Involutions on sheaves of endomorphisms of modules over ringed spaces

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Declaration

I, Richard Ng'ambi the undersigned, declare that the thesis, which I hereby submit for the degree of Doctor of Philosophy at the University of Pretoria is my own work and has not previously been submitted by me for any degree at this or any other tertiary institution.

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Dedication

This work is dedicated to my son Raymond, my dear wife Olivia, my parents Weston Siwale Ng'ambi and Medrine Nakamba, my sister Oradia Ng'ambi.

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Contents

Abstract

The study of Azumaya algebras over schemes has had a comparatively formidable reputation in algebraic geometry over the past decades. In this thesis, we conduct a study on Azumaya algebras with involution and on involutions on sheaves of endomorphisms of $\mathscr{O}_X$-modules over a scheme $X = \mathrm{Spec}(R)$. To this end, given a commutative ring R, our focus derives from the classical study of properties and characterizations of Azumaya R-algebras with R-progenerator modules at the core of this investigation. As a consequence, we note that, given an Azumaya R-algebra A, the Azumaya R-algebra Brauer equivalent to it is of the form $\mathrm{End}_A(P)^{\mathrm{o}}$ where P is both an A-module and an R-progenerator or P is a free left A-Azumaya algebra of finite rank such that $\mathrm{End}_A(P)$ is a simple left R-module. Then, we introduce and show the existence of the concept of an Azumaya quadratic pair (A, σ, f) over a commutative ring or PID where A is an Azumaya R-algebra over a commutative ring R, endowed with an involution σ of the first kind, and where $f : \mathrm{Sym}(A, \sigma) \to R$ is a linear map of R-modules, subject to the following condition;

$$f(x+\sigma(x)) = \mathrm{Trd}_A(x),$$

for all $x \in A$, with $\mathrm{Trd}_A(x)$, the reduced trace of x and $\mathrm{Sym}(A, \sigma)$, the set of all symmetric elements on A relative to the involution σ. Finally, the main contribution of this thesis constitutes obtaining a generalization of the results of Knus-Parimala-Srinivas on trivial Azumaya algebras with involution in the context of $\mathscr{O}_X$-algebras over a scheme X via a Morita equivalence on their respective category of sheaves. To begin, we show that for a coherent $\mathscr{O}_X$-algebra $\mathscr{F}$ whose affine restriction to an open covering of $\mathscr{U} = (U_i)_{i\in I} \subset X$ is associated with some faithful finitely generated projective R_i-algebra A_i, then $\mathscr{F}$ admits a unique standard involution $\widetilde{\sigma}$, which commutes with all automorphisms and anti-automorphisms of $\mathscr{F}$ if σ_i is an anti-automorphism of A_i such that $x\sigma_i(x) \in A_i$ for all $x \in A_i$. Subsequently, given a locally finitely presented $\mathscr{O}_X$-module $\mathscr{E}$ on an affine scheme X, and an involution of the first kind σ on the sheaf of endomorphisms $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$, there exist an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and isomorphisms $\varphi : \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L} \xrightarrow{\sim} \mathscr{E}^*$ and $\Phi : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}^*)$ such that, locally, $\sigma \otimes \mathrm{id} = \Phi \circ m$, where m is the natural isomorphism $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{L}) \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$ on any open U in X. Under the conditions that $(X, \mathscr{O}_X)$ is a locally ringed space, $\mathscr{E}$ a locally finitely presented $\mathscr{O}_X$-module, and σ an involution of the first kind on $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$, for any $x \in X$, there is $u \in \mathscr{L}_x$ such that $\sigma_x(f) = u^{-1} \circ f^* \circ u$, for any $f \in \mathrm{End}_{\mathscr{O}_{X,x}}(\mathscr{E}_x)$. Moreover, given a local gauge V of $\mathscr{L}$ at x, there is a unit $\varepsilon \in \mathscr{O}_X(V)$ such that $\varepsilon_x u(p)(q) = u(p)(q)$, for all $p, q \in \mathscr{E}_x$.

The summary of this thesis is as follows :

In Chapter 1, we review various concepts and results on categories and functors, localizations of rings and of modules as well as concepts, constructions and characterizations on classical Azumaya R-algebras.

In Chapter 2, we start off by looking at modules and their properties over Azumaya algebras. Then, we consider the Brauer group and Brauer equivalence over Azumaya R-algebras. The statements in Lemma 2.1.1.3, Lemma 2.1.1.5 and Corollary 2.1.1.4 give an extension of [KMRT98, Proposition 1.10] in the context of Azumaya R-algebras. Particularly, we observe that $\mathrm{End}_A(M)$ is R-Azumaya from the transitivity of the property of being an R-progenerator or provided $\mathrm{End}_R(M)$ is a simple R-module and $A \otimes \mathrm{End}_A(M) \cong \mathrm{End}_R(M)$. Further, it is shown that any Azumaya R-algebra Brauer equivalent to the Azumaya R-algebra A is of the form $\mathrm{End}_A(M)^{\circ}$. For the remainder of this chapter, we consider hermitian forms and involutions on Azumaya R-algebras as well as introduce Azumaya quadratic pairs as a generalized case of [KMRT98, §5]. An Azumaya quadratic pair on an Azumaya R-algebra A is a triple (A, σ, f) where σ is the adjoint involution corresponding to a non-singular bilinear form on A and $f : \mathrm{Sym}(A, \sigma) \longrightarrow R$ is a linear map of R-modules. Lastly, in Theorem 2.3.0.8, it is indicated that given a non-singular quadratic R-module (M, q) of finite rank $2n$, there is a unique linear map $f_q \equiv f$ such that (σ_q, f_q) is a quadratic pair on $End_R(M)$ where

$$\varphi_q : M \otimes_R M \longrightarrow End_R(M)$$

is an isomorphism and σ_q is the adjoint involution corresponding to a non-singular bilinear form b_q of q uniquely determined up to a unit in R.

In Chapter 3, attention is drawn to sheaves of modules and of algebras over schemes. Initially, we extend the context and generality of involutions on Azumaya algebras to involutions on sheaves of Azumaya algebras over schemes. Notably, we obtain a generalization of the extended result in [KMRT98, Theorem, Chapter 1, p.1] to classical Azumaya R-algebras in the contexts of Azumaya $\mathscr{O}_X$-algebras over a scheme X. Theorem 2.2.1.6 and the proposition in [Bos13, Proposition 2, p.258] act as a precursor to obtaining the bijective correspondence between adjoint anti-automorphisms σ on $\mathrm{End}_R(A)$ and adjoint anti-automorphisms $\widetilde{\sigma}$ on associated sheaves of the form $\widetilde{\mathrm{End}_R(A)}$. Additionally, we remark that, for a ring R, the scheme $X = \mathrm{Spec}(R)$ and A an R-algebra, $\widetilde{A}$ has a standard involution if and only if the involution on A is standard. Particularly, for an R-algebra that is faithful finitely generated and projective as an R-module, a standard involution on the R-algebra A induces a unique standard involution on $\widetilde{A}$ that will commute with all anti-automorphisms and automorphisms of the $\mathscr{O}_X$-algebra $\widetilde{A}$. Besides, for a locally projective quasi-coherent $\mathscr{O}_X$-module $\mathscr{E}$ of constant rank 2 on a scheme $(X, \mathscr{O}_X)$, that is, the $\mathscr{O}_X$-module $\mathscr{E}$ is associated with a projective R-module of constant rank

2, $\mathscr{E}$ turns out to be a commutative $\mathscr{O}_X$-algebra, endowed with a unique standard involution.

In Chapter 4, we put into context the results of Knus-Parimala-Srinivas in [KPS90] on classical trivial Azumaya algebras over a commutative local ring R in the framework of Azumaya algebras over a scheme $(X, \mathscr{O}_X)$. Indeed, given a locally finitely presented $\mathscr{O}_X$-module $\mathscr{E}$ on an affine scheme X, and σ an involution of the first kind on $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$, there exist an invertible $\mathscr{O}_X$-module $\mathscr{L}$, a sheaf isomorphism φ of $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L}$ onto $\mathscr{E}^*$, and an isomorphism $\Phi : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}^*)$ such that, on some appropriate open U in X, $\sigma \otimes \mathrm{id} = \Phi \circ m$, where m is the natural isomorphism $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{L}) \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$ on U, and for any open V in U, $\Phi_{VV}(s) = \varphi_V^{-1} s^* \varphi_V$, for any section $s \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(V)$. An immediate consequence is that, given a vector sheaf $\mathscr{E}$ of finite rank n on a scheme X, σ an involution of the first kind $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module sheaf such that $\mathscr{E} \otimes \mathscr{L} \simeq \mathscr{L}^*$ is an isomorphism φ with $\sigma(s) \otimes 1 = \varphi^{-1} s^* \varphi$, for any $s \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}(U))$, where U is any open subset of X such that $\mathscr{L}|_U \simeq \mathscr{O}_X|_U$ and $\mathscr{E}|_U \simeq \mathscr{O}_X^n \simeq (\mathscr{O}_X|U)^n$, then identifying $\mathscr{E}|_U$ with $(\mathscr{E}|_U)^* \simeq \mathscr{E}^*|_U$ with the help of some section $u \in \mathscr{L}(U)$, where $\sigma(f) = \sigma(s) \otimes 1 = u^{-1} \circ f^* \circ u$, for any $f \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$, and identifying $\mathscr{E} \otimes \mathscr{E}^*$ with $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}), \sigma(r \otimes s) = \varepsilon u^{-1}(s) \otimes u(r)$, for $\varepsilon \in \mathscr{O}_X^\bullet(U)$, $r \in \mathscr{E}(U)$ and $s \in \mathscr{E}^*(U)$.

Chapter 1

Preliminary concepts and results

This chapter combines classical results to notions in categories and functors, localizations of rings and of modules, progenerator modules, separable and Azumaya algebras. This chapter is meant to provide the necessary background of the theory and concepts to be employed in later chapters in our investigations. The main prerequisite texts to this chapter are the referenced ones.

1.1 Categories and functors

Let R be a ring (not necessarily commutative) and suppose M and N are R-modules. We shall denote the set of all R-module homomorphisms from M to N by $\mathrm{Hom}_R(M,N)$ and the categories of all left and right R-modules together with R-module homomorphisms respectively by ${}_R\mathfrak{M}$ and $\mathfrak{M}_R$. We shall only consider categories $\mathfrak{K}$ whose objects and classes of morphisms are sets.

Definition 1.1.0.1. *A* covariant *functor from a category $\mathfrak{C}$ to a category $\mathfrak{D}$ is a correspondence $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ which is a function on objects $M \mapsto \mathfrak{F}(M)$ and for any pair of objects $M, N \in \mathfrak{C}$, each f in $\mathrm{Hom}_{\mathfrak{C}}(M,N)$ is mapped to a morphism $\mathfrak{F}(f)$ in $\mathrm{Hom}_{\mathfrak{D}}\left(\mathfrak{F}(M),\mathfrak{F}(N)\right)$ such that the following are satisfied*

a. If $1_{\mathfrak{C}} : M \to M$ is the identity map in $\mathfrak{C}$, then $\mathfrak{F}(1_{\mathfrak{C}}) : \mathfrak{F}(M) \to \mathfrak{F}(M)$ is the identity map in $\mathfrak{D}$. That is, $\mathfrak{F}(1_M) = 1_{\mathfrak{F}(M)}$.

b. Given a commutative triangle

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[ur, "f"] \arrow[r, "gf"'] & C \end{tikzcd}$$

in $\mathfrak{C}$ *, the triangle*

$$\begin{tikzcd} & \mathfrak{F}(B) \arrow[d, "\mathfrak{F}(g)"] \\ \mathfrak{F}(A) \arrow[ur, "\mathfrak{F}(f)"] \arrow[r, "\mathfrak{F}(gf)"] & \mathfrak{F}(C) \end{tikzcd}$$

is commutative in $\mathfrak{D}$*. That is,* $\mathfrak{F}(g \circ f) = \mathfrak{F}(g) \circ \mathfrak{F}(f)$ *for any two composable morphisms* f *and* g*.*

Example 1.1.0.2. For a fixed ring map $R \longrightarrow S$, the assignments

$$M \mapsto M \otimes_R S$$

and

$$\left(f : M \longrightarrow N\right) \mapsto \left(f \otimes id_S : M \otimes S \longrightarrow N \otimes S\right)$$

defines a covariant functor

$$\mathfrak{M}_R \longrightarrow_S \mathfrak{M}.$$

In the case of a *contravariant* functor, the arrows are reversed.

Definition 1.1.0.3. *Given a covariant functor* $\mathfrak{F} : \mathfrak{C} \longrightarrow \mathfrak{D}$*. The functor* $\mathfrak{F}$ *is said to be:*

a. *Faithful, if for all objects* A *and* B *of* $\mathfrak{C}$*, the map*

$$\mathrm{Hom}_{\mathfrak{C}}\left(A, B\right) \longrightarrow \mathrm{Hom}_{\mathfrak{D}}\left(\mathfrak{F}(A), \mathfrak{F}(B)\right),$$

$\beta \mapsto \mathfrak{F}(\beta)$ *between Hom-sets is injective.*

b. *Fully faithful, if the map*

$$\mathrm{Hom}_{\mathfrak{C}}\left(A, B\right) \longrightarrow \mathrm{Hom}_{\mathfrak{D}}\left(\mathfrak{F}(A), \mathfrak{F}(B)\right),$$

between the Hom-sets is bijective.

c. *Essentially surjective, if for every object* $D \in \mathfrak{D}$*, there exists an object* $A \in \mathfrak{C}$ *and an isomorphism* $\mathfrak{F}(A) \cong D$*.*

d. *Left exact, if for every short exact sequence*

$$0 \longrightarrow M \xrightarrow{\alpha} M' \xrightarrow{\beta} M'' \longrightarrow 0$$

in $\mathfrak{C}$*, the corresponding sequence*

$$0 \longrightarrow \mathfrak{F}(M) \xrightarrow{\mathfrak{F}(\alpha)} \mathfrak{F}(M') \xrightarrow{\mathfrak{F}(\beta)} \mathfrak{F}(M'')$$

is exact in $\mathfrak{D}$*. A functor that is both left and right exact is called exact.*

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two parallel covariant functors from a category of modules $\mathfrak{C}$ to another category of modules $\mathfrak{D}$. Functors $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *naturally equivalent* if, for every module M in $\mathfrak{C}$, there is an isomorphism φ_M in $\mathrm{Hom}_{\mathfrak{D}}(\mathfrak{A}(M), \mathfrak{B}(M))$ such that, for any N in $\mathfrak{C}$ and any morphism $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(M,N)$, the diagram

$$\begin{tikzcd} \mathfrak{A}(M) \arrow[r, "\mathfrak{A}(\alpha)"] \arrow[d, "\varphi_M"] & \mathfrak{A}(N) \arrow[d, "\varphi_N"] \\ \mathfrak{B}(M) \arrow[r, "\mathfrak{B}(\alpha)"] & \mathfrak{B}(N) \end{tikzcd}$$

is commutative.

The two categories $\mathfrak{C}$ and $\mathfrak{D}$ are equivalent if there is a functor $\mathfrak{A} : \mathfrak{C} \longrightarrow \mathfrak{D}$ and a functor $\mathfrak{B} : \mathfrak{D} \longrightarrow \mathfrak{C}$ such that $\mathfrak{A} \circ \mathfrak{B}$ is naturally equivalent to $1_{\mathfrak{D}}$ and $\mathfrak{B} \circ \mathfrak{A}$ is naturally equivalent to $1_{\mathfrak{C}}$. The functors $\mathfrak{A}$ and $\mathfrak{B}$ are then referred to as inverse equivalences.

Proposition 1.1.0.4. [Bas68, Proposition 1.1, p.4] *Given a covariant functor $\mathfrak{F} : \mathfrak{C} \longrightarrow \mathfrak{D}$. The functor $\mathfrak{F}$ establishes an equivalence of categories between $\mathfrak{C}$ and $\mathfrak{D}$ if and only if it is fully faithful and essentially surjective.*

Example 1.1.0.5. Let R be a commutative ring and R° its opposite ring. Any right R-module M can be made into a left R°-module by action $m * r = rm$. This evidently gives a covariant functor and there is an equivalence of categories $\mathfrak{M}_R$ and ${}_{R^\circ}\mathfrak{M}$.

Definition 1.1.0.6. *Given any two categories $\mathfrak{C}$ and $\mathfrak{D}$ and a functor (Covariant), $\mathfrak{F} : \mathfrak{C} \longrightarrow \mathfrak{D}$. A functor $\mathfrak{G} : \mathfrak{D} \longrightarrow \mathfrak{C}$ is right adjoint to $\mathfrak{F}$ and $\mathfrak{F}$ left adjoint to $\mathfrak{G}$ if for any objects $X \in \mathfrak{C}$ and $Y \in \mathfrak{D}$, there is a bijective correspondence*

$$Hom_{\mathfrak{C}}(X, \mathfrak{G}(Y)) \cong Hom_{\mathfrak{G}}(\mathfrak{F}(X), Y),$$

functorial in both X and Y. If $\mathfrak{F}$ is an equavalence of categories, then, any functor $\mathfrak{G} : \mathfrak{D} \longrightarrow \mathfrak{C}$ such that $\mathfrak{G} \circ \mathfrak{F} \cong 1_{\mathfrak{C}}$ and $\mathfrak{F} \circ \mathfrak{G} \cong 1_{\mathfrak{D}}$ is right adjoint and left adjoint to $\mathfrak{F}$. [GW10, (A.2), p.542].

Definition 1.1.0.7. *An object Z in a category $\mathfrak{C}$ is called final if the set* $\mathrm{Hom}_{\mathfrak{C}}(X,Z)$ *has exactly one element for all objects X in $\mathfrak{C}$. On the other hand, the object Z is initial in $\mathfrak{C}$ when the set* $\mathrm{Hom}_{\mathfrak{C}}(Z,X)$ *has exactly one element for all objects X.*

1.2 Localizations of rings and of modules

Localization process is a technique that involves embedding a given ring or module into another ring or module so its properties can be easily studied. Ideally, this procedure reduces the problems on rings and modules to that involving local rings. The familiar example is the embedding of the integral domain $\mathbb{Z}$ into its field of fractions $\mathbb{Q}$, (see [Bos13, pp.18-19]). Moreover, if an ideal contains a unit, then it is the whole ring; so, if a commutative ring is not local, the process of localization enlarges it to a local ring by adjoining inverses of some of its elements thereby reducing the number of maximal ideals. In essence, for a commutative ring R, the object of a localization is to find a larger ring B in which the elements of a given multiplicatively closed subset Ω of R become invertible. For instance, in this way, denominators are introduced to a ring R forming a new ring B that will consist of equivalence classes of fractions $\left[\frac{r}{w}\right]$ where $r \in R$, $w \in \Omega$. This section adopts texts largely from [Rot09, Chapter 4, pp.188-203] and [Bas68, pp.104-110].

1.2.1 Localization of rings

Recall that a commutative ring R is said to be a local ring if it has a unique maximal ideal $\mathfrak{m}$ and the quotient $R/\mathfrak{m}$ is a field called the residue field of R.

Definition 1.2.1.1. *Let R be a commutative ring. A subset $\Omega \subseteq R$ is multiplicatively closed in R if $1_R \in \Omega$ and Ω is closed under finite products, that is, $w \cdot w_1 \in \Omega$ for all $w_1, w \in \Omega$.*

Examples of multiplicatively closed sets include:

a. The set $\Omega = R^{\times}$ of units in an integral domain R is multiplicative.

b. For any prime ideal $\mathfrak{p}$ in R, the set theoretic complement $\Omega = R - \mathfrak{p}$ gives a multiplicatively closed set. Indeed, since $0 \in \mathfrak{p}$, then $0 \notin R - \mathfrak{p}$. For $a, b \in R - \mathfrak{p}$, if $ab \in \mathfrak{p}$, then $\mathfrak{p}$ isn't prime. So $ab \in R - \mathfrak{p}$.

c. If $w \in R$ is not nilpotent, then the set $\Omega = \{w^n \,|\, n \geq 0\}$ of non-negative powers of w in R is multiplicative.

Definition 1.2.1.2. *Let R be a commutative ring. An R-algebra A is a ring together with a homomorphism of rings $\theta : R \longrightarrow Z(A)$ defined by $\theta(r) = r \cdot 1_A$ mapping R into the centre of A.*

The structure homomorphism θ endows A with an R-module structure via the action $r \cdot a = \theta(r)a$. As such, an R-algebra is a ring with an induced module structure by θ. And an R algebra A is said to be central over R if it has centre $Z(A) = R$ and it is faithful as a $Z(A)$-module.

Construction of a localization; Our approach to localization follows the exposition of M. Artin in [Art91, Chapter 10, Section 6] and J.J Rotman in [Rot09, Chapter 4, §4.7, pp.188-203]. A localization $\Omega^{-1}R$ over a multiplicative set Ω is constructed up to isomorphism as a unique solution to the universal mapping problem of embedding a given commutative ring R as a subring into its field or ring of fractions outlined in Definition 1.2.1.3; the field of fractions being the set of equivalence classes of fractions. To start, the existence of a localization is developed together with some basic properties with which we show that the equivalence relations between fractions in the field of fractions occur naturally.

Definition 1.2.1.3. *Let R be a commutative ring and $\Omega \subseteq R$ a multiplicatively closed subset . Consider ordered pairs (A, φ), where A is commutative R-algebra and $\varphi : R \longrightarrow A$ a linear map such that $\varphi(w)$ is a unit in A for all $w \in \Omega$. An ordered pair $(\Omega^{-1}R,\, f_\Omega)$ where $f_\Omega : R \longrightarrow \Omega^{-1}R$ is a ring homomorphism is called a localization or ring of fractions of R with respect to Ω if it is a solution to the universal mapping problem below*

$$\begin{tikzcd} R \arrow[r, "f_\Omega"] \arrow[dr, "\varphi"'] & \Omega^{-1}R \arrow[d, dashed, "\tilde{\varphi}"] \\ & A; \end{tikzcd}$$

that is, the localization pair $(\Omega^{-1}R,\, f_\Omega)$ is initial among all pairs $(A,\, \varphi)$, such that there is a unique R-algebra map $\tilde{\varphi} : \Omega^{-1}R \longrightarrow A$ satisfying $\varphi = \tilde{\varphi} \circ f_\Omega$.

Remark 1.2.1.4. i. By [Rot09, Theorem 4.68, p.190], the localization

$$\Omega^{-1}R = \left\{ \frac{a}{w} \,\middle|\, a \in R,\; w \in \Omega \right\}$$

exists and is unique up to a unique isomorphism $\tilde{\varphi}$. Moreover, every element $u \in \Omega^{-1}R$ has a unique fractional factorisation

$$u = f_\Omega(a) f_\Omega(w)^{-1}.$$

Particularly, $\dfrac{a}{1} = f_\Omega(a)$ and $\dfrac{a}{w} = f_\Omega(a) f_\Omega(w)^{-1}$ for $a \in R$, $w \in \Omega$. Indeed, for every $a \in R$, the uniqueness assertion in Definition 1.2.1.3 ensures that $\tilde{\varphi}$ is uniquely defined by φ, that is,

$$\varphi(a) = \tilde{\varphi}\left(\frac{a}{1}\right) = \tilde{\varphi}\left(\frac{a}{w} \cdot \frac{w}{1}\right) = \tilde{\varphi}\left(\frac{a}{w}\right) \varphi(w),$$

thereby giving a well defined mapping

$$\tilde{\varphi}\left(\frac{a}{w}\right) = \varphi(a)\varphi(w)^{-1}$$

as $\varphi(\Omega) \subseteq (A)^{\times}$.

ii. Given a multiplicatively closed set $\Omega \subseteq R$, there is equality

$$\frac{a}{w} = \frac{a'}{w'}$$

in $\Omega^{-1}R$ if and only if there is an element $w'' \in \Omega$ such that

$$w''(aw' - a'w) = 0.$$

iii. For the localization map f_Ω, the kernel is given by

$$ker(f_\Omega) = \left\{a \in R \;\middle|\; \frac{a}{w} = \frac{0}{w'}\right\} = \left\{a \in R \;\middle|\; aw' = 0, \, for\ some\ w' \in \Omega\right\}.$$

Indeed, given an element $a \in Ker(f_\Omega)$, then

$$f_\Omega(a) = f_\Omega(a) f_\Omega(w) f_\Omega(w)^{-1} = 0$$

as $f_\Omega(w)$ is a unit. So, $f_\Omega(a) f_\Omega(w) = 0$ in the localization $\Omega^{-1}R$ if $aw = 0$. In particular, the map f_Ω is injective only when $w \in \Omega$ is a non zero divisor of R. Hence, $\dfrac{a}{w} \in \ker(\tilde{\varphi})$, only for $\tilde{\varphi}(a)\varphi(w)^{-1} = 0$. Finally, $\tilde{\varphi}(a) = 0$ implies that

$$\ker \tilde{\varphi} = \Omega^{-1} (\ker \varphi).$$

See [Bas68, §4, p.104].

Let us recall that for a commutative ring R, the collection of all proper prime ideals of R is the spectrum of R denoted $\mathrm{Spec}(R)$.

Example 1.2.1.5.

a. Given a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$. Taking $\Omega = R - \mathfrak{p}$ as a multiplicative subset of a commutative ring with unity R, we denote the localization at a prime $\mathfrak{p}$ by $\Omega^{-1}R = R_\mathfrak{p}$. With the assumption that $\Omega = R - \mathfrak{p}$, we invert by elements $f \in R$ not in $\mathfrak{p}$.
Suppose R is an integral domain, by definition, $ab = 0$ if and only if $a = 0$ or $b = 0$ for any elements $a, b \in R$. Since the zero ideal satisfies the aforestated property and it is not equal to R, it is therefore a prime ideal and we have

$$R_{\langle 0 \rangle} = \Omega^{-1}R = K(R)$$

where $K(R)$ is the quotient field of R.

b. If $\Omega = \{1\}$ is a multiplicatively closed set, then $\Omega^{-1}R \cong R$.

Remark 1.2.1.6. Given $\mathfrak{p} \in \mathrm{Spec}(R)$, the ring $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ and residue field denoted $\kappa_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ which is also the quotient field of the integral domain $R/\mathfrak{p}$ as a localization at the image of Ω. If R is an integral domain with quotient field $K(R)$, then all localizations of R can be viewed as subrings of $K(R)$. In this regard, taking $X = Spm(R)$, the set of all maximal ideals of R, we can recover the ring R from the intersection of localizations at primes $\mathfrak{m}$ ranging over maximal ideals of R; that is,

$$R = \bigcap R_{\mathfrak{m}}.$$

Indeed, the inclusion $R \subseteq \bigcap R_{\mathfrak{m}}$ is evident from the canonical injective map $R \longrightarrow R_{\mathfrak{m}}$ in Definition 1.2.1.3. The converse is established in [Mat86, (1.G), Lemma 2, p.8] by way of considering an ideal $J = \{r \in R \,|\, rx \in R\}$ consisting of all possible denominators of $x \in K(R)$ when written as a fraction of elements of R together with 0. Since $x \in R_{\mathfrak{p}}$ at some prime $\mathfrak{p} \in \mathrm{Spec}(R)$ implies it is a unit, so $J \not\subset \mathfrak{p}$. Hence, if $x \in R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$, then the ideal J contains units. That is, $1 \in J$ and $R = J$.

Remark 1.2.1.7.

i. [Bos13, Remark 4., p.20] $\Omega^{-1}R \neq \{0\}$ at Ω if and only if $0 \notin \Omega$. That is, Ω doesn't contain nilpotent elements. By definition, $f_{\Omega}(a) = 0$ in $\Omega^{-1}R$ only when $\frac{a}{1} = \frac{0}{1}$. So, for some $w \in \Omega$, $aw = 0$ and $a \in Ker(f_{\Omega})$.
Conversely, the localization $\Omega^{-1}R = 0$ if the equality $1_{\Omega^{-1}R} = 0$ is satisfied in $\Omega^{-1}R$, i.e. $\frac{1}{1} = \frac{0}{1}$. This occurs only if there exists some element $z \in \Omega$ so that $z \cdot 1 = 0$, and $0 \in \Omega$.

ii. For a multiplicative set $\Omega \subseteq R$, any ideal J of R can be extended to an ideal $\Omega^{-1}I$ of $\Omega^{-1}R$ generated by $f_{\Omega}(I)$ in the localization for some ideal I of R, that is,

$$J = f_{\Omega}(I)[\Omega^{-1}R] = \left\{ \sum_{i}^{n} a_i f_{\Omega}(w_i) \,\middle|\, a_i \in \Omega^{-1}R,\, w_i \in I, n \in \mathbb{N} \right\}$$

for some ideal $I \subset R$. However, if $\Omega \cap I \neq \emptyset$, then $\Omega^{-1}I = \Omega^{-1}R$ as I contains units. In fact, if R is a domain and $I = J \cap R$, then $J = \Omega^{-1}I$. Therefore, the necessary condition for $J = \Omega^{-1}I$ is that the ideal I of R be the inverse image of $f_{\Omega}(R) \cap J$. So, $I = f_{\Omega}^{-1}\left(f_{\Omega}(R) \cap J\right)$. See [Rot09, Corollary 4.74, p.193].

iii. [Rot09, Theorem 4.75, p.194], If $\mathfrak{p}$ is a prime ideal in R such that $\mathfrak{p} \cap \Omega = \varnothing$, then

$$\mathfrak{p}(\Omega^{-1}R) = \left\{ \frac{a}{b} \in \Omega^{-1}R \,\middle|\, a \in \mathfrak{p}, b \in \Omega \right\}$$

is a prime ideal in $\Omega^{-1}R$ generated by $f_{\Omega}(\mathfrak{p})$.
Indeed, if $\frac{a}{b} \notin \Omega^{-1}R$, then $a \notin \mathfrak{p}$ or $b \notin \Omega$, so $\frac{b}{a} \in \Omega^{-1}R$ is the inverse of $\frac{a}{b}$. Since $\Omega_1 = R - \mathfrak{p}(\Omega^{-1}R)$

will consist of units of R, the ideal $\mathfrak{p}(\Omega^{-1}R)$ contains every non-unit of R. By [AM94, Corollary 1.5, p.4], $\mathfrak{p}(\Omega^{-1}R)$ is maximal and consequently $R/\mathfrak{p}(\Omega^{-1}R)$ is a field. Since every field is an integral domain, so $\mathfrak{p}(\Omega^{-1}R)$ is a prime ideal in $\Omega^{-1}R$.

1.2.2 Localization of modules

The results, generality and flexibility of localization of rings can be transferred to localization of modules. In the current case, we will obtain a module of fractions denoted $\Omega^{-1}M$ with preferably a left $\Omega^{-1}R$-module structure induced by a scalar multiplication

$$\left(\frac{r}{w}\right)\left(\frac{a}{t}\right) = \frac{(ra)}{(wt)}$$

where $\left(\frac{r}{w}\right) \in \Omega^{-1}R$, $\left(\frac{a}{t}\right) \in \Omega^{-1}M$. Moreover, there is a canonical map $f_M : M \to \Omega^{-1}M$ given by

$$f_M(a) = \frac{aw}{w}$$

for some $w \in \Omega$.

Definition 1.2.2.1. *Let M be an R-module and $\Omega \subseteq R$ a multiplicatively closed subset. A localization of M is a pair $(\Omega^{-1}M, f_M)$, where $\Omega^{-1}M$ is an $\Omega^{-1}R$-module and $f_M : M \to \Omega^{-1}M$ is an R-linear map that is a solution to the universal mapping problem*

$$\begin{tikzcd} M \arrow[r, "f_M"] \arrow[dr, "\varphi"'] & \Omega^{-1}M \arrow[d, dashed, "\tilde{\varphi}"] \\ & M_1 \ ; \end{tikzcd}$$

If M_1 is an $\Omega^{-1}R$-module and $\varphi : M \to M_1$ is an R-linear map, then there exists a unique $\Omega^{-1}R$-map $\tilde{\varphi} : \Omega^{-1}M \to M_1$ satisfying $\varphi = \tilde{\varphi} \circ f_M$.

Remark 1.2.2.2. From Definition 1.2.2.1, we have the following properties:

i. Localization $\Omega^{-1}(-)$ is an additive functor between the categories ${}_R\mathfrak{M}$ and ${}_{\Omega^{-1}R}\mathfrak{M}$. Moreover, localization is a special case of extension of scalars. Thus, for any homomorphism $f \in \mathrm{Hom}_{_R\mathfrak{M}}(M, N)$, the diagram

$$\begin{tikzcd} \Omega^{-1}M \arrow[r, "\Omega^{-1}f"] \arrow[d, "\varphi_M"] & \Omega^{-1}N \arrow[d, "\varphi_N"] \\ \Omega^{-1}R \otimes M \arrow[r, "1_{\Omega^{-1}R} \otimes f"] & \Omega^{-1}R \otimes N \end{tikzcd},$$

commutes where φ_M, the component of φ at M is an isomorphism defined by

$$\varphi_M\left(\frac{a}{w}\right)=\left(\frac{1}{w}\right)\otimes a$$

with an inverse map

$$\varphi_M^{-1}\left(\frac{r}{w}\otimes a\right)=\frac{ra}{w}.$$

From the naturality square above,

$$\begin{aligned}\left(\varphi_N\circ\Omega^{-1}f\right)\left(\frac{a}{w}\right)=&\varphi_N\left(\Omega^{-1}f\left(\frac{a}{w}\right)\right)\\ &=\varphi_N\left(\frac{f(a)}{w}\right)\\ &=\left(\frac{1}{w}\otimes f(a)\right),\end{aligned}$$

and

$$\begin{aligned}\left(1_{\Omega^{-1}}\otimes f\right)\left(\varphi_M\left(\frac{a}{w}\right)\right)=&(1_{\Omega^{-1}}\otimes f)\left(\frac{1}{w}\otimes a\right)\\ &=\left(\frac{1}{w}\otimes f(a)\right).\end{aligned}$$

So, any R-linear map $f:M\longrightarrow N$ induces an $\Omega^{-1}R$-homomorphism $\Omega^{-1}R\otimes M\longrightarrow\Omega^{-1}R\otimes N$ thereby giving a natural isomorphism of functors $\Omega^{-1}R\otimes M\longrightarrow\Omega^{-1}M$. (c.f. [Bas68, §4, pp.104-110].)

ii. The functors $\Omega^{-1}(-)$ and $\Omega^{-1}R\otimes -$ are both left adjoint to the restriction of scalars functor from the localized ring $\Omega^{-1}R$ to the base ring R. That is, given an R-module M and an $\Omega^{-1}R$-module N, since $\Omega^{-1}R$ is an R-algebra, there is an $(R,\Omega^{-1}R)$-isomorphism

$$\mathrm{Hom}_{\Omega^{-1}R}(M\otimes_{\Omega^{-1}R}R,N)\cong\mathrm{Hom}_R(M,N).$$

(See [KA13, Theorem 12.9, p.62] and [KA13, Corollary 12.12, p.62].)

iii. For any $M,N\in_R\mathfrak{M}$, localization preserves tensor products i.e.

$$\Omega^{-1}(M\otimes_R N)=\Omega^{-1}M\otimes_{\Omega^{-1}R}\Omega^{-1}N.$$

When M is of finite presentation, we have

$$\Omega^{-1}\mathrm{Hom}_R(M,N)\cong\mathrm{Hom}_{\Omega^{-1}R}\left(\Omega^{-1}M,\Omega^{-1}N\right).$$

See [Mat80, (1.G)].

iv. [Rot09, Corollary 4.81] Localization functor,

$$M \mapsto \Omega^{-1}M \cong \Omega^{-1}R \otimes M$$

is exact since $\Omega^{-1}R$ is a flat R-module.

Taking $\Omega = R - \mathfrak{p}$, we denote by $M_{\mathfrak{p}}$ the localization of the R-module M at a prime ideal $\mathfrak{p}$ of R. The lemma that follows is as a result of Remark 1.2.1.6;

Lemma 1.2.2.3. *Let R be an integral domain with quotient field $K(R)$ and let M be a torsion free R-module. Then*

$$M = \bigcap_{\mathfrak{m}} M_{\mathfrak{m}}$$

where $\mathfrak{m}$ ranges over maximal ideals of R.

Proof. Given M a torsion free R-module, the inclusion $M \subseteq \bigcap_{\mathfrak{m}} M_{\mathfrak{m}}$ is immediate from Definition 1.2.2.1 and the definition provided in [TS21, Chapter 15, Definition.15.22.1] to justify that the map $M \longrightarrow M_{\mathfrak{m}}$ is an injection. Conversely, suppose $x \in \bigcap_{\mathfrak{m}} M_{\mathfrak{m}}$. Then $x \in M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ and $x \notin \mathfrak{m}$ i.e. $x = \dfrac{a}{b}$ for some $a, b \in R$ associated with $\mathfrak{m}$ where $b \notin \mathfrak{m}$. We now consider the fractional ideal $I = \{r \in R |\ rx \in M\}$ consisting of all possible denominators of x (See [AM94, Chapter 9, p.96] for definitions). Clearly, I is not contained in any maximal ideal $\mathfrak{m}$ of R. Otherwise, it would mean the following; $I \subseteq \mathfrak{m}$ if and only if $x \in \mathfrak{m}$. So I must contain units and $I = R$. Thus, $x = 1 \cdot x \in M$ since $1 \in I$. (c.f. [Rei75, Exercise 1, p.43].) □

Proposition 1.2.2.4. *Let A be an R-algebra over a commutative ring R, M and N left A-modules where M is finitely presented over A. Then, the $\Omega^{-1}R$- homomorphism*

$$\alpha : \Omega^{-1}R \otimes_R \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_{\Omega^{-1}A}\left(\Omega^{-1}R \otimes M, \Omega^{-1}R \otimes N\right)$$

is an isomorphism.

Proof. This result is a direct consequence of flatness of localization $\Omega^{-1}R$ and the isomorphism in [Rei75, Theorem 2.38, p.25]. Since M is finitely presented, there is an exact sequence of A-modules $R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$ for some integers $m, n > 0$. As $\mathrm{Hom}_A(-, N)$ is left exact functor, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_A(R^n, N) \longrightarrow \mathrm{Hom}_A(R^m, N)$$

and from flatness of the localization functor,

$$0 \longrightarrow \Omega^{-1}R \otimes \mathrm{Hom}_A(M, N) \longrightarrow \Omega^{-1}R \otimes \mathrm{Hom}_A(R^n, N) \longrightarrow \Omega^{-1}R \otimes \mathrm{Hom}_A(R^m, N).$$

The above sequence by definition induces an exact sequence of $\Omega^{-1}R$-modules ,

$$0 \to \mathrm{Hom}_{\Omega^{-1}A}\left(\Omega^{-1}M, \Omega^{-1}N\right) \to \mathrm{Hom}_{\Omega^{-1}A}\left(\Omega^{-1}R^n, \Omega^{-1}N\right) \to \mathrm{Hom}_{\Omega^{-1}A}\left(\Omega^{-1}R^m, \Omega^{-1}N\right).$$

Distributing the Hom-functor over finite sums, then

$$\begin{aligned}\Omega^{-1}R \otimes \mathrm{Hom}_A\left(R^m, N\right) &\cong \Omega^{-1}R \otimes \oplus^m \mathrm{Hom}_A\left(R, N\right)\\ &\cong \Omega^{-1}R \otimes N^m\\ &\cong \left(\Omega^{-1}R \otimes N\right)^m\\ &\cong \mathrm{Hom}_{\Omega^{-1}R\otimes A}\left([\Omega^{-1}R]^m, \Omega^{-1}R \otimes N\right)\\ &\cong \mathrm{Hom}_{\Omega^{-1}R\otimes A}\left([\Omega^{-1}R]^m, \Omega^{-1}N\right).\end{aligned}$$

As a result, all the sequences above are exact only in the case when we have isomorphisms

$$\mathrm{Hom}_A\left(M, N\right) \cong \Omega^{-1}R \otimes \mathrm{Hom}_A\left(M, N\right) \cong \mathrm{Hom}_{\Omega^{-1}A}\left(\Omega^{-1}M, \Omega^{-1}N\right)$$

and

$$\mathrm{Hom}_A\left(R^m, N\right) \cong \Omega^{-1}R \otimes \mathrm{Hom}_A\left(R^m, N\right) \cong \mathrm{Hom}_{\Omega^{-1}A}\left([\Omega^{-1}R]^m, \Omega^{-1}N\right).$$

Hence, this completes the proof. $\square$

1.3 Azumaya algebras

An Azumaya algebra over a commutative ring R is a generalization of a central simple algebra over some field F. The term Azumaya originates from the work done by Goro Azumaya in his 1951 paper titled *On maximally central algebras*, (see [Azu51]). He showed that an R-algebra A that is free as an R-module is central separable if and only if A admits a generating subset $\{a_1, \ldots, a_n\}$ such that the square matrix $\left[(a_j a_i)\right]$ is non-singular in A. This notion of an Azumaya algebra coincided with a study by Auslander and Goldman on central separable algebras in their paper, "Brauer group of a commutative ring" (see [AG60]). Over the years, such algebras have evolved (see [KO74] and [DI71]) and have taken centre stage in new directions of research such as polynomial identity theory, cohomological algebra and many others. In this section, we reflect on properties and characterizations of Azumaya algebras, whose proofs can be found in the current literature (see for instance [For17] and [DI71]). In the sequel, we maintain the convention that R will always denote a commutative ring with 1 and all modules considered over R shall be projective and finitely generated unless specified otherwise.

1.3.1 Projective modules and rank

In this subsection, we look at the projective module, the most fundamental type of module relevant to our discourse and its rank over a commutative ring.

Proposition 1.3.1.1. *Let R be a ring and P an R-module, then the following conditions are equivalent:*

a. *P is isomorphic as an R-module to a direct summand of a free R-module.*

b. *The sequence $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ is split exact in the category of R-modules.*

c. *For any diagram*

$$\begin{tikzcd} & P \arrow[dl, dotted, "\varphi"'] \arrow[d, "\alpha"] & \\ M \arrow[r, "\xi"] & N \arrow[r] & 0 \end{tikzcd}$$

with the bottom row exact, there exists an R-module homomorphism $\varphi : P \longrightarrow M$ such that $\xi\varphi = \alpha$.

d. *P has a dual basis $\{(x_i, f) \mid i \in I\}$ for a suitable index set I consisting of $x_i \in P$, $f_i \in Hom_R(P, R)$ satisfying*

   i. *for all $x \in P, f_i(x) = 0$ for all but finitely many $i \in I$, and*

   ii. *for all $x \in P$, $x = \sum_{i \in I} f_i(x)x_i$.*

Proof. See [DI71, Lemma 1.3, p.3] for (a) $\Leftrightarrow$ (d). We now show that (c) $\Rightarrow$ (b). Indeed, given the commutative diagram

$$\begin{tikzcd} & & & P \arrow[dl, dotted, "\varphi"'] \arrow[d, equal, "1_P"] & \\ 0 \arrow[r] & M \arrow[r] & N \arrow[r, "\xi"] & P \arrow[r] & 0, \end{tikzcd}$$

the surjective map $\xi : N \longrightarrow P$ and identity map $1_P : P \longrightarrow P$ yield a section $P \longrightarrow N$, that is, a linear map φ such that $\xi\varphi = 1_P$. So, the sequence is split.

For (b)$\Rightarrow$(a), by [For17, Lemma 1.1.4, p.5], every R-module P is a homomorphic image of a free R-module i.e. for a suitable index set I, there is a surjective homomorphism $\rho : R^{(I)} \longrightarrow P$. By (b) and [Bru02, Proposition 2.5, p.45], we can fit $R^{(I)}$ into a split exact sequence as follows

$$0 \longrightarrow Ker(\rho) \longrightarrow R^{(I)} \longrightarrow P \longrightarrow 0$$

such that

$$R^{(I)} \cong \rho(P) \oplus Ker(\rho),$$

where P is identified by the image of ρ.

To complete the proof, we need to solve the equivalence (a)$\Rightarrow$(c). Let the given free module be $R^{(I)} \cong P \oplus P'$. Let $\pi : P \oplus P' \longrightarrow P$ be a projection defined by $\pi(p, p') = p$ and

$$i : P \longrightarrow P \oplus P'$$

its section with $i(p) = (p, 0)$. We need to show the existence of a map φ such that $\xi\varphi = \alpha$ where $\alpha : P \longrightarrow N$. Consider the diagram

$$\begin{tikzcd} & M \arrow[d, two heads, "\xi"] \\ P \oplus P' \arrow[ur, dotted, "\psi"] \arrow[r, "\rho"] & N, \end{tikzcd}$$

where $\rho = \alpha\pi$ and construct a map ψ that will make the diagram commutative. Let $\{p_i \mid i \in I\}$ be a basis of $P \oplus P'$. For each $i \in I$, set $n_i = \alpha\pi(p_i)$. As ξ is surjective, every $n_i \in N$ is such that $n_i = \xi(p_i)$. We then define a map ψ on the basis $\{p_i \mid i \in I\}$ by $\psi(p_i) = m_i$ extended linearly. This construction yields

$$n_i = \xi(\psi(p_i)) = \alpha(\pi(p_i)), (\forall i \in I).$$

Since $\pi \circ i = 1_P$, we can define $\varphi = \psi i$. □

Definition 1.3.1.2. *An R-module P is **projective** if it satisfies any of the equivalent conditions of Proposition 1.3.1.1.*

Proposition 1.3.1.3. *Given a local ring R and a finitely generated projective R-module M, then M is a free module over R. In particular, $M \cong R^s$, where*

$$s = \dim_{R/\mathfrak{m}}(M/\mathfrak{m}M) = \dim_{R/\mathfrak{m}}(M \otimes_R R/\mathfrak{m})$$

where $\mathfrak{m}$ is a maximal ideal of R.

Proof. Since M is projective and finitely generated, then there is a finitely generated projective R-module N such that the sequence

$$0 \longrightarrow N \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

of R-modules is exact. Moreover, the R-modules R^n is also finitely generated and $M \oplus N \cong R^n$. By [Mat80, (1.N), p.12], given a maximal ideal $\mathfrak{m}$ of R, we have that $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are canonically finite dimensional vector spaces over $R/\mathfrak{m}$. Then, $[R/\mathfrak{m}]^s \oplus [R/\mathfrak{m}]^t \cong [R/\mathfrak{m}]^n$ where $M/\mathfrak{m}M \cong (R/\mathfrak{m})^s$ and $N/\mathfrak{m}N \cong (R/\mathfrak{m})^t$ for integers s and t. Since $\mathfrak{m}$, a maximal ideal of R is in the Jacobson radical of R, application of [Bos13, Corollary 12, p.37] gives two subsets $\{m_i\}_{i=1}^s$ and $\{n_j\}_{j=1}^t$ of M and N mapping respectively to bases of $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$. The elements m_i and n_j determine the R-linear map

$$[M/\mathfrak{m}M]^s \oplus [N/\mathfrak{m}N]^t \longrightarrow R^n.$$

From [Bru02, Proposition 1.26], identifying R^m with $R^{1\times m}$, the map above is a multiplication by a matrix $T \in R^{n\times n}$ whose reduction $\bar{T} \in (R/\mathfrak{m})^{n\times n}$ is invertible. Moreover, we have an isomorphism $\mathrm{Hom}(R^s \oplus R^t, R^n) \cong R^{(s+t)\times n}$. Thus, $\{m_1,\dots,m_s,n_1,\dots,m_t\}$ is a basis for $M \oplus N$, and it follows that $\{m_i\}_{i=1}^s$ is a free basis for M. (c.f. [Wei13, Lemma 2.2., p.11].) □

Remark 1.3.1.4. i. [For17, Proposition 2.3.2, p.65] For a commutative ring R and a finitely generated R-module P, the free rank $\mathrm{Rank}_{\mathfrak{p}}$ of P at a prime ideal $\mathfrak{p} \subset R$ is

$$\mathrm{Rank}_{\mathfrak{p}} P = \dim_{K_{\mathfrak{p}}} P_{\mathfrak{p}}/(\mathfrak{p}P_{\mathfrak{p}}).$$

Since $P_{\mathfrak{p}}/(\mathfrak{p}P_{\mathfrak{p}}) \cong \Big(R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})\Big)^{\mathrm{Rank}_{\mathfrak{p}}(P)}$, $\mathrm{Rank}_{\mathfrak{p}}(P)$ is the minimal number of generators of $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$.

ii. [For17, Theorem 2.3.5, p.67] For a finitely generated projective R-module P, the rank $\mathrm{Rank}_{\mathfrak{p}}$ is a continuous locally constant function from $\mathrm{Spec}(R)$ to the discrete topological space $\mathbb{N} \subset \mathbb{Z}$. If $\mathrm{Rank}_{\mathfrak{p}}(P) = n$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$, then P is said to be of constant rank.

iii. [For17, Proposition 2.3.4] The rank of a finitely generated projective R-module P is invariant under base change. In fact,

$$\mathrm{Rank}_R(P) = \mathrm{Rank}_S(P \otimes S)$$

for a commutative ring R and a commutative R-algebra S.

1.3.2 Progenerator R-modules

In this section, our aim is to show that over a commutative ring and for projective modules of finite type, the condition of being a progenerator just says that the module is nowhere zero locally . This condition happens to be a local condition. For this sequel, we follow the literature in [For17, Chapter 1].

Definition 1.3.2.1. *The trace of a finitely generated projective module M is the image of the map*

$$\mathfrak{T}_R : M \otimes \mathrm{Hom}_R(M,R) \longrightarrow R,$$

given by $m \otimes f \mapsto f(m)$ *for* $m \in M$ *and* $f \in \mathrm{Hom}_R(M,R)$.
The image of the trace map is a two-sided ideal called the trace ideal denoted $\mathfrak{T}_R(M)$. *This ideal is generated by the homomorphic images of* M *in* R.

For the trace ideal of a finitely generated projective$R$-module, we have the following properties:

Proposition 1.3.2.2. *Let* M *be a finitely generated projective* R*-module, then*

i. $\mathfrak{T}_R(M)$ *is a finitely generated ideal of* R.

ii. $M(\mathfrak{T}_R(M)) = M$, $[\mathfrak{T}_R]^2 = \mathfrak{T}_R$ *and* $annih_R(M) = annih_R[\mathfrak{T}(M)]$.

iii. for an R*-module homomorphism* $R \longrightarrow S$*, we have*

$$\mathfrak{T}_S(M \otimes S) = \mathfrak{T}_R(M) \otimes_R S,$$

for any finitely presented R*-module* M *and any commutative flat* R*-algebra* S*. In particular, taking the trace ideal commutes with localization and completion. Moreover,*

$$\mathfrak{T}_R(M \oplus N) = \mathfrak{T}_R(M) + \mathfrak{T}_R(N).$$

Proof. i. Since M is finitely generated and projective, then M has a finite dual basis $\{x_i, \vartheta | 1 \leq i \leq n\}$. For $f \in M^*$, we have $f = \sum_i a_i f_i$ for some $a_i \in R$. As $a = \sum f_i(a) a_i$ for any $a \in M$,

$$f(a) = f\left(\sum_i a_i f_i(a)\right) = \sum_i (f(a_i) f_i(a))$$

rendering every element of $\mathfrak{T}_R(M)$ a finite linear combination, (see [Lam99, Proposition 2.40, p.51]).

From [For17, Lemma 1.3.17(2), p.29], we see that the ring R is a right R left R bi-module and induces on the dual $M^* = \mathrm{Hom}_R(M,R)$, a right R-module structure via the action

$$(fr)(p) = f(pr) = f(p))r$$

for any $r \in R$ and $p \in M$ where $fr \in M^*$. As a result,

$$[frs]\,(p_1 + p_2) = [(f(p_1) + f(p_2))\,r]\,s$$

and

$$[f(r_1 + r_2)]\,(p) = (fr_1 + fr_2))(p) = (f(p)r_1 + f(p)r_1.$$

are satisfied. Thus, we have

$$\left[\sum_i f_i(p)\right] r = \sum_i f_i(pr) = \sum_i (f_i r)(p)$$

which shows that $\mathfrak{T}_R(M)r \subset \mathfrak{T}_R(M)$. Equally, by symmetry, for any right R-module M, the dual $M^* = \mathrm{Hom}_R(M,R)$ will be a left R module with the action

$$r(f(p)) = f(rp) = (rf)(p)$$

thereby inducing an inclusion $r\mathfrak{T}_R(M) \subset \mathfrak{T}_R(M)$.

ii. See [Lam99, Proposition 2.40, p.51] and [Lin17, Proposition 2.8]

iii. See [Lin17, Proposition 2.8]

Note that (ii) and (iii) imply that the trace ideal is generated by an idempotent.

□

Definition 1.3.2.3. *A module M over a ring R (not necessarily commutative) is called a progenerator, or R-progenerator, when it is finitely generated and projective, and its trace is R, viz*

$$\mathfrak{T}_R(M) = \left\{ \sum_{i=1}^{n} f_i(m_i) |\ n \geq 1,\ f_i \in Hom_R(M,R),\ m_i \in M \right\} = R.$$

Equivalently, a finitely generated and projective R-module M is a progenerator if every left R-module is a homomorphic image of a direct sum P^I of copies of P for a suitable index set I, (see [For17, Ex.1.1.11]). We deduce by [Lam99, §18B, pp.483-485] that every finitely generated and projective R-module M is a generator in $\mathfrak{M}_R$ if the functor $\mathfrak{F} = \mathrm{Hom}_R(M,-)$ is a faithful functor from $\mathfrak{M}_R$ to the category of Abelian groups. This implies that, $\mathfrak{F} = \mathrm{Hom}_R(M,-)$ takes non-zero morphisms of $\mathfrak{M}_R$ to non-zero morphisms of the category of Abelian groups. So being a progenerator is a categorical property. Thus, under an equivalence of categories $F : \mathfrak{M}_S \longrightarrow \mathfrak{M}_T$, M is an S-progenerator if and only if $F(M)$ is an T-progenerator.

Remark 1.3.2.4. Let R be a PID and M a torsion-free R-progenerator module such that the map

$$M \longrightarrow M^{**}$$

is an isomorphism. Then, there is an R-algebra isomorphism

$$Z(\mathrm{End}_R(M)) \cong \mathrm{End}_R(\mathfrak{T}_R(M)).$$

Since $\mathfrak{T}_R(M) = R$, then $Z(\mathrm{End}_R(M)) \cong R$. (c.f. [Lin17, Remark 3.22].)

Theorem 1.3.2.5. *For any finitely generated projective module P over a* commutative *ring R, the following are equivalent:*

a. P is an R-progenerator;

b. *P is faithful;*

c. *For every maximal ideal $\mathfrak{m}$ of R, the module $P/\mathfrak{m}P$ is nonzero;*

d. *For any connected component of R, the module P is nonzero over that connected component. In other words, if S is a factor of R, then $S \otimes_R P$ is nonzero;*

e. $\mathrm{Rank}(P_{\mathfrak{p}}) \neq 0$, *for every* $\mathfrak{p} \in \mathrm{Spec}(R)$*;*

f. $\mathrm{Rank}(P_{\mathfrak{p}}) \neq 0$, *for every closed point* $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. See [For17, Corollary 1.1.16] for (a) $\Leftrightarrow$ (b). We now show that (b) $\Rightarrow$ (c). Indeed, since $\mathrm{annih}_R(P) = 0$, then for every maximal ideal $\mathfrak{m}$, $\mathfrak{m}P \neq P$. See [For17, Lemma 1.1.13]. For (c) $\Rightarrow$ (b), assuming (c) we derive (b) by one more use of the Nakayama lemma viz. [For17, Lemma 1.1.13]. We thereby note that $\mathfrak{m}P \neq P \Leftrightarrow \mathfrak{m} + \mathrm{annih}_R(M) \neq R$, of which, since $\mathrm{annih}_R(P)$ is a two-sided ideal in R, implies that $\mathrm{annih}_R(P) = 0$. Therefore, P is faithful. As observed in Remark 1.3.1.4, since the rank of P is a locally constant map on $\mathrm{Spec}(R)$, it is therefore constant on any connected component. Thus, $P \neq 0$ on any connected component iff its rank is not zero on it. Thus, (d) $\Leftrightarrow$ (e). Now, since $\mathrm{Rank}(P_{\mathfrak{p}}) = \mathrm{Rank}(P_{\mathfrak{m}})$, for all maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$, (e) $\Leftrightarrow$ (f). Finally, by virtue of the fact that $\mathrm{Rank}(P_{\mathfrak{p}}) = \dim_{K_{\mathfrak{p}}} P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$, where $K_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, (f) $\Leftrightarrow$ (c). □

Further, for any commutative ring R, Theorem 1.3.2.5 characterizes an R-progenerator as a finitely generated R-module which is projective, an R-generator and faithful over R. In other literature, R-progenerators are referred to as faithfully projective R-modules ([Bas68, Proposition 1.2, p.53]).

1.3.3 Separable Algebras

In this section, we review the characterizations of separable algebras over a commutative ring based on [For17, Chapter 4, pp.115-158] and [Row88, Chapter 5, pp.415-433]. We take the classical approach of separability over a commutative ring R by examining the module structure of an associative R-algebra A over the ring $A \otimes A^o$ as in [AG60].

Let R denote a commutative ring and A an R-algebra. We write $A^e = A \otimes A^o$ for the enveloping algebra where A^o denotes the opposite algebra. The algebra A^o is the ring with $1_{A^o} = 1_A$ having the same underlying R-module structure as A with multiplication $*$ defined by $a * b = ba$ where $a, b \in A$. The R-algebra A^e induces a left A^e-module structure on A via the action

$$\left(\sum a_i \otimes b_i\right) \cdot x = \sum a_i x b_i$$

for $a,x \in A,\ b \in A^{o}$.

In addition, any left A^{e}-module M, can be viewed as an $A-A$-bimodule by ensuring that the left and right A-actions agree, that is,

$$ax = (a \otimes 1) \cdot x = (1 \otimes a) \cdot x = xa$$

and

$$x(a \otimes b) = (ax)b = a(xb).$$

These actions above induce an isomorphism between categories ${}_{A^e}\mathfrak{M}$ and ${}_{A}\mathfrak{M}_{A}$, See [Pie82, §10.1, Proposition, p.180].

In addition, by [Row91, Remark 5.3.1, p.417], A is a cyclic A^{e}-module and there is an A^{e}-module epimorphism

$$\psi_1 : A \otimes A^{o} \longrightarrow A$$

defined by $\psi_1(a \otimes b) = ab$. We shall denote the kernel of the map ψ_1 by J_{ψ_1} and set

$$M^{A} = \{m \in M | am = ma,\ for\, all\, a \in A\}$$

for any $A-A$-bimodule M .

Definition 1.3.3.1. *An R-algebra A is considered to be separable over R if it is projective as a left A^{e}-module.*

We can characterize separable R-algebras by the following equivalent conditions;

Proposition 1.3.3.2. *Let A be an R-algebra. Then , the following conditions are equivalent :*

a. A is a separable R-algebra.

b. The functor $(-)^{A}$ is a right exact functor.

c. There is an element

$$e \in (A^{e})^{A} = \{z \in A^{e} | za = az,\ for\, all\, a \in A\}$$

such that ψ_1 is an epimorphism and $\psi_1(e) = 1_A$.

d. The sequence

$$0 \longrightarrow J_{\psi_1} \longrightarrow A^{e} \longrightarrow A \longrightarrow 0$$

is split exact.

Proof. (a) $\Leftrightarrow$ (b). This follows directly from [For17, Proposition 1.3.20] and [For17, Lemma 4.1.4]. Now, we derive the implication (d) $\Leftrightarrow$ (c). The statement of [Pie82, §10.2, Proposition, p.182] implies that the map ψ_1 is split by a linear map $\gamma : A \to A^e$ and $\psi_1[\gamma(1_A)] = 1_A$. Suppose $e = \gamma(1_A)$. Indeed

$$\psi_1(e) = \psi[\gamma(1_A)] = 1_A$$

and

$$\begin{aligned} ze = z\gamma(1_A) =& \gamma(z1_A) \\ &= \gamma(1_A z) \\ &= \gamma(1_A)z \\ &= ez \end{aligned}$$

for any $z \in A$.

(c) $\Rightarrow$ (a), Since ψ_1 is surjective, by applying [DI71, Chapter II, Proposition 1.1], we can fit it into a sequence

$$A^e \to A \to 0.$$

This sequence will be exact if there is an A^e-homomorphism $\gamma : A \to A^e$ that splits ψ_1. We first set $\gamma(z) = (1 \otimes z) \cdot e = (z \otimes 1) \cdot e$ as a map taking every $A-A$-bimodule to an A^e-module. Henceforth,

$$\begin{aligned} \gamma\left((x \otimes y) \cdot z\right) =& \gamma(xzy) \\ &= \gamma\left(xzy \otimes 1\right) \\ &= \gamma(xz \otimes 1)\left((y \otimes 1) \cdot e\right) \\ &= \gamma\left(xz \otimes 1\right)\left((1 \otimes y) \cdot e\right) \\ &= \gamma\left(xz \otimes y\right) \cdot e \\ &= \gamma(x \otimes y)(z \otimes 1) \cdot e \\ &= \gamma(x \otimes y)(1 \otimes z) \cdot e \\ &= (x \otimes y)\gamma(z), \end{aligned}$$

showing that γ is an A^e-linear map. Let $e = \sum_i x_1 \otimes x_2$. For any element $z \in A$,

$$\begin{aligned}
\psi_1 \circ \gamma(z) =& \psi_1 (z \otimes 1 \cdot e)\\
&= \psi_1 \Big((z \otimes 1) (\sum_i (x_1 \otimes x_2) \Big)\\
&= \psi_1 \Big(\sum_i z x_1 \otimes x_2 \Big)\\
&= z\psi_1 \Big(\sum_i x_1 \otimes x_2 \Big)\\
&= z\psi_1(e)\\
&= z \cdot 1_A = z.
\end{aligned}$$

Hence, A is a left A^e-module under the action of γ. See for instance also [Pie82, p.182, §10.2, Lemma]. (a) $\Rightarrow$ (d) follows from the definition in 1.3.1.1 and [DI71, Chapter II, Proposition 1.1]. □

The element e is necessarily an idempotent called the separability idempotent. As seen in Proposition 1.3.3.2, the idempotent e arises as the image of $1 \in A$ under the splitting of ψ_1. Indeed, for $e = \sum a_i \otimes b_i$

$$\begin{aligned}
e^2 &= \sum (a_i \otimes 1)\,(1 \otimes b_i)\, e\\
&= \sum (a_i \otimes 1)\,(b_i \otimes 1)\, e\\
&= \sum (a_i b_1 \otimes 1)\, e = (1 \otimes 1) e = e.
\end{aligned}$$

Lemma 1.3.3.3. [Row91, Proposition 5.3.2] *The kernel J_{ψ_1} of the map $\psi_1 : A \otimes A^\circ \to A$ is generated as a right ideal of A^e by elements of the form $a \otimes 1 - 1 \otimes a$. That is,*

$$J_{\psi_1} = \sum \Big(a \otimes 1 - 1 \otimes a \Big) A^e.$$

Proof. Suppose $x = \sum s_i \otimes t_i \in J_{\psi_i}$. Then we have image $\sum s_i t_i = 0$ and so, every such element has the form

$$\begin{aligned}
x &= \sum s_i \otimes t_i\\
&= \sum (s_i \otimes 1)(1 \otimes t_i)\\
&= \sum (s_i \otimes 1)(1 \otimes t_i) - 1 \otimes \sum s_i t_i\\
&= \sum (s_i \otimes 1 - 1 \otimes s_i)(1 \otimes t_i).
\end{aligned}$$

Conversely, since ψ_1 is an A^e-linear map, we have

$$\begin{aligned}\psi_1(x) =&\psi_1[(\sum(s_i\otimes 1-1\otimes s_i)(1\otimes t_i))]\\ =&[(\sum\psi_1(s_i\otimes 1)-(1\otimes s_i)]\psi_1(1\otimes t_i)\\ =&0.\end{aligned}$$

. □

Example 1.3.3.4. For R a commutative ring and A a separable R-algebra. The R-algebra $A\oplus A$ is separable with separability idempotent $e=e_1\otimes e_1+e_2\otimes e_2$. Indeed, taking $e=(1,0)$ and $e_2=(0,1)$ to be primitive orthogonal idempotents in A. We have $\psi_1(e)=1$ and

$$(x+y\otimes 1-1\otimes y+x)e=[(x\otimes 1-1\otimes x)+(y\otimes 1-1\otimes y)]e=0.$$

(See [For08].*)*

The notion of an R-algebra being separable is a local property. Thus,

Proposition 1.3.3.5. *Given an R-separable algebra A which is finitely generated as an R-module, the following assertions are equivalent,*

i. *$A\otimes R_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-separable for any prime ideal $\mathfrak{p}$ in R.*

ii. *$A\otimes R_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-separable for any maximal ideal $\mathfrak{m}$ in R.*

iii. *$A\otimes R/\mathfrak{m}$ is a separable $R/\mathfrak{m}$-algebra for any maximal ideal $\mathfrak{m}$ in R.*

Proof. The statements follow directly from the proofs of [For17, Theorem 8.1.22] and [For17, Theorem 8.1.24]. □

Remark 1.3.3.6. Any separable R-algebra A is projective and finitely generated as an R-module. [DI71, Theorem 3.8, p.55]

1.3.4 Azumaya algebras

Applying the results of Section 1.3.3 on separable algebras, we can define Azumaya algebras as follows:

Definition 1.3.4.1. *An R-algebra A is an Azumaya algebra if it satisfies any of the following equivalent properties:*

a. A is a central R-algebra and separable over R.

b. A is an R-progenerator and the natural representation

$$\mu_A : A \otimes_R A^{\mathrm{o}} \longrightarrow End_R(A),$$

defined by $\mu_A(a \otimes_R b^{\mathrm{o}})(x) = axb^{\mathrm{o}}$, *where* $a,\ b^{\mathrm{o}} \in A$ *and* $x \in R$, *is an isomorphism.*

c. For any maximal ideal $\mathfrak{m}$ *of* R, *the quotient* $A/\mathfrak{m}A$ *is a central simple* $R/\mathfrak{m}$*-algebra.*

From Definition 1.3.4.1 above, we infer the following basic properties of Azumaya R-algebras:

a. If A and B are Azumaya R-algebras, then $A \otimes B$, their tensor product is also Azumaya over R. Indeed, the tensor product of separable R-algebras is separable and $A \otimes B$ is R-central. See [For17, Proposition 7.1.3].

b. For any R-progenerator P, $A = \mathrm{End}_R(P)$ is an Azumaya R-algebra satisfying condition (b) in Definition 1.3.4.1 by [DI71, Proposition 4.1, p.56]

c. If A is an Azumaya R-algebra and S a faithfully flat étale R-algebra i.e. S is a finitely generated commutative R-algebra which is flat, faithful and separable. Since étale is preserved by a change of base, $A \otimes S$ is an étale S-algebra. See [Mil80, Proposition 3.3, p.22].
Conversely, if S is étale over R and $A \otimes S$ an Azumaya S-algebra, then A is separable over R. By condition (c) of Definition 1.3.4.1, A is a finite R-algebra. On the strengths of Lemma 8.6 and Lemma 8.7 of [FD93], there is a commutative diagram

$$\begin{array}{ccc} \mu_A \otimes 1_S : (A \otimes A^{\mathrm{o}}) \otimes S & \longrightarrow & \mathrm{End}_R(A) \otimes S \\ \downarrow \cong & & \downarrow \cong \\ \mu_{(A\otimes S)} : (A \otimes S) \otimes (A \otimes S)^{\mathrm{o}} & \longrightarrow & \mathrm{End}_S(A \otimes S) \end{array}$$

that satisfies the condition (b) in Definition 1.3.4.1. Hence $A \otimes S$ is an Azumaya S-algebra.

Note in particular that, from [For17, Corollary 4.5.4], every central simple K-algebra is an Azumaya algebra over the given field K.

Remark 1.3.4.2. Let A be an Azumaya R-algebra. For a maximal ideal $\mathfrak{m} \in \mathrm{Spec}(R)$, $A/\mathfrak{m}A$ is a central simple $R/\mathfrak{m}$-algebra. Then

$$\begin{aligned} \mathrm{Rank}(A) &= \dim_{R/\mathfrak{m}}(A/\mathfrak{m}A) \\ &= \dim_{R/\mathfrak{m}}(A \otimes R/\mathfrak{m}). \end{aligned}$$

So the degree of A is locally the element $\sqrt{\dim_{R/\mathfrak{m}}(A \otimes R/\mathfrak{m})}$ resulting from the map $\mathrm{Spec}(R) \rightarrow \mathbb{N}$. Hence, the rank of an Azumaya R-algebra is a square. (c.f. [Sch85, Corollary 8.4.9].)

Example 1.3.4.3. [For17, Theorem 7.1.10] When M is a progenerator module over a commutative ring R, the algebra $End_R(M)$ is an Azumaya algebra over R; in particular, when $M = R^n$, we have that $\mathrm{M}_n(R^o)$ is an Azumaya algebra over R.

Theorem 1.3.4.4. *Let A be an R-algebra. The following properties are equivalent:*

(a). A is an Azumaya R-algebra.

(b). A is an R-central A^e-progenerator.

(c). The functors

$$(-)\otimes_R A : \mathfrak{M}_R \longrightarrow {}_{A^e}\mathfrak{M}$$

and

$$(*)^A : {}_{A^e}\mathfrak{M} \longrightarrow \mathfrak{M}_R$$

are inverse equivalences of the categories $\mathfrak{M}_R$ and $\mathfrak{M}_{A^e}$ with projective modules corresponding to projective modules.

(d). There is a faithfully flat étale R-algebra S such that $A \otimes S \cong \mathrm{End}_R(P)$ for some R-progenerator P.

Proof. The proof of this theorem is a combination of parts from [For17, Theorem 7.1.4], [Knu91, Theorem 5.1.1] and [KO74, Theoreme 5.1].
(a)$\Leftrightarrow$ (b) ; See [For17, Theorem 7.1.4]. (c)$\Rightarrow$(a), follows from [KO74, Theoreme 5.1]. (a)($\Leftrightarrow$ (d): See [Knu91, Theorem 5.1.1]
(a) $\Rightarrow$ (c), follows from [For17, Theorem 7.1.4] and [For17, Exercise 1.5.6]. □

A non-trivial example of an Azumaya R-algebra is given below:

Example 1.3.4.5. Let R be any commutative ring in which 2 is invertible. The ring of Quaternions is a free R-module

$$\Lambda = \left\{ r_1 \cdot 1 + r_2 i + r_3 j + r_4 ij \;\middle|\; r_k \in R, 1 \leq k \leq 4 \right\}$$

having a basis $\{1, i, j, ij\}$ endowed with an R-bilinear multiplication defined by extending relations

$$i^2 = j^2 = (ij)^2 = -1, \; ij = -ji$$

by associativity and distributivity. We maintain the following observations:

a. Suppose $\Lambda_0 = r_1 \cdot 1 + r_2 i + r_3 j + r_4 ij \in Z(\Lambda)$. Then we have,

$$i\Lambda_0 = \Lambda_0 i,$$

and we get $\Lambda_0 = r_1 + r_2 j$. Furthermore,

$$j\Lambda_0 = \Lambda_0 j$$

yields $\Lambda_0 = r_1$ as an element of $R = Z(\Lambda)$. That is, $Z(\Lambda) = R$.

b. There is an isomorphism of R-algebras $\Lambda \cong \Lambda^\circ$ with mapping

$$(r_1 + r_2 i + r_3 j + r_4 ij) \mapsto (r_1 - r_2 i - r_3 j - r_4 ij).$$

c. By Theorem 3 in [Sze78], the R-algebra Λ is shown to be separable with separability idempotent taken as the element

$$e = \frac{1}{4}\Big(1 \otimes 1 - i \otimes i - j \otimes j - ij \otimes ij\Big).$$

Thus, Λ is an Azumaya R-algebra.

Proposition 1.3.4.6. *Every Azumaya R-algebra A is a finite direct product of Azumaya algebras of constant rank.*

Proof. Since A is projective over R as an R- module, by [Row91, Proposition 2.12.22], there are R orthogonal idempotents $\{e_1, \ldots, e_n\}$ such that $A_i = e_i A$ is a projective $R_i = e_i R$ module. Furthermore, by [Wei13, Remark 2.3.1, p.13] , since A is R-projective and finitely generated,

$$A \cong A_1 \times A_2 \times \cdots \times A_n$$

and

$$R \cong R_1 \times R_2 \times \cdots \times R_n.$$

So by [Row91, Proposition 2.12.5], Spec(R) is covered by a finite number of open and closed sets each corresponding to ranks of localizations of A in a bijective correspondence with the set of idempotents of R. By Definition 1.3.4.1, A is central separable and $A \cong \prod_i^n A_i$ with A_i an Azumaya algebra over R_i. □

Another characterization of Azumaya algebras is through the Artin-Procesi theorem (See [Pro72]) which provides a theorem that remarkably characterizes Azumaya algebras without prior reference to its centre. Kaplansky in [Kap48] first proved that primitive polynomial identity rings with coefficients in its centre is an Azumaya algebra over the centre. Further, Artin classified rings with a polynomial identity that are finite modules over their centres to be Azumaya algebras of constant rank n^2 . This identification establishes a bijective correspondence between classes of rings which satisfy a polynomial identity and are finite as R-modules over their centres with Azumaya algebras of constant rank n^2. The generalizations of the Artin-procesi theorem by Braun in [Bra82] and by Dicks in [Dic88], yielded the following intrinsic characterization theorem;

Theorem 1.3.4.7. *Let R be a commutative ring. An algebra R-algebra A is Azumaya if and only if there is an element $e = \sum_i a_i \otimes b_i \in A^e$ such that $\psi_1(\sum_i a_i \otimes b_i) = 1$ and $\psi_{A^e}(\sum_i a_i \otimes b_i) = \sum_i a_i A^e b_i \subseteq R$ where ψ_1 is the map $\psi_1 : A \otimes A^o \longrightarrow A$*

Proof. See [Bra82, Theorem 4.1]. □

1.3.5 Reduced trace and norm of an Azumaya R-algebra

When an Azumaya algebra is viewed as a form of a matrix via [For17, Corollary 3.3.10], its elements can be associated to invariants particular to matrices namely, characteristic polynomial, norm and trace. This section is adopted from the literature in [EW67, §3], [Bou58] and [KO74, §2, p.108].

Remark 1.3.5.1. [Bou58, §12.1, p.132] Given an S-basis $\{m_1, m_2, \ldots, m_n\}$ of $M \otimes S$. Every element $\alpha \in \mathrm{End}_R(M)$, can be regarded as an element of $\mathrm{End}_S(M \otimes S)$ via $(\alpha \otimes 1)$ by setting $\alpha(m_k) = \sum_{i=1}^n m_i s_{ik}$ for some $s_{ik} \in S$. The characteristic polynomial $\mathrm{Char}.poly_M(\alpha)$ of α is defined to be the characteristic polynomial of the matrix of α with respect to any basis of M. This is given by $\mathrm{Char}.poly_M(\alpha) = \det(s_{ik} - X\delta_{ik}) \in R[X]$ for some indeterminate X. Moreover, its trace and norm are given respectively by $Tr_M(\alpha) = \sum_i^n s_{ii}$ and $N_M(\alpha) = \det(s_{ik})$.

Since any R-algebra A can act on itself as a ring of R-module homomorphisms via a right or left multiplication by an element x viz. $R_x : A \longrightarrow A$ where $R_x(a) = ax$. In this way, each element $x \in A$ is mapped to R_x. So, for A finitely generated as an R-module, the characteristic polynomial of an element $x \in A$ is defined to be the characteristic polynomial of the endomorphism R_x.

An Azumaya R-algebra A of rank n^2, $n \geq 1$ is *split* if there exists a commutative faithfully flat ètale R-algebra S such that $A \otimes_R S \cong M_n(S)$. By [For17, Corollary 10.3.9], there is an S-progenerator module N of rank n and an S-algebra isomorphism $\delta_S : A \otimes S \cong \mathrm{End}_S(N)$. Therefore,

$$M_n(S) \cong A \otimes S \cong \mathrm{End}_S(N).$$

Definition 1.3.5.2. *Given a split Azumaya R-algebra A of rank $n^2, n \geq 1$. The reduced characteristic polynomial of an element $a \in A$ is defined to be the characteristic polynomial of its image $\delta(a \otimes 1_S) \in$*

$\mathrm{End}_S(N)$.*i.e. an element of* $R[X]$ *for an indeterminate* X*, independent of the choice of* S*,* N *and* δ_S *given by*

$$\begin{aligned}\mathrm{Char}_{red}.poly_A(\delta(a\otimes 1_s)) &= \det(XI_n - \delta(a\otimes 1_S))\\ &= X^n - Trd_A(a)X^{n-1} + \ldots (-1)^n \mathcal{N}rd_A(a)\end{aligned}$$

where

$$Trd_A(a) = Tr_A(\delta(a\otimes 1_S))$$

is the reduced trace and

$$\mathcal{N}rd_A(a) = \det(\delta(a\otimes 1_S))$$

the reduced norm of an element $a \in A$.

Proposition 1.3.5.3. *The reduced trace and norm satisfy the following properties :*

a. *The reduced norm is a multiplicative semi-group map from* A *to* R*, that is,*

$$\mathcal{N}rd_A(a\cdot b) = \mathcal{N}rd_A(a)\mathcal{N}rd_A(b)$$

and

$$\mathcal{N}rd_A(\kappa a) = \kappa^n \mathcal{N}rd_A(a).$$

b. *The reduced trace is an* R*-linear map from* A *to* R *satisfying*

$$\kappa Trd_A(a+b) = Trd_A(\kappa a + b)$$

and

$$Trd_A(ab) = Trd_A(ba).$$

for $a, b \in A$ *and* $\kappa \in A$.

Proof. The properties above follow immediately from the multiplicative properties of the determinant and trace on matrix algebras. (a) The reduced norm is multiplicative i.e. $\mathcal{N}rd_A(a\cdot b) = \mathcal{N}rd_A(a)\mathcal{N}rd_A(b)$. Likewise,

$$\begin{aligned}\mathcal{N}rd_A(a\cdot b) &= \det(\delta(ab\otimes 1_S)\\ &= \det(\delta[(a\otimes 1_S)(b\otimes 1_S)])\\ &= \det(\delta(a\otimes 1_S)\det(\delta(b\otimes 1_S))\\ &= \mathcal{N}rd_A(a)\mathcal{N}rd_A(b)\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}rd_A(\delta(\lambda\otimes 1_S)) &= \mathcal{N}rd_A(\lambda\delta(1_A\otimes 1_S))\\ &= \det(\lambda I_n)\\ &= (\lambda)^n.\end{aligned}$$

(b) By Definition 1.3.5.2, we have

$$\begin{aligned}Trd_A(ab)) &= Tr_A(ab\otimes 1_S)\\ &= Tr_A[(a\otimes 1_S)(b\otimes 1_S)].\end{aligned}$$

Then, from Proposition 10.3.9 of [For17], considering the matrix representations of the endomorphisms of the elements a and b respectively, [Bou89, Chapter II, §10.11] facilitates the desired result.

□

Remark 1.3.5.4. i. The reduced characteristic polynomial commutes with localizations. Indeed, given an Azumaya R-algebra A and an R- alegbra S. For any maximal ideal $\mathfrak{m}$ in R, we have a splitting $S/\mathfrak{m}S$-algebra isomorphism

$$(A\otimes_R S)_{\mathfrak{m}} \cong (A/\mathfrak{m}A\otimes_{R/\mathfrak{m}} S/\mathfrak{m}S) \cong M_n(S/\mathfrak{m}S).$$

Thus, $A/\mathfrak{m}A$ is Azumaya over the field $R/\mathfrak{m}$ which is a central simple $R/\mathfrak{m}$-algebra. By [BO13, Remark IV.2.8, p.66], we have

$$[\mathrm{Char}.poly_A(a)]_{\mathfrak{m}} = \mathrm{Char}.poly_{A/\mathfrak{m}A}(a_{\mathfrak{m}})$$

where $a_{\mathfrak{m}} \in A/\mathfrak{m}A$ and $\mathrm{Char}.poly_{A/\mathfrak{m}A}(a_{\mathfrak{m}}) \in (R/\mathfrak{m})[X]$. Moreover, we have,

$$Trd_A = Trd_{A/\mathfrak{m}A}$$

and

$$\mathcal{N}rd_A = \mathcal{N}rd_{A/\mathfrak{m}A}.$$

(c.f. [EW67, §3].)

ii. For any Azumaya R-algebra A of rank n^2, by [Bou58, Proposition 8, p.143],

$$\mathrm{Char}.poly_A(a) = [\mathrm{Char}_{red}.poly_A(a)]^n.$$

Consequently, we get the following connections between the trace and reduced trace, and the norm and reduced norm of an element of A :

$$Trd_A(1) = n,\ \ Trd_A(a) = \frac{1}{n}\cdot Tr_A(a)$$

and

$$\mathcal{N}r_A(a) = [\mathcal{N}rd_A(a)]^n.$$

where Tr_A and $\mathcal{N}r_A$ are the regular trace and norm of an element in A.

iii. By definition, the reduced trace is an element of $\mathrm{End}_R(A)$. Since

$$\mu : A \otimes A^{\mathrm{o}} \cong End_R(A)$$

is a bijection by Definition 1.3.4.1, there is a unique element $\sum a_i \otimes b_i \in A \otimes A$ such that

$$\mu\Big(\sum a_i \otimes b_i\Big)(x) = \sum a_i x b_i = \mathrm{Trd}_A(x)$$

for some $x \in A$.

(c.f. [KO74, §4, p.112].)

Chapter 2

Modules over Azumaya algebras, the Brauer group and quadratic pairs

In this chapter, we find results on the Brauer group, hermitian forms and involutions on Azumaya R-algebras. Also considered are Azumaya quadratic pairs. Eventually, we learn what can be said about the category of modules over Azumaya algebras by looking at indecomposable R-modules which are a weaker form of R-simple modules. For Brauer equivalence on Azumaya algebras, we exploit its connection with Morita equivalence of their categories of modules to obtain a general form of an Azumaya algebra Brauer equivalent to a given one. Concluding this chapter, we introduce the notion of Azumaya quadratic pairs and consequently obtain results in the context of Azumaya R-algebras analogous to classical ones on central simple algebras as in ([KMRT98]).

2.1 Modules over Azumaya algebras

Given a base ring that is a field, an Azumaya algebra is a simple ring and any module over an Azumaya algebra defined over a field decomposes as a direct sum of simple ones; since simple modules are indecomposable, there is only one simple module up to isomorphism (see [Bou58, Ã, §7, 2, Proposition 2]). Here, we consider the category of modules in the case when the base ring is more general.

Proposition 2.1.0.1 ([For17, Ex. 7.6.7] or [DeM69, Theorem 1]). *If A is Azumaya over a* local *ring R, then any two indecomposable finitely generated projective A-modules are isomorphic.*

Proof. Since every local ring is a semi-local ring, the proof of [DeM69, Theorem 1] will be a particular case here. Indeed, if M and N are indecomposable finitely generated projective A-modules, then by

[For17, Proposition 2.3.2, p.65], M and N are free of finite rank. That is, for the maximal ideal $\mathfrak{m}$ of R, there are bases $\{x_1 + \mathfrak{m}M | 1 \leq i \leq n\}$ and $\{y_1 + \mathfrak{m}N | 1 \leq i \leq m\}$ respectively of vector spaces $M/\mathfrak{m}M$ and $N/\mathfrak{m}M$ such that $Rank_R(M) = n$ and $Rank_R(N) = m$ for some positive intergers m and n. Suppose that $Rank_R(M) \geq Rank_R(N)$, then there is an A-module epimorphism

$$\varphi : M/\mathfrak{m} \longrightarrow N/\mathfrak{m}M$$

which can be lifted to an isomorphism $\psi : M \longrightarrow N$ since M and N are projective and finitely generated.

□

2.1.1 Brauer equivalence on Azumaya algebras

This section explores Brauer equivalence on Azumaya algebras over commutative rings and some points relating to this equivalence.

Definition 2.1.1.1. *Two Azumaya algebras A and B over R are said to be* Brauer-equivalent *provided there are progenerator (i.e. nowhere zero) R-modules P and Q such that*

$$A \otimes_R End_R(P) \simeq B \otimes_R End_R(Q).$$

As $\mathrm{E}nd_R(P \otimes_R Q) \simeq \mathrm{E}nd_R(P) \otimes_R \mathrm{E}nd_R(Q)$, the Brauer equivalence turns out to be an equivalence relation compatible with the tensor product (over R) of algebras. Additionally, this relation induces a monoid structure on equivalence classes. As such, isomorphic algebras will belong to the same equivalence class. Actually, this monoid is a group, with neutral element $[R]$. Assuming that the group operation is given by setting $[A][B] := [A \otimes B]$, the inverse of $[A]$ is $[A^o]$, since

$$[A \otimes_R A^o] = [\mathrm{E}nd_R(A)] = [R]$$

and A is an R-progenerator. See [For17, Theorem 7.1.4].

Proposition 2.1.1.2. *a. An Azumaya algebra is neutral if and only if there is a progenerator P such that $A \simeq \mathrm{E}nd_R(P)$.*

b. Azumaya algebras A and B are Brauer-equivalent if and only if there is an R-progenerator P such that $A \otimes B^o \simeq \mathrm{E}nd_R(P)$.

Proof. This is Proposition 7.3.4 of [For17]. □

We now draw attention to the fact that Brauer equivalence relates to Morita equivalence. Ultimately, two algebras A and B will be *Morita equivalent* if their corresponding module categories are equivalent, that is, there are additive functors S and T

$$\begin{tikzcd} \mathfrak{M}_A \arrow[r, shift left, "T"] & \mathfrak{M}_B \arrow[l, shift left, "S"] \end{tikzcd}$$

such that $ST \simeq 1_{\mathfrak{M}_A}$, and $TS \simeq 1_{\mathfrak{M}_B}$. It follows that $T \simeq - \otimes_A P$, $S \simeq - \otimes_B Q$, where $P = T(A)$ and $Q = S(B)$. See, for instance, [Bas68, p.60]. Given a semilocal and connected ring R, H. Bass in [Bas64, Corollary 17.2] shows that two Azumaya R-algebras A and B are Brauer equivalent if and only if they are Morita equivalent as R-algebras.

Given that A is an Azumaya R-algebra, P a left A-module and $B = \mathrm{E}nd_A(P)$. The action of B on P by evaluation of functions induces on P a right B-module structure commuting with that of A; in sum, P becomes an $A \otimes_R B^{\mathrm{o}}$-module. In particular, for all $a \in A$, $\varphi \in B$, and $p \in P$, $(a \otimes \varphi)p = ap\varphi$ with endomorphisms written on the right of arguments.

With a focus on the next lemma, recall from [For17, Proposition 1.1.8] the fact that, a progenerator over a progenerator is a progenerator, that is, given a ring homomorphism $R \to S$ such that S is an R-progenerator, then any S-progenerator is also an R-progenerator.

Lemma 2.1.1.3. *Let A be an Azumaya right R-algebra, and let P be an Azumaya left A-algebra. Then $B := \mathrm{E}nd_A(P)$ is an Azumaya R-algebra. Furthermore, the natural morphism*

$$\begin{array}{rcl} A \otimes_R B & \longrightarrow & \mathrm{E}nd_R(P) \\ a \otimes \varphi & \longmapsto & (p \mapsto \varphi(ap) = a\varphi(p)) \end{array}$$

is an isomorphism; therefore, B is Brauer equivalent to the opposite Azumaya R-algebra A^{o}.

Proof. Since A is an R-progenerator, and $\mathrm{End}_A(P)$ is an A-progenerator, then $\mathrm{End}_A(P)$ is an R-progenerator. Moreover, by [Kap54, Exercise 95] and [Lin17, Lemma 3.12.], $\mathrm{End}_A(P)$, as an R-algebra, is R-central; therefore, $\mathrm{End}_A(P)$ is an Azumaya R-algebra (see [For17, Theorem 7.1.4]), and consequently, the tensor product $A \otimes_R B$ is an Azumaya R-algebra. Finally, that the map in the statement of Lemma 2.1.1.3 is an isomorphism of Azumaya R-algebras results from Proposition 2.1.1.2. □

By way of symmetry, if A is an Azumaya left R-algebra and P a right A-module, then $B := \mathrm{E}nd_A(P)$ is an Azumaya right R-algebra and we have, $B \otimes_R A \simeq \mathrm{E}nd_R(P)$.

Corollary 2.1.1.4. *Subject to the hypotheses of Lemma 2.1.1.3, the R-algebra $E := \mathrm{E}nd_B(P)$, where $B = \mathrm{E}nd_A(P)$ is Azumaya, is isomorphic to A, that is, $A \simeq \mathrm{E}nd_B(P)$.*

Proof. On account of Lemma 2.1.1.3, B is an Azumaya left R-algebra; since P is an Azumaya right B-algebra, it follows that $E := \mathrm{E}nd_B(P)$ is an Azumaya right R-algebra. Further, as $E \otimes_R B \simeq \mathrm{E}nd_R(P) \simeq A \otimes_R B$, it follows that $A \simeq \mathrm{E}nd_B(P)$, thereby completing the proof. □

As a converse of Lemma 2.1.1.3, we have

Lemma 2.1.1.5. *Let R be a commutative ring, and A, B Azumaya R-algebras. Then, if B is Brauer equivalent to A, B is of the form $\mathrm{End}_A(P)^{\mathrm{o}}$, where P is both an A-module and R-progenerator.*

Proof. In view of [For17, Proposition 7.3.4], there is an isomorphism $\phi : A \otimes_R B^{\mathrm{o}} \simeq \mathrm{E}nd_R(P)$, where P is an R-progenerator. Plainly, by setting $(a \otimes 1)(x) = ax$, for all $a \in A$ and $x \in P$, P ends up being an A-module. From [For17, Theorem 7.2.3], it follows that the commutant of A identified to its image via ϕ in $\mathrm{E}nd_R(P)$ is exactly B. In other words, since

$$\mathrm{E}nd_R(P)^{\phi(A)} \equiv \mathrm{E}nd_R(P)^A = \{\varphi \in \mathrm{E}nd_R(P) |\ \varphi a = a\varphi,\ a \in A\} = \mathrm{E}nd_A(P),$$

it follows that

$$B^{\mathrm{o}} \simeq \phi(B^{\mathrm{o}}) = \mathrm{E}nd_A(P).$$

□

We note that, premising on Lemma 2.1.1.5 and the assumption that $End_R(P)$ is simple, we have the following lemma:

Lemma 2.1.1.6. *Let R be a commutative ring and let P be a free left Azumaya algebra of finite rank over an Azumaya R-algebra A such that $\mathrm{End}_R(P)$ is a simple left R-module. Then, $B := \mathrm{End}_A(P)$ is an Azumaya R-algebra, which is Brauer equivalent to the opposite algebra A^{o}.*

Proof. Suppose that $\mathrm{rank}_A(P) = n$; therefore,

$$\mathrm{H}om_A(P,P) \simeq M_n(A^{\mathrm{o}}),$$

where A^{o} is the opposite algebra of A (see [AW92, Corollary 3.9, p.219]). By setting, for all $a \in A$ and $X \in M_n(A^{\mathrm{o}})$,

$$a \cdot X = Xa^{\mathrm{o}},$$

$M_n(A^{\mathrm{o}})$ acquires a left A-module structure. As is shown in [For17, Example 4.2.1], $\mathrm{M}_n(A^{\mathrm{o}})$ is A-separable. By [For17, Theorem 4.4.2], $\mathrm{M}_n(A^{\mathrm{o}})$ is R-separable. Since A is R-central, it turns out that the center of $\mathrm{M}_n(A^{\mathrm{o}})$ is isomorphic to R. Thus, $\mathrm{M}_n(A^{\mathrm{o}})$ is an Azumaya R-algebra.

On applying [For17, Proposition 7.3.4], we have

$$\mathrm{Hom}_A(P,P)\otimes_R \left(A^{\mathrm{o}}\right)^{\mathrm{o}} \subseteq \mathrm{Hom}_A(P,P)\otimes_A A \simeq \mathrm{Hom}_A(P,P).$$

But

$$\mathrm{Hom}_A(P,P) \simeq \mathrm{Hom}_A(P,P)\otimes 1 \subseteq \mathrm{Hom}_A(P,P)\otimes_R A,$$

therefore,

$$\mathrm{Hom}_A(P,P)\otimes_R A \simeq \mathrm{Hom}_A(P,P).$$

On the other hand, since $\mathrm{Hom}_A(P,P) \neq \emptyset$, $\mathrm{Hom}_A(P,P) \subseteq \mathrm{Hom}_R(P,P)$, and $\mathrm{Hom}_R(P,P)$ is simple, then

$$\mathrm{Hom}_A(P,P) = \mathrm{Hom}_R(P,P);$$

and since P is a progenerator module over R, there is a Brauer equivalence between $\mathrm{Hom}_A(P,P)$ and A^{o}. □

2.2 Involutions on algebras

This section borrows directly from the literature in [KO74] and [KMRT98].

Definition 2.2.0.1. *Let A be an Azumaya R-Algebra where R is a commutative local ring. An involution σ on A is a ring anti-automorphism of A of order at most 2. That is, an anti-automorphism $\sigma : A \longrightarrow A$ of A for all $x, y \in A$ satisfying the following properties;*

i. $\sigma(x+y) = \sigma(x)+\sigma(y)$

ii. $\sigma(1) = 1$

iii. $\sigma(xy) = \sigma(y)\sigma(x)$

iv. $\sigma(\sigma(x)) = x.$

We shall denote an Azumaya R-algebra with involution by (A,σ). Further, an involution σ on an Azumaya R-algebra A is said to be of the first kind if it is an R-linear map that leaves the centre $Z(A)$ of the ring A invariant. Otherwise, σ is an involution of the second kind.
Since A is a ring with involution, restricting this involution to the centre, $\sigma = \sigma|_{Z(A)}$ is an involution on $Z(A)$ and A is a $Z(A)$-algebra. Thus, the involution σ on A is an extension of the involution σ_R on R. That is, $\sigma(ra) = \sigma_R(r)\sigma(a)$ for $a \in A$, $r \in R$.

Definition 2.2.0.2. *Let (A,σ_1) and (B,σ_2) be rings with involutions. A morphism of rings with involutions*

$$\varphi : (A,\sigma_1) \longrightarrow (B,\sigma_2)$$

is a homomorphism of rings $\varphi : A \longrightarrow B$ such that it preserves involutions and $\sigma_2(\varphi(a)) = \varphi(\sigma_1(a))$ for all $a \in A$.

For instance, given a two sided ideal $J \subset A$ such that $\sigma_1(J) \subset J$, the natural projection $A \longrightarrow A/J$ is a morphism of rings with involution.

Given a local ring R, an Azumaya R-algebra A of constant rank n^2, endowed with an involution σ of the first kind, and a faithfully flat splitting ring S of A, the isomorphism $\alpha : S \otimes_R A \simeq M_n(S)$ induces an involution $\widetilde{\sigma}$: $\widetilde{\sigma} = \alpha(1 \otimes \sigma)\alpha^{-1}$ on $M_n(S)$, [Knu91, Lemma 8.1.1, p.170]. By [Knu91, (5.1), p.134], $S \otimes_R A$ is an Azumaya algebra over a local ring R, so every R-algebra automorphism of $S \otimes_R A \simeq M_n(S)$ is inner, (see [For17, Corollary 7.8.15, p.280,]). That is, if λ is an R-algebra automorphism of $M_n(S)$, then λ is associated with some invertible $u \in M_n(S)$ in the sense that $\lambda(x) \equiv \lambda_u(x) = uxu^{-1}$ for every $x \in M_n(S)$. It is evident that if $\tau : M_n(S) \longrightarrow M_n(S)$ is an involution, the map $\tau^0 : M_n(S) \longrightarrow M_n(S)$, defined by setting $\tau^0(x) = \tau(x)$ and $\tau^0(xy) = \tau^0(x)\tau^0(y)$, for all x, $y \in M_n(S)$, is an R-algebra automorphism of $M_n(S)$; therefore $\tau^0(x) \equiv \tau^0_u(x) = uxu^{-1}$, for some invertible $u \in M_n(S)$. Hence, τ is inner. In the same vein, the map $x \mapsto \widetilde{\sigma}(x^t)$, where x^t is the transpose of x, is an automorphism of $M_n(S)$. Thus, $\widetilde{\sigma}(x) = ux^tu^{-1}$ for some invertible u of $M_n(S)$. Since $\widetilde{\sigma}^2 = 1$, $u^t = \varepsilon u$, for some $\varepsilon \in \mu_2(S)$, where $\mu_2(S) = \{x \in S|\ x^2 = 1\}$, (see [Knu91, p. 122]). This element ε is called the type of the involution.

Lemma 2.2.0.3. *Let R be a local ring and (A,σ) an Azumaya R-algebra with an involution of the first kind. There exists a faithfully flat splitting $\alpha : S \otimes_R A \simeq M_n(S)$ of A such that $\alpha(1 \otimes \sigma)\alpha^{-1} = \sigma_u$, where $u \in M_n(S)$ is invertible and satisfies $u^t = \varepsilon u$, where $\varepsilon \in \mu_2(R)$. Furthermore, the element ε is independent of the splitting α of A.*

Proof. See the proof of [Knu91, Lemma (8.1.1), p.170]. □

Remark 2.2.0.4. Let A be an Azumaya R-algebra of constant rank n^2, $n \geq 1$ over a local ring R with an involution of the first kind σ. For every element $a \in A$, the reduced traces of the elements a and $\sigma(a)$ are equal. Indeed, assuming $A = M_n(R)$, in the light of Lemma 2.2.0.3, there is an involution $\sigma_u(x) = ux^tu^{-1} = \sigma$ on $M_n(S)$ for some $u \in Gl_n(S)$ induced by the involution σ on A. By application of Proposition 1.3.5.3, we have,

$$Trd_A(\sigma(x)) \quad = Trd_A(\sigma_u(x)) = Trd_A(ux^tu^{-1}) = Trd_A((x^tu^{-1})u) = Trd_A(x^t) = Trd_A(x)$$

for some $x \in A, \ \varepsilon \in \mu_2(S)$.

On an Azumaya R-algebra A with involution σ of the first kind (A,σ), we have the subsets of symmetric, skew-symmetric, symmetrized and alternating elements in A with respect to σ defined as follows:

a. $Sym(A,\sigma) = \{a \in A | \sigma(a) = a\}$

b. $Skew(A,\sigma) = \{a \in A | \sigma(a) = -a\}$

c. $Symd(A,\sigma) = \{a + \sigma(a) | a \in A\}$

d. $Alt(A,\sigma) = \{a - \sigma(a) | a \in A\}$.

Moreover, these subsets are projective and finitely generated as R-modules provided σ is an involution of the first kind (see [Knu91, Lemma 8.1.4, p.171]).

Definition 2.2.0.5. *Let R be a commutative ring and (A,σ) an R-algebra with involution σ. The trace tr and the norm $\mathcal{N}$ with respect to the involution σ are given respectively by $tr(a) = a + \sigma(a)$ and $\mathcal{N} = a \cdot \sigma(a)$ for some element $a \in A$. The involution σ is said to be a standard involution on A if*

1. $\sigma(R) = R$, that is, it fixes R.

2. $tr(a) \in R$ and $\mathcal{N}(a) \in R$ for all $a \in A$.

Remark 2.2.0.6. Given a standard involution σ on A, we have

$$\mathcal{N}(1+a) \equiv \mathcal{N}(a) + 1 + tr(a).$$

As $tr(a) \in R$ for all $a \in A$, it follows also that $\mathcal{N}(a) \in R$ for all $a \in A$. So, for every $a \in A$,

$$atr(a) = tr(a)a$$

and

$$a^2 - tr(a)a + \mathcal{N}(a) = 0.$$

It can be seen that $\mathcal{N}(a) = a\sigma(a) = a\sigma(a)$ for all $a \in A$.

2.2.1 Hermitian forms and involutions on Azumaya algebras

In this section, we give an extension and generality of the result in [KMRT98, Theorem, Chapter 1, p.1] on central simple algebras to the context of classical Azumaya R-algebras.

Definition 2.2.1.1. *Let R be a ring with involution(conjugation) ϑ and A be an R-algebra. A sesquilinear form on A over (R,ϑ) is a biadditive map $b : A\times A \longrightarrow R$ such that*

$$b(xs, yt) = \vartheta(s)b(x, y)t$$

for $s,t \in R$ and $x, y \in A$.

The bilinear form b on A induces an R-linear map

$$h_b : A \longrightarrow A^*$$

given by $h_b(x)(y) = b(x, y)$ for $x, y \in A$ where $A^* = \mathrm{Hom}_R(A,R)$ is an additive group with a right R-module structure given by $(f \cdot a)(x) = \vartheta(a)f(x)$ where $a \in R, f \in A^*$ and $x \in A$. Indeed,

$$\begin{aligned} h_b(xr)(y) =& b(xr, y) \\ =& \vartheta(r)b(x, y) \\ =& (\vartheta(r)(h_b(x)))(y) \\ =& (h_b(x)r)(y). \end{aligned}$$

Conversely, the map $h : A \longrightarrow A^*$ defines a sesquilinear form

$$b_h(x, y) = h(x, y),$$

for $x, y \in A$. The bilinear form h_b is called the adjoint of b. (c.f. [Knu91, §2, p.5].)

Definition 2.2.1.2. *A hermitian form on A with respect to the conjugation ϑ is a sesquilinear map $h : A\times A \longrightarrow R$ such that $h(y,x) = \vartheta(h(x, y))$, for all $x,\ y \in A$. The map h is called skew-hermitian if $h(y,x) = -\vartheta(h(x, y))$, for all $x,\ y \in A$.*

Remark 2.2.1.3. Let R be a *local* ring and A an Azumaya R-algebra of finite rank as an R-module. A bilinear form $b : A\times A \longrightarrow R$ is *nonsingular* if the induced map

$$\hat{b} : A \longrightarrow A^* := \mathrm{H}om_R(A,R),$$

defined by

$$\hat{b}(x)(y) = b(x, y),$$

for all $x,\ y \in A$, is an R-linear isomorphism. For any $f \in \mathrm{E}nd_R(A)$, we define an endomorphism $\sigma_b(f) \in \mathrm{E}nd_R(A)$ by setting

$$\sigma_b(f) = \hat{b}^{-1} \circ f^t \circ \hat{b}, \tag{2.1}$$

where $f^t \in \mathrm{E}nd_R(A^*)$ is the transpose of f. In addition, the standard approach is to define each element $\sigma_b(f)$ by requiring that it verifies the property:

$$b\big(x, f(y)\big) = b\big(\sigma_b(f)(x), y\big),$$

for all $x,\ y \in A$. From this relation above, σ_b turns out to be an anti-automorphism of $\mathrm{E}nd_R(A)$; it is called the *adjoint anti-automorphism* with respect to the nonsingular bilinear form b. (c.f. [KMRT98, Chapter 1, p.1].)

Definition 2.2.1.4. *A quadratic form on a finitely generated projective module M is a map $q : M \longrightarrow A$ with the following properties*

i. $q(\lambda x) = \lambda^2 q(x)$ *for all* $x \in M,\ \lambda \in A$

ii. $b_q(x, y) = q(x+y) - q(x) - q(y)$ *defines on M a bilinear form* $b_q : M \times M \longrightarrow A$.

The pair (M, q) is a quadratic module over A and (M, b_q) the associated bilinear form. If (M, b_q), the associated bilinear module is non-singlar, call (M, q) a no-singular quadratic module.

Remark 2.2.1.5. i. If $\frac{1}{2} \in A$, then every quadratic form (M, q_b) identifies a symmetric bilinear form (M, b) in such a way that $q_b(x) = b(x, x)$ for all $x \in M$.

ii. If A has characteristic 2, then $b_q(x, x) = 2q(x) = 0$ for every quadratic form q over A, ı.e b_q is an alternating form and the rank of every module (M, q) is even.

iii. For every quadratic module (M, q), the following are equivalent:

  i. (M, q) is non singular

  ii. $(M_{\mathfrak{m}}, q_{\mathfrak{m}})$ is non singular for all maximal ideals $\mathfrak{m} \in \mathrm{Spec}(R)$.

Now we consider the relationship between classes of non-singular hermitian forms over Azumaya R-algebras and R-linear involutions on $\mathrm{End}_R(A)$;

Theorem 2.2.1.6. *Let R be a local ring and A an Azumaya R-algebra of finite rank. Moreover, let Λ be the map that sends each nonsingular bilinear form $b : A \times A \longrightarrow R$ onto its adjoint anti-automorphism σ_b. Then, Λ induces a bijection $\widetilde{\Lambda}$ between the set of equivalence classes of nonsingular bilinear forms on A modulo multiplication by a unit of R and the set of adjoint anti-automorphisms of* $\mathrm{End}_R(A)$. *Under the map $\widetilde{\Lambda}$, the R-linear involutions of* $\mathrm{End}_R(A)$ *correspond to nonsingular bilinear forms which are either symmetric or skew-symmetric.*

Proof. The proof is standard; see, for instance, [KMRT98, Theorem, pp 1-2,]. Indeed, from relation (2.1) in Remark 2.2.1.3,

$$\begin{aligned}(\alpha b)(\sigma_{\alpha b}(f)(x), y) &= \alpha b(x, f(y)) \\ &= \alpha(b(x, f(y))) \\ &= \alpha(b(\sigma_{\alpha b}(f)(x), y)).\end{aligned}$$

Rearranging the brackets on the left hand side above, we obtain

$$b(\sigma_{\alpha b}(f)(x), y) = b(\sigma_b(f)(x), y)$$

of which

$$b\big((\sigma_{\alpha b}(f) - \sigma_b(f))(x), y\big) = 0$$

and finally,

$$\sigma_{\alpha b}(f) = \sigma_b(f).$$

We see that for any unit α in R, $\sigma_{\alpha b} = \sigma_b$; therefore, the map Λ induces a well-defined map $\widetilde{\Lambda} : [b] \mapsto \sigma_b$, where $[b]$ denotes the equivalence class containing b.

Now, let's show that $\widetilde{\Lambda}$ is one-to-one. To this end, note that if b, b' are nonsingular bilinear forms on A, the isomorphism $v \equiv \hat{b}^{-1} \circ \hat{b}'$ is such that

$$\begin{aligned}b(v(x), y) =& \hat{b}(v(x))(y) \\ =& \hat{b}((\hat{b}^{-1} \circ \hat{b}')(x))(y) \\ =& \hat{b}(\hat{b}^{-1}(\hat{b}'(x))(y) \\ =& \hat{b}'(x)(y) \\ =& b'(x, y)\end{aligned}$$

for all x, $y \in A$; whence, one has, for all $f \in \mathrm{E}nd_R(A)$,

$$\begin{aligned}\sigma_b(f) =& \hat{b}^{-1} \circ f^t \circ \hat{b} \\ =& \hat{b}^{-1} \circ \hat{b}' \circ \hat{b}'^{-1} \circ f^t \circ \hat{b}' \circ \hat{b}'^{-1} \circ \hat{b} \\ =& \hat{b}^{-1} \circ \hat{b}' \circ \sigma_{b'}(f) \circ \hat{b}'^{-1} \circ \hat{b}' \\ =& v \circ \sigma_{b'}(f) \circ v^{-1}\end{aligned}$$

which can be rewritten as

$$\sigma_b = \mathrm{I}nt(v) \circ \sigma_{b'},$$

where

$$\mathrm{I}nt(v)(f) = v \circ f \circ v^{-1},$$

for all $f \in \mathrm{E}nd_R(A)$. Therefore, if $\sigma_b = \sigma_{b'}$, then v is central i.e.

$$\sigma_b \circ v = v \circ \sigma_{b'}$$

where v is a unit in R, and $[b] = [b']$.

Next, let us fix a non-singular bilinear form b on A. It follows that for any linear anti-automorphism v of $\mathrm{E}nd_R(A)$, $\sigma_b \circ v^{-1}$ is an R-linear automorphism of $\mathrm{E}nd_R(A)$. Since $\mathrm{E}nd_R(A)$ is an Azumaya R-algebra (see [For17, Theorem 7.1.4] together with [For17, Proposition 7.1.10]) and R is local, by the Skolem-Noether theorem ([For17, Corollary 7.8.15]), $\sigma_b \circ v^{-1}$ is an inner automorphism, that is, $\sigma_b \circ v^{-1} = \mathrm{I}nt(u)$, for some R-linear isomorphism $u \in \mathrm{E}nd_R(A)$. Then, v is an adjoint anti-automorphism for the bilinear form b' defined by

$$b'(x, y) = b(u(x), y),$$

which ends the proof of the first part of the theorem.

Finally, if b is a nonsingular bilinear form on A with adjoint anti-automorphism σ_b, then the nonsingular bilinear form b':

$$b'(x, y) = b(y, x) \quad \text{for all } x, y \in A$$

satisfies the equation

$$\sigma_{b'} = \sigma_b^{-1}.$$

Therefore, b and b' are scalar multiples of each other if and only if $\sigma_b^2 = 1$; it follows that if $b' = \varepsilon b$, for some unit ε, then $\varepsilon^2 = 1$. Hence, b is symmetric or skew-symmetric. □

Lemma 2.2.1.7. *Let b and b' be two non-singular bilinear forms. For any two R-progenerator modules P and Q,*

$$\left(End_R(P \otimes Q), \sigma_b \otimes \sigma'_{b_1}\right) \simeq \left(End_R(P), \sigma_b\right) \otimes \left(End_R(Q), \sigma'_{b_1}\right).$$

Moreover,

$$\left(End_R(P), \sigma_b\right) = \left(End_R(P), \sigma'_{b_1}\right)$$

if and only if b and b' belong to the same class.

Proof. Let (P,b) and (Q,b') be two non-singular bilinear forms on P and Q respectively. Also, let $f \in End_R(P)$, $f' \in End_R(Q)$ and $x,x' \in P$, $y,y' \in Q$. Then we have,

$$\begin{aligned}(b\otimes b')\Big(\sigma_b\otimes\sigma_{b'}(f\otimes f')(x\otimes x'),y\otimes y'\Big) &= (b\otimes b')\Big(\sigma_b(f)(x)\otimes\sigma_{b'}(f')(x'),y\otimes y'\Big)\\ &= b\big(\sigma_b(f)(x),y\big)\otimes b'\big(\sigma_{b'}(f')(x'),y'\big)\\ &= b\big(x,f(y)\big)\otimes b'\big(x',f'(y')\big)\\ &= b\otimes b'\big(x\otimes x',(f\otimes f')(y\otimes y')\big).\end{aligned}$$

Therefore, $\sigma_b\otimes\sigma_{b'}$ is well defined on $End_R(P\otimes Q)$ and from bilinearity of $b\otimes b'$, we get the above equality. Further, the natural R-module isomorphism

$$\mathrm{Hom}(P\otimes Q,P\otimes Q)\simeq \mathrm{Hom}(P,P)\otimes Hom(Q,Q)$$

induces an indentifcation

$$\big(End_R(P\otimes Q),\sigma_b\otimes\sigma_{b_1}\big)\simeq (End_R(P),\sigma_b)\otimes(End_R(Q),\sigma'_{b_1})$$

of algebras with involutions.

Since b and b' belong to the same class, by Theorem 2.2.1.6, $b'=\alpha b$ for some unit $\alpha\in R^\star$. Therefore,

$$(\alpha b)(\sigma_{\alpha b}(f)(x),y)=\alpha(b(\sigma_{\alpha b}(f)(x),y)).$$

Hence, we obtain

$$\sigma_{\alpha b}(f)=\sigma_b(f)$$

as required. □

Remark 2.2.1.8. More generally, given R a local ring and a PID, A an Azumaya R-algebra, M a finitely free right A-module, and $\vartheta: A\to A$ an involution (of any kind). Then, *for every nonsingular hermitian or skew-hermitian form $h: M\times M\to A$, there exists a unique involution $\sigma_h:\mathrm{End}_A(M)\to\mathrm{End}_A(M)$ such that*

$$h(x,f(y))=h(\sigma_h(f)(x),y),$$

for all $x,\ y\in M$. The *involution σ_h* is the adjoint involution with respect to h. (c.f. [KMRT98, Proposition 4.1].)

2.3 Azumaya quadratic pairs

On considering the equivalence in [For17, Theorem 10.3.9], we have a faithfully flat étale splitting R-algebra S for an Azumaya R-algebra A, such that there is an S-progenerator module P satisfying the isomorphism $S \otimes_R A \simeq End_S(P)$ as S-algebras. It is worth noting that P has rank n if A is an Azumaya R-algebra of constant rank n^2. Particularly, the faithfully flat étale splitting ring S satisfies an isomorphism $S \otimes_R A \simeq M_n(S)$. On the other hand, since $End_S(P)$ is an Azumaya R-algebra, if R is a local ring, then by Theorem 2.2.1.6, the involution $\sigma_S \equiv 1 \otimes \sigma$ is then the adjoint involution σ_b with respect to some nonsingular symmetric or skew-symmetric bilinear form b on P. Furthermore, since P is finite-free over R, then by means of some basis of P, we may identify P with S^n. If $g \in GL_n(S)$ denotes the Gram matrix of b with respect to the chosen basis, then

$$b(x,y) = x^t \cdot g \cdot y,$$

where x, y are considered as column vectors and $g^t = g$ if b is symmetric, $g^t = -g$ if b is skew-symmetric. Consequently, we note that the involution σ_S may be identified with the involution $\sigma_b \equiv \sigma_g$, where

$$\sigma_g(m) = g^{-1} \cdot m^t \cdot g,$$

for all $m \in M_n(S)$.

The argument above can be summarized in the following lemma.

Lemma 2.3.0.1. *Let R be a local ring and (A,σ) be an Azumaya R-algebra of constant rank n^2, endowed with an involution σ of the first kind. Moreover, let S be a faithfully flat etale splitting ring of A and P some S-progenerator of rank n such that $S \otimes_R A \simeq \mathrm{End}_S(P)$. Then, there is a nonsingular symmetric or skew-symmetric bilinear form on P with a Gram representing matrix $g \in GL_n(P)$ such that*

$$(A_S, \sigma_S) \simeq (\mathrm{End}_S(P), \sigma_b) \simeq (M_n(S), \sigma_g).$$

Definition 2.3.0.2. *The involution σ of the first kind on A is said to be orthogonal when the corresponding bilinear form b is symmetric.*

As a precursor to results related to *Azumaya quadratic pairs,* which are to be defined later, we allow a digression about a skew-hermitian version of the Gram-Schmidt process. See [CdS01, Theorem 1.1].

Lemma 2.3.0.3. *Let R be a PID, A an Azumaya R-algebra with no zero divisors and such that $rank_R A < \infty$, M a right A-module of finite rank, ϑ an involution (of any kind) on A, and $h : M \times M \longrightarrow A$ a hermitian or skew-hermitian map on M with respect to the involution ϑ, such that, for all $u \in M$, $h(u,u) = 0$, and, for some $v \in M$, $h(u,v) = 1$. Then, there is an R-basis $(u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n)$ of M such that*

$$\begin{aligned} h(u_i, x) &= 0, && \textit{for all } i \textit{ and all } x \in M \\ h(e_i, e_j) &= 0 = h(f_i, f_j), && \textit{for all } i,\ j, \\ h(e_i, f_j) &= 1, && \textit{for all } i,\ j. \end{aligned}$$

Proof. We will discuss the case where h is skew-hermitian. The argument analogously holds for hermitian maps as well.

Let $N := \{x \in M|\ h(x,y) = 0, \text{for all } y \in M\}$. It can be seen that N is an R-submodule of M, and therefore N is free and of finite rank as an R-module. Additionally, as long as $x \in N$ and $x = za$, for some $a \neq 0$ in A and $z \in M$, it follows that $z \in N$, and in consequence, N is a pure R-submodule of M. As a result, N is complemented in M, that is,

$$M = N \oplus T,$$

for some R-submodule T. Now, take a nonzero $e_1 \in T$. Then, there is $f_1 \in T$ such that $h(e_1, f_1) = 1$. Let

$$T_1 := [e_1, f_1],$$

that is, T_1 is the submodule spanned by e_1 and f_1, and

$$T_1^h := \{t \in T|\ h(t,y) = 0, \text{ for all } y \in T_1\}.$$

The elements e_1 and f_1 are R-linearly independent, for if $e_1 = f_1 a$, with $a \in R$, then

$$0 \neq h(e_1, f_1) = h(f_1 a, f_1) = \vartheta(a) h(f_1, f_1) = 0,$$

a *contradiction.* So (e_1, f_1) is an R-basis of T_1. Furthermore, we prove that

$$(i)\ T_1 \cap T_1^h = 0, \quad \text{and } (ii)\ T_1 + T_1^h = T.$$

Indeed, suppose that $t = e_1 a + f_1 b \in T_1 \cap T_1^h$, where $a, b \in R$. It will follow that

$$0 = h(t, e_1) = h(e_1 a + f_1 b, e_1) = \vartheta(b) h(f_1, e_1),$$

$$0 = h(t, f_1) = h(e_1 a + f_1 b, f_1) = \vartheta(a) h(e_1, f_1);$$

therefore $a = 0 = b$ since A has no zero divisors, so $T_1 \cap T_1^h = 0$. Next, for every $t \in T$, one has

$$t = -f_1 \vartheta(h(t, e_1)) + e_1 \vartheta(h(t, f_1)) + (t + f_1 \vartheta(h(t, e_1)) - e_1 \vartheta(h(t, f_1)))$$

with

$$-f_1 \vartheta(h(t, e_1)) + e_1 \vartheta(h(t, f_1)) \in T_1$$

and

$$t + f_1 \vartheta(h(t, e_1)) - e_1 \vartheta(h(t, f_1)) \in T_1^h.$$

Thus,

$$M = N \oplus T_1 \oplus T_1^h.$$

Now, let $e_2 \in T_1^h$ with $e_2 \neq 0$. There is $f_2 \in T_1^h$ such that $h(e_2, f_2) = 1$. As above, let T_2 = span of e_2, f_2. Etc ...

This process will eventually terminate as $\operatorname{rank}(M) < \infty$. Hence, we obtain

$$M = N \oplus T_1 \oplus T_2 \oplus \cdots \oplus T_n,$$

or more accurately,

$$M = N \perp T_1 \perp T_2 \perp \cdots \perp T_n,$$

where T_i has basis (e_i, f_i) with $h(e_i, f_i) = 1$. □

For the hermitian maps case, in order to show that $T = T_1 + T_1^h$, for any $t \in T$, consider the following decomposition

$$t = -f_1 h(t, e_1) + e_1 h(t, f_1) + \big(t + f_1 h(t, e_1) - e_1 h(t, f_1)\big).$$

Surely, $-f_1 h(t, e_1) + e_1 h(t, f_1) \in T_1$; and using the face that

$$0 = h(e_1 + t, e_1 + t) = h(e_1, t) + h(t, e_1),$$

one has that

$$\vartheta(h(t, e_1)) = -\vartheta(h(e_1, t)).$$

Therefore,

$$h(t, e_1) + \vartheta(h(t, e_1))h(f_1, e_1) - \vartheta(h(t, f_1))h(e_1, e_1) = h(t, e_1) + \vartheta(h(t, e_1))$$

$$= h(t, e_1) - \vartheta(h(e_1, t)) = h(t, e_1) - h(t, e_1) = 0;$$

thus,

$$t + f_1 h(t, e_1) - e_1 h(t, f_1) \in T_1^h.$$

It is readily understood that $\operatorname{rank}_R(N) = 0$ if h is non-singular, and consequently $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ is an R-basis of M, satisfying

$$h(e_i, f_j) = \delta_{ij} \quad \text{and} \quad h(e_i, e_j) = 0 = h(f_i, f_j).$$

The R-basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ is called a *symplectic* basis.

Now, suppose that A is a finitely generated Azumaya R-algebra, where R is *both a local ring and a PID,* and suppose that ϑ is an involution (of any kind) on A. Moreover, suppose also that M is a right

A-module of finite rank and a torsion-free R-module; then we associate with M the left A-module, denoted ${}^{\vartheta}M$, defined as follows: M and ${}^{\vartheta}M$ have the same underlying set. With a view to differentiate ${}^{\vartheta}M$ from M, whenever we consider an element x in M as an element of ${}^{\vartheta}M$, we shall use the notation ${}^{\vartheta}x$. The module operations on ${}^{\vartheta}M$ are given by assuming that

$$ {}^{\vartheta}x +{}^{\vartheta} y ={}^{\vartheta} (x+y) \quad \text{and} \quad a\cdot{}^{\vartheta} x ={}^{\vartheta} (x.\vartheta(a)), $$

for any $x,\, y \in M$ and $a \in A$. Clearly, $M \otimes {}^{\vartheta}M$ is a finitely generated torsion-free R-module, and

$$ \operatorname{rank}_R(M \otimes_A {}^{\vartheta}M) = (\operatorname{rank}_A M)^2 \operatorname{rank}_R A = \operatorname{rank}_R(\mathrm{E}nd_A(M)) $$

Next, let $h : M \times M \longrightarrow A$ be a nonsingular hermitian or skew-hermitian form on M with respect to the conjugation ϑ. On considering the R-linear map

$$ \varphi_h : M \otimes^{\vartheta} M \longrightarrow \mathrm{E}nd_A(M) $$

such that

$$ \varphi_h(x \otimes^{\vartheta} y)(u) = x \cdot h(y,u), $$

a counterpart result of [KMRT98, Theorem 5.1] is given in the form of the following theorem.

Theorem 2.3.0.4. *The map φ_h is an isomorphism satisfying the equality*

$$ \sigma_h(\varphi_h(x \otimes^{\vartheta} y)) = \delta\varphi_h(y \otimes^{\vartheta} x), \tag{2.1} $$

for all $x,\ y \in M$, where σ_h stands for the adjoint involution induced by h, and where $\delta = 1$ if h is hermitian and $\delta = -1$ if h is skew-hermitian. Moreover,

$$ \varphi_h(x_1 \otimes^{\vartheta} y_1) \circ \varphi_h(x_2 \otimes^{\vartheta} y_2) = \varphi_h(x_1 h(y_1,x_2) \otimes^{\vartheta} y_1). \tag{2.2} $$

Proof. First, note that, since M is a right A-module of finite rank, where A is an Azumaya R-algebra, with R both a local ring and a PID, as in [KMRT98, Proposition 4.1], for every hermitian or skew-hermitian form h on M, there exists a unique involution σ_h on $\mathrm{E}nd_A(M)$ such that $h(x, f(y)) = h(\sigma_h(f)(x), y)$, for all $x,\ y \in M$, and $f \in \mathrm{E}nd_A(M)$. On the strength of Lemma 2.3.0.3 and the paragraph directly thereafter, the proof of [KMRT98, Theorem 5.1] applies here as well with the only difference that M is regarded as an R-module with finite rank. □

By setting

$$ (x_1 \otimes {}^{\vartheta} y_1) \circ (x_2 \otimes {}^{\vartheta} y_2) = x_1 h(y_1,x_2) \otimes {}^{\vartheta} y_2, $$

for $x_1, x_2, y_1, y_2 \in M$, $M \otimes^{\vartheta} M$ is made into an R-algebra isomorphic to $\mathrm{E}nd_A(M)$, whose algebra structure is given by Equation (2.2). On considering the involution σ on $M \otimes^{\vartheta} M$, induced by σ_h, that is, such that $\sigma_h \circ \varphi_h = \varphi_h \circ \sigma$, one has

$$\sigma(x \otimes^{\vartheta} y) = \delta y \otimes^{\vartheta} x,$$

for $x, y \in M$, where $\delta = +1$ (-1, resp.) if h is hermitian (skew-hermitian, resp.).

For purposes of the lemma below, we state again that, given any Azumaya R-algebra A of constant rank n^2, the reduced trace map Trd_A gives rise to a symmetric non-degenerate R-bilinear form $T_A : A \times A \longrightarrow R$, by setting

$$T_A(\alpha, \beta) = \mathrm{Trd}_A(\alpha\beta), \quad \alpha, \beta \in A.$$

See [For17, Corollary 11.1.6, p.410]. On the other hand, if σ is an involution of the first kind on an Azumaya R-algebra A, then every $x \in A$ has the same reduced trace as its corresponding image $\sigma(x)$. Precisely, one has: $\mathrm{Trd}_A(\sigma(x)) = \mathrm{Trd}_A(x)$, for every $x \in A$.

Lemma 2.3.0.5. *Let R be a PID, and A an Azumaya R-algebra of constant rank. Then, the rank of A is equal to some n^2, where $n \geq 1$. Moreover,*

$$\mathrm{rank}_R \mathrm{Sym}(A, \sigma) + \mathrm{rank}_R \mathrm{Alt}(A, \sigma) = n^2.$$

Furthermore,

$$\mathrm{Alt}(A, \sigma) = \mathrm{Sym}(A, \sigma)^{\perp}$$

with respect to the bilinear form T_A on A, induced by the reduced trace Trd_A. *Similarly,*

$$\mathrm{rank}_R \mathrm{Skew}(A, \sigma) + \mathrm{rank}_R \mathrm{Symd}(A, \sigma) = n^2$$

and

$$\mathrm{Sym}(A, \sigma) = \mathrm{Skew}(A, \sigma)^{\perp}$$

with respect to T_A.

Proof. [For17, Corollary 10.3.10, p.395] ensures that A is of rank n^2 for some $n \geq 1$.

The first displayed relation follows from the fact that R is a PID, A is an R-module of finite rank, and $\mathrm{Alt}(A, \sigma)$ is the image of the linear endomorphism $\mathrm{Id} - \sigma$ of A, with $\mathrm{Sym}(A, \sigma) = \ker(\mathrm{Id} - \sigma)$, (see [AW92, Proposition 8.8, p.173]). Next, for the sake of easy referencing, we recall that a submodule of a finitely generated module over a PID is *pure* if and only if it is complemented, (see [AW92, Proposition 8.2, p.171]). Therefore, since T_A is nondegenrate (see [Rei75, p. 116, Theorem]), A an R-module of finite rank and $\mathrm{Sym}(A,\sigma)$ a pure submodule with $T_A|_{\mathrm{Sym}(A,\sigma)}$ and $T_A|_{\mathrm{Sym}(A,\sigma)^\perp}$ both nonsingular, it follows that (see [AW92, Corollary 2.34, p.361])

$$\left(\mathrm{Sym}(A,\sigma)^\perp\right)^\perp = \mathrm{Sym}(A,\sigma)$$

and

$$\mathrm{Sym}(A,\sigma) \oplus \mathrm{Sym}(A,\sigma)^\perp = A;$$

hence $\mathrm{Sym}(A,\sigma)^\perp$ is isomorphic to $\mathrm{Alt}(A,\sigma)$. But, for all $s \in \mathrm{Sym}(A,\sigma)$,

$$T_A(x-\sigma(x),s) = \mathrm{Trd}_A(sx) - \mathrm{Trd}_A(\sigma(sx)) = 0,$$

therefore

$$\mathrm{Alt}(A,\sigma) \subseteq \mathrm{Sym}(A,\sigma)^\perp;$$

hence,

$$\mathrm{Alt}(A,\sigma) = \mathrm{Sym}(A,\sigma)^\perp.$$

□

Lemma 2.3.0.6. *Let (A,σ) be an Azumaya R-algebra with involution of the first kind, on a local ring R. If $\deg A > 2$, the set $\mathrm{Alt}(A,\sigma) = \{x-\sigma(x) |\ x \in A\}$ of alternating elements of A generates A as an associative algebra.*

Proof. Application of Lemma 2.3.0.3 and the existence of a symplectic basis over an Azumaya algebra A in Lemma 2.3.0.1 suffices to conclude that the proof here assumes the same lines as that of [KMRT98, p. 29, Lemma 2.26]. □

Let's now introduce the notion of *quadratic pair,* defined on an Azumaya R-algebra, where R is an arbitrary commutative ring with unity of characteristic other than 2. First, suppose that $\mathrm{Rank}_R(A) \equiv \mathrm{Rank}(A) = n^2$, where n is some integer ≥ 1. Next, there is a commutative faithfully flat étale R-algebra S such that

$$\alpha : A \otimes_R S \xrightarrow{\sim} \mathrm{M}_n(S),$$

where $n \geq 1$ and *reduced trace*

$$\mathrm{Trd}_A(a) = \mathrm{Tr}(\alpha(a \otimes 1)),$$

where, for $M \in \mathrm{M}_n(S)$, $\mathrm{Tr}(M)$ is the trace of M.

Definition 2.3.0.7. *An Azumaya quadratic pair is a triple (A,σ,f), where A is an Azumaya R-algebra over a commutative ring R, endowed with an orthogonal involution σ of the first kind, and where $f : Sym(A,\sigma) \longrightarrow R$ is a linear map of R-modules, subject to the following condition*

$$f(x+\sigma(x)) = Trd_A(x),$$

for all $x \in A$, with $Trd_A(x)$ being the reduced trace of x.

Theorem 2.3.0.8. *Let R be a commutative ring, (M,q) a quadratic R-module of finite rank $2n$ such that q is non-singular, φ_q denote the isomorphism $\varphi_q : M \otimes_R M \to \mathrm{End}_R(M)$, given by $\varphi_q(u \otimes v)(x) = u \cdot b_q(v,x)$ (Theorem 2.3.0.4), and $\sigma_q := \sigma_{b_q}$ denote the adjoint involution corresponding to the non-singular bilinear form b_q of q. Then, there is a unique linear map $f_q : \mathrm{Sym}(\mathrm{End}_R(M), \sigma_q) \to R$ such that*

$$(f_q \circ \varphi_q)(v \otimes v) = q(v),$$

and the pair (σ_q, f_q) is a quadratic pair on $\mathrm{End}_R(M)$. Conversely, for any quadratic pair (σ, f) on $\mathrm{End}_R(M)$, there is a quadratic R-module (M,q), where q is non-singular and such that $\sigma_q = \sigma$ and $f_q = f$, and q is uniquely determined up to a factor in $R^\times$.

Proof. (c.f. [KMRT98, Ch.I, (5.11)].) Let M be a right A-module of finite rank and a torsion free projective R-module. Suppose $(e_1, e_2, \ldots, e_n)$ is a basis of M. Indeed, the elements $\varphi_q(e_i \otimes e_j)$ and $\varphi_q(e_i \otimes e_j + e_j \otimes e_i)$ for $i, j \in \{1, \ldots, n\}$ generate elements of

$$Sym(End_R(M), \sigma_q) = \varphi_q\big(Sym(M \otimes M, \sigma_q)\big).$$

Define $f_q(\varphi_q(e_i \otimes e_j)) = q(e_i)$, $f_q(\varphi_q(e_i \otimes e_j + e_j \otimes e_i)) = b_q(e_i, e_j)$ and extend linearly to the mapping

$$f_q : Sym(End_R(M), \sigma_q) \to R.$$

For some element $m = \sum_{i=1}^n e_i \beta_i \in M$, we have

$$f_q \circ \varphi(m \otimes m) = f_q \circ \varphi\left(\sum_{i=1}^n (e_i \otimes e_i){\beta_i}^2 + \sum_{i=1}^n (e_i \otimes e_j + e_j \otimes e_i)\beta_i\beta_j\right).$$

The above equality simplifies to give a well defined map f_q on the elements of M as

$$f_q \circ \varphi(m \otimes m) = \sum_{i=1}^n q(e_i){\beta_i}^2 + \sum_{i=1}^n b(e_i, e_j)\beta_i\beta_j = q(m).$$

As $Sym(End_R(M), \sigma_q)$ is spanned by unique elements of the form $\varphi_q(m \otimes m)$, the uniqueness of the map f_q will follow. Next, we show that (σ_q, f_q) is a quadratic pair for σ_q symmetric. By definition, we have a semi trace map

$$\varphi_q(x + \sigma(x)) = \mathrm{Trd}_{End_R(M)}(x)$$

where $x \in End_R(M)$. Let $x = \varphi_q(m \otimes t)$, where $m, t \in M$. So, the left side of f_q becomes

$$\begin{aligned} f_q(\varphi(m \otimes t)) =& f_q(\varphi(m \otimes t) + \varphi(t \otimes m)) \\ &= f_q \circ \varphi((m+t) \otimes (t+m)) - f_q \circ \varphi(m \otimes m) - f_q \circ \varphi(t \otimes t) \\ &= b_q(m,t). \end{aligned}$$

From Theorem 2.3.0.4, we have

$$b_q(m,t) = b_q(t,m) = \mathrm{Trd}(\varphi(m \otimes t)).$$

Conversely, suppose we have a quadratic pair (σ, f) on $End_R(A)$. By Theorem 2.2.1.6, the involution σ is the adjoint involution with respect to some nonsingular symmetric bilinear form $b : M \times M \to R$ which is uniquely determined up to a factor in $R^{\times}$. Since $\varphi_b(M \otimes M) \cong (End_R(M), \sigma_b)$, where $\varphi_q(m \otimes t)(x) = m \cdot b_q(t,x)$ with $\sigma(m \otimes t) = m \otimes t$. We can define the associated quadratic form $q : M \to R$ such that

$$q(m) = f \circ \varphi_q)(m \otimes m).$$

It is evident from the Definition (2.2.1.4) that

$$q(\beta m) = f \circ \varphi_q)(\beta m \otimes \beta m) = q(m)\beta^2.$$

Further,

$$\begin{aligned} q(m+t) - q(m) - q(t) =& f_q \circ \varphi(m \otimes t) + f_q \circ \varphi(t \otimes m)) \\ &= f_q\big(\varphi(m \otimes t) + \sigma(\varphi(t \otimes m))\big) \\ &= Trd_{End_R M}(m \otimes t). \end{aligned}$$

As (σ, f) is a quadratic pair, then we have

$$\mathrm{Trd}(\varphi(m \otimes t)) = b_q(m,t) = b_q(t,m).$$

Owing to the fact that q is a quadratic with associated bilinear form b, $\sigma_b \equiv \sigma$ and $f \equiv f_q$. Since b is uniquely determined, then so is q.

□

Chapter 3

Sheaves of Modules and Algebras

In this chapter, we consider elements from the theory of sheaves and schemes directly relevant to the sequel. Standard definitions and basic results are reviewed, including a supply of proofs of results to be employed within this chapter and the next. Our texts are adopted from: [Mal98, Chapter 1, pp 2-79], [GW10, Chapters 2-3, pp.40-88] and [Bos13, Part B, §6.1-§6.9]. We start our exploration by considering sheaves on topological spaces.

3.1 Preliminaries on sheaves and schemes

Definition 3.1.0.1. *A sheaf of sets is a triple $(\mathcal{S}, \pi, X)$ where $\mathcal{S}, X$ are topological spaces and $\pi : \mathcal{S} \longrightarrow X$ a surjective local homeomorphism with the property that : for every element $z \in \mathcal{S}$, there is an open neighbourhood V of $z \in \mathcal{S}$ such that $\pi(V)$ is an open neighbourhood of $\pi(z)$ in X and the restriction map*

$$\pi|_V : V \longrightarrow \Big(\pi|_V\Big)(V) = \pi(V)$$

is a homeomorphism. In the triple $(\mathcal{S}, \pi, X)$, we refer to $\mathcal{S}$ simply as a sheaf over X or sheaf space, X as the base space of the sheaf concerned, while the onto map π as a projection of the sheaf space $\mathcal{S}$ on X.

Remark 3.1.0.2. i. For any open subset U of X, the open set $\pi^{-1}(U) \subseteq \mathcal{S}$ defines a subsheaf (subspace) of $\mathcal{S}$ which by restriction to U gives a sheaf $\Big(\pi^{-1}(U),\, \pi|_{\pi^{-1}(U)}, U\Big)$ over U with $\mathcal{S} := \pi^{-1}(U)$. [Mal98, Chapter.1,§pp.1-10]

ii. For every $x \in X$, where x is in the image of π, the set

$$\mathcal{S}_x := \pi^{-1}\Big(\{x\}\Big) \equiv \pi^{-1}(x)$$

is called the fiber or stalk of $\mathcal{S}$ at the point $x \in X$. In this regard, $\mathcal{S}$ is a partition of fibers and thus a disjoint union of sets, ı.e,

$$\mathcal{S} := \sum_{x\in X} \pi^{-1}(x) \equiv \sum_{x\in X} \mathcal{S}_x,$$

for every $x \in X$ and the corresponding stalk $\mathcal{S}_x$ of a given sheaf is a discrete space when endowed with the relative topology of $\mathcal{S}$. Every element z in $\mathcal{S}_x$ is an open neighbourhood V as in Definition 3.1.0.1 satisfying $V \cap \mathcal{S}_x = \{z\}$. We shall denote by $\mathcal{S}h_X$, the category of sheaves of sets on X.

iii. For a sheaf $(\mathcal{S}, \pi, X)$, the open subsets of $\mathcal{S}$ to which the restriction of π is a homeomorphism constitutes a basis for the topology of $\mathcal{S}$. [Mal98, Lemma 1.1, p.4]

Definition 3.1.0.3. *Let $(\mathcal{E}, \rho, X)$ and $(\mathcal{S}, \pi, X)$ be two sheaves of sets on X. A morphism of $\mathcal{E}$ into $\mathcal{S}$ is a stalk preserving continuous map $\varphi : \mathcal{E} \to \mathcal{S}$ for which the diagram below is commutative;*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\varphi"] \arrow[dr, "\rho"'] & \mathcal{S} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

which by definition is the relation $\pi \circ \varphi = \rho$ where φ_x is continuous with $\varphi(\mathcal{E}_x) \subseteq \mathcal{S}_x$ for every $x \in X$, such that

$$\varphi_x := \varphi|_{\mathcal{E}_x} : \mathcal{E} \to \mathcal{S}.$$

Additionally,

$$\varphi(z) := \varphi_x(z)$$

where $x = \pi(z)$ for some $z \in \mathcal{E}$.

Suppose $(\mathcal{S}, \pi, X)$ is a sheaf and $U \subseteq X$ an arbitrary open subset. A section s of $\mathcal{S}$ over the open subset U is a continuous map $s : U \to \mathcal{S}$ such that $\pi \circ s = id_U$. Any element $s \in \mathcal{S}$ is a *local* section of $\mathcal{S}$ over U whenever there is a proper containment $U \subset X$ and we denote the set of all local sections of $\mathcal{S}$ over U by

$$\mathcal{S}(U) \equiv \Gamma(U, \mathcal{S}).$$

Similarly, the elements of $\mathcal{S}(X) \equiv \Gamma(X, \mathcal{S})$ are called *global* sections $\mathcal{S}$ over X for the particular case $U = X$.

Next, we now consider the definition equivalent to that of a sheaf given in Definition 3.1.0.1. We move by considering a presheaf of sets on a topological space.

Let X be a topological space. Define a category $\mathfrak{O}(X)$ with open sets as objects on X as follows; for any two objects $V, U \in \mathfrak{O}(X)$, define morphisms by setting

$$\mathrm{Hom}_{\mathfrak{O}(X)}(V, U) = \left\{ i_U^V \right\},$$

a set with exactly one element, the inclusion map $i_V^U : V \to U$ for any pair (V,U) if $V \subseteq U$ and otherwise,

$$\mathrm{Hom}_{\mathfrak{O}(X)}(V,U) = \emptyset.$$

Definition 3.1.0.4. *A presheaf $\mathcal{F}$ of sets of an algebraic structure (resp. groups, rings, modules over a ring,...) on X with values in the category **Set** is a contravariant functor*

$$\mathcal{F} : \mathfrak{O}(X) \to \textbf{\textit{Set}}$$

such that we have the following data:

*F.1 For every open set $U \in \mathfrak{O}(X)$, the functor $\mathcal{F}$ associates it to an object $\mathcal{F}(U)$ in **Set**.*

F.2 For each pair of open sets $V \subseteq U$, we get a restriction map ρ_V^U of U to V viz. $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ which is a morphism such that,

i. $\rho_U^U = id_{\mathcal{F}(U)}$

ii. for any chain inclusion of open sets $V_1 \subset V_2 \subset U$, we have a commutative diagram

$$\begin{tikzcd}
 & \mathcal{F}(V_1) \arrow[d, "\rho_{V_2}^{V_1}"] \\
\mathcal{F}(U) \arrow[ur, "\rho_{V_1}^U"] \arrow[r, "\rho_{V_2}^U"'] & \mathcal{F}(V_2)
\end{tikzcd}$$

which by definition is $\rho_{V_2}^U = \rho_{V_2}^{V_1} \circ \rho_{V_1}^U$.

We shall denote by $\mathcal{PS}h_X$, the category of presheaves of sets on X. The equivalent formulation of Definition 3.1.0.4 can be obtained as an inductive system of sets $(\mathcal{F}(U))_{U \in \mathfrak{O}(X)}$, where morphisms are of type ρ_V^U and $\mathfrak{O}(X)$ is a set preordered by inclusion. See [Mal98, Chapter 1, p.28]

Proposition 3.1.0.5. [Bos13, Proposition 2, §6.4] *Inductive and projective limits exist in the categories of sets, groups, rings, and modules (over a given ring R).*

Definition 3.1.0.6. *A presheaf $\mathcal{F}$ of an algebraic structure (resp. groups, rings , modules over a ring,...) on a topological space X is called a sheaf or a complete presheaf if for every open subset $U \subset X$ and every open covering $\mathcal{U} = (U_\lambda)_{\lambda \in \Lambda}$ by open subsets $U_\lambda \subset U$, the following are satisfied:*

Sh.1 Given any two sections $f, g \in \mathcal{F}(U)$ such that

$$\rho_{U_\lambda}^U(f) \equiv (f|_{U_\lambda}) = f_\lambda = g_\lambda = (g|_{U_\lambda}) \equiv \rho_{U_\lambda}^U(g)$$

for all $\lambda \in \Lambda$. Then $f = g$.

Sh.2 If $f_\lambda \in \mathcal{F}(U_\lambda)$, $\lambda \in \Lambda$ *satisfies*

$$\rho^{U_\lambda}_{U_\lambda \cap U_{\lambda'}}(f_\lambda) \equiv f_\lambda|_{U_\lambda \cap U_{\lambda'}} = f_{\lambda'}|_{U_\lambda \cap U_{\lambda'}} \equiv \rho^{U_{\lambda'}}_{U_\lambda \cap U_{\lambda'}}(f_{\lambda'})$$

for $\lambda, \lambda' \in \Lambda$, *then there exists a section* $f \in \mathcal{F}(U)$ *such that* $f|_{U_\lambda} = f_\lambda$.

Definition 3.1.0.7. *Let* $(\mathcal{F}(U), (\rho)_{U,V})$ *and* $(\mathcal{E}(U), (\beta)_{U,V})$ *be presheaves on* X, *a map* $\alpha : \mathcal{E} \to \mathcal{F}$ *is a set of maps*

$$\alpha(U) : \mathcal{E}(U) \to \mathcal{F}(U)$$

for each open set $U \subset X$, *such that for any* $V \subset U \subset X$, *the diagram below commutes*

$$\begin{array}{ccc} \mathcal{E}(U) & \xrightarrow{\alpha(U)} & \mathcal{F}(U) \\ \big\downarrow \beta^U_V & & \big\downarrow \rho^U_V \\ \mathcal{E}(V) & \xrightarrow{\alpha(V)} & \mathcal{F}(V); \end{array}$$

that is,

$$\rho^U_V \circ \alpha(U) = \alpha(V) \circ \beta^U_V$$

with every $\alpha \equiv \prod_{U \in \mathfrak{O}(X)} Hom(\mathcal{E}, \mathcal{F})$. (c.f. [Mal98, Chapter 1, §6].)

Remark 3.1.0.8. Given a sheaf of algebras $\mathcal{A}$, a vector sheaf is a locally free $\mathcal{A}$-module of finite rank n where $n \in \mathbb{Z}$, with $n > 1$ or rank 1 respectiviely.

Remark 3.1.0.9. [GW10, Proposition 2.7, p.52] Let $\mathcal{F}$ be a presheaf on a topological space X. There exists a pair $(\mathcal{F}^+, \alpha_\mathcal{F})$ where $\mathcal{F}^+$ is a sheaf on X and $\alpha_\mathcal{F} : \mathcal{F} \to \mathcal{F}^+$ is a morphism of presheaves satisfying the following property; if $\mathcal{G}$ is a sheaf on X and $\vartheta : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then there exists a unique morphism of sheaves $\theta : \mathcal{F}^+ \to \mathcal{G}$ such that $\vartheta = \theta \circ \alpha_\mathcal{F}$. The pair $(\mathcal{F}^+, \alpha_\mathcal{F})$ is unique up to isomorphism. Additionally,

a. For all $x \in X$, the morphism

$$\alpha_{\mathcal{F}_x} : \mathcal{F}_x \to \mathcal{F}^+_x$$

is a bijection.

b. For every presheaf $\mathcal{G}$ on X and every morphism of preseheaves $\varphi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ making the diagram below commutative

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\alpha_\mathcal{F}} & \mathcal{F}^+ \\ \big\downarrow \varphi & & \big\downarrow \varphi^+ \\ \mathcal{G} & \xrightarrow{\alpha_\mathcal{G}} & \mathcal{G}^+. \end{array}$$

Particularly, the functor $\mathfrak{S} : \mathcal{PS}h_X \to \mathcal{S}h_X$ with assignment $\mathcal{F} \mapsto \mathcal{F}^+$ is a covariant functor from the category of presheaves of sets on X to the the category of sheaves of sets on X. The sheaf $\mathcal{F}^+$ is called the sheaf associated to $\mathcal{F}$ or a sheafification of $\mathcal{F}$. The sheafification functor is left adjoint to the inclusion functor $\mathfrak{I} : \mathcal{S}h_X \to \mathcal{PS}h_X$ of the category of sheaves into the category of presheaves, that is, there is a bijection

$$Hom_{\mathcal{S}h_X}(\mathfrak{S}(\mathcal{F}), \mathcal{G}) \cong Hom_{\mathcal{PS}h_X}(\mathcal{F}, \mathcal{G}),$$

where $\mathcal{F}$ is a presheaf and $\mathcal{G}$ a sheaf on X. See also [Mal98, Chapter 1,§8, pp.33-35].

3.2 Structure sheaf of rings

Generally, the spectrum $X := \text{Spec}(R)$ of a commutative ring R endowed with a Zariski topology may not contain sufficient information about the structure of R. So, $X = \text{Spec}(R)$ is enlarged by introducing an additional structure on it called its structure sheaf from which the ring R can be recovered .
Let R be a ring, $X = \text{Spec}(R)$ the spectrum of R and denote by $D(X)$ the category of distinguished open subsets $D(f) = \{\mathfrak{p} \in \text{Spec}(R) | f \notin \mathfrak{p}\}$ for $f \in R$ with inclusions as morphisms. A structure presheaf of rings on $X = \text{Spec}(R)$ is a contravariant functor

$$\mathcal{O}_X : D(X) \to \textbf{Ring}$$

with assignments :

a. $D(f) \mapsto R_f$ where R_f is the localization of R at some ring element f of the set $\Omega = \{f^n | n \geq 0\}$.

b. To each inclusion map $D(f) \subset D(g)$, the functor $\mathcal{O}_X$ associates a well defined map $R_g \to R_f$.

By way of [MO15, pp.5-7], the sections of the structure sheaf $\mathcal{O}_X(D(f))$ over distinguished open sets $D(f)$ are defined as localizations of the ring R at the multiplicative set $\Omega = \{f^n | \, n \geq 0\}$ or the ring of fractions $\dfrac{a}{f^n}$, where $a \in R, n \in \mathbb{Z}$. We notice from [MO15, Proposition 1.1.9] that,

$$\text{Spec}(R) = \bigcup_{f \in \Omega} \text{Spec}(R)_f$$

if and only if the unity element belongs to the ideal $\sum_{f \in \Omega} f \cdot R$ generated by Ω, that is, $1 \in \sum_{f \in \Omega} f \cdot R$. Therefore, there are finite sets of elements $f_1, f_2, .., f_n \in \Omega$ and $g_1, ..., g_n \in R$ such that $1 = \sum g_i f_i$. Since every open cover in $\text{Spec}(R)$ admits a finite sub-cover (see [MO15, Corollary 1.1.10]), $D(f) \subset$

$\bigcup_{i=1}^{n} D(g_i)$ and there is an integer $m \geq 1$, and some element $a_i \in R$ such that $f^m = \sum a_i g_i$. In particular, for an inclusion $D(f) \subset D(g)$, we have $f^m = \sum a_i g_i$ and the map $R_g \longrightarrow R_f$ defined by

$$\frac{b}{g^n} \mapsto \frac{b}{\left(\frac{f^m}{a}\right)^n} = \frac{ba^n}{f^{mn}}.$$

Thereupon, the equality $D(f) = D(g)$ gives an identification $D(f) \equiv D(g)$ as $R_g \longrightarrow R_f$ and $R_f \longrightarrow R_g$ will be inverse mappings. Thus, we have the presheaf of rings

$$\mathcal{O}_X(D(f)) := R_f$$

on the base $\{D(f) \big| f \in R\}$ of $X = \operatorname{Spec}(R)$ such that whenever $D(f) \subseteq D(g)$, there is a canonical restriction map $\rho^{D(g)}_{D(f)} : R_g \longrightarrow R_f$. In addition, the chain of inclusions $D(f) \subseteq D(g) \subseteq D(h)$ yields a composition map

$$\rho^{D(h)}_{D(f)} = \rho^{D(h)}_{D(g)} \circ \rho^{D(g)}_{D(f)}.$$

Lemma 3.2.0.1. *Given a ring R and a topological space $X = \operatorname{Spec}(R)$. The presheaf $\mathcal{O}_X$ of rings is a sheaf on the basis $\left\{D(f) \middle| f \in R\right\}$ when we restrict to open coverings $D(f) = \cup_{\lambda \in \Lambda} D(f_\lambda)$.*

Proof. See [Bos13, Proposition 2, p.241] or [GW10, Theorem 2.33, p.58.]. Indeed for any element such that $\frac{b}{f^m} \mapsto 0$ in each R_{g_i}, by construction of $\mathcal{O}_X(D(f))$, there is an element $a \in R$ such that $a = a(\sum b_i g_i) = 0$ since $X = \operatorname{Spec}(R)$ is quasi-compact and $1 = \sum b_i g_i$. Thus, for any section $s \in \mathcal{O}_X(D(f))$ such that $s|_{D(f_i)} = 0$ for all i, we have $s = 0$.
Secondly, for all elements $\frac{b_k}{g_k^{m_k}} \in R_{g_k}$ such that $\frac{b_k}{g_k^{m_k}} = \frac{b_j}{g_j^{m_j}}$ in $R_{g_k} \cdot R_{g_j}$, there is an element $\frac{b}{f^m} \in R_f$ mapping to $\frac{b_k}{g_k^{m_k}} \in R_{g_k}$ for every k. See also [MO15, lemma 1.1.13]. □

Remark 3.2.0.2. If the value of each element of $\mathcal{O}_X(U)$ is known on each $U \subset X$, we can define $\mathcal{O}_X$ as a functor on every open subset V of X. That is, the functor $\mathcal{O}_X$ on $\{D(f)| f \in R\}$ can be extended to a functor $\widetilde{\mathcal{O}}_X$ defined on all $X = \operatorname{Spec}(R)$ by defining its sections as follows; for all open subsets $U \subset X$

$$\widetilde{\mathcal{O}}_X(U) = \varprojlim_{D(f) \subset U} \mathcal{O}_X(D(f)) = \varprojlim_{f \in R,\, D(f) \subset U} R_f,$$

where the projective limit runs over all open subsets $D(f) \subset X$ that are contained in U for all $f \in R$.

By [Bos13, Theorem 3, p.244], the universal property of projective limits confirms $\widetilde{\mathcal{O}}_X$ to be a functor on the category of open sets and it yields a sheaf of rings on X.

Definition 3.2.0.3. *Let* $\mathfrak{D}(\mathfrak{p}) = \left\{D(f) | \mathfrak{p} \in D(f)\right\}$ *be a collection of all open sets containing* $\mathfrak{p} \in \operatorname{Spec}(R)$ *such that* $\mathfrak{D}(\mathfrak{p})$ *is preordered by inclusion on opens* $D(f)$*. The inductive limit*

$$\varinjlim_{D(f)\in\mathfrak{D}(\mathfrak{p})} R_f$$

of the system $\left(R_f, \rho^{D(f)}_{D(f)}\right)_{D(f)\in\mathfrak{D}(\mathfrak{p})}$ *is defined by way of the association*

$$\frac{a}{f^n} \in R_f \mapsto \left(\frac{a}{f^n}, D(f)\right).$$

We first let $\mathcal{R}$ *be a set of pairs* $\left(\frac{a}{f^n}, D(f)\right)$ *such that*

$$\mathcal{R} = \left\{\left(\frac{a}{f^n}, D(f)\right) | \ D(f) \in \mathfrak{D}(\mathfrak{p})\right\}.$$

Then, an equivalence relation $\sim$ *is introduced on* $\mathcal{R}$ *as follows ;*

$$\left(\frac{a}{f^r}, D(f)\right) \sim \left(\frac{a'}{g^t}, D(g)\right)$$

if and only if there is an open set $D(h) \in \mathfrak{D}(\mathfrak{p})$ *such that* $D(h) \subset D(f)$ *and* $D(h) \subset D(g)$ *satisfying*

$$\rho^{D(f)}_{D(h)}\left(\frac{a}{f^r}\right) = \rho^{D(f)}_{D(h)}\left(\frac{a'}{g^t}\right).$$

Thus,

$$\varinjlim_{D(f)\in\mathfrak{D}(\mathfrak{p})} R_f$$

is the quotient $\mathcal{R}/\sim$ *consisting of classes* $\left[\left(\frac{a}{f^n}, D(f)\right)\right]$*. See* [KK99a, pp.58-59].

Proposition 3.2.0.4. *Let* R *be a commutative ring and* $\mathfrak{p}$ *a prime ideal in* $\operatorname{Spec}(R)$*. There is an isomorphism*

$$\varinjlim_{D(f)\ni\mathfrak{p}} R_f \cong R_{\mathfrak{p}}$$

for all $f \in R$*.*

Proof. See [KK99b, Proposition 2.14, p.62]. □

Remark 3.2.0.5. If $\mathcal{O}_X$ is a sheaf on a topological space X. The stalk of the sheaf of rings $\mathcal{O}_X$ at x is the inductive limit of all sections of $\mathcal{O}_X$ taken over all open sets $U \subset X$ containing x and the system of maps ρ^U_V for $V \subset U$;

$$\mathcal{O}_{X,x} = \varinjlim_{U\ni x} \mathcal{O}_X(U) = \varinjlim_{D(f)\ni x} \mathcal{O}_X(D(f)) = \varinjlim_{f\in\Omega} R_f = R_{\mathfrak{p}}$$

where $f \in \Omega = R - \mathfrak{p}$ and $R_{\mathfrak{p}}$ is the localization of the ring R at a point $\mathfrak{p} \in X = \operatorname{Spec}(R)$. By Remark 1.2.1.6, the stalk $\mathscr{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{p}_x R_{\mathfrak{p}_x}$ and there is a natural map

$$R = \mathscr{O}_X(X) \longrightarrow \mathscr{O}_{X,x} \longrightarrow \mathfrak{p}_x/\mathfrak{p}_x R_{\mathfrak{p}_x}$$

where $\mathfrak{p}_x/\mathfrak{p}_x R_{\mathfrak{p}_x}$ is the quoitent field of $R/\mathfrak{p}$.
Also, the stalk at the point $x \in X$ is the set of pairs

$$\mathscr{O}_{X,x} = \Big\{(s,U)\Big| x \in U, s \in \mathscr{O}_X(U)\Big\},$$

where U is an open neighbourhood of x factored over an equivalence relation where $(s_1,U_1) \equiv (s_2,U_2)$ if there exists an open set $U_3 \subseteq U_1 \cap U_2$ such that $x \in U_3$ and $s_1|_{U_3}(x) = s_2|_{U_3}(x)$. For this reason, for every $x \in U$, there is a natural map

$$\mathscr{O}_X(U) \longrightarrow \mathscr{O}_{X,x}$$

sending a section $s \in \mathscr{O}_X(U)$ to the equivalence class of (s,U) denoted $s_x \equiv s(x)$ called the germ at x. Ultimately, each section of a sheaf is completely determined by its germs, that is, the natural map

$$\mathscr{O}_X(U) \longrightarrow \prod_{x\in U} \mathscr{O}_{X,x}$$

is injective.

Definition 3.2.0.6. *A pair $(X,\mathscr{O}_X)$ of a topological space X and a sheaf of rings $\mathscr{O}_X$ on it is called a ringed space.*

A morphism of ringed spaces $(X,\mathscr{O}_X)$ and $(Y,\mathscr{O}_Y)$ is a pair $(f,f^\sharp)$ where $f : X \longrightarrow Y$ is a continuous map and $f^\sharp : \mathscr{O}_Y \longrightarrow f_*(\mathscr{O}_X)$ a morphism of sheaves of rings on Y. Here $f_*(O_X)$ is a sheaf on Y given by

$$V \longmapsto \mathscr{O}_X(f^{-1}(V)) \equiv f_*(\mathscr{O}_X)$$

and canonical restriction maps. Thus, $f^\sharp$ consists of a family of ring homomorphisms

$$f : \mathscr{O}_Y(V) \longrightarrow \mathscr{O}_X(f^{-1}(V))$$

where V is an open subset of Y compatible with restriction morphisms; so, the diagram

$$\begin{tikzcd}
\mathscr{O}_Y(V) \arrow[r] \arrow[d, "\rho^V_W"] & \mathscr{O}_X(f^{-1}(V)) \arrow[d, "\rho^{f^{-1}(V)}_{f^{-1}(W)}"] \\
\mathscr{O}_Y(W) \arrow[r] & \mathscr{O}_X(f^{-1}(W))
\end{tikzcd}$$

is commutative for open subsets $U, W \subset Y$ such that $W \subseteq V$. The pair $(f, f^\sharp)$ is an isomorphism if f is a homeomorphism and each map $f_V^\sharp : \mathcal{O}_Y(V) \to f_*(\mathcal{O}_X)(V)$ is a ring isomorphism. This morphism of ringed spaces in a natural way induces a ring homomorphism

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

for all $x \in X$ and open subset V in Y such that $f(x) \in V$. This gives way to a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r] \arrow[d, "\rho_y^V"] & \mathcal{O}_X(f^{-1}(V)) \arrow[d, "\rho_x^{f^{-1}(V)}"] \\
\mathcal{O}_{Y,f(x)} \arrow[r] & \mathcal{O}_{X,x}.
\end{tikzcd}$$

The composition

$$\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_{X,x}$$

is compatible with restrictions on $\mathcal{O}_Y$ as

$$\mathcal{O}_{Y,f(x)} \equiv \varinjlim_{V \subseteq Y} \mathcal{O}_Y(V) \to \mathcal{O}_{X,x},$$

(see [Bos13, p.248, Remark 7]). As observed earlier, every stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{p}_x R_{\mathfrak{p}_x}$. In this case, the associated ringed space $(X, \mathcal{O}_X)$ is a locally ringed space.

Definition 3.2.0.7. *An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that there is an isomorphism of ringed spaces $(X, \mathcal{O}_X) \simeq (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for some ring R. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open covering $(X_i)_{i \in I}$ of X for which each pair $(X_i, \mathcal{O}_X|_{X_i})$ is an affine scheme for all $i \in I$ where $\mathcal{O}_X|_{X_i}$ is the restriction of the sheaf $\mathcal{O}_X$ to the open subset $X_i \subset X$.*

When there is no confusion of terminology, we shall write X to refer to the scheme $(X, \mathcal{O}_X)$ and a morphism of schemes is just a morphism of locally ringed spaces.

3.3 Quasi-coherent $\mathcal{O}_X$-modules

In this section, we consider basic properties of quasi-coherent and coherent sheaves. We move by looking at definitions and results on sheaves of modules over ringed spaces.

Definition 3.3.0.1. *An $\mathcal{O}_X$-module $\widetilde{M}$ is an additive sheaf of abelian groups on X together with a law of composition. That is, on $\widetilde{M}$, we have two morphisms of sheaves defined as follows; addition,*

$$\sigma : \widetilde{M}(U) \times \widetilde{M}(U) \to \widetilde{M}(U),$$

$(s,s') \mapsto s+s'$ for an open subset $U \subset X$ with sections $s,s' \in \widetilde{M}(U)$ and multiplication

$$\mu : \mathscr{O}_X(U) \times \widetilde{M}(U) \longrightarrow \widetilde{M}(U)$$

such that, for any open subset $U \subset X$, the map $\mu(U) : \mathscr{O}_X(U) \times \widetilde{M}(U) \longrightarrow \widetilde{M}(U)$ defines an $\mathscr{O}_X(U)$-module structure on $\widetilde{M}(U)$ by the action $(a,s) \mapsto as$ where $a \in \mathscr{O}_X(U), s \in \widetilde{M}(U)$.

The construction of the sheaf of rings $\mathscr{O}_X$ in Section 3.2 can be extended to construct a sheaf of $\mathscr{O}_X$-modules $\widetilde{M}$ on an affine scheme $X = \operatorname{Spec}(R)$ associated to every R-module M. In this instant, to every open set $D(f)$, we assign a localization M_f of M at a multiplicative set Ω. So

$$\widetilde{M}(D(f)) := M \otimes_R R_f = M_f$$

yields an $\mathscr{O}_X(D(f)) = R_f$-module structure on M_f. For open subsets $D(f) \subset D(g) \subset X$, we get a restriction map $\rho^{D(g)}_{D(f)} : M_g \longrightarrow M_f$. Following the result in [Bos13, Theorem 5, p.247], we can extend the map $D(f) \mapsto M_f$ to a sheaf of groups defined by $U \mapsto \widetilde{M}(U)$ for every open set U containing $D(f)$. With this construction, we can recover M from $\widetilde{M}$ by setting

$$\widetilde{M}(U) := \varprojlim M_f$$

where the limit runs over all $f \in R$ such that $D(f) \subset U$. Then $\widetilde{M}$ is a module over $\mathscr{O}_X(U)$. Any inclusion $U \subset V$ defines a homomorphism $\rho^U_V : \widetilde{M}(U) \longrightarrow \widetilde{N}(U)$ mimicking the case of the structure sheaf $\mathscr{O}_X$ for $M = R$. Therefore, the system $\left(\widetilde{M}(U),\, \rho^U_V\right)$ defines a sheaf of modules $\widetilde{M}$ over $\mathscr{O}_X$. A homomorphism $\varphi : M \longrightarrow N$ of R-modules determines a homomorphism $\varphi_f : M_f \longrightarrow N_f$ for every $f \in R$, and on passing to the limits, we obtain a homomorphism of sheaves $\widetilde{\varphi} : \widetilde{M} \longrightarrow \widetilde{N}$.
If $\varphi : M \longrightarrow N$ and $\psi : N \longrightarrow Q$ are two homomorphisms, then $\widetilde{\varphi \circ \psi} = \widetilde{\varphi} \circ \widetilde{\psi}$. This enables us to recover M from $\widetilde{M}$ and we get $\widetilde{M}(X) = M$. See [Sha13, Chapter 6,§3, pp.85-86].
The stalk at a point $x \in X = \operatorname{Spec}(R)$ is an inductive limit

$$\widetilde{M}_x = \varinjlim_{\mathfrak{p} \in D(f)} M_f(D(f) := M_{\mathfrak{p}_x} = M \otimes_R R\mathfrak{p}$$

where $M_{\mathfrak{p}_x}$ denotes the localization of M at the prime ideal $\mathfrak{p}_x$ and the set $D(f)$ runs through the open neighborhoods of $\mathfrak{p} \in X = \operatorname{Spec}(R)$.

As seen in Section 3.2, for open sets $V \subset U \subset X$, the restriction maps satisfy the commutative diagram

$$\begin{tikzcd}
\mathscr{O}_X(U) \times \widetilde{M}(U) \arrow[r] \arrow[d] & \widetilde{M}(U) \arrow[d] \\
\mathscr{O}_X(V) \times \widetilde{M}(V) \arrow[r] & \widetilde{M}(V),
\end{tikzcd}$$

thereby rendering $\widetilde{M}$ a presheaf of $\mathcal{O}_X$-modules that satisfies the sheaf properties. See [Bos13, Theorem 5, p.247]. For any R-linear map $\varphi : M \to N$, and $f \in R$, we have an R_f-linear map $\varphi_f : M_f \to N_f$ such that for an inclusion of open sets $D(f) \subset D(f)$, the induced diagram

$$\begin{array}{ccc} M_f & \longrightarrow & N_f \\ \downarrow & & \downarrow \\ M_g & \longrightarrow & N_g, \end{array}$$

is commutative. Additionally, for $\mathfrak{p} \in \mathrm{Spec}(R)$, the R_f-linear map induces a map on stalks $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$. Consequently, for all $U \subset X$, the induced map

$$\widetilde{\varphi}_U : \widetilde{M}(U) \to \widetilde{N}(U)$$

can be extended to a homomorphism

$$\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$$

of sheaves of $\mathcal{O}_X$-modules. In this way, we obtain a functor $M \mapsto \widetilde{M}$ from the category of R-modules to the category of $\mathcal{O}_X$ -modules for any given R-linear map.
Conversely, for the inverse, an $\mathcal{O}_X$-linear map $\psi : \widetilde{\mathcal{M}} \to \mathcal{G}$ induces an R-linear map on global sections of modules over $R = \mathcal{O}_X(X)$ given by

$$\psi(X) : \mathcal{M} \to \mathcal{G}(X).$$

From the diagram below,

$$\begin{array}{ccc} M & \xrightarrow{\psi(X)} & \mathcal{G}(X) \\ \downarrow & & \downarrow \\ M_f & \xrightarrow{\psi(D(f))} & \mathcal{G}(D(f)), \end{array}$$

the map $\psi(X)$ is a right inverse to ψ. Furthermore, since

$$\psi(X)|_{(D(f))} = \psi(D(f))$$

for any $f \in R$, the R_f-morphism $\psi(D(f))$ is uniquely determined by $\psi(X)$. Thereupon, we get a functor

$$\mathcal{M} \mapsto \widetilde{\mathcal{M}}(X)$$

between the category of $\mathcal{O}_X$-modules and the category of R-modules.

Theorem 3.3.0.2. *Let R be a commutative ring, M and N be R-modules and $X = \mathrm{Spec}(R)$ the spectrum of R. Then, the two maps*

$$\mathrm{Hom}_R(M,N) \leftarrow \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$$

and

$$\mathrm{Hom}_R(M,N) \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$$

are mutually inverse. Particularly, the functor $M \mapsto \widetilde{M}$ is exact, fully faithful and there is an equivalence of categories of R-modules to that of $\mathcal{O}_X$-modules of the form $\widetilde{M}$.

Proof. See [KK99b, Proposition 4.20, p.25] and [Har77, Proposition 5.2, p.110]. □

The result in Theorem 3.1.1.1 enables us to translate much of the theory of R-modules into the theory of sheaves on $\mathrm{Spec}(R)$, and brings various geometric ideas into the theory of R-modules.

Definition 3.3.0.3. *Let $\mathcal{F}$ be an $\mathcal{O}_X$ module over a scheme X.*

i. If for each point $x \in X$, there is an open neighbourhood U of x so that the sequence of $\mathcal{O}_U$-modules

$$\mathcal{O}_X^{\oplus J}|_U \rightarrow \mathcal{O}_X^{\oplus I}|_U \rightarrow \mathcal{F}|_U \rightarrow 0$$

is exact and $\mathcal{F}|_U \simeq \widetilde{M}$ for some open affine set $U = D(f) \subset X$. Note that I and J need not be finite sets.

ii. For every open affines $V \subset U$, the mapping $\mathcal{F}(U) \otimes_{\mathcal{O}_X} \mathcal{O}_X(X) \rightarrow \mathcal{F}(X)$ is an isomorphism.

Then, the $\mathcal{O}_X(U)$-module $\mathcal{F}$ is said to be a quasi-coherent or a sheaf of $\mathcal{O}_X$-modules.

If for each point $x \in X$, there is an open neighbourhood U of x so that the sequence of $\mathcal{O}_U$-modules

$$\mathcal{O}_X^{\oplus n}|_U \rightarrow \mathcal{F}|_U \rightarrow 0$$

is exact, then $\mathcal{F}$ is said to be a finitely generated $\mathcal{O}_X$-module.

Example 3.3.0.4. a. The structure sheaf of rings $\mathcal{O}_X$ is finitely generated quasi-coherent $\mathcal{O}_X$-module since for every affine open set $U = \mathrm{Spec}(R_f)$, $\mathcal{O}_X|_U \cong \tilde{R}$. cf.[Har77, Example.5.2.1, p.111]

b. The dual $\mathcal{F}^\vee$ of a quasi-coherent sheaf $\mathcal{F}$ is also quasi-coherent. Additionally,

$$\widetilde{(M^\vee)} \cong (\widetilde{M})^\vee$$

where M is some R-module such that $\widetilde{M} \simeq \mathcal{F}|_{D(f)}$. Indeed, by application of Theorem 3.3.0.2, we have

$$\begin{aligned}
\mathrm{Hom}_R\, \widetilde{(M,R)}(D(f)) = & \mathrm{Hom}_R\,(M,R) \otimes_R R_f \simeq \mathrm{Hom}_{R_f}(M_f, R_f),\\
& = \mathrm{Hom}_{\mathcal{O}_{D(f)}}(\widetilde{M_f}|_{D(f)}, \widetilde{R_f}|_{D(f)})\\
& = \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{R})\\
& = (\widetilde{M})^\vee.
\end{aligned}$$

For properties of Quasi-coherent $\mathcal{O}_X$-modules , consider the theorem below;

Theorem 3.3.0.5. *Let $(X, \mathcal{O}_X)$ be an affine scheme determined by a commutative ring R.*

a. The $\mathcal{O}_X$-modules $\widetilde{M}$ induced by an R-module M is quasi-coherent, and for any open set $D(f)$ of X,

$$\widetilde{M}(D(f)) = M_f,$$

and in particular

$$\widetilde{M}(X) = M.$$

b. For an R-module homomorphism $\varphi : M \longrightarrow N$, the map

$$\Phi : \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$$

assigning an $\mathcal{O}_X$-module homomorphism $\tilde{\varphi}$ is an isomorphism of R-modules.

c. For R-modules M and N, we have isomorphisms of $\mathcal{O}_X$- modules

$$\widetilde{M} \oplus \widetilde{N} \simeq (M \oplus N)\widetilde{\ }$$

and

$$\widetilde{M} \otimes \widetilde{N} \simeq (M \otimes N)\widetilde{\ }.$$

Furthermore, if M is a finitely presented R-module, then there is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \simeq \mathrm{Hom}_R(M,N)\widetilde{\ }$$

Proof. See [KK99b, Proposition 4.20, p.25]. □

Example 3.3.0.6. Every $\mathcal{O}_X$-module is quasi-coherent for an affine scheme $X = \mathrm{Spec}(R)$. Indeed, every R-module can be recovered from an $\mathcal{O}_X$-module $\widetilde{M}$ and every R-linear map $\varphi : M \to N$ induces an $\mathcal{O}_X$-linear map $\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$. Upon considering the stalks, the map $\varphi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is a localized $R_\mathfrak{p}$-linear map. As localization preserves exactness, a short exact sequence of R-modules

$$M \to M_1 \to M_2$$

will induce a short exact sequence of $\mathcal{O}_X$-modules

$$\widehat{M} \to \widetilde{M_1} \to \widetilde{M_2}.$$

Thus, the $\mathcal{O}_X$-module $\widetilde{M}$ determining an R-module M is quasi-coherent.

Definition 3.3.0.7. *An $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is locally of finite type if every point $x \in X$ admits an open neighborhood $U \subset X$ together with an exact sequence of type*

$$(\mathcal{O}_X|_U)^{(J)} \to \mathcal{F}|_U \to 0$$

where I is finite. Additionally, if $\mathcal{F}$ is an $\mathcal{O}_X$-module is such that for every open set $U \subset X$ open and $\mathcal{O}_X$-module morphism $\phi : (\mathcal{O}_X|_U)^{(J)} \to \mathcal{F}|_U$ with the kernel $\mathrm{Ker}\,\phi$ *locally of finite type, then every such $\mathcal{F}$ where J is finite is said to be a coherent $\mathcal{O}_X(U)$-module .*

Definition 3.3.0.8. *An $\mathcal{O}_X$-module $\mathcal{E}$ is said to be a locally free $\mathcal{O}_X$-module of rank n if there exists an open covering $(U_j)_{(j\in J)}$ of X such that the restriction $\mathcal{E}|_{U_J}$ of $\mathcal{E}$ to U_j is a free module of rank n over $\mathcal{O}_{U_j} = \mathcal{O}_X|_{U_j}$. A locally free $\mathcal{O}_X$-module of rank n is also called a locally free sheaf of rank n or a Local gauge of $\mathcal{E}$.*

3.4 Involutions on Azumaya algebras over schemes

In this section, we discuss involutions on Azumaya algebras over schemes which are underpinned by generalizing the result in the theorem in [KMRT98, Chapter 1, p.1] to classical Azumaya R-algebras. On considering the framework of Azumaya algebras over schemes, Theorem 2.2.1.6 can be restated as follows:

Theorem 3.4.0.1. *Let R be a local ring and A an Azumaya R-algebra of finite rank. The map that sends each nonsingular bilinear form $b : A \times A \to R$ onto its adjoint anti-automorphism*

$$\widetilde{\sigma_b} : \widetilde{\mathrm{End}_R(A)} \to \widetilde{\mathrm{End}_R(A)}$$

is a bijection. Moreover, the $\mathcal{O}_X$-linear involutions of $\mathrm{End}_{\mathcal{O}_X}(\widetilde{\mathrm{End}_R(A)})$ correspond to nonsingular bilinear forms which are either symmetric or skew-symmetric.

Proof. The existence and fact that $\widetilde{\sigma_b}$ is a sheaf anti-automorphism evidently derives from the bijective map

$$\mathrm{H}om_R(\mathrm{E}nd_R(A),\mathrm{E}nd_R(A)) \to \mathrm{H}om_{\mathscr{O}_X}(\widetilde{\mathrm{E}nd_R(A)},\widetilde{\mathrm{E}nd_R(A)}), \quad \sigma \to \widetilde{\sigma}$$

in Theorem 3.3.0.2. □

Now, let A be an R-algebra not necessarily Azumaya, and $X = \mathrm{Spec}(R)$ an affine scheme. We consider *involutions* on $\mathscr{O}_X$-algebras $\mathscr{F}$ associated with A classically denoted by $\widetilde{A}$. An involution on an $\mathscr{O}_X$-algebra $\mathscr{F}$ is an $\mathscr{O}_X$-anti-automorphism of order 2, that is an $\mathscr{O}_X$-endomorphism of $\mathscr{F}$ such that, for any given sections s, t of $\mathscr{F}$ over some open subset U of X, $\sigma(st) = \sigma(t)\sigma(s)$ and $\sigma^2 = \mathrm{id}$.

Definition 3.4.0.2. *Let R be a ring, A an R-algebra such that the canonical morphism $R \longrightarrow A$ is injective, $X = Spec(R)$ an affine scheme, and $\mathscr{F}|_U \simeq \widetilde{A}$ the $\mathscr{O}_X$-algebra associated with A. An involution σ of $\mathscr{F}$ is a standard $\mathscr{O}_X$-involution provided that, for every open U in X, $\sigma(U)$ is a standard involution, that is, the morphism $\mathscr{O}_X(U) \longrightarrow \mathscr{F}(U)$ is injective, and $a\sigma(U)(a) \equiv a\sigma(a) \in \mathscr{O}_X(U)$ for all $a \in \mathscr{F}(U)$; the scalar $a\sigma(a)$ is the norm of a denoted by $\mathscr{N}(U)(a) \equiv \mathscr{N}(a)$. The trace of $a \in \mathscr{F}(U)$ is the element $a + \sigma(a) = \mathscr{N}(a+1) - \mathscr{N}(a) - 1 \in \mathscr{O}_X(U)$ usually denoted by $tr_U(a) \equiv tr(a)$.*

When the case at hand is clear, we shall write σ for any component $\sigma(U)$ of a sheaf morphism σ, so that the condition $a\sigma(U)(a) \in \mathscr{O}_X(U)$ of Definition 3.4.0.2 becomes $a\sigma(a) \in \mathscr{O}_X$.

We notice from Theorem 3.3.0.2 that the mapping $\mathrm{H}om_R(A,A) \to \mathrm{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{F})$, sending any endomorphism φ of A onto its corresponding endomorphism $\widetilde{\varphi}$ of $\mathscr{F}$, where for any $f \in R$, $\widetilde{\varphi}(D(f)) = \varphi \otimes 1_{R_f}$, is a bijection. On the strength of this bijection, it goes without mentioning that, *an endomorphism $\sigma : A \longrightarrow A$ is an involution if and only if its image $\widetilde{\sigma}$ is an involution of $\mathscr{F}$.* Indeed, let $\frac{a}{f^m}$, $\frac{b}{f^n} \in A_f$; Evidently, we have that $\widetilde{\sigma}\left(\frac{a}{f^m}\frac{b}{f^n}\right) = \frac{\sigma(ab)}{f^m f^n} = \widetilde{\sigma}\left(\frac{b}{f^n}\right)\widetilde{\sigma}\left(\frac{a}{f^m}\right)$. Further to this, since the correspondence $A \mapsto \widetilde{A}$ yields an exact fully faithful functor from the category of R-modules to the category of $\mathscr{O}_X$-modules (c.f. Theorem 3.3.0.2), it follows that $\widetilde{\sigma}^2 = 1$.
For any open set U in X, the U-th component of $\widetilde{\sigma}$ can be recovered from σ by setting

$$\widetilde{\sigma}(U) = \varprojlim_{D(f)\subseteq U} \widetilde{\sigma}(D(f)) = \varprojlim_{f\in R \text{ with } D(f)\subseteq U} \sigma_f,$$

which is an involution on $\mathscr{F}(U)$. In view of this bijective correspondence, we shall time and again identify σ with $\widetilde{\sigma}$ whenever there is no confusion.

We now make a note that, the natural morphism $\iota : R \longrightarrow A$ gives rise to the sheaf morphism $\widetilde{\iota} : \mathscr{O}_X \longrightarrow \mathscr{F}$, where, if $\iota_f : R_f \longrightarrow A_f$ denotes the localization of ι at $f \in R$, then, for any open set U in X,

$$\widetilde{\iota}(U) = \varprojlim_{D(f)\subseteq U} \widetilde{\iota}(D(f)) = \varprojlim_{f\in R \text{ with } D(f)\subseteq U} \iota_f.$$

By [Bos13, Definition 4, p.226], it is seen from the universal property of projective limits that $\widetilde{\iota}$ is a functor on the category of open subsets of X. Now, since injectiveness (of morphisms of modules) is a local property, and since, for all $x \in X$, $\mathcal{O}_{X,x} \simeq R_x$ (c.f. [Bos13, Proposition 9, p.248]) and

$$\mathcal{F}_x = \varinjlim_{D(f)\ni x} \mathcal{F}(D(f)) = A \otimes_R \varinjlim_{D(f)\ni x} R_f = A \otimes_R R_x = A_x,$$

it follows that the natural morphism $\iota : R \to A$ is injective if and only if the induced morphism $\widetilde{\iota} : \mathcal{O}_X \to \mathcal{F}$ is injective; thus, we have the following:

Lemma 3.4.0.3. *Let R be a ring, $X = \mathrm{Spec}(R)$ an affine scheme, and A an R-algebra. Then, an endomorphism σ of A is a standard involution if and only if $\widetilde{\sigma}$ is a standard involution of $\mathcal{F} = \widetilde{A}$.*

Proof. Let U be an open subset of X; as a projective limit of the projective system $\{\mathcal{F}(D(f)) : f \in R \text{ and } D(f) \subseteq U\}$, $\mathcal{F}(U)$ is contained in $\prod_{D(f)\subseteq U} \mathcal{F}(D(f))$ (c.f. [Bos13, Proposition 5, p.227]). Moreover, for $x \in \mathcal{F}(U)$, we have $\rho_f^U(x) = \dfrac{a}{f^n} \in A_f$, where $\rho_f^U : \mathcal{F}(U) \to \mathcal{F}(D(f))$ is a restriction map for the $\mathcal{O}_X$-module $\mathcal{F}$; at the same time, $\rho_f^U = \mathrm{pr}_f$, where pr_f is the natural projection of $\prod_{D(f)\subseteq U} \mathcal{F}(D(f))$ onto $\mathcal{F}(D(f))$. In fact , $\rho_f^U(\widetilde{\sigma}(x)) = \sigma_f(\dfrac{a}{f^n})$; consequently, $\rho_f^U(x\widetilde{\sigma}(x)) = \frac{a}{f^n}\sigma_f(\dfrac{a}{f^n}) \in R_f = \mathcal{O}_X(D(f))$. It follows that $x\widetilde{\sigma}(x) \in \mathcal{O}_X(U)$ for all $x \in \mathcal{F}(U)$; plainly put, $\widetilde{\sigma}$ is a standard involution of $\mathcal{F}$ whenever σ is a standard involution of A. The converse will follow from the reverse argument. □

Remark 3.4.0.4. For algebras that are faithful and finitely generated projective R-modules, the converse of Lemma 3.4.0.3 holds true. Recall that the faithfulness of A as a module is equivalent to the injectiveness of the natural morphism $R \to A$, which, in turn, as seen in Lemma 3.4.0.3, is equivalent to the induced morphism $\mathcal{O}_X \to \mathcal{F}$ being injective.

The result can be stated in the following lemma.

Corollary 3.4.0.5. *Let R be a ring, $X = \mathrm{Spec}(R)$ an affine scheme, A an R-algebra whose underlying R-module is faithful, finitely generated, and projective; and let σ be an anti-automorphism of A such that $x\sigma(x) \in R$ for all $x \in A$. Then, σ induces an involution $\widetilde{\sigma}$ on the $\mathcal{O}_X$-algebra $\mathcal{F}$ associated with A; it is, in addition, the only standard involution of $\mathcal{F}$. Moreover, $\widetilde{\sigma}$ commutes with all automorphisms and anti-automorphisms of $\mathcal{F}$.*

Proof. By [HM08, Lemma 1.13.8, p.40], σ turns out to be the only standard involution of A, and commutes with all automorphisms and anti-automorphisms of A. Moreover, by Lemma 3.4.0.3, $\widetilde{\sigma}$ is the only standard involution of $\mathcal{F}$; since the mapping $\sim$ is a functor, $\widetilde{\sigma}$ commutes with all automorphisms and anti-automorphisms of $\mathcal{F}$.

We make a note that, for all $f \in R$, the localization $\sigma_f : A_f \longrightarrow A_{\sigma(f)} = A_f$ is an involution such that, for every $x \in A_f$, $x\sigma_f(x) \in R_f$. Now consider, for all $f \in R$, the following diagram

$$\begin{tikzcd} & A \arrow[d, "q"] & \\ & A_f \arrow[d, "\varphi"] & \\ M \arrow[r, "\psi"] & M'' \arrow[r] & 0, \end{tikzcd}$$

where M, M'' are R_f-modules, φ a R_f-morphism, and q the natural R-morphism. Since M, M'' can also be viewed as R-modules, and the underlying R-morphism of ψ is surjective, it follows that, as A is projective, there is a morphism $\lambda : A \longrightarrow M$ such that $\varphi \circ q = \psi \circ \lambda$. Plainly, there is a unique R_f-morphism $\vartheta : A_f \longrightarrow M$ such that $\lambda = \vartheta \circ q$, therefore $\varphi = \psi \circ \vartheta$, and, hence, A_f is projective. It is also evident from localization that, for all $f \in R$, A_f is faithful and finitely generated. By applying [HM08, Lemma 1.13.8, p.40] again, σ_f commutes with all automorphisms and anti-automorphisms of A_f. Now, let us consider any open set U in X; since the distinguished sets $D(f)$, $f \in R$, form a basis for the Zariski topology on X, $U = \cup_{f\in T} D(f)$, for some set T. Furthermore, by virtue of the isomorphism $\mathcal{H}om_R(A, A) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ (c.f. Theorem 3.3.0.2), let $\widetilde{\sigma}$ be the endomorphism of $\mathcal{F}$ corresponding to the involution σ; one observes that, given any open set U in X, then, for any basic open subset $D(f) \subseteq U$, it follows that $\rho^U_f \circ \widetilde{\sigma}(U) = \sigma_f \circ \rho^U_f$, where $\rho^U_f : \mathcal{F}(U) \longrightarrow A_f$ is the corresponding restriction map. But since $\mathcal{F}(U) = \varprojlim_{D(f)\subseteq U} \mathcal{F}(D(f)) = \varprojlim_{D(f)\subseteq U} A_f$, $\widetilde{\sigma}(U) = \varprojlim_{D(f)\subseteq U} \sigma_f$, and, for every $f \in R$, ρ^U_f is a projection on A_f, every such σ_f is an involution, therefore $\widetilde{\sigma}(U)$ is an involution. Finally, let $s \in \mathcal{F}(U)$, so $s|_{D(f)} \in \mathcal{F}(D(f)) = A_f$, for any $D(f) \subseteq U$. But $s|_{D(f)}\sigma_f(s|_{D(f)}) \in R_f = \mathcal{O}_X(D(f))$ and $s|_{D(ff')}\sigma_{ff'}(s|_{D(ff')}) \in \mathcal{O}_X(D(ff'))$; since $\mathcal{O}_X$ is a sheaf, it follows that $s\widetilde{\sigma}(U)(s) \equiv s\widetilde{\sigma}(s) \in \mathcal{O}_X(U)$ for all $s \in \mathcal{F}(U)$, as required. □

Corollary 3.4.0.6. *Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-algebra such that if $\mathcal{U} := (U_i)_{i\in I}$ is a covering of X by open affine subsets $U_i = \operatorname{Spec}(R_i)$, then, for each i, the restriction $\mathcal{F}|_{U_i}$ is associated with some faithful finitely generated projective R_i-algebra A_i. Moreover, let σ_i, $i \in I$, be an anti-automorphism of A_i such that $x\sigma_i(x) \in R_i$, for all $x \in A_i$. Then, $\mathcal{F}$ admits exactly one standard involution $\widetilde{\sigma}$; in addition, $\widetilde{\sigma}$ commutes with all automorphisms and anti-automorphisms of $\mathcal{F}$.*

Proof. According to Corollary 3.4.0.5, let $\widetilde{\sigma_i}$ be the only standard involution of the $\mathcal{O}|_{U_i}$-algebra $\mathcal{F}|_{U_i}$, where, by hypothesis, $\mathcal{F}|_{U_i}$ is the $\mathcal{O}|_{U_i}$-algebra associated with the faithful finitely generated projective R_i-algebra A_i. The morphism $\widetilde{\sigma} : \mathcal{F} \longrightarrow \mathcal{F}$ is such that $\widetilde{\sigma}|_{U_i} = \widetilde{\sigma_i}$ is well defined. Indeed, for all i, j such that $U_i \cap U_j \neq \emptyset$, $\widetilde{\sigma_i}|_{U_i\cap U_j} = \widetilde{\sigma_j}|_{U_i\cap U_j}$. Evidently, $\widetilde{\sigma}$ is the only standard involution on $\mathcal{F}$, and it commutes with all automorphisms and anti-automorphisms of $\mathcal{F}$. □

Corollary 3.4.0.7. *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{I}$ be an $\mathcal{O}_X$-ideal generated by nowhere-zero global sections $(f_1, \dots, f_n)$. The direct product $\mathcal{L} = \prod_{i=1}^n \mathcal{O}_{X,f_i}$ of the sheaves of rings of fractions $\mathcal{O}_{X,f_i}$ is faithfully flat if and only if $\mathcal{I} = \mathcal{O}_X$. Whenever $\mathcal{I} = \mathcal{O}_X$, the sheaf of rings $\mathcal{L}$ is called a* Zariski extension of $\mathcal{O}_X$.

Proof. For all $i = 1, \dots, n$, the ring sheaf extension $\mathcal{O}_X \longrightarrow \mathcal{O}_{X,f_i}$ is flat; therefore $\mathcal{L}$ is flat. Now, suppose that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} = 0$ for some $\mathcal{O}_X$-module $\mathcal{E}$; since $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} = 0$ if and only if $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E})_x = \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{E}_x = 0_x = 0$, for all $x \in X$, it is sufficient to show that $\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{E}_x = 0$ implies $\mathcal{E}_x = 0$. But then $\mathcal{I}_x$ is the ideal of $\mathcal{O}_{X,x}$ generated by germs $(f_{1,x}, \dots, f_{n,x})$, $\mathcal{L}_x = \left(\prod_{i=1}^n \mathcal{O}_{X,f_i}\right)_x = \prod_{i=1}^n \left(\mathcal{O}_{X,x}\right)_{f_{i,x}}$ is faithfully flat if and only if $\mathcal{I}_x = \mathcal{O}_{X,x}$ (see [HM08, Corollary 1.10.6, p. 24]). □

Note that the notation $f_{i,x}$ in the above proof means the germ defined by the section f_i at the point $x \in X$. On the other hand, $\mathcal{O}_{X,f_i}$ is the sheaf obtained by sheafifying the presheaf, given by the assignment

$$U \mapsto \mathcal{O}_{X,f_i}(U),$$

where, for any open subset U of X,

$$\mathcal{O}_{X,f_i}(U) \equiv \mathcal{O}_X(U)_{f_i} = \left\{ \frac{s}{\rho_U^X(f_i)^n} = \frac{s}{(f_i|_U)^n} = \frac{s}{f_i^n|_U};\ s \in \mathcal{O}_X(U),\ n \geq 0 \right\}.$$

In [RG71, 3.1, 2nd part], it is shown that a quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ is locally projective if and only if, for all open affine subschemes $U = \operatorname{Spec}(R) \subseteq X$, the restriction $\mathcal{H}|_U$ is isomorphic to some associated sheaf $\widetilde{P}$, where P is a projective R-module. We remark that a *locally projective quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ will be of constant rank n if, for any open affine subscheme U of X, the associated R-module P of $\mathcal{H}|_U$ is of constant rank n.*

Also, supplement to the proof of Theorem 3.4.0.8 below is a result concerned with glueing of sheaves. Indeed, given a topological space X, an open covering $(U_i)_{i\in I}$ of X and, a sheaf $\mathcal{F}_i$ on U_i for each i such that, for each $i, j \in I$, there is given an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$ satisfying properties: (1) $\varphi_{ii} = \mathrm{id}$, for all i, and (2) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$, for all $i, j, k \in I$. Then, there is a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ such that, for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. See [Har77, p.69].

Theorem 3.4.0.8. *Let X be a scheme and $\mathcal{E}$ a locally projective quasi-coherent $\mathcal{O}_X$-module of constant rank 2. Then, $\mathcal{E}$ is a commutative $\mathcal{O}_X$-algebra, endowed with a unique standard involution.*

Proof. Let $(U_i)_{i\in I} \equiv (U_i, \mathcal{O}_X|_{U_i})$ be an affine open covering of X. For $i \in I$, let P_i be a projective R-module with the property that $\mathcal{E}|_{U_i} \simeq \widetilde{P}_i$. Since P_i is a projective module of constant rank 2, it is a known fact that P_i is a commutative algebra, endowed with a unique standard involution σ_i, (c.f.

[HM08, Theorem 1.13.10, p.42]). By Lemma 3.4.0.3, $\widetilde{\sigma_i}$ is a standard involution of $\widetilde{P}_i$, $i \in I$. But then, from $\widetilde{\sigma_i} \in \mathscr{H}om_{\mathscr{O}_X}(\widetilde{P}_i, \widetilde{P}_i)(U_i)$, it follows that, for any pair (i, j) in $I \times I$ with $i \neq j$, $\widetilde{\sigma_i}|_{U_i \cap U_j} = \widetilde{\sigma_j}|_{U_i \cap U_j}$ is the unique involution on $\widetilde{P}_i|_{U_i \cap U_j} \simeq \widetilde{P_j}|_{U_i \cap U_j}$. The collection $(\widetilde{P}_i, \varphi_{ij})$, where φ_{ij} is the isomorphism $\widetilde{P}_i|_{U_i \cap U_j} \simeq \widetilde{P_j}|_{U_i \cap U_j}$, is a glueing data for sheaves of sets with respect to the covering $X = \cup_{i \in I} U_i$. Thus, there is a sheaf of sets $\mathscr{F}$ on X together with isomorphisms

$$\varphi_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \widetilde{P}_i,$$

that is,

$$\mathscr{F} \simeq \mathscr{E}.$$

Since $\widetilde{\sigma_i}|_{U_i \cap U_j} = \widetilde{\sigma_j}|_{U_i \cap U_j}$, there is a unique standard involution $\widetilde{\sigma}$ on $\mathscr{E}$ such that $\widetilde{\sigma}|_{U_i} = \widetilde{\sigma_i}$, $i \in I$. $\square$

Chapter 4

Involutions on sheaves of endomorphisms of $\mathscr{O}_X$-algebras

* In this chapter, we discuss involutions of the first kind on $\mathscr{O}_X$-algebras $\mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})$, where $\widetilde{M}$ is the sheaf of modules associated with an R-module M on an affine scheme $X = \operatorname{Spec}(R)$ (see [NN21]). Let N be another R-module and assume that $\varphi : \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M}) \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{N})$ a sheaf isomorphism. For any open U of X, set

$$\alpha_U : \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})(U) \times \widetilde{N}(U) \longrightarrow \widetilde{N}(U)$$

by

$$\alpha_U(f, s) = \varphi_{UU}(f_U)(s) \equiv \varphi(f)(s),$$

for any $f \equiv (f_V)_{U \supseteq V,\, open} \in \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})(U)$ and $s \in \widetilde{N}(U)$. The sheaf morphism $\alpha \equiv (\alpha_U)_{X \supseteq U,\, open}$ defines a left $\mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})$-module structure on $\widetilde{N}$; we denote $\widetilde{N}$ endowed with the left $\mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})$-module structure by ${}_{\varphi}\widetilde{N}$. In the similar way, we define ${}_{\varphi^{-1}}\widetilde{M}$. See [Knu91, (8.2), p.171].

In line with the sequel, we recall the following (see [NY14]): Let X be a topological space, $\mathscr{A} \equiv (\mathscr{A}, \pi, X)$ a sheaf of unital and commutative algebras and $\mathscr{S} \equiv (\mathscr{S}, \pi|_{\mathscr{S}}, X)$ a sheaf of submonoids in $\mathscr{A}$. A *sheaf of algebras of fractions of $\mathscr{A}$ by $\mathscr{S}$* is a sheaf of algebras, denoted $\mathscr{S}^{-1}\mathscr{A}$, such that, for every $x \in X$, the corresponding stalk $(\mathscr{S}^{-1}\mathscr{A})_x$ is an algebra of fractions of $\mathscr{A}_x$ by $\mathscr{S}_x$.

In this context, we also recall the following:

Theorem 4.0.0.1. [NY14] *For all $\mathscr{A}$-modules $\mathscr{E}$ and $\mathscr{F}$ on a topological space X, the $(\mathscr{S}^{-1}\mathscr{A})$-morphism*

$$\vartheta : \mathscr{S}^{-1}\mathscr{H}om_{\mathscr{A}}(\mathscr{E}, \mathscr{F}) \longrightarrow \mathscr{H}om_{\mathscr{S}^{-1}\mathscr{A}}(\mathscr{S}^{-1}\mathscr{E}, \mathscr{S}^{-1}\mathscr{F}),$$

*The content of this chapter to appear soon in the Mediteranean journal of Mathematics.

given by

$$\vartheta_x(f/s)(e/t) = f(e)/st,$$

where $x \in X$, $s,t \in \mathcal{S}_x$, $e \in \mathcal{S}^{-1}\mathcal{E}_x$, $f \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})_x$, *is an* $(\mathcal{S}^{-1}\mathcal{A})$*-isomorphism , whenever* $\mathcal{E}$ *is a locally finitely presented* $\mathcal{A}$*-module.*

Similarly, persuant to our need for the proof counterpart to [Knu91, (8.2), p.171], we recall the isomorphism in [CF15, Lemme 2.4.1.6, p.33].

Lemma 4.0.0.2. *The natural map*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})(U) \to \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{E}|_U(U),\mathcal{F}|_U(U)) = \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{E}(U),\mathcal{F}(U)),$$

where U *is open in* X*, is an isomorphism of modules if and only if the* $\mathcal{O}_X$*-modules* $\mathcal{E}$ *and* $\mathcal{F}$ *are free or locally free of finite type and* $X = \mathrm{Spec}(R)$*.*

In the above context, for every $x \in X$, the canonical homomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})_x \longrightarrow Hom_{\mathcal{O}_X}(\mathcal{E}_x,\mathcal{F}_x),$$

is an isomorphism. In general, this isomorphism holds for every ringed space $(X,\mathcal{O}_X)$, any $\mathcal{O}_X$-module $\mathcal{F}$ of finite presentation, and any $\mathcal{O}_X$-module $\mathcal{E}$. (see [GW10, Proposition 7.27, p.190]).

Lemma 4.0.0.3. *Let* M *be a locally of finite presentation* R*-module. Then,*

$$\widetilde{\mathrm{End}_R(M)} \xrightarrow{\sim} \mathcal{E}nd_{\mathcal{O}_X}(\widetilde{M}),$$

where $X = \mathrm{Spec}(R)$ *and* $\widetilde{M}$ *the sheaf of modules associated with* M*.*

Proof. For any $f \in R$, by Theorem 4.0.0.1,

$$\widetilde{\mathrm{End}_R(M)}(D(f)) = \mathrm{End}_{R_f}(M_f),$$

and

$$\mathcal{E}nd_{\mathcal{O}_X}(\widetilde{M})(D(f)) = \mathrm{End}_{\mathcal{O}_X|_{D(f)}}(\widetilde{M}|_{D(f)}),$$

whence we have $\widetilde{\mathrm{End}_R(M)}(D(f)) \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_X|_{D(f)}}(\widetilde{M}|_{D(f)})$ by Lemma 4.0.0.2. Moreover, since the $D(f)$ form a basis for the Zariski topology on X, the sought isomorphism follows thereby completing the proof. □

From Lemma 4.0.0.3, we follow through with the statement below:

Lemma 4.0.0.4. *Let M and N be locally finitely presented progenerator R-modules such that sheaves $\mathscr{E}nd_{\mathcal{O}_X}(\widetilde{M})$ and $\mathscr{E}nd_{\mathcal{O}_X}(\widetilde{N})$ are isomorphic (via an isomorphism φ), where $\widetilde{M}$ ($\widetilde{N}$, resp.) is the associated sheaf of R-modules for M (N, resp.) on the affine scheme $X = \operatorname{Spec}(R)$. Then , there exist an invertible R-module L and an isomorphism $\widetilde{\rho} : \widetilde{M} \otimes \widetilde{L} \xrightarrow{\sim} {}_{\varphi}\widetilde{N}$ such that $\widetilde{\varphi_X(f)} = \widetilde{\rho}(\widetilde{f \otimes 1})\widetilde{\rho}^{-1}$, for every $f \in \operatorname{End}_R(M)$.*

Proof. Let M and N be locally finitely presented progenerator R-modules such that $\varphi : \mathscr{E}nd_{\mathcal{O}_X}(\widetilde{M}) \to \mathscr{E}nd_{\mathcal{O}_X}(\widetilde{N})$ is an isomorphism, so the component $\varphi_X : \operatorname{End}_R(M) \to \operatorname{End}_R(N)$ is an R-module isomorphism. By [Knu91, Lemma 8.2.1, p.181], there exist an invertible R-module L and an isomorphism $\rho : M \otimes L \to {}_{\varphi_X}N$ of $\operatorname{End}_R(M)$-modules such that $\varphi_X(f) = \rho(f \otimes 1)\rho^{-1}$, for every $f \in \operatorname{End}_R(M)$. (The $\operatorname{End}_R(M)$-structure on $M \otimes L$ is given by the assignment $(f, m \otimes l) \mapsto f(m) \otimes l$.) By the isomorphism ([Bos13, Proposition 2, p.258,])

$$\operatorname{Hom}_R(\operatorname{End}_R(M), \operatorname{End}_R(N)) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{\operatorname{End}_R(M)}, \widetilde{\operatorname{End}_R(N)})$$

given by $\alpha \mapsto \widetilde{\alpha}$, one has $\widetilde{\varphi_X} : \widetilde{\operatorname{End}_R(M)} \to \widetilde{\operatorname{End}_R(N)}$. But then, by virtue of Lemmas 4.0.0.2 and 4.0.0.3, $\widetilde{\varphi_X} = \varphi$; thereafter, by [Har77, Proposition 5.2, p.110], the isomorphism $\widetilde{\rho} : \widetilde{M} \otimes \widetilde{L} \xrightarrow{\sim} {}_{\varphi}\widetilde{N}$ is such that

$$\widetilde{\varphi_X(f)} = \widetilde{\rho}(\widetilde{f \otimes 1})\widetilde{\rho}^{-1},$$

for every $f \in \operatorname{End}_R(M)$. □

The result in the above lemma can be generalized to the following context: Let $\mathscr{E}$, $\mathscr{F}$ be locally finitely presented progenerator $\mathcal{O}_X$-modules on an affine scheme $X = \operatorname{Spec}(R)$, and $\varphi : \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E}) \xrightarrow{\sim} \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{F})$. For any open subset $U \subseteq X$, $\mathscr{F}(U)$ carries a left $\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})(U)$-module structure; in fact, by Lemma 4.0.0.2, $\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})(U)$ is isomorphic to $\operatorname{End}_{\mathcal{O}_X(U)}(\mathscr{E}(U))$, and the action of $\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})(U)$ on $\mathscr{F}(U)$ into $\mathscr{F}(U)$, is given by $(f, s) \mapsto \varphi_U(f)(s)$, for any $f \in \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})(U)$ and $s \in \mathscr{F}(U)$. Hence, $\mathscr{F}$ will assume a left $\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})$-module structure on X which we denote as ${}_{\varphi}\mathscr{F}$. In a similar way, ${}_{\varphi^{-1}}\mathscr{E}$ denotes $\mathscr{E}$ endowed with the right $\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{F})$-structure obtained through φ^{-1}.

The sought generalization can now be formulated as follows:

Lemma 4.0.0.5. *Let $\mathscr{E}$ and $\mathscr{F}$ be locally finitely presented progenerator $\mathcal{O}_X$-modules, where $X = \operatorname{Spec}(R)$, and let $\varphi : \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E}) \xrightarrow{\sim} \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{F})$. Then, there exist an invertible $\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})$-module $\mathscr{L}$ and an isomorphism $\rho : \mathscr{E} \otimes \mathscr{L} \xrightarrow{\sim} {}_{\varphi}\mathscr{F}$ of $\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})$-modules such that, for any open $U \subseteq X$ such that $\mathscr{L}|_U \simeq \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})|_U$, $\varphi_U(f) = \rho_U(f \otimes 1)\rho_U^{-1}$, for all $f \in \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})(U) = \operatorname{End}_{\mathcal{O}_X|_U}(\mathscr{E}|_U)$.*

Proof. In line with a variant of the well-known Morita equivalence for $\mathcal{O}_X$-stacks (see [KS06, Theorem 19.5.4, p.475]), functors $(\ ) \otimes_{\mathcal{O}_X} \mathscr{E} : \mathfrak{M}_{\mathcal{O}_X} \to {}_{\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})}\mathfrak{M}$ and $\mathscr{H}om_{\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})}(\mathscr{E}, \ ) : {}_{\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})}\mathfrak{M} \to \mathfrak{M}_{\mathcal{O}_X}$

are inverse equivalences; for this reason, the $\mathscr{O}_X$-module $\mathscr{L} = \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\mathscr{E}, {}_{\varphi}\mathscr{F})$ is invertible with inverse $\mathscr{L}^{-1} = \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\varphi\mathscr{F}, \mathscr{E})$. That is, Since $\mathscr{E}$ and $\mathscr{F}$ are finite locally free, it follows that $\mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}({}_{\varphi}\mathscr{F}, \mathscr{E})$ is finite locally free, and therefore (see [GW10, p. 177, Proposition 7.7])

$$\begin{aligned}
\mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\mathscr{E}, {}_{\varphi}\mathscr{F}) \otimes_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})} \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}({}_{\varphi}\mathscr{F}, \mathscr{E}) \\
\simeq \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\mathscr{E}, {}_{\varphi}\mathscr{F} \otimes \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}({}_{\varphi}\mathscr{F}, \mathscr{E})) \\
\simeq \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\mathscr{E}, \mathscr{E}) \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}),
\end{aligned}$$

which implies that

$$\mathscr{L}^{-1} = \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}({}_{\varphi}\mathscr{F}, \mathscr{E}),$$

or $\mathscr{L}$ is invertible. So, for any open set $U \subseteq X$ such that $\mathscr{L}|_U \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})|_U$ and any $f \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$, one has

$$\varphi_U(f) = \rho_U(f \otimes 1)\rho_U^{-1}.$$

We thus obtain an $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$-isomorphism $\rho : \mathscr{L} \otimes \mathscr{E} \xrightarrow{\sim} {}_{\varphi}\mathscr{F}$ as in the classical case (see [Knu91, Lemma 8.2.1, p. 171]). □

Lemma 4.0.0.5 does not hold at the level of sections in general as the sheaf $\mathscr{E} \otimes \mathscr{L}$ is generated by the presheaf $(U \mapsto \mathscr{E}(U) \otimes \mathscr{L}(U))_{X \supseteq U,\, open}$, and $\mathscr{E}(U) \otimes \mathscr{L}(U)$ is not in general bijective to ${}_{\varphi_U}\mathscr{F}(U)$. However, section-wise, one may relax the conditions on progenerator $\mathscr{O}_X$-modules $\mathscr{E}$ and $\mathscr{F}$ to obtain the following lemma.

Lemma 4.0.0.6. *Let $\mathscr{E}$ and $\mathscr{F}$ be locally finitely free progenerator $\mathscr{O}_X$-modules on $X = \mathrm{Spec}(R)$, and let $\varphi : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})$. Then, there exist an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and an isomorphism ρ : $\mathscr{E} \otimes \mathscr{L} \xrightarrow{\sim} {}_{\varphi}\mathscr{F}$ of $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$-modules such that, for every open set U in X, $\varphi_{UU}(s) = \rho_U(\widetilde{s \otimes 1})\rho_U^{-1}$, for all $s \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$, and where $\widetilde{s \otimes 1}$ stands for the section of $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{L})$ over U, corresponding to $s \otimes 1$ through sheafification.*

Before we proceed to look at different types of involutions on sheaves of Azumaya algebras, let us first recall the concept of Azumaya $\mathscr{O}_X$-algebra with involution on an affine scheme $X = \mathrm{Spec}(R)$.

Definition 4.0.0.7. *An Azumaya $\mathscr{O}_X$-algebra $(\mathscr{A}, \sigma)$ with involution of the first kind is a sheaf of Azumaya R-algebras on a scheme $X = \mathrm{Spec}(R)$ with an $\mathscr{O}_X$-linear involution σ.*

Remark 4.0.0.8. If (M, σ) is an R-module with involution of the first kind σ, it is easy to see that $\widetilde{\sigma}$ is an $\mathscr{O}_X$-linear involution on the corresponding sheaf of R-modules $\widetilde{M}$. In fact, for any $f \in R$, $m \in M$, and $p \in \mathbb{N}$, $\widetilde{\sigma}_{D(f)}\left(\dfrac{m}{f^p}\right) = \dfrac{\sigma(m)}{f^p}$.

Let *A be an Azumaya R-algebra of constant rank n^2* and with involution σ of the first kind. By [For17, Corollary 10.3.10, p. 395], *there exists a commutative faithfully flat étale R-algebra S such that $A \otimes_R S$ is isomorphic to* $\mathrm{M}_n(S)$. Let φ be an isomorphism $A \otimes_R S \xrightarrow{\sim} \mathrm{M}_n(S)$ that makes S into a faithfully flat splitting R-algebra of A, it induces an involution $\kappa = \varphi \circ (\sigma \otimes 1) \circ \varphi^{-1}$ on $\mathrm{M}_n(S)$. On considering the sheaves associated with $A \otimes_R S$ and $\mathrm{M}_n(S)$, respectively, on $X = \mathrm{Spec}(R)$, we have $\widetilde{A} \otimes_{\mathcal{O}_X} \widetilde{S} \equiv \widetilde{A} \otimes_{\widetilde{R}} \widetilde{S} \xrightarrow{\sim} \widetilde{\mathrm{M}_n(S)}$. By virtue of [Har77, Proposition 5.2, p. 110], $\widetilde{\kappa} = \widetilde{\varphi} \circ (\widetilde{\sigma} \otimes 1) \circ \widetilde{\varphi^{-1}} = \widetilde{\varphi} \circ (\widetilde{\sigma} \otimes 1) \circ (\widetilde{\varphi})^{-1}$ is the induced involution on $\widetilde{\mathrm{M}_n(S)}$. The map $\Gamma : \widetilde{\mathrm{M}_n(S)} \longrightarrow \widetilde{\mathrm{M}_n(S)}$, given by $\Gamma_U(s) = \widetilde{\kappa}_U(s^t)$, where $s \in \widetilde{\mathrm{M}_n(S)}(U)$ and s^t means the transpose of s, is clearly an automorphism of $\widetilde{\mathrm{M}_n(S)}$ and corresponds to the automorphism $x \mapsto \kappa(x^t)$ of $\mathrm{M}_n(S)$. By choosing S such that $\kappa(x) = vx^tv^{-1}$, for any $x \in \mathrm{M}_n(S)$ and for some $v \in \mathrm{GL}_n(S)$, for any open U in X, $\kappa_U(s) = us^tu^{-1}$, where $s \in \widetilde{\mathrm{M}_n(S)}(U)$ and $u \in \widetilde{\mathrm{GL}_n(S)}(U)$. In [Knu91, p. 170], there is $\varepsilon \in \mu_2(S)$ ($\mu_2(S) = \{x \in S |\ x^2 = 1\}$) such that $v^t = \varepsilon v$. Next, let us show that the correspondence

$$U \mapsto \mu_2(\widetilde{S}(U)) \tag{4.1}$$

yields a complete presheaf (of groups). That the correspondence given above in (4.1) is a presheaf is evident from Definition 3.1.0.4. In order to show the *completeness* of this presheaf or that it is a sheaf, let U be an open subset of X, and $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ an open covering of U; moreover, let $s, t \in \mu_2(\widetilde{S}(U))$ such that

$$\rho^U_{U_\alpha}(s) \equiv s|_{U_\alpha} \equiv s_\alpha = t_\alpha \equiv t|_{U_\alpha} \equiv \rho^U_{U_\alpha}(t), \quad \alpha \in I,$$

where the $(\rho^U_{U_\alpha})_{\alpha \in I}$ are the restriction maps of the sheaf $\widetilde{S}$. Since $\mu_2(\widetilde{S}(U)) \subseteq \widetilde{S}(U)$ and $\widetilde{S}$ is the $\mathcal{O}_X$-module attached to the R-algebra S, $s = t$.

On the other hand, consider any sequence

$$(s_\alpha) \in \prod_{\alpha \in I} \mu_2(\widetilde{S}(U_\alpha)) \subseteq \prod_{\alpha \in I} \widetilde{S}(U_\alpha)$$

such that

$$s_\alpha|_{U_\alpha \cap U_\beta} = s_\alpha|_{U_\alpha \cap U_\beta},$$

for any $\alpha, \beta \in I$, with $U_\alpha \cap U_\beta \neq \emptyset$. There is an element $s \in \widetilde{S}(U)$ such that

$$s|_{U_\alpha} = s_\alpha, \quad \alpha \in I.$$

Thus,

$$(s^2)|_{U_\alpha} = \rho^U_{U\alpha}(s^2) = \rho^U_{U_\alpha}(s)\rho^U_{U_\alpha}(s) = 1|_{U_\alpha}, \quad \alpha \in I.$$

One infers that $s^2 = 1 \in \widetilde{S}(U)$, so that $s \in \mu_2(\widetilde{S}(U))$. Hence, the presheaf is complete hereby completing the proof.

Going back to the involution $\widetilde{\kappa}$, it follows that, given any open U in X, the equation $\kappa_U(s) = us^t u^{-1}$, where $u \in \widetilde{\mathrm{GL}_n(S)}(U)$, entails, for some $\varepsilon \in \mu_2(\widetilde{S}(U))$, $u^t = \varepsilon u$. As in the classical case, an involution $s \mapsto us^t u^{-1}$ of the $\mathscr{O}_X$-module $\widetilde{M_n(S)}$, where $u^t = \varepsilon u$, $s \in \widetilde{M_n(S)}(U)$, $u \in \widetilde{\mathrm{GL}_n(S)}(U)$, and $\varepsilon \in \mu_2(\widetilde{S}(U))$ is said to be *of type ε on the open subset U*, and κ_U is denoted κ_u.

Lemma 4.0.0.9. *Let $\mathscr{E}$ be a sheaf of modules over a scheme X, and σ an $\mathscr{O}_X$-endomorphism of $\mathscr{E}$. Then, σ is an involution if and only if, for every $x \in X$, $\sigma_x : \mathscr{E}_x \to \mathscr{E}_x$ is an involution.*

Proof. It is known that σ is bijective if and only if σ_x is bijective for all $x \in X$. (See, for instance, [Bos13, Proposition 3, p. 233].) Therefore, we need only show that σ is an anti-isomorphism if and only if σ_x is an anti-isomorphism, for all $x \in X$. The *only-if* part is easily seen from the characterization of stalks. To settle the *if* part, observe that, if U is an open neighbourhhod of x, and $\mathscr{U}(x)$ denotes the set of all open sets containing x, and $f, g \in \mathscr{E}(U)$,

$$\varinjlim_{V \in \mathscr{U}(x)} \sigma_V(f \cdot g) = \sigma_x(f_x \cdot g_x) = \sigma_x(g_x)\sigma_x(f_x) = \varinjlim_{V \in \mathscr{U}(x)} \sigma_V(g) \cdot \varinjlim_{V \in \mathscr{U}(x)} \sigma_V(f)$$
$$= \varinjlim_{V \in \mathscr{U}(x)} \sigma_V(g)\sigma_V(f),$$

which entails that, for some open neigbourhood V^x of x in U, $\sigma_{V^x}(f|_{V^x} \cdot g|_{V^x}) = \sigma_{V^x}(g|_{V^x})\sigma_{V^x}(f|_{V^x})$. By Sheaf Axiom (Sh.1), $\sigma_U(f \cdot g) = \sigma_U(g)\sigma_U(f)$. For the last displayed equality, (see [Bou68, (35), p. 211]). □

Definition 4.0.0.10. *Let X be a scheme. An Azumaya $\mathscr{O}_X$-algebra $\mathscr{E}$ with involution of the first kind is a sheaf of Azumaya algebras with involution of the first kind on X, i.e., an involution that leaves the centre elementwise invariant.*

It is evident that, given an open set U in X and sections f, g of an $\mathscr{O}_X$-algebra $\mathscr{E}$ over U, if $f_x \cdot g_x = g_x \cdot f_x$, for all $x \in U$, $f \cdot g = g \cdot f$. It follows that an involution σ of $\mathscr{E}$ fixes the centre of $\mathscr{E}$ elementwise if and only if , for every $x \in X$, σ_x keeps fixed the centre of $\mathscr{E}_x$ elementwise. Hence, σ is an involution of the first kind on $\mathscr{E}$ if and only if, for every $x \in X$, σ_x is an involution of $\mathscr{E}_x$ of the first kind.

Lemma 4.0.0.11. *Let $\mathscr{E}$ be a locally finitely presented $\mathscr{O}_X$-module on an affine scheme $X = \mathrm{Spec}(R)$, and let σ be an involution of the first kind on $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$. Then, there exist an invertible $\mathscr{O}_X$-module $\mathscr{L}$, a sheaf isomorphism φ of $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L}$ onto $\mathscr{E}^*$, and an $\mathscr{O}_X$-isomorphism $\Phi : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}^*)$ such that, on every open U in X where $\mathscr{L}|_U \simeq \mathscr{O}_X|_U$,*

$$\sigma \otimes \mathrm{Id} = \Phi \circ m, \tag{4.2}$$

where m is the natural isomorphism $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{L}) \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \otimes_{\mathscr{O}_X} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{L}) \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}_X) \otimes_{\mathscr{O}_X} \mathscr{O}_X \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$ on U, and, for any open V in U, and any section s of $\mathscr{E}nd_{\mathscr{O}_X}\mathscr{E}$ over V, $\Phi(s) \equiv \Phi_{VV}(s) = \varphi_V^{-1} s^* \varphi_V \equiv \varphi^{-1} s^* \varphi$, and s^* is the image of s through the natural morphism $\mathscr{E}nd_{\mathscr{O}_X}\mathscr{E} \to \mathscr{E}nd_{\mathscr{O}_X}\mathscr{E}^*$.

Proof. As in the proof of Lemma 4.0.0.5, by letting $\Phi = \tau\sigma$, where τ is the anti-automorphism $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}^*)$, and by virtue of Morita theory, the $\mathcal{O}_X$-algebra $\mathcal{L} = \mathcal{H}om_{\mathcal{E}nd_{\mathcal{O}_X}}(\mathcal{E}, {}_{\Phi}\mathcal{E}^*)$ is invertible. So, locally, Φ will satisfy the conditions of Equation (4.2). □

Going forward, we recall the sheaf-theoretic notion of a centre of a group. Particularly, let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathcal{E}$ a vector sheaf on X of constant rank n. The $\mathcal{O}_X$-module $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ is also locally free and of constant rank n^2 (see [Mal98, Equation (6.26), p. 138]). On considering the correspondence

$$U \mapsto Z(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(U)) \simeq Z(\mathcal{O}_X{}^{n^2}|_U) \simeq Z((\mathcal{O}_X|_U)^{n^2})$$

(where $Z(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(U))$ consists of all $\vartheta \in \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(U)$ such that $\vartheta \circ \varphi = \varphi \circ \vartheta$, for all $\varphi \in \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(U)$) together with the obvious restriction maps yields a complete presheaf, called the *(pre)sheaf of centres of groups*. On any local gauge U of the vector sheaf $\mathcal{E}$, one has

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})|_U \simeq \mathcal{O}_X{}^{n^2}|_U \simeq \mathrm{M}_n(\mathcal{O}_X)|_U;$$

therefore,

$$Z(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(U)) \simeq Z(\mathrm{M}_n(\mathcal{O}_X(U))).$$

Lemma 4.0.0.12. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{E}$ a locally finitely presented $\mathcal{O}_X$-module with involution of the first kind σ on $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$. Then, for any local gauge U,*

$$(\varphi^*\eta \otimes 1)\varphi^{-1} \in Z(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}^*)(U)),$$

where η is the canonical $\mathcal{O}_X$-isomorphism $\mathcal{E} \longrightarrow \mathcal{E}^{**}$, and φ is the $\mathcal{O}_X$-isomorphism $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L} \xrightarrow{\sim} \mathcal{E}^*$ of Lemma 4.0.0.11. Furthermore, for some $\varepsilon \in \mu_2(\mathcal{O}_X(U))$,

$$\varepsilon\varphi^*\eta \otimes 1 = \varphi. \tag{4.3}$$

(N.B. For any open open V in X, $\eta(s)(u) \equiv s^{**}(u) := u(s)$, where $s \in \mathcal{E}(V)$, and $u \in \mathcal{E}^*(V) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})(V) = \mathrm{Hom}_{\mathcal{O}_X|_V}(\mathcal{E}|_V, \mathcal{E}|_V)$.)

Proof. From Equation (4.2),

$$\sigma(s) \otimes 1 = \varphi^{-1}s^*\varphi,$$

for all $s \in \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(U)$, where U is a local gauge of $\mathcal{L}$. Since $\sigma^2 = 1$, it follows that

$$s \otimes 1 = \varphi^{-1}\sigma(s)^*\varphi. \tag{4.4}$$

On transposing (4.4), we obtain

$$s^* \otimes 1 = \varphi^*\sigma(s)^{**}(\varphi^{-1})^*.$$

But then

$$\sigma^{**}(s^{**}) = \sigma(s)^{**} = \eta\sigma(s)\eta^{-1},$$

so

$$(\varphi^*\eta)^{-1} \circ (s^* \otimes 1) \circ (\varphi^*\eta) = \sigma(s). \tag{4.5}$$

Tensoring (4.5) with 1 yields, under the assumption $\mathscr{L}^* \otimes \mathscr{L} \simeq \mathscr{O}_X$, which allows one to identify $s^* \otimes 1 \otimes 1$ with s^*,

$$(\varphi^*\eta \otimes 1)^{-1} \circ s^* \circ (\varphi^*\eta \otimes 1) = \sigma(s) \otimes 1 = \varphi^{-1}s^*\varphi.$$

Hence,

$$(\varphi^*\eta \otimes 1)\varphi^{-1} \in Z(\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)).$$

By virtue of (4.2), and since $Z(\mathrm{M}_n(\mathscr{O}_X(U))) \simeq \mathscr{O}_X(U)$,

$$(\varphi^*\eta \otimes 1)\varphi^{-1} = \varepsilon$$

for some $\varepsilon \in \mathscr{O}_X(U)$. It is clear that ε must be invertible, that is, $\varepsilon \in \mathscr{O}_X(U)^\bullet = \mathscr{O}_X^\bullet(U)$, with $\mathscr{O}_X(U)^\bullet$ the group of units of the unital ring $\mathscr{O}_X(U)$. ($\mathscr{O}_X^\bullet$ is the sheaf on X generated by the presheaf defined by the correspondence

$$U \mapsto \mathscr{O}_X^\bullet \simeq \mathscr{O}_X(U)^\bullet,$$

where U varies over the Zariski topology of X, (See [Mal98, Lemma 1.1, p. 282]).) □

Corollary 4.0.0.13. *The section $\varepsilon \in \mathscr{O}_X(U)$ satisfies the condition: $\varepsilon^2 = 1$.*

Proof. First, note that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{E}(U) \otimes \mathscr{L}(U) \arrow[r, "\varphi_U"] \arrow[d, "\eta_U \otimes 1_{\mathscr{L}(U)}"'] \arrow[dr, phantom, "\varepsilon"] & \mathscr{E}^*(U) \otimes \mathscr{O}_X(U) \simeq \mathscr{E}^*(U) \, , \\
\mathscr{E}^{**}(U) \otimes \mathscr{L}(U) \arrow[r, "\varphi_U^* \otimes 1_{\mathscr{L}(U)}"'] & \mathscr{E}^*(U) \otimes \mathscr{L}^*(U) \otimes \mathscr{L}(U) \arrow[u, "1_{\mathscr{E}^*(U)} \otimes \mu_U"']
\end{tikzcd}$$

where μ is the canonical isomorphism $\mathscr{L}^* \otimes \mathscr{L} \xrightarrow{\sim} \mathscr{O}_X$, and ε, in the centre of the diagram, means that the diagram commutes up to a factor ε.

Note that, on U, $\mathscr{E}^*(U) = \mathscr{E}(U)^*$, $\mathscr{E}^{**}(U) = \mathscr{E}(U)^{**}$, and $\mathscr{L}^*(U) = \mathscr{L}(U)^*$. On transposing the diagram above, one obtains:

$$\begin{tikzcd}
\mathscr{E}(U)^{**} \arrow[r, "\varphi_U^*"] \arrow[d, "1_{\mathscr{E}(U)^{**}} \otimes \mu_U^*"'] \arrow[dr, phantom, "\varepsilon"] & \mathscr{E}(U)^* \otimes \mathscr{L}(U)^* \\
\mathscr{E}(U)^{**} \otimes \mathscr{L}(U)^{**} \otimes \mathscr{L}(U)^* \arrow[r, "\varphi_U^{**} \otimes 1_{\mathscr{L}(U)^*}"'] & \mathscr{E}(U)^{***} \otimes \mathscr{L}(U)^* . \arrow[u, "\eta_U^* \otimes 1_{\mathscr{L}(U)^*}"']
\end{tikzcd}$$

Tensoring with $\mathscr{L}(U)$ and taking into account the isomorphism $\mathscr{L}(U)^* \otimes \mathscr{L}(U) \simeq \mathcal{O}_X(U)$ yields:

$$\begin{tikzcd}
\mathscr{E}(U)^{**} \otimes \mathscr{L}(U) \arrow[r, "\varphi_U^* \otimes 1_{\mathscr{L}(U)}"] \arrow[d, "1_{\mathscr{E}(U)^{**}} \otimes \mu_U^* \otimes 1_{\mathscr{L}(U)}"'] \arrow[dr, phantom, "\varepsilon"] & \mathscr{E}(U)^* \\
\mathscr{E}(U)^{**} \otimes \mathscr{L}(U)^{**} \otimes \mathscr{L}(U)^* \otimes \mathscr{L}(U) \arrow[r, "\varphi_U^{**}"'] & \mathscr{E}(U)^{***}. \arrow[u, "\eta_U^*"']
\end{tikzcd}$$

Superposing the first diagram with the last one, one obtains

$$\begin{tikzcd}
\mathscr{E}(U) \otimes \mathscr{L}(U) \arrow[r, "\varphi_U"] \arrow[d, "\eta_U \otimes 1_{\mathscr{L}(U)}"'] \arrow[dr, phantom, "\varepsilon"] & \mathscr{E}^*(U) \otimes \mathcal{O}_X(U) \simeq \mathscr{E}^*(U) \; . \\
\mathscr{E}^{**}(U) \otimes \mathscr{L}(U) \arrow[r, "\varphi_U^* \otimes 1_{\mathscr{L}(U)}"'] \arrow[d, "1_{\mathscr{E}(U)^{**}} \otimes \mu_U^* \otimes 1_{\mathscr{L}(U)}"'] & \mathscr{E}^*(U) \otimes \mathscr{L}^*(U) \otimes \mathscr{L}(U) \arrow[u, "1_{\mathscr{E}(U)^*}"'] \\
\mathscr{E}(U)^{**} \otimes \mathscr{L}(U)^{**} \otimes \mathscr{L}(U)^* \otimes \mathscr{L}(U) \arrow[r, "\varphi_U^{**}"'] & \mathscr{E}(U)^{***} \arrow[u, "\eta_U^*"']
\end{tikzcd}$$

From the outer contour, one has: $\eta_U^* \varphi_U^{**} \eta_U = \varepsilon^2 \varphi_U$ or, equivalently, $\varphi_U^{**} \eta_U = \varepsilon^2 \eta_U \varphi_U$. But then, $\varphi_U^{**} \eta_U = \eta_U \varphi_U$, hence, $\varepsilon^2 = 1$. □

Theorem 4.0.0.14. *Let $(X, \mathcal{O}_X)$ be a locally ringed space, $\mathscr{E}$ a locally finitely presented $\mathcal{O}_X$-module, and $\sigma : \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E}) \longrightarrow \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})$ an involution of the first kind. Moreover, let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module and φ an isomorphism $\mathscr{E} \otimes \mathscr{L} \xrightarrow{\sim} \mathscr{E}^*$ such that $\sigma \otimes Id = \Phi \circ m$, where $\Phi = \tau\sigma$ with τ the anti-$\mathcal{O}_X$-automorphism $\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E}) \longrightarrow \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E}^*)$, and $m : \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{L})|_U \xrightarrow{\sim} \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})|_U$, with U a local gauge of $\mathscr{L}$. Then, for any $x \in X$, there is $u \in \mathscr{L}_x$ such that*

$$\sigma_x(f) = u^{-1} \circ f^* \circ u,$$

for any $f \in \mathrm{End}_{\mathcal{O}_{X,x}}(\mathscr{E}_x)$, i.e., $\sigma_x = \sigma_{x,u^{-1}}$. Furthermore, for any local gauge V of $\mathscr{L}$ at x, there is a unit $\varepsilon \in \mathcal{O}_X(V)$ such that

$$\varepsilon_x u(q)(p) = u(p)(q),$$

for all p, $q \in \mathscr{E}_x$. [NN21]

Proof. For all $x \in X$, observe that $\mathscr{E}_x$ is a finitely presented $\mathcal{O}_{X,x}$-module (see [Mal98, (1.54) and (1.55), p. 101]); since $\mathscr{L}_x$ is invertible over a local ring $\mathcal{O}_{X,x}$, it is necessarily free. Therefore, $\mathscr{L}_x \simeq u\mathcal{O}_{X,x} \simeq \mathcal{O}_{X,x}$ for some $u \in \mathscr{L}_x$. By Lemma 4.0.0.11, $\mathscr{L} = \mathscr{H}om_{\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})}(\mathscr{E}, {}_{\Phi}\mathscr{E}^*)$, where $\Phi = \tau\sigma$. Since $\mathscr{E}$ is locally finitely presented, $\mathscr{L}_x = \mathscr{H}om_{\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})}(\mathscr{E}, {}_{\Phi}\mathscr{E}^*)_x \simeq \mathrm{Hom}_{\mathrm{E}nd_{\mathcal{O}_{X,x}}(\mathscr{E}_x)}(\mathscr{E}_x, {}_{\Phi_x}(\mathscr{E}_x)^*)$. Since $\sigma(s) \otimes 1 = \varphi^{-1} s^* \varphi$, for any section s of $\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E})$, or equivalently, $\varphi \circ (\sigma(s) \otimes 1) = s^* \circ \varphi$. Stalk-wise, we have that, for any $x \in X$, $\varphi_x \circ (\sigma_x(s_x) \otimes 1) = s_x^* \circ \varphi_x \in \mathrm{Hom}_{\mathrm{E}nd_{\mathcal{O}_{X,x}}(\mathscr{E}_x)}(\mathscr{E}_x \otimes \mathscr{L}_x, \mathscr{E}_x^*)$, where $\varphi_x(p \otimes u) = u(p)$, for all $p \in \mathscr{E}_x$; hence, $(s_x^* \circ \varphi_x)(p \otimes u) = (s_x^* \circ u)(p) \in \mathscr{E}_x^*$. On the other hand, $(\varphi_x \circ (\sigma_x(s_x) \otimes 1))(p \otimes u) = u(\sigma_x(s_x)(p))$. Thus, $s_x^* \circ u = u \circ \sigma_x(s_x)$ and $\sigma_x(s_x) = u^{-1} \circ s_x^* \circ u$.

If $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$ is represented by the matrix sheaf $\mathrm{M}_n(\mathscr{O}_X) \simeq \mathscr{O}_X^{n^2}$ (rank$\mathscr{E} = n$), then $\mathscr{E}nd_{\mathscr{O}_{X,x}}(\mathscr{E}_x) \simeq \mathscr{O}_{X,x}^{n^2}$, we have $\sigma_x = \sigma_{x,u^{-1}}$.

Now, $\varphi_x^*\eta_x \otimes 1 : \mathscr{E}_x \otimes \mathscr{L}_x \longrightarrow (\mathscr{E}_x \otimes \mathscr{L}_x)^* \otimes \mathscr{L}_x \simeq \mathscr{E}_x^*$ maps $p \otimes u$ onto $\varphi_x^*(\eta_x(p)) \otimes u$. Since $\mathscr{L}_x$ is free of rank 1, we may use u to identify $\mathscr{E}_x \otimes \mathscr{L}_x$ with $\mathscr{E}_x$; then $\varphi_x^*\eta_x : \mathscr{E}_x \longrightarrow \mathscr{E}_x^*$ maps p onto $\varphi_x^*(\eta_x(p)) \in \mathscr{E}_x^*$, which is the mapping $q \mapsto \eta_x(p)(u(q)) = u(q)(p)$. On the other hand, since $\mathscr{E}_x \otimes \mathscr{L}_x \xrightarrow{\sim} \mathscr{E}_x$, we may assume φ_x to be an isomorphism $\mathscr{E}_x \longrightarrow \mathscr{E}_x^*$; therefore, $\varphi_x(p)(q) = u(p)(q)$. It follows that

$$\varepsilon_x u(q)(p) = u(p)(q),$$

for all $p, q \in \mathscr{E}_x$. □

Note that one arrives at a similar result section-wise when one considers any vector sheaf $\mathscr{E}$ of finite rank on a locally ringed space $(X, \mathscr{O}_X)$.

Theorem 4.0.0.15. *Let $(X, \mathscr{O}_X)$ be a locally ringed space, $\mathscr{E}$ a vector sheaf of finite rank n on X, and σ an involution of the first kind on $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$. Moreover, let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and $\varphi \equiv (\varphi_V)_{X \supseteq V,\ open}$ an isomorphism $\mathscr{E} \otimes \mathscr{L} \xrightarrow{\sim} \mathscr{E}^*$ such that $\sigma(s) \otimes 1 = \varphi^{-1} s^* \varphi$, for any $s \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$ and any $\varphi_U : (\mathscr{E} \otimes \mathscr{L})(U) \xrightarrow{\sim} \mathscr{E}^*(U)$ or $\varphi_U : \mathscr{E}(U) \xrightarrow{\sim} \mathscr{E}^*(U)$, where the open subset U of X is chosen such that both $\mathscr{L}|_U \simeq \mathscr{O}_X|_U$ and $\mathscr{E}|_U \simeq \mathscr{O}_X^n|_U$ are satisfied. Then, there is $u \in \mathscr{L}(U)$ such that*

$$\sigma(f) = u^{-1} \circ f^* \circ u,$$

for any $f \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U) = \mathrm{End}_{\mathscr{O}_X|_U}(\mathscr{E}|_U)$. Furthermore, there is a unit $\varepsilon \in \mathscr{O}_X(U)$ such that

$$\varepsilon u(t)(r) = u(r)(t), \tag{4.6}$$

for all $r, t \in \mathscr{E}(U)$. [NN21]

Proof. Let $x \in X$, V an open neighborhood of x such that $\mathscr{E}|_V \simeq \mathscr{O}_X^n|_V$, and W a local gauge of $\mathscr{L}$ at x, i.e., $\mathscr{L}|_W \simeq \mathscr{O}_X|_W$. Define: $U = V \cap W$. Since $\mathscr{O}_X(U)$ is a local ring and $\mathscr{L}(U)$ is invertible over $\mathscr{O}_X(U)$, $\mathscr{L}(U)$ is free. Therefore, $\mathscr{L}(U) \simeq u\mathscr{O}_X(U) \simeq \mathscr{O}_X(U)$, for some $u \in \mathscr{L}(U)$. For any section $s \in \mathscr{E}(U)$, $\varphi \circ (\sigma(s) \otimes 1) = s^* \circ \varphi \in \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\mathscr{E} \otimes \mathscr{L}, \mathscr{E}^*)(U) = \mathrm{Hom}_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})|_U}((\mathscr{E} \otimes \mathscr{L})|_U, \mathscr{E}^*|_U) = \mathrm{Hom}_{\mathscr{E}nd_{\mathscr{O}_X|_U}(\mathscr{E}|_U)}(\mathscr{E}|_U \otimes \mathscr{L}|_U, \mathscr{E}^*|_U) = \mathrm{Hom}_{\mathscr{E}nd_{\mathscr{O}_X|_U}(\mathscr{E}|_U)}(\mathscr{E}|_U, \mathscr{E}^*|_U)$. For any $r \in \mathscr{E}(U)$, $\varphi(r \otimes u) = u(r)$; therefore, $(s^* \circ \varphi)(r \otimes u) = (s^* \circ u)(r) \in \mathscr{E}^*(U)$. On another side, $(\varphi \circ (\sigma(s) \otimes 1))(r \otimes u) = \varphi(\sigma(s)(r) \otimes u) = u(\sigma(s)(r))$. Thus, $s^* \circ u = u \circ \sigma(s)$ or $\sigma(s) = u^{-1} \circ s^* \circ u$.

Since $\mathscr{L}^* \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{O}_X$, one has

$$\varphi^*\eta \otimes 1 : \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L} \xrightarrow{\sim} \mathscr{E}^* \otimes_{\mathscr{O}_X} \mathscr{L}^* \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{E}^*;$$

therefore, for any open U in X with $\mathscr{L}|_U \simeq \mathscr{O}_X|_U$ and $\mathscr{E}|_U \simeq \mathscr{O}_X{}^n|_U$, any section $r \otimes u$ of the $\mathscr{O}_X$-module $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L}$ maps onto $\varphi^*(\eta(r)) \otimes u$. Since $\mathscr{L}|_U$ is free of rank 1, we may use a suitably chosen u, namely any nowhere-zero section, as an isomorphism $(\mathscr{E} \otimes \mathscr{L})|_U \xrightarrow{\sim} \mathscr{E}|_U$; then $\varphi^*\eta|_U : \mathscr{E}|_U \to \mathscr{E}^*|_U \simeq (\mathscr{E}|_U)^*$ maps r onto $\varphi^*(\eta(r)) \in \mathscr{E}^*(U) = \mathscr{E}(U)^*$, which in turn maps a section t in $\mathscr{E}^*(U)$ onto $\eta(r)(u(t)) = u(t)(r)$. On the other hand, since $\mathscr{E}|_U \otimes_{\mathscr{O}_X|_U} \mathscr{L}|_U \xrightarrow{\sim} \mathscr{E}|_U$, we may assume φ to be an isomorphism $\mathscr{E}|_U \to \mathscr{E}^*|_U$; therefore, $\varphi(r)(t) = u(r)(t)$. From Equation (4.3), it follows that, for some $\varepsilon \in \mathscr{O}_X(U)^\bullet$,

$$\varepsilon u(t)(r) = u(r)(t),$$

for all $r, t \in \mathscr{E}(U)$. □

Corollary 4.0.0.16. *Let R be a commutative ring such that the induced ringed space $(X, \mathscr{O}_X)$ is a locally ringed space; let $\mathscr{E}$ be a vector sheaf of finite rank n on X, σ an involution of the first kind on the vector sheaf $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module such that $\mathscr{E} \otimes \mathscr{L} \xrightarrow{\sim} \mathscr{E}^*$ is an isomorphism φ with $\sigma(s) \otimes 1 = \varphi^{-1}s^*\varphi$, for any $s \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$, where U is any open subset of X such that $\mathscr{L}|_U \simeq \mathscr{O}_X|_U$ and $\mathscr{E}|_U \simeq \mathscr{O}_X{}^n|_U$. Then, on identifying $\mathscr{E}|_U$ with $(\mathscr{E}|_U)^* = \mathscr{E}^*|_U$ with the help of some section u of $\mathscr{L}$, where $\sigma(f) = u^{-1} \circ f^* \circ u$, for any $f \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$, and identifying $\mathscr{E} \otimes \mathscr{E}^*$ with $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$,*

$$\sigma(r \otimes s) = \varepsilon u^{-1}(s) \otimes u(r),$$

for $\varepsilon \in \mathscr{O}_X^\bullet$, $r \in \mathscr{E}(U)$ and $s \in \mathscr{E}^*(U)$.

Proof. For the sake of containedness, we recall that, given any $\mathscr{O}_X$-modules $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ with $\mathscr{E}$ or $\mathscr{G}$ being locally finitely free, the functorial homomorphism

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \tag{4.7}$$

is an isomorphism, (see [GW10, p. 177, Proposition 7.7]). In particular, for any vector sheaf $\mathscr{E}$ of finite rank on X,

$$\mathscr{E}^* \otimes_{\mathscr{O}_X} \mathscr{E} \xrightarrow{\sim} \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) = \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}).$$

It follows that since, for some section u of $\mathscr{L}$, one has: $u : \mathscr{E}|_U \xrightarrow{\sim} \mathscr{E}^*|_U$, and $\sigma(r \otimes s) = u^{-1} \circ (r \otimes s)^* \circ u$, where $r \otimes s \in \mathscr{E}(U) \otimes \mathscr{E}^*(U) = (\mathscr{E} \otimes \mathscr{E}^*)(U) \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U) = \mathrm{End}_{\mathscr{O}_X|_U}(\mathscr{E}|_U)$. The transpose $(r \otimes s)^* : \mathscr{E}^*|_U \to \mathscr{E}^*|_U$ is such that $(r \otimes s)^*(u(t)) = u(t) \circ (r \otimes s)$, for any section t of $\mathscr{E}$ on U. It is clear that, for any $z \in \mathscr{E}(U)$,

$$(u(t) \circ (r \otimes s))(z) = u(t)(s(z)r) = u(t)(r)s(z),$$

viz.

$$u(t) \circ (r \otimes s) = u(t)(r)s.$$

Consequently, on using (4.6), one has

$$\sigma(r \otimes s) = u(t)(r)u^{-1}(s) = \varepsilon u(r)(t)u^{-1}(s).$$

Thus,

$$\sigma(r \otimes s) = \varepsilon u^{-1}(s) \otimes u(r),$$

for $r \in \mathscr{E}(U)$ and $s \in \mathscr{E}^*(U)$. $\square$

Bibliography

[AW92] W. A. Adkins and S.H. Weintraub, *Algebra, An Approach via Module Theory*, Gratuate Texts in Mathematics, **136**, Springer-Verlag, Berlin, 1992. ↑41, 54, 55

[Art91] M. Artin, *Algebra*, Prentice Hall, 1991. ↑13

[AM94] M.F. Atiyah and I.G. MacDonald, *Introduction To Commutative Algebra*, Addison-Wesley series in mathematics, Avalon Publishing, 1994. ↑17, 19

[AG60] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Transactions of the American Mathematical Society **97** (1960), no. 3, 367–409. ↑20, 26

[Azu51] G. Azumaya, *On maximally central algebras*, Nagoya Mathematical Journal **2** (1951), 119–150. ↑20

[Bas64] H. Bass, *K-theory and stable algebra,* Publications mathématiques de l' I.H.È.S. **22** (1964), 5–60. ↑40

[Bas68] ______, *Algebraic K-theory.*, W. A. Benjamin, Inc., New York, 1968. ↑12, 13, 15, 18, 26, 40

[BO13] G. Berhuy and F. Oggier, *An introduction to central simple algebras and their applications to wireless communication.*, American Mathematical Society, Providence, RI, 2013. ↑36

[Bos13] S. Bosch, *Algebraic Geometry and Commutative Algebra*, Vol. 1, Springer-Verlag, 2013. ↑8, 13, 16, 23, 58, 60, 63, 66, 67, 68, 73, 79, 82

[Bou58] N. Bourbaki, *Éléments de mathématique. Algèbre. Chapitre 8*, Hermann, Paris, 1958. ↑34, 36, 38

[Bou68] ______, *Elements of Mathematics, Theory of Sets*, Hermann, Paris, 1968. ↑82

[Bou89] ______, *Elements of mathematics I,Chapters 1-3*, Springer-Verlag Berlin Heidelberg, New York, 1989. ↑36

[Bru02] A. M Bruce, *An Algebraic Introduction to K-Theory*, Cambridge University Press Online publication, 2002. ↑21, 23

[Bra82] A. Braun, *On Artin's theorem and Azumaya algebras*, Journal of Algebra **77** (1982), no. 2, 323–332. ↑34

[CF15] B. Calmès and J. Fasel, *Groupes classiques*, Autours des schémas en groupes. Vol. II, 2015, pp. 1–133. MR3525594 ↑78

[CdS01] A. Cannas da Silva, *Lectures on Symplectic Geometry*, Springer-Verlag, Berlin, 2001. ↑50

[DeM69] F. R. DeMeyer, *Projective modules over central separable algebras*, Canadian Journal of Mathematics **21** (1969), 39–43. MR234987 ↑38

[DI71] F. R. DeMeyer and E. Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Mathematics, Vol. 181, Springer-Verlag, Berlin-New York, 1971. ↑20, 21, 28, 29, 30, 31

[Dic88] W. Dicks, *On a characterization of Azumaya algebras*, Publicacions Matemàtiques **32** (1988), no. 2, 165 –166. ↑34

[EW67] S. Endo and Y. Watanbe, *On Separable Algebras over a Commutative Ring*, Osaka J. Math. **4** (1967), 233–242. MR0227211 (37 #2796) ↑34, 36

[FD93] B. Farb and R. K. Dennis, *Noncommutative Algebra*, Graduate texts in mathematics, Springer-Verlag, 1993. ↑31

[For08] T. J Ford, *Rings and Modules [lecture notes]*, Florida Atlantic University, 2008. ↑30

[For17] T. J. Ford, *Separable algebras*, Graduate Studies in Mathematics, vol. 183, American Mathematical Society, Providence, RI, 2017. MR3618889 ↑20, 21, 23, 24, 25, 26, 28, 30, 31, 32, 34, 36, 38, 39, 40, 41, 42, 43, 48, 49, 54, 81

[GW10] U. Görtz and T. Wedhorn, *Algebraic Geometry: Part I: Schemes. With Examples and Exercises*, Advanced Lectures in Mathematics, Vieweg+Teubner Verlag, 2010. ↑12, 58, 61, 63, 78, 80, 87

[Har77] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977. ↑69, 75, 79, 81

[Kap48] I. Kaplansky, *Rings with a polynomial identity*, Project Euclid **54** (1948), 575–580. ↑33

[Kap54] ______, *Infinite Abelian Groups*, University of Michigan Press, Ann Arbor, 1954. ↑40

[KS06] M. Kashiwara and P. Schapira, *Categories and Sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2006. ↑79

[KK99a] U. Kenji and G. Kato, *Algebraic Geometry 1:From algebraic varieties to schemes*, Iwanami series in modern mathematics, vol. 185, American Mathematical Society, 1999. ↑64

[KK99b] ______, *Algebraic Geometry 2:Sheaves and Cohomology*, Algebraic Geometry, American Mathematical Society, 1999. ↑64, 69, 70

[KA13] S. Kleiman and A. Altman, *A Term of Commutative Algebra*, Worldwide Center of Mathematics, LLC, 2013. ↑18

[KO74] M. A. Knus and M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lecture Notes in Mathematics, Vol. 389, Springer-Verlag, Berlin, 1974. MR0417149 (54 #5209) ↑20, 32, 34, 37, 42

[KPS90] M. A. Knus, R. Parimala, and V. Srinivas, *Azumaya algebras with involutions*, Journal of Algebra. **130** (1990). ↑9

[Knu91] M. A. Knus, *Quadratic and Hermitian forms over rings*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 294, Springer-Verlag, Berlin, 1991. ↑32, 43, 44, 45, 77, 78, 79, 80, 81

[KMRT98] M. A. Knus, A.S. Merkurjev, M. Rost, and J. P. Tignol, *The Book of Involutions*, Amer. Math. Soc. Colloquium Publications, Amer. Math. Soc., 1998. ↑8, 38, 42, 45, 46, 47, 49, 53, 55, 56, 71

[Lam99] T. Y. Lam, *Lectures on Modules and Rings*, Graduate texts in Mathematics, Vol. 189, Springer, Berlin-New York, 1999. ↑24, 25

[Lin17] H. Lindo, *Trace ideals and centers of endomorphism rings of modules over commutative rings* **482** (2017), 102 –130. ↑25, 40

[Mal98] A. Mallios, *Geometry of Vector Sheaves: An Axiomatic Approach to Differential Geometry*, Geometry of Vector Sheaves: An Axiomatic Approach to Differential Geometry, Kluwer Academic Publishers, 1998. ↑58, 59, 60, 61, 62, 83, 84, 85

[Mat80] H. Matsumura, *2 Commutative Algebra*, The Benjamin/Cummings Publishing Company, Inc., New York, 1980. ↑18, 22

[Mat86] ______, *Commutative ring theory*, 1986. ↑16

[HM08] J. Helmstetter and A. Micali, *Quadratic Mappings and Clifford Algebras*, Birkhauser, 2008. ↑73, 74, 75, 76

[Mil80] J.S. Milne, *Etale Cohomology (PMS-33)*, Princeton Legacy Library, Princeton University Press, 1980. ↑31

[MO15] D. Mumford and T. Oda, *Algebraic Geometry II*, Hindustan Book Agency Series, Hindustan Book Agency, 2015. ↑62, 63

[NN21] R. Ng'ambi and P. P Ntumba, *Involutions on sheaves of endomorphisms of locally finitely presented $\mathscr{O}_X$-modules* (2021). MRMJOM-D-20-01008 ↑77, 85, 86

[NY14] P.P. Ntumba and B. Yizengaw, *On the commutativity of the Clifford and "extension of scalars" functors*, Topology and its Applications **168** (2014), 159–179. MR3196848 ↑77

[Pie82] R. S. Pierce, *Associative Algebras*, Graduate Texts in Mathematics, Springer-Verlag New York, 1982. ↑27, 28, 29

[Pro72] C. Procesi, *On a theorem of M. Artin*, Journal of Algebra **22** (1972), no. 2, 309–315. ↑33

[RG71] M. Raynaud and L. Gruson, *Critères de platitude et de projectivité. Techniques de platification d' un module*, Inventiones Math. **13** (1971), 1–89. MR0308104 ↑75

[Rei75] I. Reiner, *Maximal Orders*, Academy Press, London, 1975. ↑19, 54

[Rot09] J. J Rotman, *An introduction to homological algebra*, Vol. 2, Springer, New York, 2009. ↑13, 14, 16, 19

[Row88] L. H. Rowen, *Ring Theory V1*, Academic Press, 1988. ↑26

[Row91] ______, *Ring Theory : Student Edition*, 2nd ed., Academic press, 1991. ↑27, 29, 33

[Sch85] W. Scharlau, *Quadratic and Hermitian Forms*, Springer-Verlag, Berlin, 1985. ↑31

[Sha13] I. R. Shafarevich, *Basic algebraic geometry 2:Schemes and complex manifolds*, Springer, Heidelberg, 2013. ↑67

[TS21] The Stacks project authors, *The Stacks project*, 2021. ↑19

[Sze78] S. Szeto, *On separable extensions of group rings and quaternion rings*, International Journal of Mathematics and Mathematical Sciences **1** (1978), 433 –438. ↑33

[Wei13] C. A. Weibel, *The K-book: An introduction to algebraic K-theory*, Vol. 145, American Mathematical Society Providence, RI, 2013. ↑23, 33

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Glossary of Notations

The list describes several symbols and notaions used within the body of the document

$\mathrm{Hom}_{\mathfrak{C}}(M,N)$ morphisms of objects in a category, page 10

$\mathfrak{M}$ category of R-modules, page 10

$\mathfrak{M}_R$ category of right R-modules, page 10

$\mathfrak{T}_R$ trace map over R, page 24

$\mathfrak{T}_R(M)$ trace of a finitely generated module M over R, page 24

R° opposite ring, page 12

$(\mathcal{S},\pi,X)$ Sheaf of sets over a topological space, page 58

$1_{\mathfrak{C}}$ identity morphism on $\mathfrak{C}$, page 10

$\mathrm{Char}.poly_M(\alpha)$ The characteristic polynomial of α in M, page 34

$\mathrm{Char}_{red}.poly_A$ reduced characteristic polynomial over A, page 35

$\dim_K$ dimesnsion over K, page 23

$Spm(R)$ set of all maximal ideals of the ring R, page 16

$\mathfrak{p}(\Omega^{-1}R)$ extension of a prime ideal $\mathfrak{p}$ to a localization of the ring, page 16

$\mathrm{Hom}_R(M,N)$ set of R-module morphisms, page 10

$\kappa_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ residue field at a prime ideal $\mathfrak{p}$, page 16

$\lim_{\longleftarrow}\mathcal{O}_X$ Project limit , page 63

$\lim_{\longrightarrow} R_f$ Inductive limit , page 64

$\mathcal{PSh}_X$ category of presheaves, page 60

$\mathcal{S}(U) \equiv \Gamma(U, \mathcal{S})$ Sections of a sheaf $\mathcal{S}$ over U, page 59

$\mathcal{S}_x$ stalk of a sheaf $\mathcal{S}$ at the point $x \in X$., page 59

$\mathcal{Sh}_X$ category of sheaves, page 62

$\mathscr{E}^*$ Dual to the module sheaf $\mathscr{E}^*$, page 82

$\mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})$, Sheaf of endomorphisms of $\mathscr{O}_X$-algebras, page 77

$\mathscr{N}rd_A(a)$ reduced norm of an element $a \in A$, page 35

$\mathscr{S}^{-1}\mathscr{A}$ sheaf of algebras of fractions of $\mathscr{A}$, page 77

$\Omega^{-1}M$ localizationat of a module M at Ω, page 17

$\Omega^{-1}R$ localization of a ring R at Ω, page 14

$\mathrm{Rank}_{\mathfrak{p}}$ $\mathfrak{p}$-rank, page 23

ρ_V^U restriction morphism on a sheaf or presheaf, page 60

$\mathrm{Spec}(R)$ spectrum of the ring R, page 15

φ^* Dual to the map φ, page 83

$\widetilde{M}$ Module sheaf associated to a module M, page 67

${}_R\mathfrak{M}$ category of left R-modules, page 10

$A^e = A \otimes A^o$ eveloping algebra, page 26

$D(f)$ distinguished open subset of $\mathrm{Spec}(R)$, page 62

f_Ω homomorphism attached to the set Ω, page 14

$K(R)$ quotient field of the integral domain R, page 15

$ker(f_\Omega)$ kernel of a homomorphism f_Ω, page 15

M^A centralizer of M in A, page 27

$M_{\mathfrak{m}}$ localization of a module M at a maximal ideal $\mathfrak{m}$, page 19

$R_{\mathfrak{m}}$ localization of the ring R at a maximal ideal $\mathfrak{m}$, page 16

$R_{\mathfrak{p}}$ localizationat of the ring R at a prime ideal $\mathfrak{p}$, page 15

$Trd_A(a)$ reduced trace of an element $a \in A$, page 35

$\mathfrak{p}R_{\mathfrak{p}}$ extension of a prime ideal $\mathfrak{p}$ to a localization of the ring, page 16

$\varprojlim \widetilde{\sigma}(D(f))$ Projective limit of σ, page 59

$\mathrm{annih}_R(P)$ annihilator of M in R, page 26

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