Second grade fluids with slip boundary conditions

by

CHRISTIAAN LE ROUX

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Title:	Second grade fluids with slip boundary conditions
Name:	Christiaan le Roux
Supervisor:	Prof. N. Sauer
Department:	Mathematics and Applied Mathematics
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Summary

It is well-known that non-Newtonian fluids such as polymers melts do not satisfy the usual adherence boundary condition. On the other hand, the available theory relies heavily on the no-slip assumption. The purpose of this work is to establish the well-posedness of the initial-boundary-value problem for flows of second grade fluids subject to general partial slip boundary conditions. It is assumed that the fluid satisfies the usual thermodynamical restrictions, that the domain of flow is bounded and simply-connected, and that the slip yield stress is zero.

The proof is based on a fixed point formulation of the problem which decomposes it into three linear ones: a Stokes type problem and two transport problems. After proving the solvability of these auxiliary problems by the Faedo-Galerkin method, the existence of a unique classical solution, local in time, is established by means of a Schauder fixed point theorem. Then global *a priori* estimates are derived to obtain a unique global classical solution for sufficiently small data and large viscosity. The solution is found to be stable under mild restrictions on the slip operator.

Titel:	Tweede-graadse vloeistowwe met gly-randvoorwaardes
Naam:	Christiaan le Roux
Promotor:	Prof. N. Sauer
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Opsomming

Dit is algemeen bekend dat nie-Newtonse vloeistowwe soos gesmelte polimere nie die gebruiklike geen-glyding randvoorwaarde bevredig nie. Nogtans steun die bestaande teorie swaar op die geen-glyding aanname. Die doel van hierdie werk is om te toon dat die begin- randwaardeprobleem vir vloeie van tweede-graadse vloeistowwe onderhewig aan algemene gly-randvoorwaardes goedgeformuleer is. Dit word aanvaar dat die vloeistof die gebruiklike termodinamiese beperkings bevredig, dat die vloeigebied begrens en enkelvoudig samehangend is, en dat die gly-drumpelspanning nul is.

Die bewys is gebaseer op 'n dekpuntformulering van die probleem wat dit ontbind in drie lineêre probleme: 'n Stokes-tipe probleem en twee transportprobleme. Die oplosbaarheid van hierdie hulpprobleme word bewys deur die Faedo-Galerkinmetode, en daarna word die bestaan van 'n unieke klassieke oplossing, lokaal in tyd, bewys met behulp van 'n Schauder-dekpuntstelling. Globale *a priori* afskattings word dan afgelei om 'n unieke globale klassieke oplossing te verkry vir klein genoeg data en groot genoeg viskositeit. Die oplossing is stabiel onder matige beperkings op die gly-operator.

To my parents

See for yourselves how little were my labours
compared with the great refreshment I have found.

Sirach 51:27

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Second Grade Fluids with Slip Boundary Conditions

C. le Roux

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Contents

Chapter 1

INTRODUCTION

Heaven knows what seeming nonsense may
not tomorrow be demonstrated truth!
A. N. Whitehead

1.1 Newtonian Fluids and Boundary Slip

The question of whether there is a relative velocity between the fluid and the obstacle when a viscous fluid flows past a fixed obstacle has a long and interesting history. In the absence of reliable experimental observations it is natural to expect that some kind of friction law will apply at the fluid-solid interface. Thus in 1827 C.L.M.H. Navier formulated a boundary condition which admits **partial slip**. According to **Navier's slip law** the tangential component of the stress in the fluid is proportional to the slip velocity, i.e.

$$\left.\begin{aligned} \boldsymbol{v}\cdot\boldsymbol{n} &= \boldsymbol{v}_{\mathrm{w}}\cdot\boldsymbol{n} \\ k(\mathcal{T}\boldsymbol{n})_{\tau} &= -(\boldsymbol{v}-\boldsymbol{v}_{\mathrm{w}})_{\tau} \end{aligned}\right\} \text{ on } \Gamma, \tag{1.1}$$

where $\boldsymbol{v}$ denotes the velocity of the fluid, $\boldsymbol{n}$ is the outward unit normal on the solid surface Γ, $\boldsymbol{v}_{\mathrm{w}}$ is the velocity of the solid,

$$\mathcal{T} = -p\boldsymbol{I} + \mu[\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T] \tag{1.2}$$

is the stress tensor in the fluid, with p the pressure and μ the coefficient of viscosity, and k is a positive constant (k is usually called the slip coefficient, and $1/k$ the coefficient of momentum transfer (or friction), but for simplicity the term *slip coefficient* will henceforth be used to refer to $-1/k$ instead). In 1845 C.G. Stokes also conjectured that the **no-slip** condition

$$\boldsymbol{v} = \mathbf{0} \quad \text{on } \Gamma \tag{1.3}$$

holds at most for small fluid velocities, and that for larger velocities the tangential force is proportional to the square of the slip velocity, i.e.

$$\left.\begin{aligned} &\boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{v}_{\mathrm{w}}\cdot\boldsymbol{n} \\ &k(\boldsymbol{\mathcal{T}n})_{\tau} = -|(\boldsymbol{v}-\boldsymbol{v}_{\mathrm{w}})_{\tau}|(\boldsymbol{v}-\boldsymbol{v}_{\mathrm{w}})_{\tau} \end{aligned}\right\} \text{ on } \Gamma. \tag{1.4}$$

However, by 1850 he had rejected this in favour of the no-slip condition on the basis of experimental observations and the argument that the presence of slip would imply an infinitely greater resistance to the sliding of one portion of fluid past another than to the sliding of fluid over a solid.
[More precisely: any slip between the particles trapped in the surface irregularities (which are large compared with the size of the fluid molecules, so that the fluid-wall interaction is essentially the same as in the fluid) and the neighbouring fluid molecules would create an infinite velocity gradient, and the resulting infinite viscous stress would eliminate the discontinuity instantaneously.]

Due to conflicting experimental results, however, the matter continued to attract much debate for several more decades. This is not surprising in the light of the considerable technical difficulties involved in acquiring accurate measurements of fluid velocities near solid boundaries (as witnessed by the fact that Ludwig Prandtl arrived at the idea of boundary layers only in 1904). For a detailed discussion of these results and related references, see [1, pp. 676–9] (which gives an account starting with observations by Daniel Bernoulli in 1738), [2] and [13, pp. 1213–4]. The relevant fact is that by the end of the nineteenth century the validity of the no-slip condition for most real fluids (as modelled by the Navier-Stokes/Stokes and continuity equations) under moderate pressures and velocities appears to have been well established. The almost universal acceptance of the no-slip condition is based on ([2]):

(a) experiments involving the variation of the physical surface,
(b) comparisons between experimental and theoretical solutions (that satisfy the no-slip condition) of simple flow problems,
(c) direct observations of fluids near surfaces,
(d) arguments involving the molecular interactions between solids (kinetic theory), and more recently,

(e) comparisons between numerical simulations and experimental results of a large array of complex flow problems.

The mathematical convenience of the no-slip condition is undoubtedly another reason why it has been adopted in the vast majority of theoretical and numerical studies to date. Modern, accurate observations [3] indicate that this may not be fully justified (for non-wetting fluids).

One significant exception to the no-slip rule concerns the motion of slightly rarefied gases. As early as 1875, experiments by Kundt and Warburg conclusively demonstrated that gases at low pressures slip past solid surfaces [7, chapter VIII]. Maxwell in 1879, followed by Knudsen and others [8], then proceeded to develop a theory of gas slippage. In particular, when the Knudsen number ($Kn = \lambda/L$, where λ is the mean free path of the gas molecules and L is some characteristic length) is sufficiently large, velocity slip and a temperature jump occur at the wall surface. Moreover, for a certain range of Knudsen numbers this phenomenon has a continuum model: the Navier-Stokes equations subject to partial slip conditions of the form

$$\left.\begin{aligned} k(\boldsymbol{T}\boldsymbol{n})_\tau - k_1\frac{\partial T}{\partial \boldsymbol{s}} &= -(\boldsymbol{v}-\boldsymbol{v}_{\mathrm{w}})_\tau, \\ k_2\frac{\partial T}{\partial n} &= -(T-T_{\mathrm{w}}) \end{aligned}\right\} \text{ on } \Gamma. \tag{1.5}$$

Here k, k_1, k_2 are positive constants, T is the temperature of the gas, T_{w} is wall temperature and $\boldsymbol{s}$ denotes the unit vector in the direction of $(\boldsymbol{v}-\boldsymbol{v}_{\mathrm{w}})_\tau$. (For a unidirectional flow $\boldsymbol{v} = (v_1(x_1,x_2),0,0)$ in the half-space $x_2 \geq 0$ this becomes

$$k\frac{\partial v_1}{\partial x_2} + k_1\frac{\partial T}{\partial x_1} = v_1, \qquad k_2\frac{\partial T}{\partial x_2} = T - T_{\mathrm{w}}, \qquad x_2 = 0.)$$

Various studies have yielded refinements of (1.5), including second-order and nonlinear slip laws ([9, chapter 3], [10, chapter VIII], [11, chapter 7], [12, 13, 14, 15, 16, 17, 18], [19, sections 1.4, 6.2], [20]). These works mostly attempt to obtain analytical/quantitative characterizations of the slip law and to establish the limitations of the continuum model, which is of practical relevance to high altitude aerodynamics, the launching of satellites, etc. [As a rocket travels upwards through the atmosphere, it moves through a continuum region (Navier-Stokes equations, no velocity slip), the so-called *slip flow region* (Navier-Stokes equations, partial slip conditions (1.5)), a transition region (kinetic theory, Maxwell-Boltzmann equation) and the region of free molecular flow (where molecular interactions are negligible).]

Not surprisingly, the gas slippage problem has been the subject of countless numerical simulations (see e.g. [21, 22]).

For ordinary fluids, the free surfaces in free boundary problems (FBPs) are generally modelled as being stress free, so that the condition of **perfect slip** (usually in combination with the condition that the normal stress is proportional to the curvature of the free surface) applies:

$$(\mathcal{T}\boldsymbol{n})_\tau = \boldsymbol{0} \quad \text{on } \Gamma. \tag{1.6}$$

The first existence results for FBPs were obtained only fairly recently, following the work of [23] on the stationary Stokes equations in a fixed domain with perfect boundary slip, i.e. the mixed boundary problem (MBP) with boundary conditions $(1.1)_1$, (1.6). These results for the MBP – which appears as an auxiliary problem in the fixed point approach to the FBP – have since been extended to the time-dependent Stokes and Navier-Stokes equations with the perfect slip condition, as well as with the inhomogeneous **traction condition**

$$(\mathcal{T}\boldsymbol{n})_\tau = \boldsymbol{\sigma}_\tau \quad \text{on } \Gamma, \tag{1.7}$$

where $\boldsymbol{\sigma}_\tau$ is some given function (combined with either $(1.1)_1$ or a condition prescribing the normal stress); see e.g. [24, 25, 26, 27, 28, 29, 30, 31]. The theory of Navier-Stokes FBPs has also grown rapidly. For more detail, see the survey [32] and the references therein.

A fundamental problem in the study of FBPs is the appearance of infinite velocity gradients at the contact lines (points in the two-dimensional case) where the free and rigid surfaces meet. These (apparent) stress singularities, which result from
(a) the presence of edges (corners) in the flow domain, and
(b) the no-slip condition on the fixed part of the boundary,
have been successfully incorporated into the mathematical treatment of FBPs by the use of function spaces with weighted norms, albeit at the expense of much added technical complexity. As the concept of infinite forces (or energy) is physically meaningless, this remains unsatisfactory however.

The inadequacy of the no-slip condition is particularly clear when one considers the motion of the contact line where a fluid-fluid interface meets a solid surface. The analysis of [5] (which also applies to non-Newtonian fluids) shows that although a no-slip condition on the solid surface is kinematically compatible with a moving contact line if the fluid-fluid interface rolls onto or off the solid surface, it necessarily gives rise to a discontinuous velocity field and unbounded gradient, irrespective of the boundary condition on the fluid interface. The natural route – as suggested earlier by [4] – is to relax the no-slip condition on the solid surface by applying a partial slip condition in a neighbourhood (which should ideally not be fixed in advance, but determined as part of the solution) of the contact line. Some numerical and analytical (infinitesimal analysis, etc.) studies – for example [33, 72, 76, 78, 89] – have focused specifically on the stress singularities and the extent to which it can be alleviated by permitting partial slip. In fact, the well-posedness of certain FBPs for the Navier-Stokes equations with partial slip conditions on the fixed boundaries has been established in [34, 35, 36, 37, 38, 39]. On the other hand, the work of [6] indicates that the effect of long-range Van der Waals forces in suppressing the singularity at the contact line may sometimes dominate over that of slippage.

It is worth noting here that the slip coefficient S in Navier's slip law, written as

$$(\boldsymbol{\mathcal{T}}\boldsymbol{n})_\tau = S(\boldsymbol{v} - \boldsymbol{v}_{\mathrm{w}})_\tau + \boldsymbol{\sigma}_\tau \quad \text{on } \Gamma, \tag{1.8}$$

may be a function of $\boldsymbol{x} \in \Gamma$. Thus condition (1.8) (with $\boldsymbol{\sigma}_\tau = \mathbf{0}$) may describe perfect slip ($S = 0$), partial slip ($-\infty < S < 0$) and no-slip (after division by $S = -\infty$) on different parts of the boundary.
(Figure 1 illustrates typical velocity profiles in a neighbourhood of the boundary for these three cases. It is assumed that $\boldsymbol{v} \equiv (v_1(x_2), 0, 0)$, $x_2 \geq 0$, so that (1.8) reduces to

$$\frac{\partial v_1}{\partial x_2} = -Sv_1 = |S|v_1.)$$

With regard to the MBPs for **incompressible** Navier-Stokes flows subject to partial slip, there are some theoretical results for both thermally conductive fluids [40] and isothermal fluids [41, 42, 44, 45, 46], as well as several numerical/analytical studies [47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58]. The well-posedness of the Navier-Stokes equations for **compressible** heat-conducting fluids has also been studied [59, 60, 62]. For the isothermal case, the work of [64, 65] is particularly interesting. They numerically solved an extrusion problem subject to a nonlinear slip law of the form

$$(\boldsymbol{\mathcal{T}}\boldsymbol{n})_\tau = -m_1\left(1 + \frac{m_2}{1 + m_3|(\boldsymbol{v} - \boldsymbol{v}_{\mathrm{w}})_\tau|^2}\right)(\boldsymbol{v} - \boldsymbol{v}_{\mathrm{w}})_\tau \quad \text{on } \Gamma, \tag{1.9}$$

where the m_i are positive material parameters. When the slip velocity is in the range between the local maximum and minimum of this S-shaped slip law (see Figure 2(b)), the flow is unstable and a hysteresis effect – self-sustained oscillations of the pressure drop and the mass flow rate at the exit – occurs.

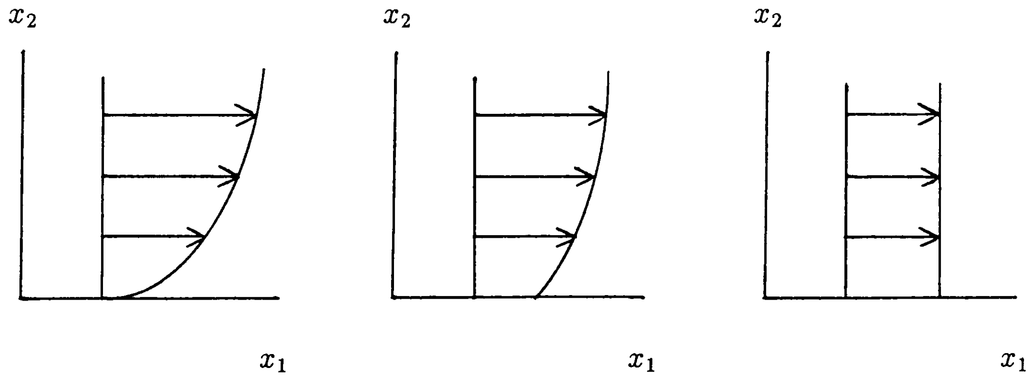
Partial slip conditions are also encountered in a wide variety of situations in which the flow is modelled by the Stokes equations or some other simplification/approximation of the Navier-Stokes equations. Firstly, for viscous flows past/through porous materials, Navier's slip law (referred to as the condition of Beavers and Joseph, or of Saffman, in this context) is applied on the permeable boundary to match the flow inside the porous material (usually described by Darcy's law or a generalization of it) with the Navier-Stokes/Stokes flow on the outside [66, 67, 68, 69, 70, 71, 72, 73]. Navier's law (here called the Maxwell condition) also appears in lubrication models of the motion of a viscous droplet over a solid surface [74, 75, 76, 77] and fluid models of glaciers [78, 79]. Models that approximate the boundary behaviour of fluids may also involve some kind of slip condition [80, 81, 82, 83, 84, 85, 86]. For example, the boundary layer analysis of [87] employs the quadratic slip law (1.4). There does not appear to be many applications to Newtonian flows of **yield stress** slip conditions, i.e. partial slip laws in which the shear stress must exceed some critical value (henceforth called the *slip yield stress*) before slip occurs, but the analytical study [89] is an example. Interestingly, for rarefied gases [88, p. 240] suggests a condition of this kind in which the slip yield stress is proportional to the normal stress, namely (1.1) with k defined by

$$k = \begin{cases} q_1\left(1 - q_2\dfrac{|(\mathcal{T}\boldsymbol{n})\cdot\boldsymbol{n}|}{|(\mathcal{T}\boldsymbol{n})_\tau|}\right) & \text{if } |(\mathcal{T}\boldsymbol{n})_\tau| > q_2|(\mathcal{T}\boldsymbol{n})\cdot\boldsymbol{n}|, \\ 0 & \text{otherwise,} \end{cases} \tag{1.10}$$

where q_1 and q_2 are positive constants. Lastly, one notes that **vorticity** boundary conditions of the type

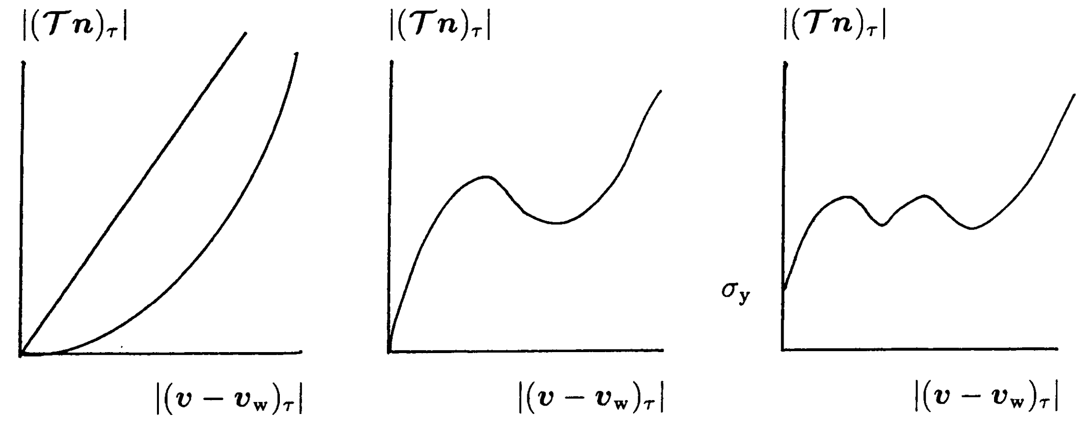
$$\left.\begin{aligned} \boldsymbol{v}\cdot\boldsymbol{n} &= \boldsymbol{v}_{\mathrm{w}}\cdot\boldsymbol{n} \\ (\mathbf{curl}\,\boldsymbol{v} - \boldsymbol{c}) \times \boldsymbol{n} &= \mathbf{0} \end{aligned}\right\} \text{ on } \Gamma,$$

where $\boldsymbol{c}$ is prescribed, also allow tangential slip [91, 92]. [90] gives other examples of such "kinematic" slip conditions.



(a) no slip (b) partial slip (c) perfect slip

Figure 1
Typical velocity profiles in a neighbourhood of a solid boundary.



(a) Navier, Stokes slip (b) non-monotone slip (c) yield stress slip

Figure 2
Examples of partial slip laws.

1.2 Nonlinear Fluids and Boundary Slip

A large variety of fluids, including biological fluids (blood, protein solutions, food, etc.), molten metals, multigrade oils, printing inks, paints, suspensions, polymer solutions and molten plastics, exhibit a wide spectrum of memory and nonlinear effects (dependence of the stress on the deformation history, shear thinning/thickening, stress relaxation, creep, normal stress differences, yield stress) which cannot be described by the linearly viscous Newtonian model (1.2). The study of these interesting substances, collectively known as **non-Newtonian** (or **nonlinear**) fluids, has intensified with the growth of the polymer and plastics industry over the last four decades. In the process a complex body of theory and models has been developed, the detail of which is not important here; see [93, 94, 95, 96] for an overview.

Experimental observations of **wall slip** in the flow of nonlinear fluids such as paint, paste, polymeric solutions, lubricants, hydraulic fracturing fluids, biological fluids, emulsions and polymer melts have been reported since the 1920's ([3]). Recent experimental measurements of polymer melts show that the shear stress at the wall is a nonlinear function (S-shaped, as in Figure 2(b)) of the slip velocity ([97, 98]). In the case of highly entangled polymers, a yield stress slip law applies, and the slip-stress function may have more than one local minimum ([99]; see Figure 2(c)). Several analytical/numerical studies have produced stochastic-mechanical molecular theories (in which the polymers are modelled as beads consisting of Hookean spring-dumbbells) to explain this nonlinear behaviour and to derive formulae for the associated macroscopic slip laws ([100, 101, 102, 103]). The relevant point is that in general non-Newtonian fluids do slip past solid surfaces, and that these slip-stress relations are nonlinear.

Despite the above-mentioned developments, Navier's slip condition – which involves only one parameter – continues to be used in numerical simulations of non-Newtonian flows (for various fluid models) with boundary slip ([104, 105, 106, 107, 108]). To mention one alternative, the slip law in [109] is defined piecewise, with the shear stress proportional to different powers of the magnitude of the slip velocity for different ranges of the slip velocity. Other simple models include perfect slip ([110]) and ones in which the slip velocity is proportional to a power, or a hyperbolic function, of the magnitude of the tangential stress ([111]).

The stick-slip problem – in which there is a sudden transition from no-slip to perfect slip along the boundary, with a resulting stress singularity – has also been analyzed for some non-Newtonian fluid models ([112, 113, 114]). In particular, [113] shows that for a **second grade fluid** (defined in the next section) the shear stress is $O(r^{-1/2})$, where r is the distance to the singularity, while the pressure and the normal stress are $O(r^{-1})$. Hence the total force on the wall behaves like $\ln r$, which is physically unreasonable (not integrable). This suggests that a partial slip condition should be applied – perhaps a yield stress slip condition of the form

$$(\boldsymbol{v} - \boldsymbol{v}_{\mathrm{w}})_\tau = \begin{cases} -f(|(\boldsymbol{\mathcal{T}n})_\tau| - \sigma_{\mathrm{y}}) \dfrac{(\boldsymbol{\mathcal{T}n})_\tau}{|(\boldsymbol{\mathcal{T}n})_\tau|} & \text{if } |(\boldsymbol{\mathcal{T}n})_\tau| > \sigma_{\mathrm{y}}, \\ \mathbf{0} & \text{otherwise,} \end{cases} \tag{1.11}$$

where the non-negative function $f(\cdot)$ and slip yield stress σ_{y} are determined experimentally, as in [114] (for a power-law fluid).

The **existence theory** for non-Newtonian fluids with partial wall slip appears to be scant; I am only aware of the following two results for incompressible nonlinearly viscous fluids: [115] considers the steady rectilinear motion in a cylindrical domain (so that $\boldsymbol{v} \equiv (v_1(x_1, x_2), 0, 0)$) of a fluid with stress tensor

$$\boldsymbol{\mathcal{T}} = -pI + M\boldsymbol{A}_1, \quad \boldsymbol{A}_1 \equiv \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T, \tag{1.12}$$

where the effective viscosity $M = M(|\boldsymbol{A}_1|^2)$, subject to a slip law of the type (1.8) with $S = S(|\boldsymbol{v}|^2, \boldsymbol{x})$, $\boldsymbol{x} \in \partial\Omega$ (which reduces to

$$M(2|\nabla v_1|^2)\frac{\partial v_1}{\partial n} = S(|v_1|^2, \boldsymbol{x})v_1, \quad \boldsymbol{x} \in \partial\Omega.)$$

Under suitable conditions on M and S, the existence of a unique weak solution is established by means of a monotone operator argument. More generally, [116] considers the nonstationary, nonisothermic motion in a bounded domain of a fluid (1.12) with $M = M(|\boldsymbol{A}_1|^2, \theta)$, where θ denotes the temperature, and slip law (1.8) with $S = S(|\boldsymbol{v}|^2, \theta)$. The Faedo-Galerkin method is used to prove the existence of a generalized solution of the initial-boundary-value problem in a given finite time interval. The restrictions imposed on S in these two papers are given in Remark 5.10 on page 95.

1.3 Second Grade Fluids

Amongst the many families of models for nonlinear fluids that were proposed during the past 50 years or so, those of **differential type** (see [94] for the precise meaning of this), also called Rivlin-Ericksen fluids, have proved to be exceptionally successful and popular. An important subclass of the fluids of differential type are the fluids of **complexity** n. The Cauchy stress tensor for an **incompressible** fluid of complexity n is of the form

$$\mathcal{T} = -p\boldsymbol{I} + \mathcal{F}(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n),$$

where the spherical stress $-pI$ reflects the assumption of incompressibility, and $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ are the first n Rivlin-Ericksen tensors ([93]), defined recursively by

$$\begin{aligned} \boldsymbol{A}_1 &= \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T, \\ \boldsymbol{A}_n &= D_t \boldsymbol{A}_{n-1} + \boldsymbol{A}_{n-1}(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^T \boldsymbol{A}_{n-1}, \quad n \geq 2, \end{aligned}$$

where D_t denotes the material time derivative. Thus, for example,

$$\begin{aligned} \boldsymbol{A}_2 &= D_t \boldsymbol{A}_1 + \boldsymbol{A}_1(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^T \boldsymbol{A}_1, \\ \boldsymbol{A}_3 &= D_t \boldsymbol{A}_2 + \boldsymbol{A}_2(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^T \boldsymbol{A}_2 \\ &= D_t^2 \boldsymbol{A}_1 + 2(D_t \boldsymbol{A}_1)(\nabla \boldsymbol{v}) + 2(\nabla \boldsymbol{v})^T (D_t \boldsymbol{A}_1) + \boldsymbol{A}_1(D_t \nabla \boldsymbol{v}) + (D_t \nabla \boldsymbol{v})^T \boldsymbol{A}_1 \\ &\quad + \boldsymbol{A}_1(\nabla \boldsymbol{v})^2 + 2(\nabla \boldsymbol{v})^T \boldsymbol{A}_1(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v}^T)^2 \boldsymbol{A}_1. \end{aligned}$$

Fluids of **grade** n are examples of fluids of complexity n. In particular, the stress tensors for fluids of grades $1, \ldots, 4$, respectively, are assumed to be of the following form ([94, p. 494]):

$$\begin{aligned} \mathcal{T}^{[1]} &= -p\boldsymbol{I} + \mu \boldsymbol{A}_1, \\ \mathcal{T}^{[2]} &= \mathcal{T}^{[1]} + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2, \\ \mathcal{T}^{[3]} &= \mathcal{T}^{[2]} + \beta_1 \boldsymbol{A}_3 + \beta_2(\boldsymbol{A}_1 \boldsymbol{A}_2 + \boldsymbol{A}_2 \boldsymbol{A}_1) + \beta_3 \mathrm{tr}(\boldsymbol{A}_1^2)\boldsymbol{A}_2, \\ \mathcal{T}^{[4]} &= \mathcal{T}^{[3]} + \gamma_1 \boldsymbol{A}_4 + \gamma_2(\boldsymbol{A}_3 \boldsymbol{A}_1 + \boldsymbol{A}_1 \boldsymbol{A}_3) + \gamma_3 \boldsymbol{A}_2^2 \\ &\quad + \gamma_4(\boldsymbol{A}_2 \boldsymbol{A}_1^2 + \boldsymbol{A}_1^2 \boldsymbol{A}_2) + \gamma_5(\mathrm{tr}\boldsymbol{A}_2)\boldsymbol{A}_2 + \gamma_6(\mathrm{tr}\boldsymbol{A}_2)\boldsymbol{A}_1^2 \\ &\quad + [\gamma_7(\mathrm{tr}\boldsymbol{A}_3) + \gamma_8 \mathrm{tr}(\boldsymbol{A}_2 \boldsymbol{A}_1)]\boldsymbol{A}_1, \end{aligned} \tag{1.13}$$

where μ, α_i, β_i and γ_i are (possibly temperature dependent) material coefficients. Hence a first grade fluid is simply a Newtonian fluid, and a second grade fluid is a generalization of it. For a short derivation of the grade 2 tensor (as an expansion to the tensor for a simple fluid with fading memory), see [117, pp. 52 – 54].

The validity of the second (and higher) grade model as an exact description of a real fluid has been the subject of some controversy over many years ([118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133]), despite the seemingly conclusive **stability** study [119]. However, the lucid analysis and survey of [132] – which discusses the somewhat messy history of the debate in depth – has finally resolved the issue. In short, when the material coefficients are consistent with the **restrictions of thermodynamics** (see section 2.1), the second (and third) grade model is a perfectly valid constitutive equation.

Special flows of second and third grade fluids have been investigated analytically and numerically by several authors, and a number of **exact solutions** have been constructed (e.g. [134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 158, 159]). Although the equations of motion for a second grade fluid involves third-order spatial derivatives, the no-slip condition suffices in these studies because of the additional assumptions. However, [161] and [162] provide examples in which the no-slip condition does not determine the solution uniquely. In [163] this is overcome by imposing additional velocity and shear rate conditions at the boundary, but physically meaningful boundary conditions of this kind have not yet been identified. The no-slip condition is also inadequate for certain problems in bounded domains; see [164] and the references therein.

The question of the existence, uniqueness and stability of solutions to the general initial-boundary-value problem for an incompressible second grade fluid in a bounded domain, with no slip, has only been addressed recently. The first step was taken when [165] proved the global (in time) existence and uniqueness of a **generalized solution** to a linearized version of the problem by formulating it in terms of $\boldsymbol{u} \equiv \boldsymbol{v} - \alpha_1 \triangle \boldsymbol{v}$ and applying the Faedo-Galerkin method. Then [166] followed a similar approach (using the quantity $\mathbf{curl}\,(\boldsymbol{v} - \alpha_1 \triangle \boldsymbol{v})$ and applying the Faedo-Galerkin method to the full problem) to obtain a unique solution, global in time if $\Omega \subset \boldsymbol{R}^2$ and local in time if $\Omega \subset \boldsymbol{R}^3$, for flows with

$$\alpha_1 \geq 0, \qquad \alpha_1 + \alpha_2 = 0. \tag{1.14}$$

Similarly, [167] established existence results for certain one-dimensional flows of a so-called *power-law* fluid of grade two (with shear-dependent viscosity).

Significantly, [168] succeeded recently in proving the global existence of the variational solution – which is also classical if the data is sufficiently smooth – by this direct approach.

The first existence and uniqueness results for **classical solutions** were established by [170], who formulated the problem as a fixed point problem in $\boldsymbol{u} \equiv \boldsymbol{v} - \alpha_1 \triangle \boldsymbol{v}$. No restriction is placed on α_2 in [170], but $\alpha_1 > 0$ must be sufficiently large for the global existence result to hold. This is somewhat counter-intuitive, especially as [171] proved that the stationary problem is well-posed for arbitrary values of $\alpha_1 > 0$ and α_2. They considered a fixed point problem in $\boldsymbol{w}$, based on the Helmholtz decomposition $\triangle \boldsymbol{v} = \boldsymbol{w} + \nabla \pi$. This approach was generalized in [172] and [173] to obtain local and global existence results for a class of complexity-2 fluids with shear-dependent viscosity which include second grade fluids that satisfy (1.14), as well as certain third grade fluids (see section 2.1). The main virtue of these fixed point methods (which were apparently adapted from [190]) is that it decomposes the nonlinear problem into linear ones. Furthermore, by taking the data sufficiently small, the nonlinearity can be controlled by the linear terms.

Employing a different fixed point argument (in which the unknown is $\boldsymbol{u} \equiv \mathbf{curl}\,(\boldsymbol{v} - \alpha_1 \triangle \boldsymbol{v})$), [174] showed that the lower bound imposed on α_1 in [170] can be removed if (1.14) holds. This result was extended to third grade fluids by a similar argument in [175]. Moreover, using multivalued fixed point theory, [176] derived a local existence result for flows subject to the Dirichlet boundary condition

$$\boldsymbol{v} = \boldsymbol{v}_* \text{ on } \partial\Omega, \tag{1.15}$$

and showed that the solution is unique if $\boldsymbol{v}_* \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.

Lastly, note that in the above-mentioned existence studies the initial condition is effectively of the form

$$(B\boldsymbol{v})(0) = \boldsymbol{v}_0 \quad \text{in } \Omega,$$

where B is a linear operator determined by the fixed point decomposition of the problem.

1.4 Thesis Problem

The techniques (specifically the evaluation of boundary integrals arising from Green's formulas) used in the stability and existence and studies mentioned above in many instances rely in an essential way on the assumption of no slip, which – as the remarks in the previous sections suggest – may be inappropriate for **second grade fluids**. To evaluate the second grade model (and, for that matter, any other nonlinear model for which no convincing experimental data is available) properly, it should be studied under conditions that allow for the possibility of slip.

The aim of this thesis is therefore to prove the existence and uniqueness of classical solutions – local and global in time – subject to general **stress-slip boundary conditions** of the form

$$\left.\begin{aligned} (\mathcal{T}\boldsymbol{n})_\tau &= \tilde{S}(|\boldsymbol{v}|)\boldsymbol{v} + \tilde{\boldsymbol{d}} \\ \boldsymbol{v}\cdot\boldsymbol{n} &= 0 \end{aligned}\right\} \text{ on } \partial\Omega \times (0,T), \tag{1.16}$$

where $\boldsymbol{n}$ denotes the outward unit normal to $\partial\Omega$, $\tilde{\boldsymbol{d}}$ is a tangential surface force, and the slip coefficient $\tilde{S}$ is assumed to be a smooth function of the magnitude of the slip velocity. (Note that the slip yield stress in (1.16) is zero, i.e. when $\tilde{\boldsymbol{d}} \equiv \boldsymbol{0}$, the slip velocity is nonzero whenever $\boldsymbol{\sigma} \equiv (\mathcal{T}\boldsymbol{n})_\tau$ is.)

It is assumed that (1.14) holds with $\alpha_1 > 0$.

Chapter 2

THE SLIP PROBLEM

Better a slip on the floor than a slip of the tongue.
Sirach 20:18

This chapter starts with an introduction of the governing equations (section 2.1), followed by a discussion of the slip boundary condition (section 2.2) and a (preliminary) formulation of the slip problem as a fixed point problem (section 2.3).

2.1 Equations of Motion

As indicated in section 1.3, the Cauchy stress tensor $\boldsymbol{\mathcal{T}}$ for an incompressible fluid of grade two is given by

$$\boldsymbol{\mathcal{T}} = -\tilde{p}\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2{\boldsymbol{A}_1}^2, \tag{2.1}$$

where $\tilde{p}$ is the pressure, μ is the viscosity, and α_1 and α_2 are the normal stress moduli. $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the first two Rivlin-Ericksen tensors, i.e.

$$\begin{aligned}\boldsymbol{A}_1 &= \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T,\\ \boldsymbol{A}_2 &= D_t\boldsymbol{A}_1 + \boldsymbol{A}_1\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\boldsymbol{A}_1,\end{aligned}$$

with $\boldsymbol{v}$ denoting the velocity field and $D_t \equiv \partial/\partial t + \boldsymbol{v}\cdot\nabla$ the material time derivative. Using the identity

$$\begin{aligned}\boldsymbol{A}_1\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\boldsymbol{A}_1 &= (\nabla\boldsymbol{v})^2 + 2(\nabla\boldsymbol{v})^T(\nabla\boldsymbol{v}) + (\nabla\boldsymbol{v}^T)^2\\ &= \boldsymbol{A}_1^2 + \boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1,\end{aligned}$$

where $\boldsymbol{W} \equiv \dfrac{1}{2}(\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^T)$, the stress tensor can also be written as

$$\boldsymbol{\mathcal{T}} = -\tilde{p}\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1(D_t\boldsymbol{A}_1 + \boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1) + (\alpha_1 + \alpha_2)\boldsymbol{A}_1^2. \tag{2.2}$$

[119] derived necessary and sufficient conditions for a fluid modelled by this relation to be **compatible with thermodynamics** (see pp. 196 – 198 of [119] for the precise meaning of this, and for the formulation of the following two inequalities). In particular, the Clausius-Duhem inequality implies that

$$\mu \geq 0, \quad \alpha_1 + \alpha_2 = 0, \tag{2.3}$$

and the assumption that the Helmholtz free energy is a minimum when the fluid is in rest requires that

$$\alpha_1 \geq 0. \tag{2.4}$$

Under these restrictions the equations of motion (conservation of linear momentum and of the total mass) for an incompressible second grade fluid in a **thermally passive** environment (so that the temperature and therefore also μ, α_1, α_2 are constant) are

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(\boldsymbol{v} - \alpha\Delta\boldsymbol{v}) - \nu\Delta\boldsymbol{v} = \nabla p - \mathbf{curl}\,(\boldsymbol{v} - \alpha\Delta\boldsymbol{v}) \times \boldsymbol{v} + \boldsymbol{g}\\ &\nabla\cdot\boldsymbol{v} = 0\end{aligned}\right\}\text{ in } \Omega \times (0,T), \tag{2.5}$$

where Ω denotes the (fixed) domain of flow in $\boldsymbol{R}^3$, $T > 0$ is a chosen length of time, $\mathbf{curl}\,\boldsymbol{v} = \nabla \times \boldsymbol{v}$, $\alpha = \alpha_1/\rho$, $\nu = \mu/\rho$, ρ is the constant density of the fluid, and the modified pressure

$$p \equiv \phi - \tilde{p}/\rho - \frac{1}{2}|\boldsymbol{v}|^2 + \alpha(\boldsymbol{v}\cdot\triangle\boldsymbol{v} + \frac{1}{4}|\boldsymbol{A}_1|^2). \tag{2.6}$$

Here the body force has been split into a conservative part $\rho\nabla\phi$ and a rotational part $\rho\boldsymbol{g}$ (i.e. $\nabla\cdot\boldsymbol{g} = 0$ in Ω, $\boldsymbol{g}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$) via the Helmholtz decomposition. The derivation of equations (2.5) – (2.6) can be found in section 2.4. Note that once $\boldsymbol{v}$ and ∇p have been determined from (2.5), $\nabla\tilde{p}$ is fixed by (2.6).

Remark 2.1 (a) It is worth pointing out that the derivation of conditions (2.3) – (2.4) in [119] does not require the choice of a boundary condition, and is therefore valid in the present situation as well. On the other hand, the no-slip condition does play a crucial role in the proofs of asymptotic stability in [119, 120], and the unboundedness and instability results (for the case $\alpha_1 < 0$) of [119, 120, 121]. In fact, seemingly insurmountable technical difficulties (due to the non-disappearance of boundary integrals containing higher-order derivatives) would otherwise appear. However, [127] showed that the rest state of a second grade fluid in a half-space (or domains with flat boundaries) with stress-free boundaries, i.e. **perfect slip**, is conditionally stable for arbitrary α_2 if $\mu > 0$, $\alpha_1 > 0$, and unstable if $\alpha_1 < 0$. As the partial slip condition is in a sense intermediate to the extremes of no and perfect slip, this provides additional support for the use of assumptions (2.3) – (2.4) here.

(b) The thermodynamic restrictions on the material moduli of a third grade fluid (see section 1.3) were shown by [120] to be

$$\begin{gathered} \mu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \\ \beta_1 = 0, \quad \beta_2 = 0, \quad \beta_3 \geq 0, \end{gathered} \tag{2.7}$$

which reduce to (2.3) – (2.4) when $\beta_3 = 0$. Under these conditions the stress tensor $(1.13)_3$ becomes

$$\mathcal{T} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 + \beta_3(\mathrm{tr}\boldsymbol{A}_1^2)\boldsymbol{A}_2. \tag{2.8}$$

The corresponding equations of motion are given on page 27 in section 2.4.

2.2 The Slip Boundary Condition

Let Ω be a bounded domain with boundary $\partial\Omega$ of class C^2. Setting $S = \tilde{S}/\rho$ and $\boldsymbol{d} = \tilde{\boldsymbol{d}}/\rho$, the tangential stress condition (1.16) on $\partial\Omega$ becomes

$$\boldsymbol{\sigma} \equiv (\mathcal{T}\boldsymbol{n})_\tau = \rho S(|\boldsymbol{v}|)\boldsymbol{v} + \rho\boldsymbol{d}, \tag{2.9}$$

which can be written as

$$\boldsymbol{\sigma} = \rho R(|\boldsymbol{v}|)\frac{\boldsymbol{v}}{|\boldsymbol{v}|} + \rho\boldsymbol{d}, \tag{2.10}$$

where $R(x) \equiv S(x)x$ corresponds to the slip-stress functions sketched in Figure 2. To simplify the analysis somewhat (in particular the estimates derived below), it will be assumed that S is not a (general) function of $\boldsymbol{y} \in \partial\Omega$. It is also convenient to assume that S is a smooth function of $|\boldsymbol{v}|^2$, but this is not a serious restriction: for any chosen $m \geq 0$ and $M > 0$, one knows from Sobolev's imbedding theorem [201] that there is a constant L such that the ball $B(0, M)$ in $H^{m+2}(\Omega)$ is contained in the ball $B(0, L)$ in $C^m(\overline{\Omega})$. Assume that $\boldsymbol{v} \in \boldsymbol{H}^{m+2}(\Omega)$ with $\|\boldsymbol{v}\|_{m+2} \leq M$, let $S \in C[0, L]$ and let K be any fixed constant. Then, by Weierstrass' theorem, for every $\varepsilon > 0$ there exists a polynomial N_ε such that

$$\|S(\sqrt{x}) - K - N_\varepsilon(x)\|_{C[0,L^2]} < \frac{\varepsilon}{L+1},$$

and thus

$$\|S(x) - (K + N_\varepsilon(x^2))\|_{C[0,L]} < \frac{\varepsilon}{L+1} < \varepsilon, \tag{2.11}$$

$$\|R(x) - (K + N_\varepsilon(x^2))x\|_{C[0,L]} < \frac{\varepsilon L}{L+1} < \varepsilon. \tag{2.12}$$

Hence, without loss of generality (at least when (2.9), (2.10) is viewed as an approximation of an empirical relation), one may assume that $S(x) = K + N(x^2)$, $x \in [0, L]$, for some constant K and function $N \in C^\infty[0, L^2]$. Actually it is sufficient here (see Remark 2.4(e) with regard to the dependence in $\boldsymbol{y}$) to assume the following:

Assumption 2.2 *The slip coefficient S is of the form*

$$S(\boldsymbol{y}, x) = K(\boldsymbol{y}) + N(x^2), \quad (\boldsymbol{y}, x) \in \partial\Omega \times [0, L], \tag{2.13}$$

with $K \in C^m(\partial\Omega)$ and $N \in C^m[0, L^2]$ for some $m \geq 0$.

It will actually be sufficient to take $N \in C^m$. The following lemma shows that N then has the boundedness and continuity properties necessary for the arguments in Sections 3.3 and 4.1:

Lemma 2.3 *Suppose that $m \geq 0$, $f \in H^m(\Omega)$, $N \in C^m[a,b]$ and that $N(f)$ is well-defined in Ω. Then*

(a) *$N(f) \in H^m(\Omega)$ and*

$$\|N(f)\|_m \leq \|N\|_{C^m}(\,|\Omega|^{1/2} + \; C(\Omega, m)[\|f\|_m + \|f\|_m^m]\,). \tag{2.14}$$

(b) *For $m = 0$ or $m \geq 2$, if $N \in C^{m+1}[a,b]$ and g is a function with the same properties as f, then*

$$\|N(f) - N(g)\|_m \leq C(\Omega, m)\|N'\|_{C^m}(1 + \|f\|_m^m + \|g\|_m^m)\|f - g\|_m. \tag{2.15}$$

For $m = 1$, if $f \in H^2(\Omega)$, $g \in H^1(\Omega)$ and $N(g)$ is well-defined, then

$$\|N(f) - N(g)\|_1 \leq C(\Omega)\|N'\|_{C^1}(1 + \|\nabla f\|_1)\|f - g\|_1. \tag{2.16}$$

Proof. The inequalities are derived from the chain rule for the derivatives of $N(f)$; see Section 2.4.

Remark 2.4 (a) It appears that to date only certain diluted polymer suspensions have been clearly identified as ones for which the second grade model is an accurate description ([132]). Moreover, there is seemingly no experimental data available on the interaction of a second grade fluid with a fixed surface – of any material or degree of roughness – that can be used to infer a suitable slip law. The model (2.9), (2.13) – which should be viewed as an additional constitutive law – was chosen because it is sufficiently general to incorporate the vast majority of the models for nonlinear fluids appearing in the literature, including stress-slip relations with multiple local extrema (e.g. the bell-shaped curves of [100] and the refinements thereof in [99]). Moreover, its relation to Navier's slip law is simple and it is mathematically convenient.

(b) One limitation of (2.9) – potentially significant (but also immaterial) in the light of the absence of empirical data – is that it excludes the possibility of a yield stress condition; due to the factor $\boldsymbol{v}$ in the right hand side, $\boldsymbol{v}$ is nonzero whenever $\boldsymbol{\sigma}$ is. As mentioned in Section 1.2, for some nonlinear fluids the modulus $|\boldsymbol{\sigma}|$ of the tangential surface stress must reach a critical value, say σ_y, before macroscopic velocity slip occurs. In the absence of evidence to the contrary, it would be desirable to have a slip model for second grade fluids which allows for this possibility. One would then have an interesting free boundary problem, since (although $\partial\Omega$ is fixed) the slip region is not known in advance, so that one does not know where to apply the slip law and the no-slip condition. In an iterative numerical method this may conceivably be overcome by incorporating a stick-slip test into the solution scheme (e.g. at each point of the boundary, use the velocity field calculated during the previous iteration to test whether $|\boldsymbol{\sigma}| < \sigma_{\mathrm{y}}$: if so, use $\boldsymbol{v} = \mathbf{0}$ there during the present iteration; else apply the slip law there), but there does not appear to be any theoretical results of this kind for nonlinear fluids. The model (2.9), (2.13) does allow for points of no-slip to occur, but it is not necessary to determine these explicitly since (2.9) applies on the whole boundary.

Furthermore, for the method of proof employed here it was necessary to express the tangential surface stress vector $\boldsymbol{\sigma}$ as a function of the slip velocity $\boldsymbol{v}$, and in the case of a yield stress condition this is not possible where $\boldsymbol{v} = \mathbf{0}$ (since one only knows that $|\boldsymbol{\sigma}| < \sigma_{\mathrm{y}}$).

(c) If $\alpha = 0$ and $N \equiv$ constant, then (2.5) and (2.9) reduce to the Navier-Stokes equations and Navier's law, the usual slip law for such fluids. It would therefore be reasonable to assume that $N \equiv N(0)$ if $\alpha = 0$, but this is not necessary here.

Note that one may always assume that $N(0) = 0$; if not, replace N and K by $N - N(0)$ and $K + N(0)$. Then, by the mean-value theorem,

$$\|N(f)\|_0 \leq \|N'\|_{C^0}\|f\|_0,$$

so that the term involving $|\Omega|$ can be dropped from (2.14).

(d) In (2.13) N is expressed as a function of $|\boldsymbol{v}(\boldsymbol{y})|^2$, rather than $|\boldsymbol{v}(\boldsymbol{y})|$, to satisfy the conditions of Lemma 2.3. The only purpose of the coefficient ρ is to help simplify equation (II)* on page 46; it could just as well be absorbed into S.

(e) For the sake of simplicity, it is assumed N is not a function of $\boldsymbol{y} \in \partial\Omega$, but such dependence can easily be incorporated into the work (by adapting the estimates for N). This would allow for possible variations in the stress-slip interaction due to local changes in the roughness or material properties of the boundary surface. As a further generalization – for situations where it cannot reasonably be assumed that the temperature is constant (for example in die casting processes) – one could consider general temperature-dependent slip laws (as in [116]).

2.3 Fixed Point Formulation

The method of [174] can be adapted as follows. Applying the curl operator to equation $(2.5)_1$ and using the identity

$$\mathbf{curl}\,(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\nabla\cdot\boldsymbol{v})\boldsymbol{u} - (\nabla\cdot\boldsymbol{u})\boldsymbol{v} \tag{2.17}$$

with

$$\boldsymbol{u} = B_1\boldsymbol{v} \equiv \mathbf{curl}\,(\boldsymbol{v} - \alpha\Delta\boldsymbol{v}), \tag{2.18}$$

so that $\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{v} = 0$, one obtains

$$\left.\begin{array}{ll} \left.\begin{array}{l} \dfrac{\partial\boldsymbol{u}}{\partial t} + \dfrac{\nu}{\alpha}(\boldsymbol{u} - \mathbf{curl}\,\boldsymbol{v}) = \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \boldsymbol{h} \\ \nabla\cdot\boldsymbol{u} = 0 \end{array}\right\} & \text{in } \Omega \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega, \end{array}\right\} \quad \text{(I)}$$

where $\boldsymbol{h} \equiv \mathbf{curl}\,\boldsymbol{g}$ and $\boldsymbol{u}_0 \equiv B_1\boldsymbol{v}_0$.

Furthermore, using an extension $\hat{\boldsymbol{n}}$ of $\boldsymbol{n}$ to Ω, extend

$$\boldsymbol{a} = B_2\boldsymbol{v} \equiv (\boldsymbol{A}_1\boldsymbol{n})_\tau \tag{2.19}$$

to Ω by defining

$$\boldsymbol{a} = \boldsymbol{A}_1(\boldsymbol{v})\hat{\boldsymbol{n}} - (\hat{\boldsymbol{n}}\cdot\boldsymbol{A}_1(\boldsymbol{v})\hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \quad \text{in } \Omega. \tag{2.20}$$

Then

$$\begin{aligned} \boldsymbol{v}\cdot\nabla\boldsymbol{a}_i &= v_k(A_{1ij}\hat{n}_j - \hat{n}_r A_{1rj}\hat{n}_j\hat{n}_i)_{,k} \\ &= v_k(A_{1ij,k}\hat{n}_j - \hat{n}_r A_{1rj,k}\hat{n}_j\hat{n}_i) \\ &\quad + v_k(A_{1ij}\hat{n}_{j,k} - A_{1rj}(\hat{n}_r\hat{n}_j\hat{n}_i)_{,k}), \ i = 1,2,3, \end{aligned}$$

so that

$$[(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_1]\hat{\boldsymbol{n}} - (\hat{\boldsymbol{n}}\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_1]\hat{\boldsymbol{n}})\hat{\boldsymbol{n}} = \boldsymbol{v}\cdot\nabla\boldsymbol{a} + \boldsymbol{b} \quad \text{in } \Omega$$

and thus

$$([(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_1]\boldsymbol{n})_\tau = \boldsymbol{v}\cdot\nabla\boldsymbol{a} + \boldsymbol{b} \quad \text{on } \partial\Omega, \tag{2.21}$$

where $\boldsymbol{b}$ is defined by its components

$$\begin{aligned} b_i &= A_{1rj}v_k(\hat{n}_r\hat{n}_j\hat{n}_i)_{,k} - A_{1ij}v_k\hat{n}_{j,k} \\ &= A_{1rj}(\boldsymbol{v}\cdot\nabla[\hat{n}_r\hat{n}_j\hat{n}_i]) - A_{1ij}(\boldsymbol{v}\cdot\nabla\hat{n}_j), \ i = 1,2,3. \end{aligned} \tag{2.22}$$

It follows that

$$\begin{aligned}\boldsymbol{\sigma} &= \mu(\boldsymbol{A}_1\boldsymbol{n})_\tau + \alpha_1([\frac{\partial \boldsymbol{A}_1}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_1 + \boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1]\boldsymbol{n})_\tau \\ &= \mu\boldsymbol{a} + \alpha_1(\frac{\partial \boldsymbol{a}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}),\end{aligned}$$

where

$$\boldsymbol{c} \equiv ([\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1]\boldsymbol{n})_\tau, \tag{2.23}$$

so that equation $(1.16)_1$ becomes

$$\left.\begin{array}{ll}\dfrac{\partial \boldsymbol{a}}{\partial t} + \dfrac{\nu}{\alpha}\boldsymbol{a} + \boldsymbol{v}\cdot\nabla\boldsymbol{a} = \dfrac{1}{\alpha}S\boldsymbol{v} - \boldsymbol{b} - \boldsymbol{c} + \dfrac{1}{\alpha}\boldsymbol{d} & \text{on } \partial\Omega\times(0,T), \\ \boldsymbol{a}(0) = \boldsymbol{a}_0 & \text{on } \partial\Omega,\end{array}\right\} \quad (\mathrm{II})^\star$$

where $\boldsymbol{a}_0 \equiv B_2\boldsymbol{v}_0$.

Hence, existence will be proved once it is shown that the mapping

$$\Phi : (\boldsymbol{\phi}, \boldsymbol{\eta}) \mapsto \boldsymbol{v} \mapsto (\boldsymbol{u}, \boldsymbol{a}), \tag{2.24}$$

where $\boldsymbol{v}$ solves

$$\left.\begin{array}{l}\left.\begin{array}{l}\mathbf{curl}\,(\boldsymbol{v} - \alpha\Delta\boldsymbol{v}) = \boldsymbol{\phi} \\ \nabla\cdot\boldsymbol{v} = 0\end{array}\right\} \text{in } \Omega\times(0,T), \\ \\ \left.\begin{array}{l}\boldsymbol{v}\cdot\boldsymbol{n} = 0 \\ (\boldsymbol{A}_1\boldsymbol{n})_\tau = \boldsymbol{\eta}\end{array}\right\} \text{on } \partial\Omega\times(0,T),\end{array}\right\} \quad (\mathrm{III})$$

and $\boldsymbol{u}$ and $\boldsymbol{a}$ are solutions of problems (I) and $(\mathrm{II})^\star$, has a fixed point.

Remark 2.5 (a) Note that $\boldsymbol{a}$ in $(\mathrm{II})^\star$ is understood to denote the trace of a function defined in Ω and that $\boldsymbol{\eta}$ can be extended as in (2.20). Moreover, if $\boldsymbol{\eta} = \boldsymbol{a}$ on $\partial\Omega$, then $\boldsymbol{v}\cdot\nabla\boldsymbol{\eta} = \boldsymbol{v}\cdot\nabla\boldsymbol{a}$ on $\partial\Omega$ since $\boldsymbol{v}$ is tangential to $\partial\Omega$. Thus $(\mathrm{III})_4$ is sufficient to ensure that (2.21) will hold whenever $(\boldsymbol{u}, \boldsymbol{a})$ is a fixed point. By the same argument, it follows from equation $(2.22)_2$ that the trace of $\boldsymbol{b}$ is independent of the choice of $\hat{\boldsymbol{n}}$.

(b) In the exact definition of Φ problem $(\mathrm{II})^\star$ will be replaced by a problem in Ω, namely problem (II) in Proposition 3.11 on page 46.

2.4 Appendix

Derivation of equations (2.5) – (2.6).
The local equations for the balance of linear momentum for a fluid described by (2.2) are

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \tilde{\boldsymbol{f}} &= \frac{1}{\rho}\nabla\cdot\boldsymbol{\mathcal{T}} \\ &= -\frac{1}{\rho}\nabla\tilde{p} + \nu\nabla\cdot\boldsymbol{A}_1 + \alpha\nabla\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{A}_1 + \frac{\partial \boldsymbol{A}_1}{\partial t} + \boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1) \\ &\quad + (\alpha+\beta)\nabla\cdot\boldsymbol{A}_1^2,\end{aligned} \tag{2.25}$$

where $\tilde{\boldsymbol{f}}$ denotes the specific body force per unit mass and $\beta = \alpha_2/\rho$. These equations can be simplified in several ways by means of the identities collected in

Lemma 2.6 *Let* $\boldsymbol{V}_m = \{\boldsymbol{v} \in \boldsymbol{H}^m(\Omega) : \nabla\cdot\boldsymbol{v} = 0\}$, $m = 1, 2, \ldots$, *and for* $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ *set* $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{v} = \nabla \times \boldsymbol{v}$ *and*

$$\boldsymbol{A}_1 = \frac{1}{2}[\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T], \quad \boldsymbol{W} = \frac{1}{2}[\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^T].$$

Furthermore, for any two second order tensors $K = [K_{ij}]$ *and* $L = [L_{ij}]$ *with components in* $H^1(\Omega)$, *and* K *symmetric, define* $\nabla\cdot K$, ∇L *and* $K : \nabla L$ *by*

$$\left.\begin{aligned}&(\nabla\cdot K)_i = K_{ji,j} = K_{ij,j}, \quad i = 1,2,3, \\ &(\nabla L)_{kji} = L_{ji,k}, \quad i,j,k = 1,2,3, \\ &(K : \nabla L)_i = K_{jk}(\nabla L)_{kji} = K_{jk}L_{ji,k}, \quad i = 1,2,3.\end{aligned}\right\} \tag{2.26}$$

Then

(a) $\nabla\cdot\boldsymbol{A}_1 = \triangle\boldsymbol{v} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_2$.

(b) $\nabla\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{A}_1) = \boldsymbol{v}\cdot\nabla(\triangle\boldsymbol{v}) + (\nabla\boldsymbol{v})^T : \nabla\boldsymbol{A}_1 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_3$.

(c) $\nabla\cdot(\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1) = (\nabla\boldsymbol{v})^T\triangle\boldsymbol{v} + 2\nabla\boldsymbol{v} : \nabla\boldsymbol{W} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_3$.

(d) $\nabla\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{A}_1 + \boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1) = 2(\triangle\boldsymbol{W})\boldsymbol{v} + \nabla(\boldsymbol{v}\cdot\triangle\boldsymbol{v} + \frac{1}{4}|\boldsymbol{A}_1|^2)$
$\forall\, \boldsymbol{v} \in \boldsymbol{V}_3$.

(e) $2(\triangle\boldsymbol{W})\boldsymbol{a} = (\triangle\boldsymbol{\omega}) \times \boldsymbol{a} \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^3(\Omega),\ \forall\, \boldsymbol{a} \in \boldsymbol{R}^3$.

(f) $\boldsymbol{v}\cdot\nabla\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v} + \frac{1}{2}\nabla|\boldsymbol{v}|^2 \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$.

(g) $\nabla\cdot\boldsymbol{A}_1^2 = 2\nabla\cdot[\nabla\boldsymbol{v}(\nabla\boldsymbol{v})^T] + \frac{1}{4}\nabla|\boldsymbol{A}_1|^2 + \boldsymbol{A}_1(\triangle\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_2$.

Proof. The arguments are elementary, but since (for this very reason) it does not appear in the literature, it is copied out here to show the precise relation between the terms in the stress tensor and those in the equations of motion. For the sake of clarity, the components of $\boldsymbol{A}_1$ will be denoted without the subscript 1. Where applicable, the relation $\nabla\boldsymbol{\cdot}\boldsymbol{v} = 0$ and the definitions (2.26) (from [210, pp. 39, 62]) are applied without mention.

(a) is self-evident, and (b) is immediate from

$$\begin{aligned}(v_k A_{ij,k})_{,j} &= v_k(v_{i,jjk} + v_{j,jik}) + v_{k,j} A_{ji,k} \\ &= v_k(\triangle v_i)_{,k} + v_{k,j}(\nabla \boldsymbol{A}_1)_{kji}, \quad i = 1,2,3.\end{aligned}$$

(c) For $i = 1,2,3$,

$$\begin{aligned}&(A_{ik}W_{kj} - W_{ik}A_{kj})_{,j} \\ &= \frac{1}{2}(v_{i,kj} + v_{k,ij})(v_{k,j} - v_{j,k}) + \frac{1}{2}(v_{i,k} + v_{k,i})(v_{k,jj} - v_{j,jk}) \\ &\quad - \frac{1}{2}(v_{i,kj} - v_{k,ij})(v_{k,j} + v_{j,k}) - \frac{1}{2}(v_{i,k} - v_{k,i})(v_{k,jj} + v_{j,jk}) \\ &= -v_{i,kj}v_{j,k} + v_{k,ij}v_{k,j} + v_{k,i}v_{k,jj} \\ &= v_{k,j}(v_{k,i} - v_{i,k})_{,j} + v_{k,i}v_{k,jj} \\ &= 2v_{k,j}(\nabla \boldsymbol{W})_{jki} + v_{k,i}\triangle v_k.\end{aligned}$$

(d) From the derivations of (b) and (c) above one has

$$\begin{aligned}&[\nabla\boldsymbol{\cdot}(\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1 + \boldsymbol{v}\boldsymbol{\cdot}\nabla\boldsymbol{A}_1)]_i \\ &= v_{k,i}v_{k,jj} + v_{k,j}(v_{k,ji} - v_{i,jk}) + v_{k,j}(v_{i,jk} + v_{j,ki}) + v_k v_{i,jjk} \\ &= v_k v_{i,jjk} + v_{k,i}v_{k,jj} + v_{k,j}A_{kj,i}, \quad i = 1,2,3.\end{aligned}$$

On the other hand,

$$\begin{aligned}&[2(\triangle\boldsymbol{W})\boldsymbol{v} + \nabla(\boldsymbol{v}\boldsymbol{\cdot}\triangle\boldsymbol{v})]_i \\ &= (v_{i,jkk} - v_{j,ikk})v_j + v_j v_{j,kki} + v_{j,i}v_{j,kk} \\ &= v_k v_{i,jjk} + v_{k,i}v_{k,jj}, \quad i = 1,2,3,\end{aligned}$$

and

$$\frac{1}{4}(\nabla|\boldsymbol{A}_1|^2)_i = \frac{1}{4}(A_{kj}A_{kj})_{,i} = \frac{1}{2}A_{kj}A_{kj,i} = v_{k,j}A_{kj,i}, \quad i = 1,2,3.$$

(e) Set $\boldsymbol{u} = \triangle \boldsymbol{v}$, then $\triangle \boldsymbol{W}(\boldsymbol{v}) = \boldsymbol{W}(\boldsymbol{u})$ and $\triangle \mathbf{curl}\, \boldsymbol{v} = \mathbf{curl}\, \boldsymbol{u}$, so that (e) reduces to the well-known identity

$$2\boldsymbol{W}(\boldsymbol{u}) \times \boldsymbol{a} = \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{a}, \quad \forall\, \boldsymbol{a} \in \boldsymbol{R}^3,$$

which follows easily via the "$\varepsilon\delta$ rule".

(f) It suffices to note that $(\boldsymbol{\omega} \times \boldsymbol{v})_i = 2W_{ij}v_j = (v_{i,j} - v_{j,i})v_j$ and $(\nabla|\boldsymbol{v}|^2)_i = (v_j v_j)_{,i} = 2v_j v_{j,i}$, $i = 1, 2, 3$.

(g) For $i = 1, 2, 3$, $[\nabla\cdot(\boldsymbol{A}_1^2)]_i = A_{ik,j}A_{kj} + A_{ik}A_{kj,j} = I + J$, with

$$I = v_{i,kj}A_{kj} = 2v_{i,kj}v_{j,k} = 2(v_{i,k}v_{j,k})_{,j} = 2(\nabla\cdot[(\nabla\boldsymbol{v})(\nabla\boldsymbol{v})^T])_i$$

and

$$\begin{aligned} J &= v_{k,ij}A_{kj} + A_{ik}A_{kj,j} = \frac{1}{2}A_{kj,i}A_{kj} + A_{ik}v_{k,jj} \\ &= \frac{1}{4}(A_{kj}A_{kj})_{,i} + A_{ik}\triangle v_k = (\frac{1}{4}\nabla|\boldsymbol{A}_1|^2 + \boldsymbol{A}_1\triangle\boldsymbol{v})_i. \end{aligned}$$

Observe that (g) also holds if the condition $\nabla\cdot\boldsymbol{v} = 0$ is replaced by $2v_{i,k}v_{j,kj} = A_{ik}v_{j,kj}$, $i = 1, 2, 3$, or, equivalently, $\boldsymbol{\omega} \times \nabla(\nabla\cdot\boldsymbol{v}) = 0$.

By the Helmholtz decomposition, $\tilde{\boldsymbol{f}}$ is of the form

$$\tilde{\boldsymbol{f}} = \nabla\phi + \boldsymbol{g}, \quad \nabla\cdot\boldsymbol{g} = 0 \text{ in } \Omega \times (0,T), \quad \boldsymbol{g}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega \times (0,T). \tag{2.27}$$

Thus, setting $\hat{p} = \phi - \tilde{p}/\rho$ and applying (a) – (b) to (2.25) yields

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) + \boldsymbol{v}\cdot\nabla(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) - \nu\triangle\boldsymbol{v} - \nabla\hat{p} - \boldsymbol{g} \\ &= \alpha[(\nabla\boldsymbol{v})^T : \nabla\boldsymbol{A}_1 + \nabla\cdot(\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1)] + (\alpha+\beta)\nabla\cdot\boldsymbol{A}_1^2 \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega \times (0,T), \tag{2.28}$$

as in [170, 171, 172]. Alternatively, using (a), (d) – (g) to express the non-linear terms in gradient form and/or in terms of $\boldsymbol{\omega}$, one obtains

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) - \nu\triangle\boldsymbol{v} + (\boldsymbol{\omega} - \alpha\triangle\boldsymbol{\omega}) \times \boldsymbol{v} - \boldsymbol{g} \\ &= \nabla p + (\alpha+\beta)(2\nabla\cdot[(\nabla\boldsymbol{v})(\nabla\boldsymbol{v})^T] + \boldsymbol{A}_1(\triangle\boldsymbol{v})) \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega \times (0,T), \tag{2.29}$$

where

$$p \equiv \phi - \tilde{p}/\rho - \frac{1}{2}|\boldsymbol{v}|^2 + \alpha\boldsymbol{v}\cdot\triangle\boldsymbol{v} + \frac{1}{4}(2\alpha+\beta)|\boldsymbol{A}_1|^2. \tag{2.30}$$

These equations seemingly lend themselves more readily to analysis than (2.28), as is witnessed by the majority of recent publications on second grade fluids; see e.g. [138, 140, 142, 144, 154, 166, 174, 176].

Lastly, noting that $\mathrm{tr}(\boldsymbol{A}_1^2) = |\boldsymbol{A}_1|^2$ and $\nabla\cdot(|\boldsymbol{A}_1|^2\boldsymbol{A}_1) = \boldsymbol{A}_1\nabla|\boldsymbol{A}_1|^2 + |\boldsymbol{A}_1|^2\triangle\boldsymbol{v}$, and comparing (2.8) with (2.1), one arrives at the corresponding form of the equations of motion for an incompressible fluid of grade 3 satisfying conditions (2.7):

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) - \nu\triangle\boldsymbol{v} + (\boldsymbol{\omega} - \alpha\triangle\boldsymbol{\omega}) \times \boldsymbol{v} \\ &= \nabla p + (\alpha+\beta)(2\nabla\cdot[(\nabla\boldsymbol{v})(\nabla\boldsymbol{v})^T] + \boldsymbol{A}_1(\triangle\boldsymbol{v})) \\ &\quad + \gamma(\boldsymbol{A}_1\nabla|\boldsymbol{A}_1|^2 + |\boldsymbol{A}_1|^2\triangle\boldsymbol{v}) + \boldsymbol{g} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega \times (0,T), \qquad (2.31)$$

with p as in (2.30) and $\gamma = \beta_3/\rho$, as in e.g. [148, 175].

The proof of Lemma 2.3 is based on the following chain rule:

Lemma 2.7 *Let $N \in C^m[a,b], m \geq 1, a < b$, and let $f \in H^m(\Omega)$, with Ω a domain in $\boldsymbol{R}^n, n \geq 1$, and suppose that $N \circ f$ is well-defined on Ω. Then, for every $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_1, \ldots, \alpha_n$ nonnegative integers and $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n \leq m$,*

$$D^{\boldsymbol{\alpha}}(N \circ f)(\boldsymbol{x}) = \sum_{i=1}^{|\boldsymbol{\alpha}|} N^{(i)}(f(\boldsymbol{x})) \left[\sum_{\substack{\{\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i\} \\ \in \mathcal{B}(i,\boldsymbol{\alpha})}} C(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i) \prod_{j=1}^{i} D^{\boldsymbol{\beta}^j} f(\boldsymbol{x}) \right], \qquad (2.32)$$

where the constants $C(\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^i)$ are positive integers and

$$\mathcal{B}(i,\boldsymbol{\alpha}) = \{\{\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i\} : |\boldsymbol{\beta}^j| \geq 1, j = 1,\ldots,i, \ \boldsymbol{\beta}^1 + \cdots + \boldsymbol{\beta}^i = \boldsymbol{\alpha}\}.$$

Proof. For a complex function $N : \mathcal{C} \mapsto \mathcal{C}$, sufficient conditions for the validity of such a chain rule, and the form of the terms, are given in [206, p. XIX]. Since the precise formula and proof are omitted, a complete proof seems in order:

It suffices to consider $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = m$. If $m = 1$, then $\boldsymbol{\alpha} = \boldsymbol{e}^h$ for some $1 \leq h \leq n$, where $e_j^h \equiv \delta_{hj}, j = 1,\ldots,n$. Hence (2.32) holds, because

$\mathcal{B}(1,\boldsymbol{\alpha}) = \{\{\boldsymbol{\alpha}\}\}$ and $N \circ f \in L^2(\Omega)$, so that $\partial_h N(f(\boldsymbol{x})) = N'(f(\boldsymbol{x}))\partial_h f(\boldsymbol{x})$ by Theorem 2.1.11 of [214, p. 48].
(One could also use the result of [206, p. XX]: If N is Lipschitz continuous and has a bounded derivative N' with at most countably many discontinuities, and $f \in H^1(\Omega)$, then $N(f) \in H^1(\Omega)$ and the above chain rule holds. Moreover, $N(H_0^1(\Omega)) \subset H_0^1(\Omega)$ if $N(0) = 0$.
Lemma 2.5 of [209, p. 219] states: Suppose that $N \in C^1(\boldsymbol{R})$, $N' \in L^\infty(\boldsymbol{R})$, Ω is an open subset of $\boldsymbol{R}^n$, f is real-valued and locally integrable over Ω and all its first-order weak derivatives on Ω exists. Then all the first-order weak derivatives on Ω of $N \circ f$ exists and the above chain rule holds.)

Suppose that (2.32) holds for a fixed $m \geq 1$ and consider any $\tilde{\boldsymbol{\alpha}}$ with $|\tilde{\boldsymbol{\alpha}}| = m+1$. Choose an $1 \leq h \leq n$ for which $\tilde{\alpha}_h \geq 1$ and set $\boldsymbol{\alpha} = \tilde{\boldsymbol{\alpha}} - \boldsymbol{e}^h$. Then $|\boldsymbol{\alpha}| = m$, so that, by (2.32),

$$D^{\tilde{\boldsymbol{\alpha}}} N(f(\boldsymbol{x})) = \partial_h D^{\boldsymbol{\alpha}} N(f(\boldsymbol{x})) = \sum_{i=1}^{m} N^{(i)}(f(\boldsymbol{x})) \times$$

$$\times \left[\sum_{\substack{\{\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i\} \\ \in \mathcal{B}(i,\boldsymbol{\alpha})}} \sum_{j=1}^{i} C(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i) \left(\prod_{p=1}^{j-1} D^{\boldsymbol{\beta}^p} f\right) \left(\partial_h D^{\boldsymbol{\beta}^j} f\right) \left(\prod_{q=j+1}^{i} D^{\boldsymbol{\beta}^q} f\right) \right]$$

$$+ \sum_{i=1}^{m} N^{(i+1)}(f(\boldsymbol{x})) \left[\sum_{\substack{\{\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^i\} \\ \in \mathcal{B}(i,\boldsymbol{\alpha})}} C(\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^i) \partial_h f(\boldsymbol{x}) \left(\prod_{j=1}^{i} D^{\boldsymbol{\gamma}^j} f(\boldsymbol{x})\right) \right]$$

$$= N'(f(\boldsymbol{x}))\partial_h D^{\boldsymbol{\alpha}} f \; + \; \sum_{i=2}^{m} N^{(i)}(f(\boldsymbol{x})) \left[\sum_{\substack{\{\boldsymbol{\delta}^1,\ldots,\boldsymbol{\delta}^i\} \\ \in \mathcal{B}_h(i,\boldsymbol{\alpha})}} C(\boldsymbol{\delta}^1,\ldots,\boldsymbol{\delta}^i) \prod_{j=1}^{i} D^{\boldsymbol{\delta}^j} f(\boldsymbol{x}) \right]$$

$$+ \; N^{(m+1)}(f(\boldsymbol{x})) \left[\sum_{\substack{\{\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m\} \\ \in \mathcal{B}(m,\boldsymbol{\alpha})}} \partial_h f(\boldsymbol{x}) \left(\prod_{j=1}^{m} D^{\boldsymbol{\gamma}^j} f(\boldsymbol{x})\right) \right]$$

where

$$\begin{aligned}
&\mathcal{B}_h(i,\boldsymbol{\alpha})\\
&\equiv \{\{\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^{j-1},\boldsymbol{\beta}^j+\boldsymbol{e}^h,\boldsymbol{\beta}^{j+1},\ldots,\boldsymbol{\beta}^i\} : \{\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i\}\in\mathcal{B}(i,\boldsymbol{\alpha}),\\
&\quad j=1,\ldots,i\}\cup\{\{\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^{i-1},\boldsymbol{e}^h\} : \{\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^{i-1}\}\in\mathcal{B}(i-1,\boldsymbol{\alpha})\}\\
&=\mathcal{B}(i,\tilde{\boldsymbol{\alpha}}),\qquad 2\le i\le m,
\end{aligned}$$

$\{\{\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m,\boldsymbol{e}^h\} : \{\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m\}\in\mathcal{B}(m,\boldsymbol{\alpha})\}=\mathcal{B}(m+1,\tilde{\boldsymbol{\alpha}})$ (since $\mathcal{B}(m,\boldsymbol{\alpha})=\{\{\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m\}\}$), and the integers $C(\boldsymbol{\delta}^1,\ldots,\boldsymbol{\delta}^i)$ are determined from the corresponding constants $C(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i)$, $C(\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^{i-1})$ by the definition of $\mathcal{B}_h(i,\boldsymbol{\alpha})$. Thus (2.32) holds for $m+1$. □

Proof of Lemma 2.3.
(a) Let

$$P(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i;f)\equiv\prod_{j=1}^{i}D^{\boldsymbol{\beta}^j}f,$$

$$Q(i,\boldsymbol{\alpha},f)\equiv\sum_{\substack{\{\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i\}\\ \in\mathcal{B}(i,\boldsymbol{\alpha})}}C(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i)P(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i;f),$$

$$R_k^2\equiv\sum_{|\boldsymbol{\alpha}|=k}\|D^{\boldsymbol{\alpha}}N(f)\|_0^2,$$

with $\mathcal{B}(i,\boldsymbol{\alpha})$ defined as in Lemma 2.7, then

$$\|N(f)\|_m^2=\sum_{k=0}^{m}R_k^2.$$

For $m=0$ and $m=1$, inequality (2.14) follows from

$$\begin{aligned}
R_0&=\|N(f)\|_0\le\|N\|_{C^0}|\Omega|^{1/2},\\
R_1&=\|\nabla N(f)\|_0=\|N'(f)\nabla f\|_0\le\|N'\|_{C^0}\|\nabla f\|_0.
\end{aligned}$$

Let $m\ge 2$, then for $k=1,\ldots,m$, using (2.32) and the inequality

$$\|x_1+\cdots+x_n\|^2\le(\|x_1\|+\cdots+\|x_n\|)^2\le n(\|x_1\|^2+\cdots+\|x_n\|^2),$$

one gets

$$
\begin{aligned}
R_k^2 &= \sum_{|\boldsymbol{\alpha}|=k} \|\sum_{i=1}^{k} N^{(i)}(f)Q(i,\boldsymbol{\alpha},f)\|_0^2 \\
&\leq k\left[\sum_{|\boldsymbol{\alpha}|=k}\sum_{i=1}^{k} \|N^{(i)}(f)Q(i,\boldsymbol{\alpha},f)\|_0^2\right] \\
&\leq k\|N'\|_{C^{k-1}}^2\left[\sum_{|\boldsymbol{\alpha}|=k}\sum_{i=1}^{k}\|\sum_{\substack{\{\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i\}\\ \in \mathcal{B}(i,\boldsymbol{\alpha})}} C(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i)P(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i;f)\|_0^2\right] \\
&\leq k\tilde{C}(k)\|N'\|_{C^{k-1}}^2\left[\sum_{|\boldsymbol{\alpha}|=k}\sum_{i=1}^{k}\sum_{\{\cdots\}}\|P(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i;f)\|_0^2\right]
\end{aligned}
$$

where

$$
\tilde{C}(k) = \max\{|\mathcal{B}(i,\boldsymbol{\alpha})|C(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i) \;:\; \{\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i\}\in\mathcal{B}(i,\boldsymbol{\alpha}), \quad 1\leq i\leq k,\, |\boldsymbol{\alpha}| = k\}.
$$

It remains to find an estimate for the term inside the summation:

If $i = 1$, then $\{\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i\} = \{\boldsymbol{\alpha}\}$, and

$$
\|P(\boldsymbol{\alpha};f)\|_0 = \|D^{\boldsymbol{\alpha}}f\|_0 \leq \|f\|_k.
$$

If $i = 2$, then $|\boldsymbol{\beta}^j| \leq k-1$, $j = 1,2$, (since $|\boldsymbol{\beta}^j| \geq 1$, $j = 1,\ldots,i$) so that from (3.48),

$$
\|P(\boldsymbol{\beta}^1,\boldsymbol{\beta}^2;f)\|_0 \leq C(\Omega)\|D^{\boldsymbol{\beta}^1}f\|_1\|D^{\boldsymbol{\beta}^2}f\|_1 \leq C(\Omega)\|f\|_k^2.
$$

If $i\geq 3$, then $|\boldsymbol{\beta}^j| \leq k-2$, $j = 1,\ldots,i$, and so by the algebra property of $H^2(\Omega)$,

$$
\|P(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i;f)\|_0 \leq C_1(\Omega)^{i-1}\prod_{j=1}^{i}\|D^{\boldsymbol{\beta}^j}f\|_2 \leq C_1(\Omega)^{i-1}\|f\|_k^i.
$$

Thus

$$R_k^2 \leq k\tilde{C}(k)\breve{C}(\Omega,k)\|N'\|^2_{C^{k-1}}\left[\sum_{|\boldsymbol{\alpha}|=k}\sum_{i=1}^{k}|\mathcal{B}(i,\boldsymbol{\alpha})|.\|f\|_k^{2i}\right]$$

$$\leq \hat{C}(\Omega,k)\|N'\|^2_{C^{m-1}}\left[\sum_{i=1}^{m}\|f\|_m^{2i}\right].$$

where $\breve{C} = \max(1, C(\Omega), C_1(\Omega)^{2k-2})$ and (using $\Omega \subset \boldsymbol{R}^3$) $\hat{C}(\Omega,k) = k(k/2+1)(k+1)\tilde{C}(k)^2\breve{C}$. Hence, using the inequality $x^2+\cdots+x^{2m} \leq m(x^2+x^{2m}) \leq m(x+x^m)^2$, summing over k and setting $C(\Omega,m)^2 = m\sum_{k=1}^{m}\hat{C}(\Omega,k)$, one obtains (2.14):

$$\|N(f)\|_m \leq \|N\|_{C^0}|\Omega|^{1/2} + C(\Omega,m)\|N'\|_{C^{m-1}}[\|f\|_m + \|f\|_m^m].$$

(b) By the mean-value theorem and the inequalities (3.47), (3.48),

$$\|N(f) - N(g)\|_0 \leq \|N'\|_{C^0}\|f-g\|_0$$

and, since $\nabla N(f) = N'(f)\nabla f$,

$$\begin{aligned}
&\|\nabla(N(f) - N(g))\|_0 \\
&\leq \|(N'(f) - N'(g))\nabla f\|_0 + \|N'(g)\nabla(f-g)\|_0 \\
&\leq C(\Omega)\|N''\|_{C^0}\|f-g\|_1\|\nabla f\|_1 + \|N'\|_{C^0}\|\nabla(f-g)\|_0.
\end{aligned}$$

This establishes (2.15) for $m = 0,1$. Now let $m \geq 2$. In addition to the notation defined in (a), set $P(\boldsymbol{\beta}^r,\ldots,\boldsymbol{\beta}^s;\cdot) \equiv 1$ if $r > s$ and let $S(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i;f,g) \equiv P(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i;f) - P(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i;g)$. By applying Lemma 2.7 and rearranging terms one obtains

$$\begin{aligned}
&D^{\boldsymbol{\alpha}}(N(f) - N(g)) \\
&= \sum_{i=1}^{|\boldsymbol{\alpha}|} N^{(i)}(f)\left[\sum_{\substack{\{\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i\}\\ \in \mathcal{B}(i,\boldsymbol{\alpha})}} C(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i)S(\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^i;f,g)\right] \\
&+ \sum_{i=1}^{|\boldsymbol{\alpha}|}[N^{(i)}(f) - N^{(i)}(g)]Q(i,\boldsymbol{\alpha},g).
\end{aligned}$$

and

$$S(\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^i; f, g)$$
$$= \sum_{s=1}^{i} P(\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^{s-1}; f) P(\boldsymbol{\beta}^s; f - g) P(\boldsymbol{\beta}^{s+1}, \ldots, \boldsymbol{\beta}^i; g).$$

As in (a) it follows that for $i = 1$ (and $|\boldsymbol{\alpha}| = k$) this reduces to

$$\|S(\boldsymbol{\alpha}; f, g)\|_0 = \|P(\boldsymbol{\alpha}; f - g)\|_0 \leq \|f - g\|_k,$$

while for $i = 2$ one has

$$\|S(\boldsymbol{\beta}^1, \boldsymbol{\beta}^2; f, g)\|_0$$
$$\leq C(\Omega)\|D^{\boldsymbol{\beta}^1}(f - g)\|_1 \|D^{\boldsymbol{\beta}^2} g\|_1 + C(\Omega)\|D^{\boldsymbol{\beta}^1} f\|_1 \|D^{\boldsymbol{\beta}^2}(f - g)\|_1$$
$$\leq 2C(\Omega)\max(\|f\|_k, \|g\|_k)\|f - g\|_k,$$

and for $i \geq 3$,

$$\|S(\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^i; f, g)\|_0 \leq iC_1(\Omega)^{i-1}\max(\|f\|_k, \|g\|_k)^{i-1}\|f - g\|_k.$$

Moreover, f and g are continuous since $m \geq 2$, so that by the mean-value theorem

$$\|N^{(i)}(f) - N^{(i)}(g)\|_{C^0} \leq C(\Omega)\|N^{(i+1)}\|_{C^0}\|f - g\|_2, \; i = 1, \ldots, m.$$

With the above inequalities in hand, proceeding as in (a) yields

$$\|N(f) - N(g)\|_m^2$$
$$\leq \|N'\|_{C^0}^2 \|f - g\|_0^2 + C(\Omega, m)^2 \|N'\|_{C^m}^2 \left[\sum_{i=1}^{m} \|g\|_m^{2i}\right] \|f - g\|_2^2$$
$$+ C(\Omega, m)^2 \|N'\|_{C^{m-1}}^2 \left[\sum_{i=1}^{m} \max(\|f\|_m, \|g\|_m)^{2i-2}\right] \|f - g\|_m^2,$$

or

$$\|N(f) - N(g)\|_m \leq C(\Omega, m)\|N'\|_{C^m}(1 + \|f\|_m^m + \|g\|_m^m)\|f - g\|_m.$$

Chapter 3

SOLVABILITY OF THE AUXILIARY PROBLEMS

The purpose of this chapter is to establish the well-posedness of the auxiliary problems (I) - (III) and to derive *a priori* estimates for their solutions. First it is necessary to introduce some **notation**:

Ω denotes a bounded and, unless stated otherwise, simply-connected domain of class C^2 in $\boldsymbol{R}^3$, and Ω_T and $\partial\Omega_T$ denote $\Omega \times (0,T)$ and $\partial\Omega \times (0,T)$, respectively. When $\partial\Omega$ is required to have additional regularity, this will be indicated.

For m a nonnegative integer and $1 < q < \infty$, $W^{m,q}(\Omega)$ is the usual Sobolev space with norm $\|\cdot\|_{m,q}$, with $L^q(\Omega)$ denoting the space $W^{m,q}(\Omega)$. For $m \geq 1$, the associated trace space is denoted by $W^{m-1/q,q}(\partial\Omega)$ and its norm by $\|\cdot\|_{m-1/q,q,\partial\Omega}$. $H^m(\Omega)$ denotes the Sobolev space $W^{m,2}(\Omega)$ of order m with inner product $(\cdot,\cdot)_m$ and norm $\|\cdot\|_m$, with $H^0(\Omega)$ denoting $L^2(\Omega)$. The inner product and norm of $H^{m-1/2}(\partial\Omega) \equiv W^{m-1/2,2}(\partial\Omega)$, $m \geq 1$, are denoted by $(\cdot,\cdot)_{m-1/2,\partial\Omega}$ and $\|\cdot\|_{m-1/2,\partial\Omega}$.

For a detailed treatment of these spaces, consult e.g. [201] or [209]. The important properties here are that $H^m(\Omega)$ is a multiplicative algebra for $m \geq 2$, i.e. there is a constant $C_1 = C_1(\Omega)$ such that if $u, v \in H^m(\Omega)$, then $u \cdot v \in H^m(\Omega)$ and

$$\|u \cdot v\|_m \leq C_1 \|u\|_m \|v\|_m, \tag{3.1}$$

and that the trace operator $\gamma_0 : H^m(\Omega) \mapsto H^{m-1/2}(\partial\Omega), m \geq 1$, is bounded, i.e.

$$\|\gamma_0(u)\|_{m-1/2,\partial\Omega} \leq C_2(\Omega, m)\|u\|_m \quad \forall\, u \in H^m(\Omega), \tag{3.2}$$

and surjective, and therefore also has a continuous right inverse. The corresponding spaces of vector fields are denoted by boldface letters, i.e.

$$\boldsymbol{W}^{m,q}(\Omega) = [W^{m,q}(\Omega)]^3,$$

etc., with their inner products and norms denoted as in the case of the scalar fields.

Certain subspaces arise naturally in the treatment of the problem: for $m \geq 0$, set

$$\begin{aligned}
\boldsymbol{V}_m &= \boldsymbol{V}_m(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^m(\Omega) : \nabla\cdot\boldsymbol{v} = 0\},\\
\boldsymbol{X}_m &= \boldsymbol{X}_m(\Omega) = \{\boldsymbol{v} \in \boldsymbol{V}_m : \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},\\
\boldsymbol{Y}_{m+1} &= \boldsymbol{Y}_{m+1}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^{m+1}(\Omega) : \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},\\
\boldsymbol{Z}_{m+1/2} &= \boldsymbol{Z}_{m+1/2}(\partial\Omega) = \{\boldsymbol{a} \in \boldsymbol{H}^{m+1/2}(\partial\Omega) : \boldsymbol{a}\cdot\boldsymbol{n} = 0\}.
\end{aligned}$$

These are Hilbert spaces with the inner products of the associated Sobolev spaces.

Furthermore, for a given time $T > 0$ and Banach space Y with norm $\|\cdot\|_Y$, $L^p(0,T;Y)$ $(1 \leq p < \infty)$ denotes the Banach space of all measurable functions $v : t \in (0,T) \mapsto v(t) \in Y$ such that the norm $\int_0^T \|v(t)\|_Y^p dt$ is finite, $L^\infty(0,T;Y)$ denotes the Banach space of all measurable, essentially bounded functions on $(0,T)$ with values in Y, and
$W^{k,p}(0,T;Y)$ $(k \geq 0, 1 \leq p \leq \infty)$ is the space of functions in $L^p(0,T;Y)$ for which the distributional time derivatives of order up to k are also in this space. In addition,
$C^k([0,T];Y)$ $(k \geq 0)$ denotes the space of k times continuously differentiable functions on the closed interval $[0,T]$ with values in Y.

The norms in $L^p(0,T;\boldsymbol{H}^m(\Omega))$ and $L^p(0,T;\boldsymbol{H}^{m-1/2}(\partial\Omega))$, $1 \leq p < \infty$ are denoted by $\|\cdot\|_{L^p,m,T}$ and $\|\cdot\|_{L^p,m-1/2,T,\partial\Omega}$, respectively.
The usual norms in $W^{k,\infty}(0,T;\boldsymbol{H}^m(\Omega))$ and $W^{k,\infty}(0,T;\boldsymbol{H}^{m-1/2}(\partial\Omega))$ are denoted by $\|\cdot\|_{k,m,T}$ and $\|\cdot\|_{k,m-1/2,T,\partial\Omega}$, respectively, and for $k = 0$ by $\|\cdot\|_{m,T}$ and $\|\cdot\|_{m-1/2,T,\partial\Omega}$.

Lastly, the constant C which appears in inequalities denotes a generic positive constant that may take different values even in the same calculation. Where necessary, constants are fixed by the addition of a subscript or superscript, or by using other letters. The quantities on which a constant may possibly depend are given in brackets.

3.1 The Stokes Problem

The first step in making the definition of the map Φ in Section 2.3 rigorous is to establish the well-posedness of the the auxiliary problem

$$\left.\begin{aligned} &\mathbf{curl}\,(\boldsymbol{v} - \alpha\Delta\boldsymbol{v}) = \boldsymbol{\phi} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{in } \Omega_T, \qquad \left.\begin{aligned} &\boldsymbol{v}\cdot\boldsymbol{n} = 0 \\ &(\boldsymbol{A}_1\boldsymbol{n})_\tau = \boldsymbol{\eta} \end{aligned}\right\} \text{on } \partial\Omega_T. \tag{III}$$

One needs the following two lemmas:

Lemma 3.1 *Suppose that Ω is a bounded, simply-connected domain of class $C^{m+2}, m \geq 0$, and let $\boldsymbol{\phi} \in W^{k,\infty}(0,T;\boldsymbol{V}_m), k \geq 0$. Then there exists a unique vector field $\boldsymbol{\psi} \in W^{k,\infty}(0,T;\boldsymbol{X}_{m+1})$ such that*

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{\phi} \quad \text{in } \Omega_T, \tag{3.3}$$

$$\|\boldsymbol{\psi}\|_{k,m+1,T} \leq C(\Omega,m)\|\boldsymbol{\phi}\|_{k,m,T}. \tag{3.4}$$

Proof. See Lemma 2.1 on p. 300 of [174]. □

Lemma 3.2 *Let Ω be an open bounded set in $\boldsymbol{R}^n, n \geq 2$, with a boundary of class $C^{m+2}, m \geq 0$, and suppose that*

$$(\boldsymbol{v},\pi) \in \boldsymbol{W}^{2,q}(\Omega) \times W^{1,q}(\Omega), \ 1 < q < \infty, \tag{3.5}$$

is a solution of the problem

$$\left.\begin{aligned} \boldsymbol{v} - \alpha\Delta\boldsymbol{v} + \nabla\pi &= \boldsymbol{\psi} \\ \nabla\cdot\boldsymbol{v} &= g \end{aligned}\right\} \text{ in } \Omega, \tag{3.6}$$

$$\left.\begin{aligned} \boldsymbol{v}\cdot\boldsymbol{n} &= h \\ (\boldsymbol{A}_1\boldsymbol{n})_\tau &= \boldsymbol{\eta} \end{aligned}\right\} \text{ on } \partial\Omega. \tag{3.7}$$

If $\boldsymbol{\psi} \in \boldsymbol{W}^{m,q}(\Omega)$, $g \in W^{m+1,q}(\Omega)$, $h \in W^{m+2-1/q,q}(\partial\Omega)$ and $\boldsymbol{\eta} \in \boldsymbol{W}^{m+1-1/q,q}(\partial\Omega)$, then

$$\boldsymbol{v} \in \boldsymbol{W}^{m+2,q}(\Omega), \ \pi \in W^{m+1,q}(\Omega), \tag{3.8}$$

and there exists a constant $C = C(\Omega, m, q, \alpha)$ such that

$$\begin{aligned} &\|\boldsymbol{v}\|_{m+2,q} + \|\nabla\pi\|_{m,q} + \inf_{r\in\mathcal{R}} \|\pi + r\|_{0,q} \\ &\leq C(\|\boldsymbol{\psi}\|_{m,q} + \|g\|_{m+1,q} + \|h\|_{m+2-1/q,q,\partial\Omega} \\ &\quad + \|\boldsymbol{\eta}\|_{m+1-1/q,q,\partial\Omega} + d_q\|\boldsymbol{v}\|_{0,q}) \end{aligned} \tag{3.9}$$

with $d_q = 1$ for $1 < q < 2, d_q = 0$ for $q \geq 2$. Moreover, if $q \geq 2$, the solution is unique.

Proof. The proof of (3.8) and (3.9) is similar to that of Proposition 2.2 in [185] for the Stokes equation with a Dirichlet boundary condition, and is given in full in Section 3.4.

The uniqueness for $q \geq 2$ is proved in the usual way: if $\boldsymbol{u}$ is a solution of the corresponding homogeneous problem, then taking the $\boldsymbol{L}^2(\Omega)$ inner product of $(3.6)_1$ with $\boldsymbol{u}$ and integrating by parts yields

$$\|\boldsymbol{u}\|_0^2 + \frac{\alpha}{2}\|\boldsymbol{A}_1(\boldsymbol{u})\|_0^2 = 0.$$

Remark 3.3 (a) From the proofs of Lemma 4.2 (on page 67) and Theorem 4.4 (on page 71) it is clear that one actually only requires the case $k = 0$ of Lemma 3.1 and Proposition 3.4 below to prove that Φ has fixed point (in the setting defined in Section 4.1), and the case $k = 1$ for deriving the additional regularity of the resulting vector field $\boldsymbol{v}$.

(b) If Ω is a region in $\boldsymbol{R}^3$ obtained by revolution around a vector $\boldsymbol{k}$ through a point $\boldsymbol{x}_0$, then problem (3.6), (3.7) differs from the corresponding Stokes slip problem (without the term $\boldsymbol{v}$ in $(3.6)_1$; see e.g. [23, 25]) in that (3.6) – (3.7) does not impose a compatibility condition on $\boldsymbol{\psi}$ and $\boldsymbol{\eta}$. Taking the $\boldsymbol{L}^2(\Omega)$ inner product of $(3.6)_1$ with $\boldsymbol{u}_0 \equiv \boldsymbol{k} \times (\boldsymbol{x} - \boldsymbol{x}_0)$ (which satisfies $\nabla\cdot\boldsymbol{u}_0 = 0,\ \boldsymbol{A}_1(\boldsymbol{u}_0) = \mathbf{O},\ \boldsymbol{u}_0\cdot\boldsymbol{n} = 0$) simply gives

$$(\boldsymbol{v}, \boldsymbol{u}_0)_0 + \alpha(\boldsymbol{\eta}, \boldsymbol{u}_0)_{0,\partial\Omega} = (\boldsymbol{\psi}, \boldsymbol{u}_0)_0.$$

(Of course one still has the condition

$$\int_\Omega g = \int_{\partial\Omega} h.)$$

Proposition 3.4 *Let Ω be a simply-connected domain in $\boldsymbol{R}^3$ of class C^{m+3}, $m \geq 0$, and, for $k \geq 0$, let*

$$\boldsymbol{\phi} \in W^{k,\infty}(0,T;\boldsymbol{V}_m), \quad \boldsymbol{\eta} \in W^{k,\infty}(0,T;\boldsymbol{Z}_{m+3/2}). \tag{3.10}$$

Then there exists a unique vector field $\boldsymbol{v} \in W^{k,\infty}(0,T;\boldsymbol{X}_{m+3})$ satisfying (III). *Moreover, there is a constant $C_3 = C_3(m,\Omega,\alpha)$ such that*

$$\|\boldsymbol{v}\|_{k,m+3,T} \leq C_3(\|\boldsymbol{\phi}\|_{k,m,T} + \|\boldsymbol{\eta}\|_{k,m+3/2,T,\partial\Omega}). \tag{3.11}$$

Proof. Since Ω is simply connected and $\boldsymbol{\phi}$ is solenoidal, one knows from Lemma 3.1 that there exists a uniquely determined $\boldsymbol{\psi} \in W^{k,\infty}(0,T;\boldsymbol{X}_{m+1})$ such that

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{\phi}, \quad \|\boldsymbol{\psi}\|_{m+1,T} \leq C(m,\Omega)\|\boldsymbol{\phi}\|_{m,T}.$$

Hence it suffices to establish the unique solvability of the problem

$$\left.\begin{aligned} &\boldsymbol{v} - \alpha\Delta\boldsymbol{v} + \nabla\pi = \boldsymbol{\psi} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{in } \Omega_T, \qquad \left.\begin{aligned} &\boldsymbol{v}\cdot\boldsymbol{n} = 0 \\ &(\boldsymbol{A}_1\boldsymbol{n})_\tau = \boldsymbol{\eta} \end{aligned}\right\} \text{on } \partial\Omega_T, \tag{3.12}$$

where $\nabla\pi$ is the irrotational part of $\boldsymbol{\psi} - \boldsymbol{v} + \alpha\Delta\boldsymbol{v}$.

First let $k = 0$. In view of Lemma 3.2 it only remains to prove that, given $\boldsymbol{\psi} \in L^2(\Omega), \boldsymbol{\eta} \in \boldsymbol{Z}_{1/2}$, there exists a solution $\boldsymbol{v} \in \boldsymbol{X}_2, \pi \in H^1(\Omega)$ to (3.12) (with t treated as a parameter). Inequality (3.11) will then follow by taking suprema over $[0,T]$ in (3.9). This result follows by simplification from [27], where the corresponding stationary Navier-Stokes slip problem is considered. The Stokes problem with perfect slip ($\boldsymbol{\eta} = \boldsymbol{0}$) is studied in [23, 25].

For $k \geq 1$, differentiating the equations (3.12) k times with respect to t, using the uniqueness of the solution – in short, replacing $\boldsymbol{\phi}, \boldsymbol{\eta}$ and $\boldsymbol{v}$ by their k-th t-derivatives in (3.12) – and applying the estimate derived for $k = 0$ shows that $\boldsymbol{v} \in W^{k,\infty}(0,T;\boldsymbol{X}_{m+3})$ and that (3.11) holds.

Remark 3.5 (a) As $W^{1,2}(0,T;\boldsymbol{H}^{m+3}(\Omega))$ is continuously imbedded in $C([0,T];\boldsymbol{H}^{m+3}(\Omega))$ (see e.g. [212, p. 480], [206, p. XIII] or [185, Chapter III, Lemma 2.1]), it follows that

$$\boldsymbol{v} \in C([0,T];\boldsymbol{H}^{m+3}(\Omega)) \tag{3.13}$$

if $k \geq 1$ in Proposition 3.4.

(b) It follows from the linearity (in both the data and solution) of the problem and the *a priori* estimate for the stationary problem that if instead of (3.10) one assumes that

$$\boldsymbol{\phi} \in C([0,T]; \boldsymbol{V}_m), \quad \boldsymbol{\eta} \in C([0,T]; \boldsymbol{Z}_{m+3/2}),$$

then $\boldsymbol{v}$ also satisfies (3.13).

3.2 The Transport Problem

One advantage of the fixed point formulation (2.24) over a more direct attack on the problem lies in the relative simplicity of problem (I) (and problem (II) in the next section): the *nonlinear third-order* equation $(2.5)_1$ is transformed to the *linear first-order* equation $(\mathrm{I})_1$. The resulting initial-value problem

$$\left.\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \frac{\nu}{\alpha}(\boldsymbol{u} - \mathbf{curl}\,\boldsymbol{v}) &= \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \boldsymbol{h} && \text{in } \Omega_T, \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega_T, \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0 && \text{in } \Omega \end{aligned}\right\} \text{(I)}$$

is solved in [174] via the Galerkin method and the following inequalities:

Lemma 3.6 (a) *For $m \geq 0$ there is a constant $C_4 = C_4(\Omega, m)$ such that if $\boldsymbol{v} \in \boldsymbol{X}_{m+2}$ and $\boldsymbol{u} \in \boldsymbol{H}^m(\Omega)$, then*

$$|(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})_m| \leq C_4\|\boldsymbol{v}\|_{m+r(m)}\|\boldsymbol{u}\|_m^2 \tag{3.14}$$

with $C_4 = 0$ if $m = 0$, and $r(1) = 2$, $r(2) = 1$ and $r(m) = 0$ for $m \geq 3$.

(b) *For $s = 1, 2, 3$, $\boldsymbol{u} \in \boldsymbol{H}^{3-s}(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{H}^s(\Omega)$,*

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{v}\|_0 \leq C_5(\Omega)\|\boldsymbol{u}\|_{3-s}\|\boldsymbol{v}\|_s. \tag{3.15}$$

(c) *For $s = 1, 2$, $\boldsymbol{u} \in \boldsymbol{H}^{3-s}(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{H}^{1+s}(\Omega)$,*

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{v}\|_1 \leq C_5(\Omega)\|\boldsymbol{u}\|_{3-s}\|\boldsymbol{v}\|_{1+s}. \tag{3.16}$$

(d) *If $m \geq 2$, $\boldsymbol{u} \in \boldsymbol{H}^m(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{H}^{m+1}(\Omega)$, then*

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{v}\|_m \leq C_5(\Omega)\|\boldsymbol{u}\|_m\|\boldsymbol{v}\|_{m+1}. \tag{3.17}$$

Proof. Inequalities (3.14) – (3.16) are proved via the Sobolev imbedding theorem as in [187], and inequality (3.17) follows from (3.1). See Section 3.4 for the detail.

Allowing for greater regularity of the initial velocity field, Lemma 2.4 of [174] can be slightly extended to

Proposition 3.7 *Let q and m satisfy one of the conditions*

(a) $q = 0, m \geq 1$, (b) $q = 1, m \geq 0$, (c) $q = 2, m \geq 0$,

and assume that Ω is class C^{m+q+3}.

If $\boldsymbol{v} \in L^\infty(0,T; \boldsymbol{X}_{m+3})$ with $\|\boldsymbol{v}\|_{m+3,T} \leq M$, $\boldsymbol{h} \in L^\infty(0,T; \boldsymbol{V}_{m+q})$ and $\boldsymbol{u}_0 \in \boldsymbol{H}^{m+q}(\Omega)$, then there exists a unique solution $\boldsymbol{u}$ to $(\mathrm{I})_1, (\mathrm{I})_3$ *such that*

$$\boldsymbol{u} \in L^\infty(0,T; \boldsymbol{H}^{m+q}(\Omega)) \cap W^{1,\infty}(0,T; \boldsymbol{H}^{m-1+q}(\Omega)),$$

$$\|\boldsymbol{u}\|_{m+q,T} + \left\|\frac{d\boldsymbol{u}}{dt}\right\|_{m-1+q,T} \leq C_6(\Omega, m, q, M, T, \frac{\nu}{\alpha}, \|\boldsymbol{h}\|_{m+q,T}, \|\boldsymbol{u}_0\|_{m+q}). \tag{3.18}$$

Moreover, if $\nabla \cdot \boldsymbol{u}_0 = 0$ in Ω, then $\nabla \cdot \boldsymbol{u} = 0$ in Ω_T.

Proof. Case (a) is Lemma 2.4 of [174] and the other two cases are proved in the same way by making use of Lemma 3.6. Note that for each $k \geq 0$, the statement for case (c) with $m = k$ implies case (b) with $m = k+1$, which in turn implies case (a) with $m = k+2$. A detailed proof can be found in Section 3.4. □

Alternatively, to highlight the dependence on the regularity of $\boldsymbol{v}$, one can write Proposition 3.7 as

Proposition 3.8 *Let $m \geq 1$, assume that Ω is of class C^{m+3}, and define $n(1) = 2$, $n(m) = 1$ for $m \geq 2$. If $\boldsymbol{v} \in L^\infty(0,T; \boldsymbol{X}_{m+n(m)})$ with $\|\boldsymbol{v}\|_{m+n(m),T} \leq M$, $\boldsymbol{h} \in L^\infty(0,T; \boldsymbol{V}_m)$ and $\boldsymbol{u}_0 \in \boldsymbol{H}^m(\Omega)$, then there exists a unique solution $\boldsymbol{u}$ to* $(\mathrm{I})_1, (\mathrm{I})_3$ *such that*

$$\boldsymbol{u} \in L^\infty(0,T; \boldsymbol{H}^{m}(\Omega)) \cap W^{1,\infty}(0,T; \boldsymbol{H}^{m-1}(\Omega)),$$

$$\|\boldsymbol{u}\|_{m,T} + \left\|\frac{d\boldsymbol{u}}{dt}\right\|_{m-1,T} \leq C_6(\Omega, m, M, T, \frac{\nu}{\alpha}, \|\boldsymbol{h}\|_{m,T}, \|\boldsymbol{u}_0\|_{m}), \tag{3.19}$$

with $\nabla \cdot \boldsymbol{u} \equiv 0$ if $\nabla \cdot \boldsymbol{u}_0 = 0$.

Proof. The statements for $m = 1$ and $m \geq 2$ follow from parts (b) (with $m = 0$) and (c) of Proposition 3.7, respectively.

Remark 3.9 (a) The condition $\nabla\cdot\boldsymbol{v} \equiv 0$ in Propositions 3.7 and 3.8 is only necessary for establishing the incompressibility of $\boldsymbol{u}$, because the boundary condition

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega_T$$

alone ensures the uniqueness of the solution (see page 56).

(b) It is shown in e.g. [212, Chapter XVIII] that the space

$$\begin{aligned} &W(0,T;\boldsymbol{H}^m(\Omega),\boldsymbol{H}^{m-1}(\Omega)) \\ &\equiv \{\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}^m(\Omega)) : \boldsymbol{u}' \in L^2(0,T;\boldsymbol{H}^{m-1}(\Omega))\}, \end{aligned}$$

which equipped with the inner product

$$(\boldsymbol{u},\boldsymbol{v})_W = (\boldsymbol{u},\boldsymbol{v})_{m,T} + (\boldsymbol{u}',\boldsymbol{v}')_{m-1,T}$$

is a Hilbert space, is continuously imbedded in $C([0,T];\boldsymbol{H}^{m-1/2}(\Omega))$. Hence (3.18) with $q = 0$ implies that

$$\boldsymbol{u} \in C([0,T];\boldsymbol{H}^{m-1/2}(\Omega)). \tag{3.20}$$

(c) Proposition 3.7(a) is sufficient for the arguments in Chapter 4 and has a simple proof (the Galerkin method), but does not establish that $\boldsymbol{u} : [0,T] \mapsto \boldsymbol{H}^m(\Omega)$ is continuous. Using the theory of [186] – [188] (which involves the use of strongly continuous groups), [193] proved the well-posedness of a class of initial-boundary-value problems for transport equations of the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \boldsymbol{\mathcal{A}}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega_T,$$

where $\boldsymbol{\mathcal{A}} = [a_{ij}]$ denotes a matrix, of which $(\mathrm{I})_1$ is a special case. From this one can extract the following result, which shows that the solution does have the *persistence property* if $\boldsymbol{v}$ is sufficiently regular.

Proposition 3.10 *For* $m \geq 0$, *let* Ω *be of class* C^r, $r = \max(1,m)$, *let* $\boldsymbol{h} \in L^1(0,T;\boldsymbol{H}^m(\Omega))$, $\boldsymbol{u}_0 \in \boldsymbol{H}^m(\Omega)$ *and define* $n(0) = 4$, $n(1) = n(2) = 3$, $n(m) = 1$ *for* $m \geq 3$, *and assume that*

$$\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{H}^{m+n(m)}(\Omega)) \cap W^{1,\infty}(0,T;\boldsymbol{H}^{m+n(m)-1}(\Omega)) \tag{3.21}$$

with

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega_T.$$

Then problem $(\mathrm{I})_1, (\mathrm{I})_3$ *has a unique solution* $\boldsymbol{u} \in C([0,T]; \boldsymbol{H}^m(\Omega))$ *and there are constants* $C = C(\Omega, m)$ *such that*

$$\|\boldsymbol{u}\|_{m,T} \leq C(\|\boldsymbol{u}_0\|_m + \frac{\nu T}{\alpha}\|\boldsymbol{v}\|_{m+1,T} + \|\boldsymbol{h}\|_{L^1,m,T})e^{C(\frac{\nu}{\alpha} + \|\boldsymbol{v}\|_{m+n(m),T})T}. \tag{3.22}$$

Moreover, if $m \geq 3$ *then*

$$\|\boldsymbol{u}\|_{m-1,T} \leq C(\|\boldsymbol{u}_0\|_{m-1} + \frac{\nu T}{\alpha}\|\boldsymbol{v}\|_{m,T} + \|\boldsymbol{h}\|_{L^1,m-1,T})e^{C(\frac{\nu}{\alpha} + \|\boldsymbol{v}\|_{m+1,T})T},$$

and if $\nu^\star$, $\alpha^\star$, $\boldsymbol{u}_0^\star$, $\boldsymbol{v}^\star$, $\boldsymbol{h}^\star$ *is another set of functions verifying the above hypotheses and* $\boldsymbol{u}^\star$ *is the corresponding solution, then*

$$\begin{aligned}
&\|\boldsymbol{u} - \boldsymbol{u}^\star\|_{m-1,T} \\
&\leq C\{\|\boldsymbol{u}_0 - \boldsymbol{u}_0^\star\|_{m-1} + T\|\frac{\nu}{\alpha}\boldsymbol{v} - \frac{\nu^\star}{\alpha^\star}\boldsymbol{v}^\star\|_{m,T} + \|\boldsymbol{h} - \boldsymbol{h}^\star\|_{L^1,m-1,T} \\
&\quad + (\|\boldsymbol{u}_0\|_m + \frac{\nu T}{\alpha}\|\boldsymbol{v}\|_{m+1,T} + \|\boldsymbol{h}\|_{L^1,m,T})(|\frac{\nu}{\alpha} - \frac{\nu^\star}{\alpha^\star}| + \|\boldsymbol{v} - \boldsymbol{v}^\star\|_{m,T})\times \\
&\quad \times e^{C(\frac{\nu}{\alpha} + \frac{\nu^\star}{\alpha^\star} + \|\boldsymbol{v}\|_{m+1,T} + \|\boldsymbol{v}^\star\|_{m+1,T})T}\}.
\end{aligned}$$

Proof. Extend $\boldsymbol{v}$ to $[-T, 0]$ by $\boldsymbol{v}(t) = \boldsymbol{v}(-t)$, extend $\boldsymbol{h}$ in the same way and set

$$\boldsymbol{\mathcal{A}} \equiv \frac{\nu}{\alpha}\boldsymbol{I} + \nabla\boldsymbol{v}, \qquad \boldsymbol{f} = \frac{\nu}{\alpha}\mathbf{curl}\,\boldsymbol{v} + \boldsymbol{h}.$$

Then, for $m \leq 2$, assumption (3.21), the imbedding mentioned in Remark 3.5(a) and the Sobolev imbedding $\boldsymbol{H}^{s+2}(\Omega) \hookrightarrow \boldsymbol{C}^s(\overline{\Omega})$ ensure that

$$\begin{aligned}
&\boldsymbol{v}, \boldsymbol{\mathcal{A}} \in L^\infty(-T, T; \boldsymbol{C}^r(\overline{\Omega})) \cap C([-T, T]; \boldsymbol{C}^{r-1}(\overline{\Omega})), \\
&\boldsymbol{f} \in L^1(-T, T; \boldsymbol{H}^m(\Omega)),
\end{aligned}$$

so that one may apply Corollary 2.3 of [193] (with $k = m$, $l = 0$, $n = N = 3$, $p = 2$) to obtain the desired result. Similarly, for $m \geq 3$, (3.21) implies that

$$\begin{aligned}
&\boldsymbol{v}, \boldsymbol{\mathcal{A}} \in L^\infty(-T, T; \boldsymbol{H}^m(\Omega)) \cap C([-T, T]; \boldsymbol{H}^{m-1}(\Omega)), \\
&\boldsymbol{f} \in L^1(-T, T; \boldsymbol{H}^m(\Omega)),
\end{aligned}$$

and therefore Corollaries 2.3* and 2.4* of [193] apply.

3.3 The Boundary Problem

The aim of this section is to formulate and solve a problem in Ω with the property that the trace of its solution is a solution of problem $(\mathrm{II})^*$ (see page 23). A natural approach is to simply extend the terms in the right hand side of $(\mathrm{II})_1^*$ to Ω by using their form on $\partial\Omega$ and an extension $\hat{\boldsymbol{n}}$ of $\boldsymbol{n}$.

Let $\partial\Omega$ be of class $C^{m+4}, m \geq 0$. Then $\boldsymbol{n} \in \boldsymbol{C}^{m+3}(\partial\Omega)$, so that it can be extended to a vector field $\hat{\boldsymbol{n}} \in \boldsymbol{H}^{m+3}(\Omega)$ (with $\|\hat{\boldsymbol{n}}\|_{m+3} \leq C\|\boldsymbol{n}\|_{m+5/2,\partial\Omega} = C(\Omega)$; in fact, according to Proposition 4.9 on p. 251 of [209] one may take $\hat{\boldsymbol{n}} \in \boldsymbol{C}^{m+3}(\overline{\Omega})$). In the same way $K \in C^{m+2}(\partial\Omega)$ can be extended to a function $\hat{K} \in C^{m+2}(\overline{\Omega})$.

Thus, given $\boldsymbol{\phi}, \boldsymbol{\eta}$ and $\boldsymbol{v}$ as in Proposition 3.4, and a constant M such that $\|\boldsymbol{v}\|_{m+3,T} \leq M$, define $\boldsymbol{a}$ as in (2.20) and $\boldsymbol{b}$ as in (2.22). Then, using the algebra property (3.1) of $\boldsymbol{H}^{m+2}(\Omega)$,

$$\begin{aligned}\|\boldsymbol{a}\|_{m+2} &\leq \|\boldsymbol{A}_1(\boldsymbol{v})\hat{\boldsymbol{n}}\|_{m+2} + \|(\hat{\boldsymbol{n}}\cdot\boldsymbol{A}_1(\boldsymbol{v})\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}\|_{m+2} \\ &\leq C\|\nabla\boldsymbol{v}\|_{m+2}(\|\hat{\boldsymbol{n}}\|_{m+2}^2 + 1)\|\hat{\boldsymbol{n}}\|_{m+2},\end{aligned}$$

while

$$\begin{aligned}\|b_i\|_{m+2} &\leq \|A_{1rj}v_k(\hat{n}_r\hat{n}_j\hat{n}_i)_{,k}\|_{m+2} + \|A_{1ij}v_k\hat{n}_{j,k}\|_{m+2} \\ &\leq C_1^2(\|A_{1rj}\|_{m+2}\|v_k\|_{m+2}\|(\hat{n}_r\hat{n}_j\hat{n}_i)_{,k}\|_{m+2} \\ &\quad + \|A_{1ij}\|_{m+2}\|v_k\|_{m+2}\|\hat{n}_{j,k}\|_{m+2}) \\ &\leq C\|\nabla\boldsymbol{v}\|_{m+2}\|\boldsymbol{v}\|_{m+2}(3\|\hat{\boldsymbol{n}}\|_{m+2}^2 + 1)\|\hat{\boldsymbol{n}}\|_{m+3}, \ i = 1,2,3,\end{aligned}$$

with C independent of $\boldsymbol{v}$ (and $m, \hat{\boldsymbol{n}}$), and thus

$$\boldsymbol{a} \in L^\infty(0,T;\boldsymbol{Y}_{m+2}), \quad \|\boldsymbol{a}\|_{m+2,T} \leq C(\Omega,m)M, \tag{3.23}$$

$$\boldsymbol{b} \in L^\infty(0,T;\boldsymbol{H}^{m+2}(\Omega)), \quad \|\boldsymbol{b}\|_{m+2,T} \leq C(\Omega,m)M^2. \tag{3.24}$$

Similarly, define $\boldsymbol{c}$ in Ω by

$$\boldsymbol{c} = (\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1)\hat{\boldsymbol{n}} - (\hat{\boldsymbol{n}}\cdot(\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1)\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}, \tag{3.25}$$

then

$$\begin{aligned}\|c_i\|_{m+2} &\leq C_1\|[\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{A}_1\boldsymbol{W}]_{ij}\|_{m+2}\|\hat{n}_j\|_{m+2} \\ &\quad + C_1^3\|[\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{A}_1\boldsymbol{W}]_{kj}\|_{m+2}\|\hat{n}_k\|_{m+2}\|\hat{n}_j\|_{m+2}\|\hat{n}_i\|_{m+2} \\ &\leq C\|\nabla\boldsymbol{v}\|_{m+2}^2(\|\hat{\boldsymbol{n}}\|_{m+2}^2 + 1)\|\hat{\boldsymbol{n}}\|_{m+2}, \ i = 1,2,3,\end{aligned}$$

with C independent of $\boldsymbol{v}$, and therefore

$$\boldsymbol{c} \in L^\infty(0,T;\boldsymbol{Y}_{m+2}), \quad \|\boldsymbol{c}\|_{m+2,T} \leq C(\Omega,m)M^2. \tag{3.26}$$

In addition, $N(\cdot)$ being independent of $\boldsymbol{y} \in \partial\Omega$, $N(|\boldsymbol{v}|)\boldsymbol{v}$ is well-defined in Ω. Moreover, for any $\boldsymbol{v} \in \boldsymbol{H}^{m+2}(\Omega)$, it follows from (3.1) that $|\boldsymbol{v}|^2 \in H^{m+2}(\Omega)$ with $\||\boldsymbol{v}|^2\|_{m+2} \leq C_1\|\boldsymbol{v}\|_{m+2}^2$, and therefore from (2.14) that

$$\|N(|\boldsymbol{v}|^2)\|_{m+2} \leq \|N\|_{C^{m+2}}(|\Omega|^{1/2} + C[C_1\|\boldsymbol{v}\|_{m+2}^2 + C_1^m\|\boldsymbol{v}\|_{m+2}^{2(m+2)}]). \tag{3.27}$$

Hence, again by (3.1), $S(|\boldsymbol{v}|)\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{Y}_{m+2})$ with

$$\begin{aligned} &\|S(|\boldsymbol{v}|)\boldsymbol{v}\|_{m+2,T} \\ &\leq C(\Omega,m)(\|\hat{K}\|_{C^{m+2}} + \|N\|_{C^{m+2}}(|\Omega|^{1/2} + M^2 + M^{2m+4}))M. \end{aligned} \tag{3.28}$$

(One can replace $m+2$ by $m+3$ here, but (3.28) suffices for $(3.30)_3$ below.) Lastly, if $\boldsymbol{d} \in L^\infty(0,T;\boldsymbol{Z}_{m+3/2})$, then, using any bounded right inverse of the trace map, $\boldsymbol{d}$ can be extended to a vector field $\hat{\boldsymbol{d}} \in L^\infty(0,T;\boldsymbol{Y}_{m+2})$ such that

$$\|\hat{\boldsymbol{d}}\|_{m+2,T} \leq C(\Omega,m)\|\boldsymbol{d}\|_{m+3/2,T,\partial\Omega}.$$

These extensions (plus the requirement on $\boldsymbol{\eta}$ in Proposition 3.4) suggest the following problem:

Given $\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{X}_{m+3})$ with $\|\boldsymbol{v}\|_{m+3,T} \leq M$, find $\boldsymbol{a} \in L^\infty(0,T;\boldsymbol{Y}_{m+2})$ satisfying

$$\left.\begin{aligned} \frac{\partial \boldsymbol{a}}{\partial t} + \frac{\nu}{\alpha}\boldsymbol{a} + \boldsymbol{v}\cdot\nabla\boldsymbol{a} = \frac{1}{\alpha}S(|\boldsymbol{v}|)\boldsymbol{v} - \boldsymbol{b} - \boldsymbol{c} + \frac{1}{\alpha}\hat{\boldsymbol{d}} \quad &\text{in } \Omega_T, \\ \boldsymbol{a}\cdot\boldsymbol{n} = 0 \quad &\text{on } \partial\Omega_T, \\ \boldsymbol{a}(0) = \hat{\boldsymbol{a}}_0 \quad &\text{in } \Omega, \end{aligned}\right\} \tag{3.29}$$

where $\hat{\boldsymbol{a}}_0 \in \boldsymbol{H}^{m+2}(\Omega)$ is defined as in (2.20), using $\boldsymbol{v}_0$ instead of $\boldsymbol{v}$.

However, due to the boundary condition $(3.29)_2$ and the absence of a "complementing" term (corresponding to $\nabla\pi$ in (3.6)) in equation $(3.29)_1$, one cannot expect this problem to be well-posed in general. Moreover, since $\boldsymbol{Y}_1$ is dense in $\boldsymbol{L}^2(\Omega)$ (so that no Helmholtz-type decomposition is possible; $\boldsymbol{Y}_0 \equiv \boldsymbol{L}^2(\Omega)$ as in [191] is the only possibility) and, to my best knowledge, the orthogonal complement of $\boldsymbol{Y}_n$ in $\boldsymbol{H}^n(\Omega)$ has not been characterised for any $n \geq 1$ ([191] only gives a characterisation of the orthogonal complement of $\boldsymbol{X}_n$ in $\boldsymbol{Y}_n$), it is not clear how a proof analogous to that in [189, 192] could be constructed. Instead I shall formulate a problem with a solenoidal solution, so that the properties of $\boldsymbol{X}_n$ can be exploited as in [189, 192].

Extension to Ω.

Given any $\boldsymbol{\phi} \in L^\infty(0,T;\boldsymbol{V}_m)$, $m \geq 0$, $\boldsymbol{\eta} \in L^\infty(0,T;\boldsymbol{X}_{m+2})$ (with the right hand side of (3.11) smaller than M so that $\|\boldsymbol{v}\|_{m+3,T} \leq M$), let $\boldsymbol{v}$ be as in Proposition 3.4, define $\boldsymbol{b}$ as in Section 2.3 and let $\boldsymbol{d} \in L^\infty(0,T;\boldsymbol{Z}_{m+3/2})$. Then, by (3.26), (3.28) and the well-known existence results for the Stokes problem ([183, 184]), there exist unique functions $\breve{\boldsymbol{s}}, \breve{\boldsymbol{c}}, \breve{\boldsymbol{d}} \in L^\infty(0,T;\boldsymbol{X}_{m+2})$ and scalar fields p_1, p_2, p_3 (unique up to a constant) such that

$$\left.\begin{aligned} \triangle\breve{\boldsymbol{s}} + \nabla p_1 &= \boldsymbol{0} \quad \text{in } \Omega_T \\ \nabla\cdot\breve{\boldsymbol{s}} &= 0 \quad \text{in } \Omega_T, \\ \breve{\boldsymbol{s}} &= S(|\boldsymbol{v}|)\boldsymbol{v} \quad \text{on } \partial\Omega_T, \\ \|\breve{\boldsymbol{s}}\|_{m+2,T} & \\ \leq C(\Omega,m)(\|\hat{K}\|_{C^{m+2}} &+ \|N\|_{C^{m+2}}(|\Omega|^{1/2} + M^2 + M^{2m+4}))M, \end{aligned}\right\} \tag{3.30}$$

$$\left.\begin{aligned} \triangle\breve{\boldsymbol{c}} + \nabla p_2 &= \boldsymbol{0} \quad \text{in } \Omega_T \\ \nabla\cdot\breve{\boldsymbol{c}} &= 0 \quad \text{in } \Omega_T, \\ \breve{\boldsymbol{c}} &= ([\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1]\boldsymbol{n})_\tau \quad \text{on } \partial\Omega_T, \\ \|\breve{\boldsymbol{c}}\|_{m+2,T} &\leq C(\Omega,m)M^2, \end{aligned}\right\} \tag{3.31}$$

and

$$\left.\begin{aligned} \triangle\breve{\boldsymbol{d}} + \nabla p_3 &= \boldsymbol{0} \quad \text{in } \Omega_T \\ \nabla\cdot\breve{\boldsymbol{c}} &= 0 \quad \text{in } \Omega_T, \\ \breve{\boldsymbol{d}} &= \boldsymbol{d} \quad \text{on } \partial\Omega_T, \\ \|\breve{\boldsymbol{d}}\|_{m+2,T} &\leq C(\Omega,m)\|\boldsymbol{d}\|_{m+3/2,T,\partial\Omega}. \end{aligned}\right\} \tag{3.32}$$

Furthermore, by (2.21) and $(\mathrm{III})_4$,

$$\int_{\partial\Omega} \boldsymbol{b}\cdot\boldsymbol{n} = -\int_{\partial\Omega} (\boldsymbol{v}\cdot\nabla\boldsymbol{a})\cdot\boldsymbol{n} = -\int_{\partial\Omega} (\boldsymbol{v}\cdot\nabla\boldsymbol{\eta})\cdot\boldsymbol{n},$$

and therefore there exists a unique vector field $\breve{\boldsymbol{b}} \in L^\infty(0,T;\boldsymbol{Y}_{m+2})$ solving the Stokes problem

$$\left.\begin{aligned} \Delta\breve{\boldsymbol{b}} + \nabla p_4 = \mathbf{0} \quad &\text{in } \Omega_T \\ \nabla\cdot\breve{\boldsymbol{b}} = -\nabla\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{\eta}) = -(\nabla\boldsymbol{v})^T : \nabla\boldsymbol{\eta} \quad &\text{in } \Omega_T, \\ \breve{\boldsymbol{b}} = \boldsymbol{b} \quad &\text{on } \partial\Omega_T, \\ \|\breve{\boldsymbol{b}}\|_{m+2,T} \le C(\Omega,m)(M + \|\boldsymbol{\eta}\|_{m+2,T})M, \quad & \end{aligned}\right\} \tag{3.33}$$

the last estimate following with the help of the trace theorem, (3.24) and the inequality

$$\|\nabla\boldsymbol{v}^T : \nabla\boldsymbol{\eta}\|_{m+1} \le \begin{cases} C(\Omega)\|\boldsymbol{v}\|_3\|\boldsymbol{\eta}\|_2 & \text{if } m = 0, \\ C(\Omega)\|\boldsymbol{v}\|_{m+2}\|\boldsymbol{\eta}\|_{m+2} & \text{if } m \ge 1, \end{cases} \tag{3.34}$$

which is easily proved by means of (3.47) and (3.48) (see the proof of (3.16), $s = 1$) and (3.1). Lastly, with $\boldsymbol{a}_0$ defined as in Section 2.3, one can find $\boldsymbol{w}_0 \in \boldsymbol{X}_{m+2}$ such that $\boldsymbol{w}_0 = \boldsymbol{a}_0$ on $\partial\Omega$ and $\|\boldsymbol{w}_0\|_{m+2} \le C(\Omega,m)\|\boldsymbol{a}_0\|_{m+3/2,\partial\Omega}$ (via a Stokes problem, for example). For any such choice of $\boldsymbol{w}_0$ one has

Proposition 3.11 *Let Ω be a bounded domain of class $C^{m+4}, m \ge 1$, and suppose that $\boldsymbol{\eta}, \boldsymbol{v}, \breve{\mathfrak{s}}, \breve{\boldsymbol{b}}, \breve{\mathfrak{c}}$ and $\breve{\boldsymbol{d}}$ are as described above. Then there exists a unique solution*

$$\boldsymbol{w} \in L^\infty(0,T;\boldsymbol{X}_{m+2}) \cap W^{1,\infty}(0,T;\boldsymbol{X}_{m+1}), \;\; \nabla q \in L^\infty(0,T;\boldsymbol{H}^{m+2}(\Omega)) \tag{3.35}$$

to the problem

$$\left.\begin{aligned} \frac{\partial\boldsymbol{w}}{\partial t} + \frac{\nu}{\alpha}\boldsymbol{w} + \boldsymbol{v}\cdot\nabla\boldsymbol{w} + \nabla q = \breve{\boldsymbol{f}} \quad &\text{in } \Omega_T, \\ \nabla\cdot\boldsymbol{w} = 0 \quad &\text{in } \Omega_T, \\ \boldsymbol{w}\cdot\boldsymbol{n} = 0 \quad &\text{on } \partial\Omega_T, \\ \boldsymbol{w}(0) = \boldsymbol{w}_0 \quad &\text{in } \Omega, \end{aligned}\right\} \text{(II)}$$

where $\breve{\boldsymbol{f}} \equiv \breve{\mathfrak{s}}/\alpha - \breve{\boldsymbol{b}} - \breve{\mathfrak{c}} + \breve{\boldsymbol{d}}/\alpha \in L^\infty(0,T;\boldsymbol{Y}_{m+2})$ and $\boldsymbol{w}_0 \in \boldsymbol{X}_{m+2}$. Moreover,

$$\begin{aligned} &\|\boldsymbol{w}\|_{m+2,T} + \|\frac{d\boldsymbol{w}}{dt}\|_{m+1,T} + \|\nabla q\|_{m+2,T} \le C_7(\Omega,m,T,\nu,\alpha,M,\ldots \\ &\ldots\|\hat{K}\|_{C^{m+2}}, \|N\|_{C^{m+2}}, \|\boldsymbol{w}_0\|_{m+2}, \|\boldsymbol{d}\|_{m+3/2,T,\partial\Omega}, \|\boldsymbol{\eta}\|_{m+2,T}). \end{aligned} \tag{3.36}$$

Proof. The proof follows that of [189, 192], and is given in Section 3.4. The restriction $m \geq 1$ is necessary for the derivation of an *a priori* estimate of the irrotational term in $(\mathrm{II})_1$.

Remark 3.12 (a) The function q in (II) is the solution (unique up to a constant) of the problem

$$\begin{aligned} \triangle q &= \nabla\cdot(\check{\boldsymbol{f}} - \boldsymbol{v}\cdot\nabla\boldsymbol{w}) = -\nabla\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{w} + \check{\boldsymbol{b}}) \\ &= \nabla\cdot(\boldsymbol{v}\cdot\nabla[\boldsymbol{\eta} - \boldsymbol{w}]) = (\nabla\boldsymbol{v})^T : \nabla(\boldsymbol{\eta} - \boldsymbol{w}) \quad \text{in } \Omega_T, \\ \frac{\partial q}{\partial n} &= (\check{\boldsymbol{f}} - \boldsymbol{v}\cdot\nabla\boldsymbol{w})\cdot\boldsymbol{n} = -(\boldsymbol{v}\cdot\nabla\boldsymbol{w} + \boldsymbol{b})\cdot\boldsymbol{n} \qquad \text{on } \partial\Omega_T, \end{aligned}$$

the second equality in both equations following from (3.30) – (3.33). Now, if $\boldsymbol{w} \equiv \boldsymbol{\eta}$, then (by $(\mathrm{III})_4$ and (2.19)) $\boldsymbol{w} = \boldsymbol{\eta} = (\boldsymbol{A}_1\boldsymbol{n})_\tau = \boldsymbol{a}$ on $\partial\Omega_T$, and thus, by Remark 2.5(a) and (2.21),

$$\boldsymbol{v}\cdot\nabla\boldsymbol{w} = \boldsymbol{v}\cdot\nabla\boldsymbol{a}, \quad \boldsymbol{v}\cdot\nabla\boldsymbol{w} + \boldsymbol{b} = ([(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_1]\boldsymbol{n})_\tau \;\text{ on } \partial\Omega_T.$$

This implies that $\nabla q \equiv 0$ and, taking the trace of equation $(\mathrm{II})_1$, that the slip boundary condition is satisfied.

(b) By the argument leading to (3.20), it follows from (3.35) that

$$\boldsymbol{w} \in C([0,T]; \boldsymbol{H}^{m+3/2}(\Omega)).$$

In fact, assuming only that $\boldsymbol{d} \in L^1(0,T;\boldsymbol{Z}_{m+3/2})$, so that $\check{\boldsymbol{d}} \in L^1(0,T;\boldsymbol{X}_{m+2})$ with

$$\|\check{\boldsymbol{d}}\|_{L^1,m+2,T} \leq C(\Omega,m)\|\boldsymbol{d}\|_{L^1,m+3/2,T,\partial\Omega},$$

one sees from $(3.30)_4$, $(3.31)_4$ and $(3.33)_4$ that $\check{\boldsymbol{f}} \in L^1(0,T;\boldsymbol{H}^{m+2}(\Omega))$ with

$$\begin{aligned} \|\check{\boldsymbol{f}}\|_{L^1,m+2,T} \;&\leq C(\Omega,m,\alpha,\|\hat{K}\|_{C^{m+2}},\|N\|_{C^{m+2}},\|\boldsymbol{v}\|_{m+3,T},\|\boldsymbol{\eta}\|_{m+2,T})T \\ &\quad + C(\Omega,m)\|\boldsymbol{d}\|_{L^1,m+3/2,T,\partial\Omega}. \end{aligned}$$

Hence, by virtue of the statements in (a), the following lemma – a slight variation of Proposition 3.10 – shows that if $(\boldsymbol{u},\boldsymbol{w})$ is a fixed point of Φ (as defined in Section 4.1), then

$$\boldsymbol{w} \in C([0,T]; \boldsymbol{H}^{m+2}(\Omega)). \tag{3.37}$$

Lemma 3.13 *For $m \geq 1$, let Ω be of class C^{m+2}, let $\boldsymbol{w}_0 \in \boldsymbol{H}^{m+2}(\Omega)$, $\boldsymbol{f} \in L^1(0,T;\boldsymbol{H}^{m+2}(\Omega))$, and assume that*

$$\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{H}^{m+2}(\Omega)) \cap W^{1,\infty}(0,T;\boldsymbol{H}^{m+1}(\Omega))$$

with

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega_T.$$

Then the initial-value problem

$$\left.\begin{aligned} \frac{\partial \boldsymbol{w}}{\partial t} + \frac{\nu}{\alpha}\boldsymbol{w} + \boldsymbol{v}\cdot\nabla\boldsymbol{w} = \boldsymbol{f} \quad &\text{in } \Omega_T, \\ \boldsymbol{w}(0) = \boldsymbol{w}_0 \quad &\text{in } \Omega, \end{aligned}\right\} \tag{3.38}$$

has a unique solution $\boldsymbol{w} \in C([0,T];\boldsymbol{H}^{m+2}(\Omega))$ and there are constants $C = C(\Omega,m)$ such that

$$\begin{aligned} \|\boldsymbol{w}\|_{m+2,T} &\leq C(\|\boldsymbol{w}_0\|_{m+2} + \|\boldsymbol{f}\|_{L^1,m+2,T})e^{C(\frac{\nu}{\alpha} + \|\boldsymbol{v}\|_{m+2,T})T}, \\ \|\boldsymbol{w}\|_{m+1,T} &\leq C(\|\boldsymbol{w}_0\|_{m+1} + \|\boldsymbol{f}\|_{L^1,m+1,T})e^{C(\frac{\nu}{\alpha} + \|\boldsymbol{v}\|_{m+2,T})T}. \end{aligned} \tag{3.39}$$

Moreover, if $\nu^\star$, $\alpha^\star$, $\boldsymbol{w}_0^\star$, $\boldsymbol{f}^\star$, $\boldsymbol{v}^\star$ is another set of functions verifying the above hypotheses and $\boldsymbol{w}^\star$ is the corresponding solution, then

$$\begin{aligned} &\|\boldsymbol{w} - \boldsymbol{w}^\star\|_{m+1,T} \leq C\{\|\boldsymbol{w}_0 - \boldsymbol{w}_0^\star\|_{m+1} + \|\boldsymbol{f} - \boldsymbol{f}^\star\|_{L^1,m+1,T} \\ &+ (\|\boldsymbol{w}_0\|_{m+2} + \|\boldsymbol{f}\|_{L^1,m+2,T})(|\frac{\nu}{\alpha} - \frac{\nu^\star}{\alpha^\star}| + \|\boldsymbol{v} - \boldsymbol{v}^\star\|_{m+1,T})\times \\ &\times e^{C(\frac{\nu}{\alpha} + \frac{\nu^\star}{\alpha^\star} + \|\boldsymbol{v}\|_{m+2,T} + \|\boldsymbol{v}^\star\|_{m+2,T})T}\}. \end{aligned}$$

Proof. As in the proof of Proposition 3.10, extend $\boldsymbol{v}$ and $\boldsymbol{f}$ symmetrically to $[-T,0]$ and set $\boldsymbol{\mathcal{A}} \equiv (\nu/\alpha)\boldsymbol{I}$, so that

$$\begin{aligned} &\boldsymbol{v}, \boldsymbol{\mathcal{A}} \in L^\infty(-T,T;\boldsymbol{H}^{m+2}(\Omega)) \cap C([-T,T];\boldsymbol{H}^{m+1}(\Omega)), \\ &\boldsymbol{f} \in L^1(-T,T;\boldsymbol{H}^{m+2}(\Omega)), \end{aligned}$$

and then apply Corollaries 2.3* and 2.4* of [193].

3.4 Appendix

Proof of Lemma 3.2.
Firstly, write equations (3.6) in the form

$$\begin{array}{lllll}
(\triangle - \frac{1}{\alpha})v_1 & & & -\frac{1}{\alpha}\pi_{,1} & = -\frac{1}{\alpha}\psi_1, \\
& (\triangle - \frac{1}{\alpha})v_2 & & -\frac{1}{\alpha}\pi_{,2} & = -\frac{1}{\alpha}\psi_2, \\
& & \ddots & & \vdots \\
& & (\triangle - \frac{1}{\alpha})v_n & -\frac{1}{\alpha}\pi_{,n} & = -\frac{1}{\alpha}\psi_n, \\
v_{1,1} & + v_{2,2} & \ldots \quad + v_{n,n} & & = g.
\end{array}$$

Let $v_{n+1} = -\pi/\alpha$ and $\mathbf{f} = (-\psi_1/\alpha, \ldots, -\psi_n/\alpha, g)$, then this becomes

$$\sum_{j=1}^{N} \ell_{ij}(\boldsymbol{\partial})v_j(\boldsymbol{x}) = f_i(\boldsymbol{x}) \text{ in } \Omega,\ i = 1, \ldots, N, \tag{3.40}$$

where $N = n + 1, \boldsymbol{\partial} = (\partial_1, \ldots, \partial_n)$, and the matrix $[\ell_{ij}(\boldsymbol{\xi})], \boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \boldsymbol{R}^n$, is given by

$$\begin{array}{l}
\ell_{ij}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2\delta_{ij} - 1/\alpha,\ |\boldsymbol{\xi}|^2 = \xi_1^2 + \cdots + \xi_n^2,\ i, j = 1, \ldots, n, \\
\ell_{nj}(\boldsymbol{\xi}) = -\ell_{jn}(\boldsymbol{\xi}) = \xi_j,\ j = 1, \ldots, n, \\
\ell_{n+1,n+1}(\boldsymbol{\xi}) = 0.
\end{array} \tag{3.41}$$

Following the proof of Proposition 2.2 in [185, p. 34] for the Stokes problem with a Dirichlet boundary condition, define two systems of weights by $s_1 = \ldots = s_n = 0, s_{n+1} = -1$, and $t_1 = \ldots t_n = 2, t_{n+1} = 1$. Then $s_i \leq 0$ and degree$(\ell_{ij}(\boldsymbol{\xi})) \leq s_i + t_j$, as required by [184, p. 38]. The matrix $[\ell'_{ij}(\boldsymbol{\xi})]$, where $\ell'_{ij}(\boldsymbol{\xi})$ consists of the terms in $\ell_{ij}(\boldsymbol{\xi})$ that are of order $s_i + t_j$ in $\boldsymbol{\xi}$, is then identical to the corresponding matrix in [185]:

$$[\ell'_{ij}(\boldsymbol{\xi})] = \begin{bmatrix}
|\boldsymbol{\xi}|^2 & & & & -\xi_1 \\
& |\boldsymbol{\xi}|^2 & & & -\xi_2 \\
& & \ddots & & \vdots \\
& & & |\boldsymbol{\xi}|^2 & -\xi_n \\
\xi_1 & \xi_2 & \ldots & \xi_n & 0
\end{bmatrix} \tag{3.42}$$

It is easy to show by induction that $\mathcal{L}(\boldsymbol{\xi}) \equiv \det[\ell'_{ij}(\boldsymbol{\xi})] = |\boldsymbol{\xi}|^{2n}$, so that $\mathcal{L}(\boldsymbol{\xi}) \neq 0$ for nonzero real $\boldsymbol{\xi}$, i.e. (3.6) is **elliptic**. In fact, (3.6) is **uniformly elliptic** in the sense of [184] (with $m = n, A = 1$ in (1.7) of [184]).

Moreover, the **supplementary condition on $\mathcal{L}$** is satisfied: $\mathcal{L}(\boldsymbol{\xi})$ is of even degree $2n$, and for every pair of linearly independent real vectors $\boldsymbol{\xi}, \boldsymbol{\xi}'$ – in particular, for each point $\boldsymbol{x}$ of $\partial\Omega$, $\boldsymbol{\xi}$ a tangent and $\boldsymbol{\xi}'$ a normal at $\boldsymbol{x}$ – the polynomial $\mathcal{L}(\boldsymbol{\xi}+\tau\boldsymbol{\xi}')$ in τ has exactly n roots with positive imaginary part, namely $\tau^+(\boldsymbol{\xi},\boldsymbol{\xi}') = i|\boldsymbol{\xi}|/|\boldsymbol{\xi}'|$:

$$\begin{aligned}\mathcal{L}(\boldsymbol{\xi}+\tau\boldsymbol{\xi}') &= [(\boldsymbol{\xi}+\tau\boldsymbol{\xi}')\cdot(\boldsymbol{\xi}+\tau\boldsymbol{\xi}')]^n \\ &= (|\boldsymbol{\xi}|^2+|\boldsymbol{\xi}'|^2\tau^2)^n \\ &= |\boldsymbol{\xi}'|^{2n}(\tau - i|\boldsymbol{\xi}|/|\boldsymbol{\xi}'|)^n(\tau + i|\boldsymbol{\xi}|/|\boldsymbol{\xi}'|)^n\end{aligned}$$

Let $\boldsymbol{\tau}^h(\boldsymbol{x}) \in \boldsymbol{C}^{m+1}(\partial\Omega), h = 1,\ldots,n$, denote a system of orthonormal vectors spanning the tangent plane at $\boldsymbol{x} \in \partial\Omega$ and set $\boldsymbol{\phi} = (\boldsymbol{\eta}\cdot\boldsymbol{\tau}^1,\ldots,\boldsymbol{\eta}\cdot\boldsymbol{\tau}^{n-1},h)$. Then the n boundary conditions (3.7) can be expressed as

$$\sum_{j=1}^{N} \mathcal{B}_{hj}(\boldsymbol{x},\boldsymbol{\partial})v_j(\boldsymbol{x}) = \phi_h(\boldsymbol{x}) \quad \text{on } \partial\Omega, \quad h = 1,\ldots,n,$$

where

$$[\mathcal{B}_{hj}(\boldsymbol{x},\boldsymbol{\xi})] = \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1n} & 0 \\ d_{21} & d_{22} & \ldots & d_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \ldots & d_{n-1,n} & 0 \\ n_1(\boldsymbol{x}) & n_2(\boldsymbol{x}) & \ldots & n_n(\boldsymbol{x}) & 0 \end{bmatrix} \tag{3.43}$$

with $d_{hj}(\boldsymbol{x},\boldsymbol{\xi}) \equiv \tau_j^h(\boldsymbol{x})(\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{\xi}) + \eta_j(\boldsymbol{x})(\boldsymbol{\tau}^h(\boldsymbol{x})\cdot\boldsymbol{\xi})$, $h = 1,\ldots,n-1$, $j = 1,\ldots,n$.

Take $r_1 = \ldots = r_n = -1$ and $r_{n+1} = -2$, then degree$(\mathcal{B}_{hj}) \le r_h + t_j$ and $[\mathcal{B}'_{hj}] = [\mathcal{B}_{hj}]$, where $\mathcal{B}'_{hj}(\boldsymbol{x},\boldsymbol{\xi})$ consists of the terms in $\mathcal{B}_{hj}(\boldsymbol{x},\boldsymbol{\xi})$ that are of order $r_h + t_j$ in $\boldsymbol{\xi}$. Now it only remains to verify the **complementing boundary condition** (which ensures that (3.6), (3.7) is coercive):

For an arbitrary $\boldsymbol{x} \in \partial\Omega$, let $\boldsymbol{n}$ denote the (outward) unit normal at $\boldsymbol{x}$, let $\boldsymbol{\xi}$ be any nonzero real tangent vector to $\partial\Omega$ at $\boldsymbol{x}$, and define $\mathcal{L}^{jk}(\cdot) \equiv \ell'_{jk}(\cdot)$, $j,k = 1,\ldots,N$. Then

$$[\mathcal{B}'_{hj}(\boldsymbol{\xi}+\tau\boldsymbol{n})][\mathcal{L}_{jk}(\boldsymbol{\xi}+\tau\boldsymbol{n})]$$
$$= \begin{bmatrix} (\tau_1^1\tau + b_1 n_1)Q(\tau) & \dots & (\tau_n^1\tau + b_1 n_n)Q(\tau) & 2b_1\tau \\ (\tau_1^2\tau + b_2 n_1)Q(\tau) & \dots & (\tau_n^2\tau + b_2 n_n)Q(\tau) & 2b_2\tau \\ \vdots & & \vdots & \vdots \\ (\tau_1^{n-1}\tau + b_{n-1} n_1)Q(\tau) & \dots & (\tau_n^{n-1}\tau + b_{n-1} n_n)Q(\tau) & 2b_{n-1}\tau \\ n_1 Q(\tau) & \dots & n_n Q(\tau) & \tau \end{bmatrix} \tag{3.44}$$

where $Q(\tau) \equiv |\boldsymbol{\xi}|^2 + \tau^2, b_h \equiv \boldsymbol{\tau}^h \cdot \boldsymbol{\xi}$.
(Since $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0, |\boldsymbol{n}| = 1$, one has
$|\boldsymbol{\xi}+\tau\boldsymbol{n}|^2 = Q(\tau),\ \ d_{hj}(\boldsymbol{x},\boldsymbol{\xi}+\tau\boldsymbol{n}) = \tau_j^h\tau + n_j(\boldsymbol{\tau}^h\cdot\boldsymbol{\xi}) = \tau_j^h\tau + b_h n_j$,
$\sum_{j=1}^n(\tau_j^h\tau + b_h n_j)(\xi_j + \tau n_j) = 2b_h\tau,\ \ \sum_{j=1}^n n_j(\xi_j+\tau n_j) = \tau$.)

Let $\tau^+ = \tau^+(\boldsymbol{\xi},\boldsymbol{n}) = i|\boldsymbol{\xi}|$, set $\mathcal{M}^+(\tau) = (\tau-\tau^+)^n$ and suppose that $\mathcal{C} = (\mathcal{C}_1,\dots,\mathcal{C}_n)$ is a constant vector with the property that, as polynomials in τ,

$$\sum_{h=1}^n \mathcal{C}_h(\sum_{j=1}^N \mathcal{B}'_{hj}\mathcal{L}^{jk}) \equiv 0 \pmod{\mathcal{M}^+},\ \ k=1,\dots,n,$$

i.e.

$$\begin{aligned}&\mathcal{C}\cdot[\tau(\tau_k^1,\dots,\tau_k^{n-1},0) + n_k(b_1,\dots,b_{n-1},1)](\tau-\tau^+)(\tau+\tau^+)\\ &\equiv 0 \pmod{\mathcal{M}^+},\ k=1,\dots,n-1,\end{aligned} \tag{3.45}$$

$$\tau\mathcal{C}\cdot(2b_1,\dots,2b_{n-1},1) \equiv 0 \pmod{\mathcal{M}^+}. \tag{3.46}$$

From (3.46) one gets $\mathcal{C}\cdot(2b_1,\dots,2b_{n-1},1) = 0$. Thus, if $n=2$, then (3.45) implies that

$$\mathcal{C}\cdot(\tau_k^1,\dots,\tau_k^{n-1},0) = -i|\boldsymbol{\xi}|n_k\mathcal{C}\cdot(b_1,\dots,b_{n-1},1) = -\frac{i}{2}|\boldsymbol{\xi}|n_k\mathcal{C}_n,$$

so that

$$(\mathcal{C}_1,\dots,\mathcal{C}_{n-1},\frac{i}{2}|\boldsymbol{\xi}|\mathcal{C}_n)\cdot(\tau_k^1,\dots,\tau_k^{n-1},n_k) = 0,\ k=1,\dots,n,$$

or

$$\mathcal{C}_1\boldsymbol{\tau}^1 + \dots + \mathcal{C}_{n-1}\boldsymbol{\tau}^{n-1} + \frac{i}{2}|\boldsymbol{\xi}|\mathcal{C}_n\boldsymbol{n} = \boldsymbol{0},$$

and therefore $\mathcal{C} = \boldsymbol{0}$. If $n \geq 3$, (3.45) implies that
$\mathcal{C}_1\boldsymbol{\tau}^1 + \dots + \mathcal{C}_{n-1}\boldsymbol{\tau}^{n-1} = \boldsymbol{0}$ and $\mathcal{C}\cdot(b_1,\dots,b_{n-1},1) = 0$, so that $\mathcal{C} = \boldsymbol{0}$. Hence the rows of $[\mathcal{B}'_{hj}][\mathcal{L}^{jk}]$ are linearly independent modulo $\mathcal{M}^+(\tau)$.

The statement of the lemma now follows by applying the classical Theorem 10.5 of [184]. The change in notation is

$$\begin{array}{ll}
N \equiv n+1 \ (n \geq 2), & p \equiv q, \\
t' \equiv \max(t_j) = 2 & m \equiv n, A = 1, \\
\ell_1 \equiv \max(0, r_h + 1) = 0, & \ell \equiv m \geq 0, \\
\ell + t_j \equiv m + 2 \ (j = 1, \dots, n), & \ell + t_{n+1} \equiv m + 1, \\
\ell - s_i \equiv m \ (i = 1, \dots, n), & \ell - s_{n+1} \equiv m + 1, \\
\ell - r_h \equiv m + 1 \ (h = 1, \dots, n), & \ell - r_{n+1} \equiv m + 2.
\end{array}$$

□

Proof of Lemma 3.6.
Recall the well-known estimates

$$\|fg\|_0 \leq C(\Omega)\|f\|_2\|g\|_0, \quad f \in H^2(\Omega), \ g \in L^2(\Omega), \tag{3.47}$$

$$\|fg\|_0 \leq C(\Omega)\|f\|_1\|g\|_1, \quad f, g \in H^1(\Omega), \tag{3.48}$$

which follow from the Cauchy-Schwarz inequality and the imbeddings $H^2(\Omega) \hookrightarrow C_B(\Omega)$ and $H^1(\Omega) \hookrightarrow L^4(\Omega)$, respectively.

(a) One has

$$(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})_m = \sum_{|\alpha| \leq m} (D^\alpha(\boldsymbol{v}\cdot\nabla\boldsymbol{u}), D^\alpha\boldsymbol{u})_0$$

with

$$D^\alpha(\boldsymbol{v}\cdot\nabla\boldsymbol{u}) = (\boldsymbol{v}\cdot\nabla)D^\alpha\boldsymbol{u} + \sum_{0 < \beta \leq \alpha} C_{\alpha,\beta}(D^\beta\boldsymbol{v}\cdot\nabla)D^{\alpha-\beta}\boldsymbol{u},$$

$$C_{\alpha,\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\binom{\alpha_3}{\beta_3}.$$

As in [187], $((\boldsymbol{v}\cdot\nabla)D^\alpha\boldsymbol{u}, D^\alpha\boldsymbol{u})_0 = 0$ since $\boldsymbol{v} \in \boldsymbol{X}_{m+2}$. Thus inequality (3.14) follows from the following inequalities for $0 < \beta \leq \alpha$ (which satisfy $|\alpha - \beta| = |\alpha| - |\beta|$):

If $|\beta| = 1$, then by (3.47),

$$\|(D^\beta\boldsymbol{v}\cdot\nabla)D^{\alpha-\beta}\boldsymbol{u}\|_0 \leq C\|D^\beta\boldsymbol{v}\|_2\|\nabla D^{\alpha-\beta}\boldsymbol{u}\|_0 \leq C\|\boldsymbol{v}\|_3\|\boldsymbol{u}\|_{|\alpha|}.$$

If $|\beta| = 2$, then by (3.48),

$$\|(D^\beta\boldsymbol{v}\cdot\nabla)D^{\alpha-\beta}\boldsymbol{u}\|_0 \leq C\|D^\beta\boldsymbol{v}\|_1\|\nabla D^{\alpha-\beta}\boldsymbol{u}\|_1 \leq C\|\boldsymbol{v}\|_3\|\boldsymbol{u}\|_{|\alpha|}.$$

If $|\beta| \geq 3$, then by (3.47),

$$\|(D^\beta \boldsymbol{v}\cdot\nabla)D^{\alpha-\beta}\boldsymbol{u}\|_0 \leq C\|D^\beta \boldsymbol{v}\|_0\|\nabla D^{\alpha-\beta}\boldsymbol{u}\|_2 \leq C\|\boldsymbol{v}\|_{|\beta|}\|\boldsymbol{u}\|_{|\alpha|-|\beta|+3}.$$

(b) The inequalities for $s = 1, 2, 3$ follow from inequalities (3.47), (3.48) and (3.47), respectively.

(c) In the light of the estimates (3.15) for $\|\boldsymbol{v}\cdot\nabla\boldsymbol{u}\|_0$ it suffices to note that, by (3.47) and (3.48), for every $1 \leq i, k \leq 3$,

$$\|u_{j,k}v_{i,j}\|_0 \leq C\|u_{j,k}\|_1\|v_{i,j}\|_1 \leq C\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2,$$
$$\|u_j v_{i,jk}\|_0 \leq C\|u_j\|_2\|v_{i,jk}\|_0 \leq C\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2,$$

and

$$\|u_{j,k}v_{i,j}\|_0 \leq C\|u_{j,k}\|_0\|v_{i,j}\|_2 \leq C\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_3,$$
$$\|u_j v_{i,jk}\|_0 \leq C\|u_j\|_1\|v_{i,jk}\|_1 \leq C\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_3.$$

(d) This is clear from the algebra property (3.1) of $\boldsymbol{H}^m(\Omega)$ for $m \geq 2$. □

For the proof of Proposition 3.7 and some of the proofs in the later chapters one needs a version of **Gronwall's lemma**. In the literature this usually refers to results of the type in (a) – (c) below, but in some papers (e.g. [170, 174, 172, 176]) the inequality in (d), also goes by this name.

Lemma 3.14 (a) *Let $f, g : [t_0, T_0] \mapsto \boldsymbol{R}$ be continuous functions and $c : [t_0, T_0] \mapsto \boldsymbol{R}$ an integrable function, with $g, c \geq 0$ on $[t_0, T_0]$, which satisfy*

$$f(t) \leq g(t) + \int_{t_0}^{t} c(s)f(s)\mathrm{d}s, \quad \forall\, t \in [t_0, T_0].$$

Then

$$f(t) \leq g(t) + \int_{t_0}^{t} g(s)c(s)e^{\int_s^t c(r)\mathrm{d}r}\mathrm{d}s, \quad \forall\, t \in [t_0, T_0].$$

(b) *Let $f, g : [t_0, T_0] \mapsto \boldsymbol{R}$ be continuous functions, with g decreasing, which for a constant $c > 0$ satisfy*

$$f(t) \leq g(t) + c\int_{t_0}^{t} f(s)\mathrm{d}s, \quad \forall\, t \in [t_0, T_0].$$

Then

$$f(t) \leq g(t)e^{c(t-t_0)}, \quad \forall\, t \in [t_0, T_0].$$

(c) *Let $f \in L^\infty(t_0, T_0)$ and suppose there are constants $c \geq 0$, b such that*

$$f(t) \leq b + c\int_{t_0}^{t} f(s)\mathrm{d}s, \quad \forall\, t \in [t_0, T_0].$$

Then

$$f(t) \leq b e^{c(t-t_0)}, \quad \text{for a.e. } t \in [t_0, T_0].$$

(d) *Let $f : [t_0, T_0] \mapsto \boldsymbol{R}$ be a nonnegative continuous function with an integrable derivative a.e. in $[t_0, T_0]$ (i.e. f is absolutely continuous) which, for constants b, c (of any sign), satisfies*

$$f'(t) \leq b + cf(t), \quad \text{for a.e. } t \in [t_0, T_0].$$

Then

$$f(t) \leq e^{c(t-t_0)} f(t_0) + \frac{b}{c}(e^{c(t-t_0)} - 1), \quad \forall\, t \in [t_0, T_0], \tag{3.49}$$

with the inequality reducing to $f(t) \leq f(t_0) + b(t - t_0)$ if $c = 0$.

Proof. (a) is given in [215, p. 508], (b) is Proposition 3.10 of [208, p. 82], (c) is given in [197, p. 124], and (d) follows by factor integration. (For other versions of the lemma, with stronger smoothness conditions on f and g, see [211, p. 436].) □

Proof of Proposition 3.7.

The proof is by the Galerkin method, of which the first step is to derive *a priori* estimates for the solution of problem (I). Let m and q satisfy any of the relations (a) – (c), then it follows from Lemma 3.6 that

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{v}\|_{m+q} \leq \begin{cases} C_5\|\boldsymbol{v}\|_3\|\boldsymbol{u}\|_1 & \text{if } m+q=1 \quad [(3.16), s=2] \\ C_5\|\boldsymbol{v}\|_{m+q+1}\|\boldsymbol{u}\|_{m+q} & \text{if } m+q \geq 2 \quad [(3.17)] \end{cases}$$

$$\leq C_5\|\boldsymbol{v}\|_{m+3}\|\boldsymbol{u}\|_{m+q}, \tag{3.50}$$

$$|(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})_{m+q}| \leq \begin{cases} C_4\|\boldsymbol{v}\|_3\|\boldsymbol{u}\|_1^2 & \text{if } m+q=1 \quad [(3.14), r=2] \\ C_4\|\boldsymbol{v}\|_3\|\boldsymbol{u}\|_2^2 & \text{if } m+q=2 \quad [(3.14), r=1] \\ C_4\|\boldsymbol{v}\|_{m+q}\|\boldsymbol{u}\|_{m+q}^2 & \text{if } m+q \geq 3 \quad [(3.14), r=0] \end{cases}$$

$$\leq C_4\|\boldsymbol{v}\|_{m+3}\|\boldsymbol{u}\|_{m+q}^2, \tag{3.51}$$

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{u}\|_{m+q-1} \leq \begin{cases} C_5\|\boldsymbol{v}\|_2\|\boldsymbol{u}\|_1 & \text{if } m+q=1 \quad [(3.15), s=1] \\ C_5\|\boldsymbol{v}\|_2\|\boldsymbol{u}\|_2 & \text{if } m+q=2 \quad [(3.16), s=1] \\ C_5\|\boldsymbol{v}\|_{m+q-1}\|\boldsymbol{u}\|_{m+q} & \text{if } m+q \geq 3 \quad [(3.17)] \end{cases}$$

$$\leq C_5 \|\boldsymbol{v}\|_{m+2} \|\boldsymbol{u}\|_{m+q}. \tag{3.52}$$

Taking the $\boldsymbol{H}^{m+q}(\Omega)$ inner product of $(\mathrm{I})_1$ with $\boldsymbol{u}$ gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2_{m+q} + \frac{\nu}{\alpha}\|\boldsymbol{u}\|^2_{m+q} &= \frac{\nu}{\alpha}(\mathbf{curl}\,\boldsymbol{v}, \boldsymbol{u})_{m+q} + (\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{u})_{m+q} \\ &\quad - (\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})_{m+q} + (\boldsymbol{h}, \boldsymbol{u})_{m+q}. \end{aligned} \tag{3.53}$$

By the triangle, Schwarz and Cauchy inequalities,

$$\begin{aligned} |(\mathbf{curl}\,\boldsymbol{v}, \boldsymbol{u})_{m+q}| &\leq \sqrt{2}\|\boldsymbol{v}\|_{m+q+1}\|\boldsymbol{u}\|_{m+q} \leq 2\|\boldsymbol{v}\|^2_{m+3} + \frac{1}{4}\|\boldsymbol{u}\|^2_{m+q}, \\ |(\boldsymbol{h}, \boldsymbol{u})_{m+q}| &\leq \|\boldsymbol{h}\|_{m+q}\|\boldsymbol{u}\|_{m+q} \leq \frac{\alpha}{\nu}\|\boldsymbol{h}\|^2_{m+q} + \frac{\nu}{4\alpha}\|\boldsymbol{u}\|^2_{m+q}, \end{aligned} \tag{3.54}$$

Hence, from (3.50), (3.51), (3.53) and (3.54) one obtains

$$\begin{aligned} &\frac{d}{dt}\|\boldsymbol{u}\|^2_{m+q} + \frac{\nu}{\alpha}\|\boldsymbol{u}\|^2_{m+q} \\ &\leq 2(C_4 + C_5)\|\boldsymbol{v}\|_{m+3}\|\boldsymbol{u}\|^2_{m+q} + \frac{4\nu}{\alpha}\|\boldsymbol{v}\|^2_{m+3} + \frac{2\alpha}{\nu}\|\boldsymbol{h}\|^2_{m+q}. \end{aligned} \tag{3.55}$$

By using $\|\boldsymbol{v}\|_{m+3,T} \leq M$ and inequality (3.49), one finds that

$$\|\boldsymbol{u}\|_{m+q,T} \leq D_1(\Omega, m+q, T, M, \frac{\nu}{\alpha}, \|\boldsymbol{h}\|_{m+q,T}, \|\boldsymbol{u}_0\|_{m+q}), \tag{3.56}$$

with

$$\begin{aligned} D_0 &= 2(C_4 + C_5)M - \frac{\nu}{\alpha}, \\ D_1^2 &= \|\boldsymbol{u}_0\|^2_{m+q} e^{D_0 T} + (\frac{4\nu M^2}{\alpha} + \frac{2\alpha}{\nu}\|\boldsymbol{h}\|_{m+q,T})\frac{1}{D_0}(e^{D_0 T} - 1). \end{aligned}$$

Furthermore, from $(\mathrm{I})_1$ and inequalities (3.50) and (3.52) it follows that

$$\begin{aligned} &\|\frac{d\boldsymbol{u}}{dt}\|_{m+q-1,T} \\ &\leq \frac{\nu}{\alpha}(\|\boldsymbol{u}\|_{m+q-1,T} + \sqrt{2}\|\boldsymbol{v}\|_{m+q,T}) + 2C_5\|\boldsymbol{v}\|_{m+3,T}\|\boldsymbol{u}\|_{m+q,T} + \|\boldsymbol{h}\|_{m+q-1,T} \\ &\leq D_2(\Omega, m+q, T, M, \frac{\nu}{\alpha}, \|\boldsymbol{h}\|_{m+q,T}, \|\boldsymbol{u}_0\|_{m+q}) \end{aligned} \tag{3.57}$$

where

$$D_2 = \frac{\nu}{\alpha}(D_1 + \sqrt{2}M) + 2C_5 D_1 M + \|\boldsymbol{h}\|_{m+q-1,T}.$$

Using the estimates (3.56) and (3.57), one can now show by the Faedo-Galerkin method (as in [185, Chapter III] or the proof of Proposition 3.11 below) that there exists a solution to problem (I) with the stated regularity properties. The uniqueness of the solution follows easily:

If $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ are any two solutions of (I), then $\boldsymbol{u} \equiv \boldsymbol{u}^1 - \boldsymbol{u}^2$ solves the homogeneous problem

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \frac{\nu}{\alpha}\boldsymbol{u} &= \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u} \quad \text{in } \Omega_T, \\ \boldsymbol{u}(0) &= \boldsymbol{0} \qquad\qquad\qquad \text{in } \Omega. \end{aligned}$$

With $f(t) = \|\boldsymbol{u}(t)\|_0^2$, using $(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})_0 = 0$, (3.15) and $\|\boldsymbol{v}\|_{3,T} \leq M$ gives

$$\frac{1}{2}f'(t) + \frac{\nu}{\alpha}f(t) \leq C_5 M f(t), \;\; f(0) = 0, \tag{3.58}$$

and thus by (3.49), $f \equiv 0$, i.e. $\boldsymbol{u}_1 \equiv \boldsymbol{u}_2$. Observe that $\boldsymbol{u}$ is unique even if $\boldsymbol{v}$ is not solenoidal, since by a simple integration by parts, the boundary condition $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on $\partial\Omega_T$ and inequality (3.47) one obtains

$$-(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})_0 = \frac{1}{2}(\nabla\cdot\boldsymbol{v}, |\boldsymbol{u}|^2)_0 \leq C(\Omega) M f(t),$$

and therefore again an equation of the form (3.58).

Lastly, the incompressibility of $\boldsymbol{u}$ is proved as in [174, p. 38]: Taking the divergence of $(\mathrm{I})_1$ and using the identity (2.17) and the incompressibility of $\boldsymbol{v}$ and $\boldsymbol{h} = \mathbf{curl}\,\boldsymbol{g}$ yields

$$\frac{\partial \zeta}{\partial t} + \frac{\nu}{\alpha}\zeta = -\nabla\cdot(\mathbf{curl}\,(\boldsymbol{u}\times\boldsymbol{v}) + \zeta\boldsymbol{v}) = -\boldsymbol{v}\cdot\nabla\zeta$$

where $\zeta \equiv \nabla\cdot\boldsymbol{u}$. Multiplying this equation by ζ, integrating over Ω and noting that $(\boldsymbol{v}\cdot\nabla\zeta, \zeta)_0 = 0$ since $\boldsymbol{v} \in \boldsymbol{X}_1$, one obtains

$$\frac{d}{dt}\|\zeta\|_0^2 = -\frac{2\nu}{\alpha}\|\zeta\|_0^2, \;\; \zeta(0) = 0,$$

and therefore $\zeta \equiv 0$ in $[0, T]$. □

The proof of Proposition 3.11 relies on the following two well-known results, the second of which is usually referred to as "Aubin's lemma".

Lemma 3.15 *Let $T > 0$ and let X and Y be Hilbert spaces or separable Banach spaces with dual spaces X' and Y'. Suppose that Y is continuously and densely imbedded in X. If*

$$\boldsymbol{u}_n \longrightarrow \boldsymbol{u} \text{ weakly}\star \text{ in } L^\infty(0,T;X')$$

and

$$\frac{d\boldsymbol{u}_n}{dt} \longrightarrow \boldsymbol{\chi} \text{ weakly}\star \text{ in } L^\infty(0,T;Y'),$$

then

$$\boldsymbol{\chi} = \frac{d\boldsymbol{u}}{dt} \text{ in } L^\infty(0,T;Y').$$

Proof. See [200, p. 68].

Lemma 3.16 (a) *Let X_0, X, X_1 be three Banach spaces, with X_0 and X_1 reflexive, such that*

$$X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1.$$

Then, for any $0 < T < \infty$, $1 < p_i < \infty$, $i = 1, 2$, the space

$$W = W(0,T;p_0,p_1;X_0,X_1) \equiv \{\boldsymbol{v} \in L^{p_0}(0,T;X_0) : \frac{d\boldsymbol{v}}{dt} \in L^{p_1}(0,T;X_1)\},$$

equipped with the norm

$$\|\boldsymbol{v}\|_W = \|\boldsymbol{v}\|_{L^{p_0}(0,T;X_0)} + \|\frac{d\boldsymbol{v}}{dt}\|_{L^{p_1}(0,T;X_1)},$$

is a Banach space. Moreover, one has the imbeddings

$$\begin{aligned} &W \hookrightarrow C([0,T];X_1), \\ &W \hookrightarrow\hookrightarrow L^{p_0}(0,T;X). \end{aligned} \tag{3.59}$$

(b) *If, in addition, X_1 is a Hilbert space, then one may take $p_1 = 1$ in* (a).

(c) *If X_0, X, X_1 in* (a) *are Hilbert spaces, then*

$$W(0,T;X_0,X_1) \equiv W(0,T;2,2;X_0,X_1)$$

is a Hilbert space with the inner product

$$(\boldsymbol{v},\boldsymbol{w})_W = (\boldsymbol{v},\boldsymbol{w})_{L^2(0,T;X_0)} + (\frac{d\boldsymbol{v}}{dt}, \frac{d\boldsymbol{w}}{dt})_{L^2(0,T;X_1)}.$$

Proof. See [199, p. 57] or [185, p. 271] for (a), and [185, pp. 274 – 278] for (b) and (c). □

Proof of Proposition 3.11.
The proof is by the Faedo-Galerkin method with a special basis, as in the proof of [189] for the Euler equations.

Basis.
Let $m \geq 0$ be fixed. For each $\boldsymbol{g} \in \boldsymbol{X}_0$, the mapping $\boldsymbol{v} \mapsto (\boldsymbol{v}, \boldsymbol{g})_0$ defines a bounded linear functional on $\boldsymbol{X}_{m+2}$ and thus, by the Lax-Milgram theorem, there exists a unique $L\boldsymbol{g} \in \boldsymbol{X}_{m+2}$ such that

$$(\boldsymbol{v}, \boldsymbol{g})_0 = (\boldsymbol{v}, L\boldsymbol{g})_{m+2} \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{m+2}. \tag{3.60}$$

In fact, as $\partial\Omega$ is of class C^{m+4} (for the construction of $\boldsymbol{n}$), it follows from Theorem 4.1 (with $k = 2$) of [191] that $L\boldsymbol{g} \in \boldsymbol{X}_{m+4}$. (It is mentioned in [192] and [191] (see (4.27) on p. 1294) that $L\boldsymbol{g} \in \boldsymbol{X}_{2(m+2)}$ if $\partial\Omega$ is of class $C^{2(m+2)}$, but this is not necessary; $\boldsymbol{v}\cdot\nabla\boldsymbol{w}_n \in \boldsymbol{H}^{m+2}(\Omega)$ if $\boldsymbol{w}_n \in \boldsymbol{H}^{m+3}(\Omega)$.)

The linear operator $L : \boldsymbol{X}_0 \mapsto \boldsymbol{X}_0$ is self-adjoint, bounded (with $\|L\|_* \leq 1$) and compact (since $\boldsymbol{H}^{m+2}(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega)$). Furthermore, $\boldsymbol{X}_0$ is an inner-product space (a closed subspace of $\boldsymbol{L}^2(\Omega)$) and $L(\boldsymbol{X}_0)$ is dense in $\boldsymbol{X}_0$ (since if $\boldsymbol{v} \in \boldsymbol{X}_0$ and $(L\boldsymbol{g}, \boldsymbol{v})_0 = 0 \ \forall \boldsymbol{g} \in \boldsymbol{X}_0$, then $(\boldsymbol{g}, L\boldsymbol{v})_0 = 0 \ \forall \boldsymbol{g} \in \boldsymbol{X}_0$, i.e. $L\boldsymbol{v} = \boldsymbol{0}$, and thus $\boldsymbol{v} = \boldsymbol{0}$ as L is injective). Hence (by Theorem 6.4-B of [196] or Theorem 7.C of [203]) L possesses a sequence of nonzero eigenvalues $(1/\lambda_i)$ (with $\lambda_i \to \infty, i \to \infty$) such that the corresponding sequence of eigenvectors $(\boldsymbol{y}_i)$ is orthonormal and complete in $\boldsymbol{L}^2(\Omega)$ and satisfies:

$$\boldsymbol{y}_i \in \boldsymbol{X}_{m+3}, \quad \lambda_i(\boldsymbol{v}, \boldsymbol{y}_i)_0 = (\boldsymbol{v}, \boldsymbol{y}_i)_{m+2}, \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{m+2}, \ i = 1, 2, \ldots \tag{3.61}$$

For each $n \geq 1$, let $Y^n = \operatorname{span}\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$, and let $P_n : \boldsymbol{L}^2(\Omega) \mapsto Y^n$ denote the corresponding orthogonal projection. (Note that, by (3.61), $(\boldsymbol{y}_i, \boldsymbol{y}_j)_{m+2} = \delta_{ij}\lambda_i, i, j = 1, 2, \ldots$ Hence the $\boldsymbol{y}_i$ are orthogonal in $\boldsymbol{X}_{m+2}$, $\|\boldsymbol{y}_i\|^2_{m+2} = \lambda_i$, and P_n is also the orthogonal projection of $\boldsymbol{X}_{m+2}$ onto Y^n.)

Approximate Problem.
For $n \geq 1$, set $\boldsymbol{w}_{0n} = P_n \boldsymbol{w}_0$ and consider the following problem:

Find

$$\boldsymbol{w}_n(t) = \sum_{j=1}^{n} g_{nj}(t)\boldsymbol{y}_j \tag{3.62}$$

satisfying

$$\begin{aligned}&(\frac{\partial \boldsymbol{w}_n}{\partial t}+\frac{\nu}{\alpha}\boldsymbol{w}_n+\boldsymbol{v}\cdot\nabla\boldsymbol{w}_n,\boldsymbol{y}_i)_0=(\check{\boldsymbol{f}},\boldsymbol{y}_i)_0,\ i=1,\dots,n,\\&\boldsymbol{w}_n(0)=\boldsymbol{w}_{0n},\end{aligned}\tag{3.63}$$

where $\check{\boldsymbol{f}}\equiv\check{\boldsymbol{s}}/\alpha-\check{\boldsymbol{b}}-\check{\boldsymbol{c}}+\check{\boldsymbol{d}}/\alpha$. The equations (3.63) are equivalent to the system of n linear first-order ordinary differential equations

$$\begin{aligned}&\boldsymbol{g}_n'(t)+\mathcal{A}_n(t)\boldsymbol{g}_n(t)=\boldsymbol{f}_n(t),\\&\boldsymbol{g}_n(0)=\boldsymbol{g}_{0n},\end{aligned}\tag{3.64}$$

where $\mathcal{A}_n(t),\boldsymbol{f}_n(t)$ and $\boldsymbol{g}_{0n}$ are defined by

$$\begin{aligned}&\mathcal{A}_{nij}(t)=\mathcal{A}_{ij}(t)=\frac{\nu}{\alpha}+(\boldsymbol{v}(t)\cdot\nabla\boldsymbol{y}_j,\boldsymbol{y}_i)_0,\\&f_{ni}(t)=f_i(t)=(\check{\boldsymbol{f}}(t),\boldsymbol{y}_i)_0,\\&g_{0ni}=g_{0i}=(\boldsymbol{w}_0,\boldsymbol{y}_i)_0,\quad i,j=1,\dots,n.\end{aligned}$$

As

$$\begin{aligned}&|\mathcal{A}_{ij}|\le\frac{\nu}{\alpha}+\|\boldsymbol{v}\|_{2,T}\|\boldsymbol{y}_j\|_1\le\frac{\nu}{\alpha}+\max(\sqrt{\lambda_1},\dots,\sqrt{\lambda_n})M,\\&|f_i|\le\|\check{\boldsymbol{f}}\|_{0,T}\le\|\check{\boldsymbol{f}}\|_{2,T}\le C(\Omega,\alpha,\|\hat{K}\|_{C^2},\|N\|_{C^2},M,\|\boldsymbol{d}\|_{3/2,T,\partial\Omega},\|\boldsymbol{\eta}\|_{2,T})\end{aligned}$$

(from (3.30) – (3.33)), the coefficients $\mathcal{A}_{ij}(t)$ and $f_i(t)$ are integrable and bounded on $(0,T)$, and therefore the classical results of [195] (see p. 74 and Problem 1 on p. 97–98), ensure that this problem has a unique solution on $[0,T]$ satisfying

$$\boldsymbol{g}_n\in C^0([0,T])^n,\quad \boldsymbol{g}_n'\in L^\infty(0,T)^n,$$

i.e.

$$\boldsymbol{w}_n\in C^0([0,T];\boldsymbol{X}_{m+4}),\quad \frac{d\boldsymbol{w}_n}{dt}\in L^\infty(0,T;\boldsymbol{X}_{m+4}).\tag{3.65}$$

(As $m+4\ge2$, the partial derivative $\frac{\partial\boldsymbol{w}_n}{\partial t}$ also exists and equals $\frac{d\boldsymbol{w}_n}{dt}$ a.e. in Ω_T.)

A Priori Estimates.
Equation (3.63) can be written as

$$\begin{aligned}&(\frac{d\boldsymbol{w}_n}{dt},\boldsymbol{y}_i)_0+(\frac{\nu}{\alpha}\boldsymbol{w}_n,\boldsymbol{y}_i)_0+(P[\boldsymbol{v}\cdot\nabla\boldsymbol{w}_n+\check{\boldsymbol{b}}],\boldsymbol{y}_i)_0\\&=(\frac{1}{\alpha}\check{\boldsymbol{s}}-\check{\boldsymbol{c}}+\frac{1}{\alpha}\check{\boldsymbol{d}},\boldsymbol{y}_i)_0,\ i=1,\dots,n,\end{aligned}\tag{3.66}$$

where P denotes the orthogonal projection of $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{X}_0$. Multiplying (3.66) by λ_i and using (3.61) gives

$$\begin{aligned}&(\frac{d\boldsymbol{w}_n}{dt}, \boldsymbol{y}_i)_{m+2} + (\frac{\nu}{\alpha}\boldsymbol{w}_n, \boldsymbol{y}_i)_{m+2} + (P[\boldsymbol{v}\cdot\nabla\boldsymbol{w}_n + \breve{\boldsymbol{b}}], \boldsymbol{y}_i)_{m+2}\\ &= (\frac{1}{\alpha}\breve{\boldsymbol{s}} - \breve{\boldsymbol{c}} + \frac{1}{\alpha}\breve{\boldsymbol{d}}, \boldsymbol{y}_i)_{m+2},\ i = 1, \ldots, n,\end{aligned} \tag{3.67}$$

and multiplying this by $g_{ni}(t)$ and adding in $i = 1, \ldots, n$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}_n\|^2_{m+2} + \frac{\nu}{\alpha}\|\boldsymbol{w}_n\|^2_{m+2}\\ &= -(P[\boldsymbol{v}\cdot\nabla\boldsymbol{w}_n + \breve{\boldsymbol{b}}], \boldsymbol{w}_n)_{m+2} + (\frac{1}{\alpha}\breve{\boldsymbol{s}} - \breve{\boldsymbol{c}}, \boldsymbol{w}_n)_{m+2}\\ &= -(\nabla q_n, \boldsymbol{w}_n)_{m+2} - (\boldsymbol{v}\cdot\nabla\boldsymbol{w}_n, \boldsymbol{w}_n)_{m+2} + (\breve{\boldsymbol{f}}, \boldsymbol{w}_n)_{m+2},\end{aligned} \tag{3.68}$$

where, for each $t \in I, q_n(t)$ satisfies the Neumann problem

$$\begin{aligned}&\triangle q_n = -\nabla\cdot[\boldsymbol{v}\cdot\nabla\boldsymbol{w}_n + \breve{\boldsymbol{b}}] = -(\nabla\boldsymbol{v})^T : \nabla\boldsymbol{w}_n - \nabla\cdot\breve{\boldsymbol{b}} \quad \text{in } \Omega,\\ &\frac{\partial q_n}{\partial n} = [\boldsymbol{v}\cdot\nabla\boldsymbol{w}_n + \boldsymbol{b}]\cdot\boldsymbol{n} \quad \text{on } \partial\Omega.\end{aligned}$$

Despite the problematic $\nabla\boldsymbol{w}_n$-term in the boundary condition (which disappears in the case of the no-slip problem) the method of Lemmas 1.1 and 1.2 in [189] (which involves a local representation of $\partial\Omega$, the classical regularity results for the Neumann problem (see [182] or e.g. [207, pp. 13 – 15]; $\partial\Omega$ is a bounded open set of class C^{m+2}), and the fact that $\boldsymbol{H}^{m+1}(\Omega)$ is an algebra for $m \geq 1$) can be used to show that

$$\|\nabla q_n(t)\|_{m+2} \leq C_8(\Omega, m)(M\|\boldsymbol{w}_n\|_{m+2} + \|\breve{\boldsymbol{b}}\|_{m+2}). \tag{3.69}$$

Furthermore, according to (3.14),

$$\begin{aligned}|(\boldsymbol{v}\cdot\nabla\boldsymbol{w}_n, \boldsymbol{w}_n)_{m+2}| &\leq C_4\|\boldsymbol{v}\|_{m+2+r(m+2)}\|\boldsymbol{w}_n\|^2_{m+2},\\ &\leq C_4M\|\boldsymbol{w}_n\|^2_{m+2},\end{aligned}$$

while

$$\|\breve{\boldsymbol{f}}\|_{m+2} \leq \frac{1}{\alpha}\|\breve{\boldsymbol{s}}\|_{m+2} + \|\breve{\boldsymbol{b}}\|_{m+2} + \|\breve{\boldsymbol{c}}\|_{m+2} + \frac{1}{\alpha}\|\breve{\boldsymbol{d}}\|_{m+2}.$$

Hence, applying the Schwarz and Cauchy inequalities in (3.68), one gets

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}_n\|^2_{m+2} + \frac{\nu}{\alpha}\|\boldsymbol{w}_n\|^2_{m+2} \\
&\leq \{\frac{1}{\alpha}\|\breve{\boldsymbol{s}}\|_{m+2} + (C_8+1)\|\breve{\boldsymbol{b}}\|_{m+2} + \|\breve{\boldsymbol{c}}\|_{m+2} + \frac{1}{\alpha}\|\breve{\boldsymbol{d}}\|_{m+2}\}\|\boldsymbol{w}_n\|_{m+2} \\
&\quad + (C_8 + C_4)M\|\boldsymbol{w}_n\|^2_{m+2} \\
&\leq \{(C_8 + C_4)M + \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)\}\|\boldsymbol{w}_n\|^2_{m+2} \\
&\quad + \frac{1}{2\varepsilon_1\alpha^2}\|\breve{\boldsymbol{s}}\|^2_{m+2} + \frac{(C_8+1)^2}{2\varepsilon_2}\|\breve{\boldsymbol{b}}\|^2_{m+2} + \frac{1}{2\varepsilon_3}\|\breve{\boldsymbol{c}}\|^2_{m+2} + \frac{1}{2\varepsilon_4\alpha^2}\|\breve{\boldsymbol{d}}\|^2_{m+2}.
\end{aligned}
$$

With e.g. $\varepsilon_1 = \ldots = \varepsilon_4 = \nu/(4\alpha)$, this gives

$$
\begin{aligned}
&\frac{d}{dt}\|\boldsymbol{w}_n\|^2_{m+2} + (\frac{\nu}{\alpha} - 2(C_4 + C_8)M)\|\boldsymbol{w}_n\|^2_{m+2} \\
&\leq \frac{4}{\nu\alpha}\|\breve{\boldsymbol{s}}\|^2_{m+2} + \frac{4\alpha}{\nu}(C_8+1)^2\|\breve{\boldsymbol{b}}\|^2_{m+2} + \frac{4\alpha}{\nu}\|\breve{\boldsymbol{c}}\|^2_{m+2} + \frac{4}{\nu\alpha}\|\breve{\boldsymbol{d}}\|^2_{m+2} \\
&\leq C_9(\Omega, m)\{\frac{M^2}{\nu}(\frac{1}{\alpha}[\|\hat{K}\|^2_{C^{m+2}} + \|N\|^2_{C^{m+2}}(|\Omega| + M^4 + M^{4m+8})] \\
&\quad + \alpha[\|\boldsymbol{\eta}\|^2_{m+2,T} + M^2]) + \frac{1}{\nu\alpha}\|\boldsymbol{d}\|^2_{m+3/2,T,\partial\Omega}\} \equiv F,
\end{aligned}
\tag{3.70}
$$

where the last inequality was derived from the estimates in (3.30) – (3.33). Thus, by Gronwall's inequality (3.49),

$$
\|\boldsymbol{w}_n\|_{m+2,T} \leq E_1, \tag{3.71}
$$

where E_1 depends only on

$$
\Omega, m, T, \alpha, \nu, M, \|\hat{K}\|_{C^{m+2}}, \|N\|_{C^{m+2}}, \|\boldsymbol{w}_0\|_{m+2}, \|\boldsymbol{\eta}\|_{m+2,T}, \|\boldsymbol{d}\|_{m+3/2,T,\partial\Omega},
$$

and is defined by

$$
E_1^2 = \|\boldsymbol{w}_0\|^2_{m+2}e^{E_0 T} + \frac{F}{E_0}(e^{E_0 T} - 1), \quad E_0 = 2(C_4 + C_8)M - \frac{\nu}{\alpha},
$$

independent of n, i.e. $\boldsymbol{w}_n$ remains bounded in $L^\infty(0,T;\boldsymbol{X}_{m+2})$ as $n \to \infty$.

Since the $\boldsymbol{y}_i$ are orthogonal in $\boldsymbol{X}_0$, $(3.63)_1$ can be written as

$$
\frac{d\boldsymbol{w}_n}{dt} + \frac{\nu}{\alpha}\boldsymbol{w}_n = P_n P(\breve{\boldsymbol{f}} - \boldsymbol{v}\cdot\nabla\boldsymbol{w}_n).
$$

Thus, using (3.71) for $m = 1$,

$$\begin{aligned}\|\frac{d\boldsymbol{w}_n}{dt}\|_{0,T} &\leq \frac{\nu}{\alpha}\|\boldsymbol{w}_n\|_{0,T} + \|\boldsymbol{v}\cdot\nabla\boldsymbol{w}_n\|_{0,T} + \|\breve{\boldsymbol{f}}\|_{0,T} \\ &\leq (\frac{\nu}{\alpha} + C(\Omega)M)E_1 + \|\breve{\boldsymbol{f}}\|_{0,T},\end{aligned}$$

i.e. $\boldsymbol{w}_n'$ remains bounded in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ as $n \longrightarrow \infty$. (One can also derive this estimate by multiplying $(3.63)_1$ by $g_{ni}'(t)$, summing over $i = 1,\ldots,n$, applying the Cauchy-Schwarz inequality and then dividing by $\|\boldsymbol{w}_n'(t)\|_0$.)

Passage to Limit.
The estimates derived above show that

$$(\boldsymbol{w}_n) \text{ is bounded in } L^\infty(0,T;\boldsymbol{X}_{m+2}), \tag{3.72}$$

$$(\frac{d\boldsymbol{w}_n}{dt}) \text{ is bounded in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)). \tag{3.73}$$

From (3.72) and the fact that $L^\infty(0,T;\boldsymbol{X}_{m+2})$ (where $\boldsymbol{X}_{m+2}$ is identified with its dual $\boldsymbol{X}_{m+2}'$ via the Riesz representation theorem) is the dual of $L^1(0,T;\boldsymbol{X}_{m+2})$, which is separable, it follows that there is a subsequence $(\boldsymbol{w}_q)$ of $(\boldsymbol{w}_n)$ and a function $\boldsymbol{w}^\star \in L^\infty(0,T;\boldsymbol{X}_{m+2})$ such that

$$\boldsymbol{w}_q \longrightarrow \boldsymbol{w}^\star \text{ weakly} \star \text{ in } L^\infty(0,T;\boldsymbol{X}_{m+2}). \tag{3.74}$$

This implies that $\boldsymbol{w}_q \longrightarrow \boldsymbol{w}^\star$ weakly$\star$ in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$. (Given any $\boldsymbol{\phi} \in L^1(0,T;\boldsymbol{L}^2(\Omega))$, applying the Riesz representation theorem to the functionals $\boldsymbol{y} \mapsto (\boldsymbol{y},\boldsymbol{\phi}(t))_0$ on $\boldsymbol{X}_{m+2}$ shows that there is a function $\boldsymbol{\xi} \in L^1(0,T;\boldsymbol{X}_{m+2})$ (with $\|\boldsymbol{\xi}(t)\|_{m+2} = \|\boldsymbol{\phi}(t)\|_0$) such that

$$\int_0^T (\boldsymbol{u}(t),\boldsymbol{\phi}(t))_0 \, \mathrm{d}t = \int_0^T (\boldsymbol{u}(t),\boldsymbol{\xi}(t))_{m+2} \, dt \quad \forall\, \boldsymbol{u} \in L^\infty(0,T;\boldsymbol{X}_{m+2}).\ )$$

Hence, using (3.73) and Lemma 3.15 (with $X = Y = \boldsymbol{L}^2(\Omega)$) in a similar argument as above, one can extract a subsequence $(\boldsymbol{w}_r)$ of $(\boldsymbol{w}_q)$ such that

$$\frac{d\boldsymbol{w}_r}{dt} \longrightarrow \frac{d\boldsymbol{w}^\star}{dt} \text{ weakly} \star \text{ in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)). \tag{3.75}$$

Furthermore, as T is finite, (3.72) – (3.73) implies that

$$(\boldsymbol{w}_r) \text{ is bounded in } L^2(0,T;\boldsymbol{X}_{m+2}),$$

$$(\frac{d\boldsymbol{w}_r}{dt}) \text{ is bounded in } L^2(0,T;\boldsymbol{L}^2(\Omega)),$$

and therefore, by Lemma 3.16, there exists a subsequence $(\boldsymbol{w}_s)$ of $(\boldsymbol{w}_r)$ and a function $\boldsymbol{w} \in W(0,T;\boldsymbol{X}_{m+2},\boldsymbol{L}^2(\Omega))$ such that

$$\boldsymbol{w}_s \longrightarrow \boldsymbol{w} \text{ weakly in } W(0,T;\boldsymbol{X}_{m+2},\boldsymbol{L}^2(\Omega)), \tag{3.76}$$

$$\boldsymbol{w}_s \longrightarrow \boldsymbol{w} \text{ strongly in } L^\infty(0,T;\boldsymbol{X}_{m+1}). \tag{3.77}$$

(3.74) means that for each $\boldsymbol{\phi} \in L^1(0,T;\boldsymbol{X}_{m+2}) \supset L^2(0,T;\boldsymbol{X}_{m+2})$,

$$\int_0^T (\boldsymbol{w}_q(t) - \boldsymbol{w}^\star(t), \boldsymbol{\phi}(t))_{m+2}\, \mathrm{d}t \longrightarrow 0, \quad q \longrightarrow \infty,$$

i.e. $\boldsymbol{w}_q \longrightarrow \boldsymbol{w}^\star$ weakly in $L^2(0,T;\boldsymbol{X}_{m+2})$. On the other hand, (3.76) implies that $\boldsymbol{w}_s \longrightarrow \boldsymbol{w}$ weakly in $L^2(0,T;\boldsymbol{X}_{m+2})$. Hence $\boldsymbol{w}^\star = \boldsymbol{w}$.

Now let $\varphi \in C^0([0,T])$ and $\boldsymbol{y} \in \boldsymbol{X}_0$. Then there is a sequence $(\boldsymbol{y}^n), \boldsymbol{y}^n \in Y^n$, such that $\boldsymbol{y}^n \longrightarrow \boldsymbol{y}$ in $\boldsymbol{L}^2(\Omega)$. Thus, defining $\boldsymbol{\psi}_n(t) \equiv \varphi(t)\boldsymbol{y}^n$ and $\boldsymbol{\psi}(t) \equiv \varphi(t)\boldsymbol{y}$,

$$\boldsymbol{\psi}_n \longrightarrow \boldsymbol{\psi} \text{ strongly in } L^2(0,T;\boldsymbol{L}^2(\Omega)). \tag{3.78}$$

From equation (3.63) one deduces

$$\int_0^T (\boldsymbol{w}_s'(t) + \frac{\nu}{\alpha}\boldsymbol{w}_s(t) + \boldsymbol{v}(t)\cdot\nabla\boldsymbol{w}_s(t), \boldsymbol{\psi}_s(t))_0\, \mathrm{d}t = \int_0^T (\check{\boldsymbol{f}}_s(t), \boldsymbol{\psi}_s(t))_0\, \mathrm{d}t, \quad \forall\, s.$$

By virtue of (3.78) and (3.76), which implies that $\boldsymbol{w}_s' \longrightarrow \boldsymbol{w}'$ weakly in $L^2(0,T;\boldsymbol{L}^2(\Omega))$,

$$\int_0^T (\boldsymbol{w}_s'(t), \boldsymbol{\psi}_s(t))_0\, \mathrm{d}t \longrightarrow \int_0^T (\boldsymbol{w}'(t), \boldsymbol{\psi}(t))_0\, \mathrm{d}t, \quad s \longrightarrow \infty. \tag{3.79}$$

Furthermore, (3.77) ensures that $\boldsymbol{w}_s \longrightarrow \boldsymbol{w}$ strongly in $L^2(0,T;\boldsymbol{L}^2(\Omega))$, so that

$$\int_0^T (\boldsymbol{w}_s(t), \boldsymbol{\psi}_s(t))_0\, \mathrm{d}t \longrightarrow \int_0^T (\boldsymbol{w}(t), \boldsymbol{\psi}(t))_0\, \mathrm{d}t, \quad s \longrightarrow \infty. \tag{3.80}$$

Similarly, since (3.77) implies that $\boldsymbol{w}_s \longrightarrow \boldsymbol{w}$ strongly in $L^2(0,T;\boldsymbol{H}^1(\Omega))$, the estimate $\|\boldsymbol{v}\cdot\nabla(\boldsymbol{w}_s - \boldsymbol{w})\|_0 \leq C_5(\Omega)\|\boldsymbol{v}\|_{2,T}\|\boldsymbol{w}_s - \boldsymbol{w}\|_1$ shows that $\boldsymbol{v}\cdot\nabla\boldsymbol{w}_s \longrightarrow \boldsymbol{v}\cdot\nabla\boldsymbol{w}$ strongly in $L^2(0,T;\boldsymbol{L}^2(\Omega))$ and therefore

$$\int_0^T (\boldsymbol{v}\cdot\nabla\boldsymbol{w}_s(t), \boldsymbol{\psi}_s(t))_0\, \mathrm{d}t \longrightarrow \int_0^T (\boldsymbol{v}\cdot\nabla\boldsymbol{w}(t), \boldsymbol{\psi}(t))_0\, \mathrm{d}t, \quad s \longrightarrow \infty. \tag{3.81}$$

Lastly, it also follows from (3.78) that

$$\int_0^T (\breve{\boldsymbol{f}}(t), \boldsymbol{\psi}_s(t))_0 \, \mathrm{d}t \longrightarrow \int_0^T (\breve{\boldsymbol{f}}(t), \boldsymbol{\psi}(t))_0 \, \mathrm{d}t, \quad s \longrightarrow \infty. \tag{3.82}$$

Hence, in the limit one obtains

$$\int_0^T (\frac{d\boldsymbol{w}}{dt}(t) + \frac{\nu}{\alpha}\boldsymbol{w}(t) + \boldsymbol{v}(t)\cdot\nabla\boldsymbol{w}(t), \boldsymbol{y})_0\varphi(t) \, \mathrm{d}t = \int_0^T (\breve{\boldsymbol{f}}(t), \boldsymbol{y})_0\varphi(t) \, \mathrm{d}t$$
$$\forall\, \boldsymbol{y} \in \boldsymbol{X}_0, \ \forall\, \varphi \in C^0([0,T]),$$

and thus, by the density of $C^0([0,T])$ in $L^2(0,T)$,

$$(\frac{d\boldsymbol{w}}{dt} + \frac{\nu}{\alpha}\boldsymbol{w} + \boldsymbol{v}\cdot\nabla\boldsymbol{w} - \breve{\boldsymbol{f}}, \boldsymbol{y})_0 = 0, \quad \forall\, \boldsymbol{y} \in \boldsymbol{X}_0, \text{ for a.e. } t \in (0,T),$$

or equivalently,

$$\frac{d\boldsymbol{w}}{dt} = -\frac{\nu}{\alpha}\boldsymbol{w} - P(\boldsymbol{v}\cdot\nabla\boldsymbol{w} + \breve{\boldsymbol{b}}) + \frac{1}{\alpha}\breve{\boldsymbol{s}} - \breve{\boldsymbol{c}} + \frac{1}{\alpha}\breve{\boldsymbol{d}} \text{ for a.e. } t \in (0,T). \tag{3.83}$$

In the light of (3.75) and the classical Helmholtz decomposition, this establishes the existence of $q \in L(0,T;\boldsymbol{H}^1(\Omega))$ satisfying equation $(\mathrm{II})_1$. In fact, as $P \in \mathcal{L}(\boldsymbol{H}^{m+1}(\Omega))$ with $\|P\|_* = C(\Omega)$ (see [185, p. 18]; $\partial\Omega \in C^{m+2}$), one gets

$$\begin{aligned} &\|\frac{d\boldsymbol{w}}{dt}\|_{m+1,T} \\ &\le C(\Omega)(C_5(\Omega,m)\|\boldsymbol{v}\|_{m+1,T}\|\boldsymbol{w}\|_{m+2,T} + \|\breve{\boldsymbol{b}}\|_{m+1,T}) \\ &\quad + \frac{\nu}{\alpha}\|\boldsymbol{w}\|_{m+1,T} + \frac{1}{\alpha}\|\breve{\boldsymbol{s}}\|_{m+1,T} + \|\breve{\boldsymbol{c}}\|_{m+1,T} + \frac{1}{\alpha}\|\breve{\boldsymbol{d}}\|_{m+1,T} \\ &\le E_2, \end{aligned} \tag{3.84}$$

where

$$\begin{aligned} &E_2(E_1,\Omega,m,\nu,\alpha,\|\hat{K}\|_{C^{m+1}},\|N\|_{C^{m+1}},M,\|\boldsymbol{d}\|_{m+1/2,T,\partial\Omega},\|\boldsymbol{\eta}\|_{m+1,T}), \\ &= C(\Omega,m)\{(\frac{\nu}{\alpha} + M)E_1 + M^2 + M\|\boldsymbol{\eta}\|_{m+1,T} \\ &+ \frac{M}{\alpha}(\|\hat{K}\|_{C^{m+1}} + \|N\|_{C^{m+1}}[|\Omega|^{1/2} + M^2 + M^{2m+2}]) + \frac{1}{\alpha}\|\boldsymbol{d}\|_{m+1/2,T,\partial\Omega}\}, \end{aligned}$$

by applying the estimates (3.15), $(3.30)_4$ – $(3.33)_4$ and (3.71). Hence

$$\frac{d\boldsymbol{w}}{dt} \in L^\infty(0,T;\boldsymbol{X}_{m+1}), \quad q \in L^\infty(0,T;\boldsymbol{H}^{m+2}(\Omega)).$$

The initial value $\boldsymbol{w}(0)$ is well-defined and belongs to $\boldsymbol{X}_0$, because $\boldsymbol{w} \in W^{1,2}(0,T;\boldsymbol{X}_0) \subset C([0,T];\boldsymbol{X}_0)$. To verify the initial condition, choose a function $\varphi \in C^1([0,T])$ with $\varphi(0) \neq 0$ and $\varphi(T) = 0$ (say $\varphi(t) = 1 - t/T$) in the definitions of $\boldsymbol{\psi}$ and $\boldsymbol{\psi}_n$ above (3.78). Then

$$\boldsymbol{\psi}_n' \longrightarrow \boldsymbol{\psi}' \text{ strongly in } L^2(0,T;\boldsymbol{L}^2(\Omega)). \tag{3.85}$$

Hence, integrating by parts (see [212, p. 477]) and using (3.79) (or (3.80) – (3.82)), (3.77) and (3.85), one finds

$$\begin{aligned}
&-\int_0^T (\boldsymbol{w}(t),\boldsymbol{\psi}'(t))_0 \,\mathrm{d}t - (\boldsymbol{w}(0),\boldsymbol{y})_0\varphi(0) \\
&= \int_0^T (\boldsymbol{w}'(t),\boldsymbol{\psi}(t))_0 \,\mathrm{d}t \\
[&= \int_0^T (-\frac{\nu}{\alpha}\boldsymbol{w}(t) - \boldsymbol{v}(t)\cdot\nabla\boldsymbol{w}(t) + \check{\boldsymbol{f}}(t),\boldsymbol{\psi}(t))_0 \,\mathrm{d}t \\
&= \lim_{s\to\infty}\int_0^T (-\frac{\nu}{\alpha}\boldsymbol{w}_s(t) - \boldsymbol{v}_s(t)\cdot\nabla\boldsymbol{w}_s(t) + \check{\boldsymbol{f}}(t),\boldsymbol{\psi}_s(t))_0 \,\mathrm{d}t\ ] \\
&= \lim_{s\to\infty}\int_0^T (\boldsymbol{w}_s'(t),\boldsymbol{\psi}_s(t))_0 \,\mathrm{d}t \\
&= -\lim_{s\to\infty}\int_0^T (\boldsymbol{w}_s(t),\boldsymbol{\psi}_s'(t))_0 \,\mathrm{d}t - \lim_{s\to\infty}(\boldsymbol{w}_s(0),\boldsymbol{y}_s)_0\varphi(0) \\
&= -\int_0^T (\boldsymbol{w}(t),\boldsymbol{\psi}'(t))_0 \,\mathrm{d}t - (\boldsymbol{w}_0,\boldsymbol{y})_0\varphi(0) \quad \forall\, \boldsymbol{y} \in \boldsymbol{X}_0,
\end{aligned}$$

and therefore $\boldsymbol{w}(0) = \boldsymbol{w}_0$ as $\boldsymbol{w}_0 \in \boldsymbol{X}_0$.

Lastly, as $\boldsymbol{w}$ is solenoidal by construction, it remains to note that $\boldsymbol{w}$ is unique by the argument in the proof of Proposition 3.7; see (3.58). (It follows that one may take $(\boldsymbol{w}_s)$ to be the whole sequence $(\boldsymbol{w}_n)$.)

Chapter 4

LOCAL SOLUTIONS

He who despises small things
will fail little by little.
Sirach 19:1

By construction, the nonlinear terms in the original problem were either transformed to terms which are linear in the unknowns of the auxiliary problems (I) – (III), or became data terms in these problems. Hence in the previous chapter the nonlinear aspect of the problem was restricted to the derivation of bounds for terms in the right hand sides of the equations, which was easily accomplished via the algebra property of the Sobolev spaces. In this chapter the *nonlinearity* of the problem, essentially contained in the mapping Φ, is addressed by means of a Schauder fixed point theorem. In this way the fixed point approach also allows one to circumvent some "hard analysis" by exploiting the "soft analysis" imbedded in a general theorem.

4.1 Existence

The existence proof is based on the following version of the Schauder Fixed-Point Theorem:

Lemma 4.1 *Let G be a nonempty, closed, convex subset of a Banach space X, and suppose $\Phi : X \supset G \to X$ is a continuous operator such that $\Phi(G) \subset G$ and $\Phi(G)$ is relatively compact. Then Φ has a fixed point.*

Proof. See e.g. [205, p. 153] or [198, p. 171].

Let $T > 0, m \geq 1$ and suppose that Ω is a bounded simply connected domain of class C^{m+4}. Given $D > 0, \boldsymbol{u}_0 \in \boldsymbol{V}_m$ with $\|\boldsymbol{u}_0\|_m \leq D$, and $\boldsymbol{w}_0 \in \boldsymbol{X}_{m+2}$ with $\|\boldsymbol{w}_0\|_{m+2} \leq D$, define the Banach space

$$X = X(T, \Omega, m) = C([0,T]; \boldsymbol{V}_{m-1}) \times C([0,T]; \boldsymbol{X}_{m+1}),$$

with the norm

$$\|(\boldsymbol{\phi}, \boldsymbol{\eta})\|_X = \max(\|\boldsymbol{\phi}\|_{m-1,T}, \|\boldsymbol{\eta}\|_{m+1,T}),$$

and the subset

$$\begin{aligned} G = G(T, \Omega, m, D, \boldsymbol{u}_0, \boldsymbol{w}_0) = \{(\boldsymbol{\phi}, \boldsymbol{\eta}) \in X :& \\ \boldsymbol{\phi} \in L^\infty(0,T; \boldsymbol{H}^m(\Omega)),\ \|\boldsymbol{\phi}\|_{m,T} \leq D,\ \boldsymbol{\phi}(0) = \boldsymbol{u}_0,& \\ \boldsymbol{\eta} \in L^\infty(0,T; \boldsymbol{H}^{m+2}(\Omega)),\ \|\boldsymbol{\eta}\|_{m+2,T} \leq D,\ \boldsymbol{\eta}(0) = \boldsymbol{w}_0\}.& \end{aligned}$$

G is clearly nonempty (take $\boldsymbol{\phi} \equiv \boldsymbol{u}_0, \boldsymbol{\eta} \equiv \boldsymbol{w}_0$), and for $\boldsymbol{d} \in L^\infty(0,T; \boldsymbol{Z}_{m+3/2})$, $\boldsymbol{h} \in L^\infty(0,T; \boldsymbol{V}_m)$, Propositions 3.4, 3.7 and 3.11 show that the map

$$\Phi : X \supset G \longmapsto X : (\boldsymbol{\phi}, \boldsymbol{\eta}) \longmapsto (\boldsymbol{v}, \boldsymbol{\eta}) \longmapsto (\boldsymbol{u}, \boldsymbol{w}),$$

where $\boldsymbol{v}$ denotes the solution of problem (III), and $\boldsymbol{u}$ and $\boldsymbol{w}$ are the solutions of the corresponding problems (I) and (II), is well-defined. Furthermore, one has

Lemma 4.2 (a) *For any T, D, $\boldsymbol{u}_0$, $\boldsymbol{w}_0$ and $\boldsymbol{h}$ that satisfy the above conditions, G is bounded, convex and closed in X, $\Phi(G)$ is relatively compact in X, and Φ is continuous.*

(b) *For arbitrary $\nu > 0$, $\alpha > 0$, $m \geq 1$, $K, N \in C^{m+2}$, $\boldsymbol{u}_0 \in \boldsymbol{V}_m$, $\boldsymbol{w}_0 \in \boldsymbol{X}_{m+2}$, $\boldsymbol{h} \in L^\infty(0,\infty; \boldsymbol{V}_m)$, $\boldsymbol{d} \in L^\infty(0,\infty; \boldsymbol{Z}_{m+3/2})$ and $D > D_\star \equiv \max(\|\boldsymbol{u}_0\|_m, \|\boldsymbol{w}_0\|_{m+2})$*

there exists a $T^\star > 0$ *such that* Φ *has a fixed point for any* $0 < T \leq T^\star$. *In particular, one may take*

$$T^\star = \begin{cases} \dfrac{1}{F_0}\ln\left(\dfrac{F_0 D^2 + F_1}{F_0 D_\star^2 + F_1}\right) & \text{if } F_0 > 0, \\ \dfrac{D^2 - D_\star^2}{F_1} & \text{if } F_0 \leq 0, \end{cases} \tag{4.1}$$

where

$$F_0 \equiv C_0 D - \frac{\nu}{\alpha}, \quad C_0 = C_0(\Omega, m, \alpha),$$

$$F_1 \equiv \max\left(\frac{4\nu C_{10}^2 D^2}{\alpha} + \frac{2\alpha}{\nu}\|\boldsymbol{h}\|_{m,\infty}^2, F\right),$$

$$F \equiv C_9\{\frac{C_{10}D}{\nu}\left(\frac{1}{\alpha}[\,\|\hat{K}\|_{C^{m+2}} + \|N\|_{C^{m+2}}(|\Omega|^{1/2} + C_{10}^2 D^2 + (C_{10}D)^{2m+4})\,]\right.$$
$$\left. + \alpha[C_{10} + 1]D\right) + \frac{1}{\nu\alpha}\|\boldsymbol{d}\|_{m+3/2,\infty,\partial\Omega}\},$$

with $C_9(\Omega, m)$ *as in (3.70) and* $C_{10}(\Omega, m, \alpha)$ *as in* (4.22).

Proof. (a) The proof of Lemma 3.1 in [174] can be adapted in a straightforward manner to the present situation. This is not surprising as problems (I) and (II) are apparently similar in many respects. The main difference is in showing that the mapping $(\boldsymbol{\phi}, \boldsymbol{\eta}) \mapsto \boldsymbol{w}$ is continuous, which is done as in the proof of Lemma 2.2 in [170] (or Theorem 2.3 in [172]). The complete proof is given in Section 4.3.

(b) In view of (a) and Lemma 4.1 it only remains to show that $\Phi(G) \subset G$. For any $(\boldsymbol{\phi}, \boldsymbol{\eta}) \in G$, $(\boldsymbol{u}, \boldsymbol{w}) = \Phi(\boldsymbol{\phi}, \boldsymbol{\eta})$ satisfies the required initial and regularity conditions according to Propositions 3.4 ($k = 1$), 3.8 and 3.11. Moreover, with C_{10} defined as in (4.22), it follows from (3.11) and the definition of G that the corresponding solution $\boldsymbol{v}$ of problem (III) satisfies $\|\boldsymbol{v}\|_{m+3,T} \leq C_{10}D$. Hence, taking $M = C_{10}D$, one sees from (3.55) and (3.70) (on pages 55 and 61) that both $\|\boldsymbol{u}\|_m^2$ and $\|\boldsymbol{w}\|_{m+2}^2$ satisfy the inequality

$$f'(t) + \frac{\nu}{\alpha}f(t) \leq F_1 + C_0 D f(t), \; C_0 \equiv 2(C_4 + \max(C_5, C_8))C_{10},$$

in f. Here $C_4(\Omega, m)$ and $C_5(\Omega)$ are the constants in Lemma 3.6, and $C_8(\Omega, m)$ is as in (3.69). Thus, setting $F_0 = C_0 D - \nu/\alpha = \max(D_0, E_0)$ and applying the Gronwall inequality (3.49) with $f(0) \leq D_\star^2$, one obtains

$$f(t) \leq (D_\star^2 + \frac{F_1}{F_0})(e^{F_0 t} - 1) + D_\star^2, \;\; \forall\, t \in [0, T]. \tag{4.2}$$

If $F_0 > 0$ the right hand side of (4.2) is less than or equal to D^2 if

$$T \leq T^* = \frac{1}{F_0}\ln(1 + \frac{D^2 - D_\star^2}{D_\star^2 + F_1/F_0}), \tag{4.3}$$

which is $(4.1)_1$. If $F_0 = 0$ one has $f(t) \leq F_1 T + D_\star^2$ and thus $(4.1)_2$. This is also sufficient if $F_0 < 0$, because (4.2) then gives

$$f(t) \leq D_\star^2 e^{F_0 t} + \frac{F_1}{F_0}(e^{F_0 t} - 1) \leq D_\star^2 + F_1 T. \tag{4.4}$$

□

In summary, one has the following local (or "small-time") existence result:

Theorem 4.3 *Let Ω be a bounded, simply-connected domain of class C^{m+4}, $m \geq 1$, and assume that $\nu > 0$, $\alpha > 0$, $K, N \in C^{m+2}$, $D > D_\star \geq 0$, $\boldsymbol{g} \in L^\infty(0,\infty; \boldsymbol{X}_{m+1})$ and $\boldsymbol{d} \in L^\infty(0,\infty; \boldsymbol{Z}_{m+3/2})$. Then there is a constant $C_\star = C_\star(\Omega, m, \alpha) > 0$ with the property that if $\boldsymbol{v}_0 \in \boldsymbol{X}_{m+3}$ and*

$$\|\boldsymbol{v}_0\|_{m+3} \leq C_\star D_\star, \tag{4.5}$$

then there exists a $T > 0$ such that the slip problem

$$\left.\begin{array}{ll}
\frac{\partial}{\partial t}(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) - \nu\triangle\boldsymbol{v} + \mathbf{curl}\,(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) = \nabla p + \boldsymbol{g} & \text{in } \Omega_T, \\
\nabla\cdot\boldsymbol{v} = 0 & \text{in } \Omega_T, \\
([\nu\boldsymbol{A}_1 + \alpha(\frac{\partial \boldsymbol{A}_1}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{A}_1 + \boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1)]\boldsymbol{n})_\tau & \\
= (K + N(|\boldsymbol{v}|^2))\boldsymbol{v} + \boldsymbol{d} & \text{on } \partial\Omega_T, \\
\boldsymbol{v}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega_T, \\
\boldsymbol{v}(0) = \boldsymbol{v}_0 & \text{in } \Omega,
\end{array}\right\} \tag{4.6}$$

has a solution

$$\begin{array}{l}
\boldsymbol{v} \in C([0,T]; \boldsymbol{H}^{m+3}(\Omega)) \;\cap\; W^{1,\infty}(0,T; \boldsymbol{H}^{m+2}(\Omega)), \\
\nabla p \in L^\infty(0,T; \boldsymbol{H}^m(\Omega)),
\end{array} \tag{4.7}$$

and there are constants C_{10} and C_{11}, depending only on Ω, m and α, such that

$$\begin{array}{l}
\|\boldsymbol{v}\|_{m+3,T} \leq C_{10} D, \\
\|\frac{d\boldsymbol{v}}{dt}\|_{m+2,T} \leq C_{11}([\nu + D + \|\hat{K}\|_{C^{m+1}} + C(D)\|N\|_{C^{m+1}}]D \\
\qquad\qquad + \|\mathbf{curl}\,\boldsymbol{g}\|_{m-1,T} + \|\boldsymbol{d}\|_{m+1/2,T,\partial\Omega}).
\end{array} \tag{4.8}$$

Moreover, if $m \geq 2$ and

$$\boldsymbol{g} \in W^{1,\infty}(0,T;\boldsymbol{H}^{m-1}(\Omega)), \quad \boldsymbol{d} \in W^{1,\infty}(0,T;\boldsymbol{H}^{m-1/2}(\partial\Omega)),$$

then

$$\frac{d^2\boldsymbol{v}}{dt^2} \in L^\infty(0,T;\boldsymbol{H}^{m+1}(\Omega)), \quad \boldsymbol{v} \in C^1([0,T];\boldsymbol{C}^{m-1}(\overline{\Omega})), \tag{4.9}$$

so that $\boldsymbol{v}$ is a classical solution if $m \geq 4$.

Proof. Let $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ be defined as before, then there is a constant $C = C(\Omega, m, \alpha)$ such that $\max(\|\boldsymbol{u}_0\|_m, \|\boldsymbol{w}_0\|_{m+2}) \leq C\|\boldsymbol{v}_0\|_{m+3}$. Thus, with $C_\star = 1/C$, $\boldsymbol{h} = \mathbf{curl}\,\boldsymbol{g}$ and $T = T^\star$ as in (4.1), Lemma 4.2 ensures the existence of a fixed point $(\boldsymbol{u}, \boldsymbol{w}) \in G$ of the associated mapping Φ. By definition of G and Proposition 3.4 (with $k = 0$ and $k = 1$, resp.), the corresponding solution $\boldsymbol{v}$ of problem (III) satisfies

$$\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{X}_{m+3}) \cap W^{1,\infty}(0,T;\boldsymbol{X}_{m+2}). \tag{4.10}$$

Moreover, from the definition of G and the proof of Lemma 4.2,

$$\max(\|\boldsymbol{u}\|_{m,T}, \|\boldsymbol{w}\|_{m+2,T}) \leq D, \quad \|\boldsymbol{v}\|_{m+3,T} \leq M = C_{10}D,$$

where $C_{10} = C_{10}(\Omega, m, \alpha)$ is as in (4.22). Setting $M = C_{10}D$ and replacing D_1 by D in inequality $(3.57)_1$ (with $q = 0$) yields

$$\|\frac{d\boldsymbol{u}}{dt}\|_{m-1,T} \leq C(\Omega, m, \alpha)(\nu + D)D + \|\boldsymbol{h}\|_{m-1,T}.$$

Similarly, replacing E_1 by D in (3.84) (with $\boldsymbol{\eta} = \boldsymbol{w}$) gives

$$\|\frac{d\boldsymbol{w}}{dt}\|_{m+1,T} \leq C(\Omega, m, \alpha)(\nu + D + \|\hat{K}\|_{C^{m+1}} + C(D)\|N\|_{m+1})D + \frac{C(\Omega, m)}{\alpha}\|\boldsymbol{d}\|_{m+1/2,T,\partial\Omega},$$

so that $(4.8)_2$ is immediate from inequality (3.11) with $k = 1$:

$$\|\frac{d\boldsymbol{v}}{dt}\|_{m+2,T} \leq C(\Omega, m, \alpha)(\|\frac{d\boldsymbol{u}}{dt}\|_{m-1,T} + \|\frac{d\boldsymbol{w}}{dt}\|_{m+1,T}).$$

Furthermore, reversing the steps in Section 2.3, one can write equation $(\mathrm{I})_1$ as

$$\mathbf{curl}\,[\frac{\partial}{\partial t}(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) - \nu\triangle\boldsymbol{v} + \mathbf{curl}\,(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) \times \boldsymbol{v} - \boldsymbol{g}] = 0 \text{ in } \Omega_T.$$

Eliminating the **curl** then yields $(4.6)_1$ for a unique $\nabla p \in L^\infty(0,T;\boldsymbol{H}^m(\Omega))$.

Concerning the initial condition, the continuity of $\boldsymbol{v}$ on $[0,T]$ guarantees that $\boldsymbol{v}(0) \in \boldsymbol{X}_{m+2}$ is well-defined. Moreover, since the map $B_1 : \boldsymbol{H}^{m+2}(\Omega) \mapsto \boldsymbol{V}_{m-1}$ is continuous for $m \geq 1$,

$$\mathbf{curl}\,(\boldsymbol{v}(0) - \alpha\triangle\boldsymbol{v}(0)) \equiv B_1\boldsymbol{v}(0) = B_1(\lim_{t\to 0}\boldsymbol{v}(t))$$
$$= \lim_{t\to 0} B_1\boldsymbol{v}(t) = \lim_{t\to 0}\boldsymbol{u}(t) = \boldsymbol{u}_0 \equiv \mathbf{curl}\,(\boldsymbol{v}_0 - \alpha\triangle\boldsymbol{v}_0)$$

in $\boldsymbol{V}_{m-1}$. Similarly, via the boundedness of $B_2 : \boldsymbol{H}^{m+2}(\Omega) \mapsto \boldsymbol{Z}_{m+1/2}$, one gets

$$(\boldsymbol{A}_1(\boldsymbol{v}(0))\boldsymbol{n})_\tau = \boldsymbol{a}_0 \equiv (\boldsymbol{A}_1(\boldsymbol{v}_0)\boldsymbol{n})_\tau \quad \text{on } \partial\Omega.$$

Thus $\boldsymbol{v}(0) - \boldsymbol{v}_0 \in \boldsymbol{X}_{m+2} \subset \boldsymbol{X}_3$ is the solution of the stationary version of problem (III) with zero data, which is identically zero, i.e. $\boldsymbol{v}(0) = \boldsymbol{v}_0 \in \boldsymbol{X}_{m+3}$.

Lastly, noting that $n(m) \leq 3$ for $m \geq 1$, it follows from (4.10), Proposition 3.10 and Lemma 3.13 that

$$\boldsymbol{u} \in C([0,T];\boldsymbol{V}_m), \quad \boldsymbol{w} \in C([0,T];\boldsymbol{X}_{m+2}),$$

which, in turn, as indicated in Remark 3.5(b), implies that

$$\boldsymbol{v} \in C([0,T];\boldsymbol{X}_{m+3}).$$

The smoothness property (4.9) can be derived by arguments analogous to those in [174, pp. 310 – 311]. See the proof in Section 4.3. □

An alternative, slightly more natural, formulation of the theorem is:

Theorem 4.4 *Let Ω, m, ν, α, K, N, $\boldsymbol{g}$ and $\boldsymbol{d}$ be as in Theorem 4.3 and let $\boldsymbol{v}_0 \in \boldsymbol{X}_{m+3}$. Then, for each $\varepsilon > 0$, there is a time $T > 0$ such that problem (4.6) has a solution with the regularity properties (4.7) and (4.9). Moreover, there are constants $C = C(\Omega, m, \alpha)$ and $\tilde{C} = \tilde{C}(\Omega, m, \alpha, \|\boldsymbol{v}_0\|_{m+3} + \varepsilon)$ such that*

$$\|\boldsymbol{v}\|_{m+3,T} \leq C(\|\boldsymbol{v}_0\|_{m+3} + \varepsilon),$$
$$\|\frac{d\boldsymbol{v}}{dt}\|_{m+2,T} \leq \tilde{C}([\nu + 1 + \|\hat{K}\|_{C^{m+1}} + \|N\|_{C^{m+1}}](\|\boldsymbol{v}_0\|_{m+3} + \varepsilon)$$
$$+ \|\mathbf{curl}\,\boldsymbol{g}\|_{m-1,T} + \|\boldsymbol{d}\|_{m+1/2,T,\partial\Omega}).$$

Proof. Set $D_\star = \|\boldsymbol{v}_0\|_{m+3}/C_\star$, $D = D_\star + \varepsilon/C_\star$ and apply Theorem 4.3.

The following remarks motivate the need for additional *a priori* estimates in order to establish global (or "large-time") existence.

Remark 4.5 (a) From the definition of F_1 in Lemma 4.2 one has the lower bound

$$F_1 \geq \max(4C_{10}\frac{\nu}{\alpha},\ C_9(C_{10}+1)\frac{\alpha}{\nu})C_{10}D^2, \tag{4.11}$$

which results from the term $(\nu/\alpha)\mathbf{curl}\,\boldsymbol{v}$ in equation (I) and the terms $\check{\boldsymbol{b}}$ and $\check{\boldsymbol{s}}$ in equation (II). This implies that, for any choice of $D_\star$, the time $T^\star$ in (4.1) is bounded with respect to D. For example, if $D \longrightarrow \infty$ then $F_0, F_1/F_0 \geq C(\Omega, m, \nu, \alpha)D$ and therefore $T^\star \longrightarrow 0$ (see (4.3)). On the other hand, if $D_\star = 0$ and $D \longrightarrow 0$, then $(4.1)_2$ gives $T^\star \leq C(\Omega, m, \nu, \alpha)$, irrespective of the values of $\boldsymbol{h}$, K, N and $\boldsymbol{d}$. Here one could attempt to improve on $(4.1)_2$ by using

$$f(t) \leq D_\star^2 e^{F_0 t} + \frac{F_1}{F_0}(e^{F_0 t} - 1) \leq D_\star^2 - \frac{F_1}{F_0}$$

instead of (4.4). However, even if $D_\star$, K, N, $\boldsymbol{d}$ and $\boldsymbol{g}$ are all zero, inequality (4.11) shows that $-F_1/F_0 > \alpha F_1/\nu \geq \overline{C}(\Omega, m, \nu, \alpha)D^2$, where

$$\overline{C} = \max(4C_{10}, C_9(C_{10}+1)\frac{\alpha^2}{\nu^2})C_{10} \tag{4.12}$$

is unknown (in general at least) and cannot be assumed to be less than 1.

(b) The boundary condition $(\boldsymbol{A}_1\boldsymbol{n})_\tau = \boldsymbol{\eta}$ in problem (III) is independent of α. Hence, unlike the situation in [170], the constant $C_3(\Omega, m, \alpha)$ of the estimate (3.11) for the Stokes problem – and therefore $C_{10}(\Omega, m, \alpha)$ in (4.22) – is not of the form $C(\Omega, m)/\alpha$. Thus, in view of the first term in (4.12), it seems impossible to deduce directly from $(4.1)_2$ the existence of a solution for an arbitrary finite T (not to mention a global solution as in [170]) by choosing α sufficiently large, and α/ν and D correspondingly small.

4.2 Uniqueness

The important relation here is Korn's (second) inequality:

Lemma 4.6 *If Ω is a bounded domain with a Lipschitz continuous boundary, then there is a constant $\kappa = \kappa(\Omega) > 0$ such that*

$$\|\boldsymbol{A}_1(\boldsymbol{v})\|_0^2 + \|\boldsymbol{v}\|_0^2 \geq \kappa \|\boldsymbol{v}\|_1^2 \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega). \tag{4.13}$$

Proof. See [202, p. 110], [207, p. 86], [48, p. 701] or [22, p. 31]. □

The following lemma (with $\Sigma = \emptyset$) shows that the solution obtained in Theorem 4.4 is unique:

Lemma 4.7 *Let Ω be a bounded domain of class C^4 with $\partial\Omega = \Gamma \cup \Sigma$, $\Gamma \cap \Sigma = \emptyset$, and suppose that $\boldsymbol{F} : \boldsymbol{X}_3 \mapsto \boldsymbol{Z}_{1/2}(\Gamma)$ is an operator with the continuity property*

$$\|\boldsymbol{F}(\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{v}')\|_{0,\Gamma} \leq C_F(\Omega, \|\boldsymbol{v}\|_3, \|\boldsymbol{v}'\|_3)\|\boldsymbol{v} - \boldsymbol{v}'\|_1 \quad \forall\, \boldsymbol{v}, \boldsymbol{v}' \in \boldsymbol{X}_3. \tag{4.14}$$

Then, for any $0 < T < \infty$, $\alpha > 0$, $\beta \equiv \alpha_2/\rho \in \boldsymbol{R}$, and arbitrary data $\boldsymbol{v}_0 \in \boldsymbol{X}_3$, $\boldsymbol{v}_\Sigma \in L^\infty(0,T;\boldsymbol{Z}_{5/2}(\Sigma))$, $\tilde{\boldsymbol{f}} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ and $\boldsymbol{d} \in L^\infty(0,T;\boldsymbol{Z}_{1/2}(\Gamma))$, the problem

$$\begin{aligned}
&\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = \nabla\cdot\tilde{\boldsymbol{\mathcal{T}}}(\boldsymbol{v},\tilde{p}) + \tilde{\boldsymbol{f}} && \text{in } \Omega_T,\\
&\nabla\cdot\boldsymbol{v} = 0 && \text{in } \Omega_T,\\
&\boldsymbol{v}\cdot\boldsymbol{n} = 0 && \text{on } \partial\Omega_T,\\
&\boldsymbol{v}_\tau = \boldsymbol{v}_\Sigma && \text{on } \Sigma_T,\\
&(\tilde{\boldsymbol{\mathcal{T}}}\boldsymbol{n})_\tau(\boldsymbol{v}) = \boldsymbol{F}(\boldsymbol{v}) + \boldsymbol{d} && \text{on } \Gamma_T,\\
&\boldsymbol{v}(0) = \boldsymbol{v}_0 && \text{in } \Omega,
\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}
\tilde{\boldsymbol{\mathcal{T}}}(\boldsymbol{v},\tilde{p}) \equiv \frac{1}{\rho}\boldsymbol{\mathcal{T}} = -\frac{\tilde{p}}{\rho}\boldsymbol{I} + \nu\boldsymbol{A}_1 + \alpha\Big(\frac{\partial}{\partial t}\boldsymbol{A}_1 + \boldsymbol{v}\cdot\nabla\boldsymbol{A}_1 + \boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1\Big)\\
+ (\alpha+\beta)\boldsymbol{A}_1^2,
\end{aligned}$$

can have at most one solution $\boldsymbol{v}, \nabla\tilde{p}$ with

$$\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{H}^3(\Omega)), \ \frac{d\boldsymbol{v}}{dt} \in L^\infty(0,T;\boldsymbol{H}^1(\Omega)).$$

Proof. Let $\boldsymbol{v}^1, p^1$ and $\boldsymbol{v}^2, p^2$ be any two solutions of problem (4.15) and set $\mathcal{T}^i = \tilde{\mathcal{T}}(\boldsymbol{v}^i, p^i), i = 1, 2$, $\boldsymbol{V} = \boldsymbol{v}^1 - \boldsymbol{v}^2, P = p^1 - p^2$. Then, subtracting the equation $(4.15)_1$ in $\boldsymbol{v}^2$ from the corresponding equation in $\boldsymbol{v}^1$, taking the $\boldsymbol{L}^2(\Omega)$ scalar product of the difference with $\boldsymbol{V}$ and integrating by parts, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{V}\|_0^2 + (\boldsymbol{v}^1\cdot\nabla\boldsymbol{v}^1 - \boldsymbol{v}^2\cdot\nabla\boldsymbol{v}^2, \boldsymbol{V})_0 \\
&= ([\mathcal{T}^1 - \mathcal{T}^2]\boldsymbol{n}, \boldsymbol{V})_{0,\partial\Omega} - (\mathcal{T}^1 - \mathcal{T}^2, \nabla\boldsymbol{V})_0 \\
&= (\boldsymbol{F}(\boldsymbol{v}^1) - \boldsymbol{F}(\boldsymbol{v}^2), \boldsymbol{V})_{0,\Gamma} - \frac{1}{2}(\mathcal{T}^1 - \mathcal{T}^2, \boldsymbol{A}_1(\boldsymbol{V}))_0.
\end{aligned} \tag{4.16}$$

For each $t \in [0, T]$, $\boldsymbol{v}^2 \in \boldsymbol{X}_3$, so that

$$(\boldsymbol{v}^2\cdot\nabla\boldsymbol{V}, \boldsymbol{V})_0 = 0$$

and thus, using (3.47),

$$|(\boldsymbol{v}^1\cdot\nabla\boldsymbol{v}^1 - \boldsymbol{v}^2\cdot\nabla\boldsymbol{v}^2, \boldsymbol{V})_0| = |(\boldsymbol{V}\cdot\nabla\boldsymbol{v}^1, \boldsymbol{V})_0| \leq C(\Omega)\|\boldsymbol{v}^1\|_{3,T}\|\boldsymbol{V}\|_0^2. \tag{4.17}$$

In the same way one has

$$(\boldsymbol{v}^2\cdot\nabla\boldsymbol{A}_1(\boldsymbol{V}), \boldsymbol{A}_1(\boldsymbol{V}))_0 = 0,$$

and so, with the help of (3.48) and Korn's inequality,

$$\begin{aligned}
&|(\boldsymbol{v}^1\cdot\nabla\boldsymbol{A}_1(\boldsymbol{v}^1) - \boldsymbol{v}^2\cdot\nabla\boldsymbol{A}_1(\boldsymbol{v}^2), \boldsymbol{A}_1(\boldsymbol{V}))_0| = |(\boldsymbol{V}\cdot\nabla\boldsymbol{A}_1(\boldsymbol{v}^1), \boldsymbol{A}_1(\boldsymbol{V}))_0| \\
&\leq C(\Omega)\|\boldsymbol{v}^1\|_{3,T}\|\boldsymbol{V}\|_1\|\boldsymbol{A}_1(\boldsymbol{V})\|_0 \leq \frac{C(\Omega)}{\sqrt{\kappa}}\|\boldsymbol{v}^1\|_{3,T}(\|\boldsymbol{A}_1(\boldsymbol{V})\|_0^2 + \|\boldsymbol{V}\|_0^2).
\end{aligned} \tag{4.18}$$

Similarly, by (3.47),

$$\begin{aligned}
&|(\boldsymbol{A}_1(\boldsymbol{v}^1)^2 - \boldsymbol{A}_1(\boldsymbol{v}^2)^2, \boldsymbol{A}_1(\boldsymbol{V}))_0| \\
&= |(\boldsymbol{A}_1(\boldsymbol{V})\boldsymbol{A}_1(\boldsymbol{v}^1) + \boldsymbol{A}_1(\boldsymbol{v}^2)\boldsymbol{A}_1(\boldsymbol{V}), \boldsymbol{A}_1(\boldsymbol{V}))_0| \\
&\leq C(\Omega)(\|\boldsymbol{v}^1\|_{3,T} + \|\boldsymbol{v}^2\|_{3,T})\|\boldsymbol{A}_1(\boldsymbol{V})\|_0^2.
\end{aligned} \tag{4.19}$$

Furthermore, since $\boldsymbol{A}_1(\boldsymbol{V})\boldsymbol{W}(\boldsymbol{v}^1):\boldsymbol{A}_1(\boldsymbol{V}) = \boldsymbol{W}(\boldsymbol{v}^1)\boldsymbol{A}_1(\boldsymbol{V}):\boldsymbol{A}_1(\boldsymbol{V}) = \boldsymbol{W}(\boldsymbol{v}^1):\boldsymbol{A}_1(\boldsymbol{V})^2 = 0$, it follows as in (4.18) via (3.47) and Korn's inequality that

$$\begin{aligned}
&|([\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1](\boldsymbol{v}^1) - [\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1](\boldsymbol{v}^2), \boldsymbol{A}_1(\boldsymbol{V}))_0| \\
&= |(\boldsymbol{A}_1(\boldsymbol{v}^2)\boldsymbol{W}(\boldsymbol{V}) - \boldsymbol{W}(\boldsymbol{V})\boldsymbol{A}_1(\boldsymbol{v}^2), \boldsymbol{A}_1(\boldsymbol{V}))_0| \\
&\leq \frac{C(\Omega)}{\sqrt{\kappa}}\|\boldsymbol{v}^2\|_{3,T}(\|\boldsymbol{A}_1(\boldsymbol{V})\|_0^2 + \|\boldsymbol{V}\|_0^2).
\end{aligned} \tag{4.20}$$

Lastly, by the trace theorem, assumption (4.14) and Korn's inequality,

$$|(\boldsymbol{F}(\boldsymbol{v}^1) - \boldsymbol{F}(\boldsymbol{v}^2), \boldsymbol{V})_{0,\Gamma}| \leq \hat{C}(\|\boldsymbol{A}_1(\boldsymbol{v})\|_0^2 + \|\boldsymbol{v}\|_0^2) \tag{4.21}$$

with $\hat{C} = C_2(\Omega)C_F(\Omega, \|\boldsymbol{v}^1\|_{3,T}, \|\boldsymbol{v}^2\|_{3,T})/\kappa$. Collecting inequalities (4.17) - (4.21) into equation (4.16) yields

$$\frac{d}{dt}(\|\boldsymbol{V}\|_0^2 + \frac{\alpha}{2}\|\boldsymbol{A}_1(\boldsymbol{V})\|_0^2) \leq \frac{C}{\min(1, \alpha/2)}(\|\boldsymbol{V}\|_0^2 + \frac{\alpha}{2}\|\boldsymbol{A}_1(\boldsymbol{V})\|_0^2)$$

with $C = C(\Omega, \|\boldsymbol{v}^1\|_{3,T}, \|\boldsymbol{v}^2\|_{3,T}, C_F, \alpha, \nu, |\alpha+\beta|)$. Since $\boldsymbol{V}(0) = \boldsymbol{0}$, Gronwall's lemma implies that $\boldsymbol{V} \equiv \boldsymbol{0}$, and consequently also $\nabla P \equiv \boldsymbol{0}$.

Remark 4.8 (a) Inequality (4.14) holds for $\boldsymbol{F}(\boldsymbol{v}) \equiv (K + N(|\boldsymbol{v}|^2))\boldsymbol{v}$, since – using the imbedding $\boldsymbol{H}^3(\Omega) \hookrightarrow \boldsymbol{C}^1(\Omega)$ and inequalities (2.16), (3.1) and (2.14) – one has

$$\|\boldsymbol{F}(\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{v}')\|_{0,\Gamma} \leq C(\Omega)(\|K\|_{C^0}\|\boldsymbol{v} - \boldsymbol{v}'\|_1 + J_1)$$

with

$$\begin{aligned}
J_1 &= \|N(|\boldsymbol{v}|^2)\boldsymbol{v} - N(|\boldsymbol{v}'|^2)\boldsymbol{v}'\|_1 \leq J_2 + J_3, \\
J_2 &= \|\{N(|\boldsymbol{v}|^2) - N(|\boldsymbol{v}'|^2)\}\boldsymbol{v}\|_1 \\
&\leq C(\Omega)\|\boldsymbol{v}\|_3\|N'\|_{C^1}(1 + \|\nabla(|\boldsymbol{v}|^2)\|_1)\|(\boldsymbol{v} + \boldsymbol{v}')\cdot(\boldsymbol{v} - \boldsymbol{v}')\|_1 \\
&\leq C(\Omega)\|N\|_{C^2}\|\boldsymbol{v}\|_3(1 + \|\boldsymbol{v}\|_2^2)(\|\boldsymbol{v}\|_3 + \|\boldsymbol{v}'\|_3)\|\boldsymbol{v} - \boldsymbol{v}'\|_1, \\
J_3 &= \|N(|\boldsymbol{v}'|^2)(\boldsymbol{v} - \boldsymbol{v}')\|_1 \\
&\leq C(\Omega)\|N\|_{C^3}(\sqrt{|\Omega|} + C(\Omega)(\||\boldsymbol{v}'|^2\|_3 + \||\boldsymbol{v}'|^2\|_3^3)\|\boldsymbol{v} - \boldsymbol{v}'\|_1 \\
&\leq C(\Omega)\|N\|_{C^3}(1 + \|\boldsymbol{v}'\|_3^2 + \|\boldsymbol{v}'\|_3^6)\|\boldsymbol{v} - \boldsymbol{v}'\|_1.
\end{aligned}$$

(b) With $\Gamma = \emptyset$, the Lemma provides a simple uniqueness proof for the Dirichlet problems (on bounded domains) considered in [166, 170, 174, 176].

4.3 Appendix

Proof of Lemma 4.2(a).
The **convexity** and **boundedness** of G is obvious.

To prove that G is **closed** in X, let $(\boldsymbol{\phi}_n, \boldsymbol{\eta}_n), n = 1, 2, \ldots,$ be a sequence in G converging to $(\boldsymbol{\phi}, \boldsymbol{\eta})$ in X. Then $\boldsymbol{\phi}(0) = \boldsymbol{u}_0$ and $\boldsymbol{\eta}(0) = \boldsymbol{w}_0$ because

$$\begin{aligned}&\max(\|\boldsymbol{\phi}(0) - \boldsymbol{u}_0\|_{m-1}, \|\boldsymbol{\eta}(0) - \boldsymbol{w}_0\|_{m+1})\\ &= \max(\|\boldsymbol{\phi}(0) - \boldsymbol{\phi}_n(0)\|_{m-1}, \|\boldsymbol{\eta}(0) - \boldsymbol{\eta}_n(0)\|_{m+1})\\ &\leq \|(\boldsymbol{\phi}, \boldsymbol{\eta}) - (\boldsymbol{\phi}_n, \boldsymbol{\eta}_n)\|_X \to 0, \; n \to \infty.\end{aligned}$$

Moreover, for a.e. $t \in [0, T]$, $(\boldsymbol{\phi}_n(t))$ is a bounded sequence in the reflexive Banach space $\boldsymbol{H}^m(\Omega)$ and therefore has a subsequence $(\boldsymbol{\phi}_{n_k}(t))$ which converges weakly to some $\boldsymbol{\psi}(t)$ in $\boldsymbol{H}^m(\Omega)$, which (by a corollary of the Hahn-Banach theorem; see e.g. [204, p. 262]) implies that

$$\|\boldsymbol{\psi}(t)\|_m \leq \liminf \|\boldsymbol{\phi}_{n_k}(t)\|_m.$$

As $\boldsymbol{H}^m(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{H}^{m-1}(\Omega)$, the subsequence converges to $\boldsymbol{\psi}(t)$ in $\boldsymbol{H}^{m-1}(\Omega)$ and hence by uniqueness of limits, $\boldsymbol{\phi}(t) = \boldsymbol{\psi}(t) \in \boldsymbol{H}^m(\Omega)$ with $\|\boldsymbol{\phi}(t)\|_m \leq D$. (Alternatively, since $\boldsymbol{H}^m(\Omega) \hookrightarrow \boldsymbol{H}^{m-1}(\Omega)$, the subsequence converges weakly to $\boldsymbol{\psi}(t)$ in $\boldsymbol{H}^{m-1}(\Omega)$ and hence by the uniqueness of weak limits, $\boldsymbol{\phi}(t) = \boldsymbol{\psi}(t)$.) In the same way it follows that $\boldsymbol{\eta} \in L^\infty(0, T; \boldsymbol{H}^{m+2}(\Omega))$ with $\|\boldsymbol{\eta}\|_{m+2,T} \leq D$. Thus $(\boldsymbol{\phi}, \boldsymbol{\eta}) \in G$.

To prove that $\overline{\Phi(G)}$ is **compact**, let $(\overline{\boldsymbol{u}}_n, \overline{\boldsymbol{w}}_n), n = 1, 2, \ldots,$ be any sequence in $\overline{\Phi(G)}$. Then there is a sequence $(\boldsymbol{\phi}_n, \boldsymbol{\eta}_n)$ in G such that $(\boldsymbol{u}_n, \boldsymbol{w}_n) \equiv \Phi(\boldsymbol{\phi}_n, \boldsymbol{\eta}_n)$ satisfies $\|(\boldsymbol{u}_n, \boldsymbol{w}_n) - (\overline{\boldsymbol{u}}_n, \overline{\boldsymbol{w}}_n)\|_X < 1/n, n = 1, 2, \ldots$ With $\boldsymbol{v}_n$ denoting the solution of problem (III) corresponding to the data $(\boldsymbol{\phi}_n, \boldsymbol{\eta}_n)$, we know from (3.11) (with $k = 1$) that

$$\begin{aligned}\|\boldsymbol{v}_n\|_{m+3,T} &\leq C_3(\Omega, m, \alpha)(\|\boldsymbol{\phi}_n\|_{m,T} + C_2(\Omega, m)\|\boldsymbol{\eta}_n\|_{m+2,T})\\ &\leq C_{10}(\Omega, m, \alpha)D,\end{aligned} \tag{4.22}$$

where $C_{10} = C_3(1 + C_2)$, and thus from (3.18) that

$$\boldsymbol{u}_n \quad \text{is bounded in } L^\infty(0, T; \boldsymbol{V}_m),$$

$$\frac{d\boldsymbol{u}_n}{dt} \quad \text{is bounded in } L^\infty(0, T; \boldsymbol{V}_{m-1}).$$

This implies that $\boldsymbol{u}_n$ is bounded in $W^{1,2}(0,T;\boldsymbol{V}_{m-1})$, which is compactly imbedded in $C([0,T];\boldsymbol{V}_{m-1})$ ([172, p. 538]), so that $(\boldsymbol{u}_n)$ has a subsequence which converges to, say, $\boldsymbol{u}$ in $C([0,T];\boldsymbol{V}_{m-1})$. By a similar argument, based on (3.36), the corresponding subsequence of $(\boldsymbol{w}_n)$ has a subsequence $(\boldsymbol{w}_{n_k})$ which converges to a $\boldsymbol{w}$ in $C([0,T];\boldsymbol{X}_{m+1})$. Hence $(\overline{\boldsymbol{u}}_{n_k},\overline{\boldsymbol{w}}_{n_k}) \to (\boldsymbol{u},\boldsymbol{w})$ in X.

For any $(\boldsymbol{\phi},\boldsymbol{\eta}),(\boldsymbol{\phi}_n,\boldsymbol{\eta}_n) \in G$, let $\boldsymbol{v},\boldsymbol{v}_n$ be the solutions of the corresponding problems (III) and set $(\boldsymbol{u},\boldsymbol{w}) = \Phi(\boldsymbol{\phi},\boldsymbol{\eta}),(\boldsymbol{u}_n,\boldsymbol{w}_n) = \Phi(\boldsymbol{\phi}_n,\boldsymbol{\eta}_n)$. Then $\boldsymbol{v}-\boldsymbol{v}_n$ is the solution of (III) with the data $(\boldsymbol{\phi}-\boldsymbol{\phi}_n,\boldsymbol{\eta}-\boldsymbol{\eta}_n)$, and as in (4.22) we have

$$\|\boldsymbol{v}\|_{m+3,T},\ \|\boldsymbol{v}_n\|_{m+3,T} \le C_{10}(\Omega,m,\alpha)D, \tag{4.23}$$

$$\|\boldsymbol{v}-\boldsymbol{v}_n\|_{m+3,T} \le C_{10}(\Omega,m,\alpha)\|(\boldsymbol{\phi},\boldsymbol{\eta})-(\boldsymbol{\phi}_n,\boldsymbol{\eta}_n)\|_X, \tag{4.24}$$

and thus by case (a) of (3.18), with $\mathrm{M} = C_{10}\mathrm{D}$,

$$\|\boldsymbol{u}\|_{m,T} \le C_6(\Omega,m,T,D,\alpha,\nu). \tag{4.25}$$

Furthermore, via (3.14) - (3.17) we obtain

$$\begin{aligned} &\|\boldsymbol{u}\cdot\nabla\boldsymbol{v} - \boldsymbol{u}_n\cdot\nabla\boldsymbol{v}_n\|_{m-1} \\ &\le \|\boldsymbol{u}\cdot\nabla(\boldsymbol{v}-\boldsymbol{v}_n)\|_{m-1} + \|(\boldsymbol{u}-\boldsymbol{u}_n)\cdot\nabla\boldsymbol{v}_n\|_{m-1} \\ &\le C_5(\|\boldsymbol{u}\|_{m-1}\|\boldsymbol{v}-\boldsymbol{v}_n\|_{m+2} + \|\boldsymbol{u}-\boldsymbol{u}_n\|_{m-1}\|\boldsymbol{v}_n\|_{m+2}) \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} &|(\boldsymbol{v}\cdot\nabla\boldsymbol{u} - \boldsymbol{v}_n\cdot\nabla\boldsymbol{u}_n, \boldsymbol{u}-\boldsymbol{u}_n)_{m-1}| \\ &\le \|(\boldsymbol{v}-\boldsymbol{v}_n)\cdot\nabla\boldsymbol{u}\|_{m-1}\|\boldsymbol{u}-\boldsymbol{u}_n\|_{m-1} + |(\boldsymbol{v}_n\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_n),\boldsymbol{u}-\boldsymbol{u}_n)_{m-1}| \\ &\le C_5\|\boldsymbol{v}-\boldsymbol{v}_n\|_{m+2}\|\boldsymbol{u}\|_m\|\boldsymbol{u}-\boldsymbol{u}_n\|_{m-1} + C_4\|\boldsymbol{v}_n\|_{m+1}\|\boldsymbol{u}-\boldsymbol{u}_n\|_{m-1}^2. \end{aligned} \tag{4.27}$$

Subtracting equation $(\mathrm{I})_1$ for $\boldsymbol{u}_n$ from the one for $\boldsymbol{u}$ gives

$$\begin{aligned} \frac{\partial}{\partial t}(\boldsymbol{u}-\boldsymbol{u}_n) + \frac{\nu}{\alpha}(\boldsymbol{u}-\boldsymbol{u}_n) &= \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \boldsymbol{u}_n\cdot\nabla\boldsymbol{v}_n - \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \boldsymbol{v}_n\cdot\nabla\boldsymbol{v}_n \\ &\quad + \frac{\nu}{\alpha}\mathbf{curl}\,(\boldsymbol{v}-\boldsymbol{v}_n). \end{aligned}$$

Let $d_n(t) \equiv \|\boldsymbol{u}-\boldsymbol{u}_n\|_{m-1}$. Then taking the inner product in $\boldsymbol{H}^{m-1}(\Omega)$ of this equation with $\boldsymbol{u}-\boldsymbol{u}_n$, and applying the Cauchy-Schwarz inequality and

(4.23) – (4.27) yields

$$
\begin{aligned}
&[d_n'(t) + \frac{\nu}{\alpha} d_n(t)] d_n(t) \\
&\le [\frac{2\nu}{\alpha} \|\boldsymbol{v} - \boldsymbol{v}_n\|_m + C_5 \|\boldsymbol{u}\|_{m-1} \|\boldsymbol{v} - \boldsymbol{v}_n\|_{m+2} + C_5 d_n(t) \|\boldsymbol{v}_n\|_{m+2} \\
&\quad + C_5 \|\boldsymbol{v} - \boldsymbol{v}_n\|_{m+2} \|\boldsymbol{u}\|_m + C_4 \|\boldsymbol{v}_n\|_{m+1} d_n(t)] d_n(t) \\
&\le [(C_4 + C_5) C_{10} D d_n(t) + 2(\frac{\nu}{\alpha} + C_5 C_6) C_{10} \|(\boldsymbol{\phi}, \boldsymbol{\eta}) - (\boldsymbol{\phi}_n, \boldsymbol{\eta}_n)\|_X] d_n(t)
\end{aligned}
$$

or

$$
\begin{aligned}
&d_n'(t) + \lambda d_n(t) \le \xi \|(\boldsymbol{\phi}, \boldsymbol{\eta}) - (\boldsymbol{\phi}_n, \boldsymbol{\eta}_n)\|_X, \; d(0) = 0, \\
&\lambda = \frac{\nu}{\alpha} - (C_4 + C_5) C_{10} D, \quad \xi = 2(\frac{\nu}{\alpha} + C_5 C_6) C_{10}.
\end{aligned}
$$

It follows that

$$
\|\boldsymbol{u} - \boldsymbol{u}_n\|_{m-1,T} \le K_1 \|(\boldsymbol{\phi}, \boldsymbol{\eta}) - (\boldsymbol{\phi}_n, \boldsymbol{\eta}_n)\|_X \tag{4.28}
$$

with $K_1 = \xi T$ if $\lambda = 0$ and $K_1 = \xi(1 - e^{-\lambda T})/\lambda$ otherwise, i.e. the map $(\boldsymbol{\phi}, \boldsymbol{\eta}) \mapsto \boldsymbol{u} : G \mapsto C([0,T]; \boldsymbol{H}^{m-1}(\Omega))$ is Lipschitz continuous.

Subtracting equation $(\mathrm{I})_1$ for $\boldsymbol{w}_n$ from the corresponding equation in $\boldsymbol{w}$ gives

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\boldsymbol{w} - \boldsymbol{w}_n) + \frac{\nu}{\alpha}(\boldsymbol{w} - \boldsymbol{w}_n) + \nabla(q - q_n) + \boldsymbol{v} \cdot \nabla \boldsymbol{w} - \boldsymbol{v}_n \cdot \nabla \boldsymbol{w}_n \\
&= \frac{1}{\alpha}(\breve{\boldsymbol{s}} - \breve{\boldsymbol{s}}_n) - \breve{\boldsymbol{b}} + \breve{\boldsymbol{b}}_n - \breve{\boldsymbol{c}} + \breve{\boldsymbol{c}}_n,
\end{aligned}
$$

where $\breve{\boldsymbol{s}}_n$ denotes $\breve{\boldsymbol{s}}(\boldsymbol{v}_n)$, etc. As in [170, 172] one can avoid the difficulty of deriving an estimate for the irrotational term by working in $\boldsymbol{L}^2(\Omega)$. Define $y_n(t) = \|\boldsymbol{w} - \boldsymbol{w}_n\|_0$ and take the $\boldsymbol{L}^2(\Omega)$ scalar product of the above equation with $\boldsymbol{w} - \boldsymbol{w}_n$. Since by (3.14) and (3.15)

$$
\begin{aligned}
&|(\boldsymbol{v} \cdot \nabla \boldsymbol{w} - \boldsymbol{v}_n \cdot \nabla \boldsymbol{w}_n, \boldsymbol{w} - \boldsymbol{w}_n)_0| \\
&\le |(\boldsymbol{v} - \boldsymbol{v}_n) \cdot \nabla \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{w}_n)_0| + |(\boldsymbol{v}_n \cdot \nabla(\boldsymbol{w} - \boldsymbol{w}_n), \boldsymbol{w} - \boldsymbol{w}_n)_0| \\
&\le C_5 \|\boldsymbol{v} - \boldsymbol{v}_n\|_2 \|\boldsymbol{w}\|_1 y_n(t),
\end{aligned}
$$

application of the Cauchy-Schwarz inequality and division by $y_n(t)$ yields

$$
\begin{aligned}
y_n'(t) + \frac{\nu}{\alpha} y_n(t) \le \frac{1}{\alpha} \|\breve{\boldsymbol{s}} - \breve{\boldsymbol{s}}_n\|_0 + \|\breve{\boldsymbol{b}} - \breve{\boldsymbol{b}}_n\|_0 + \|\breve{\boldsymbol{c}} - \breve{\boldsymbol{c}}_n\|_0 \\
+ C_5 \|\boldsymbol{v} - \boldsymbol{v}_n\|_2 \|\boldsymbol{w}\|_1.
\end{aligned}
\tag{4.29}
$$

By construction, $\breve{\boldsymbol{s}}-\breve{\boldsymbol{s}}_n$ satisfies the Stokes equation with zero body force and is solenoidal in Ω_T, and is equal to $S(|\boldsymbol{v}|)\boldsymbol{v} - S(|\boldsymbol{v}_n|)\boldsymbol{v}_n$ on $\partial\Omega_T$. Hence, for each t, it follows from the well-known results of [183] (see Theorem VII), the extension (3.28), the trace theorem, the algebra property (3.1) and Lemma 2.3 (with $f = |\boldsymbol{v}|^2, g = |\boldsymbol{v}_n|^2$, so that $\|f-g\|_2 \leq C_1\|\boldsymbol{v}+\boldsymbol{v}_n\|_2\|\boldsymbol{v}-\boldsymbol{v}_n\|_2$) that

$$\begin{aligned}
&\|\breve{\boldsymbol{s}} - \breve{\boldsymbol{s}}_n\|_2 \\
&\leq C(\Omega)\|S(|\boldsymbol{v}|)\boldsymbol{v} - S(|\boldsymbol{v}_n|)\boldsymbol{v}_n\|_{3/2,\partial\Omega} \\
&\leq C(\Omega)(\|\hat{K}\|_{C^2}\|\boldsymbol{v}-\boldsymbol{v}_n\|_2 + C_1\|N(|\boldsymbol{v}|^2)\|_2\|\boldsymbol{v}-\boldsymbol{v}_n\|_2 \\
&\quad + C_1\|N(|\boldsymbol{v}|^2) - N(|\boldsymbol{v}_n|^2)\|_2\|\boldsymbol{v}_n\|_2) \\
&\leq C(\Omega)\|\boldsymbol{v}-\boldsymbol{v}_n\|_2 \cdot \\
&\quad \cdot\{\|\hat{K}\|_{C^2} + \|N\|_{C^2}(|\Omega|^{1/2} + C(\Omega)[C_1\|\boldsymbol{v}\|_2^2 + C_1^2\|\boldsymbol{v}\|_2^4]) \\
&\quad + C(\Omega)\|N'\|_{C^2}(1 + C_1^2\|\boldsymbol{v}\|_2^4 + C_1^2\|\boldsymbol{v}_n\|_2^4)C_1(\|\boldsymbol{v}\|_2 + \|\boldsymbol{v}_n\|_2)\}.
\end{aligned} \tag{4.30}$$

One could also use the estimate of [183] for $\|\breve{\boldsymbol{s}} - \breve{\boldsymbol{s}}_n\|_1$, but the $\boldsymbol{H}^2(\Omega)$-estimate is convenient and suffices since $m \geq 1$. In the same way, with $\boldsymbol{c}$ and $\boldsymbol{c}_n$ denoting the extension (3.25) for $\boldsymbol{v}$ and $\boldsymbol{v}_n$ respectively, one obtains

$$\begin{aligned}
\|\breve{\boldsymbol{c}} - \breve{\boldsymbol{c}}_n\|_2 &\leq C(\Omega)\|([\boldsymbol{A}_1(\boldsymbol{v})\boldsymbol{W}(\boldsymbol{v}) - \boldsymbol{W}(\boldsymbol{v})\boldsymbol{A}_1(\boldsymbol{v})]\boldsymbol{n})_\tau \\
&\qquad - ([\boldsymbol{A}_1(\boldsymbol{v}_n)\boldsymbol{W}(\boldsymbol{v}_n) - \boldsymbol{W}(\boldsymbol{v}_n)\boldsymbol{A}_1(\boldsymbol{v}_n)]\boldsymbol{n})_\tau\|_{3/2,\partial\Omega} \\
&\leq C(\Omega)\|\boldsymbol{c} - \boldsymbol{c}_n\|_2 \\
&\leq C(\Omega)(\|\boldsymbol{v}\|_3 + \|\boldsymbol{v}_n\|_3)\|\boldsymbol{v}-\boldsymbol{v}_n\|_3.
\end{aligned} \tag{4.31}$$

Furthermore, with $\boldsymbol{b}$ and $\boldsymbol{b}_n$ defined as in (2.22),

$$\begin{aligned}
&\triangle(\breve{\boldsymbol{b}} - \breve{\boldsymbol{b}}_n) + \nabla(\pi - \pi_n) = 0 && \text{in } \Omega_T, \\
&\nabla\cdot(\breve{\boldsymbol{b}} - \breve{\boldsymbol{b}}_n) = (\nabla\boldsymbol{v}_n)^T : \nabla\boldsymbol{\eta}_n - (\nabla\boldsymbol{v})^T : \nabla\boldsymbol{\eta} && \text{in } \Omega_T, \\
&\breve{\boldsymbol{b}} - \breve{\boldsymbol{b}}_n = \boldsymbol{b} - \boldsymbol{b}_n && \text{on } \partial\Omega_T,
\end{aligned}$$

and thus, by [183] and the trace theorem,

$$\|\breve{\boldsymbol{b}} - \breve{\boldsymbol{b}}_n\|_2 \leq C(\Omega)(\|(\nabla\boldsymbol{v})^T : \nabla\boldsymbol{\eta} - (\nabla\boldsymbol{v}_n)^T : \nabla\boldsymbol{\eta}_n\|_1 + \|\boldsymbol{b}-\boldsymbol{b}_n\|_2) \tag{4.32}$$

with

$$\begin{aligned}
&\|(\nabla\boldsymbol{v})^T : \nabla\boldsymbol{\eta} - (\nabla\boldsymbol{v}_n)^T : \nabla\boldsymbol{\eta}_n\|_1 \\
&\leq \|\nabla(\boldsymbol{v}-\boldsymbol{v}_n)^T : \nabla\boldsymbol{\eta}\|_1 + \|\nabla(\boldsymbol{v}_n)^T : \nabla(\boldsymbol{\eta}-\boldsymbol{\eta}_n)\|_1 \\
&\leq C(\Omega)(\|\boldsymbol{v}-\boldsymbol{v}_n\|_3\|\boldsymbol{\eta}\|_2 + \|\boldsymbol{v}_n\|_3\|\boldsymbol{\eta}-\boldsymbol{\eta}_n\|_2)
\end{aligned}$$

according to $(3.34)_1$, and, for $i = 1, 2, 3$,

$$\begin{aligned} &\|b_i - b_{ni}\|_2 \\ &\le \|A_1(\boldsymbol{v})_{rj}((\boldsymbol{v} - \boldsymbol{v}_n)\cdot\nabla[\hat{n}_r\hat{n}_j\hat{n}_i])\|_2 + \|A_1(\boldsymbol{v} - \boldsymbol{v}_n)_{rj}(\boldsymbol{v}_n\cdot\nabla[\hat{n}_r\hat{n}_j\hat{n}_i])\|_2 \\ &\quad + \|A_1(\boldsymbol{v})_{ij}((\boldsymbol{v} - \boldsymbol{v}_n)\cdot\nabla\hat{n}_j)\|_2 + \|A_1(\boldsymbol{v} - \boldsymbol{v}_n)_{ij}(\boldsymbol{v}_n\cdot\nabla\hat{n}_j)\|_2 \\ &\le C(\Omega)(\|\boldsymbol{v}\|_3\|\boldsymbol{v} - \boldsymbol{v}_n\|_2 + \|\boldsymbol{v} - \boldsymbol{v}_n\|_3\|\boldsymbol{v}_n\|_2). \end{aligned}$$

Collecting the inequalities (4.30) – (4.32) in (4.29) and using (3.36) (with $m = 1$) and (4.23) – (4.24) one gets

$$\begin{aligned} &y_n'(t) + \frac{\nu}{\alpha}y_n(t) \le \gamma\|(\boldsymbol{\phi}, \boldsymbol{\eta}) - (\boldsymbol{\phi}_n, \boldsymbol{\eta}_n)\|_X,\ y(0) = 0, \\ &\gamma = \gamma(\Omega, m, T, \alpha, \nu, D, \|\hat{K}\|_{C^3}, \|N\|_{C^3}), \end{aligned}$$

and thus

$$\|\boldsymbol{w} - \boldsymbol{w}_n\|_{0,T} \le K_2\|(\boldsymbol{\phi}, \boldsymbol{\eta}) - (\boldsymbol{\phi}_n, \boldsymbol{\eta}_n)\|_X \tag{4.33}$$

with $K_2 = \alpha\gamma(1 - e^{-\nu T/\alpha})/\nu$. Now let $(\boldsymbol{\phi}_n, \boldsymbol{\eta}_n) \longrightarrow (\boldsymbol{\phi}, \boldsymbol{\eta})$ in X, then it follows from (4.28) and (4.33) that

$$\boldsymbol{u}_n \longrightarrow \boldsymbol{u} \text{ in } C([0,T]; \boldsymbol{H}^{m-1}(\Omega)), \quad \boldsymbol{w}_n \longrightarrow \boldsymbol{w} \text{ in } C([0,T]; \boldsymbol{L}^2(\Omega)). \tag{4.34}$$

Suppose that $(\boldsymbol{w}_n)$ does not converge to $\boldsymbol{w}$ in $C([0,T]; \boldsymbol{H}^{m+1}(\Omega))$, then there exists an $\varepsilon > 0$ and a subsequence $(\boldsymbol{w}_{n_k})$ such that

$$\|\boldsymbol{w}_{n_k} - \boldsymbol{w}\|_{m+1,T} \ge \varepsilon, \quad k = 1, 2, \ldots \tag{4.35}$$

By the precompactness of $\Phi(G)$ in X, $(\boldsymbol{u}_{n_k}, \boldsymbol{w}_{n_k})$ has a subsequence $(\boldsymbol{u}_{m_k}, \boldsymbol{w}_{m_k})$ which converges to, say, $(\boldsymbol{u}^\star, \boldsymbol{w}^\star)$ in X. This and $(4.34)_2$ implies that $\boldsymbol{w}_{m_k} \longrightarrow \boldsymbol{w} = \boldsymbol{w}^\star$ in $C([0,T]; \boldsymbol{H}^{m+1}(\Omega))$, contradicting (4.35). Hence $(\boldsymbol{u}_n, \boldsymbol{w}_n) \longrightarrow (\boldsymbol{u}, \boldsymbol{w})$ in X, i.e. Φ is **continuous**. □

Proof of Theorem 4.4 (contd.).
Differentiating equation $(\mathrm{I})_1$ with respect to t gives

$$\frac{\partial^2\boldsymbol{u}}{\partial t^2} + \frac{\nu}{\alpha}(\frac{\partial\boldsymbol{u}}{\partial t} - \mathbf{curl}\,\frac{\partial\boldsymbol{v}}{\partial t}) = \frac{\partial\boldsymbol{u}}{\partial t}\cdot\nabla\boldsymbol{v} + \boldsymbol{u}\cdot\nabla\frac{\partial\boldsymbol{v}}{\partial t} - \frac{\partial\boldsymbol{v}}{\partial t}\cdot\nabla\boldsymbol{u} - \boldsymbol{v}\cdot\nabla\frac{\partial\boldsymbol{u}}{\partial t} + \frac{\partial\boldsymbol{h}}{\partial t},$$

where $\boldsymbol{h} \equiv \mathbf{curl}\,\boldsymbol{g}$, so that

$$\frac{\partial\boldsymbol{h}}{\partial t} \in L^\infty(0,T; \boldsymbol{H}^{m-2}(\Omega)).$$

According to Propositions 3.8 and 3.4, for $m \geq 1$,

$$\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}^m(\Omega)), \quad \frac{d\boldsymbol{u}}{dt} \in L^\infty(0,T;\boldsymbol{H}^{m-1}(\Omega)),$$
$$\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{H}^{m+3}(\Omega)), \quad \frac{d\boldsymbol{v}}{dt} \in L^\infty(0,T;\boldsymbol{H}^{m+2}(\Omega)),$$

and therefore, for $m \geq 2$, one can use Lemma 3.6 to obtain the estimates

$$\|\frac{\partial \boldsymbol{u}}{\partial t}\cdot\nabla\boldsymbol{v}\|_{m-2,T} \leq C_5\|\frac{d\boldsymbol{u}}{dt}\|_{m-2,T}\|\boldsymbol{v}\|_{m+1,T} < \infty,$$
$$\|\boldsymbol{u}\cdot\nabla\frac{\partial \boldsymbol{v}}{\partial t}\|_{m-2,T} \leq C_5\|\boldsymbol{u}\|_{m-2,T}\|\frac{d\boldsymbol{v}}{dt}\|_{m+1,T} < \infty,$$
$$\|\frac{\partial \boldsymbol{v}}{\partial t}\cdot\nabla\boldsymbol{u}\|_{m-2,T} \leq C_5\|\frac{d\boldsymbol{v}}{dt}\|_{m,T}\|\boldsymbol{u}\|_{m-1,T} < \infty,$$
$$\|\boldsymbol{v}\cdot\nabla\frac{\partial \boldsymbol{u}}{\partial t}\|_{m-2,T} \leq C_5\|\boldsymbol{v}\|_{m,T}\|\frac{d\boldsymbol{u}}{dt}\|_{m-1,T} < \infty,$$

and hence

$$\boldsymbol{u} \in W^{2,\infty}(0,T;\boldsymbol{H}^{m-2}(\Omega)). \tag{4.36}$$

In the same fashion, differentiating $(\mathrm{III})_1$ with respect to t yields

$$\frac{\partial^2 \boldsymbol{w}}{\partial t^2} + \frac{\nu}{\alpha}\frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial \boldsymbol{v}}{\partial t}\cdot\nabla\boldsymbol{w} + \boldsymbol{v}\cdot\nabla\frac{\partial \boldsymbol{w}}{\partial t} = \frac{1}{\alpha}\frac{\partial \check{\boldsymbol{s}}}{\partial t} - \frac{\partial \check{\boldsymbol{b}}}{\partial t} - \frac{\partial \check{\boldsymbol{c}}}{\partial t} + \frac{1}{\alpha}\frac{\partial \check{\boldsymbol{d}}}{\partial t}.$$

Now, from Proposition 3.11 one has

$$\boldsymbol{w} \in L^\infty(0,T;\boldsymbol{X}_{m+2}), \quad \frac{d\boldsymbol{w}}{dt} \in L^\infty(0,T;\boldsymbol{H}^{m+1}(\Omega)),$$

and therefore, from Lemma 3.6 for $m \geq 2$,

$$\|\frac{\partial \boldsymbol{v}}{\partial t}\cdot\nabla\boldsymbol{w}\|_{m,T} \leq C_5\|\frac{d\boldsymbol{v}}{dt}\|_{m,T}\|\boldsymbol{w}\|_{m+1,T} < \infty,$$
$$\|\boldsymbol{v}\cdot\nabla\frac{\partial \boldsymbol{w}}{\partial t}\|_{m,T} \leq C_5\|\boldsymbol{v}\|_{m,T}\|\frac{d\boldsymbol{w}}{dt}\|_{m+1,T} < \infty.$$

In addition, differentiating the Stokes problem defining $\check{\boldsymbol{s}}$ with respect to t and using the boundedness of the trace map $\gamma_0 : \boldsymbol{H}^m(\Omega) \mapsto \boldsymbol{H}^{m-1/2}(\partial\Omega)$, one finds

$$\left.\begin{aligned} \triangle\frac{\partial \check{\boldsymbol{s}}}{\partial t} + \nabla\frac{\partial p_1}{\partial t} &= 0 \\ \nabla\cdot\frac{\partial \check{\boldsymbol{s}}}{\partial t} &= 0 \end{aligned}\right\} \quad \text{in } \Omega_T,$$

$$\left.\begin{aligned} \frac{\partial \check{\boldsymbol{s}}}{\partial t} &= \frac{\partial}{\partial t}\gamma_0(\hat{K}\boldsymbol{v} + N(|\boldsymbol{v}|^2)\boldsymbol{v}) \\ &= 2N'(|\boldsymbol{v}|^2)(\boldsymbol{v}\cdot\frac{\partial \boldsymbol{v}}{\partial t})\boldsymbol{v} + (K + N(|\boldsymbol{v}|^2))\frac{\partial \boldsymbol{v}}{\partial t} \end{aligned}\right\} \quad \text{on } \partial\Omega_T,$$

and thus, using the *a priori* estimate for the Stokes problem (with t treated as a parameter) and the algebra property (3.1),

$$\begin{aligned}&\|\frac{\partial \breve{\boldsymbol{s}}}{\partial t}\|_{m,T}\\ &\le C(\Omega, m)(\|\hat{K}\|_{C^m} + \|N(|\boldsymbol{v}|^2)\|_{m,T} + \|N'(|\boldsymbol{v}|^2)\|_{m,T}\|\boldsymbol{v}\|^2_{m,T})\|\frac{d\boldsymbol{v}}{dt}\|_{m,T} < \infty\end{aligned}$$

since, as in (3.27),

$$\begin{aligned}\|N(|\boldsymbol{v}|^2)\|_{m,T} &\le C(\Omega, m, \|\boldsymbol{v}\|_{m,T})\|N\|_{C^m},\\ \|N'(|\boldsymbol{v}|^2)\|_{m,T} &\le C(\Omega, m, \|\boldsymbol{v}\|_{m,T})\|N\|_{C^{m+1}}.\end{aligned}$$

By similar reasoning it follows that

$$\left.\begin{aligned}\triangle\frac{\partial \breve{\boldsymbol{b}}}{\partial t} + \nabla\frac{\partial p_4}{\partial t} = 0\\ \nabla\cdot\frac{\partial \breve{\boldsymbol{b}}}{\partial t} = -(\nabla\frac{\partial \boldsymbol{v}}{\partial t})^T : \nabla \boldsymbol{w} - (\nabla \boldsymbol{v})^T : \nabla\frac{\partial \boldsymbol{w}}{\partial t}\end{aligned}\right\} \text{ in } \Omega_T,$$

$$\left.\begin{aligned}\frac{\partial \breve{b}_i}{\partial t} = {}&(\frac{\partial}{\partial t}A_{1rj}(\boldsymbol{v}))(\boldsymbol{v}\cdot\nabla[\breve{n}_r\breve{n}_j\breve{n}_i]) + A_{1rj}(\boldsymbol{v})(\frac{\partial \boldsymbol{v}}{\partial t}\cdot\nabla[\breve{n}_r\breve{n}_j\breve{n}_i])\\ &- (\frac{\partial}{\partial t}A_{1ij}(\boldsymbol{v}))(\boldsymbol{v}\cdot\nabla\breve{n}_j) - A_{1ij}(\boldsymbol{v})(\frac{\partial \boldsymbol{v}}{\partial t}\cdot\nabla\breve{n}_j),\ i = 1, 2, 3,\end{aligned}\right\} \text{ on } \partial\Omega_T,$$

and consequently, with the help of the estimate (3.34), that

$$\begin{aligned}\|\frac{\partial \breve{\boldsymbol{b}}}{\partial t}\|_{m,T} \le C(\Omega, m)(&\|\frac{d\boldsymbol{v}}{dt}\|_{m+1,T}(\|\boldsymbol{w}\|_{m,T} + \|\boldsymbol{v}\|_{m,T})\\ &+ \|\boldsymbol{v}\|_{m+1,T}(\|\frac{d\boldsymbol{w}}{dt}\|_{m,T} + \|\frac{d\boldsymbol{v}}{dt}\|_{m,T})) < \infty.\end{aligned}$$

One also has

$$\left.\begin{aligned}\triangle\frac{\partial \breve{\boldsymbol{c}}}{\partial t} + \nabla\frac{\partial p_2}{\partial t} = 0\\ \nabla\cdot\frac{\partial \breve{\boldsymbol{c}}}{\partial t} = 0\end{aligned}\right\} \text{ in } \Omega_T,$$

$$\frac{\partial \breve{\boldsymbol{c}}}{\partial t} = \frac{\partial}{\partial t}[(\boldsymbol{A}_1(\boldsymbol{v})\boldsymbol{W}(\boldsymbol{v}) - \boldsymbol{W}(\boldsymbol{v})\boldsymbol{A}_1(\boldsymbol{v}))\boldsymbol{n}]_\tau \quad \text{on } \partial\Omega_T,$$

and as a result, via (3.1),

$$\|\frac{\partial \breve{\boldsymbol{c}}}{\partial t}\|_{m,T} \le C(\Omega, m)\|\frac{d\boldsymbol{v}}{dt}\|_{m+1,T}\|\boldsymbol{v}\|_{m,T} < \infty.$$

In the same way it follows that

$$\|\frac{\partial \check{\boldsymbol{d}}}{\partial t}\|_{m,T} \leq C(\Omega, m)\|\frac{\partial \boldsymbol{d}}{\partial t}\|_{m-1/2,T,\partial\Omega} < \infty.$$

Collecting all the bounds, one obtains

$$\boldsymbol{w} \in W^{2,\infty}(0,T;\boldsymbol{H}^m(\Omega)). \tag{4.37}$$

In conclusion, from (4.36), (4.37) and Proposition 3.4 one deduces that

$$\boldsymbol{v} \in W^{2,\infty}(0,T;\boldsymbol{H}^{m+1}(\Omega)),$$

i.e.

$$\boldsymbol{v},\ \frac{d\boldsymbol{v}}{dt} \in W^{1,\infty}(0,T;\boldsymbol{H}^{m+1}(\Omega)) \hookrightarrow C([0,T];\boldsymbol{H}^{m+1}(\Omega)),$$

or

$$\boldsymbol{v} \in C^1([0,T];\boldsymbol{H}^{m+1}(\Omega)) \hookrightarrow C^1([0,T];\boldsymbol{C}^{m-1}(\overline{\Omega})).$$

Chapter 5

GLOBAL SOLUTIONS

The aim of this chapter is to construct a global solution by repeated application of the local existence theorem. The main hurdle is to derive appropriate time-independent estimates in order to establish that the solution always remains in the same ball as the initial data. This is done for an arbitrary $\alpha > 0$ and slip coefficient $S(\cdot)$ under the assumption that the initial data and the force fields are sufficiently small and that ν is sufficiently large. However, the existence and stability results do not apply when Ω is rotationally symmetric and $S(\cdot)$ is allowed to have nonnegative values.

5.1 Existence

The following result is a slight variation of Lemma 2.5 of [174].

Lemma 5.1 *Let $T > 0$ and suppose that $y(t)$ is a non-negative, continuous function on $[0, T]$ with an integrable derivative in $(0, T)$ satisfying the inequality*

$$y'(t) + [k_1 - G(y(t))]y(t) \leq F(t) \quad \forall\, t \in [0, T], \tag{5.1}$$

where $k_1 > 0$, G is a non-negative, continuous function such that $G(x) \leq k_2$ for all $x \in [0, \varepsilon]$ for some $k_2 \leq k_1$ and $\varepsilon > 0$, and F is a non-negative, integrable function. If

$$y(0) + \int_0^T F(t)\,\mathrm{d}t < \varepsilon$$

then

$$y(t) + (k_1 - k_2)\int_0^T y(s)\,\mathrm{d}s \leq y(0) + \int_0^T F(s)\,\mathrm{d}s \quad \forall\, t \in [0, T]. \tag{5.2}$$

Proof. Assume there is a $\bar{t}$ such that $y(\bar{t}) = \varepsilon$ and $y(t) < \varepsilon$ for all $t \in [0, \bar{t})$. Then $k_1 - G(y(t)) \geq k_1 - k_2$ and thus $y'(t) \leq F(t)$ for all $t \in [0, \bar{t})$. Thus, integrating over $[0, \bar{t})$ and using the non-negativity of $F(t)$, one gets

$$y(\bar{t}) \leq y(0) + \int_0^{\bar{t}} F(t)\,\mathrm{d}t < \varepsilon,$$

a contradiction. Hence $y(t) < \varepsilon$ for all $t \in [0, \varepsilon)$, so that inequality (5.2) follows from integrating (5.1) over $[0, T]$. □

The following three versions of the **Poincaré-Morrey inequality**, the first two of which will be used later, highlights a difference between the no-slip and slip boundary conditions.

Lemma 5.2 *Let Ω be a bounded domain with a Lipschitz continuous boundary. Set*

$$\mathcal{S} = \operatorname{span}\{\boldsymbol{\beta} \times \boldsymbol{x} : \ \boldsymbol{\beta} \in \boldsymbol{R}^3,\ |\boldsymbol{\beta}| = 1,\ \boldsymbol{\beta} \textit{ is a symmetry axis of } \Omega\}$$

and let $\|\cdot\|_{\mathcal{S}}$ denote the norm in $\boldsymbol{L}^2(\Omega)/\mathcal{S}$. In addition, for an arbitrary subsurface $\Sigma \subset \partial\Omega$ with $\operatorname{meas}(\Sigma) > 0$, define

$$\boldsymbol{H}^1_\Sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Sigma\}.$$

Then there exists a constant $C_P = C_P(\Omega)$ such that

$$\begin{aligned} \|\boldsymbol{A}_1(\boldsymbol{v})\|_0^2 + \|\boldsymbol{v}\|_{0,\partial\Omega}^2 &\geq C_P\|\boldsymbol{v}\|_0^2 && \forall\ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \\ \|\boldsymbol{A}_1(\boldsymbol{v})\|_0^2 + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{0,\partial\Omega}^2 &\geq C_P\|\boldsymbol{v}\|_S^2 && \forall\ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \\ \|\boldsymbol{A}_1(\boldsymbol{v})\|_0^2 &\geq C_P\|\boldsymbol{v}\|_0^2 && \forall\ \boldsymbol{v} \in \boldsymbol{H}_\Sigma^1(\Omega). \end{aligned} \tag{5.3}$$

Proof. See [22, p. 31], [48, p. 701] and [213, p. 115] (or [202, p. 115]), respectively.

Under the

Assumption 5.3 *Ω has no axes of symmetry.*

one can use the previous two lemmas to derive global *a priori* estimates:

Lemma 5.4 *Let Ω be a bounded simply-connected domain without symmetry axes, with $\partial\Omega$ of class C^{m+4}, $m \geq 1$, $\alpha > 0$ and $K, N \in C^{m+2}$. Then there is a constant*

$$\nu^\star = \nu^\star(\Omega, m, \alpha, \|\hat{K}\|_{C^{m+2}}, \|N\|_{C^{m+2}})$$

such that for each $\nu \geq \nu^\star$ there exist positive constants δ_0, δ_1, γ_0, γ_1, which depend at most on Ω, m, ν, α, $\|\hat{K}\|_{C^{m+2}}$ and $\|N\|_{C^{m+2}}$, and have the following property:

(a) *If, for any given $T > 0$, initial values $\boldsymbol{u}_0 \in \boldsymbol{V}_m$, $\boldsymbol{w}_0 \in \boldsymbol{X}_{m+2}$ and force fields $\boldsymbol{g} \in L^2(0,T;\boldsymbol{X}_{m+1})$, $\boldsymbol{d} \in L^2(0,T;\boldsymbol{Z}_{m+3/2})$ with*

$$\|\boldsymbol{u}_0\|_m + \|\boldsymbol{w}_0\|_{m+2} < \delta_0, \quad \|\boldsymbol{g}\|_{L^2,m+1,T} + \|\boldsymbol{d}\|_{L^2,m+3/2,T,\partial\Omega} < \gamma_0, \tag{5.4}$$

$(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})$ is a solution of the corresponding problems (I) - (III) *with $(\boldsymbol{\phi},\boldsymbol{\eta}) = (\boldsymbol{u},\boldsymbol{w})$ (and thus $\nabla q = \boldsymbol{0}$) and satisfies*

$$\begin{aligned} &\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}^m(\Omega)) \cap W^{1,\infty}(0,T;\boldsymbol{H}^{m-1}(\Omega)), \\ &\boldsymbol{w} \in L^\infty(0,T;\boldsymbol{X}_{m+2}) \cap W^{1,\infty}(0,T;\boldsymbol{X}_{m+1}), \\ &\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{X}_{m+3}), \end{aligned}$$

then

$$\begin{aligned} &\|\boldsymbol{u}\|_{m,T}^2 + \|\boldsymbol{w}\|_{m+2,T}^2 + \frac{\nu}{2\alpha}\int_0^T (\,\|\boldsymbol{u}(s)\|_m^2 + \|\boldsymbol{w}(s)\|_{m+2}^2\,)\,\mathrm{d}s \\ &\leq \delta_1(\,\|\boldsymbol{u}_0\|_m^2 + \|\boldsymbol{w}_0\|_{m+2}^2\,) + \gamma_1(\,\|\boldsymbol{g}\|_{L^2,m+1,T}^2 + \|\boldsymbol{d}\|_{L^2,m+3/2,T,\partial\Omega}^2\,), \end{aligned} \tag{5.5}$$

(b) *If, in addition,* $\boldsymbol{g} \in L^\infty(0,T;\boldsymbol{X}_m)$ *and* $\boldsymbol{d} \in L^\infty(0,T;\boldsymbol{Z}_{m+1/2})$, *then*

$$\begin{aligned}
&\|\frac{d\boldsymbol{u}}{dt}\|_{m-1,T} + \|\frac{d\boldsymbol{w}}{dt}\|_{m+1,T} \\
&\leq C(\Omega,m,\alpha)(\|\boldsymbol{g}\|_{m,T} + \|\boldsymbol{d}\|_{m+1/2,T,\partial\Omega}) + C(\Omega,m,\nu,\alpha,\|\hat{K}\|_{C^{m+2}}, \\
&\quad \|N\|_{C^{m+2}}, \|\boldsymbol{u}_0\|_m + \|\boldsymbol{w}_0\|_{m+2}, \|\boldsymbol{g}\|_{L^2,m+1,T} + \|\boldsymbol{d}\|_{L^2,m+3/2,T,\partial\Omega}).
\end{aligned} \tag{5.6}$$

Proof. (a) First it is necessary to derive some

Estimates of $\boldsymbol{v}$.
From the proof of Proposition 3.4 and inequality (4.22) it is clear that for every fixed t one has the estimate

$$\|\boldsymbol{v}(t)\|_{m+3}^2 \leq C_{10}(\Omega,\alpha)^2(\|\boldsymbol{u}(t)\|_m^2 + \|\boldsymbol{w}(t)\|_{m+2}^2). \tag{5.7}$$

Furthermore, applying the identity (2.17) to equation $(\mathrm{I})_1$ with $\boldsymbol{u} = \mathbf{curl}\,(\boldsymbol{v} - \alpha\triangle\boldsymbol{v})$ gives

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbf{curl}\,(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) + \frac{\nu}{\alpha}(\mathbf{curl}\,(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) - \mathbf{curl}\,\boldsymbol{v}) + \mathbf{curl}\,\boldsymbol{g} \\
&= -\mathbf{curl}\,(\mathbf{curl}\,(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) \times \boldsymbol{v}) \quad \text{in } \Omega_T,
\end{aligned}$$

which implies that

$$\frac{\partial}{\partial t}(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) - \nu\triangle\boldsymbol{v} = \nabla p - \mathbf{curl}\,(\boldsymbol{v} - \alpha\triangle\boldsymbol{v}) \times \boldsymbol{v} + \mathbf{curl}\,\boldsymbol{g} \quad \text{in } \Omega_T$$

for some $p \in L^\infty(0,T;H^m(\Omega))$, since Ω is simply-connected. By proceeding as in Section 4.2, i.e. writing the above equation in the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = \nabla\cdot\boldsymbol{\mathcal{T}}(\boldsymbol{v},p)/\rho + \boldsymbol{g},$$

taking the $\boldsymbol{L}^2(\Omega)$-inner product with $\boldsymbol{v}$, applying a standard Green's formula, and noting that

$$(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v})_0 = (p\boldsymbol{I},\boldsymbol{A}_1)_0 = (\boldsymbol{v}\cdot\nabla\boldsymbol{A}_1,\boldsymbol{A}_1)_0 = (\boldsymbol{A}_1\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1,\boldsymbol{A}_1)_0 = 0,$$

one arrives at

$$\begin{aligned}
&\frac{d}{dt}(\|\boldsymbol{v}\|_0^2 + \frac{\alpha}{2}\|\boldsymbol{A}_1\|_0^2) + \nu\|\boldsymbol{A}_1\|_0^2 \\
&= 2(\boldsymbol{g},\boldsymbol{v})_0 + 2(S(|\boldsymbol{v}|),|\boldsymbol{v}|^2)_{0,\partial\Omega} + 2(\boldsymbol{d},\boldsymbol{v})_{0,\partial\Omega} \\
&\leq 2C_2(\Omega)^2\|S\|_{C^0}\|\boldsymbol{v}\|_1^2 + 2(\|\boldsymbol{g}\|_0 + C_2(\Omega)\|\boldsymbol{d}\|_{0,\partial\Omega})\|\boldsymbol{v}\|_1. \\
&\leq 2(C_2(\Omega)^2\|S\|_{C^0} + \varepsilon)\|\boldsymbol{v}\|_1^2 + \frac{1}{\varepsilon}(\|\boldsymbol{g}\|_0^2 + C_2(\Omega)^2\|\boldsymbol{d}\|_{0,\partial\Omega}^2), \quad \varepsilon > 0.
\end{aligned} \tag{5.8}$$

Assuming that Ω has no axes of symmetry, it follows from $(5.3)_2$ and Korn's inequality (4.13) that

$$\|\boldsymbol{A}_1\|_0^2 \geq \frac{C_P}{C_P+1}(\|\boldsymbol{A}_1\|_0^2 + \|\boldsymbol{v}\|_0^2) \geq \frac{\kappa C_P}{C_P+1}\|\boldsymbol{v}\|_1^2. \tag{5.9}$$

Setting $\varepsilon = \nu\kappa C_P/(4C_P+4)$, integrating over $[0,t]$, applying (5.9) and

$$\begin{aligned} &\min(1,\frac{\alpha}{2})\kappa\|\boldsymbol{v}(t)\|_1^2 \leq \|\boldsymbol{v}(t)\|_0^2 + \frac{\alpha}{2}\|\boldsymbol{A}_1(t)\|_0^2, \\ &\|\boldsymbol{v}_0\|_0^2 + \frac{\alpha}{2}\|\boldsymbol{A}_1(\boldsymbol{v}_0)\|_0^2 \leq \max(1,2\alpha)\|\boldsymbol{v}_0\|_1^2, \end{aligned} \tag{5.10}$$

and dividing by $\min(1,\alpha/2)\kappa$, one obtains

$$\begin{aligned} &\|\boldsymbol{v}\|_{1,T}^2 + \lambda_1\int_0^T \|\boldsymbol{v}(s)\|_1^2\,\mathrm{d}s \leq \lambda_2\|\boldsymbol{v}_0\|_1^2 + \frac{\lambda_3}{\nu}\int_0^T (\|\boldsymbol{g}(s)\|_0^2 + \|\boldsymbol{d}(s)\|_{0,\partial\Omega}^2)\,\mathrm{d}s, \\ &\lambda_1 = \max(1,\frac{2}{\alpha})(\frac{\nu C_P}{2C_P+2} - \frac{2}{\kappa}C_2(\Omega)^2\|S\|_{C^0}), \\ &\lambda_2 = \frac{1}{\kappa}\max(1,\frac{2}{\alpha})\max(1,2\alpha), \\ &\lambda_3 = \max(1,C_2(\Omega)^2)\max(1,\frac{2}{\alpha})\frac{4(C_P+1)}{\kappa^2 C_P}, \end{aligned} \tag{5.11}$$

if

$$\nu > \nu_0 \equiv \frac{4(C_P+1)}{\kappa C_P}C_2(\Omega)^2\|S\|_{C^0}. \tag{5.12}$$

[Alternatively, setting $f(t) = \|\boldsymbol{v}(t)\|_0^2 + (\alpha/2)\|\boldsymbol{A}_1(t)\|_0^2$ and applying

$$\|\boldsymbol{A}_1\|_0^2 \geq \frac{2C_P}{\alpha C_P+2}(\|\boldsymbol{v}\|_0^2 + \frac{\alpha}{2}\|\boldsymbol{A}_1\|_0^2)$$

and $(5.10)_1$ to inequality (5.8) with $\varepsilon = \min(1,\alpha/2)\nu\kappa C_P/(2\alpha C_P+4)$ leads to

$$\begin{aligned} &f'(t) \leq -\lambda_{1\star}f(t) + \max(1,C_2(\Omega)^2)\max(1,\frac{2}{\alpha})\frac{2\alpha C_P+4}{\nu\kappa C_P}(\|\boldsymbol{g}\|_0^2 + \|\boldsymbol{d}\|_{0,\partial\Omega}^2), \\ &\lambda_{1\star} = \frac{\nu C_P}{\alpha C_P+2} - \max(1,\frac{2}{\alpha})\frac{2}{\kappa}C_2(\Omega)^2\|S\|_{C^0}, \end{aligned}$$

and thus, via Gronwall's lemma and the inequalities (5.10),

$$\begin{aligned} \|\boldsymbol{v}(t)\|_1^2 &\leq \lambda_2 e^{-\lambda_{1\star}t}\|\boldsymbol{v}_0\|_1^2 + \frac{\lambda_{3\star}}{\nu\lambda_{1\star}}(1-e^{-\lambda_{1\star}t})(\|\boldsymbol{g}(t)\|_0^2 + \|\boldsymbol{d}(t)\|_{0,\partial\Omega}^2) \\ &\leq \lambda_2\|\boldsymbol{v}_0\|_1^2 + \frac{\lambda_{3\star}}{\nu\lambda_{1\star}}(\|\boldsymbol{g}(t)\|_0^2 + \|\boldsymbol{d}(t)\|_{0,\partial\Omega}^2) \quad \forall\, t \in [0,T], \end{aligned} \tag{5.13}$$

where $\lambda_{3\star} = \max(1, C_2(\Omega)^2)\max(1, \frac{2}{\alpha})^2 \frac{2\alpha C_P + 4}{\kappa^2 C_P}$, and

$$\begin{aligned}
&\int_0^T \|\boldsymbol{v}(s)\|_1^2 \,\mathrm{d}s \\
&\le \frac{\lambda_2}{\lambda_{1\star}}(1 - e^{\lambda_{1\star}T})\|\boldsymbol{v}_0\|_1^2 + \frac{\lambda_{3\star}}{\nu\lambda_{1\star}}\int_0^T (1 - e^{-\lambda_{1\star}s})(\|\boldsymbol{g}(s)\|_0^2 + \|\boldsymbol{d}(s)\|_{0,\partial\Omega}^2)\,\mathrm{d}s \\
&\le \frac{\lambda_2}{\lambda_{1\star}}\|\boldsymbol{v}_0\|_1^2 + \frac{\lambda_{3\star}}{\nu\lambda_{1\star}}\int_0^T (\|\boldsymbol{g}(s)\|_0^2 + \|\boldsymbol{d}(s)\|_{0,\partial\Omega}^2)\,\mathrm{d}s
\end{aligned} \tag{5.14}$$

for any given $T > 0$, if

$$\nu > \nu_{0\star} \equiv \max(1, \frac{2}{\alpha})\frac{2\alpha C_P + 4}{\kappa C_P} C_2(\Omega)^2 \|S\|_{C^0}. \;]$$

Note that the right hand sides of the estimates (5.11) and (5.13) – (5.14) are independent of T. The remainder of the proof is a direct generalization of the proof of Lemma 2.6 in [174], and is given in Section 5.3.

An alternative to Assumption 5.3 is

Assumption 5.5 *There is a constant $S_0 > 0$ such that*

$$S(x) \le -S_0 \quad \forall\, x \ge 0.$$

Instead of Lemma 5.4 one then has

Lemma 5.6 *Let Ω be a bounded, simply-connected domain of class C^{m+4}, $m \ge 1$, $\alpha > 0$ and $K, N \in C^{m+2}$, with S_0 as above. Then there is a constant*

$$\nu^\star = \nu^\star(\Omega, m, \alpha, S_0, \|\hat{K}\|_{C^{m+2}}, \|N\|_{C^{m+2}})$$

such that for each $\nu \ge \nu^\star$ there exist positive constants δ_0, δ_1, γ_0, γ_1, which depend only on Ω, m, ν, α, $\|\hat{K}\|_{C^{m+2}}$ and $\|N\|_{C^{m+2}}$, and have the property described in Lemma 5.4(a).

Proof. Inequality (5.7) remains unchanged, but inequality (5.8) becomes

$$\begin{aligned}
&\frac{d}{dt}(\|\boldsymbol{v}\|_0^2 + \frac{\alpha}{2}\|\boldsymbol{A}_1\|_0^2) + \nu\|\boldsymbol{A}_1\|_0^2 + 2S_0\|\boldsymbol{v}\|_{0,\partial\Omega}^2 \\
&\le 2(\boldsymbol{g}, \boldsymbol{v})_0 + 2(\boldsymbol{d}, \boldsymbol{v})_{0,\partial\Omega} \\
&\le \frac{C_P S_0}{2}\|\boldsymbol{v}\|_0^2 + \frac{S_0}{2}\|\boldsymbol{v}\|_{0,\partial\Omega}^2 + \frac{2}{C_P S_0}\|\boldsymbol{g}\|_0^2 + \frac{2}{S_0}\|\boldsymbol{d}\|_{0,\partial\Omega}^2.
\end{aligned} \tag{5.15}$$

Furthermore, by the Poincaré-Morrey inequality $(5.3)_1$ and Korn's inequality,

$$
\begin{aligned}
&(C_P + \frac{3}{2})\|\boldsymbol{A}_1(t)\|_0^2 + \frac{3}{2}\|\boldsymbol{v}(t)\|_{0,\partial\Omega}^2 \\
&\geq C_P\|\boldsymbol{A}_1(t)\|_0^2 + \frac{3C_P}{2}\|\boldsymbol{v}(t)\|_0^2 \qquad (5.16) \\
&\geq \kappa C_P\|\boldsymbol{v}(t)\|_1^2 + \frac{C_P}{2}\|\boldsymbol{v}(t)\|_0^2 \quad \forall\, t \in [0,T].
\end{aligned}
$$

Thus, integrating (5.15) over $[0,t]$, applying inequalities (5.7) and (5.16), taking the supremum over $[0,T]$ and dividing by $\min(1,\alpha/2)\kappa$, one gets

$$
\begin{aligned}
&\|\boldsymbol{v}\|_{1,T}^2 + \eta_1\int_0^T \|\boldsymbol{v}(s)\|_1^2\,\mathrm{d}s + \eta_2\int_0^T \|\boldsymbol{A}_1(s)\|_0^2\,\mathrm{d}s \\
&\leq \eta_3\|\boldsymbol{v}_0\|_1^2 + \eta_4\int_0^T (\|\boldsymbol{g}(s)\|_0^2 + \|\boldsymbol{d}(s)\|_{0,\partial\Omega})\,\mathrm{d}s,
\end{aligned} \qquad (5.17)
$$

where

$$
\begin{aligned}
\eta_1 &= C_P S_0 \max(1,\frac{2}{\alpha}), & \eta_2 &= (\nu-\nu_0)\frac{1}{\kappa}\max(1,\frac{2}{\alpha}) \\
\eta_3 &= \frac{1}{\kappa}\max(1,\frac{2}{\alpha})\max(1,2\alpha), & \eta_4 &= \frac{2}{\kappa S_0}\max(1,\frac{2}{\alpha})\max(1,\frac{1}{C_P}),
\end{aligned}
$$

if

$$
\nu \geq \nu_0 \equiv (C_P + \frac{3}{2})S_0. \qquad (5.18)
$$

[One could also set $f(t) \equiv \|\boldsymbol{v}(t)\|_0^2 + \frac{\alpha}{2}\|\boldsymbol{A}_1(t)\|_0^2$ and use

$$
(\frac{\alpha}{2}C_P + \frac{3}{2})\|\boldsymbol{A}_1(t)\|_0^2 + \frac{3}{2}\|\boldsymbol{v}(t)\|_{0,\partial\Omega}^2 \geq C_P f(t) + \frac{C_P}{2}\|\boldsymbol{v}(t)\|_0^2
$$

to obtain

$$
\begin{aligned}
&f'(t) + C_P S_0 f(t) + (\nu - \nu_{0\star})\|\boldsymbol{A}_1(t)\|_0^2 \\
&\leq \frac{2}{S_0}\max(1,\frac{1}{C_P})(\|\boldsymbol{g}(t)\|_0^2 + \|\boldsymbol{d}(t)\|_{0,\partial\Omega}^2) \qquad \forall\, t \in [0,T]
\end{aligned} \qquad (5.19)
$$

for

$$
\nu \geq \nu_{0\star} \equiv (\alpha C_P + 3)S_0/2. \qquad (5.20)
$$

It then follows (again from Gronwall's lemma and (5.7)) that

$$
\begin{aligned}
&\|\boldsymbol{v}(t)\|_1^2 + \frac{\eta_2}{C_P S_0}(1 - e^{-C_P S_0 t})\|\boldsymbol{A}_1(t)\|_0^2 \\
&\leq \eta_3 e^{-C_P S_0 t}\|\boldsymbol{v}_0\|_1^2 + \eta_4(1 - e^{-C_P S_0 t})(\|\boldsymbol{g}(t)\|_0^2 + \|\boldsymbol{d}(t)\|_{0,\partial\Omega}^2)
\end{aligned}
$$

for all $t \in [0,T]$, and (by integrating (5.19) over $[0,t]$, etc.) that

$$\|\boldsymbol{v}\|_{1,T}^2 + C_P S_0 \int_0^T \|\boldsymbol{v}(s)\|_1^2 \, \mathrm{d}s + \eta_2 \int_0^T \|\boldsymbol{A}_1(s)\|_0^2 \, \mathrm{d}s$$

$$\leq \eta_3 \|\boldsymbol{v}_0\|_1^2 + \eta_4 \int_0^T (\|\boldsymbol{g}(s)\|_0^2 + \|\boldsymbol{d}(s)\|_{0,\partial\Omega}^2) \, \mathrm{d}s. \;]$$

The lower bound (5.18) [or (5.20)] on ν was imposed in order to simplify the expressions $\eta_1, \ldots, \eta_4$, but is not really necessary here. Requiring only that $\nu > 0$, one can derive estimates in terms of $\min(\nu, 2S_0)$, as will be done in the next section (see the proof of Proposition 5.9(c)).

The proof now proceeds exactly as in Section 5.3 for Lemma 5.4, the only difference being that in the case $m = 0$ one must define

$$\delta_1 \equiv 1 + \frac{4\nu\eta_3 C_{10}^2}{\alpha\eta_1} = 1 + \frac{4\nu C_{10}^2 \max(2, 1/\alpha)}{\kappa C_P S_0}$$

$$\gamma_1 \equiv \frac{C_F}{\nu} + \frac{4\nu\eta_4}{\alpha\eta_1} = \frac{C_F}{\nu} + \frac{8\nu}{\alpha\kappa \min(1, C_P) C_P S_0^2}.$$

□

Now one can establish the existence of global classical solutions:

Theorem 5.7 *Let $m \geq 1$ and let Ω, α, K, N and $\nu^\star$ be as in Lemma 5.4 or Lemma 5.6. Then, for every $\nu > \nu^\star$, there are positive constants δ and γ, depending only on Ω, m, ν, α, $\|\hat{K}\|_{C^{m+2}}$ and $\|N\|_{C^{m+2}}$, such that if*

$$\begin{gathered} \boldsymbol{v}_0 \in \boldsymbol{X}_{m+3}, \quad \|\boldsymbol{v}_0\|_{m+3} < \delta, \\ \boldsymbol{g} \in L^\infty(0,\infty;\boldsymbol{X}_{m+1}) \cap L^2(0,\infty;\boldsymbol{X}_{m+1}), \\ \boldsymbol{d} \in L^\infty(0,\infty;\boldsymbol{Z}_{m+3/2}) \cap L^2(0,\infty;\boldsymbol{Z}_{m+3/2}), \\ \|\boldsymbol{g}\|_{L^2,m+1,\infty} + \|\boldsymbol{d}\|_{L^2,m+3/2,\infty,\partial\Omega} < \gamma, \end{gathered} \tag{5.21}$$

then the slip problem (4.6) has a unique solution $\boldsymbol{v}, \nabla p$ for all $t \in [0,\infty)$, satisfying the regularity conditions (4.7) for every $0 < T < \infty$.

Proof. With δ_0, δ_1, γ_0 and γ_1 as in Lemma 5.4 (Lemma 5.6, resp.) and $C_\star$ as in Theorem 4.4, assume that

$$\begin{gathered} \|\boldsymbol{v}_0\|_{m+3} < \delta \equiv C_\star \min(1, \frac{1}{2\sqrt{\delta_1}}) \frac{\delta_0}{2}. \\ \|\boldsymbol{g}\|_{L^2,m+1,\infty} + \|\boldsymbol{d}\|_{L^2,m+3/2,\infty,\partial\Omega} < \gamma \equiv \min(\gamma_0, \frac{\delta_0}{\sqrt{8\gamma_1}}). \end{gathered} \tag{5.22}$$

Then, defining $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ as before, it follows from the definition of $C_\star$ (see the first paragraph in the proof of Theorem 4.4) that

$$\max(\|\boldsymbol{u}_0\|_m, \|\boldsymbol{w}_0\|_{m+2}) \leq \frac{1}{C_\star}\|\boldsymbol{v}_0\|_{m+3} < \min(1, \frac{1}{\sqrt{2\delta_1}})\frac{\delta_0}{2}. \tag{5.23}$$

Thus, by Theorem 4.3 (with $D_\star = \delta_0/2$ and, for example, $D = \delta_0$) there exists a unique solution $\boldsymbol{v}^1$ on $[0, T]$. Moreover, from the inequalities (5.5) (of Lemma 5.4 or 5.6), (5.23) and $(5.22)_2$ one has

$$\begin{aligned}
&\max(\|\boldsymbol{u}^1(T)\|_m, \|\boldsymbol{w}^1(T)\|_{m+2})^2 \\
&\leq 2\delta_1 \max(\|\boldsymbol{u}_0\|_m, \|\boldsymbol{w}_0\|_{m+2})^2 + \gamma_1(\|\boldsymbol{g}\|^2_{L^2,m+1,\infty} + \|\boldsymbol{d}\|^2_{L^2,m+3/2,\infty,\partial\Omega}) \\
&< \frac{\delta_0^2}{8} + \frac{\delta_0^2}{8} = (\frac{\delta_0}{2})^2.
\end{aligned}$$

Hence one may again apply Theorem 4.3 and Lemma 5.4 (Lemma 5.6, resp.) to deduce the existence of a unique solution $\boldsymbol{u}^2, \boldsymbol{v}^2, \boldsymbol{w}^2$ on $[T, 2T]$ with $\boldsymbol{v}^2(T) = \boldsymbol{v}^1(T)$, etc. From Propositions 3.7 and 3.11 and (4.7) it is clear that the resulting vector functions $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ on $[0, 2T]$ satisfy the conditions of Lemma 5.4 (Lemma 5.6, resp.) with $2T$ instead of T, and so it follows as above from (5.5) that

$$\max(\|\boldsymbol{u}(2T)\|_m, \|\boldsymbol{w}(2T)\|_{m+2}) < \frac{\delta_0}{2}.$$

By repeating this procedure one obtains a solution on $[0, \infty)$ with the stated regularity properties, the uniqueness being ensured by Lemma 4.7. □

5.2 Stability of the Rest State

The quantity

$$E(t) \equiv \|\boldsymbol{v}(t)\|_0^2 + \frac{\alpha}{2}\|\boldsymbol{A}_1(t)\|_0^2,$$

which was encountered in the previous section, can be interpreted as the sum of the averaged kinetic and stretching energy in the fluid at time t. For flows that satisfy the no-slip condition on a portion of the boundary, [119, pp. 221 – 222] showed that $E(t)$ cannot decay to zero in a finite time. Not surprisingly, this is also the case here:

Proposition 5.8 *Let Ω be a bounded domain, $\nu \geq 0$, $\alpha > 0$ and suppose that*

$$\underline{S} \equiv \inf_{x \geq 0} S(x) > -\infty.$$

Then there is a constant $C = C(\Omega)$ such that any global solution $\boldsymbol{v}$ of the slip problem (4.9) *under a conservative body force (i.e. $\boldsymbol{g} \equiv \boldsymbol{0}$, $\boldsymbol{d} \equiv \boldsymbol{0}$) satisfies*

$$\max(1, 2\alpha)\|\boldsymbol{v}(t)\|_1^2 \geq E(t) \geq E(0)^{-\lambda t} \geq \min(1, \frac{\alpha}{2})\kappa\|\boldsymbol{v}_0\|_1^2 e^{-\lambda t} \tag{5.24}$$

for all $t \geq 0$, with

$$\lambda = \max(\frac{2}{\alpha}(\nu + C(\Omega)|\min(0, \underline{S})|),\ C(\Omega)|\min(0, \underline{S})|\,) \geq 0. \tag{5.25}$$

Moreover, if Ω is not rotationally symmetric, or if there is a fixed subsurface $\Sigma \subset \partial\Omega$ of nonzero measure such that $\boldsymbol{v} = \boldsymbol{0}$ on Σ for all $t \geq 0$, then

$$\|\boldsymbol{A}_1(t)\|_0 \geq C(\Omega, \alpha)\|\boldsymbol{v}_0\|_1^{-\lambda t/2} \quad \forall\, t \geq 0. \tag{5.26}$$

Proof. Since $S(|\boldsymbol{v}|) \geq \underline{S} \geq \min(0, \underline{S}) = -|\min(0, \underline{S})|$, it follows from equation $(5.8)_1$, the trace theorem and Korn's inequality that

$$\begin{aligned} E'(t) &\geq -\nu\|\boldsymbol{A}_1(t)\|_0^2 - |\min(0, \underline{S})|C_2(\Omega)^2\|\boldsymbol{v}(t)\|_1^2 \\ &\geq -(\nu + C(\Omega)|\min(0, \underline{S})|)\|\boldsymbol{A}_1(t)\|_0^2 - C(\Omega)|\min(0, \underline{S})|.\|\boldsymbol{v}(t)\|_0^2 \\ &\geq -\lambda E(t) \qquad \forall\, t \geq 0, \end{aligned}$$

where $C = C_2(\Omega)^2/\kappa$ and λ is as in (5.25). This yields (5.24) upon integration, and (5.26) is then immediate from (5.9), which follows from the Poincaré-Morrey inequality $(5.3)_2$ or $(5.3)_3$.

Proposition 5.9 *Let Ω be a bounded domain, $\nu \geq 0$, $\alpha > 0$, $\boldsymbol{g} \equiv \mathbf{0}$, $\boldsymbol{d} \equiv \mathbf{0}$, and assume that*

$$\overline{S} \equiv \sup_{x \geq 0} S(x) < \infty.$$

Then, for any global solution $\boldsymbol{v}$ of the slip problem (4.9), with any initial value $\boldsymbol{v}_0$, one has:

(a) *If $\overline{S} \leq 0$, then*

$$E'(t) \leq 0 \quad \forall\, t \geq 0, \tag{5.27}$$

i.e. the null solution is (monotonically) stable in the Lyapounov sense: For any $t_0 \geq 0$ and $\varepsilon > 0$, if

$$\|\boldsymbol{v}(t_0)\|_1^2 < \kappa \min(1, \frac{\alpha}{2}) \min(1, \frac{1}{2\alpha}) \varepsilon^2,$$

then $\|\boldsymbol{v}(t)\|_1 \leq \varepsilon$ for all $t \geq t_0$.

(b) *If Ω has no axes of symmetry, or if there is a subsurface $\Sigma \subset \partial\Omega$ of nonzero measure such that $\boldsymbol{v} = \mathbf{0}$ on Σ for all $t \geq 0$, then there exist positive constants C, C_λ, C_ν, depending only on Ω and α, such that*

$$\min(1, \frac{\alpha}{2}) \kappa \|\boldsymbol{v}(t)\|_1^2 \leq E(t) \leq E(0) e^{-\lambda t} \leq C \|\boldsymbol{A}_1(\boldsymbol{v}_0)\|_0^2 e^{-\lambda t} \tag{5.28}$$

for all $t \geq 0$ and

$$\lambda = -C_\lambda(\nu - C_\nu \overline{S}),$$

so that the null solution is exponentially stable if $\nu > C_\nu \overline{S}$.

(c) *If $\overline{S} = -S_0 < 0$ and $\nu > 0$, then*

$$\min(1, \frac{\alpha}{2}) \kappa \|\boldsymbol{v}(t)\|_1^2 \leq E(t) \leq E(0) e^{-\lambda t} \leq \max(1, 2\alpha) \|\boldsymbol{v}_0\|_1^2 e^{-\lambda t} \tag{5.29}$$

for all $t \geq 0$ and

$$\lambda = \frac{2\nu C_P \min(\nu, 2S_0)}{2\nu + \alpha C_P \min(\nu, 2S_0)} > 0,$$

i.e. the null solution is exponentially stable.

Proof. From equation (5.8),

$$E'(t) + \nu \|\boldsymbol{A}_1(t)\|_0^2 = 2(S(|\boldsymbol{v}|), |\boldsymbol{v}|^2)_{0,\partial\Omega} \leq 2\overline{S} \|\boldsymbol{v}(t)\|_{0,\partial\Omega}^2, \tag{5.30}$$

which establishes (a) in view of (5.10).

Inequality (5.30) also shows that one may substitute $\max(0, \overline{S})$ for $\|S\|_{C^0}$ in the derivation of inequality $(5.13)_1$ (page 88), so that (b) holds with

$$C_\lambda = \frac{C_P}{\alpha C_P + 2}, \quad C_\nu = \max(1, \frac{2}{\alpha})\frac{2\alpha C_P + 4}{\kappa C_P} C_2(\Omega)^2.$$

To verify (c), set

$$\varepsilon = \frac{2\nu}{2\nu + \alpha C_P \min(\nu, 2S_0)},$$

then inequalities (5.30) and $(5.3)_1$ give

$$\begin{aligned}
&-E'(t)\\
&\geq \nu\|\boldsymbol{A}_1(t)\|_0^2 + 2S_0\|\boldsymbol{v}(t)\|_{0,\partial\Omega}^2\\
&\geq (\nu - \varepsilon\min(\nu, 2S_0))\|\boldsymbol{A}_1(t)\|_0^2 + \varepsilon\min(\nu, 2S_0)(\|\boldsymbol{A}_1(t)\|_0^2 + \|\boldsymbol{v}(t)\|_{0,\partial\Omega}^2)\\
&\geq (1-\varepsilon)\nu\|\boldsymbol{A}_1(t)\|_0^2 + \varepsilon C_P \min(\nu, 2S_0)\|\boldsymbol{v}(t)\|_0^2\\
&= \varepsilon C_P \min(\nu, 2S_0) E(t) \qquad \forall\, t \geq 0.
\end{aligned}$$

Remark 5.10 (a) The assumption in Proposition 5.9(c) – that S is bounded from above by a negative number – excludes the important (for free surface flows) case of perfect slip, but is not unreasonable in the light of the restrictions imposed by the existence proofs of [115] and [116] (for simpler fluids). With the notation as on page 11, [115] assumes that

$$S(v, \boldsymbol{y}), \frac{\partial S}{\partial v}(v, \boldsymbol{y}) \in C^0([0,\infty) \times \partial\Omega),$$

and

$$\left.\begin{aligned}
-S_1 \leq S \leq -S_0\\
S + v\frac{\partial S}{\partial v} \leq -S_2\\
\frac{\partial S}{\partial v} \geq 0
\end{aligned}\right\} \quad \forall\, (v, \boldsymbol{y}) \in [0,\infty) \times \partial\Omega, \tag{5.31}$$

where S_0, S_1, S_2 are positive constants, while [116] requires that

$$S(v, \theta), \frac{\partial S}{\partial v}(v, \theta) \in C^0([0,\infty) \times \boldsymbol{R}),$$

in addition to (5.31) with $(5.31)_3$ replaced by

$$v\left|\frac{\partial S}{\partial v}\right| \leq S_3 \quad \forall\, (v, \boldsymbol{y}) \in [0,\infty) \times \boldsymbol{R},$$

for some constant $S_3 > 0$. (In both cases, identical regularity and growth conditions (with M in place of $-S$) are also imposed on the viscosity function M.)

(b) The stability result of [127] for flows with perfect slip in unbounded domains assumes that the boundary is *flat*, in which case the condition

$$(\mathcal{T}\boldsymbol{n})_\tau = \mathbf{0} \quad \text{on } \partial\Omega$$

becomes equivalent to

$$(\boldsymbol{A}_1\boldsymbol{n})_\tau = \mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \partial\Omega.$$

For incompressible second grade fluids that satisfy the no-slip condition this holds on smooth boundaries of arbitrary shape, as can be seen from the formula derived in [181]:

$$\mathcal{T}\boldsymbol{n} = (-p + (2\alpha_1 + \alpha_2)|\mathbf{curl}\,\boldsymbol{v}|^2)\boldsymbol{n} + (\mu\mathbf{curl}\,\boldsymbol{v} + \alpha_1\frac{\partial}{\partial t}\mathbf{curl}\,\boldsymbol{v}) \times \boldsymbol{n}.$$

(c) As a final comment, I point out that the arguments (involving eigenvalues) employed in [119, section 9] and [121], section 7, to prove the stability of arbitrary base flows rely on the no-slip condition to a degree that seems impossible to circumvent in any obvious manner.

5.3 Appendix

Proof of Lemma 5.4 (contd.).
The case $m = 0$.
Taking the $\boldsymbol{L}^2(\Omega)$-scalar product of equation $(\mathrm{I})_1$ with $\boldsymbol{u}$ gives

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_0^2 + \frac{\nu}{\alpha}\|\boldsymbol{u}\|_0^2 = \frac{\nu}{\alpha}(\mathbf{curl}\,\boldsymbol{v}, \boldsymbol{u})_0 + (\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{u})_0(\mathbf{curl}\,\boldsymbol{g}, \boldsymbol{u})_0$$
$$\leq \sqrt{2}(\frac{\nu}{\alpha}\|\boldsymbol{v}\|_1 + \|\boldsymbol{g}\|_1)\|\boldsymbol{u}\|_0 + \|\boldsymbol{v}\cdot\nabla\boldsymbol{u}\|_0\|\boldsymbol{u}\|_0$$

as $(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})_0 = 0$. Hence, after using inequality (3.15) (with $s = 3$) and the Cauchy inequality as in $(3.55)_1$, one obtains

$$\frac{d}{dt}\|\boldsymbol{u}\|_0^2 + \frac{\nu}{\alpha}\|\boldsymbol{u}\|_0^2 \leq 2C_5\|\boldsymbol{v}\|_3\|\boldsymbol{u}\|_0^2 + \frac{4\nu}{\alpha}\|\boldsymbol{v}\|_1^2 + \frac{4\alpha}{\nu}\|\boldsymbol{g}\|_1^2. \tag{5.32}$$

Similarly, taking the scalar product of equation $(\mathrm{II})_1$ (see page 46) and $\boldsymbol{w}$ in $\boldsymbol{H}^2(\Omega)$ and applying inequality (3.14) gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_2^2 + \frac{\nu}{\alpha}\|\boldsymbol{w}\|_2^2 &= -(\boldsymbol{v}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_2 + (\check{\boldsymbol{f}}, \boldsymbol{w})_2 \\ \leq C_4(\Omega)\|\boldsymbol{v}\|_3\|\boldsymbol{w}\|_2^2 &+ (\frac{1}{\alpha}\|\check{\boldsymbol{s}}\|_2 + \|\check{\boldsymbol{b}}\|_2 + \|\check{\boldsymbol{c}}\|_2 + \frac{1}{\alpha}\|\check{\boldsymbol{d}}\|_2)\|\boldsymbol{w}\|_2,\end{aligned} \tag{5.33}$$

while the *a priori* estimate for the Stokes problem, the steps leading to (3.24), (3.26) and (3.28), and inequality $(3.34)_1$ imply that

$$\begin{aligned}&\|\check{\boldsymbol{s}}\|_2 \leq C(\Omega)(\|\hat{K}\|_{C^2} + \|N\|_{C^2}[\,|\Omega|^{1/2} + \|\boldsymbol{v}\|_2^2 + \|\boldsymbol{v}\|_2^4\,])\|\boldsymbol{v}\|_2,\\ &\|\check{\boldsymbol{b}}\|_2 \leq C(\Omega)\|\boldsymbol{v}\|_3(\|\boldsymbol{v}\|_2 + \|\boldsymbol{w}\|_2),\\ &\|\check{\boldsymbol{c}}\|_2 \leq C(\Omega)\|\boldsymbol{v}\|_3^2,\\ &\|\check{\boldsymbol{d}}\|_2 \leq C(\Omega)\|\boldsymbol{d}\|_{3/2,\partial\Omega}.\end{aligned} \tag{5.34}$$

From $(5.34)_4$ and Cauchy's inequality one also gets

$$\|\check{\boldsymbol{d}}\|_2\|\boldsymbol{w}\|_2 \leq \frac{\nu}{2}\|\boldsymbol{u}\|_2^2 + \frac{C(\Omega)}{2\nu}\|\boldsymbol{d}\|_{3/2,\partial\Omega}^2. \tag{5.35}$$

Thus, by adding (5.32) and (5.33) and using (5.34) – (5.35), one arrives at

$$\begin{aligned}&\frac{d}{dt}(\|\boldsymbol{u}\|_0^2 + \|\boldsymbol{w}\|_2^2) + \frac{\nu}{\alpha}(\|\boldsymbol{u}\|_0^2 + \|\boldsymbol{w}\|_2^2)\\ &\leq C(\Omega)\{\frac{1}{\alpha}(\|\hat{K}\|_{C^2} + \|N\|_{C^2}[\,|\Omega|^{1/2} + \|\boldsymbol{v}\|_3^2 + \|\boldsymbol{v}\|_3^4\,])\|\boldsymbol{v}\|_3\\ &\quad + \|\boldsymbol{v}\|_3^2\}\|\boldsymbol{w}\|_2 + C(\Omega)\|\boldsymbol{v}\|_3(\|\boldsymbol{u}\|_0^2 + \|\boldsymbol{w}\|_2^2)\\ &\quad + \frac{4\nu}{\alpha}\|\boldsymbol{v}\|_1^2 + \frac{4\alpha}{\nu}\|\boldsymbol{g}\|_1^2 + \frac{C(\Omega)}{\nu\alpha}\|\boldsymbol{d}\|_{3/2,\partial\Omega}^2.\end{aligned} \tag{5.36}$$

With $y(t) \equiv \|\boldsymbol{u}(t)\|_0^2 + \|\boldsymbol{w}(t)\|_2^2$, it follows from (5.7) that $\|\boldsymbol{v}\|_3 \leq C_{10}\sqrt{y}$, and hence from (5.36) that

$$y'(t) + [\frac{\nu}{\alpha} - G(y(t))]y(t) \leq F(t) \quad \forall\, t \in [0,T], \tag{5.37}$$

where $G(x) \equiv C_G(\|\hat{K}\|_{C^2} + \|N\|_{C^2}(1+x+x^2)+\sqrt{x}\,), x \geq 0$, for some constant $C_G = C_G(\Omega, \alpha)$ and

$$F(t) \equiv \frac{4\nu}{\alpha}\|\boldsymbol{v}(t)\|_1^2 + \frac{C_F}{\nu}(\|\boldsymbol{g}(t)\|_1^2 + \|\boldsymbol{d}\|_{3/2,\partial\Omega}^2)$$

with $C_F = \max(4\alpha, C(\Omega)/\alpha)$. Assume that

$$\nu > \nu^\star \equiv \max(\nu_0,\, 2\alpha C_G(\Omega,\alpha)(\|\hat{K}\|_{C^2} + \|N\|_{C^2})). \tag{5.38}$$

Then, since G is continuous and $G(0) = C_G(\|\hat{K}\|_{C^2} + \|N\|_{C^2})$, there exists an $\varepsilon = \varepsilon(\Omega, \nu, \alpha, \|\hat{K}\|_{C^2}, \|N\|_{C^2}) > 0$ such that $G(x) \leq \nu/(2\alpha)$ for all $x \in [0,\varepsilon]$. Moreover, by (5.11) and (5.7) (for $m = 0$, $t = 0$)

$$y(0) + \int_0^T F(t)\,\mathrm{d}t \leq \delta_1 y(0) + \gamma_1 \int_0^T (\|\boldsymbol{g}(t)\|_1^2 + \|\boldsymbol{d}(t)\|_{3/2,\partial\Omega}^2)\,\mathrm{d}t$$

with

$$\delta_1 \equiv 1 + \frac{4\nu\lambda_2 C_{10}^2}{\alpha\lambda_1} = 1 + \frac{8\nu C_{10}^2(C_P+1)\max(2,1/\alpha)}{\nu C_P\kappa - 4C_2^2(C_P+1)\|S\|_{C^0}},$$

$$\gamma_1 = \frac{C_F}{\nu} + \frac{4\lambda_3}{\alpha\lambda_1} = \frac{C_F}{\nu} + \frac{32(C_P+1)^2\max(1,C_2^2)}{\alpha\kappa C_P[\nu C_P\kappa - 4C_2^2(C_P+1)\|S\|_{C^0}]}.$$

Hence, if

$$\begin{aligned} \|\boldsymbol{u}_0\|_0 + \|\boldsymbol{w}_0\|_2 &< \delta_0 \equiv \sqrt{\frac{\varepsilon}{2\delta_1}}, \\ \|\boldsymbol{g}\|_{L^2,1,T} + \|\boldsymbol{d}\|_{L^2,3/2,T,\partial\Omega} &< \gamma_0 \equiv \sqrt{\frac{\varepsilon}{2\gamma_1}}, \end{aligned} \tag{5.39}$$

then according to Lemma 5.1 (with $k_1 = 2k_2 = \nu/\alpha$),

$$\begin{aligned} &\|\boldsymbol{u}\|_{0,T}^2 + \|\boldsymbol{w}\|_{2,T}^2 + \frac{\nu}{2\alpha}\int_0^T (\,\|\boldsymbol{u}(s)\|_0^2 + \|\boldsymbol{w}(s)\|_2^2\,)\,\mathrm{d}s \\ &\leq \delta_1(\,\|\boldsymbol{u}_0\|_0^2 + \|\boldsymbol{w}_0\|_2^2\,) + \gamma_1(\,\|\boldsymbol{g}\|_{L^2,1,T}^2 + \|\boldsymbol{d}\|_{L^2,3/2,T,\partial\Omega}^2\,). \end{aligned} \tag{5.40}$$

[Note that the bounds δ_0, γ_0 on the magnitude of the data can be made arbitrarily large by taking $\nu^\star$ sufficiently large; when $\nu \longrightarrow \infty$, then $\varepsilon \longrightarrow \infty$ and $\delta_1 \longrightarrow \overline{\delta}_1$, say, $\gamma_1 \longrightarrow 0$, so that $\delta_0 \longrightarrow \infty$, $\gamma_0 \longrightarrow \infty$.

This also shows that δ_0, δ_1, γ_0 and γ_1 can be chosen independent of ν: Choose a $\nu^\star > \max(\nu_0, 2\alpha G(0))$ such that $\delta_1(\nu) \leq \overline{\delta}_1 + 1$ and $\gamma_1(\nu) \leq 1$ for all $\nu \geq \nu^\star$. Then there is an $\varepsilon > 0$, depending only on $\nu^\star$ and $G(\cdot)$, such that $G(x) \leq \nu^\star/(2\alpha)$ for all $x \in [0, \varepsilon]$. Hence it suffices to replace δ_1 and γ_1 by $\overline{\delta}_1 + 1$ and 1.]

The case $m = 1$.
Taking $m + q = 1$ in (3.50) – (3.55), one gets

$$\frac{d}{dt}\|\boldsymbol{u}\|_1^2 + \frac{\nu}{\alpha}\|\boldsymbol{u}\|_1^2 \leq 2(C_4(\Omega) + C_5(\Omega))\|\boldsymbol{v}\|_3\|\boldsymbol{u}\|_1^2 + \frac{4\nu}{\alpha}\|\boldsymbol{v}\|_2^2 + \frac{4\alpha}{\nu}\|\boldsymbol{g}\|_2^2. \tag{5.41}$$

Moreover, for any $m \geq 1$, arguing as in (5.33) – (5.34) yields

$$\begin{aligned}
&\frac{d}{dt}\|\boldsymbol{w}\|_{m+2}^2 + \frac{\nu}{\alpha}\|\boldsymbol{w}\|_{m+2}^2 \\
&\leq C(\Omega)\{\frac{1}{\alpha}(\|\hat{K}\|_{C^2} + \|N\|_{C^{m+2}}[\,|\Omega|^{1/2} + \|\boldsymbol{v}\|_{m+2}^2 + \|\boldsymbol{v}\|_{m+2}^{2m+4}\,])\|\boldsymbol{v}\|_{m+2} \\
&\quad + \|\boldsymbol{v}\|_{m+2}(\|\boldsymbol{v}\|_{m+3} + \|\boldsymbol{w}\|_{m+2}) + \|\boldsymbol{v}\|_{m+3}^2\}\|\boldsymbol{w}\|_{m+2} \\
&\quad + 2C_4(\Omega)\|\boldsymbol{v}\|_{m+2}\|\boldsymbol{w}\|_{m+2}^2 + \frac{C(\Omega)}{\nu\alpha}\|\boldsymbol{d}\|_{m+3/2,\partial\Omega}^2.
\end{aligned} \tag{5.42}$$

Hence, setting $y_1(t) \equiv \|\boldsymbol{u}(t)\|_1^2 + \|\boldsymbol{w}(t)\|_3^2$, adding (5.41) to (5.42) with $m = 1$ and using the estimates $\|\boldsymbol{w}\|_3 \leq \sqrt{y_1}$, $\|\boldsymbol{v}\|_4 \leq C_{10}\sqrt{y_1}$, one obtains

$$\begin{aligned}
&y_1'(t) + [\frac{\nu}{\alpha} - G_1(y_1(t))]y_1(t) \leq F_1(t) \qquad \forall\, t \in [0, T], \\
&F_1(t) \equiv \frac{4\nu}{\alpha}\|\boldsymbol{v}(t)\|_2^2 + \frac{1}{\nu}C_{F_1}(\Omega, \alpha)(\|\boldsymbol{g}(t)\|_2^2 + \|\boldsymbol{d}(t)\|_{5/2,\partial\Omega}^2),
\end{aligned} \tag{5.43}$$

where G_1 is a function of the same form as G. If conditions (5.38) and (5.39) are satisfied, then by (5.7) and (5.40),

$$\frac{4\nu}{\alpha}\int_0^T \|\boldsymbol{v}(t)\|_2^2\,\mathrm{d}t \leq 8C_{10}^2[\,\delta_1 y(0) + \gamma_1(\|\boldsymbol{g}\|_{L^2,1,T}^2 + \|\boldsymbol{d}\|_{L^2,3/2,T,\partial\Omega}^2)\,].$$

(One could apply Cauchy's inequality to the term

$$\frac{1}{\alpha}C(\Omega)(\|\hat{K}\|_{C^2} + \|N\|_{C^3}|\Omega|^{1/2})\|\boldsymbol{v}\|_3\|\boldsymbol{w}\|_3$$

in (5.42) and take the resulting $\|\boldsymbol{v}\|_3^2$-term to the right hand side of (5.43), but there is seemingly not much to gain from this.)

Therefore, in the same way as before, it follows from Lemma 5.1 that there exist positive constants ν_1^*, $\delta_{0,1}$, $\gamma_{0,1}$, $\delta_{1,1}$ and $\gamma_{1,1}$ such that if $\nu > \nu_1^*$ and

$$\|\boldsymbol{u}_0\|_1 + \|\boldsymbol{w}_0\|_3 < \delta_{0,1}, \quad \|\boldsymbol{g}\|_{L^2,2,T} + \|\boldsymbol{d}\|_{L^2,5/2,T,\partial\Omega} < \gamma_{0,1},$$

then

$$\begin{aligned} &\|\boldsymbol{u}\|_{1,T}^2 + \|\boldsymbol{w}\|_{3,T}^2 + \frac{\nu}{2\alpha}\int_0^T (\,\|\boldsymbol{u}(s)\|_1^2 + \|\boldsymbol{w}(s)\|_3^2\,)\,\mathrm{d}s \\ &\leq \delta_{1,1}(\,\|\boldsymbol{u}_0\|_0^2 + \|\boldsymbol{w}_0\|_3^2\,) + \gamma_{1,1}(\,\|\boldsymbol{g}\|_{L^2,2,T}^2 + \|\boldsymbol{d}\|_{L^2,5/2,T,\partial\Omega}^2\,). \end{aligned} \tag{5.44}$$

The general case $m \geq 2$.
Let $k \geq 2$ and assume that $(5.5)_1$ holds for $m = k-1$. From (3.50) – (3.55) one has

$$\begin{aligned} \frac{d}{dt}\|\boldsymbol{u}\|_k^2 + \frac{\nu}{\alpha}\|\boldsymbol{u}\|_k^2 &\leq 2(C_4(\Omega) + C_5(\Omega))\|\boldsymbol{v}\|_{k+1}\|\boldsymbol{u}\|_k^2 \\ &\quad + \frac{4\nu}{\alpha}\|\boldsymbol{v}\|_{k+1}^2 + \frac{4\alpha}{\nu}\|\boldsymbol{g}\|_{k+1}^2. \end{aligned} \tag{5.45}$$

Hence, by adding (5.45) and (5.42) (with $m = k$) and using the estimate (5.7), one again obtains a differential inequality of the type (5.1) in

$$y_k(t) \equiv \|\boldsymbol{u}(t)\|_k^2 + \|\boldsymbol{w}(t)\|_{k+2}^2,$$

with

$$F_k(t) \equiv \frac{4\nu}{\alpha}\|\boldsymbol{v}(t)\|_{k+1}^2 + \frac{1}{\nu}C_{F_k}(\Omega,\alpha,k)(\,\|\boldsymbol{g}(t)\|_{k+1}^2 + \|\boldsymbol{d}(t)\|_{k+3/2,\partial\Omega}^2\,).$$

As above, from (5.7) and the induction hypothesis one has

$$\begin{aligned} &\frac{4\nu}{\alpha}\int_0^T \|\boldsymbol{v}(t)\|_{k+1}^2\,\mathrm{d}t \\ &\leq 8C_{10}(\Omega,\alpha)^2[\,\delta_{1,k-1}y_{k-1}(0) + \gamma_{1,k-1}(\|\boldsymbol{g}\|_{L^2,k,T}^2 + \|\boldsymbol{d}\|_{L^2,k+1/2,T,\partial\Omega})\,], \end{aligned}$$

and therefore, by the same reasoning as before, it follows that inequality $(5.5)_1$ holds for $m = k$, and thus for all $m \geq 2$.

(b) By adding inequalities (3.57) (with $q = 0$) and (3.84) (with $E_1, M, \boldsymbol{\eta}$ replaced by $\|\boldsymbol{w}\|_{m+2,T}, \|\boldsymbol{v}\|_{m+3,T}, \boldsymbol{w}$, respectively) and using (5.7) one obtains

$$\begin{aligned} \|\frac{d\boldsymbol{u}}{dt}\|_{m-1,T} + \|\frac{d\boldsymbol{w}}{dt}\|_{m+1,T} &\leq \sqrt{2}\|\boldsymbol{g}\|_{m,T} + \frac{C(\Omega,m)}{\alpha}\|\boldsymbol{d}\|_{m+1/2,T,\partial\Omega} \\ &\quad + C(\Omega,\nu,\alpha,\|\hat{K}\|_{C^2},\|N\|_{C^{m+1}},\|\boldsymbol{u}\|_{m,T},\|\boldsymbol{w}\|_{m+2,T}), \end{aligned}$$

and therefore (5.6) by virtue of (5.5).

5.4 List of Constants

Constant	**Equation**	**Page**
C_1	(3.1)	33
C_2	(3.2)	34
C_3	(3.11)	37
C_4	(3.14)	39
C_5	(3.15)	39
C_6	(3.18)	40
C_7	(3.36)	46
$D_0,\ D_1$	(3.56)	55
D_2	(3.57)	55
C_8	(3.69)	60
$C_9,\ F$	(3.70)	61
$E_0,\ E_1$	(3.71)	61
E_2	(3.84)	64
$D,\ D_\star$		67
$T^\star,\ F_0,\ F_1,\ F$	(4.1)	68
$C_\star$	(4.5)	69
M		70
κ	(4.13)	73
C_{10}	(4.22)	76
C_P	(5.3)	86

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