = \frac{(1-r)a\left[(1-d_2)\,k_2 - (1-r)a\right](T-t_3)}{(1-d_2)\,k_2}$$

Hence, the maximum backlog can be written as:

$$S = \frac{(1-r)a\left[(1-d_2)\,k_2 - (1-r)a\right]}{(1-d_2)\,k_2}\left[T - \frac{I_2}{a} + \frac{\theta I_2^2}{2a^2} - \frac{I_2 - I_1}{\left[(1-d_2)\,k_2 - a\right]} - \frac{\theta\left(I_2^2 - I_1^2\right)}{2\left[(1-d_2)\,k_2 - a\right]^2} - \frac{I_1}{\left[(1-d_1)\,k_1 - a\right]} - \frac{\theta I_1^2}{2\left[(1-d_1)\,k_1 - a\right]^2}\right]$$

$$= \frac{(1-r)a\left[(1-d_2)\,k_2-(1-r)a\right]}{(1-d_2)\,k_2}\left[T-BI_2+\frac{\theta}{2}CI_2^2+EI_1+\frac{\theta}{2}FI_1^2\right] \quad (5.44)$$

With:

$$(1-d_1)\,k_1-a=\rho_1$$
$$(1-d_2)\,k_2-a=\rho_2$$
$$\frac{1}{a}+\frac{1}{\rho_2}=B$$
$$\frac{1}{a^2}-\frac{1}{\rho_2^2}=C$$
$$\frac{1}{\rho_2}-\frac{1}{\rho_1}=E$$
$$\frac{1}{\rho_2^2}-\frac{1}{\rho_1^2}=F$$

Thus, S can be written in terms of I_1, I_2 and T. Therefore, S is not a decision variable.

5.3 Cost components involved in the mathematical formulation

To find the optimal quantities, we first calculate the total cost per inventory cycle which is the sum of deteriorating cost, production cost, production setup cost, inventory holding cost, cost of loss of sales, shortage costs and cost of disposing defective items scrap. The cost components are as follows:

5.3.1 Setup Cost (SUC)

The modelling starts by formulating the setup cost. This cost is considered fixed and represented by:

$$\text{SUC} = \text{G} \quad (5.45)$$

5.3.2 Deteriorating cost (AC)

The number of deteriorating items is equal to the number of total items produced minus the number of total demand. The total number of deteriorating items over $[0, T]$ is:

$$\int_0^{t_1}\left[(1-d_1)\,k_1-a\right]dt+\int_{t_1}^{t_2}\left[(1-d_2)\,k_2-a\right]dt-\int_{t_2}^{t_3}a\,dt \quad (5.46)$$

$$\int_0^{t_1} \left[(1-d_1)\,k_1 - a\right] dt$$

$$= \left[(1-d_1)\,k_1 - a\right] t_1$$

$$= \left[(1-d_1)\,k_1 - a\right]\left[\frac{I_1}{(1-d)k_1 - a} + \frac{\theta I_1^2}{2\left[(1-d)k_1 - a\right]^2}\right] \tag{5.46a}$$

$$= \left[I_1 + \frac{\theta I_1^2}{2\left[(1-d_1)\,k_1 - a\right]}\right]$$

$$\int_{t_1}^{t_2} \left[(1-d_2)\,k_2 - a\right] dt$$

$$= \left[(1-d_2)\,k_2 - a\right](t_2 - t_1)$$

$$= \left[(1-d_2)\,k_2 - a\right]\left[\frac{I_2 - I_1}{(1-d)k_2 - a} + \frac{\theta\,(I_2^2 - I_1^2)}{2\left[(1-d)k_2 - a\right]^2}\right] \tag{5.46b}$$

$$= \left[I_{\max} - I_1 + \frac{\theta\,(I_{\max}^2 - I_1^2)}{2\left[(1-d_2)\,k_2 - a\right]}\right]$$

$$\int_{t_2}^{t_3} a dt = a\,(t_3 - t_2)$$

$$= \left[I_2 - \frac{\theta I_2^2}{2a}\right] \tag{5.46c}$$

The total cost of deteriorating items over [0, T] can be expressed as:

$$AC = C_a\left\{\left[I_1 + \frac{\theta I_1^2}{2\left[(1-d_1)\,k_1 - a\right]}\right] + \left[I_2 - I_1 + \frac{\theta\,(I_2^2 - I_1^2)}{2\left[(1-d_2)\,k_2 - a\right]}\right] - \left[I_2 - \frac{\theta I_2^2}{2a}\right]\right\}$$

$$= C_a\left[\frac{\theta I_1^2}{2\rho_1} + \frac{\theta\,(I_2^2 - I_1^2)}{2\rho_2} + \frac{\theta I_2^2}{2a}\right] \tag{5.47}$$

5.3.3 Inventory Carrying Cost (ICC)

According to Figure (5.1), it can be seen that the total holding cost over $[0, T]$ can be summarized as follow:

$$ICC = h \times \left\{ \int_0^{t_1} I(t)dt + \int_{t_1}^{t_2} I(t)dt + \int_{t_2}^{t_3} I(t)dt \right\} \tag{5.48}$$

For the solution approach of equation (5.48), the problem has been divided into 3 time periods, each with a Number of Items Kept in Stock (NIKS) held in inventory Thus, the total number of items kept in inventory during $[0, t_1]$ is:

$$NIKS_1 = \int_0^{t_1} I(t)dt$$

$$= \int_0^{t_1} \left\{ \left[\frac{(1-d_1)\,k_1}{\theta} - \frac{a}{\theta} \right] \left(1 - e^{-\theta t}\right) \right\} dt$$

$$= \left[\frac{(1-d_1)\,k_1}{\theta} - \frac{a}{\theta} \right] \left[t + \frac{1}{\theta} e^{-\theta t} \right]_0^{t_1} \tag{5.49}$$

$$= \left[\frac{(1-d_1)\,k_1}{\theta} - \frac{a}{\theta} \right] \left(t_1 + \frac{1}{\theta} e^{-\theta t_1} \right) - \frac{1}{\theta} \left[\frac{(1-d)k_1}{\theta} - \frac{a}{\theta} \right]$$

$$= \frac{1}{\theta} \left[(1-d_1)\,k_1 - a \right] \left(t_1 + \frac{1}{\theta} e^{-\theta t_1} - \frac{1}{\theta} \right)$$

Recall from equation (5.25) that:

$$\frac{1}{\theta}\left(-1 + e^{-\theta t_1}\right) = -\frac{I_1}{\left[(1-d_1)\,k_1 - a\right]} \tag{5.50}$$

And that,

$$t_1 = \frac{\mathrm{I}_1}{(1-d_1)\,\mathrm{k}_1 - \mathrm{a}} + \frac{\theta I_1^2}{2\left[(1-d_1)\,\mathrm{k}_1 - \mathrm{a}\right]^2} + \frac{\theta^2}{3\left[(1-d_1)\,\mathrm{k}_1 - \mathrm{a}\right]^3} I_1^3 \tag{5.51}$$

Substituting (5.50) and (5.50) in (5.49), leads to:

$$\left[(1-d_1)\,k_1 - a\right] \left[\frac{\mathrm{I}_1}{\theta\left[(1-d_1)\,\mathrm{k}_1 - \mathrm{a}\right]} + \frac{\theta I_1^2}{2\theta\left[(1-d_1)\,\mathrm{k}_1 - \mathrm{a}\right]^2} + \frac{\theta^2}{3\theta\left[(1-d_1)\,\mathrm{k}_1 - \mathrm{a}\right]^3} I_1^3 - \frac{l_1}{\theta\left[(1-d_1)\,k_1 - a\right]} \right] \tag{5.52}$$

Hence, the total number of items kept in inventory ($NIKS_1$) during $[0, t_1]$ is:

$$NIKS_1 = \frac{l_1^2}{2\left[\left(1-d_1\right)k_1 - a\right]} + \frac{\theta I_1^3}{3\left[\left(1-d_1\right)\mathrm{k}_1 - \mathrm{a}\right]^2} \tag{5.53}$$

The total number of items kept in inventory ($NIKS_2$) during $[t_1, t_2]$ is:

$$NIKS_2 = \int_{t_1}^{t_2} I(t)dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\left(1-d_2\right)k_2}{\theta} - \frac{a}{\theta}\right] + \left\{I_1 - \left[\frac{\left(1-d_2\right)k_2}{\theta} - \frac{a}{\theta}\right]\right\} e^{-\theta(t-t_1)}dt \tag{5.54}$$

$$= \left\{\left[\frac{\left(1-d_2\right)k_2}{\theta} - \frac{a}{\theta}\right] t - \frac{1}{\theta}\left\{I_1 - \left[\frac{\left(1-d_2\right)k_2}{\theta} - \frac{a}{\theta}\right]\right\} e^{-\theta(t-t_1)}\right\}_{t_1}^{t_2}$$

$$= \frac{1}{\theta}\left[\left(1-d_2\right)k_2 - a\right]\left(t_2 - t_1\right) - \frac{1}{\theta}\left[\frac{\left(1-d_2\right)k_2}{\theta} - \frac{a}{\theta}\right] + \frac{1}{\theta}I_1$$
$$- \frac{1}{\theta}\left\{I_1 - \left[\frac{\left(1-d_2\right)k_2}{\theta} - \frac{a}{\theta}\right]\right\} e^{-\theta(t_2-t_1)}$$

With:

$$I_2 = \left[\frac{\left(1-d_2\right)k_2}{\theta} - \frac{a}{\theta}\right] + \left\{I_1 - \left[\frac{\left(1-d_2\right)k_2}{\theta} - \frac{a}{\theta}\right]\right\} e^{-\theta(t_2-t_1)} \tag{5.55}$$

Dividing both sides of equation (5.55) by $-\frac{1}{a}$, the following is obtained

$$-\frac{1}{\theta}\left[\frac{\left(1-d_2\right)k_2}{\theta} - \frac{a}{\theta}\right] - \frac{1}{\theta}\left\{I_1 - \left[\frac{\left(1-d_2\right)k_2}{\theta} - \frac{a}{\theta}\right]\right\} e^{-\theta(t_2-t_1)} = -\frac{1}{\theta}I_2 \tag{5.56}$$

Substituting equation (5.56) into equation (5.54), leads to:

$$\frac{1}{\theta}\left[(1-d_2)\,k_2-a\right](t_2-t_1)-\frac{1}{\theta}I_2+\frac{1}{\theta}I_1$$

$$=\frac{1}{\theta^2}\left[(1-d_2)\,k_2-a\right]\left\{-\frac{\theta I_1}{(1-d_2)\,k_2-a}-\frac{\theta^2 I_1^2}{2\left[(1-d_2)\,k_2-a\right]^2}-\frac{2\theta^3}{6\left[(1-d_2)\,k_2-a\right]^3}I_1^3\right\}$$
$$-\frac{1}{\theta^2}\left[(1-d_2)\,k_2-a\right]\left\{-\frac{\theta I_2}{(1-d_2)\,k_2-a}-\frac{\theta^2 I_2^2}{2\left[(1-d_2)\,k_2-a\right]^2}\right.$$
$$\left.-\frac{2\theta^3}{6\left[(1-d_2)\,k_2-a\right]^3}I_2^3\right\}-\frac{1}{\theta}(I_2-I_1)$$

$$=\frac{(I_2-I_1)}{\theta}+\frac{(I_2^2-I_1^2)}{2\left[(1-d_2)\,k_2-a\right]}+\frac{\theta\,(I_2^3-I_1^3)}{3\left[(1-d_2)\,k_2-a\right]^2}-\frac{1}{\theta}(I_2-I_1)$$

Thus, the total number of items kept in inventory ($NIKS$) during $[t_1,t_2]$ is:

$$NIKS_2=\frac{(I_2^2-I_1^2)}{2\left[(1-d_2)\,k_2-a\right]}+\frac{\theta\,(I_2^3-I_1^3)}{3\left[(1-d_2)\,k_2-a\right]^2} \tag{5.57}$$

$$NIKS_3=\int_{t_2}^{t_3}I(t)dt=\int_{t_2}^{t_3}\left[-\frac{a}{\theta}+\left(I_2+\frac{a}{\theta}\right)e^{-\theta(t-t_2)}\right]dt$$

$$=\left[-\frac{a}{\theta}t-\frac{1}{\theta}\left(I_2+\frac{a}{\theta}\right)e^{-\theta(t-t_2)}\right]_{t_2}^{t_3}$$

$$=-\frac{a}{\theta}t_3-\frac{1}{\theta}\left(I_2+\frac{a}{\theta}\right)e^{-\theta(t_3-t_2)}+\frac{a}{\theta}t_2+\frac{1}{\theta}\left(I_2+\frac{a}{\theta}\right)e^{-\theta(t_2-t_2)} \tag{5.58}$$

$$=-\frac{a}{\theta}(t_3-t_2)-\frac{1}{\theta}\left(I_2+\frac{a}{\theta}\right)e^{-\theta(t_3-t_2)}+\frac{1}{\theta}\left(I_2+\frac{a}{\theta}\right)$$

$$=-\frac{a}{\theta}(t_3-t_2)-\frac{1}{\theta}\left[-\frac{a}{\theta}+\left(I_2+\frac{a}{\theta}\right)e^{-\theta(t_3-t_2)}\right]+\frac{1}{\theta}I_2$$

With:

$$-\frac{a}{\theta}+\left(I_2+\frac{a}{\theta}\right)e^{-\theta(t_3-t_2)}=0 \tag{5.59}$$

$$t_3-t_2=\frac{I_2}{a}-\frac{\theta I_2^2}{2a^2}+\frac{\theta^2 I_2^3}{3a^3} \tag{5.60}$$

Substituting the equations (5.59) and (5.59) into equation (5.58) leads to the following:

$$-\frac{a}{\theta}(t_3-t_2)-\frac{1}{\theta}0+\frac{1}{\theta}I_2=-\frac{a}{\theta}\left[\frac{I_2}{a}-\frac{\theta I_2^2}{2a^2}+\frac{\theta^2 I_2^3}{3a^3}\right]+\frac{1}{\theta}I_2$$

$$=-\frac{I_2}{\theta}+\frac{I_2^2}{2a}-\frac{\theta I_2^3}{3a^2}+\frac{1}{\theta}I_2$$

Thus, the total number of items kept in inventory $NIKS$) over the period $[t_2,t_3]$ is:

$$NIKS_3=\frac{I_2^2}{2a}-\frac{\theta I_2^3}{3a^2} \tag{5.61}$$

Therefore, the total inventory carrying cost over the period [0, T] is given by:

$$ICC=h\times\left[\frac{I_1^2}{2\left[(1-d_1)k_1-a\right]}+\frac{\theta I_1^3}{3\left[(1-d_1)k_1-a\right]^2}+\frac{(I_2^2-I_1^2)}{2\left[(1-d_2)k_2-a\right]}+\right.$$
$$\left.\frac{\theta\left(I_2^3-I_1^3\right)}{3\left[(1-d_2)k_2-a\right]^2}+\frac{I_2^2}{2a}-\frac{\theta I_2^3}{3a^2}\right] \tag{5.62}$$

$$=h\times\left[\frac{I_1^2}{2\rho_1}+\frac{\theta I_1^3}{3\rho_1^2}+\frac{(I_2^2-I_1^2)}{2\rho_2}+\frac{\theta\left(I_2^3-I_1^3\right)}{3\rho_2^2}+\frac{I_2^2}{2a}-\frac{\theta I_2^3}{3a^2}\right]$$

5.3.4 Shortage Cost (SC)

Backordered demand occurs in $[t_3,t_4]$ and $[t_4,T]$, the shortage cost over the period $[t_3,T]$, SC, can be obtained in the form of equation (65) :

$$SC=C_s\times\left\{\int_{t_3}^{t_4}-a(1-r)(t-t_3)\,dt+\int_{t_4}^{\tau}\left[-S+(1-d)k_2-a(1-r)\right](t-t_4)\,dt\right\} \tag{5.63}$$

Equation (5.63) can be summarized as follows:

$$SC=C_s\times\left[\int_{t_3}^{t_4}QB_1dt+\int_{t_4}^{T}QB_2dt\right] \tag{5.64}$$

While the Quantity Backordered (QB_1) is allowed in the interval $[t_3,t_4]$, the interval $[t_4,T]$ is the period of time needed to eliminate the backlog quantity QB_2.

$$QB_1=\int_{t_3}^{t_4}-a(1-r)(t-t_3)\,dt=\left[-a(1-r)\left(\frac{1}{2}t^2-tt_3\right)\right]_{t_3}^{t_4} \tag{5.65}$$

$$= -\,a(1-r)\left(\frac{1}{2}t_4^2 - t_4t_3\right) + a(1-r)\left(\frac{1}{2}t_3^2 - t_3^2\right)$$

$$= -\,a(1-r)\left(\frac{1}{2}t_4^2 - t_4t_3\right) - a(1-r)\frac{1}{2}t_3^2$$

$$= -\,\frac{1}{2}a(1-r)\left(t_4 - t_3\right)^2$$

$$= -\frac{1}{2}\frac{S^2}{(1-r)a} \tag{5.66}$$

With:

$$t_4 - t_3 = \frac{S}{a(1-r)} \tag{5.67}$$

$$QB_2 = \int_{t_4}^{T} \{-S + [(1-d_2)\,k_2 - a(1-r)]\,(t - t_4)\}\,dt \tag{5.68}$$

$$= \left\{-St + [(1-d_2)\,k_2 - a(1-r)]\left(\frac{1}{2}t^2 - tt_4\right)\right\}_{t_4}^{T}$$

$$= -S\,(T - t_4) + \frac{1}{2}\,[(1-d_2)\,k_2 - a(1-r)]\,(T - t_4)^2$$

$$= -S\frac{s}{[(1-d_2)\,k_2 - a(1-r)]} + \frac{1}{2}\,[(1-d_2)\,k_2 - a(1-r)]\left[\frac{s}{[(1-d_2)\,k_2 - a(1-r)]}\right]^2$$

$$= -\frac{s^2}{[(1-d_2)\,k_2 - a(1-r)]} + \frac{1}{2}\frac{s^2}{[(1-d_2)\,k_2 - a(1-r)]}$$

$$= -\frac{s^2}{2\,[(1-d_2)\,k_2 - a(1-r)]} \tag{5.69}$$

Hence, the total quantity backordered over the period $[t_3, T]$ is given by:

$$-\frac{S^2}{2a(1-r)} - \frac{s^2}{2\,[(1-d_2)\,k_2 - a(1-r)]} \tag{5.70}$$

$$= \frac{-S^2\,(1-d_2)\,k_2}{2a(1-r)\,[(1-d_2)\,k_2 - a(1-r)]}$$

Therefore, the shortage cost over the period $[t_3, T]$ is:

$$SC = C_s \frac{S^2 (1-d_2) k_2}{2a(1-r)\left[(1-d_2) k_2 - a(1-r)\right]} \tag{5.71}$$

With:

$$S = \frac{(1-r)a\left[(1-d_2) k_2 - (1-r)a\right]}{(1-d_2) k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]$$

$$S^2 = \frac{(1-r)^2 a^2\left[(1-d_2) k_2 - (1-r)a\right]^2}{(1-d_2)^2 k_2^2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]^2 \tag{5.72}$$

Substituting equation (5.3.4) into equation (5.71),

$$SC = C_s \frac{\frac{(1-r)^2 a^2[(1-d_2)k_2-(1-r)a]^2}{(1-d_2)^2 k_2^2}\left(T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right)^2 (1-d_2) k_2}{2a(1-r)\left[(1-d_2) k_2 - a(1-r)\right]}$$

$$= C_s \frac{1}{2a(1-r)\left[(1-d_2) k_2 - a(1-r)\right]} \frac{(1-r)^2 a^2\left[(1-d_2) k_2 - (1-r)a\right]^2}{(1-d_2)^2 k_2^2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]^2 (1-d_1) k_2$$

$$SC = C_s \frac{a(1-r)\left[(1-d_2) k_2 - a(1-r)\right]}{2(1-d_2) k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]^2 \tag{5.73}$$

5.3.5 Disposal Cost (DC)

At the end of each production cycle, a proportion d_i of defective items is produced and a cost is incurred by the company to dispose those imperfect items. This cost can be represented as follows

$$DC = C_b \times \left[d_1 k_1 t_1 + d_2 k_2 (t_2 - t_1) + d_2 k_2 (T - t_4)\right] \tag{5.74}$$

$$= Cb \times \left\{ d_1 k_1 \left[\frac{I_1}{(1-d_1) k_1 - a} + \frac{\theta I_1^2}{2\left[(1-d_1) k_1 - a\right]^2}\right] + d_2 k_2 \left[\frac{I_2 - I_1}{(1-d_2) k_2 - a} + \frac{\theta (I_2^2 - I_1^2)}{2\left[(1-d_2) k_2 - a\right]^2}\right] + d_2 k_2 \frac{S}{\left[(1-d_1) k_1 - (1-r)a\right]} \right\}$$

$$= Cb \times \left\{ d_1 k_1 \left[\frac{I_1}{\rho_1} + \frac{\theta I_1^2}{2\rho_1^2}\right] + d_2 k_2 \left[\frac{I_2 - I_1}{\rho_2} + \frac{\theta (I_2^2 - I_1^2)}{2\rho_2^2}\right] + d_2 k_2 \frac{1}{\left[(1-d_2) k_2 - (1-r)a\right]} \frac{(1-r)a\left[(1-d_2) k_2 - (1-r)a\right]}{(1-d_2) k_2} \times \left(T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right) \right\}$$

$$= C_b \times \left\{ d_1 k_1 \left[\frac{I_1}{\rho_1} + \frac{\theta I_1^2}{2\rho_1^2} + d_2 k_2 \left[\frac{I_2 - I_1}{\rho_2} + \frac{\theta \left(I_2^2 - I_1^2\right)}{2\rho_2^2} \right] + \right. \right.$$
$$\frac{(1-r)a}{(1-d_2)} d_2 \left[T - BI_2 + \frac{\theta}{2} CI_2^2 + EI_1 + \frac{\theta}{2} FI_1^2 \right] \tag{5.75}$$

5.3.6 Lost Sale Cost (LC)

Out of stock situations can have two effects:

- Either the unrealized sale is postponed to the next period. This delay is assumed to be evaluated financially (responses to customer reminders, possible penalties to be paid to customers, favours granted, etc.). In theory, this cost is a function of the number of missing units and the duration of the shortage.
- Alternatively, the unrealized sale is definitely lost: in this case, the penalty cost corresponds to the loss of profit linked to the item requested but not supplied. This shortfall is made up of the unit margin on purchase cost usually realized on the product and the depreciation of the company's image.

Gothi et al. (2017) assumed that demand during the time $[t_4, T]$ is satisfied at a time as the production has already started at time $t = t_4$ and so loss of sales cost during this interval is not considered. But in reality, some customers are not willing to wait when the company is not able to meet all the outstanding demand instantaneously. Thus, the expression for the loss of sale LC per cycle is determined by:

$$LC = C_p \times r \left[\int_{t_3}^{t_4} a \times dt + \int_{t_4}^{T} a \times dt \right]$$
$$= C_p \times r \times a\,(T - t_3) \tag{5.76}$$

with:

$$T - t_3 = \frac{S\,(1-d_2)\,k_2}{(1-r)a\left[(1-d_2)\,k_2 - (1-r)a\right]} \tag{5.77}$$

Substituting equations (5.44) and (5.77) in (5.76) gives:

$$C_p \times r \times a \frac{(1-r)a\left[(1-d_2)\,k_2 - (1-r)a\right]}{(1-d_2)\,k_2} \times \left[T - \frac{I_2}{a} + \frac{\theta I_2^2}{2a^2} - \frac{I_2 - I_1}{\rho_2} - \right.$$
$$\left. \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2^2} - \frac{I_1}{\rho_1} - \frac{\theta I_1^2}{2\rho_1^2} \times \frac{(1-d_2)\,k_2}{(1-r)a\left[(1-d_2)\,k_2 - (1-r)a\right]} \right.$$

$$= C_p \times r \times a \left[T - BI_2 + \frac{\theta}{2} CI_2^2 + EI_1 + \frac{\theta}{2} FI_1^2 \right] \tag{5.78}$$

5.3.7 Production Cost (PC)

The following definitions are made: k_i are the production rate, such that k_1 is the production rate during the first production-consumption cycle and k_2 the production rate during the second production-consumption cycle; p_{c1} and p_{c2}, are the production cost per item in the first and second consumption cycle respectively. For the production process of interest, the following is the computation of the total production cost over the cycle [0, T]

$$PC = p_{c1}k_1t_1 + p_{c2}k_2\left(t_2 - t_1\right) + p_{c2}k_2\left(T - t_4\right) \tag{5.79}$$

$$= p_{c1}k_1\left[\frac{I_1}{\left(1-d_1\right)k_1 - a} + \frac{\theta I_1^2}{2\left[\left(1-d_1\right)k_1 - a\right]^2}\right] + p_{c2}k_2\left[\frac{I_2 - I_1}{\left(1-d_2\right)k_2 - a} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\left[\left(1-d_2\right)k_2 - a\right]^2}\right] + p_{c2}k_2\left[\frac{2}{\left[\left(1-d_1\right)k_2 - (1-r)a\right]}\right]$$

$$= p_{c1}k_1\left[\frac{I_1}{\left(1-d_1\right)k_1 - a} + \frac{\theta I_1^2}{2\left[\left(1-d_1\right)k_1 - a\right]^2}\right] + p_{c2}k_2\left[\frac{I_2 - I_1}{\left(1-d_2\right)k_2 - a} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\left[\left(1-d_2\right)k_2 - a\right]^2}\right] + p_{c2}k_2\left[\frac{\frac{(1-r)a\left[(1-d_2)k_2 - (1-r)a\right]}{(1-d_2)k_2}\left(T - BI_2 + \frac{\theta}{2}Cl_2^2 + El_1 + \frac{\theta}{2}FI_1^2\right)}{\left[\left(1-d_2\right)k_2 - (1-r)a\right]}\right]$$

$$= p_{c1}k_1\left[\frac{I_1}{\left(1-d_1\right)k_1 - a} + \frac{\theta I_1^2}{2\left[\left(1-d_1\right)k_1 - a\right]^2}\right] + p_{c2}k_2\left[\frac{I_2 - I_1}{\left(1-d_2\right)k_2 - a} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\left[\left(1-d_2\right)k_2 - a\right]^2}\right] + p_{c2}\frac{(1-r)a}{(1-d_2)}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]$$

$$= p_{c1}k_1\left[\frac{I_1}{\rho_1} + \frac{\theta I_1^2}{2\rho_1^2}\right] + p_{c2}k_2\left[\frac{I_2 - I_1}{\rho_2} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2^2}\right] + p_{c2}\frac{(1-r)a}{(1-d_2)}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right] \tag{5.80}$$

5.4 The total cost and optimality conditions

5.4.1 Total cost

The aim of the proposed system is to optimize the company's Total cost (CT), whose components are: production, loss of sale, disposal, shortage, holding, deterioration and setup costs, denoted by PC, LC, DC, SC, ICC, AC and SUC respectively. The total cost of the system per unit time is the cost divided by the cycle time, T, hence,

$$\begin{aligned}
CT = & \frac{1}{T}(PC + LC + SC + DC + ICC + AC + SUC) \qquad (5.81)\\
= & \frac{1}{T}\left\{p_{c1}k_1\left[\frac{I_1}{\rho_1}+\frac{\theta I_1^2}{2\rho_1^2}\right]+p_{c2}k_2\left[\frac{I_2-I_1}{\rho_2}+\frac{\theta\left(I_2^2-I_1^2\right)}{2\rho_2^2}\right]\right.\\
& + p_{c2}\frac{(1-r)a}{(1-d_2)}\left[T-BI_2+\frac{\theta}{2}CI_2^2+EI_1+\frac{\theta}{2}FI_1^2\right]\\
& + C_p\times r\times a\left[T-BI_2+\frac{\theta}{2}CI_2^2+EI_1+\frac{\theta}{2}FI_1^2\right]\\
& + C_b\left\{d_1k_1\left[\frac{I_1}{\rho_1}+\frac{\theta I_1^2}{2\rho_1^2}\right]+d_2k_2\left[\frac{I_2-I_1}{\rho_2}+\frac{\theta\left(I_2^2-I_1^2\right)}{2\rho_2^2}\right]\right.\\
& \left.+\frac{(1-r)a}{(1-d_2)}d_2\left[T-BI_2+\frac{\theta}{2}CI_2^2+EI_1+\frac{\theta}{2}FI_1^2\right]\right\}\\
& + C_s\frac{a(1-r)\left[(1-d_2)\,k_2-a(1-r)\right]}{2\,(1-d_2)\,k_2}\left[T-BI_2+\frac{\theta}{2}CI_2^2+EI_1+\frac{\theta}{2}FI_1^2\right]^2\\
& + h\left[\frac{I_1^2}{2\rho_1}+\frac{\theta I_1^3}{3\rho_1^2}+\frac{\left(I_2^2-I_1^2\right)}{2\rho_2}+\frac{\theta\left(I_2^3-I_1^3\right)}{3\rho_2^2}+\frac{I_2^2}{2a}-\frac{\theta I_2^3}{3a^2}\right]\\
& \left.+Ca\left[\frac{\theta I_1^2}{2\rho_1}+\frac{\theta\left(I_2^2-I_1^2\right)}{2\rho_2}+\frac{\theta I_2^2}{2a}\right]+G\right\}
\end{aligned}$$

$$\begin{aligned}
CT = & \frac{1}{T}\left\{\left[p_{c1}k_1+C_bd_1k_1\right]\left[\frac{I_1}{\rho_1}+\frac{\theta I_1^2}{2\rho_1^2}\right]+\left[p_{c2}k_2+C_bd_2k_2\right]\left[\frac{I_2-I_1}{\rho_2}+\frac{\theta\left(I_2^2-I_1^2\right)}{2\rho_2^2}\right]\right.\\
& +\left[p_{c2}\frac{(1-r)a}{(1-d_2)}+C_bd_2\frac{(1-r)a}{(1-d_2)}+C_p\times r\times a\right]\left[T-BI_2+\frac{\theta}{2}CI_2^2+EI_1+\frac{\theta}{2}FI_1^2\right]\\
& + C_s\frac{a(1-r)\left[(1-d_2)\,k_2-a(1-r)\right]}{2\,(1-d_2)\,k_2}\left[T-BI_2+\frac{\theta}{2}CI_2^2+EI_1+\frac{\theta}{2}FI_1^2\right]^2\\
& + C\left[\frac{I_1^2}{2\rho_1}+\frac{\theta I_1^3}{3\rho_1^2}+\frac{\left(I_2^2-I_1^2\right)}{2\rho_2}+\frac{\theta\left(I_2^3-I_1^3\right)}{3\rho_2^2}+\frac{I_2^2}{2a}-\frac{\theta I_2^3}{3a^2}\right]\\
& \left.+Ca\left[\frac{\theta I_1^2}{2\rho_1}+\frac{\theta\left(I_2^2-I_1^2\right)}{2\rho_2}+\frac{\theta I_2^2}{2a}\right]+G\right\}
\end{aligned} \qquad (5.82)$$

5.4.2 Optimality conditions for MSS problem

Two important axioms are assumed to be satisfied by the production system under study. The first axiom is that there is an economic region in which a decrease in the total cost value cannot result from an increased in value of the decisions variables. This axiom is equivalent to the condition of non-input factor waste. This leads to an important factor called the average cost, $f(I_1, I_2, T)$, which is the first partial derivative or gradient of the total cost function. We have:

$$\begin{aligned}\frac{\partial CT}{\partial I_1} = \frac{1}{T}\Bigg\{&(p_{c1}k_1 + C_b d_1 k_1)\left[\frac{1}{\rho_1} + \frac{\theta I_1}{\rho_1^2}\right] - (p_{c2}k_2 + C_b d_2 k_2)\left[\frac{1}{\rho_2} + \frac{\theta I_1}{\rho_2^2}\right]\\ &+ \left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right][E + \theta F I_1]\\ &+ C_s \frac{a(1-r)\left[(1-d_2)k_2 - a(1-r)\right]}{(1-d_2)k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]\\ &\times [E + \theta F I_1] + h \times \left[\frac{I_1}{\rho_1} + \frac{\theta I_1^2}{\rho_1^2} - \frac{I_1}{\rho_2} - \frac{\theta I_1^2}{\rho_2^2}\right] + C_a\left[\frac{\theta I_1}{\rho_1} - \frac{\theta I_1}{\rho_2}\right]\Bigg\}\end{aligned} \tag{5.83}$$

$$\begin{aligned}\frac{\partial CT}{\partial I_2} = \frac{1}{T}\Bigg\{&(p_{c2}k_2 + C_b d_2 k_2)\left[\frac{1}{\rho_2} + \frac{\theta I_2}{\rho_2^2}\right]\\ &+ \left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right] \times [-B + C\theta I_2]\\ &+ C_s \frac{a(1-r)\left[(1-d_2)k_2 - a(1-r)\right]}{(1-d_2)k_2}\\ &\times \left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right] \times [-B + C\theta I_2]\\ &+h \times \left[\frac{I_2}{\rho_2} + \frac{\theta I_2^2}{\rho_2^2} + \frac{I_2}{a} - \frac{\theta I_2^2}{a^2}\right] + C_a\left[\frac{\theta I_2}{\rho_2} + \frac{\theta I_2}{a}\right]\Bigg\}\end{aligned} \tag{5.84}$$

$$
\begin{aligned}
\frac{\partial CT}{\partial T} =& \frac{1}{T}\left\{\left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right]\right. \\
&+ C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]\left.\right\} \\
&- \frac{1}{T^2}\left\{(p_{c1}k_1 + C_b d_1 k_1)\left(\frac{I_1}{\rho_1} + \frac{\theta I_1^2}{2\rho_1^2}\right) + (p_{c2}k_2 + C_b d_2 k_2)\left[\frac{I_2 - I_1}{\rho_2} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2^2}\right]\right. \\
&+ \left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right]\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right] \\
&+ C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{2\,(1-d_2)\,k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]^2 + h \\
&\times \left[\frac{I_1^2}{2\rho_1} + \frac{\theta I_1^3}{3\rho_1^2} + \frac{\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{\theta\left(I_2^3 - I_1^3\right)}{3\rho_2^2} + \frac{I_2^2}{2a} - \frac{\theta I_2^3}{3a^2}\right] + C_a\left[\frac{\theta I_1^2}{2\rho_1} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{\theta I_2^2}{2a}\right] \\
&+ G\left.\right\}
\end{aligned}
\tag{5.85}
$$

When the data is available, optimum values of the decisions variables can be obtained by solving equations (5.83), (5.84), and (5.85). Sahoo et al. (2019) argued that Newton-Raphson's method could solve such a problem. The optimum values of decision variables I_1, I_2 and T could then be obtained by setting the average costs in (5.83), (5.84), and (5.85) to zero.

$$\frac{\partial CT}{\partial I_1} = 0, \quad \frac{\partial CT}{\partial I_2} = 0, \quad \frac{\partial CT}{\partial T} = 0 \tag{5.86}$$

The second cost function axiom is that the second derivative of the total cost function, or Hessian Matrix (H_M), is positive (semi)definite or surely nonnegative to be more precise. Thus, we have in the feasible region matrices expressions represented in (5.87) that are non-negative.

$$\Delta\left|H_1\right| = \frac{\partial^2 \mathrm{CT}}{\partial I_1^2} \geq 0 \tag{5.87a}$$

$$\Delta\left|H_2\right| = \begin{bmatrix} \frac{\partial^2 \mathrm{CT}}{\partial I_1^2} & \frac{\partial^2 \mathrm{CT}}{\partial I_1 \partial I_2} \\ \frac{\partial^2 \mathrm{CT}}{\partial I_2 \partial I_1} & \frac{\partial^2 \mathrm{CT}}{\partial I_2^2} \end{bmatrix} \geq 0 \tag{5.87b}$$

$$\Delta\left|H_3\right| = \begin{bmatrix} \frac{\partial^2 \mathrm{CT}}{\partial I_1^2} & \frac{\partial^2 \mathrm{CT}}{\partial I_1 \partial I_2} & \frac{\partial^2 \mathrm{CT}}{\partial I_1 \partial T} \\ \frac{\partial^2 \mathrm{CT}}{\partial I_2 \partial I_1} & \frac{\partial^2 \mathrm{CT}}{\partial I_2^2} & \frac{\partial^2 \mathrm{CT}}{\partial I_2 \partial T} \\ \frac{\partial^2 \mathrm{CT}}{\partial T \partial I_1} & \frac{\partial^2 \mathrm{CT}}{\partial T \partial I_2} & \frac{\partial^2 \mathrm{CT}}{\partial T^2} \end{bmatrix} \geq 0 \tag{5.87c}$$

With: H_M : The Hessian Matrix

$$
\begin{aligned}
\frac{\partial^2 CT}{\partial I_1^2} = \frac{1}{T}\{(p_{c1}k_1 \; &+ C_b d_1 k_1)\frac{\theta}{\rho_1^2} - (p_{c2}k_2 + C_b d_2 k_2)\frac{\theta}{\rho_2^2} + \left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right]\theta F \\
&+ C_s \frac{a(1-r)\left[(1-d_2)k_2 - a(1-r)\right]}{(1-d_2)k_2}\left[E + \theta F I_1\right]^2 \\
&+ C_s \frac{a(1-r)\left[(1-d_2)k_2 - a(1-r)\right]}{(1-d_2)k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]\theta F + h \\
&\times \left[\frac{1}{\rho_1} + \frac{2\theta I_1}{\rho_1^2} - \frac{1}{\rho_2} - \frac{2\theta I_1}{\rho_2^2}\right] + C_a\left[\frac{\theta}{\rho_1} - \frac{\theta}{\rho_2}\right]\Bigg\}
\end{aligned}
\tag{5.88}
$$

$$
\frac{\partial^2 CT}{\partial I_1 \partial I_2} = \frac{1}{T} C_s \frac{a(1-r)\left[(1-d_2)k_2 - a(1-r)\right]}{(1-d_2)k_2}\left[-B + \theta C I_2\right]\left[E + \theta F I_1\right]
\tag{5.89}
$$

$$
\begin{aligned}
\frac{\partial^2 CT}{\partial I_1 \partial T} =& \frac{1}{T}\left\{C_s \frac{a(1}{T}\frac{-r)\left[(1-d_2)k_2 - a(1-r)\right]}{(1-d_2)k_2}\left[E + \theta F I_1\right]\right\} \\
&- \frac{1}{T^2}\left\{(p_{c1}k_1 + C_b d_1 k_1)\left(\frac{1}{\rho_1} + \frac{\theta I_1}{\rho_1^2}\right) - (p_{c2}k_2 + C_b d_2 k_2)\left[\frac{1}{\rho_2} + \frac{\theta I_1}{\rho_2^2}\right]\right. \\
&+ \left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_1)}\right]\left[E + \theta F I_1\right] \\
&+ C_s \frac{a(1-r)\left[(1-d_2)k_2 - a(1-r)\right]}{(1-d_2)k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]\left[E + \theta F I_1\right] + h \\
&\times \left[\frac{I_1}{\rho_1} + \frac{\theta I_1^2}{\rho_1^2} - \frac{I_1}{\rho_2} - \frac{\theta I_1^2}{\rho_2^2}\right] + C_a\left[\frac{\theta I_1}{\rho_1} - \frac{\theta I_1}{\rho_2}\right]\Bigg\}
\end{aligned}
\tag{5.90}
$$

$$
\begin{aligned}
\frac{\partial^2 CT}{\partial I_2^2} = \frac{1}{T}\Big\{&(p_{c2}k_2 \ +C_b d_2 k_2)\,\frac{\theta}{\rho_2^2} + \left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right] C\theta \\
&+ C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2}\left[-B + C\theta I_2\right]^2 \\
&+ C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right] C\theta + h \\
&\times \left[\frac{1}{\rho_2} + \frac{2\theta I_2}{\rho_2^2} + \frac{1}{a} - \frac{2\theta I_2}{a^2}\right] + C_a\left[\frac{\theta}{\rho_2} + \frac{\theta}{a}\right]\Big\}
\end{aligned}
\tag{5.91}
$$

$$
\frac{\partial^2 CT}{\partial I_2 \partial I_1} = \frac{1}{T} C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2}\left[E + \theta F I_1\right]\left[-B + C\theta I_2\right]
\tag{5.92}
$$

$$
\begin{aligned}
\frac{\partial^2 CT}{\partial I_2 \partial T} =&\frac{1}{T} C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2}\left[-B + C\theta I_2\right] \\
&- \frac{1}{T^2}\Bigg\{(p_{c2}k_2 + C_b d_2 k_2)\left[\frac{1}{\rho_2} + \frac{\theta I_2}{\rho_2^2}\right] \\
&+ \left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right]\left[-B + C\theta I_2\right] \\
&+ C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]\left[-B + C\theta I_2\right] + h \\
&\times \left[\frac{I_2}{\rho_2} + \frac{\theta I_2^2}{\rho_2^2} + \frac{I_2}{a} - \frac{\theta I_2^2}{a^2}\right] + C_a\left[\frac{\theta I_2}{\rho_2} + \frac{\theta I_2}{a}\right]\Bigg\}
\end{aligned}
\tag{5.93}
$$

$$
\begin{aligned}
\frac{\partial^2 CT}{\partial T^2} =& \frac{1}{T} C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2} \\
&- \frac{2}{T^2}\left\{\left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right]\right. \\
&\left. + C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2}\left[T - BI_2 + \frac{\theta}{2} C I_2^2 + E I_1 + \frac{\theta}{2} F I_1^2\right]\right\} \\
&+ \frac{2}{T^3}\left\{(p_{c1} k_1 + C_b d_1 k_1)\left(\frac{I_1}{\rho_1} + \frac{\theta I_1^2}{2\rho_1^2}\right) + (p_{c2} k_2 + C_b d_2 k_2)\left[\frac{I_2 - I_1}{\rho_2} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2^2}\right]\right. \\
&+ \left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right]\left[T - BI_2 + \frac{\theta}{2} C I_2^2 + E I_1 + \frac{\theta}{2} F I_1^2\right] \\
&+ C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{2\,(1-d_2)\,k_2}\left[T - BI_2 + \frac{\theta}{2} C I_2^2 + E I_1 + \frac{\theta}{2} F I_1^2\right]^2 + h \\
&\times \left[\frac{I_1^2}{2\rho_1} + \frac{\theta I_1^3}{3\rho_1^2} + \frac{\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{\theta\left(I_2^3 - I_1^3\right)}{3\rho_2^2} + \frac{I_2^2}{2a} - \frac{\theta I_2^3}{3a^2}\right] + C_a \left[\frac{\theta I_1^2}{2\rho_1} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{\theta I_2^2}{2a}\right] \\
&\left. + G\right\}
\end{aligned}
\tag{5.94}
$$

$$
\begin{aligned}
\frac{\partial^2 CT}{\partial T \partial I_1} =& \frac{1}{T}\left\{C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2}\left[E + \theta F I_1\right]\right\} \\
& - \frac{1}{T^2}\left\{(p_{c1}k_1 + C_b d_1 k_1)\left(\frac{1}{\rho_1} + \frac{\theta I_1}{\rho_1^2}\right) - (p_{c2}k_2 + C_b d_2 k_2)\left[\frac{1}{\rho_2} + \frac{\theta I_1}{\rho_2^2}\right]\right. \\
& + \left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right]\left[E + \theta F I_1\right] \\
& + C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]\left[E + \theta F I_1\right] + h \\
& \left. \times \left[\frac{I_1}{\rho_1} + \frac{\theta I_1^2}{\rho_1^2} - \frac{I_1}{\rho_2} - \frac{\theta I_1^2}{\rho_2^2}\right] + C_a\left[\frac{\theta I_1}{\rho_1} - \frac{\theta I_1}{\rho_2}\right]\right\}
\end{aligned}
\tag{5.95}
$$

$$
\begin{aligned}
\frac{\partial^2 CT}{\partial T \partial I_2} =& \frac{1}{T}\left\{C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2}\left[-B + \theta C I_2\right]\right\} \\
& - \frac{1}{T^2}\left\{(p_{c2}k_2 + C_b d_2 k_2)\left[\frac{1}{\rho_2} + \frac{\theta I_2}{\rho_2^2}\right]\right. \\
& + \left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right]\left[-B + \theta C I_2\right] \\
& + C_s \frac{a(1-r)\left[(1-d_2)\,k_2 - a(1-r)\right]}{(1-d_2)\,k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]\left[-B + \theta C I_2\right] + h \\
& \left. \times \left[\frac{I_2}{\rho_2} + \frac{\theta I_2^2}{\rho_2^2} + \frac{I_2}{a} - \frac{\theta I_2^2}{a^2}\right] + C_a\left[\frac{\theta I_2}{\rho_2} + \frac{\theta I_2}{a}\right]\right\}
\end{aligned}
\tag{5.96}
$$

The cost function CT is therefore strictly convex in the relevant economic region if the two axioms are met. In the relevant economic region, we can show numerically that the main diagonal components of the Hessian matrix are positive. This is physically a very significant result. Many researchers such as Salamah (2019), Uthayakumar and Sekar (2017) and Sana (2007) use these two axioms for solving inventory models. If the solutions obtained from equation (5.86) do not satisfy the Hessian Matrix, it may be concluded that no possible solution will be optimal for the set of parameters considered in this chapter. Such a situation will imply that the parameter values are inconsistent or there is some error in their estimation.

Now, to prove that CT is positive (semi) definite, The determinants need to satisfy the following condition: $\Delta|H_1| \geq 0,\ \Delta|H_2| \geq 0$ and $\Delta|H_3| \geq 0$.

To derive the conditions for $\Delta|H_1| \geq 0$, substitute (5.88) on the relevant determinant in (5.87a) such that:

$$\begin{aligned}
&\frac{1}{T}\left\{\left(p_{c1}k_1 + C_b d_1 k_1 \frac{\theta}{\rho_1^2} - \left(p_{c2}k_2 + C_b d_2 k_2\right)\frac{\theta}{\rho_2^2} + \right.\right.\\
&\left[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_p \times r \times a + C_b d_2 \frac{(1-r)a}{(1-d_2)}\right]\theta F\\
&+ C_s \frac{a(1-r)\left[(1-d_2)k_2 - a(1-r)\right]}{(1-d_2)k_2}\left[E + \theta F I_1\right]^2\\
&+ C_s \frac{a(1-r)\left[(1-d_2)k_2 - a(1-r)\right]}{(1-d_2)k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]\theta F\\
&\left.+h \times \left[\frac{1}{\rho_1} + \frac{2\theta I_1}{\rho_1^2} - \frac{1}{\rho_2} - \frac{2\theta I_1}{\rho_2^2}\right] + C_a\left[\frac{\theta}{\rho_1} - \frac{\theta}{\rho_2}\right]\right\} \geq 0
\end{aligned}$$

$$\begin{aligned}
&= \left(p_{c1}k_1 + C_b d_1 k_1\right)\frac{\theta}{\rho_1^2} - \left(p_{c2}k_2 + C_b d_2 k_2\right)\frac{\theta}{\rho_2^2} + \left[p_{c2}\omega + C_p \times r \times a + C_b d_2 \omega\right]\theta F\\
&\quad + C_s\varphi\left[E + \theta F I_1\right]^2 + C_s\varphi\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]\theta F\\
&\quad + h \times \left[\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + 2\theta I_1\left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right)\right] + C_a\theta\left[\frac{1}{\rho_1} - \frac{1}{\rho_2}\right] \geq 0
\end{aligned}$$

$$\begin{aligned}
&= \left(p_{c1}k_1 + C_b d_1 k_1\right)\frac{\theta}{\rho_1^2} + \left[p_{c2}\omega + C_p \times r \times a + C_b d_2 \omega\right]\theta F + C_s\varphi\left[E + \theta F I_1\right]^2\\
&\quad + C_s\varphi\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]\theta F - h\left(E + 2\theta I_1 F\right) - C_a\theta E\\
&\quad - \left(p_{c2}k_2 + C_b d_2 k_2\right)\frac{\theta}{\rho_2^2} \geq 0
\end{aligned}$$

$$M+(Q\omega + C_p \times r \times a + C_s\varphi\sigma)\,\theta F+(C_s\varphi E - h)\,(E + 2\theta I_1 F) > C_a\theta E+N \quad (5.97)$$

It is known that $P_{c2} > P_{c1}, k_1 > k_2$ and $d_1 > d_2$; therefore when $M+(Q\omega + C_p \times r \times a+ C_s\varphi\sigma)\,\theta F + (C_s\varphi E - h)\,(E + 2\theta I_1 F) > C_a\theta E + N$ then the condition $\Delta\,|H_1| > 0$ holds.

with:

$$\varphi = \frac{a(1-r)\,[(1-d_2)\,k_2 - a(1-r)]}{(1-d_2)\,k_2} \quad (5.98a)$$

$$\omega = \frac{(1-r)a}{(1-d_2)} \quad (5.98b)$$

$$M = (p_{c1}k_1 + C_b d_1 k_1)\,\frac{\theta}{\rho_1^2} \quad (5.98c)$$

$$Q = (p_{c2} + C_b d_2) \quad (5.98d)$$

$$N = (p_{c2}k_2 + C_b d_2 k_2)\,\frac{\theta}{\rho_2^2} \quad (5.98e)$$

$$\sigma = \left[T - BI_2 + EI_1 + \frac{\theta}{2}\left(FI_1^2 + CI_2^2\right)\right] \quad (5.98f)$$

To derive the conditions for $\Delta\,|H_2| \geq 0$. The determinant is given by:

$$\Delta\,|H_2| = \left[\frac{\partial^2 \mathrm{CT}}{\partial I_1^2}\frac{\partial^2 \mathrm{CT}}{\partial I_2^2}\right] - \left[\frac{\partial^2 \mathrm{CT}}{\partial I_1 \partial I_2}\frac{\partial^2 \mathrm{CT}}{\partial I_2 \partial I_1}\right] = \left[\frac{\partial^2 \mathrm{CT}}{\partial I_1^2}\frac{\partial^2 \mathrm{CT}}{\partial I_2^2}\right] - \left[\frac{\partial^2 \mathrm{CT}}{\partial I_1 \partial I_2}\right]^2 \quad (5.99)$$

Substitute (5.88), (5.89), (5.91) and (5.92) into $\Delta\,|H_2|$. Now the first term is:

$$\begin{aligned}
&\frac{1}{T}\left\{M + (Q\omega + C_p \times r \times a + C_s\varphi\sigma)\,\theta F + (C_s\varphi E - h)\,(E + 2\theta I_1 F) - C_a\theta E - N\right\}\\
&\times \frac{1}{T}\left\{N + [(p_{c2} + C_b d_2)\,\omega + C_p \times r \times a]\,C\theta + (C_s\varphi B + h) \times (B - 2\theta I_2 C)\right.\\
&\left.+ C_s\varphi\sigma C\theta + C_a\theta B\right\}
\end{aligned} \quad (5.100)$$

And the second term of the determinant is:

$$\frac{1}{T}C_s\varphi\left(E+\theta FI_1\right)\left(-B+C\theta I_2\right)\frac{1}{T}C_s\varphi\left(E+\theta FI_1\right)\left(-B+C\theta I_2\right) \tag{5.101}$$

In order for $\Delta\left|\mathrm{H}_2\right|$ To satisfy the condition, equation (100) and (101) must satisfy the following condition:

$$\begin{aligned}&\frac{1}{T^2}\{M+(Q\omega+C_p\times r\times a+C_s\varphi\sigma)\,\theta F+(C_s\varphi E-h)\,(E+2\theta I_1F)-C_a\theta E-N\}\,\{N\\&\quad+[(p_{c2}+C_bd_2)\,\omega+C_p\times r\times a]\,C\theta+(C_s\varphi B+h)\times(B-2\theta I_2C)+C_s\varphi\sigma C\theta\\&\quad+C_a\theta B\}-\frac{1}{T^2}C_s^2\varphi^2\left(E+\theta FI_1\right)^2\left(-B+C\theta I_2\right)^2\geq 0\end{aligned}$$

$$\begin{aligned}&(Q\omega\ +C_p\times r\times a+C_s\varphi\sigma)\,\theta F\,\{N+(C_s\varphi B+h)\times(B-2\theta I_2C)\}\,(C_s\varphi E\\&\quad-h)\,\{[N+((p_{c2}+C_bd_2)\,\omega+C_p\times r\times a)\,C\theta\\&\quad+(C_s\varphi B+h)\times(B-2\theta I_2C)+(C_s\varphi\sigma C\theta+C_a\theta B)]\,E\\&\quad+[N+(C_s\varphi B+h)\,B]\,2\theta FI_1\}-C_a\theta E\,\{N+(C_s\varphi B+h)\,B\}\\&\quad\geq C_s^2\varphi^2EB\,[E\,(B-2BC\theta I_2)+2B\theta FI_1]\\&\quad-(N-M)\,\{N+((p_{c2}+C_bd_2)\,\omega+C_p\times r\times a)\,C\theta+(C_s\varphi B+h)\times(B-2\theta I_2C)\\&\quad+C_s\varphi\sigma C\theta+C_a\theta B\}\end{aligned} \tag{5.102}$$

To derive the conditions for $\Delta\left|H_3\right|$, under which $\Delta\left|H_3\right|$ satify the proof of optimality, equation (5.103) must be non negative.

$$\begin{aligned}\Delta\left|H_3\right|=&\frac{\partial^2\mathrm{CT}}{\partial I_1^2}\left[\frac{\partial^2\mathrm{CT}}{\partial I_2^2}\cdot\frac{\partial^2\mathrm{CT}}{\partial T^2}-\frac{\partial^2\mathrm{CT}}{\partial I_2\partial T}\cdot\frac{\partial^2\mathrm{CT}}{\partial T\partial I_2}\right]\\&-\frac{\partial^2\mathrm{CT}}{\partial I_2\partial I_1}\left[\frac{\partial^2\mathrm{CT}}{\partial I_1\partial I_2}\cdot\frac{\partial^2\mathrm{CT}}{\partial T^2}-\frac{\partial^2\mathrm{CT}}{\partial I_1\partial T}\cdot\frac{\partial^2\mathrm{CT}}{\partial T\partial I_2}\right]\\&+\frac{\partial^2\mathrm{CT}}{\partial T\partial I_1}\left[\frac{\partial^2\mathrm{CT}}{\partial I_1\partial I_2}\cdot\frac{\partial^2\mathrm{CT}}{\partial I_2\partial T}-\frac{\partial^2\mathrm{CT}}{\partial I_2^2}\cdot\frac{\partial^2\mathrm{CT}}{\partial I_1\partial T}\right]\end{aligned} \tag{5.103}$$

Therefore, $\Delta\left|H_3\right|\geq 0$, when

$$\frac{\partial^2\mathrm{CT}}{\partial I_1^2}\left[\frac{\partial^2\mathrm{CT}}{\partial I_2^2}\cdot\frac{\partial^2\mathrm{CT}}{\partial T^2}-\frac{\partial^2\mathrm{CT}}{\partial I_2\partial T}\cdot\frac{\partial^2\mathrm{CT}}{\partial T\partial I_2}\right]-\frac{\partial^2\mathrm{CT}}{\partial I_2\partial I_1}\left[\frac{\partial^2\mathrm{CT}}{\partial I_1\partial I_2}\cdot\frac{\partial^2\mathrm{CT}}{\partial T^2}-\frac{\partial^2\mathrm{CT}}{\partial I_1\partial T}\cdot\frac{\partial^2\mathrm{CT}}{\partial T\partial I_2}\right]$$
$$+\frac{\partial^2\mathrm{CT}}{\partial T\partial I_1}\left[\frac{\partial^2\mathrm{CT}}{\partial I_1\partial I_2}\cdot\frac{\partial^2\mathrm{CT}}{\partial I_2\partial T}-\frac{\partial^2\mathrm{CT}}{\partial I_2^2}\cdot\frac{\partial^2\mathrm{CT}}{\partial I_1\partial T}\right]\geq 0 \tag{5.104}$$

Equations (5.102) and (5.104) being complex, proving the optimality conditions would be challenging to find analytically. Therefore, a numerical approach is considered to be more appropriate.

5.5 Numerical examples and sensitivity analysis

5.5.1 Numerical results

A numerical example and a sensitivity analysis will be presented to illustrate the feasibility of the proposed model. The Newton Raphson method is used to solve the problem since it is difficult to find analytical solution. A deterministic deteriorating production system consisting of a single unreliable machine in which the production rate dependents on the failure state is analysed in this scenario. It is assumed that when a breakdown occurs, the failure mode is activated and the production rate is switched to a production rate $k_2, k_2 < k_1$

Consider a production process for making a single item where the demand rate for the item is 25 units per day, the first and second production rates are 80 units and 55 units per day respectively. Defective rates in the two production consumptions cycles are 7% and 14% respectively. The setup cost is $R2700$. The holding cost of one unit of the finished product $R0.5$ per day. The production costs of one unit of the finished item during the two production cycles are $R21$ and $R20$ respectively. The cost of a unit deteriorated items is $R18$ per day. The imperfect quality items screened may be disposed of at the end of screening period at a unit cost of $R3$, the portion of stock out demand sale lost is 20%, the rate at which items deteriorate is 0.002, the shortage cost is $R5$ per unit per time and the cost of loss of sale is $R11$ per unit.
Note from the results above that cost function is positive (semi)definite since the following conditions are satisfied:
$\Delta\,|H_1| = 0, \quad \Delta\,|H_2| = 0, \quad \Delta\,|H_3| = 0$

Table 5.3: summary of the results from the numerical example

t_1	t_2	t_3	t_4	T	I_1^*	I_2^*	S
4.6	8.95	21.6	24.1	25.92	224.18	319.88	50.06
G	^{AC}	ICC	BC	DC	LC	PC	CT^*
2700	134	1860	543	220	239	14511	20207

Based on the results of the numerical example summarised in Table (5.3) the company reaches a maximum inventory of 201.49 units in the first production-consumption cycle and stops production once a stock level of 315.46 units is reached. A maximum backlog of 49.91 units is allowed. This maximum backlog is reached at $t_4 = 24.28$ days, and it takes 1.93 days for the company to clear this backlog in the system. The cycle total time is 26.21 days, after which the cycle starts again. Deterioration, inventory holding, shortage, disposal, loss of profit, setup and production costs per cycle are $R2700$, $R134$, $R1861$, $R539$, $R232$, $R238$ and $R14643$ respectively. With this optimal replenishment policy, the company will incur a total cost $R20348$ per unit time.

Graphical illustrations are shown in Figures (5.2) and (5.3), which depict a typical inventory system curve and the variation of the inventory levels, cycle time and shortage for the model under study. Figures (5.2) and (5.3) also illustrate the average total cost that optimizes such a system.

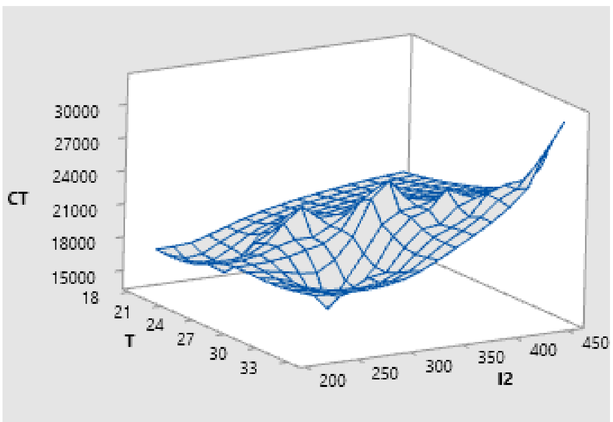

Figure 5.2: Graph of total cost per unit time versus cycle time and inventory level

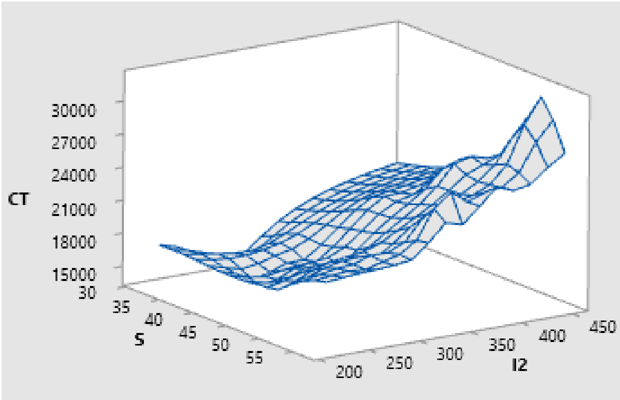

Figure 5.3: Graph of total cost per unit time VS shortage and inventory level

5.5.2 Sensitivity analysis

Sensitivity analysis depicts the extent to which the optimal solution of the model is affected by the changes in its input parameter values. Here, the sensitivity analysis is performed for the inventory levels, backlog level, cycle time and the total cost per cycle by changing values of parameters $C_a, h, C_b, C_s, p_{c1}, p_{c2}, C_p, d_1, d_2, r, p_c, \theta, a$ and G. The sensitivity analysis is performed by considering different values for each of the above parameters while keeping all other parameters as fixed. The results are presented in Tables (5.4) to (5.18).Moreover, the summary graphs for the percentage change in cycle time and total cost with respect to changes in parameter are presented from Figure (5.4) to Figure (5.8) respectively.

Table 5.4: The effect of changing c_a while keeping other parameters constant.

	% Change					
c_a	Change in parameter	T	I_1	I_2	S	CT
11.7	−35%	0.99%	0.75%	1.22%	−0.72%	−0.23%
14.4	−20%	0.56%	0.43%	0.69%	−0.41%	−0.13%
17.1	−5%	0.14%	0.11%	0.17%	−0.10%	−0.03%
18	0%	0.00%	0.00%	0.00%	−0.00%	−0.00%
19.8	10%	−0.28%	−0.21%	−0.34%	0.20%	0.07%
22.5	25%	−0.69%	−0.53%	−0.85%	0.51%	0.16%
23.4	30%	−0.82%	−0.64%	−1.01%	0.61%	0.19%

Table 5.5: The effect of changing h while keeping other parameters constant.

	% Change					
h	Change in parameter	T	I_1	I_2	S	CT
0.325	−35%	17.76%	12.32%	21.27%	−11.28%	−3.62%
0.4	−20%	8.87%	6.73%	10.83%	−6.10%	−1.96%
0.475	−5%	1.98%	1.61%	2.45%	−1.45%	−0.47%
0.5	0%	0.00%	0.00%	0.00%	−0.00%	−0.00%
0.55	10%	−3.57%	−3.06%	−4.48%	2.77%	0.89%
0.625	25%	−8.14%	−7.29%	−10.33%	6.63%	2.13%
0.65	30%	−9.49%	−8.61%	−12.08%	7.85%	2.52%

Table 5.6: The effect of changing c_b while keeping other parameters constant.

	% Change					
c_b	Change in parameter	T	I_1	I_2	S	CT
1.95	−35%	−0.33%	5.39%	0.78%	0.20%	−0.38%
2.4	−20%	−0.19%	3.07%	0.44%	0.11%	−0.22%
2.85	−5%	−0.05%	0.77%	0.11%	0.03%	−0.05%
3	0%	0.00%	0.00%	0.00%	0.00%	0.00%
3.3	10%	0.10%	−1.53%	−0.22%	−0.05%	0.11%
3.75	25%	0.25%	−3.82%	−0.53%	−0.12%	0.27%
3.9	30%	0.31%	−4.58%	−0.64%	−0.15%	0.33%

Table 5.7: The effect of changing c_s while keeping other parameters constant.

	% Change					
c_s	Change in parameter	T	I_1	I_2	S	CT
3.25	−35%	3.70%	−8.31%	−5.74%	47.27%	−1.37%
4	−20%	1.73%	−3.99%	−2.75%	22.44%	−0.66%
4.75	−5%	0.37%	−0.86%	−0.59%	4.80%	−0.14%
5	0%	0.00%	0.00%	0.00%	0.00%	0.00%
5.5	10%	−0.64%	1.51%	1.04%	−8.39%	0.25%
6.25	25%	−1.41%	3.37%	2.32%	−18.61%	0.56%
6.5	30%	−1.63%	3.91%	2.69%	−21.53%	0.64%

Table 5.8: The effect of changing p_{c1} and p_{c2} while keeping other parameters constant.

			% Change				
p_{c1}	p_{c2}	Change in parameter	T	I_1	I_2	S	CT
13.5	13	-35%	1.23%	10.41%	5.62%	-14.27%	-25.44%
16.8	16	-20%	0.40%	11.39%	3.97%	-7.57%	-14.35%
19.95	19	-5%	0.10%	2.83%	0.99%	-1.89%	-3.59%
21	15	0%	0.00%	0.00%	0.00%	0.00%	0.00%
23.1	22	10%	-0.21%	-5.62%	-1.97%	3.76%	7.17%
26.25	25	25%	-0.54%	-13.97%	-4.92%	9.36%	17.91%
27.3	26	30%	-0.66%	-16.73%	-5.89%	11.22%	21.48%

From Table (5.4), it is evident that T, I_1, I_2, S and CT are insensitive to changes in C_a. Table (5.5) reveals that T, I_1, I_2, S and CT are highly sensitive to changes in the values of h. As the unit holding cost increases T, I_1 and I_2 decrease while S increases. The total cost CT increases moderately. When trying to optimise the production and inventory control, managers should keep the holding cost as low as possible because this enables them to produce and keep more items at a lower cost, which in turn increases customer satisfaction levels by reducing the possibility of running out of stock.

From Table (5.6), it can be seen that T, I_2, S and CT are insensitive to changes in C_b. From Table (5.7), (5.8) and (5.9), it's evident that T, I_1, I_2 and S are highly sensitive to changes in values of parameters C_s, p_{c1} and p_{c2}. When C_s increases, the backlog S decreases while the inventories I_1 and I_2 increases. As the shortage cost C_s increases the cycle time T decreases while on the other hand the total cost CT increases slightly. Similarly, when p_{c1} and p_{c2} increases, the cycle time T decreases while the backlog and the total cost increase significantly.Therefore, when trying to optimise the production and inventory control, managers should consider working with small values of shortage cost.

From Table (5.10) and (5.11), it's evident that CT are insensitive to changes in values of parameters C_p and θ. However, the inventory increases I_1 slightly increases with the increase in C_p while the backlog decreases moderately; the inventory increases I_2 slightly decreases with the increase in C_p. When θ increases the inventory and the cycle time decrease while the backlog increases slightly as seen in Table (5.11).

Table 5.9: The effect of changing c_s while keeping other parameters constant.

	% Change					
c_s	Change in parameter	T	I_1	I_2	S	CT
3.25	−35%	3.70%	−8.31%	−5.74%	47.27%	−1.37%
4	−20%	1.73%	−3.99%	−2.75%	22.44%	−0.66%
4.75	−5%	0.37%	−0.86%	−0.59%	4.80%	−0.14%
5	0%	0.00%	0.00%	0.00%	0.00%	0.00%
5.5	10%	−0.64%	1.51%	1.04%	−8.39%	0.25%
6.25	25%	−1.41%	3.37%	2.32%	−18.61%	0.56%
6.5	30%	−1.63%	3.91%	2.69%	−21.53%	0.64%

Table 5.10: The effect of changing c_p while keeping other parameters constant.

	% Change					
c_p	Change in parameter	T	I_1	I_2	S	CT
7.15	−35%	−0.26%	−2.61%	−1.80%	6.35%	−0.43%
8.8	−20%	−0.14%	−1.47%	−1.02%	3.64%	−0.24%
10.45	−5%	−0.03%	−0.36%	−0.25%	0.91%	−0.06%
11	0%	0.00%	0.00%	0.00%	0.00%	0.00%
12.1	10%	0.06%	0.72%	0.49%	−1.83%	0.12%
13.75	25%	0.13%	1.77%	1.22%	−4.59%	0.29%
14.3	30%	0.13%	1.92%	1.32%	−5.01%	0.32%

Table 5.11: The effect of changing θ while keeping other parameters constant.

	% Change					
θ	Change in parameter	T	I_1	I_2	S	CT
1.3E − 3	−35%	2.34%	1.09%	2.70%	−1.81%	−0.58%
1.6E − 3	−20%	1.32%	0.62%	1.52%	−1.02%	−0.33%
1.9E − 3	−5%	0.32%	0.15%	0.37%	−0.25%	−0.08%
2E − 3	0%	0.00%	0.00%	0.00%	0.00%	0.00%
2.2E − 3	10%	−0.64%	−0.30%	−0.74%	0.50%	0.16%
2.4E − 3	25%	−1.26%	−0.60%	−1.46%	1.00%	0.32%
2.6E − 3	30%	−1.88%	−0.89%	−2.17%	1.50%	0.48%

Table 5.12: The effect of changing k_2 while keeping other parameters constant.

	% Change					
k_2	Change in parameter	T	I_1	I_2	S	CT
35.75	−35%	3.58%	26.50%	−7.35%	−6.21%	−1.99%
40	−20%	1.74%	24.59%	−4.68%	−3.85%	−1.24%
47.5	−5%	0.39%	15.92%	−1.94%	−1.56%	−0.50%
55	0%	0.00%	0.00%	0.00%	0.00%	0.00%
60.5	10%	0.14%	−18.72%	1.33%	1.05%	0.34%
68.75	25%	1.19%	−70.33%	3.85%	2.98%	0.96%
71.5	30%	1.94%	−100.%	5.08%	3.91%	1.26%

Table 5.13: The effect of changing a while keeping other parameters constant.

	% Change					
a	Change in parameter	T	I_1	I_2	S	CT
16.25	−35%	13.59%	−28.83%	−8.07%	−14.72%	−28.49%
20	−20%	6.14%	−14.70%	−3.37%	−7.49%	−15.99%
23.75	−5%	1.22%	−3.27%	−0.56%	−1.66%	−3.93%
25	0%	0.00%	0.00%	0.00%	0.00%	0.00%
27.5	10%	−1.94%	5.79%	0.60%	2.93%	7.73%
31.25	25%	−3.82%	13.06%	0.16%	6.56%	19.76%
32.5	30%	−4.08%	14.50%	−0.21%	7.27%	22.71%

Table 5.14: The effect of changing d_1 while keeping other parameters constant.

	% Change					
d_1	Change in parameter	T	I_1	I_2	S	CT
0.0455	−35%	2.73%	−32.71%	−3.36%	−2.50%	−0.80%
0.056	−20%	1.43%	−19.43%	−2.13%	−1.59%	−0.51%
0.665	−5%	0.32%	−5.06%	−0.59%	−0.44%	−0.14%
0.07	0%	0.00%	0.00%	0.00%	0.00%	0.00%
0.077	10%	−0.55%	10.55%	1.31%	0.97%	0.31%
0.0875	25%	−1.14%	27.53%	3.62%	2.69%	0.86%
0.091	30%	−1.27%	33.52%	4.49%	3.34%	1.07%

Table 5.15: The effect of changing d_2 while keeping other parameters constant.

	% Change					
d_2	Change in parameter	T	I_1	I_2	S	CT
0.091	−35%	**	**	**	**	**
0.112	−20%	−2.09%	50.65%	8.43%	4.39%	−0.75%
0.133	−5%	−0.77%	11.65%	1.87%	0.91%	−0.26%
0.14	0%	0.00%	0.00%	0.00%	0.00%	0.00%
0.154	10%	2.03%	−21.43%	−3.30%	−1.47%	0.66%
0.175	25%	6.14%	−49.29%	−7.34%	−2.96%	1.96%
0.182	30%	7.77%	−57.52%	−8.47%	−3.28%	2.47%

Table 5.16: The effect of changing G while keeping other parameters constant.

	% Change					
G	Change in parameter	T	I_1	I_2	S	CT
1755	−35%	−18.81%	−31.70%	−21.90%	−16.32%	−5.23%
2160	−20%	−10.21%	−17.22%	−11.89%	−8.86%	−2.84%
2565	−5%	−2.45%	−4.14%	−2.86%	−2.13%	−0.68%
2700	0%	0.00%	0.00%	0.00%	0.00%	0.00%
2970	10%	4.73%	8.00%	5.52%	4.11%	1.32%
3375	25%	11.45%	19.38%	13.36%	9.96%	3.19%
3510	30%	13.60%	23.03%	15.88%	11.83%	3.79%

Table 5.17: The effect of changing r while keeping other parameters constant.

	% Change					
r	Change in parameter	T	I_1	I_2	S	CT
0.13	−35%	0.05%	3.07%	2.12%	−7.33%	0.51%
0.16	−20%	0.04%	1.81%	1.25%	−4.16%	0.30%
0.19	−5%	−0.28%	−0.21%	−0.34%	0.20%	0.07%
0.2	0%	0.00%	0.00%	0.00%	0.00%	0.00%
0.22	10%	−0.02%	−0.97%	−0.67%	2.05%	−0.16%
0.25	25%	−0.07%	−2.51%	−1.73%	5.07%	−0.41%
0.26	30%	−0.09%	−3.05%	−2.11%	6.07%	−0.50%

Table 5.18: The effect of changing k_1 while keeping other parameters constant.

	% Change					
k_1	Change in parameter	T	I_1	I_2	S	CT
52	−35%	*	*	*	*	*
64	−20%	7.66%	−41.18%	−4.92%	−3.66%	−1.18%
76	−5%	1.38%	−6.01%	−1.09%	−0.82%	−0.26%
80	0%	0.00%	0.00%	0.00%	0.00%	0.00%
88	10%	−2.19%	8.72%	1.90%	1.42%	0.45%
100	25%	−4.56%	17.29%	4.17%	3.11%	1.00%
104	30%	−5.19%	19.44%	4.80%	3.58%	1.15%

Careful study of Tables (5.12) and (5.13) reveal that the decision variables T, I_1, I_2, and S are highly sensitive to changes in k_2 and a. The inventory I_1 decreases drastically as the production rate k_2 increases whereas the inventory I_2 moderately increases as with the increase in the production rate. The backlog increases significantly with the increase in values of k_2. When the production rate k_2 increases the total cost slightly increases as seen from Table (5.12). When demand rate increases, the inventory, the backlog and the total cost increase. The cycle time decreases drastically with the increase in demand as seen from Table (5.13) and Figure (5.4).

Careful analysis of Tables (5.14), (5.15) and (5.16)reveals that the inventory, the backlog and the cycle time are sensitive to changes in d_1, d_2 and G. Moreover, it has been observed that the inventory I_1 decreases with the increase in r and d_2, and increases with the increase in d_1 and G. In addition, the backlog increases with the increase in r, d_1 and G, and decreases with the increase in d_2. The cycle time is insensitive to changes in r. The cycle time and the total cost increase with the increase in G. The inventories increase drastically with the increase in G. the backlog follows a similar trend as the inventories I_1 and I_2.

Table (5.17) indicates the effect of varying r on the decision variables of the MSS under study. This Table shows that when r decreases by 35%, the stock levels of finished products increase by 3.07% and 2.12% and the backlog moderately decreases by 7.33%. If r increases by 30%, the stock levels of finished products decrease by 3.05% and 2.11% and the backlog increases by 6.07%. The results of Table (5.17) also show that the total cost and the cycle time are insensitive regardless of the level

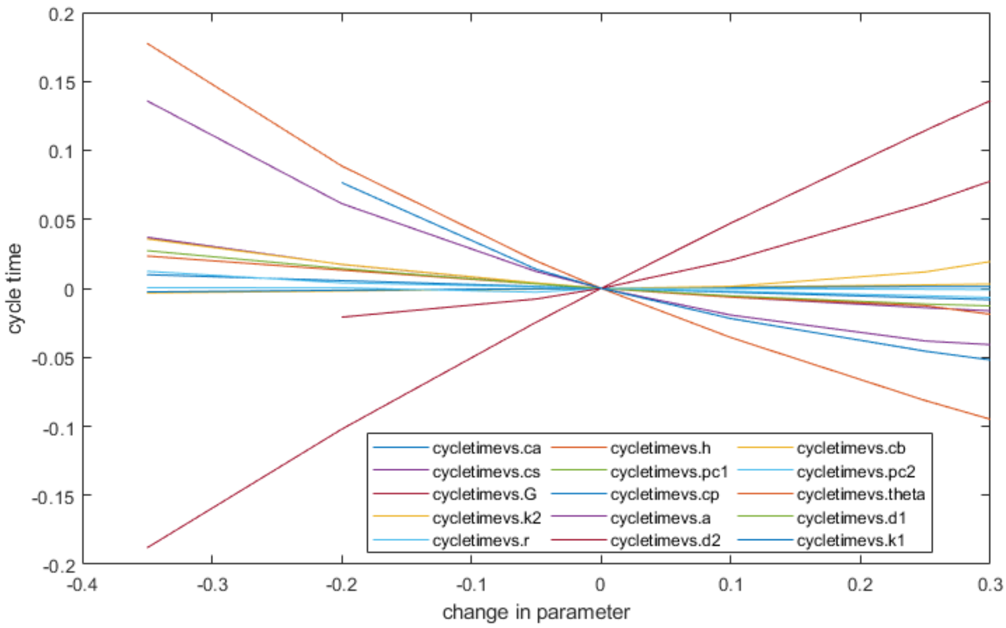

Figure 5.4: Changes in cycle time due to parameter changes

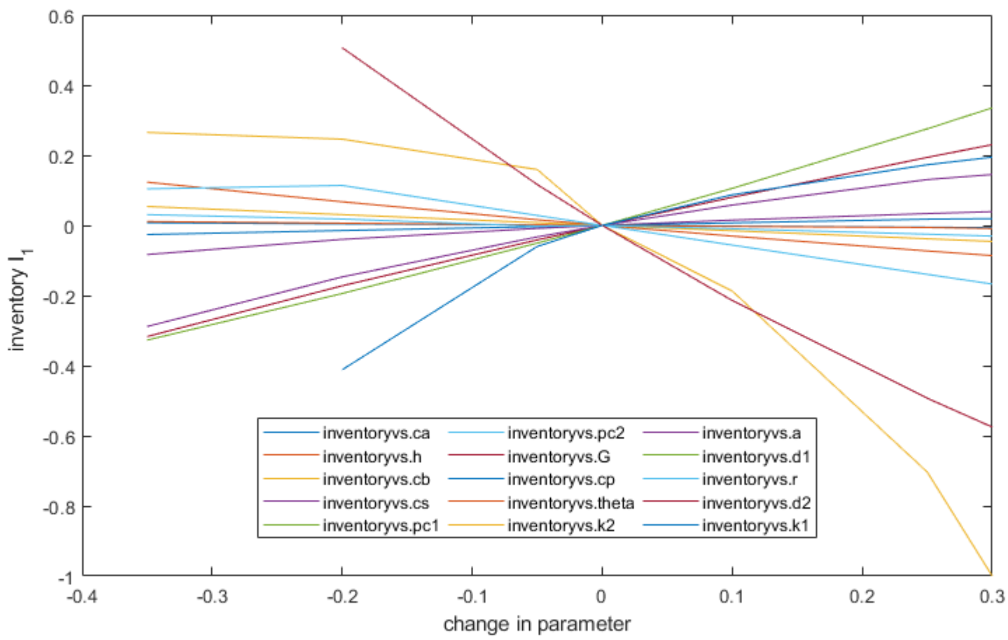

Figure 5.5: Changes in I_1 due to parameter changes

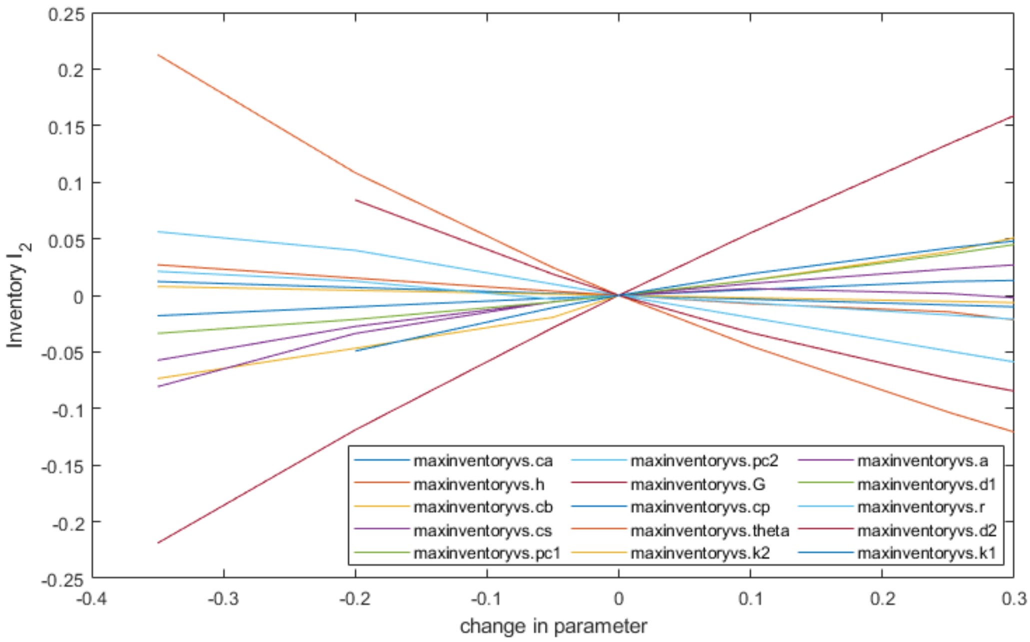

Figure 5.6: Changes in I_2 due to parameter changes

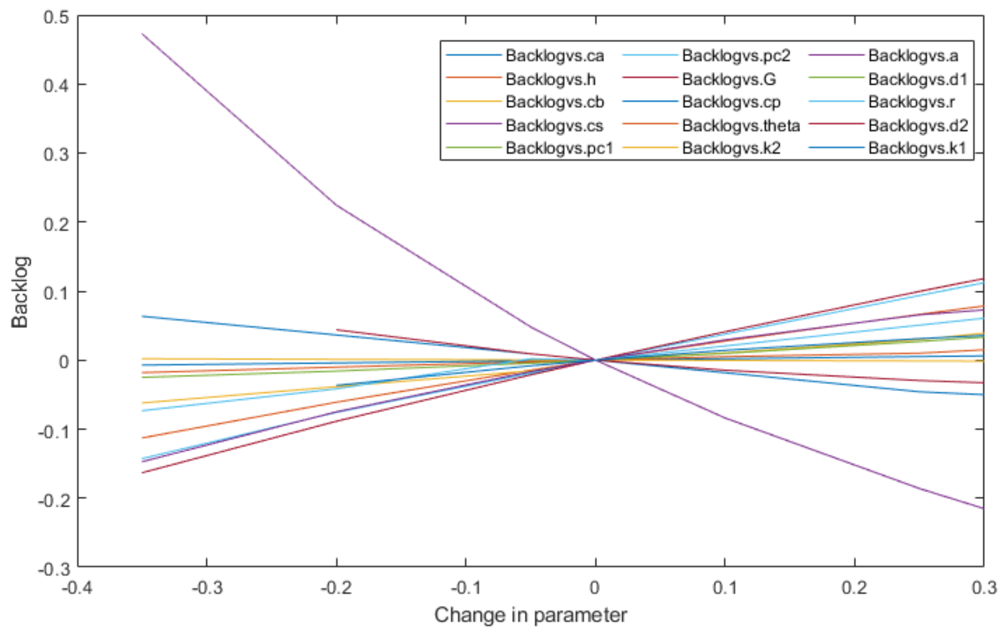

Figure 5.7: Changes in S due to parameter changes

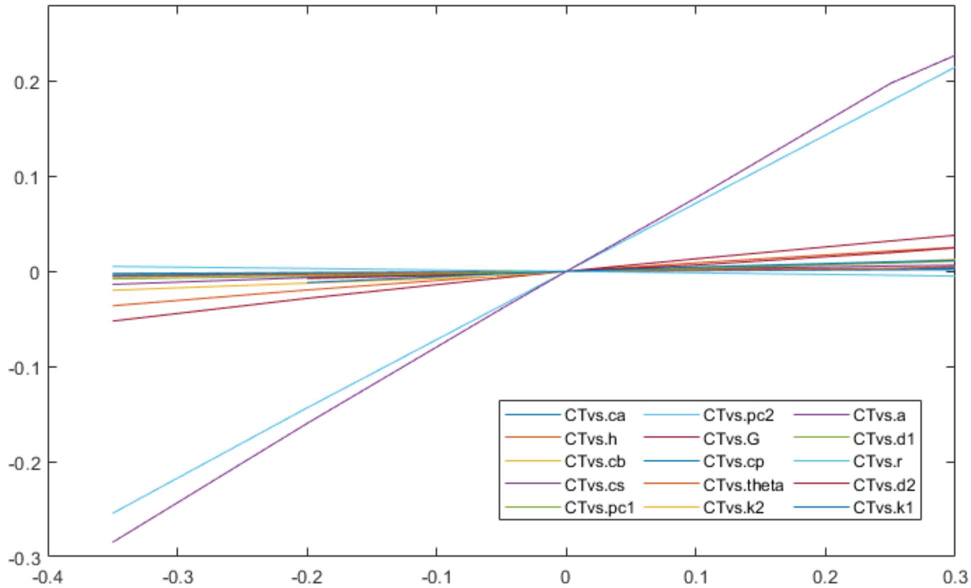

Figure 5.8: Changes in CT due to parameter changes

of r. The results summarised in Table (5.18) show that the inventory levels and the backlog increase with the increases in k_1. However, this increase has an opposite effect on the total cost and the cycle. Therefore, based on the results in Table (5.18) it can be concluded that, higher values of k_1 lead to higher values of I_1^*, I_2^*, S^* and lower values of T^* and CT^*.

In summary, it is observed that:

1. T is insensitive to changes in the values of c_a, c_b, c_p and r, moderately sensitive to changes in values of $c_s, p_{c1}, p_{c2}, k_1, k_2, \theta, d_1$ and d_2, highly sensitive to changes in the values of h, a and G.

2. I_1 is insensitive to changes in the values of c_a and θ, moderately sensitive to changes in values of c_b, c_s, c_p, and r, highly sensitive to changes in the values of $h, p_{c1}, p_{c2}, k_1, k_2, d_1, d_2 a$ and G.

3. I_2 is insensitive to changes in the values of c_b, moderately sensitive to changes in values of $c_s, c_a, c_p, p_{c1}, p_{c2}, a, r, d_1$, and d_2, highly sensitive to changes in the values of h, and G.

4. S is insensitive to changes in the values of c_a and c_b, moderately sensitive to changes in values of c_p, a, k_1, r, d_1 and d_2, highly sensitive to changes in the values of h, c_s, p_{c1}, p_{c2} and G.

These results indicate that the managers should consider working with small values of the parameters in order to minimize the total cost.

5.6 Conclusion

This study presents an imperfect EPQ model for a multi-state system with deteriorating items and alternating production rates, while allowing shortages leading to partial backlogging and lost sale. In this model, we assumed that the equipment's deterioration affects the quantity of the outputs, that the system operates in a degraded state. In such systems, the productivity is assumed to be dependent on the equipment's speed. It is also considered that the system produces both good and poor quality items. After screening, the imperfect items are disposed as a batch after the production process is completed, whilst the perfect quality items are used to meet customer demand. A portion of stock-out demand is allowed in the model formulation. The demand for the item is considered constant, and the deterioration rate follows an exponential function. It is assumed that the deterioration starts right from when finished inventory begins to accumulate after production. Shortages are allowed and they are partially backlogged and partially lost. A numerical example was presented to illustrate the solution procedure and a sensitivity analysis of various parameters was conducted to understand the relative implications of changes in input parameters on the model's important variables.

Chapter 6

Conclusion

6.1 Summary

The existing manufacturing system models in which only the binary approaches with a constant production rate and perfect quality outputs during the production processes present certain shortcomings such as flexibility, availability and reliability. This dissertation proposed a model that considers flexible production systems that continue to operate once non-essential equipment breaks down to deal with these shortcomings. Extensions have been developed from this type of system to cover more practical situations. This dissertation has been developed in five (5) chapters. The problem statement of this research was described in Chapter one. Recent scientific journals relevant to the manufacturing system were also described and critiqued. The motivations and objectives of this research were also discussed.

Chapter two and three presented a general overview of the technical background. some essential notions concerning inventory management as well as the classical EOQ model were discussed. A literature review on some primary research for deteriorating inventory replenishment were also presented. Classification of works on deteriorating inventory is also provided. A summary of some of the reviewed studies was provided at the end of Chapter three.

In Chapter four, a review of deteriorating manufacturing systems is described. In this chapter, the modelling philosophy of the performance of multi-state systems (MSS) whose operating characteristics are subject to different conditions were addressed and treated. This characterization process was undertaken assuming that the studied systems integrate reconfiguration mechanisms (hardware and/or software). Thus, based on the characteristics of the systems under study, strategies for optimizing their performance were identified. The study also focused on methodologies for evaluating and optimizing these systems' performance measures (inventories, cycle time, shortages and total cost). Finally, a conclusion and several research gaps were identified. We presented the objective and the methodology to adopt in the following chapter.

Chapter five addressed the problem of optimizing a manufacturing system operating in degraded mode. The system consisted of a single machine producing a single type of product subject to deterioration, with a variable production rate and shortage. The deterministic model obtained in this chapter demonstrated that the optimal policies are characterized by three decision variables: the stock level in the first cycle, the stock level in the second cycle, and the cycle time. The criteria for performance was the total cost of the manufacturing system. The optimization model studied was to minimize this total cost. Thus, the overall approach consisted of implementing strategies that favour the manufacturing process in degraded mode. A numerical example was provided to illustrate the usefulness of the proposed model, and sensitivity analyses conducted confirmed the model's reliability. It can be seen that, from the numerical work and sensitivity analysis it is possible to integrate the functioning of machines in degraded mode in a production system to meet customers' demands. this dissertation is a contribution to the literature on the production control of flexible manufacturing systems, where, at the lower level, the optimal policies are determined for a manufacturing system of deteriorating inventories and machines that are subject to failures.

6.2 Possible practical applications of findings

Applications of the results of the present research could be, under certain assumptions and extensions of the models, applied in:

- Assembly lines for automotive parts such as car seats: proposing optimal production policies in degraded mode subject to non-essential equipment failures
- Manufacturing plants for mechanical parts with an assembly line of equipment configured in series and buffer stock: joint optimization of a production system for mechanical parts, availability and reliability of cutting tools.
- Hydrometallurgical plants for production of metals such as copper, cobalt, zinc: optimization of the production with size reduction of ores, availability and reliability of size reduction machines (mills, crushers) and classification machines (cyclones).

- Other production systems configured in line with the models proposed in this dissertation.

6.3 Possible areas for future research

For years, the replenishment of deteriorating inventory has been investigated from different perspectives, as described in the previous sections; however, the scope remains vast, and several relaxations can be made out of this dissertation. Some research gaps in this field are as follows:

1. This work addressed the possibility of carrying out quality control of products after their production, which would prevent all non-conforming products from going to the customer, causing damage on several levels to the company. However, the cost of quality control was not considered, nor was the time required for inspection.

2. The model considers a system consisting of a single machine operating in degraded mode with reduced of the performance once the machine breakdowns. However, the model did not take into account preventive maintenance in order to avoid breakdowns. And also the model can be generalized into both multi-production and rework periods model and multiproduct and multi-machine system.

3. To cope with a real industrial environment case, this model can be extended to the case of manufacturing systems involving multiple products and multiple machines in which the percentage of rejects depends on the deterioration of the machine can be beneficial for several organizations.

4. Further investigation can be done in the case of manufacturing systems with multiple failure rates, i.e. with more than two failure rates, where the speed of production significantly influences the tool wear of the machines.

5. This model can be extended by assuming more reasonable assumptions such as considering the demand rate, production rate and the deterioration rate as fuzzy random variables also increased as carbon emission constraint.

6. Most of the inventory articles are developed with constant deterioration, but deterioration also increases with time as stress of units in a heaped stock causes damage on others. To the best of the authors' knowledge, a set of few articles have been published incorporating time varying deterioration. Moreover, the above-mentioned inventory models are developed with either crisp or fuzzy

inventory costs. Till now, none has considered time-dependent deterioration and fuzzy/rough/fuzzy-rough inventory costs in an inventory model under a two-level partial credit period in a multi-state system.

7. Considering a model with demand rate as a time- and price-dependent function or a time- and stock-dependent function and time varying in parameters such the deterioration rate, lead-time, has not attracted enough attention in the existing literature.

8. In many studies and the model developed in this dissertation shortages are partially backordered, and the rate of partial backorders is generally modelled by a constant value. Developing models to consider polynomial fractions could therefore be a promising area for further research.

9. Multi- deteriorating products models with nonlinear holding cost, stochastic lead times could be studied for further research.

10. In this study, the optimization problem is done through minimizing total cost. Applying other approaches for optimization, such as goal programming, stochastic programming or risk measures have been less studied and is strongly recommended for further research.

11. This model can also be extended to cases considering time-dependent deterioration and fuzzy/rough/fuzzy-rough inventory costs in an inventory model under a two-level partial credit period.

12. Lastly, a stochastic deteriorating production system consisting of a single machine or multiple parallel machines with the productivity-dependent failure rates of the main machine could also be investigated. A stochastic model could be proposed to evaluate and analyse the availability of a reconfigurable system subject to random failures with production dependant on both the demand and the age of the machines.

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