Left-invariant sub-Riemannian structures on a five-dimensional two-step nilpotent Lie group: isometries, classification, and geodesics

by

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Abstract

In this dissertation we study a five-dimensional two-step nilpotent matrix Lie group. Some basic group properties are investigated. The structure of the Lie algebra's subspaces is investigated; a complete set of scalar invariants is given for the Lie algebra's subspace structure. Following this, we classify the left-invariant sub-Riemannian structures on this Lie group up to isometry. The normal geodesics of the rank three left-invariant sub-Riemannian structure are determined as an illustrative case.

Declaration

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own independent work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Chapter 1

Introduction

Around 1870, Sophus Lie was inspired by Galois theory to develop an analogous theory of differential equations and their "symmetries", which generally form continuous groups[20]. The great idea of Sophus Lie was to look at elements "infinitesimally close to the identity" in a Lie group, and to use them to infer the behaviour of ordinary elements. The modern version of Lie's idea is to infer properties of the Lie group from properties of its tangent space[20]. In principal, Lie's theory reduces problems on Lie groups, of an analytical nature, to algebraic problems on Lie algebras[5].

A sub-Riemannian structure on a smooth manifold consists of a bracket generating distribution with a Riemannian metric on this distribution. Sub-Riemanninan structures often occur in the study of constrained systems in mechanics, such as the ball and plate problem where the motion of a ball that rolls without slipping is considered. The smooth manifold often represents the configuration space of the system, with the restriction — the non-slip condition in the ball and plate case — modelled by the distribution[9][12]. Smooth motion from one state to the next may be realized as a curve on the manifold. Curves that in addition respect the restrictions on the system are termed admissible curves. Additionally, the distribution's Riemannian metric allows us to assign length to these curves.

The Chow-Raschevskii Theorem (see, e.g.[6]) states that the bracket generating condition suffices to guarantee the existence of an admissible curve between any two points q_0 and q_1 of a sub-Riemannian manifold M. Admissible curves need not be unique. Given admissible curves joining q_0 and q_1 the question of the existence of length minimizers naturally arises—the geodesic problem. From a geometric control point of view, necessary conditions to characterizing length minimizes are given by Pontryagin's Maximum Principle [6].

A Lie group is an abstract group that is also a smooth manifold. The group and smooth manifold structures are compatible in the sense that the group operation and group inverse map are smooth maps. The group structure endows the symmetry of a group onto the manifold. Given a sub-Riemannian structure on a Lie group, we may require that the distribution and

Riemannian metric also be compatible with the group's symmetry. In this case we have a left-invariant sub-Riemannian structure on the Lie group.

Given a mathematical structure such as this, a standard problem is to classify all such structures up to some equivalence. The classification of left-invariant sub-Riemannian structures has taken two approaches. The first is to conduct a systematic study on low-dimensional Lie groups while the second is to classify the structures of infinite families of sufficiently well behaved Lie groups. Here we conduct a study on a low-dimensional nilpotent Lie group.

We are primarily interested in investigating the left-invariant sub-Riemannian structures on the matrix Lie group

$$\mathsf{T} = \left\{ \begin{bmatrix} 1 & x_1 & x_4 & x_5 \\ 0 & 1 & x_2 & x_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\},$$

a contribution to the above mentioned classification effort. This matrix Lie group is the lowest dimensional two-step nilpotent Lie group beyond the Heisenberg group. The left-invariant sub-Riemannian structures of the family of $(2n+1)$-dimensional Heisenberg groups have been investigated by various authors (see[4] and the references within). This is a family of two-step nilpotent Lie groups with one-dimensional commutator subgroups, thus the two-dimensional commutator subgroup of T represents a step up in complexity.

Chapter 2 establishes basic group properties of T such as the centre, nilpotency and simplicity. The Lie group T is established as a connected matrix Lie group. That is, a connected closed subgroup of the general linear group $\mathsf{GL}(4, \mathbb{R})$. We also compute the Lie algebra $\mathfrak{t}$ of T, its Lie bracket operation and the exponential map from $\mathfrak{t}$ to T.

In Chapter 3 we investigate the subspace structure of the Lie algebra. We compute the automorphism group $\mathrm{Aut}(\mathfrak{t})$ of invertible Lie bracket preserving linear maps from $\mathfrak{t}$ to itself. An equivalence relation on subspaces of $\mathfrak{t}$ is defined by regarding subspaces $\mathfrak{s}$ and $\mathfrak{w}$ as equivalent if they are related be an automorphism. That is, $\varphi \cdot \mathfrak{s} = \mathfrak{w}$ for some $\varphi \in \mathrm{Aut}(\mathfrak{t})$. The subspaces of the Lie algebra are given up to this equivalence. Additionally, we obtain a characterizing set of scalar invariants for the subspace structure of $\mathfrak{t}$.

In Chapter 4 we classify the left-invariant sub-Riemannian structures on T up to isometry. As isometries on nilpotent metric Lie groups are affine, that is the composition of an automorphisim and a left translation [10], the problem is reduced to finding sub-Riemannian structures up to a Lie group automorphism. A Lie group-Lie algebra correspondence result then allows use of the results of Chapter 3 on the Lie algebra $\mathfrak{t}$ for the computation of the sub-Riemannian structures of the Lie group T. We end this chapter by computing the isotropy groups of the left-invariant sub-Riemannian structures.

Lastly, in Chapter 5, we compute the normal geodesics of the rank 3 left-invariant sub-Riemannian structure on T. Here we utilize a result of Biggs and Nagy [4] based on Pontryagin's Maximum Principle.

The appendix collects standard results of linear algebra, abstract algebra, topology and smooth manifold theory for the readers reference. The definitions of particular objects that appear in the main body of the text, such as the Heisenberg group, are also given here. An alphabetical index is provided at the end. This includes an index of the notation used.

Chapter 2

Lie group and Lie algebra properties

In this chapter we introduce a five-dimensional two-step nilpotent Lie group, denoted by T. This will be our object of study throughout this dissertation. In the first section we give some basic properties of T. The second section includes computation of the Lie algebra $\mathfrak{t}$ corresponding to T as well as the exponential map from $\mathfrak{t}$ to T.

2.1 The Lie group T

Consider the collection of 5×5 real matrices

$$\mathsf{T} = \left\{ \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : p_1, p_2, p_3, p_4, p_5 \in \mathbb{R} \right\}.$$

Given any $P, Q \in \mathsf{T}$,

$$\begin{aligned} PQ &= \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & q_1 & q_4 & q_5 \\ 0 & 1 & q_2 & q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & p_1 + q_1 & p_4 + p_1 q_2 + q_4 & p_5 + p_1 q_3 + q_5 \\ 0 & 1 & p_2 + q_2 & p_3 + q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathsf{T}. \end{aligned}$$

For an arbitrary element P of T, we have

$$P^{-1} = \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -p_1 & p_1p_2 - p_4 & p_1p_3 - p_5 \\ 0 & 1 & -p_2 & -p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathsf{T}$$

with $PI_4 = I_4P = P$. We thus have that T, with the associative matrix multiplication, is indeed an abstract group.

Proposition 2.1.1. *The centre of the group* T *is*

$$Z(\mathsf{T}) = \left\{ \begin{bmatrix} 1 & 0 & z_1 & z_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : z_1, z_2 \in \mathbb{R} \right\}.$$

Proof. Suppose $P = \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in Z(\mathsf{T})$ and $Q = \begin{bmatrix} 1 & q_1 & q_4 & q_5 \\ 0 & 1 & q_2 & q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is an arbitrary element of T. Then $PQ - QP = \mathbf{0}$. That is,

$$\begin{bmatrix} 0 & 0 & p_1q_2 - p_2q_1 & p_1q_3 - p_3q_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$$

Thus $p_1q_2 - p_2q_1 = 0$ and $p_1q_3 - p_3q_1 = 0$ for all $q_1,q_2,q_3 \in \mathbb{R}$. This is true only if $p_1 = p_2 = p_3 = 0$. Thus $P = \begin{bmatrix} 1 & 0 & p_4 & p_5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and elements of the centre are of the desired form. □

As $Z(\mathsf{T}) \neq \mathsf{T}$, T is a non-commutative group.

Proposition 2.1.2. *The commutator subgroup of* T *is* $Z(\mathsf{T})$.

Proof. Let $P = \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & q_1 & q_4 & q_5 \\ 0 & 1 & q_2 & q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be elements of T, then a general commutator of T is an element of the form

$$PQP^{-1}Q^{-1} = \begin{bmatrix} 1 & 0 & p_1p_2 - (p_1+q_1)\,p_2 + p_1q_2 & p_1p_3 - (p_1+q_1)\,p_3 + p_1q_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & z_1 & z_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $z_1 = p_1p_2 - (p_1+q_1)\,p_2 + p_1q_2$ and $z_2 = p_1p_3 - (p_1+q_1)\,p_3 + p_1q_3$. With appropriate choices for P and Q, z_1 and z_2 may vary independently over $\mathbb{R}$. Therefore, elements of the form $PQP^{-1}Q^{-1}$ coincide with elements of the centre $Z(\mathsf{T})$. We thus have that the commutator subgroup of T coincides with the centre $Z(\mathsf{T})$ of T. That is, $\mathsf{T}' = Z(\mathsf{T})$. □

Proposition 2.1.3. *The derived series of* T *is* $\mathsf{T} \geq Z(\mathsf{T}) \geq \{I_4\}$.

Proof. By the definition of the derived series of a group (Definition A.2.3), we have $\mathsf{T}^{(0)} = \mathsf{T}$ and $\mathsf{T}^{(1)} = \mathsf{T}' = Z(\mathsf{T})$. Now, $\mathsf{T}^{(2)} = \mathsf{T}^{(1)\prime} = Z(\mathsf{T})'$. That is, the subgroup generated by the commutators of $Z(\mathsf{T})$. These commutators are elements of the form $PQP^{-1}Q^{-1}$ where P,$Q \in Z(\mathsf{T})$. We have,

$$PQP^{-1}Q^{-1} = PP^{-1}QQ^{-1} = I_4,$$

as P and Q are central. Therefore $\mathsf{T}^{(2)} = \{I_4\}$. □

Proposition 2.1.4. *The lower central series of* T *is* $\mathsf{T} \triangleright Z(\mathsf{T}) \triangleright \{I_4\}$.

Proof. By Definition A.2.2 $\gamma_1(\mathsf{T}) = \mathsf{T}$ and $\gamma_2(\mathsf{T}) = [\gamma_1(\mathsf{T}), \mathsf{T}] = [\mathsf{T}, \mathsf{T}] = Z(\mathsf{T})$. Now, $\gamma_3(\mathsf{T}) = [\gamma_2(\mathsf{T}), \mathsf{T}] = [Z(\mathsf{T}), \mathsf{T}]$. That is, $\gamma_3(\mathsf{T})$ is generated by elements of the form $PQP^{-1}Q^{-1}$ where $P \in Z(\mathsf{T})$ and $Q \in \mathsf{T}$. We have,

$$PQP^{-1}Q^{-1} = QPP^{-1}Q^{-1} = QI_4Q^{-1} = I_4,$$

as P is central. Thus $\gamma_3(\mathsf{T}) = \{I_4\}$. □

Proposition 2.1.5. T *has no maximal torus.*

Proof. Suppose $\mathbb{T}^{max}$ is a maximal torus for T. As every torus contains $\mathbb{T}^1 = \mathbb{S}^1$ as a subgroup and $\mathbb{T}^1$ contains $\{+\mathbf{1}, -\mathbf{1}\}$ as a two element cyclic subgroup, it follows that there exists a two element cyclic subgroup of $\mathbb{T}^{max}$ and thus of T. Let $P = \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be an arbitrary element of T. We suppose that P is an element of T that generates a two element cyclic subgroup and thus $P^3 = P$. That is,

$$\begin{bmatrix} 1 & 3p_1 & 3p_1p_2 + 3p_4 & 3p_1p_3 + 3p_5 \\ 0 & 1 & 3p_2 & 3p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This implies that $3p_1 = p_1$, $3p_2 = p_2$, $3p_3 = p_3$. This is true if and only if $p_1 = p_2 = p_3 = 0$. Further, $3p_1p_2 + 3p_4 = p_4$ and $3p_1p_3 + 3p_5 = p_5$ give $3p_4 = p_4$ and $3p_5 = p_5$. This is true if and only if $p_4 = p_5 = 0$. Thus $P = I_4$. However, as I_4 generates a one element subgroup of T, this contradicts the supposition that P generates a two element subgroup. Therefore no element of T generates a two element subgroup. This contradicts the supposition that T has a maximal torus $\mathbb{T}^{max}$. □

Proposition 2.1.6. T *is not a simple group.*

Proof. The centre $Z(\mathsf{T})$ of T is a non-trivial normal subgroup of T. It follows from Definition A.2.4 that T is not a simple group. □

Proposition 2.1.7. T *is a path-connected group with respect to the relative topology inherited from* $\mathsf{GL}(4, \mathbb{R})$ *(see Definition B.2.16).*

Proof. T is a path-connected group as the function $f : [0, 1] \longrightarrow \mathsf{T}$ defined by $f(t) = \begin{bmatrix} 1 & tp_1 & tp_4 & tp_5 \\ 0 & 1 & tp_2 & tp_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ defines a continuous path from I_4 to $\begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ in T, and thus any two points of T can be connected by a continuous path in T. □

Proposition 2.1.8. T *is a closed subgroup of* $\mathsf{GL}(4, \mathbb{R})$.

Proof. Let $(P_n) = \left(\begin{bmatrix} 1 & p_{1n} & p_{4n} & p_{5n} \\ 0 & 1 & p_{2n} & p_{3n} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$ be a convergent sequence in $\mathsf{GL}(4, \mathbb{R})$ contained in T. The component sequences (p_{in}), for $i = 1, 2, 3, 4, 5$, are convergent sequences in $\mathbb{R}$ and thus, respectively, converge to real numbers p_i, for $i = 1, 2, 3, 4, 5$. It follows that the sequence (P_n) converges to $P = \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. As $P \in \mathsf{T}$, T contains all its limit points, thus Theorem A.3.2 implies that T is a closed subset of $\mathsf{GL}(4, \mathbb{R})$. □

It follows from Definition B.2.16 that T is indeed a matrix Lie group.

Proposition 2.1.9. *The quotient of* T *by any one-dimensional central subgroup is isomorphic, as an abstract group, to the product group* $\mathsf{H}_3 \times \mathbb{R}$ *(see Example B.2.17).*

Proof. Suppose G is an arbitrary one-dimensional central subgroup of T then G is of the form

$$\mathsf{G} = \left\{ \begin{bmatrix} 1 & 0 & tx_1 & tx_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : t \in \mathbb{R} \right\}, \text{ were } x_1 \text{ and } x_2 \text{ are fixed real numbers, } x_1^2 + x_2^2 \neq 0.$$

We show that G is the kernel of some onto homomorphism from T to $\mathsf{H}_3 \times \mathbb{R}$ and obtain the result using Theorem A.2.6. Assuming $x_1 \neq 0$, consider the function $f : \mathsf{T} \longrightarrow \mathsf{H}_3 \times \mathbb{R}$ defined by

$$f\left(\begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = \left(\begin{bmatrix} 1 & p_1 & -x_2p_4 + x_1p_5 \\ 0 & 1 & -x_2p_2 + x_1p_3 \\ 0 & 0 & 1 \end{bmatrix}, p_2 \right).$$

Given $P = \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & q_1 & q_4 & q_5 \\ 0 & 1 & q_2 & q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathsf{T}$

$$\begin{aligned} f(P \cdot Q) &= f\left(\begin{bmatrix} 1 & p_1 + q_1 & p_4 + p_1q_2 + q_4 & p_5 + p_1q_3 + q_5 \\ 0 & 1 & p_2 + q_2 & p_3 + q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} 1 & p_1 + q_1 & -x_2(p_4 + p_1q_2 + q_4) + x_1(p_5 + p_1q_3 + q_5) \\ 0 & 1 & -x_2(p_2 + q_2) + x_1(p_3 + q_3) \\ 0 & 0 & 1 \end{bmatrix}, p_2 + q_2 \right) \\ &= \left(\begin{bmatrix} 1 & p_1 & -x_2p_4 + x_1p_5 \\ 0 & 1 & -x_2p_2 + x_1p_3 \\ 0 & 0 & 1 \end{bmatrix}, p_2 \right) * \left(\begin{bmatrix} 1 & q_1 & -x_2q_4 + x_1q_5 \\ 0 & 1 & -x_2q_2 + x_1q_3 \\ 0 & 0 & 1 \end{bmatrix}, q_2 \right) \\ &= f(P) * f(Q), \end{aligned}$$

where $*$ is the product group operation on $\mathsf{H}_3 \times \mathbb{R}$. Thus f is a group homomorphism. Further, given $m = \left(\begin{bmatrix} 1 & m_1 & m_2 \\ 0 & 1 & m_3 \\ 0 & 0 & 1 \end{bmatrix}, m_4 \right) \in \mathsf{H}_3 \times \mathbb{R}$ there exists $M \in \mathsf{T}$, namely $M =$

$\begin{bmatrix} 1 & m_1 & 0 & \frac{m_2}{x_1} \\ 0 & 1 & m_4 & \frac{m_3+x_2m_4}{x_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, such that $f(M) = m$. Thus f is onto. We now find the kernel of f;

$$\ker(f) = \left\{ P \in \mathsf{T} : f(P) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 0 \right) \right\}$$
$$= \left\{ \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : \left(\begin{bmatrix} 1 & p_1 & -x_2p_4 + x_1p_5 \\ 0 & 1 & -x_2p_2 + x_1p_3 \\ 0 & 0 & 1 \end{bmatrix}, p_2 \right) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 0 \right) \right\}.$$

This implies that for $P \in \ker(f)$, $p_1 = 0$, $p_2 = 0$, $(-x_2p_4 + x_1p_5) = 0$ and $(-x_2p_2 + x_1p_3) = 0$. We thus have that $(-x_2 \cdot 0 + x_1p_3) = 0$ and from the assumption $x_1 \neq 0$ we have $p_3 = 0$. It remains that $(-x_2p_4 + x_1p_5) = 0$ and thus $p_5 = \frac{x_2}{x_1}p_4, p_4 \in \mathbb{R}$. That is, the pairs (p_4, p_5) define a one-dimensional subspace of $\mathbb{R}^2$. We thus have that $(p_4, p_5) = t(x_1, x_2), t \in \mathbb{R}$. Therefore,

$$\ker(f) = \left\{ \begin{bmatrix} 1 & 0 & tx_1 & tx_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \mathsf{G}.$$

It follows from Theorem A.2.6 that we have the group isomorphism $\mathsf{T}/\mathsf{G} \approxeq \mathsf{H}_3 \times \mathbb{R}$.

If, on the other hand, $x_1 = 0$, then $x_2 \neq 0$ as G is one-dimensional. Consider the function $f : \mathsf{T} \longrightarrow \mathsf{H}_3 \times \mathbb{R}$ defined by

$$f\left(\begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = \left(\begin{bmatrix} 1 & p_1 & -x_2p_4 + x_1p_5 \\ 0 & 1 & -x_2p_2 + x_1p_3 \\ 0 & 0 & 1 \end{bmatrix}, p_3 \right) = \left(\begin{bmatrix} 1 & p_1 & -x_2p_4 \\ 0 & 1 & -x_2p_2 \\ 0 & 0 & 1 \end{bmatrix}, p_3 \right).$$

Similar to the previous case, f may be shown to be a group homomorphism. Given $m = \left(\begin{bmatrix} 1 & m_1 & m_2 \\ 0 & 1 & m_3 \\ 0 & 0 & 1 \end{bmatrix}, m_4 \right) \in \mathsf{H}_3 \times \mathbb{R}$, there exists $M \in \mathsf{T}$ with $f(M) = m$, namely $M = \begin{bmatrix} 1 & m_1 & -\frac{m_2}{x_2} & 0 \\ 0 & 1 & -\frac{m_3}{x_2} & m_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

We therefore have that f is onto. For the kernel of f, we have

$$\ker(f) = \left\{ \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : \left(\begin{bmatrix} 1 & p_1 & -x_2p_4 \\ 0 & 1 & -x_2p_2 \\ 0 & 0 & 1 \end{bmatrix}, p_3 \right) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 0 \right) \right\}.$$

This implies that for $P \in \ker(f)$, $p_1 = 0$, $p_3 = 0$, $-x_2p_4 = 0$ and $-x_2p_2 = 0$. As $x_2 \neq 0$, we have $p_2 = p_4 = 0$. We, more conveniently, parametrize $p_5 \in \mathbb{R}$ as $p_5 = tx_2, t \in \mathbb{R}$. Therefore,

$$\ker(f) = \left\{ \begin{bmatrix} 1 & 0 & 0 & tx_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 1 & 0 & tx_1 & tx_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \mathsf{G}.$$

It follows from Theorem A.2.6 that we have the group isomorphism $\mathsf{T}/\mathsf{G} \approxeq \mathsf{H}_3 \times \mathbb{R}$. Thus, in general $\mathsf{T}/\mathsf{G} \approxeq \mathsf{H}_3 \times \mathbb{R}$. □

The group epimorphism $f : \mathsf{T} \longrightarrow \mathsf{H}_3 \times \mathbb{R}$ is in fact a Lie group epimorphism. For this, we show that f is in addition a smooth map.

In the case where $x_1 \neq 0$, the function f was given by

$$f\left(\begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = \left(\begin{bmatrix} 1 & p_1 & -x_2p_4 + x_1p_5 \\ 0 & 1 & -x_2p_2 + x_1p_3 \\ 0 & 0 & 1 \end{bmatrix}, p_2\right).$$

This is indeed a smooth map as its representative $\hat{f}$ with respect to the smooth structures of T and $(\mathsf{H}_3 \times \mathbb{R})$ is a smooth map. That is, where the smooth structure of T is given by the global smooth chart $\phi : \mathsf{T} \longrightarrow \mathbb{R}^5$ with

$$\phi : \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto (p_1, p_2, p_3, p_4, p_5)$$

and the smooth structure of $\mathsf{H}_3 \times \mathbb{R}$ is given by the global smooth chart $\rho : \mathsf{H}_3 \times \mathbb{R} \longrightarrow \mathbb{R}^4$ with

$$\rho : \left(\begin{bmatrix} 1 & r_1 & r_3 \\ 0 & 1 & r_2 \\ 0 & 0 & 1 \end{bmatrix}, r_4\right) \mapsto (r_1, r_2, r_3, r_4).$$

To verify, $\hat{f} = \rho \circ f \circ \phi^{-1} : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$. For $p = (p_1, p_2, p_3, p_4, p_5)$

$$\begin{aligned} \hat{f}(p) &= \rho \circ f \left(\begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \\ &= \rho\left(\begin{bmatrix} 1 & p_1 & -x_2p_4 + x_1p_5 \\ 0 & 1 & -x_2p_2 + x_1p_3 \\ 0 & 0 & 1 \end{bmatrix}, p_2\right) \\ &= (p_1, -x_2p_2 + x_1p_3, -x_2p_4 + x_1p_5, p_2). \end{aligned}$$

Clearly a smooth map.

In the case where $x_1 = 0$, f similarly has coordinate representation

$$(p_1, p_2, p_3, p_4, p_5) \mapsto (p_1, -x_2p_2 + x_1p_3, -x_2p_4 + x_1p_5, p_3)$$

which is a smooth map. Hence, in all cases, $f : \mathsf{T} \longrightarrow \mathsf{H}_3 \times \mathbb{R}$ is a smooth map.

Proposition 2.1.10. *The quotient of* T *by any one-dimensional central subgroup is Lie group isomorphic to the Lie group* $\mathsf{H}_3 \times \mathbb{R}$ *(see Example B.2.17).*

Proof. As the map $f : \mathsf{T} \longrightarrow \mathsf{H}_3 \times \mathbb{R}$ defined above is an epimorphism of Lie groups, Theorem B.2.3 gives a Lie group isomorphism

$$\mathsf{T}/N \cong \mathsf{H}_3 \times \mathbb{R},$$

where $N = \ker(f)$ is a (normal) one-dimensional central subgroup of T. □

The quotient of T with its centre $Z(\mathsf{T})$, $\mathsf{T}/Z(\mathsf{T})$, has as its elements cosets of the form

$$\begin{aligned} PZ(\mathsf{T}) = \{PC : C \in Z(\mathsf{T})\} &= \left\{ \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & z_1 & z_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} 1 & p_1 & p_4 + z_1 & p_5 + z_2 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \end{aligned}$$

where $P \in \mathsf{T}$.

As z_1 and z_2 range through $\mathbb{R}$, this gives

$$PZ(\mathsf{T}) = \left\{ \begin{bmatrix} 1 & p_1 & x_1 & x_2 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}.$$

We note that $\mathsf{T}/Z(\mathsf{T})$ is isomorphic, as an abstract group, to $(\mathbb{R}^3, +, \mathbf{0})$. This result follows from Theorem A.2.6 and the observation that the function $f : (\mathsf{T}, \cdot, I_4) \longrightarrow (\mathbb{R}^3, +, \mathbf{0})$ defined by

$$f\left(\begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = (p_1, p_2, p_3)$$

is an onto group homomorphism with $\ker(f) = Z(\mathsf{T})$. We state this as the following proposition.

Proposition 2.1.11. *The quotient of* T *by its centre* $Z(\mathsf{T})$ *is group isomorphic to* $\mathbb{R}^3$*. That is,*

$$\mathsf{T}/Z(\mathsf{T}) \cong (\mathbb{R}^3, +, \mathbf{0}).$$

Proof. We show that the function f above has the desired properties. Let $P = \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & q_1 & q_4 & q_5 \\ 0 & 1 & q_2 & q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ where $P,Q \in \mathsf{T}$. Then

$$\begin{aligned} f(P \cdot Q) &= f\left(\begin{bmatrix} 1 & p_1+q_1 & p_4+p_1q_2+q_4 & p_5+p_1q_3+q_5 \\ 0 & 1 & p_2+q_2 & p_3+q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \\ &= (p_1+q_1, p_2+q_2, p_3+q_3) \\ &= (p_1,p_2,p_3)+(q_1,q_2,q_3) \\ &= f(P)+f(Q). \end{aligned}$$

Thus f is a group homomorphism.

Given $(r_1, r_2, r_3) \in \mathbb{R}^3$, there exist $R \in \mathsf{T}$ such that $f(R) = (r_1, r_2, r_3)$. Namely, $R = \begin{bmatrix} 1 & r_1 & 0 & 0 \\ 0 & 1 & r_2 & r_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Thus f is onto.

We now compute the kernel of f.

$$\begin{aligned} \ker(f) &= \{P \in \mathsf{T} : f(P) = (0,0,0)\} \\ &= \left\{ \begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : (p_1,p_2,p_3) = (0,0,0) \right\} \\ &= \left\{ \begin{bmatrix} 1 & 0 & p_4 & p_5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : p_4, p_5 \in \mathbb{R} \right\} \\ &= Z(\mathsf{T}). \end{aligned}$$

From Theorem A.2.6, we have the group isomorphism $\mathsf{T}/Z(\mathsf{T}) \approx \mathbb{R}^3$. □

The group epimorrphism $f : (\mathsf{T}, \cdot, I_4) \longrightarrow (\mathbb{R}^3, +, \mathbf{0})$ defined by

$$f\left(\begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = (p_1,p_2,p_3)$$

with $\ker(f) = Z(\mathsf{T})$ is in fact an epimorphism of Lie groups as f is smooth. This can be seen form the fact that its representation $\hat{f} : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ with respect to the global charts of the smooth structures of T and $\mathbb{R}^3$ is a smooth map. That is, for $p = (p_1, p_2, p_3, p_4, p_5) \in \mathbb{R}^5$

$$\begin{aligned}\hat{f}(p) &= \mathrm{Id}_{\mathbb{R}^3} \circ f \circ \phi^{-1}(p) \\ &= f\left(\begin{bmatrix} 1 & p_1 & p_4 & p_5 \\ 0 & 1 & p_2 & p_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \\ &= (p_1, p_2, p_3)\end{aligned}$$

is a smooth map.

Proposition 2.1.12. *The quotient of* T *with its centre* $Z(\mathsf{T})$ *is Lie group isomorphic to* $\mathbb{R}^3$*.*

Proof. As the map $f : \mathsf{T} \longrightarrow \mathbb{R}^3$ defined above is a smooth epimorphism of Lie groups with $\ker(f) = Z(\mathsf{T})$, Theorem B.2.3 gives a Lie group isomorphism

$$\mathsf{T}/Z(\mathsf{T}) \cong \mathbb{R}^3.$$

□

2.2 The Lie algebra $\mathfrak{t}$

As T is a matrix Lie group, its smooth structure is given by a global coordinate chart defined by its entries. Here we choose the global chart $\phi : \mathsf{T} \longrightarrow \mathbb{R}^5$ with

$$\phi : \begin{bmatrix} 1 & q_1 & q_4 & q_5 \\ 0 & 1 & q_2 & q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto (q_1, q_2, q_3, q_4, q_5).$$

As, by construction, ϕ is smooth when considered as a map between the manifolds T and $\mathbb{R}^5$, we have that the differential $d_{\mathbf{1}}\phi$ is a linear isomorphism between the tangent spaces at identity $T_{\mathbf{1}}\mathsf{T} = \mathfrak{t}$ and $T_{\mathbf{0}}\mathbb{R}^5$. We then have that $\mathfrak{t}$ has a basis

$$\left\{ d_{\mathbf{0}}\phi^{-1} \cdot \left.\frac{\partial}{\partial x_i}\right|_{\mathbf{0}} \right\}_{i=1}^{5}.$$

We note that $d_{\mathbf{0}}\phi^{-1}$, with respect to the smooth structure (T, ϕ) and its choice of coordinates, is represented by the identity linear map in these coordinates. The tangent space $\mathfrak{t}$ therefore has basis $\left\{ \left.\frac{\partial}{\partial x_i}\right|_{\mathbf{0}} \right\}_{i=1}^{5}$, with respect to the given smooth structure of T.

We note that the curve $\gamma_1 : (-\epsilon, \epsilon) \longrightarrow \mathsf{T}$ given by

$$\gamma_1(t) = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a smooth curve based at identity ($\gamma_1(0) = I_4$) with $\frac{d}{dt}\big|_{t=0} \gamma_1(t)$ given by

$$\frac{d}{dt}\bigg|_{t=0} \gamma_1(t) := \frac{d}{dt}\bigg|_{t=0} \phi \circ \gamma_1(t) = \frac{d}{dt}\bigg|_{t=0} (t, 0, 0, 0, 0) = (1, 0, 0, 0, 0) = \frac{\partial}{\partial x_1}.$$

That is, the tangent vector of γ_1 at $t = 0$ coincides with the first basis vector of the tangent space $\mathfrak{t}$ — where the basis for $\mathfrak{t}$ and the tangent vector of γ_1 are with respect to the global smooth chart (ϕ, T). Note that the smooth curve $\gamma_1(t)$ is in fact defined for all $t \in \mathbb{R}$.

We shall use the notation $[q_1, q_2, q_3, q_4, q_5]$ to denote the element

$$\begin{bmatrix} 1 & q_1 & q_4 & q_5 \\ 0 & 1 & q_2 & q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

of T. With this, define the smooth curves

$$\begin{aligned} \gamma_2(t) &= [0, t, 0, 0, 0], \\ \gamma_3(t) &= [0, 0, t, 0, 0], \\ \gamma_4(t) &= [0, 0, 0, t, 0] \text{ and} \\ \gamma_5(t) &= [0, 0, 0, 0, t]. \end{aligned}$$

We then similarly have that

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} \gamma_2(t) &:= \frac{d}{dt}\bigg|_{t=0} \phi \circ \gamma_2(t) = \frac{d}{dt}\bigg|_{t=0} (0, t, 0, 0, 0) = (0, 1, 0, 0, 0) = \frac{\partial}{\partial x_2}, \\ \frac{d}{dt}\bigg|_{t=0} \gamma_3(t) &:= \frac{d}{dt}\bigg|_{t=0} \phi \circ \gamma_3(t) = \frac{d}{dt}\bigg|_{t=0} (0, 0, t, 0, 0) = (0, 0, 1, 0, 0) = \frac{\partial}{\partial x_3}, \\ \frac{d}{dt}\bigg|_{t=0} \gamma_4(t) &:= \frac{d}{dt}\bigg|_{t=0} \phi \circ \gamma_4(t) = \frac{d}{dt}\bigg|_{t=0} (0, 0, 0, t, 0) = (0, 0, 0, 1, 0) = \frac{\partial}{\partial x_4} \text{ and} \\ \frac{d}{dt}\bigg|_{t=0} \gamma_5(t) &:= \frac{d}{dt}\bigg|_{t=0} \phi \circ \gamma_5(t) = \frac{d}{dt}\bigg|_{t=0} (0, 0, 0, 0, t) = (0, 0, 0, 0, 1) = \frac{\partial}{\partial x_5}. \end{aligned}$$

We rename the ordered basis

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5} \right)$$

of $\mathfrak{t}$ to (I, J, K, L, M). The vectors of the tangent space at identity $\mathfrak{t}$ may be extended to left-invariant vector fields on T. That is, given $X(\mathbf{1}) \in \mathfrak{t}$ we may define the vector field

$$X : \mathsf{T} \longrightarrow T\mathsf{T}$$

by

$$X(q) = d_{\mathbf{1}} L_q \cdot X(\mathbf{1}).$$

We will identify a left-invariant vector field X with the vector $X(\mathbf{1}) \in \mathfrak{t}$.

Further, for any left-invariant vector field X on T with

$$X(\mathbf{1}) = x_1 I + x_2 J + x_3 K + x_4 L + x_5 M = (x_1, x_2, x_3, x_4, x_5) \in \mathfrak{t},$$

we have the Cauchy problem

$$\begin{cases} \dot{q}(t) = X(q(t)) = d_{\mathbf{1}} L_{q(t)} \cdot X(\mathbf{1}) \\ q(0) = q_0. \end{cases}$$

Transforming this differential equation on T into one on $\mathbb{R}^5$. That is, writing it in coordinates by using the global smooth structure of T, we have

$$\dot{q}(t) = d_{\mathbf{1}} L_{q(t)} \cdot X(\mathbf{1}) \in T_{q(t)} \mathsf{T}$$
$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \dot{q}_4(t) \\ \dot{q}_5(t) \end{bmatrix} = [d_{\mathbf{1}} L_{q(t)}] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in T_{\phi(q(t))} \mathbb{R}^5,$$

where $[d_{\mathbf{1}} L_{q(t)}] : T_{\mathbf{0}} \mathbb{R}^5 \longrightarrow T_{\phi \circ L_{q(t)} \circ \phi^{-1}(\mathbf{0})} \mathbb{R}^5$ is the matrix of the linear map $d_{\mathbf{1}} L_{q(t)}$ with respect to the coordinate basis given by the global smooth chart (T, ϕ). That is, $[d_{\mathbf{1}} L_{q(t)}]$ is the Jacobian matrix of the representation $\hat{L}_{q(t)} = \phi \circ L_{q(t)} \circ \phi^{-1} : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$. Writing this out, we have

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \dot{q}_4(t) \\ \dot{q}_5(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & q_1(t) & 0 & 1 & 0 \\ 0 & 0 & q_1(t) & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 + q_1(t) x_2 \\ x_5 + q_1(t) x_3 \end{bmatrix}.$$

This gives the solutions

$$q_1(t) = t x_1 + q_1(0)$$
$$q_2(t) = t x_2 + q_2(0)$$
$$q_3(t) = t x_3 + q_3(0),$$

leaving

$$\dot{q}_4 = q_1(t) x_2 + x_4 = (t x_1 + q_1(0)) x_2 + x_4$$
$$\dot{q}_5 = q_1(t) x_3 + x_5 = (t x_1 + q_1(0)) x_3 + x_5,$$

and finally giving

$$q_4(t) = \frac{1}{2}t^2x_1x_2 + tx_4 + tq_1(0)x_2 + q_4(0)$$
$$q_5(t) = \frac{1}{2}t^2x_1x_3 + tx_5 + tq_1(0)x_3 + q_5(0).$$

That is, the coordinate form of our Cauchy problem has solution

$$\begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \\ q_5(t) \end{bmatrix} \in \mathbb{R}^5.$$

Transforming this back to the manifold T gives the solution $q(t) = [q_1(t), q_2(t), q_3(t), q_4(t), q_5(t)]$. That is,

$$q(t) = \begin{bmatrix} 1 & tx_1 + q_1(0) & \frac{1}{2}t^2x_1x_2 + tx_4 + tq_1(0)x_2 + q_4(0) & \frac{1}{2}t^2x_1x_3 + tx_5 + tq_1(0)x_3 + q_5(0) \\ 0 & 1 & tx_2 + q_2(0) & tx_3 + q_3(0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$q(t) = \begin{bmatrix} 1 & q_1(0) & q_4(0) & q_5(0) \\ 0 & 1 & q_2(0) & q_3(0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & tx_1 & \frac{1}{2}t^2x_1x_2 + tx_4 & \frac{1}{2}t^2x_1x_3 + tx_5 \\ 0 & 1 & tx_2 & tx_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$q(t) = q_0 \cdot \begin{bmatrix} 1 & tx_1 & \frac{1}{2}t^2x_1x_2 + tx_4 & \frac{1}{2}t^2x_1x_3 + tx_5 \\ 0 & 1 & tx_2 & tx_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As for every $q_0 \in \mathsf{T}$ the integral curve $q(t)$ to the left-invariant vector field X is defined for all $t \in \mathbb{R}$, every left-invariant vector filed X on T is **complete**.

Given the basis (I, J, K, L, M) for the Lie algebra $\mathfrak{t}$, using Definition B.2.14 we now compute the Lie bracket on $\mathfrak{t}$. Firstly, with reference to the above Cauchy problem, we note that the flow of the left-invariant vector field X with $X(\mathbf{1}) = (x_1, x_2, x_3, x_4, x_5)$ on the manifold T is given by

$$e^{tX}(q) = q \cdot \exp(tX),$$

with

$$\exp(tX) = \begin{bmatrix} 1 & tx_1 & \frac{1}{2}t^2x_1x_2 + tx_4 & \frac{1}{2}t^2x_1x_3 + tx_5 \\ 0 & 1 & tx_2 & tx_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

That is, the flow e^{tX} of the vector field X operates on the points q of T by right multiplication with the curve $exp(tX) \in \mathsf{T}$. Noting that

$$\begin{aligned}
\exp(tI) &= [t,0,0,0,0] \\
\exp(tJ) &= [0,t,0,0,0] \\
\exp(tK) &= [0,0,t,0,0] \\
\exp(tL) &= [0,0,0,t,0] \\
\exp(tM) &= [0,0,0,0,t]
\end{aligned}$$

for $q \in \mathsf{T}$, Definition B.2.14 gives

$$\begin{aligned}
[I,J]|_q &= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} e^{-tI} \circ e^{sJ} \circ e^{tI}(q) \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} q \cdot [t,0,0,0,0] \cdot [0,s,0,0,0,0] \cdot [-t,0,0,0,0] \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} q \cdot [t,0,0,0,0] \cdot [-t,s,0,0,0] \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} q \cdot [0,s,0,ts,0] \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} [q_1, q_2+s, q_3, q_4+ts+q_1 s, q_5] \\
&= (0,0,0,1,0) \in T_q\mathsf{T} \\
&= d_{\mathbf{1}}L_q \cdot L(\mathbf{1}) \\
&= L(q) = L|_q,
\end{aligned}$$

and thus $[I,J] = L$.

Similarly,

$$\begin{aligned}
[I,K]|_q &= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} e^{-tI} \circ e^{sK} \circ e^{tI}(q), \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} q \cdot [t,0,0,0,0] \cdot [0,0,s,0,0] \cdot [-t,0,0,0,0] \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} q \cdot [t,0,0,0,0] \cdot [-t,0,s,0,0] \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} q \cdot [0,0,s,0,ts] \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} [q_1, q_2, q_3+s, q_4, q_5+ts+q_1 s] \\
&= (0,0,0,0,1) \in T_q\mathsf{T} \\
&= M|_q
\end{aligned}$$

and

$$\begin{aligned}
[I, L]|_q &= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} e^{-tI} \circ e^{sL} \circ e^{tI}(q), \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} q \cdot [t, 0, 0, 0, 0] \cdot [0, 0, 0, s, 0] \cdot [-t, 0, 0, 0, 0] \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} q \cdot [t, 0, 0, 0, 0] \cdot [-t, 0, 0, s, 0] \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} q \cdot [0, 0, 0, s, 0] \\
&= \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} [q_1, q_2, q_3, q_4 + s, q_5] \\
&= (0, 0, 0, 0, 0) \in T_q \mathsf{T} \\
&= \mathbf{0}|_q,
\end{aligned}$$

the zero vector field. Similar computations show that all other Lie brackets involving the basis vectors of $\mathfrak{t}$ give the zero vector field. Therefore, the non-trivial Lie brackets on the Lie algebra $\mathfrak{t}$ of the Lie group T are determined by

$$[I, J] = L \text{ and } [I, K] = M.$$

We note that the centre $\mathfrak{z}$ of $\mathfrak{t}$ is the subspace of $\mathfrak{t}$ spanned by L and M.

Lemma 2.2.1. *Given any vectors $A = a_1 I + a_2 J + a_3 K + a_4 L + a_5 M$ and $B = b_1 I + b_2 J + b_3 K + b_4 L + b_5 M$ of $\mathfrak{t}$ we have that*

$$[A, B] = (a_1 b_2 - a_2 b_1) L + (a_1 b_3 - a_3 b_1) M.$$

Proof. Bilinearity of the Lie bracket and the fact that $\mathfrak{z} = \langle L, M \rangle$ is the centre of $\mathfrak{t}$ give

$$\begin{aligned}
[A, B] &= [A, b_1 I + b_2 J + b_3 K + b_4 L + b_5 M] \\
&= [A, b_1 I + b_2 J + b_3 K] + [A, b_4 L + b_5 M] \\
&= [A, b_1 I + b_2 J + b_3 K] + \mathbf{0} \\
&= [a_1 I + a_2 J + a_3 K + a_4 L + a_5 M, b_1 I + b_2 J + b_3 K] \\
&= [a_1 I + a_2 J + a_3 K, b_1 I + b_2 J + b_3 K] + [a_4 L + a_5 M, b_1 I + b_2 J + b_3 K] \\
&= [a_1 I + a_2 J + a_3 K, b_1 I + b_2 J + b_3 K] + \mathbf{0} \\
&= [a_1 I + a_2 J + a_3 K, b_1 I + b_2 J + b_3 K] \\
&= a_1 b_1 [I, I] + a_1 b_2 [I, J] + a_1 b_3 [I, K] \\
&+ a_2 b_1 [J, I] + a_2 b_2 [J, J] + a_2 b_3 [J, K] \\
&+ a_3 b_1 [K, I] + a_3 b_2 [K, J] + a_3 b_3 [K, K] \\
&= \mathbf{0} + a_1 b_2 [I, J] + a_1 b_3 [I, K] + a_2 b_1 [J, I] + \mathbf{0} + \mathbf{0} + a_3 b_1 [K, I] + \mathbf{0} + \mathbf{0} \\
&= a_1 b_2 L + a_1 b_3 M - a_2 b_1 L - a_3 b_1 M \\
&= (a_1 b_2 - a_2 b_1) L + (a_1 b_3 - a_3 b_1) M.
\end{aligned}$$

□

Chapter 3

Subspace classification

In this chapter our aim is to classify the subspaces of the Lie algebra $\mathfrak{t}$. That is, we wish to categorize the subspaces of $\mathfrak{t}$ up some equivalence. In the first section we compute the automorphism group of $\mathfrak{t}$. Regarding a subspace $\mathfrak{s}$ of $\mathfrak{t}$ as equivalent to its image $\varphi \cdot \mathfrak{s}$ by an automorphism φ of $\mathfrak{t}$ gives an equivalence relation on the subspaces of $\mathfrak{t}$. It is up to this equivalence that we classify subspaces of $\mathfrak{t}$ into subalgebras, ideals, fully characteristic ideals and generating subspaces.

3.1 Preliminaries

3.1.1 The automorphism group $\mathrm{Aut}(\mathfrak{t})$

We now present the group of automorphisms of the Lie algebra $\mathfrak{t}$. Throughout, we represent elements of $\mathfrak{t}$ as column vectors with respect to the ordered basis (I, J, K, L, M). The notation $\langle E_1, ..., E_n \rangle$ will denote the linear span of the vectors $E_1, ..., E_n$. We will identify an automorphism $\varphi \in \mathrm{Aut}(\mathfrak{t})$ (see Definition B.2.6) with its 5×5 matrix representation with respect to this basis.

Proposition 3.1.1. *Let $\varphi : \mathfrak{t} \longrightarrow \mathfrak{t}$ be a linear map, then $\varphi \in \mathrm{Aut}(\mathfrak{t})$ if and only if the matrix representation of φ with respect to the ordered basis (I, J, K, L, M) is a real matrix of the form*

$$\varphi = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & j_3 & k_3 & 0 & 0 \\ i_4 & j_4 & k_4 & i_1 j_2 & i_1 k_2 \\ i_5 & j_5 & k_5 & i_1 j_3 & i_1 k_3 \end{bmatrix},$$

with $i_1 \neq 0$ and $j_2 k_3 - j_3 k_2 \neq 0$.

Proof. Suppose $\varphi \in \mathrm{Aut}(\mathfrak{t})$. That is, $\varphi : \mathfrak{t} \longrightarrow \mathfrak{t}$ is a linear isomorphism that preserves Lie brackets. Let

$$\varphi = \begin{bmatrix} i_1 & j_1 & k_1 & l_1 & m_1 \\ i_2 & j_2 & k_2 & l_2 & m_2 \\ i_3 & j_3 & k_3 & l_3 & m_3 \\ i_4 & j_4 & k_4 & l_4 & m_4 \\ i_5 & j_5 & k_5 & l_5 & m_5 \end{bmatrix}$$

be the matrix representation of φ relative to the ordered basis (I, J, K, L, M).

As φ preserves Lie brackets, using Lemma 2.2.1 we have

$$\begin{aligned} \varphi \cdot [I, J] &= [\varphi \cdot I, \varphi \cdot J] \\ \varphi \cdot L &= [i_1 I + i_2 J + i_3 K + i_4 L + i_5 M \ , j_1 I + j_2 J + j_3 K + j_4 L + j_5 M], \end{aligned}$$

and thus

$$l_1 I + l_2 J + l_3 K + l_4 L + l_5 M = (i_1 j_2 - i_2 j_1)L + (i_1 j_3 - i_3 j_1)M.$$

This implies that $l_1 = l_2 = l_3 = 0$, $l_4 = (i_1 j_2 - i_2 j_1)$ and $l_5 = (i_1 j_3 - i_3 j_1)$.

Similarly,

$$\begin{aligned} \varphi \cdot [I, K] &= [\varphi \cdot I, \varphi \cdot K] \\ \varphi \cdot M &= [i_1 I + i_2 J + i_3 K + i_4 L + i_5 M, k_1 I + k_2 J + k_3 K + k_4 L + k_5 M], \end{aligned}$$

and so

$$m_1 I + m_2 J + m_3 K + m_4 L + m_5 M = (i_1 k_2 - i_2 k_1)L + (i_1 k_3 - i_3 k_1)M.$$

This implies that $m_1 = m_2 = m_3 = 0$, $m_4 = (i_1 k_2 - i_2 k_1)$ and $m_5 = (i_1 k_3 - i_3 k_1)$.

Further,

$$\begin{aligned} \varphi \cdot [J, K] &= [\varphi \cdot J, \varphi \cdot K] \\ \varphi \cdot \mathbf{0} &= [j_1 I + j_2 J + j_3 K + j_4 L + j_5 M, k_1 I + k_2 J + k_3 K + k_4 L + k_5 M] \\ \mathbf{0} &= (j_1 k_2 - j_2 k_1)L + (j_1 k_3 - j_3 k_1)M. \end{aligned}$$

This gives $(j_1 k_2 - j_2 k_1) = 0$ and $(j_1 k_3 - j_3 k_1) = 0$. We thus have that

$$\varphi = \begin{bmatrix} i_1 & j_1 & k_1 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & j_3 & k_3 & 0 & 0 \\ i_4 & j_4 & k_4 & (i_1 j_2 - i_2 j_1) & (i_1 k_2 - i_2 k_1) \\ i_5 & j_5 & k_5 & (i_1 j_3 - i_3 j_1) & (i_1 k_3 - i_3 k_1) \end{bmatrix}.$$

Now, as φ is invertible, $\det(\varphi) \neq 0$. That is,

$$\det(\varphi) = i_1(i_3(j_1 k_2 - j_2 k_1) + i_2(j_3 k_1 - j_1 k_3) + i_1(j_2 k_3 - j_3 k_2))^2 \neq 0.$$

Using $(j_1k_2 - j_2k_1) = 0$ and $(j_1k_3 - j_3k_1) = 0$, we have

$$\begin{aligned} i_1(i_1(j_2k_3 - j_3k_2))^2 &\neq 0 \\ i_1^3(j_2k_3 - j_3k_2)^2 &\neq 0. \end{aligned}$$

That is $i_1 \neq 0$ and $j_2k_3 - j_3k_2 \neq 0$.

We have that $(j_1k_2 - j_2k_1) = 0$, $(j_1k_3 - j_3k_1) = 0$ and $j_2k_3 - j_3k_2 \neq 0$. Suppose $j_1 \neq 0$, then $k_2 = \frac{j_2k_1}{j_1}$, $k_3 = \frac{j_3k_1}{j_1}$ and $j_2k_3 \neq j_3k_2$. Therefore,

$$\begin{aligned} j_2k_3 &\neq j_3k_2 \\ j_2\left(\frac{j_3k_1}{j_1}\right) &\neq j_3\left(\frac{j_2k_1}{j_1}\right). \end{aligned}$$

However, this is a contradiction. Thus $j_1 = 0$. Similarly, suppose $k_1 \neq 0$, then $\frac{j_1k_2}{k_1} = j_2$, $\frac{j_1k_3}{k_1} = j_3$ and $j_2k_3 \neq j_3k_2$. Therefore,

$$\begin{aligned} j_2k_3 &\neq j_3k_2 \\ \left(\frac{j_1k_2}{k_1}\right)k_3 &\neq \left(\frac{j_1k_3}{k_1}\right)k_2. \end{aligned}$$

This is clearly a contradiction, thus $k_1 = 0$.

Now, $j_1 = k_1 = 0$ reduces the three relations $(j_1k_2 - j_2k_1) = 0$, $(j_1k_3 - j_3k_1) = 0$ and $j_2k_3 - j_3k_2 \neq 0$ to just $j_2k_3 - j_3k_2 \neq 0$. We therefore have that if $\varphi \in \mathrm{Aut}(\mathfrak{t})$, then φ is represented by a matrix of the form

$$\varphi = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & j_3 & k_3 & 0 & 0 \\ i_4 & j_4 & k_4 & i_1j_2 & i_1k_2 \\ i_5 & j_5 & k_5 & i_1j_3 & i_1k_3 \end{bmatrix},$$

with $i_1 \neq 0$ and $j_2k_3 - j_3k_2 \neq 0$.

Conversely, suppose $\varphi : \mathfrak{t} \longrightarrow \mathfrak{t}$ is represented by a matrix of this form with respect to the ordered basis (I, J, K, L, M). We show that φ preserves Lie brackets. Using Lemma 2.2.1, we have

$$\begin{aligned} [\varphi \cdot I, \varphi \cdot J] &= [i_1I + i_2J + i_3K + i_4L + i_5M, 0 \cdot I + j_2J + j_3K + j_4L + j_5M] \\ &= (i_1j_2 - i_2 \cdot 0)L + (i_1j_3 - i_3 \cdot 0)M \\ &= i_1j_2L + i_1j_3M \\ &= \varphi \cdot L \\ &= \varphi \cdot [I, J], \end{aligned}$$

$$\begin{aligned}[\varphi \cdot I, \varphi \cdot K] &= [i_1 I + i_2 J + i_3 K + i_4 L + i_5 M, 0 \cdot I + k_2 J + k_3 K + k_4 L + k_5 M] \\ &= (i_1 k_2 - i_2 \cdot 0)L + (i_1 k_3 - i_3 \cdot 0)M \\ &= i_1 k_2 L + i_1 k_3 M \\ &= \varphi \cdot M \\ &= \varphi \cdot [I, K],\end{aligned}$$

and

$$\begin{aligned}[\varphi \cdot J, \varphi \cdot K] &= [0 \cdot_1 I + j_2 J + j_3 K + j_4 L + j_5 M, 0 \cdot I + k_2 J + k_3 K + k_4 L + k_5 M] \\ &= (0 \cdot k_2 - i_2 \cdot 0)L + (0 \cdot k_3 - i_3 \cdot 0)M \\ &= \mathbf{0} \\ &= \varphi \cdot \mathbf{0} \\ &= \varphi \cdot [J, K].\end{aligned}$$

Lie brackets involving central elements vanish. As

$$\varphi(a_1 L + a_2 M) = (a_1 i_1 j_2 + a_2 i_1 k_2)L + (a_1 i_1 j_3 + a_2 i_1 k_3)M,$$

images of central elements of $\mathfrak{t}$ under φ are themselves central elements of $\mathfrak{t}$. Lie brackets involving the images of central elements under φ therefore also vanish. It follows that φ preserves Lie brackets involving central elements, in particular those involving the basis vectors L and M. It follows that φ preserves the Lie bracket on $\mathfrak{t}$.

Now, φ is an invertible map as $\det(\varphi) = i_1^2(j_3k_2 - j_2k_3)^2$, which is nonzero by the conditions $i_1 \neq 0$ and $j_2k_3 - j_3k_2 \neq 0$. Thus indeed $\varphi \in \mathrm{Aut}(\mathfrak{t})$. □

3.1.2 Lie algebra subspaces

Here we give definitions and results relating to various kinds of subspaces of a Lie algebra that are of interest. In the following sections we use these results to classify the subspaces of the Lie algebra $\mathfrak{t}$, up to equivalence, into these categories. The relation of two subspaces by an automorphism provides our notion of equivalence.

Definition 3.1.2. *Let $\mathfrak{s}$ and $\mathfrak{w}$ be subspaces of a Lie algebra $\mathfrak{g}$. Then $\mathfrak{s}$ and $\mathfrak{w}$ are said to be* ***equivalent****, denoted $\mathfrak{s} \sim \mathfrak{w}$, if there exists $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\varphi \cdot \mathfrak{s} = \mathfrak{w}$.*

Lemma 3.1.3. *The relation $\sim$ of Definition3.1.2, is an equivalence relation on the collection of all subspaces of the Lie algebra $\mathfrak{g}$.*

Proof. Suppose $\mathfrak{s}$, $\mathfrak{w}$ and $\mathfrak{u}$ are subspace of $\mathfrak{g}$. As the identity automorphism $\mathrm{id}_\mathfrak{g}$ is an element of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{id}_\mathfrak{g} \cdot \mathfrak{s} = \mathfrak{s}$, we have that $\mathfrak{s} \sim \mathfrak{s}$. Thus $\sim$ is reflexive.

If $\mathfrak{s} \sim \mathfrak{w}$, then there exists $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\varphi \cdot \mathfrak{s} = \mathfrak{w}$. This implies that $\varphi^{-1} \cdot (\varphi \cdot \mathfrak{s}) = \varphi^{-1} \cdot \mathfrak{w}$, and thus $\varphi^{-1} \cdot \mathfrak{w} = \mathfrak{s}$. Now, $\varphi^{-1} \in \mathrm{Aut}(\mathfrak{g})$ as $\varphi \in \mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{g})$ is a group, thus $\mathfrak{w} \sim \mathfrak{s}$. This proves symmetry.

If $\mathfrak{s} \sim \mathfrak{w}$ and $\mathfrak{w} \sim \mathfrak{u}$, then there is φ, $\psi \in \mathrm{Aut}(\mathfrak{g})$ such that $\varphi \cdot \mathfrak{s} = \mathfrak{w}$ and $\psi \cdot \mathfrak{w} = \mathfrak{u}$. Thus $\psi \cdot (\varphi \cdot \mathfrak{s}) = \psi \cdot \mathfrak{w} = \mathfrak{u}$. That is, $(\psi \circ \varphi) \cdot \mathfrak{s} = \mathfrak{u}$, were $\psi \circ \varphi \in \mathrm{Aut}(\mathfrak{g})$ as $\mathrm{Aut}(\mathfrak{g})$ is a group under composition, so $\mathfrak{s} \sim \mathfrak{u}$. This proves transitivity, therefore $\sim$ is an equivalence relation. □

Definition 3.1.4. *Let $\mathfrak{g}$ be a Lie algebra with $\mathfrak{s}$ and $\mathfrak{w}$ as subspaces, we define*

$$[\mathfrak{s}, \mathfrak{w}] = \mathrm{span}\left(\{[V, W] : V \in \mathfrak{s}, W \in \mathfrak{w}\}\right).$$

Definition 3.1.5. *Let $\mathfrak{g}$ be a Lie algebra with $V \in \mathfrak{g}$ and $\mathfrak{s}$ a subspace, we define*

$$[V, \mathfrak{s}] = \{[V, W] : W \in \mathfrak{s}\}.$$

It follows from the bilinearity of the Lie bracket on $\mathfrak{g}$ that the collection $[V, \mathfrak{s}]$ is a subspace of $\mathfrak{g}$.

Definition 3.1.6. *Let $\mathfrak{g}$ be a Lie algebra, with Lie bracket $[\cdot, \cdot]$, and $\mathfrak{s}$ be a vector subspace of $\mathfrak{g}$, then:*

- *$\mathfrak{s}$ is called a **subalgebra** if $[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{s}$,*
- *an **ideal** if $[\mathfrak{g}, \mathfrak{s}] \subseteq \mathfrak{s}$,*
- *a **fully characteristic ideal** if it is an ideal with $\varphi \cdot \mathfrak{s} = \mathfrak{s}$ for any automorphism φ of $\mathfrak{g}$,*
- ***generating** if the smallest subalgebra of $\mathfrak{g}$ containing $\mathfrak{s}$ is $\mathfrak{g}$ itself.*

Remark 3.1.7. *For a Lie algebra $\mathfrak{g}$, every fully characteristic ideal is an ideal and every ideal is a subalgebra.*

Consider a Lie algebra $\mathfrak{g}$, by the **subspace structure** of $\mathfrak{g}$ we refer to the classification of the subspaces of $\mathfrak{g}$, up to automorphism, into subalgebras, ideals, fully characteristic ideals, generating subspaces and general subspaces. For convenience we introduce the following convention for presenting the subspace structure of a Lie algebra:

SA: non-ideal subalgebras
I: non-fully characteristic ideals
FCI: fully characteristic ideals
S: subspaces that are neither generating nor subalgebras
Gen: generating subspaces.

The following lemmas verify that the position of a subspace in this classification is invariant up to equivalence. We note that a proper subalgebra of a Lie algebra cannot be generating. It thus follows that every proper subspace of a Lie algebra falls into exactly one of the above categories. That is, the only subspace of a Lie algebra that is both generating and a subalgebra is, trivially, the entire Lie algebra.

Lemma 3.1.8. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be equivalent subspaces of $\mathfrak{g}$. Then $\mathfrak{s}_1$ is a subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{s}_2$ is a subalgebra of $\mathfrak{g}$.*

Proof. Suppose $\mathfrak{s}_1$ is a subalgebra of $\mathfrak{g}$. As $\mathfrak{s}_1 \sim \mathfrak{s}_2$, there exists $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\varphi \cdot \mathfrak{s}_1 = \mathfrak{s}_2$.

If $W_\alpha, W_\beta \in \mathfrak{s}_2$, then $W_\alpha = \varphi \cdot V_\alpha$ and $W_\beta = \varphi \cdot V_\beta$ for some V_α, $V_\beta \in \mathfrak{s}_1$. We then have that

$$\begin{aligned}[W_\alpha, W_\beta] &= [\varphi \cdot V_\alpha, \varphi \cdot V_\beta] \\ &= \varphi \cdot [V_\alpha, V_\beta] \\ &= \varphi \cdot V_\kappa,\end{aligned}$$

where $V_\kappa = [V_\alpha, V_\beta] \in \mathfrak{s}_1$, as $\mathfrak{s}_1$ is a subalgebra. Thus $\varphi \cdot V_\kappa \in \mathfrak{s}_2$ and $\mathfrak{s}_2$ is a subalgebra.

The converse follows from the symmetry of the equivalence relation $\sim$. □

Lemma 3.1.9. *Let $\mathfrak{g}$ be a Lie algebra with equivalent subspaces $\mathfrak{s}_1$ and $\mathfrak{s}_2$, then $\mathfrak{s}_1$ is an ideal of $\mathfrak{g}$ if and only if $\mathfrak{s}_2$ is an ideal of $\mathfrak{g}$.*

Proof. Suppose $\mathfrak{s}_1$ is an ideal of $\mathfrak{g}$. As $\mathfrak{s}_1 \sim \mathfrak{s}_2$, there exists $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\varphi \cdot \mathfrak{s}_1 = \mathfrak{s}_2$.

Let $W_\alpha \in \mathfrak{s}_2$ and $Y \in \mathfrak{g}$, there exists $V_\alpha \in \mathfrak{s}_1$ such that $\varphi \cdot V_\alpha = W_\alpha$ and $X \in \mathfrak{g}$ such that $\varphi \cdot X = Y$. Now,

$$\begin{aligned}[W_\alpha, Y] &= [\varphi \cdot V_\alpha, \varphi \cdot X] \\ &= \varphi \cdot [V_\alpha, X] \\ &= \varphi \cdot V_\kappa,\end{aligned}$$

where $V_\kappa = [V_\alpha, X] \in \mathfrak{s}_1$ as $\mathfrak{s}_1$ is an ideal. Therefore, $\varphi \cdot V_\kappa \in \mathfrak{s}_2$ and $\mathfrak{s}_2$ is an ideal of $\mathfrak{g}$.

The converse follows from the symmetry of the equivalence relation $\sim$. □

Lemma 3.1.10. *If $\mathfrak{s}_1$ is a fully characteristic ideal of a Lie algebra $\mathfrak{g}$, then $\mathfrak{s}_1 \sim \mathfrak{s}_2$ if and only if $\mathfrak{s}_2$ is a fully characteristic ideal of $\mathfrak{g}$.*

Proof. Suppose $\mathfrak{s}_1 \sim \mathfrak{s}_2$ with $\varphi \cdot \mathfrak{s}_1 = \mathfrak{s}_2$ for some $\varphi \in \mathrm{Aut}(\mathfrak{g})$. As $\mathfrak{s}_1$ is a fully characteristic ideal, $\varphi \cdot \mathfrak{s}_1 = \mathfrak{s}_1$. Therefore, $\mathfrak{s}_1 = \mathfrak{s}_2$ and $\mathfrak{s}_2$ is a fully characteristic ideal. □

Definition 3.1.11. *Let $\mathfrak{s}$ be a subspace of the Lie algebra $\mathfrak{g}$. We recursively define the following sequence of subspaces,*

$$\mathfrak{s}^{(0)} = \mathfrak{s},$$

$$\mathfrak{s}^{(k)} = \mathfrak{s}^{(k-1)} + \left[\mathfrak{s}^{(k-1)}, \mathfrak{s}^{(k-1)}\right], \textit{for } k \geq 1.$$

That is, $\mathfrak{s}^{(k)}$ is the vector subspace of $\mathfrak{g}$ generated by $\mathfrak{s}^{(k-1)}$ and all Lie brackets of $\mathfrak{s}^{(k-1)}$.

The following lemma regarding generating subspaces is required before we can show that a generating subspace may only be equivalent to another generating subspace.

Lemma 3.1.12. *Let $\mathfrak{g}$ be a Lie algebra, then a subspace $\mathfrak{s}$ is generating if and only if*

$$\bigcup_{k=0}^{\infty} \mathfrak{s}^{(k)} = \mathfrak{g} = \mathfrak{s}^{(n)}$$

for some natural number n. Further, n is bounded above, with $n \leq \dim(\mathfrak{g})$.

Proof. Let $\mathfrak{s} \subseteq \mathfrak{g}$ be generating. For every natural number $k \geq 1$, $\mathfrak{s}^{(k-1)} \subseteq \mathfrak{s}^{(k)}$; therefore, $\dim\left(\mathfrak{s}^{(k-1)}\right) \leq \dim\left(\mathfrak{s}^{(k)}\right)$. If $\dim\left(\mathfrak{s}^{(k-1)}\right) < \dim\left(\mathfrak{s}^{(k)}\right)$ for every $k \geq 1$, then $\dim\left(\mathfrak{s}^{(k)}\right) \geq \dim\left(\mathfrak{s}^{(k-1)}\right) + 1$. Induction on k gives $\dim\left(\mathfrak{s}^{(k)}\right) \geq \dim\left(\mathfrak{s}\right) + k$ for $k \geq 1$. Therefore

$$\dim\left(\mathfrak{s}^{(n_g+1)}\right) \geq \dim\left(\mathfrak{s}\right) + (n_g + 1) \geq 0 + (n_g + 1) = (n_g + 1),$$

were $n_g = \dim(\mathfrak{g})$. This is a contradiction as $\mathfrak{s}^{(n_g+1)} \subseteq \mathfrak{g}$, and so $\dim(\mathfrak{s}^{(n_g+1)}) \leq \dim\left(\mathfrak{g}\right) = n_g < n_g + 1$. Thus, for some natural number $n \geq 1$, $\dim\left(\mathfrak{s}^{(n-1)}\right) = \dim\left(\mathfrak{s}^{(n)}\right)$. We then have that $\mathfrak{s}^{(n-1)} = \mathfrak{s}^{(n)}$ and thus $\mathfrak{s}^{(n-1)}$ is a subalgebra of $\mathfrak{g}$. Now,

$$\begin{aligned}
\bigcup_{k=0}^{\infty} \mathfrak{s}^{(k)} &= \bigcup_{k=0}^{n} \mathfrak{s}^{(k)} \cup \bigcup_{k=n+1}^{\infty} \mathfrak{s}^{(k)} \\
&= \bigcup_{k=0}^{n} \mathfrak{s}^{(k)} \cup \bigcup_{k=n+1}^{\infty} \mathfrak{s}^{(n)} \\
&= \bigcup_{k=0}^{n} \mathfrak{s}^{(k)} \cup \mathfrak{s}^{(n)} \\
&= \bigcup_{k=0}^{n} \mathfrak{s}^{(k)}.
\end{aligned}$$

However, as $\mathfrak{s}^{(k-1)} \subseteq \mathfrak{s}^{(k)}$ for every $k \in \mathbb{N}$, we have

$$\bigcup_{k=0}^{\infty} \mathfrak{s}^{(k)} = \bigcup_{k=0}^{n} \mathfrak{s}^{(k)} = \mathfrak{s}^{(n)}.$$

To prove $\mathfrak{s}^{(n)} = \mathfrak{g}$ we observe that for every $k \in \mathbb{N}$, $\mathfrak{s}^{(k)}$ is a subspace of $\mathfrak{g}$ and that $\bigcup_{k=0}^{\infty} \mathfrak{s}^{(k)}$ is by definition closed under Lie brackets. Suppose

$$\mathfrak{g} \not\subseteq \bigcup_{k=0}^{\infty} \mathfrak{s}^{(k)} = \mathfrak{s}^{(n)},$$

as $\mathfrak{s}^{(n)}$ is closed under Lie brackets it is a subalgebra of $\mathfrak{g}$. That is, $\mathfrak{s}^{(n)}$ is a proper subalgebra of $\mathfrak{g}$ containing $\mathfrak{s}$. This contradicts the fact that $\mathfrak{s}$ is generating. Thus

$$\mathfrak{g} \subseteq \bigcup_{k=0}^{\infty} \mathfrak{s}^{(k)} = \mathfrak{s}^{(n)}.$$

As, clearly, $\mathfrak{s}^{(n)} \subseteq \mathfrak{g}$ we have that $\mathfrak{g} = \mathfrak{s}^{(n)}$.

Conversely, if the above relation holds, then $\mathfrak{g} = \mathfrak{s}^{(n)}$ is a subalgebra containing $\mathfrak{s}$. Suppose there exists a subalgebra $\mathfrak{w}$ such that $\mathfrak{w} \subsetneq \mathfrak{g}$ with $\mathfrak{s} \subseteq \mathfrak{w}$. As $\mathfrak{w}$ is closed under Lie brackets and $\mathfrak{s} \subseteq \mathfrak{w}$,

$$s^{(1)} = \mathfrak{s} + [\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{w}.$$

An induction argument on k gives, $\mathfrak{s}^{(k)} \subseteq \mathfrak{w}$ for every $k \in \mathbb{N}$. Therefore,

$$\mathfrak{g} = \mathfrak{s}^{(n)} = \bigcup_{k=0}^{\infty} \mathfrak{s}^{(k)} \subseteq \mathfrak{w}.$$

That is, $\mathfrak{g} \subseteq \mathfrak{w}$ and thus $\mathfrak{g} = \mathfrak{w}$. This contradicts the supposition that $\mathfrak{w} \subsetneq \mathfrak{g}$, therefore no such subalgebra $\mathfrak{w}$ exists and thus $\mathfrak{s}$ is generating.

As $n \in \mathbb{N}$ is determined by the property $\dim(\mathfrak{s}^{(n-1)}) = \dim(\mathfrak{s}^{(n)})$ and consecutive subspaces $\mathfrak{s}^{(k-1)}$ and $\mathfrak{s}^{(k)}$ that do not satisfy this property differ in dimension by at least 1; with $\mathfrak{s}^{(0)}$ having dimension at least 1, it follow that the sequence of subspaces

$$\mathfrak{s}^{(0)} \subseteq \mathfrak{s}^{(1)} \subseteq \cdots \subseteq \mathfrak{s}^{(k-1)} \subseteq \mathfrak{s}^{(k)} \subseteq \cdots$$

has at most $\dim(\mathfrak{g}) - 1$ distinct subspaces. That is, subspaces with $\dim(\mathfrak{s}^{(k)}) = k + 1$ for $k < \dim(\mathfrak{g})$ and $\dim(\mathfrak{s}^{(k)}) = \dim(\mathfrak{g})$ for $k \geq \dim(\mathfrak{g})$. Therefore, n can be at most $\dim(\mathfrak{g})$. □

Lemma 3.1.13. *Let $\mathfrak{g}$ be a Lie algebra with equivalent subspaces $\mathfrak{s}$ and $\mathfrak{w}$, then $\mathfrak{s}$ is generating if and only if $\mathfrak{w}$ is generating.*

Proof. Suppose $\mathfrak{s}$ is generating and $\varphi \cdot \mathfrak{s} = \mathfrak{w}$ for some $\varphi \in \mathrm{Aut}(g)$, then $\varphi \cdot \mathfrak{s}^{(0)} = \mathfrak{w}^{(0)}$.

Suppose $\varphi \cdot \mathfrak{s}^{(k)} = \mathfrak{w}^{(k)}$ for some $k \geq 0$, then

$$\begin{aligned}
\mathfrak{s}^{(k+1)} &= \mathfrak{s}^{(k)} + \left[\mathfrak{s}^{(k)}, \mathfrak{s}^{(k)}\right] \\
\varphi \cdot \mathfrak{s}^{(k+1)} &= \varphi \cdot \left(\mathfrak{s}^{(k)} + \left[\mathfrak{s}^{(k)}, \mathfrak{s}^{(k)}\right]\right) \\
&= \varphi \cdot \mathfrak{s}^{(k)} + \varphi \cdot \left[\mathfrak{s}^{(k)}, \mathfrak{s}^{(k)}\right] \\
&= \varphi \cdot \mathfrak{s}^{(k)} + \left[\varphi \cdot \mathfrak{s}^{(k)}, \varphi \cdot \mathfrak{s}^{(k)}\right] \\
&= \mathfrak{w}^{(k)} + \left[\mathfrak{w}^{(k)}, \mathfrak{w}^{(k)}\right] \\
&= \mathfrak{w}^{(k+1)}.
\end{aligned}$$

Thus an induction argument on k implies that

$$\varphi \cdot \mathfrak{s}^{(k)} = \mathfrak{w}^{(k)}$$

for every $k \geq 0$. As, for some natural number n, $\mathfrak{g} = \mathfrak{s}^{(n)}$ we have that $\mathfrak{g} = \varphi \cdot \mathfrak{g} = \varphi \cdot \mathfrak{s}^{(n)} = \mathfrak{w}^{(n)}$. That is, $\mathfrak{w}^{(n)} = \mathfrak{g}$ and thus Lemma 3.1.12 implies that $\mathfrak{w}$ is generating. The converse follows by the symmetry of the relation $\sim$. □

3.2 Invariants

Certain properties of a subspace $\mathfrak{s} \subseteq \mathfrak{t}$ remain invariant under automorphisms of $\mathfrak{t}$. A simple invariant is the dimension of a subspace. That is, given a subspace $\mathfrak{s}$ and $\varphi \in \mathrm{Aut}(\mathfrak{t})$ we have $\dim(\mathfrak{s}) = \dim(\varphi \cdot \mathfrak{s})$. Another invariant is the the dimension of the intersection of a subspace with a fully characteristic ideal. In light of this, we establish the fully characteristic ideals of $\mathfrak{t}$ ahead of the rest of the subspace classification. An additional invariant, particular to the Lie algebra $\mathfrak{t}$, is established. These scalar invariants are used in the next section to help classify the subspaces of $\mathfrak{t}$. It is later shown, in the last section of this chapter, that the established invariants fully characterize the subspaces of $\mathfrak{t}$ up to equivalence.

Lemma 3.2.1. *The centre* $\mathrm{Z}(\mathfrak{g})$ *of a Lie algebra* $\mathfrak{g}$ *is a fully characteristic ideal.*

Proof. For any elements $C \in \mathrm{Z}(\mathfrak{g})$ and $V \in \mathfrak{g}$ we have that $[C, V] = \mathbf{0} \in \mathfrak{g}$, thus $[\mathfrak{g}, \mathrm{Z}(\mathfrak{g})] = \{\mathbf{0}\} \subseteq \mathrm{Z}(\mathfrak{g})$, and thus $\mathrm{Z}(\mathfrak{g})$ is an ideal. Suppose $\varphi \in \mathrm{Aut}(\mathfrak{g})$ and $\mathfrak{g}$ has n-dimensional centre $\mathrm{Z}(\mathfrak{g}) = \langle B_1, ..., B_n \rangle$, then $\varphi \cdot \mathrm{Z}(\mathfrak{g}) = \langle \varphi \cdot B_1, ..., \varphi \cdot B_n \rangle$. Lemma B.2.8 implies that $\varphi \cdot B_i \in \mathrm{Z}(\mathfrak{g})$ for $i = 1, ..., n$. Therefore, as φ is an automorphism, $\varphi \cdot \mathrm{Z}(\mathfrak{g})$ is an n-dimensional subspace of $\mathfrak{g}$ contained in $\mathrm{Z}(\mathfrak{g})$, and thus $\varphi \cdot \mathrm{Z}(\mathfrak{g}) = \mathrm{Z}(\mathfrak{g})$ as required. □

Lemma 3.2.2. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{w}$ *be a fully characteristic ideal of* $\mathfrak{g}$*. Then for any subspace* $\mathfrak{s} \subseteq \mathfrak{g}$ *and any automorphism* $\varphi \in \mathrm{Aut}(\mathfrak{g})$*,*

$$\dim(\mathfrak{s} \cap \mathfrak{w}) = \dim((\varphi \cdot \mathfrak{s}) \cap \mathfrak{w}).$$

Proof. As $\mathfrak{w}$ is a fully characteristic ideal, we have that

$$\begin{aligned}(\varphi \cdot \mathfrak{s}) \cap \mathfrak{w} &= (\varphi \cdot \mathfrak{s}) \cap (\varphi \cdot \mathfrak{w}) \\ &= \varphi \cdot (\mathfrak{s} \cap \mathfrak{w}).\end{aligned}$$

Further, using the fact the φ is an automorphism, we get

$$\begin{aligned}\dim((\varphi \cdot \mathfrak{s}) \cap \mathfrak{w}) &= \dim(\varphi \cdot (\mathfrak{s} \cap \mathfrak{w})) \\ &= \dim(\mathfrak{s} \cap \mathfrak{w}).\end{aligned}$$

□

Corollary 3.2.3. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathrm{Z}(\mathfrak{g})$ *its centre. Then for any subspace* $\mathfrak{s} \subseteq \mathfrak{g}$ *and any automorphism* $\varphi \in \mathrm{Aut}(\mathfrak{g})$*,*

$$\dim(\mathfrak{s} \cap \mathrm{Z}(\mathfrak{g})) = \dim(\varphi \cdot \mathfrak{s} \cap \mathrm{Z}(\mathfrak{g})).$$

Lemma 3.2.4. *The subspace* $\langle J, K, L, M \rangle$ *is a fully characteristic ideal of* $\mathfrak{t}$.

Proof. Let $V \in \langle J, K, L, M \rangle$ and $W \in \mathfrak{t}$, then using Lemma 2.2.1

$$\begin{aligned}[V, W] &= (0 \cdot w_2 - v_2 w_1) L + (0 \cdot w_3 - v_3 w_1) M \\ &= -v_2 w_1 L - v_3 w_1 M \in \langle J, K, L, M \rangle.\end{aligned}$$

Thus $\langle J, K, L, M \rangle$ is an ideal. It is clear from the matrix representation of $\varphi \in \mathrm{Aut}(\mathfrak{t})$ in Proposition 3.1.1 that $\varphi \cdot \langle J, K, L, M \rangle = \langle J, K, L, M \rangle$. □

Corollary 3.2.5. *For any subspace $\mathfrak{s} \subseteq \mathfrak{t}$ and any automorphism $\varphi \in \mathrm{Aut}(\mathfrak{t})$,*

$$\dim\left(\mathfrak{s} \cap \langle J, K, L, M\rangle\right) = \dim\left(\varphi \cdot \mathfrak{s} \cap \langle J, K, L, M\rangle\right).$$

Proof. Follows from Lemmas 3.2.2 and 3.2.4. □

We now move towards establishing a scalar invariant dependent on the structure of the Lie algebra $\mathfrak{t}$, particularly with regards to the fully characteristic ideals $\mathfrak{c} = \langle J, K, L, M\rangle$ and its centre $\mathfrak{z} = \langle L, M\rangle$.

Lemma 3.2.6. *If $\mathfrak{s}$ is a subspace of the Lie algebra $\mathfrak{t}$ and $\varphi \in \mathrm{Aut}(\mathfrak{t})$, then*

$$\dim\left(\mathfrak{s} \cap \mathfrak{z} \cap [I, \mathfrak{s}]\right) = \dim\left(\varphi \cdot \mathfrak{s} \cap \mathfrak{z} \cap [\varphi \cdot I, \varphi \cdot \mathfrak{s}]\right).$$

Proof. Suppose $\mathfrak{s}$ is a subspace of the Lie algebra $\mathfrak{t}$ and $\varphi \in \mathrm{Aut}(\mathfrak{t})$. Using the facts that φ preserves Lie brackets and $\mathfrak{z}$ is a fully characteristic ideal, we have:

$$\begin{aligned}
\dim\left(\varphi \cdot \mathfrak{s} \cap \mathfrak{z} \cap [\varphi \cdot I, \varphi \cdot \mathfrak{s}]\right) &= \dim\left(\varphi \cdot \mathfrak{s} \cap \varphi \cdot \mathfrak{z} \cap \varphi \cdot [I, \mathfrak{s}]\right) \\
&= \dim\left(\varphi \cdot (\mathfrak{s} \cap \mathfrak{z}) \cap \varphi \cdot [I, \mathfrak{s}]\right) \\
&= \dim\left(\varphi \cdot (\mathfrak{s} \cap \mathfrak{z} \cap [I, \mathfrak{s}])\right).
\end{aligned}$$

As φ is an invertible linear map, it preserves dimension and thus

$$\dim\left(\varphi \cdot (\mathfrak{s} \cap \mathfrak{z} \cap [I, \mathfrak{s}])\right) = \dim\left(\mathfrak{s} \cap \mathfrak{z} \cap [I, \mathfrak{s}]\right).$$

We thus have the required

$$\dim\left(\varphi \cdot \mathfrak{s} \cap \mathfrak{z} \cap [\varphi \cdot I, \varphi \cdot \mathfrak{s}]\right) = \dim\left(\mathfrak{s} \cap \mathfrak{z} \cap [I, \mathfrak{s}]\right).$$

□

Corollary 3.2.7. *If $\mathfrak{s}$ is a subspace of $\mathfrak{t}$ and $\varphi \in \mathrm{Aut}(\mathfrak{t})$ is such that $\varphi \cdot I = \lambda I$, $\lambda \neq 0$, then*

$$\dim\left(\mathfrak{s} \cap \mathfrak{z} \cap [I, \mathfrak{s}]\right) = \dim\left(\varphi \cdot \mathfrak{s} \cap \mathfrak{z} \cap [I, \varphi \cdot \mathfrak{s}]\right).$$

Proof. This follows from the fact that $[\varphi \cdot I, \mathfrak{s}] = [\lambda I, \mathfrak{s}] = \lambda[I, \mathfrak{s}] = [I, \mathfrak{s}]$. This results from $[I, \mathfrak{s}]$ being a subspace of $\mathfrak{t}$ (see Definition 3.1.5) and thus being invariant with respect to nonzero scaling. □

Lemma 3.2.8. *Suppose $\mathfrak{s}$ is a subspace of $\mathfrak{t}$ with $\mathfrak{s} \subseteq \langle J, K, L, M\rangle$. For any $\varphi \in \mathrm{Aut}(\mathfrak{t})$,there exists $\psi \in \mathrm{Aut}(\mathfrak{t})$ with $\psi|_{\mathfrak{s}} = \varphi|_{\mathfrak{s}}$ and $\psi \cdot I = \lambda I$, $\lambda \neq 0$.*

Proof. Suppose $\varphi \in \mathrm{Aut}(\mathfrak{t})$ and $\mathfrak{s} \subseteq \langle J, K, L, M\rangle$. We identify φ with its matrix representation with respect to the ordered basis (I, J, K, L, M), given by

$$\varphi = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_1 & 0 & 0 \\ i_3 & j_3 & k_2 & 0 & 0 \\ i_4 & j_4 & k_3 & i_1 j_2 & i_1 k_2 \\ i_5 & j_5 & k_4 & i_1 j_3 & i_1 k_3 \end{bmatrix},$$

in accordance with Proposition 3.1.1. The automorphism $\psi \in \mathrm{Aut}(\mathfrak{t})$ given by

$$\psi = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ 0 & j_2 & k_1 & 0 & 0 \\ 0 & j_3 & k_2 & 0 & 0 \\ 0 & j_4 & k_3 & i_1 j_2 & i_1 k_2 \\ 0 & j_5 & k_4 & i_1 j_3 & i_1 k_3 \end{bmatrix},$$

also with respect to the ordered basis (I, J, K, L, M) of $\mathfrak{t}$, coincides with φ on $\langle J, K, L, M\rangle$ and thus on $\mathfrak{s}$. We also have that $\psi \cdot I = i_1 I$. That is, $\psi \in \mathrm{Aut}(\mathfrak{t})$ with $\psi|_{\mathfrak{s}} = \varphi|_{\mathfrak{s}}$ and $\psi \cdot I = \lambda I$, were $\lambda = i_1 \neq 0$, by Proposition 3.1.1. □

Proposition 3.2.9. *Let $\mathfrak{s}$ be a subspace of $\mathfrak{t}$ and $\mathfrak{c} = \langle J, K, L, M\rangle$. For any $\varphi \in \mathrm{Aut}(\mathfrak{t})$ we have*

$$\dim(\mathfrak{s} \cap \mathfrak{z} \cap [I, \mathfrak{s} \cap \mathfrak{c}]) = \dim(\varphi \cdot \mathfrak{s} \cap \mathfrak{z} \cap [I, \varphi \cdot \mathfrak{s} \cap \mathfrak{c}]).$$

Proof. As $\mathfrak{z} \subseteq \mathfrak{c}$ and $\mathfrak{c}$ is a fully characteristic ideal of $\mathfrak{t}$,

$$\begin{aligned} \dim(\varphi \cdot \mathfrak{s} \cap \mathfrak{z} \cap [I, \varphi \cdot \mathfrak{s} \cap \mathfrak{c}]) &= \dim((\varphi \cdot \mathfrak{s} \cap \mathfrak{c}) \cap \mathfrak{z} \cap [I, \varphi \cdot \mathfrak{s} \cap \mathfrak{c}]) \\ &= \dim((\varphi \cdot \mathfrak{s} \cap \varphi \cdot \mathfrak{c}) \cap \mathfrak{z} \cap [I, (\varphi \cdot \mathfrak{s} \cap \varphi \cdot \mathfrak{c})]). \end{aligned}$$

Now, $\mathfrak{s} \cap \mathfrak{c}$ is a subspaces of $\mathfrak{t}$ that is contained in $\mathfrak{c} = \langle J, K, L, M\rangle$. Lemma 3.2.8 gives the existence of $\psi \in \mathrm{Aut}(\mathfrak{t})$ with $\psi|_{\mathfrak{s}\cap\mathfrak{c}} = \varphi|_{\mathfrak{s}\cap\mathfrak{c}}$ — thus $\psi\cdot(\mathfrak{s}\cap\mathfrak{c}) = \varphi\cdot(\mathfrak{s}\cap\mathfrak{c})$ and $(\psi\cdot\mathfrak{s}\cap\psi\cdot\mathfrak{c}) = (\varphi\cdot\mathfrak{s}\cap\varphi\cdot\mathfrak{c})$ — and $\psi \cdot I = \lambda I$ for some scalar $\lambda \neq 0$. This gives,

$$\begin{aligned} &\dim((\varphi \cdot \mathfrak{s} \cap \varphi \cdot \mathfrak{c}) \cap \mathfrak{z} \cap [I, (\varphi \cdot \mathfrak{s} \cap \varphi \cdot \mathfrak{c})]) \\ =&\dim((\psi \cdot \mathfrak{s} \cap \psi \cdot \mathfrak{c}) \cap \mathfrak{z} \cap [I, (\psi \cdot \mathfrak{s} \cap \psi \cdot \mathfrak{c})]). \end{aligned}$$

Corollary 3.2.7 implies that

$$\begin{aligned} &\dim((\psi \cdot \mathfrak{s} \cap \psi \cdot \mathfrak{c}) \cap \mathfrak{z} \cap [I, (\psi \cdot \mathfrak{s} \cap \psi \cdot \mathfrak{c})]) \\ =&\dim((\mathfrak{s} \cap \mathfrak{c}) \cap \mathfrak{z} \cap [I, (\mathfrak{s} \cap \mathfrak{c})]) \\ =&\dim(\mathfrak{s} \cap (\mathfrak{c} \cap \mathfrak{z}) \cap [I, \mathfrak{s} \cap \mathfrak{c}]) \\ =&\dim(\mathfrak{s} \cap \mathfrak{z} \cap [I, \mathfrak{s} \cap \mathfrak{c}]), \end{aligned}$$

proving the result. □

Proposition 3.2.10. *Let $\mathfrak{s}$ and $\mathfrak{w}$ be subspaces of $\mathfrak{t}$ and $\mathfrak{c} = \langle J, K, L, M\rangle$. If subspaces $\mathfrak{s}$ and $\mathfrak{w}$ are equivalent then*

$$\begin{aligned} \dim(\mathfrak{s}) &= \dim(\mathfrak{w}), \\ \dim(\mathfrak{s} \cap \mathfrak{z}) &= \dim(\mathfrak{w} \cap \mathfrak{z}), \\ \dim(\mathfrak{s} \cap \mathfrak{c}) &= \dim(\mathfrak{w} \cap \mathfrak{c}) \textit{ and} \\ \dim(\mathfrak{s} \cap \mathfrak{z} \cap [I, \mathfrak{s} \cap \mathfrak{c}]) &= \dim(\mathfrak{w} \cap \mathfrak{z} \cap [I, \mathfrak{w} \cap \mathfrak{c}]). \end{aligned}$$

Proof. For any subspace $\mathfrak{u} \subseteq \mathfrak{t}$, define the map

$$S(\mathfrak{u}) = (\dim(\mathfrak{u}), \dim(\mathfrak{u} \cap \mathfrak{z}), \dim(\mathfrak{u} \cap \mathfrak{c}), \dim(\mathfrak{u} \cap \mathfrak{z} \cap [I, \mathfrak{u} \cap \mathfrak{c}])) \in \mathbb{R}^4.$$

As the components of $S(\cdot)$ are invariant under automorphism, $S(\cdot)$ itself is invariant under automorphisms. We prove the equivalent statement that if $\mathfrak{s} \sim \mathfrak{w}$ then $S(\mathfrak{s}) = S(\mathfrak{w})$.

Suppose $\mathfrak{s}$ and $\mathfrak{w}$ are equivalent subspaces of $\mathfrak{t}$, then there exists $\varphi \in \mathrm{Aut}(\mathfrak{t})$ with $\varphi \cdot \mathfrak{s} = \mathfrak{w}$. The fact that automorphisms preserve dimension, Corollary 3.2.3, Corollary 3.2.5 and Proposition 3.2.9 imply(componentwise) that $S(\mathfrak{s}) = S(\mathfrak{w})$. □

Lemma 3.2.11. *Let $P_I : \mathfrak{t} \longrightarrow \mathbb{R}$ be the map defined by*

$$P_I : V = v_1 I + v_2 J + v_3 K + v_4 L + v_5 M \mapsto v_1.$$

For any $\mathfrak{s} \subseteq \mathfrak{t}$ we have that

$$\dim(P_I \cdot \mathfrak{s}) = \dim(\mathfrak{s}) - \dim(\mathfrak{s} \cap \langle J, K, L, M\rangle).$$

Proof. Suppose $\mathfrak{s}$ is a subspace of $\mathfrak{t}$. Now,

$$\mathfrak{s} \cap \langle J, K, L, M\rangle = \{V \in \mathfrak{s} : V = v_1 \cdot I + v_2 J + v_3 K + v_4 L + v_5 M, v_1 = 0\}.$$

If $\mathfrak{s} \subseteq \langle J, K, L, M\rangle$, then $\mathfrak{s} = \mathfrak{s} \cap \langle J, K, L, M\rangle$ and thus $P_I \cdot \mathfrak{s} = \{0\}$. That is,

$$\begin{aligned} \dim(\mathfrak{s}) &= \dim(\mathfrak{s} \cap \langle J, K, L, M\rangle) \\ 0 &= \dim(\mathfrak{s}) - \dim(\mathfrak{s} \cap \langle J, K, L, M\rangle) \\ \dim(P_I \cdot \mathfrak{s}) &= \dim(\mathfrak{s}) - \dim(\mathfrak{s} \cap \langle J, K, L, M\rangle). \end{aligned}$$

If $\mathfrak{s} \nsubseteq \langle J, K, L, M\rangle$, then there exists $W = w_1 I + w_2 J + w_3 K + w_4 L + w_5 M \in \mathfrak{s}$ with $w_1 \neq 0$. As $P_I \cdot W \neq 0$ and the range of the linear map P_I is a subspace of $\mathbb{R}$, it follows that $1 \leq \dim(P_I \cdot \mathfrak{s})$. Now, $\dim(P_I \cdot \mathfrak{s}) = 1$ as it is a subspace of $\mathbb{R}$. Let $\mathfrak{s} = \langle W, X_1, ..., X_n\rangle$ with $X_i = \begin{bmatrix} x_{i1} & x_{i2} & x_{i3} & x_{i4} & x_{i5}\end{bmatrix}^\top$, then $\mathfrak{s} = \langle W, (X_1 - \frac{x_{11}}{w_1} W), ..., (X_n - \frac{x_{n1}}{w_1} W)\rangle$ with $(X_i - \frac{x_{i1}}{w_1} W) \in \langle J, K, L, M\rangle$ for $i = 1, 2, ..., n$. It follows that $\dim(\mathfrak{s}) = n + 1$ and $\dim(\mathfrak{s} \cap \langle J, K, L, M\rangle) = n$ giving

$$\dim(P_I \cdot \mathfrak{s}) = \dim(\mathfrak{s}) - \dim(\mathfrak{s} \cap \langle J, K, L, M\rangle).$$

□

Corollary 3.2.12. *If $\mathfrak{s}$ is a subspace of $\mathfrak{t}$ and $\varphi \in \mathrm{Aut}(\mathfrak{t})$, then*

$$\dim(P_I \cdot \mathfrak{s}) = \dim(P_I \cdot (\varphi \cdot \mathfrak{s})).$$

Proof. Follows from the fact that $\dim(P_I \cdot \mathfrak{s})$ is a function of $\dim(\mathfrak{s})$ and $\dim(\mathfrak{s} \cap \langle J, K, L, M\rangle)$ which are invariant under automorphisms of $\mathfrak{t}$ — as seen in Proposition 3.2.10. □

3.3 Subspace classification

We now proceed with the computation of the subspace structure of the Lie algebra $\mathfrak{t}$. We use the ordered basis (I, J, K, L, M) of $\mathfrak{t}$ in our computations and denote the column vector $\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5\end{bmatrix}^\top$ by the corresponding upper case V, unless specified otherwise.

3.3.1 One-dimensional subspace structure

Proposition 3.3.1. *Every one-dimensional subspace of* $\mathfrak{t}$ *is equivalent to exactly one of the subspaces*

$$\langle I \rangle , \langle J \rangle \ \textit{or} \ \langle L \rangle .$$

Proof. Suppose $\langle V \rangle$ is a one-dimensional subspace of $\mathfrak{t}$. We have that $\dim(\langle V \rangle \cap \mathfrak{z}) \in \{0,1\}$ and $\dim(\langle V \rangle \cap \langle J,K,L,M \rangle) \in \{0,1\}$. If $\dim(\langle V \rangle \cap \mathfrak{z}) = 0$ and $\dim(\langle V \rangle \cap \langle J,K,L,M \rangle) = 0$, then $V \notin \langle J,K,L,M \rangle$. Therefore $v_1 \neq 0$ and

$$\varphi = \begin{bmatrix} v_1 & 0 & 0 & 0 & 0 \\ v_2 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 0 & 0 \\ v_4 & 0 & 0 & v_1 & 0 \\ v_5 & 0 & 0 & 0 & v_1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle I \rangle = \langle V \rangle$.

If $\dim(\langle V \rangle \cap \mathfrak{z}) = 0$ and $\dim(\langle V \rangle \cap \langle J,K,L,M \rangle) = 1$, then $V \in \langle J,K,L,M \rangle$ and $V \notin \langle L,M \rangle$. Therefore $v_1 = 0$ and $v_2^2 + v_3^2 \neq 0$, so

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & v_2 & v_3 & 0 & 0 \\ 0 & v_3 & -v_2 & 0 & 0 \\ 0 & v_4 & 0 & v_2 & v_3 \\ 0 & v_5 & 0 & v_3 & -v_2 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle J \rangle = \langle V \rangle$.

If $\dim(\langle V \rangle \cap \mathfrak{z}) = 1$, then $V \in \mathfrak{z}$. Therefore, $v_1 = v_2 = v_3 = 0$ with $v_4^2 + v_5^2 \neq 0$ and

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & v_4 & v_5 & 0 & 0 \\ 0 & v_5 & -v_4 & 0 & 0 \\ 0 & 0 & 0 & v_4 & v_5 \\ 0 & 0 & 0 & v_5 & -v_4 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle L \rangle = \langle V \rangle$. Therefore every one-dimensional subspace $\langle V \rangle$ of $\mathfrak{t}$ is equivalent to at least one of the stated subspaces. It is left to show that $\langle V \rangle$ is equivalent to exactly one of these subspaces. For this we show that the stated subspaces are mutually nonequivalent.

Now, $\dim(\langle I \rangle \cap \mathfrak{z}) = 0$, $\dim(\langle J \rangle \cap \mathfrak{z}) = 0$ and $\dim(\langle L \rangle \cap \mathfrak{z}) = 1$, so by Corollary 3.2.3 $\langle I \rangle \nsim \langle L \rangle$ and $\langle J \rangle \nsim \langle L \rangle$. Further $\dim(\langle I \rangle \cap \langle J,K,L,M \rangle) = 0$ and $\dim(\langle J \rangle \cap \langle J,K,L,M \rangle) = 1$, therefore Corollary 3.2.5 implies $\langle I \rangle \nsim \langle J \rangle$. □

Remark 3.3.2. *We thus have the following equivalence classes:*

$$\begin{aligned} [\langle I \rangle] &= \{\langle V \rangle : \dim(\langle V \rangle \cap \mathfrak{z}) = 0 \ \textit{and} \ \dim(\langle V \rangle \cap \langle J,K,L,M \rangle) = 0\} \\ [\langle J \rangle] &= \{\langle V \rangle : \dim(\langle V \rangle \cap \mathfrak{z}) = 0 \ \textit{and} \ \dim(\langle V \rangle \cap \langle J,K,L,M \rangle) = 1\} \\ [\langle L \rangle] &= \{\langle V \rangle : \dim(\langle V \rangle \cap \mathfrak{z}) = 1\} . \end{aligned}$$

Equivalently,

$$[\langle I\rangle] = \{\langle V\rangle : v_1 \neq 0\}$$
$$[\langle J\rangle] = \left\{\langle V\rangle : v_1 = 0 \text{ and } v_2^2 + v_3^2 \neq 0\right\}$$
$$[\langle L\rangle] = \left\{\langle V\rangle : v_1 = v_2 = v_3 = 0 \text{ and } v_4^2 + v_5^2 \neq 0\right\}.$$

Corollary 3.3.3. *Given $V \in \mathfrak{t}$ and $\varphi \in \mathrm{Aut}(\mathfrak{t})$ with $\varphi \cdot V = W$, then*

1. *$v_1 \neq 0$ if and only if $w_1 \neq 0$,*
2. *$v_1 = 0$ and $v_2^2 + v_3^2 \neq 0$ if and only if $w_1 = 0$ and $w_2^2 + w_3^2 \neq 0$ and*
3. *$v_1 = v_2 = v_3 = 0$ and $v_4^2 + v_5^2 \neq 0$ if and only if $w_1 = w_2 = w_3 = 0$ and $w_4^2 + w_5^2 \neq 0$.*

Proof. Follows from Proposition 3.3.1, the closure of the equivalence classes $[\langle I\rangle], [\langle J\rangle]$ and $[\langle L\rangle]$ under automorphisms and their defining conditions. □

Proposition 3.3.4. *The one-dimensional subspace structure of $\mathfrak{t}$ is given by*

SA: $\langle I\rangle, \langle J\rangle$
I: $\langle L\rangle$.

Proof. We prove that every one-dimensional subspace $\langle V\rangle$ is a subalgebra of $\mathfrak{t}$. Suppose $\langle V\rangle$ is a one-dimensional subspace of $\mathfrak{t}$ and $V_1 = aV$, $V_2 = bV$ are vectors in $\langle V\rangle$ with $a,b \in \mathbb{R}$. Then, bilinearity and anti-symmetry of the Lie bracket give

$$[V_1, V_2] = [aV, bV] = ab\,[V, V] = -ab\,[V, V],$$

thus,

$$2ab\,[V, V] = \bar{0}.$$

As this holds for arbitrary $a, b \in \mathbb{R}$, it follows that $[V, V] = \bar{0}$ and

$$[V_1, V_2] = \bar{0} \in \langle V\rangle.$$

Thus $\langle V\rangle$ is closed under the Lie bracket operation and is thus a subalgebra of $\mathfrak{t}$.

The subspace $\langle I\rangle$ is not an ideal as $I \in \langle I\rangle$, $J \in \mathfrak{t}$, but $[I, J] = L \notin \langle I\rangle$. The subspace $\langle J\rangle$ is not an ideal as $J \in \langle J\rangle$, $I \in \mathfrak{t}$, but $[I, J] = L \notin \langle J\rangle$. However, $\langle L\rangle$ is an ideal as given any $X = aL \in \langle L\rangle$ and $V \in \mathfrak{t}$, we have

$$[X, V] = [aL, V] = a\,[L, V] = \bar{0} \in \langle L\rangle$$

from the fact that L is a central element.

The ideal $\langle L\rangle$ is not a fully characteristic ideal as $\varphi\cdot\langle L\rangle = \langle M\rangle \neq \langle L\rangle$ where φ is the automorphism given by

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

□

3.3.2 Two-dimensional subspace structure

Proposition 3.3.5. *Every two-dimensional subspace of* $\mathfrak{t}$ *is equivalent to exactly one of the following subspaces*

$$\langle I, J\rangle,\ \langle I, L\rangle,\ \langle J, K\rangle,\ \langle J, L\rangle,\ \langle J, M\rangle \text{ or } \langle L, M\rangle.$$

Proof. Let $\langle V, W\rangle$ be a two-dimensional subspace of $\mathfrak{t}$.

Case 1: Suppose $\dim(\langle V, W\rangle \cap \mathfrak{z}) = 0$, then $\langle V, W\rangle$ contains no central elements. If $\dim(\langle V, W\rangle \cap \langle J, K, L, M\rangle) = 1$, then we may assume $W \in \langle J, K, L, M\rangle$ and $V \notin \langle J, K, L, M\rangle$. That is, $w_1 = 0$, $w_2^2 + w_3^2 \neq 0$ and $v_1 \neq 0$. We therefore have that

$$\varphi = \begin{bmatrix} v_1 & 0 & 0 & 0 & 0\\ v_2 & w_2 & w_3 & 0 & 0\\ v_3 & w_3 & -w_2 & 0 & 0\\ v_4 & w_4 & 0 & v_1w_2 & v_1w_3\\ v_5 & w_5 & 0 & v_1w_3 & -v_1w_2 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi\cdot\langle I, J\rangle = \langle V, W\rangle$. Thus $\langle V, W\rangle \sim \langle I, J\rangle$.

If $\dim(\langle V, W\rangle \cap \langle J, K, L, M\rangle) = 2$, then $\langle V, W\rangle \subseteq \langle J, K, L, M\rangle$ and so $v_1 = w_1 = 0$. As V and W are non-central elements of $\mathfrak{t}$, we have that the vectors I, V, W, L, M are linearly independent. That is, the matrix $[I|V|W|L|M]$ with I, V, W, L and M for its columns is invertible with determinant $v_2w_3 - w_2v_3 \neq 0$. Thus

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & v_2 & w_2 & 0 & 0\\ 0 & v_3 & w_3 & 0 & 0\\ 0 & v_4 & w_4 & v_2 & w_2\\ 0 & v_5 & w_5 & v_3 & w_3 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi\cdot\langle J, K\rangle = \langle V, W\rangle$, hence $\langle V, W\rangle \sim \langle J, K\rangle$.

Case 2: Suppose that $\dim(\langle V, W\rangle \cap \mathfrak{z}) = 1$. We may, without loss of generality assume that W is central and that V is not central. That is, we may assume that $w_1 = w_2 = w_3 = 0$ with $w_4^2 + w_5^2 \neq 0$ and that v_1, v_2 and v_3 are not all zero.

If $\dim(\langle V, W\rangle \cap \langle J, K, L, M\rangle) = 1$, then clearly $V \notin \langle J, K, L, M\rangle$, as $W \in \langle L, M\rangle \subseteq \langle J, K, L, M\rangle$, therefore $v_1 \neq 0$. We thus have that

$$\varphi = \begin{bmatrix} v_1 & 0 & 0 & 0 & 0 \\ v_2 & w_4 & w_5 & 0 & 0 \\ v_3 & w_5 & -w_4 & 0 & 0 \\ v_4 & 0 & 0 & v_1 w_4 & v_1 w_5 \\ v_5 & 0 & 0 & v_1 w_5 & -v_1 w_4 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle I, L\rangle = \langle V, W\rangle$, hence $\langle I, L\rangle \sim \langle V, W\rangle$.

If $\dim(\langle V, W\rangle \cap \langle J, K, L, M\rangle) = 2$, then $\langle V, W\rangle \subseteq \langle J, K, L, M\rangle$ and thus $v_1 = 0$ with $v_2^2 + v_3^2 \neq 0$ (as V is not central). If in addition $\dim(\langle V, W\rangle \cap \mathfrak{z} \cap [I, \langle V, W\rangle \cap \langle J, K, L, M\rangle]) = 1$, then

$$\begin{aligned} &\dim(\langle V, W\rangle \cap \mathfrak{z} \cap [I, \langle V, W\rangle \cap \langle J, K, L, M\rangle]) \\ &= \dim(\langle W\rangle \cap [I, \langle V, W\rangle]) \\ &= 1. \end{aligned}$$

It follows that $\langle W\rangle = [I, \langle V, W\rangle]$. Therefore $[I, V] = \lambda W$ were $\lambda \in \mathbb{R}$ — this follows from the bilinearity of the Lie bracket and the fact that W is central. Now, $\lambda \neq 0$ for if so, then $[I, \langle V, W\rangle] = \{\bar{0}\} \neq \langle W\rangle$. The nontrivial Lie brackets of $\mathfrak{t}$ determine that $[I, V] = \lambda W$ if and only if $v_2 = \lambda w_4$ and $v_3 = \lambda w_5$. We thus have that

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda w_4 & w_5 & 0 & 0 \\ 0 & \lambda w_5 & -w_4 & 0 & 0 \\ 0 & v_4 & 0 & \lambda w_4 & w_5 \\ 0 & v_5 & 0 & \lambda w_5 & -w_4 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle J, L\rangle = \langle V, W\rangle$, hence $\langle J, L\rangle \sim \langle V, W\rangle$.

However, if in addition $\dim(\langle V, W\rangle \cap \mathfrak{z} \cap [I, \langle V, W\rangle \cap \langle J, K, L, M\rangle]) = 0$. We have that

$$\begin{aligned} &\dim(\langle V, W\rangle \cap \mathfrak{z} \cap [I, \langle V, W\rangle \cap \langle J, K, L, M\rangle]) \\ &= \dim(\langle W\rangle \cap [I, \langle V, W\rangle]) \\ &= 0. \end{aligned}$$

This implies that $[I, \langle V, W\rangle] \neq \langle W\rangle$. The bilinearity of the Lie bracket, the definition of the Lie bracket on $\mathfrak{t}$ (and the fact that W is central) imply that it is not the case that ($v_2 = \lambda w_4$ and $v_3 = \lambda w_5$) for $0 \neq \lambda \in \mathbb{R}$. We thus have that the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & v_2 & w_4 \\ 0 & v_3 & w_5 \end{bmatrix}$$

has linearly independent columns and is thus invertible with determinant $v_2 w_5 - v_3 w_4 \neq 0$. It follows that

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & v_2 & w_4 & 0 & 0 \\ 0 & v_3 & w_5 & 0 & 0 \\ 0 & v_4 & 0 & v_2 & w_4 \\ 0 & v_5 & 0 & v_3 & w_5 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle J, M\rangle = \langle V, W\rangle$, hence $\langle J, M\rangle \sim \langle V, W\rangle$.

Case 3: Supposing $\dim(\langle V, W\rangle \cap \mathfrak{z}) = 2$ clearly implies that $\langle V, W\rangle = \langle L, M\rangle = \mathfrak{z}$, thus $\langle L, M\rangle \sim \langle V, W\rangle$.

We have shown that every two-dimensional subspace of $\mathfrak{t}$ is equivalent to at least one of the stated subspaces. Mutual non-equivalence of these subspaces follows from the fact that they have been distinguished by the scalar invariants of Proposition 3.2.10. Thus every two-dimensional subspace of $\mathfrak{t}$ is equivalent to exactly one of the stated subspaces. □

Proposition 3.3.6. *The two-dimensional subspace structure of* $\mathfrak{t}$ *is given by*

SA: $\langle J, K\rangle, \langle I, L\rangle, \langle J, M\rangle$
I: $\langle J, L\rangle$
FCI: $\langle L, M\rangle$
S: $\langle I, J\rangle$

Proof. Let $V, W \in \mathfrak{t}$. We consider the two-dimensional subalgebras. $\langle I, J\rangle$ is not a subalgebra as $I, J \in \langle I, J\rangle$, but $[I, J] = L \notin \langle I, J\rangle$. Suppose $V, W \in \langle J, K\rangle$, then Lemma 2.2.1 gives

$$\begin{aligned}[V, W] &= (0w_2 - v_2 0)\, L + (0w_3 - v_3 0)\, M \\ &= \bar{0} \in \langle J, K\rangle.\end{aligned}$$

Thus $\langle J, K\rangle$ is a subalgebra. Similarly, suppose $V, W \in \langle I, L\rangle$, then

$$\begin{aligned}[V, W] &= (v_1 \cdot 0 - 0 \cdot w_1)\, L + (v_1 \cdot 0 - 0 \cdot w_1)\, M \\ &= \bar{0} \in \langle I, L\rangle.\end{aligned}$$

Thus $\langle I, L\rangle$ is a subalgebra. Suppose $V, W \in \langle J, M\rangle$, then

$$\begin{aligned}[V, W] &= (0 \cdot w_2 - v_2 \cdot 0)\, L + (0 \cdot 0 - 0 \cdot 0)\, M \\ &= \bar{0} \in \langle J, M\rangle.\end{aligned}$$

Thus $\langle J, M\rangle$ is a subalgebra.

Now we look at the ideals. Subspace $\langle J, K\rangle$ is not an ideal as $J \in \langle J, K\rangle$, $I \in \mathfrak{t}$, but $[I, J] = L \notin \langle J, K\rangle$. Subspace $\langle I, L\rangle$ is not an ideal as $I \in \langle I, L\rangle$, $K \in \mathfrak{t}$, but $[I, K] = M \notin \langle I, L\rangle$. Suppose $V \in \langle J, L\rangle$ and $W \in \mathfrak{t}$, then

$$\begin{aligned}[V, W] &= (v_1 w_2 - v_2 w_1)\, L + (v_1 w_3 - v_3 w_1)\, M \\ &= (0w_2 - v_2 w_1)\, L + (0w_3 - 0w_1)\, M \\ &= -v_2 w_1 L \in \langle J, L\rangle.\end{aligned}$$

Thus $\langle J, L\rangle$ is an ideal. Subspace $\langle J, M\rangle$ is not an ideal as $J \in \langle J, M\rangle$, $I \in \mathfrak{t}$, but $[I, J] = L \notin \langle J, M\rangle$.

Next we find the fully characteristic ideals. The ideal $\langle J, L\rangle$ is not fully characteristic as $\varphi \cdot \langle J, L\rangle = \langle K, M\rangle \neq \langle J, L\rangle$, where φ is the automorphism given by

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

By Lemma 3.2.1 $\langle L, M\rangle$ is a fully characteristic ideal as it is the centre $\mathfrak{z}$.

Finally we find the generating subspaces. Subspace $\langle I, J\rangle$ is not generating as $[I, J] = L$, thus $\langle I, J\rangle$ generates the vector subspace $\langle I, J, L\rangle$. Given $V, W \in \langle I, J, L\rangle$,

$$\begin{aligned}[V, W] &= (v_1 w_2 - v_2 w_1)\, L + (v_1 w_3 - v_3 w_1)\, M \\ &= (v_1 w_2 - v_2 w_1)\, L + (v_1 0 - 0 w_1)\, M \\ &= (v_1 w_2 - v_2 w_1)\, L \in \langle I, J, L\rangle.\end{aligned}$$

Thus no larger subspace than $\langle I, J, L\rangle \neq \mathfrak{t}$ can be generated by $\langle I, J\rangle$. Therefore $\langle I, J\rangle$ is not generating. □

3.3.3 Three-dimensional subspace structure

Proposition 3.3.7. *Every three-dimensional subspace of $\mathfrak{t}$ is equivalent to exactly one of the subspaces*

$$\langle I, J, K\rangle, \langle I, J, L\rangle, \langle I, J, M\rangle, \langle I, L, M\rangle, \langle J, K, L\rangle \text{ or } \langle J, L, M\rangle.$$

Proof. Let $\langle V, W, X\rangle$ be a three-dimensional subspace of $\mathfrak{t}$.

Case 1: Suppose $\dim(\langle V, W, X\rangle \cap \mathfrak{z}) = 2$, then without loss of generality we may assume that $\langle V, W, X\rangle = \langle V, L, M\rangle$.

If $\dim(\langle V, L, M\rangle \cap \langle J, K, L, M\rangle) = 2$, then there exists some element of $\langle V, L, M\rangle$ with nonzero I-component. Without loss of generality, we may let this element be V — that is, $v_1 \neq 0$. Now,

$$\varphi = \begin{bmatrix} v_1 & 0 & 0 & 0 & 0 \\ v_2 & v_1 & 0 & 0 & 0 \\ v_3 & 0 & v_1 & 0 & 0 \\ v_4 & 0 & 0 & v_1 & 0 \\ v_5 & 0 & 0 & 0 & v_1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle I, L, M\rangle = \langle V, L, M\rangle$. Therefore $\langle V, W, X\rangle = \langle V, L, M\rangle \sim \langle I, L, M\rangle$.

If $\dim(\langle V, L, M\rangle \cap \langle J, K, L, M\rangle) = 3$, then $\langle V, L, M\rangle \subseteq \langle J, K, L, M\rangle$ and thus $V \in \langle J, K, L, M\rangle$

so $v_1 = 0$. As V cannot be central, it follows that $v_2^2 + v_3^2 \neq 0$. Now,

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & v_2 & v_3 & 0 & 0 \\ 0 & v_3 & -v_2 & 0 & 0 \\ 0 & v_4 & 0 & v_2 & v_3 \\ 0 & v_5 & 0 & v_3 & -v_2 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle J, L, M \rangle = \langle V, L, M \rangle = \langle V, W, X \rangle$. It follows that $\langle V, W, X \rangle \sim \langle J, L, M \rangle$.

Case 2: Suppose $\dim(\langle V, W, X \rangle \cap \mathfrak{z}) = 1$, we may then assume that X is central. That is, $x_1 = x_2 = x_3 = 0$ and $x_4^2 + x_5^2 \neq 0$. Now, Corollary A.1.5 gives

$$2 \leq \dim(\langle V, W, X \rangle \cap \langle J, K, L, M \rangle) \leq 3$$

and so $\langle V, W, X \rangle$ contains at least two linear independent elements in $\langle J, K, L, M \rangle$. As $X \in \langle J, K, L, M \rangle$, we may further assume that $W \in \langle J, K, L, M \rangle$ and thus $w_1 = 0$. By Proposition 3.3.1 we have that $\beta \cdot W = J$ for some $\beta \in \mathrm{Aut}(\mathfrak{t})$. Therefore,

$$\beta \cdot \langle V, W, X \rangle = \langle \beta \cdot V, J, \beta \cdot X \rangle = \langle \bar{V}, J, \bar{X} \rangle,$$

where Lemma B.2.8 gives that $\bar{X}$ is central. As $\langle \bar{V}, J, \bar{X} \rangle \sim \langle V, W, X \rangle$, Proposition 3.2.10 allows us to proceed by examining the invariants of $\langle \bar{V}, J, \bar{X} \rangle$.

If $\dim(\langle \bar{V}, J, \bar{X} \rangle \cap \langle J, K, L, M \rangle) = 2$, it follows that $\bar{V} \notin \langle J, K, L, M \rangle$ as $J, \bar{X} \in \langle J, K, L, M \rangle$ and so $\bar{v}_1 \neq 0$.

If, in addition, we have

$$\dim(\langle \bar{V}, J, \bar{X} \rangle \cap \mathfrak{z} \cap [I, \langle \bar{V}, J, \bar{X} \rangle \cap \langle J, K, L, M \rangle]) = 1,$$

then

$$\begin{aligned} \dim(\langle \bar{X} \rangle \cap [I, \langle J, \bar{X} \rangle]) &= 1 \\ \dim(\langle \bar{X} \rangle \cap [I, \langle J \rangle]) &= 1 \\ \dim(\langle \bar{X} \rangle \cap \langle L \rangle) &= 1. \end{aligned}$$

This implies that $\langle \bar{X} \rangle = \langle L \rangle$ and thus $\bar{X} = \lambda L$ for some real scalar $\lambda \neq 0$. Without loss of generality, we may assume $\bar{X} = L$ and thus $\langle \bar{V}, J, \bar{X} \rangle = \langle \bar{V}, J, L \rangle$. The automorphism

$$\varphi = \begin{bmatrix} \bar{v}_1 & 0 & 0 & 0 & 0 \\ \bar{v}_2 & \bar{v}_1 & 0 & 0 & 0 \\ \bar{v}_3 & 0 & \bar{v}_1 & 0 & 0 \\ \bar{v}_4 & 0 & 0 & \bar{v}_1 & 0 \\ \bar{v}_5 & 0 & 0 & 0 & \bar{v}_1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle I, J, L \rangle = \langle \bar{V}, J, L \rangle \sim \langle V, W, X \rangle$. Therefore $\langle V, W, X \rangle \sim \langle I, J, L \rangle$.

If however, we have

$$\dim(\langle \bar{V}, J, \bar{X} \rangle \cap \mathfrak{z} \cap [I, \langle \bar{V}, J, \bar{X} \rangle \cap \langle J, K, L, M \rangle]) = 0,$$

we have that

$$\dim(\langle \bar{X} \rangle \cap \langle L \rangle) = 0.$$

That is, $\bar{X} = \bar{x}_4 L + \bar{x}_5 M$ with $\bar{x}_5 \neq 0$. It follows that

$$\varphi = \begin{bmatrix} \bar{v}_1 & 0 & 0 & 0 & 0 \\ \bar{v}_2 & 1 & \bar{x}_4 & 0 & 0 \\ \bar{v}_3 & 0 & \bar{x}_5 & 0 & 0 \\ \bar{v}_4 & 0 & 0 & \bar{v}_1 & \bar{v}_1\bar{x}_4 \\ \bar{v}_5 & 0 & 0 & 0 & \bar{v}_1\bar{x}_5 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle I, J, M \rangle = \langle \bar{V}, J, \bar{X} \rangle \sim \langle V, W, X \rangle$. That is, $\langle V, W, X \rangle \sim \langle I, J, M \rangle$.

If $\dim(\langle \bar{V}, J, \bar{X} \rangle \cap \langle J, K, L, M \rangle) = 3$, then $\langle \bar{V}, J, \bar{X} \rangle \subseteq \langle J, K, L, M \rangle$ and thus $V \in \langle J, K, L, M \rangle$, so $\bar{v}_1 = 0$. The subspace $\langle J, \bar{X} \rangle \subseteq \langle \bar{V}, J, \bar{X} \rangle \subseteq \mathfrak{t}$ is a two-dimensional subspace with

$$\dim(\langle J, \bar{X} \rangle \cap \mathfrak{z}) = 1$$

and

$$\dim(\langle J, \bar{X} \rangle \cap \langle J, K, L, M \rangle) = 2.$$

It follows from the proof of Proposition 3.3.5 that $\langle J, \bar{X} \rangle \sim \langle J, L \rangle$ or $\langle J, \bar{X} \rangle \sim \langle J, M \rangle$.

If $\beta \cdot \langle J, \bar{X} \rangle = \langle J, L \rangle$ for some $\beta \in \mathrm{Aut}(\mathfrak{t})$, then $\beta \cdot \langle \bar{V}, J, \bar{X} \rangle = \langle \beta \cdot \bar{V}, J, L \rangle$ where $\beta \cdot \bar{V} = \bar{\bar{V}}$, with $\bar{\bar{v}}_1 = 0$. Now $\langle \bar{\bar{V}}, J, L \rangle = \langle (\bar{\bar{V}} - \bar{\bar{v}}_2 J), J, L \rangle$ and thus we may assume that $\bar{\bar{v}}_2 = 0$. Since $\bar{\bar{V}}$ is not central we have that $\bar{\bar{v}}_3 \neq 0$. The automorphism

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \bar{\bar{v}}_3 & 0 & 0 \\ 0 & 0 & \bar{\bar{v}}_4 & 1 & 0 \\ 0 & 0 & \bar{\bar{v}}_5 & 0 & \bar{\bar{v}}_3 \end{bmatrix}$$

is such that $\varphi \cdot \langle J, K, L \rangle = \langle \bar{\bar{V}}, J, L \rangle$. Therefore,

$$\langle J, K, L \rangle \sim \langle \bar{\bar{V}}, J, L \rangle \sim \langle \bar{V}, J, \bar{X} \rangle \sim \langle V, W, X \rangle$$

and so $\langle J, K, L \rangle \sim \langle V, W, X \rangle$.

On the other hand, if $\beta \cdot \langle J, \bar{X} \rangle = \langle J, M \rangle$ for some $\beta \in \mathrm{Aut}(\mathfrak{t})$, then $\beta \cdot \langle \bar{V}, J, \bar{X} \rangle = \langle \beta \cdot \bar{V}, J, M \rangle$ where $\beta \cdot \bar{V} = \bar{\bar{V}}$, with $\bar{\bar{v}}_1 = 0$. Now $\langle \bar{\bar{V}}, J, M \rangle = \langle (\bar{\bar{V}} - \bar{\bar{v}}_2 J), J, M \rangle$ and thus we may assume that $\bar{\bar{v}}_2 = 0$. Since $\bar{\bar{V}}$ is not central we have that $\bar{\bar{v}}_3 \neq 0$. The automorphism

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \bar{\bar{v}}_3 & 0 & 0 \\ 0 & 0 & \bar{\bar{v}}_4 & 1 & 0 \\ 0 & 0 & \bar{\bar{v}}_5 & 0 & \bar{\bar{v}}_3 \end{bmatrix}$$

is such that $\varphi \cdot \langle J, K, M \rangle = \langle \bar{\bar{V}}, J, M \rangle$. Therefore,

$$\langle J, K, M \rangle \sim \langle \bar{\bar{V}}, J, M \rangle \sim \langle \bar{V}, J, \bar{X} \rangle \sim \langle V, W, X \rangle$$

and so $\langle J, K, M\rangle \sim \langle V, W, X\rangle$. Now, $\langle J, K, L\rangle \sim \langle J, K, M\rangle$ as $\varphi \cdot \langle J, K, L\rangle = \langle J, K, M\rangle$ where

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}).$$

We may thus conclude that if $\dim(\langle \bar{V}, J, \bar{X}\rangle \cap \langle J, K, L, M\rangle) = 3$, then $\langle V, W, X\rangle \sim \langle \bar{V}, J, \bar{X}\rangle \sim \langle J, K, L\rangle$.

Case 3: Suppose $\dim(\langle V, W, X\rangle \cap \mathfrak{z}) = 0$. CorollaryA.1.5 gives that

$$2 \leq \dim(\langle V, W, X\rangle \cap \langle J, K, L, M\rangle) \leq 3.$$

There thus exists at least two linearly independent elements of $\langle V, W, X\rangle$ in $\langle J, K, L, M\rangle$. Without loss of generality, we may let these elements be W and X, we thus have $w_1 = x_1 = 0$. As W and X cannot be central, we have $w_2^2 + w_3^2 \neq 0$ and $x_2^2 + x_3^2 \neq 0$. As W and X are linearly independent, $[I|W|X|L|M]$ is a 5×5 matrix with linearly independent columns I,W,X,L and M. It follows that $[I|W|X|L|M]$ is invertible with determinant $w_2x_3 - x_2w_3 \neq 0$.

Now, $\dim(\langle V, W, X\rangle \cap \langle J, K, L, M\rangle) = 2$; for if $\dim(\langle V, W, X\rangle \cap \langle J, K, L, M\rangle) = 3$ then V, W, X, L and M would be five linearly independent vectors of the four-dimensional space $\langle J, K, L, M\rangle$ — a contradiction. Thus $\langle V, W, X\rangle$ contains at least one element not in $\langle J, K, L, M\rangle$, we let this be V and thus $v_1 \neq 0$. We therefore have that

$$\varphi = \begin{bmatrix} v_1 & 0 & 0 & 0 & 0 \\ v_2 & w_2 & x_2 & 0 & 0 \\ v_3 & w_3 & x_3 & 0 & 0 \\ v_4 & w_4 & x_4 & v_1w_2 & v_1x_2 \\ v_5 & w_5 & x_5 & v_1w_3 & v_1x_3 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle I, J, K\rangle = \langle V, W, X\rangle$ and thus $\langle I, J, K\rangle \sim \langle V, W, X\rangle$.

We have shown that every three-dimensional subspace of $\mathfrak{t}$ is equivalent to at least one of the stated subspaces. Mutual non-equivalence of these subspaces follows from the fact that they have been distinguished by the scalar invariants of Proposition 3.2.10. Thus every three-dimensional subspace of $\mathfrak{t}$ is equivalent to exactly one of the stated subspaces. □

Proposition 3.3.8. *The three-dimensional subspace structure of* $\mathfrak{t}$ *is given by*

SA: $\langle J, K, L\rangle, \langle I, J, L\rangle$
I: $\langle I, L, M\rangle, \langle J, L, M\rangle$
S: $\langle I, J, M\rangle$
Gen: $\langle I, J, K\rangle$.

Proof. We first find the three-dimensional subalgebras. The subspace $\langle I, J, K\rangle$ is not a subalgebra as $I, J \in \langle I, J, K\rangle$, but $[I, J] = L \notin \langle I, J, K\rangle$. The subspace $\langle I, J, M\rangle$ is not a subalgebra as $I, J \in \langle I, J, M\rangle$, but $[I, J] = L \notin \langle I, J, M\rangle$. If $V, W \in \langle I, J, L\rangle$, then

$$\begin{aligned}[V, W] &= (v_1 w_2 - v_2 w_1)\, L + (v_1 w_3 - v_3 w_1)\, M \\ &= (v_1 w_2 - v_2 w_1)\, L + (v_1 \cdot 0 - 0 \cdot w_1)\, M \\ &= (v_1 w_2 - v_2 w_1)\, L \in \langle I, J, L\rangle.\end{aligned}$$

Thus, $\langle I, J, L\rangle$ is a subalgebra. Suppose $V, W \in \langle J, K, L\rangle$, then

$$\begin{aligned}[V, W] &= (0 \cdot w_2 - v_2 \cdot 0)\, L + (0 \cdot w_3 - v_3 \cdot 0)\, M \\ &= \bar{0} \in \langle J, K, L\rangle.\end{aligned}$$

Thus $\langle J, K, L\rangle$ is a subalgebra.

We now find the ideals. We have that $I \in \langle I, J, L\rangle$ and $K \in \mathfrak{t}$, however $[I, K] = M \notin \langle I, J, L\rangle$. Thus $\langle I, J, L\rangle$ is not an ideal. If $V \in \langle I, L, M\rangle$ and $W \in \mathfrak{t}$, then

$$\begin{aligned}[V, W] &= (v_1 w_2 - 0 \cdot w_1)\, L + (v_1 w_3 - 0 \cdot w_1)\, M \\ &= v_1 w_2 L + v_1 w_3 M \in \langle I, L, M\rangle.\end{aligned}$$

Thus $\langle I, L, M\rangle$ is an ideal. Suppose $V \in \langle J, L, M\rangle$ and $W \in \mathfrak{t}$, then

$$\begin{aligned}[V, W] &= (0 \cdot w_2 - v_2 w_1)\, L + (0 \cdot w_3 - 0 \cdot w_1)\, M \\ &= -v_2 w_1 L \in \langle J, L, M\rangle.\end{aligned}$$

Thus $\langle J, L, M\rangle$ is an ideal. Now, $K \in \langle J, K, L\rangle$ and $I \in \mathfrak{t}$, however $[I, K] = M \notin \langle J, K, L\rangle$, so $\langle J, K, L\rangle$ is not an ideal.

We prove that there is no three-dimensional fully characteristic ideal of $\mathfrak{t}$. The ideal $\langle J, L, M\rangle$ is not a fully characteristic ideal as $\varphi \cdot \langle J, L, M\rangle = \langle K, L, M\rangle \neq \langle J, L, M\rangle$ where φ is the automorphism

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The ideal $\langle I, L, M\rangle$ is not a fully characteristic ideal as $\varphi \cdot \langle I, L, M\rangle \neq \langle I, L, M\rangle$, as $\varphi \cdot I \notin \langle I, L, M\rangle$ where φ is the automorphism given by

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Finally we consider the generating subspaces. Consider the subspace $\langle I, J, K\rangle$. As $[I, J] = L$ and $[I, K] = M$, the subspace $\langle I, J, K\rangle$ generates all of $\mathfrak{t}$ and is thus generating. Consider the subspace $\langle I, J, M\rangle$. As $[I, J] = L$, $\langle I, J, M\rangle$ generates the subspace $\langle I, J, M, L\rangle$. Let V and W be elements of $\langle I, J, M, L\rangle$, then

$$\begin{aligned}[V, W] &= (v_1 w_2 - v_2 w_1)\, L + (v_1 \cdot 0 - 0 \cdot w_1)\, M \\ &= (v_1 w_2 - v_2 w_1)\, L \in \langle I, J, L, M\rangle.\end{aligned}$$

Thus $\langle I, J, L, M\rangle$ is a subalgebra of $\mathfrak{t}$ and $\langle I, J, M\rangle$ can generate no larger subspace of $\mathfrak{t}$. It follows that $\langle I, J, M\rangle$ is not a generating subspace. □

3.3.4 Four-dimensional subspace structure

Proposition 3.3.9. *Every four-dimensional subspace of* $\mathfrak{t}$ *is equivalent to exactly one of the subspaces*

$$\langle J, K, L, M\rangle, \langle I, J, L, M\rangle \text{ or } \langle I, J, K, L\rangle.$$

Proof. Let $\langle V, W, X, Y\rangle$ be a subspace of $\mathfrak{t}$.
Case 1: Suppose $\dim(\langle V, W, X, Y\rangle \cap \langle J, K, L, M\rangle) = 4$, then clearly $\langle V, W, X, Y\rangle = \langle J, K, L, M\rangle$.

Case 2: Suppose $\dim(\langle V, W, X, Y\rangle \cap \langle J, K, L, M\rangle) = 3$.

If $\dim(\langle V, W, X, Y\rangle \cap \mathfrak{z}) = 1$, then $\mathfrak{z} = \langle L, M\rangle \nsubseteq \langle V, W, X, Y\rangle$. Without loss of generality, suppose $\mathfrak{z} \cap \langle V, W, X, Y\rangle = \langle Y\rangle$. The subspace $\langle V, W, X\rangle$ therefore contains no nonzero central elements. That is,

$$\dim(\langle V, W, X\rangle \cap \mathfrak{z}) = 0.$$

The proofs of Proposition 3.3.7 and Proposition 3.2.10 imply that $\langle V, W, X\rangle \sim \langle I, J, K\rangle$ as it is the only three-dimensional subspace of $\mathfrak{t}$, up to equivalence, with no nonzero central elements. That is, there exists some $\beta \in \mathrm{Aut}(\mathfrak{t})$ with $\beta \cdot \langle V, W, X\rangle = \langle I, J, K\rangle$.

Now, $\beta \cdot \langle V, W, X, Y\rangle = \langle I, J, K, \beta \cdot Y\rangle$ where $\beta \cdot Y = \bar{Y} \in \mathfrak{z}$ by Lemma B.2.8. We thus have that $\bar{y}_1 = \bar{y}_2 = \bar{y}_3 = 0$ and $\bar{y}_4^2 + \bar{y}_5^2 \neq 0$. Hence, $\langle V, W, X, Y\rangle \sim \langle I, J, K, \bar{Y}\rangle$. The automorphism

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \bar{y}_4 & \bar{y}_5 & 0 & 0 \\ 0 & \bar{y}_5 & -\bar{y}_4 & 0 & 0 \\ 0 & 0 & 0 & \bar{y}_4 & \bar{y}_5 \\ 0 & 0 & 0 & \bar{y}_5 & -\bar{y}_4 \end{bmatrix}$$

is such that $\varphi \cdot \langle I, J, K, L\rangle = \langle I, J, K, \bar{Y}\rangle \sim \langle V, W, X, Y\rangle$. Therefore, $\langle V, W, X, Y\rangle \sim \langle I, J, K, L\rangle$.

If $\dim(\langle V, W, X, Y\rangle \cap \mathfrak{z}) = 2$, then $\mathfrak{z} \subseteq \langle V, W, X, Y\rangle$ and without loss of generality we may assume that $\langle V, W, X, Y\rangle = \langle V, W, L, M\rangle$. In addition, as $\dim(\langle V, W, X, Y\rangle \cap \langle J, K, L, M\rangle) = 3$, we have that $\langle V, W, L, M\rangle$ contains a non-central element of $\langle J, K, L, M\rangle$. Letting this element be W, we have $w_1 = 0$ and $w_2^2 + w_3^2 \neq 0$. As $\langle V, W, L, M\rangle \cap \langle J, K, L, M\rangle = \langle W, L, M\rangle$, we have that

$V \notin \langle J, K, L, M \rangle$ and thus $v_1 \neq 0$. The automorphism

$$\varphi = \begin{bmatrix} v_1 & 0 & 0 & 0 & 0 \\ v_2 & w_2 & w_3 & 0 & 0 \\ v_3 & w_3 & -w_2 & 0 & 0 \\ v_4 & w_4 & 0 & v_1 w_2 & v_1 w_3 \\ v_5 & w_5 & 0 & v_1 w_3 & -v_1 w_2 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t})$$

is such that $\varphi \cdot \langle I, J, L, M \rangle = \langle V, W, L, M \rangle$ and thus $\langle V, W, X, Y \rangle \sim \langle I, J, L, M \rangle$.

We have shown that every four-dimensional subspace of $\mathfrak{t}$ is equivalent to at least one of the stated subspaces. Mutual non-equivalence of these subspaces follows from the fact that they have been distinguished by the scalar invariants of Proposition 3.2.10. Thus every four-dimensional subspace of $\mathfrak{t}$ is equivalent to exactly one of the stated subspaces. □

Proposition 3.3.10. *The four-dimensional subspace structure of* $\mathfrak{t}$ *is given by*

I: $\langle I, J, L, M \rangle$
FCI: $\langle J, K, L, M \rangle$
Gen: $\langle I, J, K, L \rangle$

Proof. We first consider the four-dimensional subalgebras. The subspace $\langle I, J, K, L \rangle$ is not a subalgebra as $I, K \in \langle I, J, K, L \rangle$, but $[I, K] = M \notin \langle I, J, K, L \rangle$.

Now, for the ideals. Suppose $V \in \langle I, J, L, M \rangle$ and $W \in \mathfrak{t}$, then Lemma 2.2.1 gives

$$\begin{aligned} [V, W] &= (v_1 w_2 - v_2 w_1)\, L + (v_1 w_3 - 0 \cdot w_1)\, M \\ &= (v_1 w_2 - v_2 w_1) L + v_1 w_3 M \in \langle I, J, L, M \rangle. \end{aligned}$$

Thus $\langle I, J, L, M \rangle$ is an ideal.

Now, for the fully characteristic ideals. The ideal $\langle J, K, L, M \rangle$ is a fully characteristic ideal, by Lemma 3.2.4. The ideal $\langle I, J, L, M \rangle$ is not a fully characteristic ideal as $\varphi \cdot \langle I, J, L, M \rangle = \langle I, K, L, M \rangle \neq \langle I, J, L, M \rangle$ where

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}).$$

Finally, we look at the generating subspaces. Consider the subspace $\langle I, J, K, L \rangle$. We have that $[I, K] = M$, thus the subspace $\langle I, J, K, L \rangle$ generates the subspace $\langle I, J, K, L, M \rangle = \mathfrak{t}$ and is thus generating. □

3.4 Summary

Table 3.1 presents the subspaces of the Lie algebra $\mathfrak{t}$, up to equivalence, along with the scalar invariants of Proposition 3.2.10. These invariants fully distinguish the subspaces of $\mathfrak{t}$, up to equivalence as the proofs of Proposition 3.3.1, Proposition 3.3.5, Proposition 3.3.7 and Proposition 3.3.9 determine these representative by running through all possible values of the invariants of Proposition 3.2.10. Proposition 3.2.10 is thus in fact bi-conditional, we give this as Proposition 3.4.1 below. Theorem 3.4.2 collects Proposition 3.3.1, Proposition 3.3.5, Proposition 3.3.7 and Proposition 3.3.9 and displays the full subspace classification of $\mathfrak{t}$.

Table 3.1: Subspace structure and invariants

Subspace $\mathfrak{s}$	$\dim(\mathfrak{s})$	$\dim(\mathfrak{s} \cap \mathfrak{c})$	$\dim(\mathfrak{s} \cap \mathfrak{z})$	$\dim(\mathfrak{s} \cap \mathfrak{z} \cap [I, \mathfrak{s} \cap \mathfrak{c}])$
$\langle I \rangle$	1	0	0	0
$\langle J \rangle$	1	1	0	0
$\langle L \rangle$	1	1	1	0
$\langle I, J \rangle$	2	1	0	0
$\langle I, L \rangle$	2	1	1	0
$\langle J, K \rangle$	2	2	0	0
$\langle J, M \rangle$	2	2	1	0
$\langle J, L \rangle$	2	2	1	1
$\langle L, M \rangle$	2	2	2	0
$\langle I, J, K \rangle$	3	2	0	0
$\langle I, J, M \rangle$	3	2	1	0
$\langle I, J, L \rangle$	3	2	1	1
$\langle I, L, M \rangle$	3	2	2	0
$\langle J, K, L \rangle$	3	3	1	1
$\langle J, L, M \rangle$	3	3	2	1
$\langle I, J, K, L \rangle$	4	3	1	1
$\langle I, J, L, M \rangle$	4	3	2	1
$\langle J, K, L, M \rangle$	4	4	2	2
$\langle I, J, K, L, M \rangle$	5	4	2	2

Proposition 3.4.1. *Let* $\mathfrak{s}$ *and* $\mathfrak{w}$ *be subspaces of* $\mathfrak{t}$ *and* $\mathfrak{c} = \langle J, K, L, M \rangle$*. The subspaces* $\mathfrak{s}$ *and* $\mathfrak{w}$ *are equivalent if and only if*

$$\begin{aligned}
\dim(\mathfrak{s}) &= \dim(\mathfrak{w}), \\
\dim(\mathfrak{s} \cap \mathfrak{z}) &= \dim(\mathfrak{w} \cap \mathfrak{z}), \\
\dim(\mathfrak{s} \cap \mathfrak{c}) &= \dim(\mathfrak{w} \cap \mathfrak{c}) \text{ and} \\
\dim(\mathfrak{s} \cap \mathfrak{z} \cap [I, \mathfrak{s} \cap \mathfrak{c}]) &= \dim(\mathfrak{w} \cap \mathfrak{z} \cap [I, \mathfrak{w} \cap \mathfrak{c}]).
\end{aligned}$$

Theorem 3.4.2. *The subspace structure of the Lie algebra* $\mathfrak{t}$ *is given by*

SA: $\langle I \rangle, \langle J \rangle, \langle J, K \rangle, \langle I, L \rangle, \langle J, M \rangle, \langle J, K, L \rangle, \langle I, J, L \rangle$

I: $\langle L \rangle, \langle J, L \rangle, \langle I, L, M \rangle, \langle J, L, M \rangle, \langle I, J, L, M \rangle$

FCI: $\langle L, M \rangle, \langle J, K, L, M \rangle$

S: $\langle I, J \rangle, \langle I, J, M \rangle$

Gen: $\langle I, J, K \rangle, \langle I, J, K, L \rangle$.

Chapter 4

Classification of sub-Riemannian structures

In this chapter we classify, up to isometry, the left-invariant sub-Riemannian structures on the Lie group T. For this we employ a Lie group-Lie algebra correspondence result. This allows us to convert results on the Lie algebra $\mathfrak{t}$, established in Chapter 3, into corresponding results on the Lie group T. With these and the affine nature of isometries on nilpotent metric Lie groups we produce the desired classification. The chapter ends with a computation of the linearised isotropy groups of the left-invariant sub-Riemannian structures obtained in the classification.

4.1 Preliminaries

Definition 4.1.1. *[cf. 4, Section 2] A* ***left-invariant sub-Riemannian structure*** *is a triple* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *where* G *is a real, finite-dimensional, connected Lie group,* $\mathcal{D}$ *is a smooth bracket generating left-invariant distribution on* G*, and* $\mathbf{g}$ *is a left-invariant Riemannian metric on* $\mathcal{D}$*. That is,* $\mathcal{D}(\mathbf{1})$ *is a linear subspace of* $\mathfrak{g}$ *with*

$$\mathcal{D}(g) = d_{\mathbf{1}}L_g \cdot \mathcal{D}(\mathbf{1}) \text{ for every } g \in \mathsf{G}$$

and $\mathbf{g}_{\mathbf{1}}$ *is a positive definite, symmetric bilinear form on* $\mathcal{D}(\mathbf{1})$ *with*

$$\mathbf{g}_g(d_{\mathbf{1}}L_g \cdot A, d_{\mathbf{1}}L_g \cdot B) = \mathbf{g}_{\mathbf{1}}(A, B) \text{ for every } A, B \in \mathcal{D}(\mathbf{1}).$$

Definition 4.1.2. *[4, Section 2] An* ***isometry*** *between two left-invariant sub-Riemannian structures* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ *is a diffeomorphism* $\phi : \mathsf{G} \longrightarrow \mathsf{G}'$ *such that*

$$\phi_*\mathcal{D} = \mathcal{D}' \quad \text{and} \quad \mathbf{g} = \phi^*\mathbf{g}',$$

where $\phi_*\mathcal{D}$ *and* $\phi^*\mathbf{g}'$ *are as in DefinitionB.1.3.*

Definition 4.1.3. *We shall denote the **group of isometries** of the structure* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *by* $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$. *The **isotropy subgroup** of* $g \in \mathsf{G}$, *denoted by* $\mathsf{Iso}_g(\mathsf{G}, \mathcal{D}, \mathbf{g})$, *is the subgroup of isometries that fix* g.

Remark 4.1.4. *As the distribution* $\mathcal{D}$ *and metric* $\mathbf{g}$ *are left-invariant, they are completely determined by their values at the identity* $\mathcal{D}(\mathbf{1})$ *and* $\mathbf{g}_{\mathbf{1}}$.

Lemma 4.1.5. *Given a left-invariant distribution* $\mathcal{D}$, *the push forward* $\phi_* \mathcal{D}$ *of* $\mathcal{D}$ *by an automorphism* ϕ *is left-invariant.*

Proof. Suppose $\mathcal{D}$ is a left-invariant distribution on a smooth manifold G. As $\mathcal{D}$ is left-invariant,

$$\mathcal{D}(x) = d_{\mathbf{1}} L_x \cdot \mathcal{D}(\mathbf{1})$$

for every $x \in \mathsf{G}$. For any automorphism ϕ of G,

$$\begin{aligned}(\phi_* \mathcal{D})(x) &= ((\phi^{-1})^* \mathcal{D})(x) \\ &= d_{\phi^{-1}(x)} \phi \cdot \mathcal{D}(\phi^{-1}(x)) \\ &= d_{\phi^{-1}(x)} \phi \cdot d_{\mathbf{1}} L_{\phi^{-1}(x)} \cdot \mathcal{D}(\mathbf{1}) \\ &= d_{\mathbf{1}} (\phi \circ L_{\phi^{-1}(x)}) \cdot \mathcal{D}(\mathbf{1}).\end{aligned}$$

As $\phi \in \mathrm{Aut}(\mathsf{G})$, it is a Lie group homomorphism and thus for any $h \in \mathsf{G}$ we have

$$\begin{aligned}\left(\phi \circ L_{\phi^{-1}(x)}\right)(h) &= \phi\left((\phi^{-1}(x))h\right) \\ &= (\phi(\phi^{-1}(x)))(\phi(h)) \\ &= x(\phi(h)) \\ &= (L_x \circ \phi)(h).\end{aligned}$$

That is, $\phi \circ L_{\phi^{-1}(x)} = L_x \circ \phi$ and thus

$$\begin{aligned}(\phi_* \mathcal{D})(x) &= d_{\mathbf{1}} (\phi \circ L_{\phi^{-1}(x)}) \cdot \mathcal{D}(\mathbf{1}) \\ &= d_{\mathbf{1}} (L_x \circ \phi) \cdot \mathcal{D}(\mathbf{1}) \\ &= d_{\mathbf{1}} L_x \cdot d_{\mathbf{1}} \phi \cdot \mathcal{D}(\mathbf{1}) \\ &= d_{\mathbf{1}} L_x \cdot (\phi_* \mathcal{D})(\mathbf{1}).\end{aligned}$$

This proves that $\phi_* \mathcal{D}$ is left-invariant. □

Lemma 4.1.6. *Given a left-invariant sub-Riemannian structure* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $\phi \in \mathrm{Aut}(\mathsf{G})$, *the pull back* $\phi^* \mathbf{g}$ *is left-invariant.*

Proof. Suppose $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is a left-invariant sub-Riemannian structure, then the Riemannian metric $\mathbf{g}$ is left-invariant, that is

$$\mathbf{g}_x(d_{\mathbf{1}} L_x \cdot A, d_{\mathbf{1}} L_x \cdot B) = \mathbf{g}_{\mathbf{1}}(A, B)$$

for every $x \in \mathsf{G}$ and $A, B \in \mathfrak{g}$. Now,

$$\begin{aligned}(\phi^*\mathbf{g})_x(d_\mathbf{1}L_x \cdot A, d_\mathbf{1}L_x \cdot B) &= \mathbf{g}_{\phi(x)}(d_x\phi \cdot d_\mathbf{1}L_x \cdot A, d_x\phi \cdot d_\mathbf{1}L_x \cdot B)\\ &= \mathbf{g}_\mathbf{1}(d_{\phi(x)}L_{\phi(x)^{-1}} \cdot (d_x\phi \cdot d_\mathbf{1}L_x \cdot A), d_{\phi(x)}L_{\phi(x)^{-1}} \cdot (d_x\phi \cdot d_\mathbf{1}L_x \cdot B))\\ &= \mathbf{g}_\mathbf{1}(d_x(L_{\phi(x)^{-1}} \circ \phi) \cdot d_\mathbf{1}L_x \cdot A, d_x(L_{\phi(x)^{-1}} \circ \phi) \cdot d_\mathbf{1}L_x \cdot B).\end{aligned}$$

As $\phi \in \mathrm{Aut}(\mathsf{G})$, it is a Lie group homomorphism and thus for any $h \in \mathsf{G}$ we have

$$\begin{aligned}(L_{\phi(x)^{-1}} \circ \phi)(h) &= L_{\phi(x)^{-1}}(\phi(h))\\ &= \phi(x)^{-1}\phi(h)\\ &= \phi(x^{-1})\phi(h)\\ &= \phi(x^{-1}h)\\ &= (\phi \circ L_{x^{-1}})(h).\end{aligned}$$

That is, $(L_{\phi(x)^{-1}} \circ \phi) = (\phi \circ L_{x^{-1}})$ and thus

$$\begin{aligned}(\phi^*\mathbf{g})_x(d_\mathbf{1}L_x \cdot A, d_\mathbf{1}L_x \cdot B) &= \mathbf{g}_\mathbf{1}(d_x(L_{\phi(x)^{-1}} \circ \phi) \cdot d_\mathbf{1}L_x \cdot A, d_x(L_{\phi(x)^{-1}} \circ \phi) \cdot d_\mathbf{1}L_x \cdot B)\\ &= \mathbf{g}_\mathbf{1}(d_x(\phi \circ L_{x^{-1}}) \cdot d_\mathbf{1}L_x \cdot A, d_x(\phi \circ L_{x^{-1}}) \cdot d_\mathbf{1}L_x \cdot B)\\ &= \mathbf{g}_\mathbf{1}(d_\mathbf{1}\phi \cdot (d_xL_{x^{-1}} \cdot d_\mathbf{1}L_x \cdot A), d_\mathbf{1}\phi \cdot (d_xL_{x^{-1}} \cdot d_\mathbf{1}L_x \cdot B))\\ &= \mathbf{g}_\mathbf{1}(d_\mathbf{1}\phi \cdot A, d_\mathbf{1}\phi \cdot B)\\ &= (\phi^*\mathbf{g})_\mathbf{1}(A, B).\end{aligned}$$

This proves that $(\phi^*\mathbf{g})$ is left-invariant. □

Proposition 4.1.7. *Given left-invariant sub-Riemannian structures* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}, \mathcal{D}', \mathbf{g}')$ *on a simply connected matrix Lie group* G*, then*

there exists $\phi \in \mathrm{Aut}(\mathsf{G})$ *such that*

$$\begin{aligned}\phi_*\mathcal{D} &= \mathcal{D}'\\ \mathbf{g} &= \phi^*\mathbf{g}'.\end{aligned}$$

if and only if there exists $\psi \in \mathrm{Aut}(\mathfrak{g})$ *such that*

$$\begin{aligned}\psi \cdot \mathcal{D}(\mathbf{1}) &= \mathcal{D}'(\mathbf{1})\\ \mathbf{g}_\mathbf{1}(A, B) &= \mathbf{g}'_\mathbf{1}(\psi \cdot A, \psi \cdot B),\end{aligned}$$

where $A, B \in \mathfrak{g}$ *and* $\mathbf{1}$ *is the unit element of* G*.*

Proof. As $\phi \in \mathrm{Aut}(\mathsf{G})$, we have that $d_\mathbf{1}\phi \in \mathrm{Aut}(\mathfrak{g})$. Now,

$$\mathcal{D}' = \phi_*\mathcal{D} = \left(\phi^{-1}\right)^*\mathcal{D}$$

thus for $x \in \mathsf{G}$ we have

$$\mathcal{D}'(x) = d_{\phi^{-1}(x)}\phi \cdot \mathcal{D}\left(\phi^{-1}(x)\right).$$

Taking $x = \mathbf{1}$ and noting that $\phi^{-1}(\mathbf{1}) = \mathbf{1}$ we get

$$\mathcal{D}'(\mathbf{1}) = d_{\mathbf{1}}\phi \cdot \mathcal{D}\,(\mathbf{1})\,.$$

As

$$\mathbf{g} = \phi^* \mathbf{g}',$$

we have that

$$\mathbf{g}_x\,(d_{\mathbf{1}}L_x \cdot A, d_{\mathbf{1}}L_x \cdot B) = \mathbf{g}'_{\phi(x)}\,(d_x\phi \cdot d_{\mathbf{1}}L_x \cdot A, d_x\phi \cdot d_{\mathbf{1}}L_x \cdot B)$$

for $x \in \mathsf{G}$ and $A, B \in \mathfrak{g}$. Taking $x = \mathbf{1}$ and noting that $\phi(\mathbf{1}) = \mathbf{1}$, we get

$$\mathbf{g}_{\mathbf{1}}\,(A, B) = \mathbf{g}'_{\mathbf{1}}\,(d_{\mathbf{1}}\phi \cdot A, d_{\mathbf{1}}\phi \cdot B)\,.$$

Therefore taking $\psi = d_{\mathbf{1}}\phi$ gives (2).

Conversely, as G is simply connected, Corollary B.2.12 implies that there exists $\phi \in \mathrm{Aut}(\mathsf{G})$ such that $d_{\mathbf{1}}\phi = \psi$. As

$$\phi_*\mathcal{D} = (\phi^{-1})^*\mathcal{D},$$

we have that

$$\phi_*\mathcal{D}(x) = d_{\phi^{-1}(x)}\phi \cdot \mathcal{D}\left(\phi^{-1}\,(x)\right)$$

for $x \in \mathsf{G}$. Taking $x = \mathbf{1}$, we have

$$\begin{aligned}\phi_*\mathcal{D}(\mathbf{1}) &= d_{\mathbf{1}}\phi \cdot \mathcal{D}\,(\mathbf{1})\\ &= \psi \cdot \mathcal{D}\,(\mathbf{1})\\ &= \mathcal{D}'(\mathbf{1}).\end{aligned}$$

By Lemma 4.1.5, $\mathcal{D}'$ and $\phi_*\mathcal{D}$ are left-invariant distributions that agree at identity, we thus have

$$\phi_*\mathcal{D} = \mathcal{D}'.$$

For $A, B \in \mathfrak{g}$ and $x \in \mathsf{G}$

$$(\phi^*\mathbf{g}')_x(d_{\mathbf{1}}L_x \cdot A, d_{\mathbf{1}}L_x \cdot B) = \mathbf{g}'_{\phi(x)}(d_x\phi \cdot d_{\mathbf{1}}L_x \cdot A, d_x\phi \cdot d_{\mathbf{1}}L_x \cdot B).$$

Taking $x = \mathbf{1}$, we have

$$\begin{aligned}(\phi^*\mathbf{g}')_{\mathbf{1}}(A, B) &= \mathbf{g}'_{\phi(\mathbf{1})}(d_{\mathbf{1}}\phi \cdot A, d_{\mathbf{1}}\phi \cdot B)\\ &= \mathbf{g}'_{\mathbf{1}}(\psi \cdot A, \psi \cdot B)\\ &= \mathbf{g}_{\mathbf{1}}(A, B).\end{aligned}$$

Therefore $\phi^*\mathbf{g}'_{\mathbf{1}} = \mathbf{g}_{\mathbf{1}}$. By Lemma 4.1.6, as $\phi^*\mathbf{g}'$ and $\mathbf{g}$ are left-invariant Riemannian metrics that agree at identity, we have that

$$\mathbf{g} = \phi^*\mathbf{g}'.$$

□

Remark 4.1.8. *The matrix Lie group* T *is simply connected as it is diffeomorphic to* $\mathbb{R}^5$ *with the diffeomorphism* $\phi : \mathsf{T} \longrightarrow \mathbb{R}^5$ *given by*

$$\phi : \begin{bmatrix} 1 & x_1 & x_4 & x_5 \\ 0 & 1 & x_2 & x_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto (x_1, x_2, x_3, x_4, x_5).$$

Theorem 4.1.9. *[10, Theorem 1.2] Isometries between nilpotent metric Lie groups are affine. That is, every isometry* ϕ *is the composition of a left translation and a Lie group homomorphism.*

Corollary 4.1.10. *For a nilpotent metric Lie groups* G*, if* $\Phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$*, then* $\Phi \in \mathrm{Aut}(\mathsf{G})$*.*

Proof. Suppose $\Phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$, then Theorem 4.1.9 implies that

$$\Phi = L_g \circ F,$$

where L_g is left translation by some $g \in \mathsf{G}$ and F is a Lie group homomorphism. Now,

$$\begin{aligned} \Phi(\mathbf{1}) &= L_g(F(\mathbf{1})) \\ \mathbf{1} &= L_g(\mathbf{1}) = g. \end{aligned}$$

Therefore $\Phi = F$, a Lie group homomorphism. As Φ is invertible, it follows that $\Phi \in \mathrm{Aut}(\mathsf{G})$. ☐

We denote by $d\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ the group

$$\{d_1\phi : \phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})\} \cong \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$$

of **linearised isotropies**. For a sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ on a simply connected nilpotent Lie group, we have that the isotropy subgroup $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is given by

$$\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \{\phi \in \mathrm{Aut}(\mathsf{G}) : d_1\phi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1}), \mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}_{\mathbf{1}}(d_{\mathbf{1}}\phi \cdot A, d_{\mathbf{1}}\phi \cdot B)\}.$$

As for $\phi \in Aut(\mathsf{G})$, $d_{\mathbf{1}}\phi = \varphi \in \mathrm{Aut}(\mathfrak{g})$, we have that $d\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \subseteq \mathrm{Aut}(\mathfrak{g})$.

Proposition 4.1.11. *If* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *is a left-invariant sub-Riemannian structure, then* $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *is a semidirect product of the subgroup of left translations* G^L *by the isotropy group at identity* $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$*. That is,*

$$\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \mathsf{G}^L \rtimes \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}).$$

Proof. Suppose $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is a left-invariant sub-Riemannian structure. The isometry group $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ has as subgroups, the subgroup of left translations of G, G^L, and subgroup $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ of isometries that fix the identity. As the only left translation that fixes the identity is the identity transformation, we have that $\mathsf{G}^L \cap \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \{\mathrm{Id}_G\}$. That is, $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is a complement, in the sense of Definition A.2.7, of G^L in $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$.

Given any left translation $L_g : h \mapsto g \cdot h$ and $b \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$, Lemma 4.1.9 gives that $b = L_x \circ a$ for some $a \in \mathrm{Aut}(\mathsf{G})$, and we have that

$$\begin{aligned}(b \circ L_g \circ b^{-1})(h) &= (L_x \circ a \circ L_g \circ a^{-1} \circ L_{x^{-1}})(h) \\ &= L_x(a(g \cdot a^{-1}(x^{-1}h))) \\ &= L_x(a(g) \cdot a(a^{-1}(x^{-1}h))) \\ &= x \cdot (a(g) \cdot x^{-1}h) \\ &= L_{x \cdot a(g) \cdot x^{-1}}(h),\end{aligned}$$

a left translation. Therefore $\mathsf{G}^L \triangleleft \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$. That is, G^L is a normal subgroup of $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$.

Given any $\Phi \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$, $L_{\Phi(\mathbf{1})} \in \mathsf{G}^L$ and $L_{\Phi(\mathbf{1})^{-1}} \circ \Phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ with $\Phi = (L_{\Phi(\mathbf{1})}) \circ (L_{\Phi(\mathbf{1})^{-1}} \circ \Phi)$. Definition A.2.8 implies that

$$\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \mathsf{G}^L \rtimes_\phi \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$$

with $\phi : \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \longrightarrow \mathrm{Aut}(\mathsf{G}^L)$ defined by $\phi(k) = \phi_k$ for $k \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ where $\phi_k : L_g \mapsto k \circ L_g \circ k^{-1}$ and product

$$\begin{aligned}(L_g, k) \cdot (L_h, k') &= (L_g \circ \phi_k(L_h), k \circ k') \\ &= (L_g \circ (k \circ L_h \circ k^{-1}), k \circ k') \\ &= (L_g \circ L_{k(h)}, k \circ k') \\ &= (L_{g \cdot k(h)}, k \circ k'),\end{aligned}$$

from Theorem A.2.11.

As the homomorphism ϕ is not trivial, by Theorem A.2.12, this semidirect product is not a direct product. □

The correspondence between Lie algebra automorphisms and Lie group automorphisms given in Proposition 4.1.7 allows us to use the results of Chapter 3 on the Lie algebra $\mathfrak{t}$ to compute the left-invariant sub-Riemannian structures of T up to Lie group automorphism. This computation is executed in the sections that follow.

4.2 Sub-Riemannian structures

Left-invariant sub-Riemannian structures $(\mathsf{T}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{T}, \mathcal{D}', \mathbf{g}')$ on the Lie group T are said to be isometric if there exists an isometry ϕ, in the sense of Definition 4.1.2, between them. We will denote such an isometry by

$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \equiv (\mathsf{T}, \mathcal{D}', \mathbf{g}').$$

If the isometry ϕ is in addition a group isomorphism of the Lie group T — that is, it is an automorphism — we will denote the automorphy between these structures by

$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \cong (\mathsf{T}, \mathcal{D}', \mathbf{g}').$$

In the following sections, utilizing the observations of the previous section, we proceed to classify the left-invariant sub-Riemannian structures $(\mathsf{T}, \mathcal{D}, \mathbf{g})$ of the Lie group T up to automorphy, before later generalising this classification to isometries.

Lemma 4.2.1. *Let $(\mathsf{G}, \mathcal{D}_1, \mathbf{g})$ be a left-invariant sub-Riemannian structure on a simply connected matrix Lie group G and $\mathcal{D}_2$ be a left-invariant distribution on G. If there exists $\phi \in \mathrm{Aut}(\mathsf{G})$ such that $\phi_* \mathcal{D}_1 = \mathcal{D}_2$, then there exists a metric $\mathbf{g}'$ on $\mathcal{D}_2$ such that*

$$(\mathsf{G}, \mathcal{D}_1, \mathbf{g}) \cong (\mathsf{G}, \mathcal{D}_2, \mathbf{g}').$$

Proof. Let $(\mathsf{G}, \mathcal{D}_1, \mathbf{g})$ be a left-invariant sub-Riemannian structure on a simply connected matrix Lie group G and $\mathcal{D}_2$ be a left-invariant distribution on G. Suppose $\phi \in \mathrm{Aut}(\mathsf{G})$ is such that $\phi_* \mathcal{D}_1 = \mathcal{D}_2$. That is, for $h \in \mathsf{G}$

$$(\phi_* \mathcal{D}_1)(h) = d_{\phi^{-1}(h)} \phi \cdot \mathcal{D}_1(\phi^{-1}(h)) = \mathcal{D}_2(h).$$

As $\mathcal{D}_1(h) = d_1 L_h \cdot \mathcal{D}_1(\mathbf{1})$, we have that

$$\mathcal{D}_2(h) = d_{\phi^{-1}(h)} \phi \cdot d_1 L_{\phi^{-1}(h)} \cdot \mathcal{D}_1(\mathbf{1}).$$

Define $\mathbf{g}' = \phi_* \mathbf{g}$. That is, for $h \in \mathsf{G}$ and $A, B \in \mathcal{D}_2(\mathbf{1}) \subseteq \mathfrak{g}$

$$\begin{aligned}\mathbf{g}'_h(d_1 L_h \cdot A, d_1 L_h \cdot B) &= (\phi_* \mathbf{g})_h (d_1 L_h \cdot A, d_1 L_h \cdot B) \\ &= \mathbf{g}_{\phi^{-1}(h)}(d_h \phi^{-1} \cdot d_1 L_h \cdot A, d_h \phi^{-1} \cdot d_1 L_h \cdot B).\end{aligned}$$

Now, if $h \in \mathsf{G}$ and $A \in \mathcal{D}_2(\mathbf{1})$, then

$$d_1 L_h \cdot A \in d_{\phi^{-1}(h)} \phi \cdot d_1 L_{\phi^{-1}(h)} \cdot \mathcal{D}_1(\mathbf{1}).$$

We then have that

$$\begin{aligned}d_h \phi^{-1} \cdot d_1 L_h \cdot A &\in d_h \phi^{-1} \cdot d_{\phi^{-1}(h)} \phi \cdot d_1 L_{\phi^{-1}(h)} \cdot \mathcal{D}_1(\mathbf{1}) \\ &= d_{\phi^{-1}(h)} (\phi^{-1} \circ \phi) \cdot d_1 L_{\phi^{-1}(h)} \cdot \mathcal{D}_1(\mathbf{1}) \\ &= d_{\phi^{-1}(h)} \mathrm{Id}_{\mathsf{G}} \cdot d_1 L_{\phi^{-1}(h)} \cdot \mathcal{D}_1(\mathbf{1}) \\ &= d_1 L_{\phi^{-1}(h)} \cdot \mathcal{D}_1(\mathbf{1}) \\ &= \mathcal{D}_1(\phi^{-1}(h)).\end{aligned}$$

We thus have that the $\mathbf{g}'$ is well defined.

Suppose $h \in \mathsf{G}$ and $A, B \in \mathcal{D}_1(\mathbf{1})$.

$$\begin{aligned}(\phi^* \mathbf{g}')_h (d_1 L_h \cdot A, d_1 L_h \cdot B) &= \mathbf{g}'_{\phi(h)}(d_h \phi \cdot d_1 L_h \cdot A, d_h \phi \cdot d_1 L_h \cdot B) \\ &= (\phi_* \mathbf{g})_{\phi(h)}(d_h \phi \cdot d_1 L_h \cdot A, d_h \phi \cdot d_1 L_h \cdot B) \\ &= \mathbf{g}_{\phi^{-1}(\phi(h))}(d_{\phi(h)} \phi^{-1} \cdot d_h \phi \cdot d_1 L_h \cdot A, d_{\phi(h)} \phi^{-1} \cdot d_h \phi \cdot d_1 L_h \cdot B) \\ &= \mathbf{g}_h(d_h(\phi^{-1} \circ \phi) \cdot d_1 L_h \cdot A, d_h(\phi^{-1} \circ \phi) \cdot d_1 L_h \cdot B) \\ &= \mathbf{g}_h(d_h \mathrm{Id}_{\mathsf{G}} \cdot d_1 L_h \cdot A, d_h \mathrm{Id}_{\mathsf{G}} \cdot d_1 L_h \cdot B) \\ &= \mathbf{g}_h(d_1 L_h \cdot A, d_1 L_h \cdot B).\end{aligned}$$

This implies that $\mathbf{g} = \phi^*\mathbf{g}'$. As we in addition have that $\phi_*\mathcal{D}_1 = \mathcal{D}_2$, Proposition 4.1.7 and Definition 4.1.2 give

$$(\mathsf{G}, \mathcal{D}_1, \mathbf{g}) \cong (\mathsf{G}, \mathcal{D}_2, \mathbf{g}').$$

□

Lemma 4.2.2. *Let $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ be a left-invariant sub-Riemannian structure on a simply connected matrix Lie group G and $\mathcal{H}$ a left-invariant distribution on G with $\varphi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{H}(\mathbf{1})$ for some $\varphi \in \mathrm{Aut}(\mathfrak{g})$. Then there exists a sub-Riemannian metric $\mathbf{g}'$ on $\mathcal{H}$ such that*

$$(\mathsf{G}, \mathcal{D}, \mathbf{g}) \cong (\mathsf{G}, \mathcal{H}, \mathbf{g}').$$

Proof. For every left-invariant generating distribution $\mathcal{D}$ on G, Lemma 3.1.13 gives that if $\mathcal{D}(\mathbf{1})$ is a generating subspace of $\mathfrak{g}$, then $\varphi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{H}(\mathbf{1})$ is a generating subspace of $\mathfrak{g}$. The proof of Proposition 4.1.7 gives that there exists $\phi \in \mathrm{Aut}(\mathsf{G})$ such that $\phi_*\mathcal{D} = \mathcal{H}$. Lemma 4.2.1 then implies that

$$(\mathsf{G}, \mathcal{D}, \mathbf{g}) \cong (\mathsf{G}, \mathcal{H}, \mathbf{g}'),$$

for some appropriately defined sub-Riemannian metric $\mathbf{g}'$. □

Definition 4.2.3. *Suppose $\mathfrak{s}$ is a subspace of a Lie algebra $\mathfrak{g}$ and let $\mathsf{GL}(\mathfrak{s})$ be the collection of all invertible linear maps from $\mathfrak{s}$ to itself. Define*

$$\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{g}) = \{\psi \in \mathsf{GL}(\mathfrak{s}) : \exists\tilde{\psi} \in \mathrm{Aut}(\mathfrak{g}), \tilde{\psi} \cdot \mathfrak{s} = \mathfrak{s} \text{ and } \tilde{\psi}|_{\mathfrak{s}} = \psi\}.$$

Definition 4.2.4. *Suppose $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is a left-invariant sub-Riemannian structure, we define the matrix $X_{\mathbf{g}}$ to be the matrix representation of $\mathbf{g_1}$ with respect to some ordered basis of $\mathcal{D}(\mathbf{1})$. The matrix $X_{\mathbf{g}}$ is a positive definite matrix.*

Lemma 4.2.5. *Let $(\mathsf{G}, \mathcal{H}, \mathbf{g})$ and $(\mathsf{G}, \mathcal{H}, \mathbf{g}')$ be left-invariant sub-Riemannian structures on a simply connected matrix Lie group G. We have that $X_{\mathbf{g}'} = \psi^\top X_{\mathbf{g}} \psi$ for some $\psi \in \mathrm{Aut}_{\mathcal{H}(\mathbf{1})}(\mathfrak{g})$ if and only if $(\mathsf{G}, \mathcal{H}, \mathbf{g}) \cong (\mathsf{G}, \mathcal{H}, \mathbf{g}')$.*

Proof. Suppose $X_{\mathbf{g}'} = \psi^\top X_{\mathbf{g}} \psi$ for some $\psi \in \mathrm{Aut}_{\mathcal{H}(\mathbf{1})}(\mathfrak{g})$. Let $A, B \in \mathcal{H}(\mathbf{1})$, then

$$\begin{aligned}
\mathbf{g}'_{\mathbf{1}}(A, B) &= A^\top X_{\mathbf{g}'} B \\
&= A^\top (\psi^\top X_{\mathbf{g}} \psi) B \\
&= (\psi \cdot A)^\top X_{\mathbf{g}} (\psi \cdot B) \\
&= \mathbf{g_1}(\psi \cdot A, \psi \cdot B).
\end{aligned}$$

As $\psi \in \mathrm{Aut}_{\mathcal{H}(\mathbf{1})}(\mathfrak{g})$,there exists $\tilde{\psi} \in \mathrm{Aut}(\mathfrak{g})$ such that $\tilde{\psi} \cdot \mathcal{H}(\mathbf{1}) = \mathcal{H}(\mathbf{1})$ and $\tilde{\psi}|_{\mathcal{H}(\mathbf{1})} = \psi$. We thus have that

$$\mathbf{g_1}(\psi \cdot A, \psi \cdot B) = \mathbf{g_1}(\tilde{\psi} \cdot A, \tilde{\psi} \cdot B), \text{ for } A, B \in \mathcal{H}(\mathbf{1}) \subseteq \mathfrak{g}.$$

Thus there exist $\tilde{\psi} \in \mathrm{Aut}(\mathfrak{g})$ such that

$$\mathbf{g}'_{\mathbf{1}}(A, B) = \mathbf{g_1}(\tilde{\psi} \cdot A, \tilde{\psi} \cdot B)$$

for $A, B \in \mathcal{H}(\mathbf{1})$ and

$$\tilde{\psi} \cdot \mathcal{H}(\mathbf{1}) = \mathcal{H}(\mathbf{1}).$$

Proposition 4.1.7 implies that $(\mathsf{G}, \mathcal{H}, \mathbf{g}) \cong (\mathsf{G}, \mathcal{H}, \mathbf{g}')$.

Conversely, suppose $(\mathsf{G}, \mathcal{H}, \mathbf{g})$ and $(\mathsf{G}, \mathcal{H}, \mathbf{g}')$ are automorphic left-invariant sub-Riemannian structures on G. Proposition 4.1.7 implies that there exists $\hat{\varphi} \in \mathrm{Aut}(\mathfrak{g})$ such that

$$\hat{\varphi} \cdot \mathcal{H}(\mathbf{1}) = \mathcal{H}(\mathbf{1})$$

and

$$\mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}'_{\mathbf{1}}(\hat{\varphi} \cdot A, \hat{\varphi} \cdot B)$$

for every $A, B \in \mathcal{H}(\mathbf{1})$.

Define $\varphi \in \mathsf{GL}(\mathcal{H}(\mathbf{1}))$, by $\varphi \cdot A = \hat{\varphi} \cdot A$ for every $A \in \mathcal{H}(\mathbf{1})$, we then have that $\varphi \in \mathrm{Aut}_{\mathcal{H}(\mathbf{1})}(\mathfrak{g})$ and in addition

$$\mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}'_{\mathbf{1}}(\hat{\varphi} \cdot A, \hat{\varphi} \cdot B) = \mathbf{g}'_{\mathbf{1}}(\varphi \cdot A, \varphi \cdot B)$$

for any $A, B \in \mathcal{H}(\mathbf{1})$.

With $X_{\mathbf{g}}$ and $X_{\mathbf{g}'}$ as the positive definite matrix representations of $\mathbf{g}_{\mathbf{1}}$ and $\mathbf{g}'_{\mathbf{1}}$ with respect to some basis of $\mathcal{H}(\mathbf{1})$, in coordinates this gives

$$\begin{aligned} A^\top X_{\mathbf{g}} B &= (\varphi \cdot A)^\top X_{\mathbf{g}'} (\varphi \cdot B) \\ &= A^\top (\varphi^\top X_{\mathbf{g}'} \varphi) B, \end{aligned}$$

for all $A, B \in \mathcal{H}(\mathbf{1})$. This implies that $X_{\mathbf{g}} = \varphi^\top X_{\mathbf{g}'} \varphi$. □

Now we specialize the previous Lemmas to left-invariant sub-Riemannian structures on our particular Lie group T. We define $\mathcal{H}_3$, $\mathcal{H}_4$ and $\mathcal{H}_5$ to be the left-invariant bracket generating distributions on T having $\mathcal{H}_3(\mathbf{1}) = \langle I, J, K \rangle$, $\mathcal{H}_4(\mathbf{1}) = \langle I, J, K, L \rangle$ and $\mathcal{H}_5(\mathbf{1}) = \langle I, J, K, L, M \rangle = \mathfrak{t}$.

Corollary 4.2.6. *Consider arbitrary left-invariant sub-Riemannian structures* $(\mathsf{T}, \mathcal{D}_3, \mathbf{g}^a)$, $(\mathsf{T}, \mathcal{D}_4, \mathbf{g}^b)$ *and* $(\mathsf{T}, \mathcal{D}_5, \mathbf{g}^c)$ *on* T*, of rank* 3, 4 *and* 5 *respectively. There exists metrics* $\mathbf{g}^{(3)}$*,* $\mathbf{g}^{(4)}$ *and* $\mathbf{g}^{(5)}$ *on* $\mathcal{H}_3$*,* $\mathcal{H}_4$ *and* $\mathcal{H}_5$ *respectively such that*

1. $(\mathsf{T}, \mathcal{D}_3, \mathbf{g}^a) \cong (\mathsf{T}, \mathcal{H}_3, \mathbf{g}^{(3)})$,
2. $(\mathsf{T}, \mathcal{D}_4, \mathbf{g}^b) \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{g}^{(4)})$ *and*
3. $(\mathsf{T}, \mathcal{D}_5, \mathbf{g}^c) \cong (\mathsf{T}, \mathcal{H}_5, \mathbf{g}^{(5)})$.

Proof. Follows from Theorem 3.4.2, Lemma 4.2.2 and the fact that $\mathcal{H}_3$, $\mathcal{H}_4$ and $\mathcal{H}_5$ are generating distributions. □

We now specify $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{t})$ when $\mathfrak{s}$ is taken to be one of the bracket generating subspaces $\mathcal{H}_3(\mathbf{1})$, $\mathcal{H}_4(\mathbf{1})$ and $\mathcal{H}_5(\mathbf{1})$ of $\mathfrak{t}$.

Lemma 4.2.7.

1. $$\mathrm{Aut}_{\langle I,J,K\rangle}(\mathfrak{t}) = \left\{ \begin{bmatrix} i_1 & 0 & 0 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{bmatrix} : i_1 \neq 0 \text{ and } j_2k_3 - k_2j_3 \neq 0 \right\}$$
   were the matrix representations of the elements φ *of* $\mathrm{Aut}_{\langle I,J,K\rangle}(\mathfrak{t})$ *are written with respect to the ordered basis* (I, J, K) *of* $\langle I, J, K\rangle$.

2. $$\mathrm{Aut}_{\langle I,J,K,L\rangle}(\mathfrak{t}) = \left\{ \begin{bmatrix} i_1 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 \\ i_3 & 0 & k_3 & 0 \\ i_4 & j_4 & k_4 & i_1j_2 \end{bmatrix} : i_1 \neq 0 \text{ and } j_2k_3 \neq 0 \right\}$$
   were the matrix representations of the elements φ *of* $\mathrm{Aut}_{\langle I,J,K,L\rangle}(\mathfrak{t})$ *are written with respect to the ordered basis* (I, J, K, L) *of* $\langle I, J, K, L\rangle$.

3. $\mathrm{Aut}_{\mathfrak{t}}(\mathfrak{t}) = \mathrm{Aut}(\mathfrak{t})$.

Proof. Let
$$\tilde{\psi} = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & j_3 & k_3 & 0 & 0 \\ i_4 & j_4 & k_4 & i_1j_2 & i_1k_2 \\ i_5 & j_5 & k_5 & i_1j_3 & i_1k_3 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}),$$
with respect to the ordered basis (I, J, K, L, M) of $\mathfrak{t}$. That is, $i_1 \neq 0$ and $j_2k_3 - k_2j_3 \neq 0$ (see, Proposition 3.1.1).

Proof of (1): If $\tilde{\psi}$ is such that $\tilde{\psi} \cdot \langle I, J, K\rangle = \langle I, J, K\rangle$, then it simplifies to the form
$$\tilde{\psi} = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & j_3 & k_3 & 0 & 0 \\ 0 & 0 & 0 & i_1j_2 & i_1k_2 \\ 0 & 0 & 0 & i_1j_3 & i_1k_3 \end{bmatrix}.$$

Define $\psi \in \mathsf{GL}(\langle I, J, K\rangle)$ by $\psi \cdot A = \tilde{\psi} \cdot A$ for all $A \in \langle I, J, K\rangle$. We then have that $\psi \in \mathrm{Aut}_{\langle I,J,K\rangle}(\mathfrak{t})$, as $\psi \cdot \langle I, J, K\rangle = \langle I, J, K\rangle$ and $\tilde{\psi} \in \mathrm{Aut}(\mathfrak{t})$ with $\tilde{\psi}|_{\langle I,J,K\rangle} = \psi$. The matrix representation of ψ with respect to the ordered basis (I, J, K) of $\langle I, J, K\rangle$ is given by
$$\psi = \begin{bmatrix} i_1 & 0 & 0 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{bmatrix}.$$

Proof of (2): If $\tilde{\psi}$ is such that $\tilde{\psi} \cdot \langle I, J, K, L \rangle = \langle I, J, K, L \rangle$, then it simplifies to the form

$$\tilde{\psi} = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & 0 & k_3 & 0 & 0 \\ i_4 & j_4 & k_4 & i_1 j_2 & i_1 k_2 \\ 0 & 0 & 0 & 0 & i_1 k_3 \end{bmatrix}.$$

Define $\psi \in \mathsf{GL}(\langle I, J, K, L \rangle)$ by $\psi \cdot A = \tilde{\psi} \cdot A$ for all $A \in \langle I, J, K, L \rangle$. We then have that $\psi \in \mathrm{Aut}_{\langle I,J,K,L \rangle}(\mathfrak{t})$, in a similar way to the proof of (1). The matrix representation of ψ with respect to the ordered basis (I, J, K, L) of $\langle I, J, K, L \rangle$ is given by

$$\psi = \begin{bmatrix} i_1 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 \\ i_3 & 0 & k_3 & 0 \\ i_4 & j_4 & k_4 & i_1 j_2 \end{bmatrix}.$$

Proof of (3): This follows immediately from the definition of $\mathrm{Aut}_{\mathfrak{t}}(\mathfrak{t})$. We have shown that elements of $\mathsf{GL}(\mathfrak{s})$ of the desired form are elements of $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{t})$. Conversely, we show that any element of $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{t})$ is of the desired form.

Converse of (1): Suppose

$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{bmatrix} \in \mathrm{Aut}_{\langle I,J,K \rangle}(\mathfrak{t})$$

were the matrix representation of φ is with respect to the ordered basis (I, J, K) of $\mathcal{H}_3(\mathbf{1}) = \langle I, J, K \rangle$. We have that $\varphi \cdot \langle I, J, K \rangle = \langle I, J, K \rangle$ and that there exists $\tilde{\varphi} \in \mathrm{Aut}(\mathfrak{t})$ with $\tilde{\varphi}|_{\langle I,J,K \rangle} = \varphi$.

Let $\tilde{\varphi}$ have matrix representation

$$\tilde{\varphi} = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & j_3 & k_3 & 0 & 0 \\ i_4 & j_4 & k_4 & i_1 j_2 & i_1 k_2 \\ i_5 & j_5 & k_5 & i_1 j_3 & i_1 k_3 \end{bmatrix}$$

with respect to the ordered basis (I, J, K, L, M) of $\mathfrak{t}$. By Proposition 3.1.1 we have that $i_1 \neq 0$ and $j_2 k_3 - k_2 j_3 \neq 0$. As $\tilde{\varphi} \cdot \langle I, J, K \rangle = \varphi \cdot \langle I, J, K \rangle = \langle I, J, K \rangle$, $\tilde{\varphi}$ preserves the subspace $\langle I, J, K \rangle$ of $\mathfrak{t}$. This implies that its matrix representation takes the form

$$\tilde{\varphi} = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & j_3 & k_3 & 0 & 0 \\ 0 & 0 & 0 & i_1 j_2 & i_1 k_2 \\ 0 & 0 & 0 & i_1 j_3 & i_1 k_3 \end{bmatrix}.$$

We have that the matrix representation of φ with respect to the ordered basis (I, J, K) of $\langle I, J, K\rangle$ is given by

$$\varphi = \begin{bmatrix} | & | & | \\ \varphi \cdot I & \varphi \cdot J & \varphi \cdot K \\ | & | & | \end{bmatrix}.$$

As $\tilde{\varphi}|_{\langle I,J,K\rangle} = \varphi$, we have

$$\begin{aligned} \varphi \cdot I &= \tilde{\varphi} \cdot I = i_1 I + i_2 J + i_3 K \\ \varphi \cdot J &= \tilde{\varphi} \cdot J = 0 \cdot I + j_2 J + j_3 K \\ \varphi \cdot K &= \tilde{\varphi} \cdot K = 0 \cdot I + k_2 J + k_3 K \end{aligned}$$

and thus

$$\varphi = \begin{bmatrix} i_1 & 0 & 0 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{bmatrix}$$

with $i_1 \neq 0$ and $j_2 k_3 - k_2 j_3 \neq 0$.

Converse of (2): Suppose

$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & \varphi_{14} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & \varphi_{24} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} & \varphi_{34} \\ \varphi_{41} & \varphi_{42} & \varphi_{43} & \varphi_{44} \end{bmatrix} \in \mathrm{Aut}_{\langle I,J,K,L\rangle}(\mathfrak{t})$$

were the matrix representation of φ is with respect to the ordered basis (I, J, K, L) of $\mathcal{H}_4(\mathbf{1}) = \langle I, J, K, L\rangle$. We have that $\varphi \cdot \langle I, J, K, L\rangle = \langle I, J, K, L\rangle$ and that there exists $\tilde{\varphi} \in \mathrm{Aut}(\mathfrak{t})$ with $\tilde{\varphi}|_{\langle I,J,K,L\rangle} = \varphi$.

Let $\tilde{\varphi}$ have matrix representation

$$\tilde{\varphi} = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & j_3 & k_3 & 0 & 0 \\ i_4 & j_4 & k_4 & i_1 j_2 & i_1 k_2 \\ i_5 & j_5 & k_5 & i_1 j_3 & i_1 k_3 \end{bmatrix}$$

with respect to the ordered basis (I, J, K, L, M) of $\mathfrak{t}$. By Proposition 3.1.1 we have that $i_1 \neq 0$ and $j_2 k_3 - k_2 j_3 \neq 0$. As $\tilde{\varphi} \cdot \langle I, J, K, L\rangle = \varphi \cdot \langle I, J, K, L\rangle = \langle I, J, K, L\rangle$, $\tilde{\varphi}$ preserves the subspace $\langle I, J, K, L\rangle$ of $\mathfrak{t}$. This implies that its matrix representation takes the form

$$\tilde{\varphi} = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & 0 & k_3 & 0 & 0 \\ i_4 & j_4 & k_4 & i_1 j_2 & i_1 k_2 \\ 0 & 0 & 0 & 0 & i_1 k_3 \end{bmatrix}.$$

We have that the matrix representation of φ with respect to the ordered basis (I, J, K, L) of $\langle I, J, K, L\rangle$ is given by

$$\varphi = \begin{bmatrix} | & | & | & | \\ \varphi \cdot I & \varphi \cdot J & \varphi \cdot K & \varphi \cdot L \\ | & | & | & | \end{bmatrix}.$$

As $\tilde{\varphi}|_{\langle I,J,K,L\rangle} = \varphi$, we have

$$\begin{aligned}
\varphi \cdot I &= \tilde{\varphi} \cdot I = i_1 I + i_2 J + i_3 K + i_4 L \\
\varphi \cdot J &= \tilde{\varphi} \cdot J = 0 \cdot I + j_2 J + 0 \cdot K + j_4 L \\
\varphi \cdot K &= \tilde{\varphi} \cdot K = 0 \cdot I + k_2 J + k_3 K + k_4 L \\
\varphi \cdot L &= \tilde{\varphi} \cdot L = 0 \cdot I + 0 \cdot J + 0 \cdot K + i_1 j_2 L
\end{aligned}$$

and thus

$$\varphi = \begin{bmatrix} i_1 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 \\ i_3 & 0 & k_3 & 0 \\ i_4 & j_4 & k_4 & i_1 j_2 \end{bmatrix}$$

with $i_1 \neq 0$ and $j_2 k_3 \neq 0$. □

4.3 Sub-Riemannian structures of T

With the above theory established, we now proceed with the computation of the left-invariant sub-Riemannian structures of the Lie group T.

Lemma 4.3.1. *Let $(\mathsf{T}, \mathcal{H}_3, \mathbf{g})$ be a left-invariant sub-Riemannian structure, then*

$$(\mathsf{T}, \mathcal{H}_3, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3)$$

where the metric $\mathbf{h}^3$ is specified by $X_{\mathbf{h}^3} = I_3$ with respect to the ordered basis (I, J, K) of $\mathcal{H}_3(\mathbf{1}) = \langle I, J, K\rangle$.

Proof. Suppose $(\mathsf{T}, \mathcal{H}_3, \mathbf{g})$ is a left-invariant sub-Riemannian structure. Concretely,

$$X_{\mathbf{g}} = \begin{bmatrix} h_1 & a_1 & a_2 \\ a_1 & h_2 & a_3 \\ a_2 & a_3 & h_3 \end{bmatrix},$$

a positive definite matrix. Let

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_1 h_3 - a_2 a_3}{a_3^2 - h_2 h_3} & 1 & 0 \\ \frac{a_2 h_2 - a_1 a_3}{a_3^2 - h_2 h_3} & -\frac{a_3}{h_3} & 1 \end{bmatrix} \in \mathrm{Aut}_{\langle I,J,K\rangle}(\mathfrak{t}).$$

The entries of ψ_1 are well defined as $(a_3^2 - h_2h_3)$ and h_3 are positive reals — following from Lemma A.1.2 and the pricipal minors $-(a_3^2 - h_2h_3)$ and h_3 of $X_{\mathbf{g}}$. Direct computation gives the positive definite matrix

$$\psi_1^\top X_{\mathbf{g}}\psi_1 = \begin{bmatrix} \frac{h_3a_1^2-2a_2a_3a_1+a_3^2h_1+h_2\left(a_2^2-h_1h_3\right)}{a_3^2-h_2h_3} & 0 & 0 \\ 0 & h_2-\frac{a_3^2}{h_3} & 0 \\ 0 & 0 & h_3 \end{bmatrix}.$$

Relabelling, we have that

$$\psi_1^\top X_{\mathbf{g}}\psi_1 = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}.$$

Lemma 4.2.5 implies that

$$(\mathsf{T}, \mathcal{H}_3, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_3, \mathbf{g}')$$

where $\mathbf{g}'$ is specified by $X_{\mathbf{g}'} = \psi_1^\top X_{\mathbf{g}}\psi_1$.

Now, let

$$\psi_2 = \begin{bmatrix} \frac{1}{\sqrt{b_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{b_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{b_3}} \end{bmatrix} \in \mathrm{Aut}_{\langle I,J,K\rangle}(\mathfrak{t}).$$

The entries of ψ_2 are well defines as b_1, b_2 and b_3 are positive by Lemma A.1.2 as they are principal minors of the matrix $X_{\mathbf{g}'} = \psi_1^\top X_{\mathbf{g}}\psi_1$. Direct computation gives

$$\psi_2^\top X_{\mathbf{g}'}\psi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

Lemma 4.2.5 thus gives that

$$(\mathsf{T}, \mathcal{H}_3, \mathbf{g}') \cong (\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3).$$

The desired result,

$$(\mathsf{T}, \mathcal{H}_3, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3)$$

follows by the transitivity of the relation $\cong$. □

Lemma 4.3.2. *Let* $(\mathsf{T}, \mathcal{H}_4, \mathbf{g})$ *be a left-invariant sub-Riemannian structure, then*

$$(\mathsf{T}, \mathcal{H}_4, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha})$$

where the Riemannian metric $\mathbf{h}^{4,\alpha}$ *is specified by* $X_{\mathbf{h}^{4,\alpha}} = \alpha I_4$, $\alpha > 0$, *with respect to the ordered basis* (I, J, K, L) *of* $\mathcal{H}_4(\mathbf{1}) = \langle I, J, K, L\rangle$. *Furthermore, for distinct* $\alpha, \beta > 0$, *the sub-Riemannian structures* $(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha})$ *and* $(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\beta})$ *are non-automorphic.*

Proof. Suppose $(\mathsf{T}, \mathcal{H}_4, \mathbf{g})$ is a left-invariant sub-Riemannian structure. Concretely,

$$X_{\mathbf{g}} = \begin{bmatrix} h_1 & a_1 & a_2 & a_3 \\ a_1 & h_2 & a_4 & a_5 \\ a_2 & a_4 & h_3 & a_6 \\ a_3 & a_5 & a_6 & h_4 \end{bmatrix},$$

a positive definite matrix. Let

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{a_3}{h_4} & -\frac{a_5}{h_4} & -\frac{a_6}{h_4} & 1 \end{bmatrix} \in \mathrm{Aut}_{\langle I,J,K,L\rangle}(\mathfrak{t}).$$

As $h_4 > 0$, the entries of ψ_1 are well defined — by Lemma A.1.2. Direct computation gives the positive definite matrix

$$\psi_1^\top X_{\mathbf{g}} \psi_1 = \begin{bmatrix} h_1 - \frac{a_3^2}{h_4} & a_1 - \frac{a_3 a_5}{h_4} & a_2 - \frac{a_3 a_6}{h_4} & 0 \\ a_1 - \frac{a_3 a_5}{h_4} & h_2 - \frac{a_5^2}{h_4} & a_4 - \frac{a_5 a_6}{h_4} & 0 \\ a_2 - \frac{a_3 a_6}{h_4} & a_4 - \frac{a_5 a_6}{h_4} & h_3 - \frac{a_6^2}{h_4} & 0 \\ 0 & 0 & 0 & h_4 \end{bmatrix}.$$

Relabelling, we have that

$$\psi_1^\top X_{\mathbf{g}} \psi_1 = \begin{bmatrix} h_1' & a_1' & a_2' & 0 \\ a_1' & h_2' & a_4' & 0 \\ a_2' & a_4' & h_3' & 0 \\ 0 & 0 & 0 & h_4' \end{bmatrix}.$$

Lemma 4.2.5 implies that

$$(\mathsf{T}, \mathcal{H}_4, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{g}')$$

where $\mathbf{g}'$ is specified by $X_{\mathbf{g}'} = \psi_1^\top X_{\mathbf{g}} \psi_1$.

Now, let

$$\psi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{a_1' h_3' - a_2' a_4'}{{a_4'}^2 - h_2' h_3'} & 1 & -\frac{a_4'}{h_2'} & 0 \\ \frac{a_2' h_2' - a_1' a_4'}{{a_4'}^2 - h_2' h_3'} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}_{\langle I,J,K,L\rangle}(\mathfrak{t}).$$

As ${a_4'}^2 - h_2' h_3' > 0$, the matrix ψ_2 is well defined — by Lemma A.1.2 as ${a_4'}^2 - h_2' h_3'$ is a principal minor of the matrix $X_{\mathbf{g}'}$. Direct computation gives the positive definite matrix

$$\psi_2^\top X_{\mathbf{g}'} \psi_2 = \begin{bmatrix} \frac{h_3' {a_1'}^2 - 2 a_2' a_4' a_1' + {a_4'}^2 h_1' + h_2' \left({a_2'}^2 - h_1' h_3'\right)}{{a_4'}^2 - h_2' h_3'} & 0 & 0 & 0 \\ 0 & h_2' & 0 & 0 \\ 0 & 0 & h_3' - \frac{{a_4'}^2}{h_2'} & 0 \\ 0 & 0 & 0 & h_4' \end{bmatrix}.$$

Relabelling, we have that

$$\psi_2^\top X_{\mathbf{g}'} \psi_2 = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{bmatrix}.$$

Lemma 4.2.5 implies that

$$(\mathsf{T}, \mathcal{H}_4, \mathbf{g}') \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{g}'')$$

where $\mathbf{g}''$ is specified by $X_{\mathbf{g}''} = \psi_2^\top X_{\mathbf{g}'}\psi_2$.

Finally, we let

$$\psi_3 = \begin{bmatrix} \frac{1}{\sqrt{b_1}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{b_2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{b_3}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{b_1}\sqrt{b_2}} \end{bmatrix} \in \mathrm{Aut}_{\langle I,J,K,L\rangle}(\mathfrak{t}).$$

This matrix is well defined as $b_1, b_2, b_3, b_4 > 0$ by LemmaA.1.2. Direct computation gives

$$\psi_3^\top X_{\mathbf{g}''}\psi_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{b_4}{b_1 b_2} \end{bmatrix}.$$

Lemma 4.2.5 implies that

$$(\mathsf{T}, \mathcal{H}_4, \mathbf{g}'') \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{g}^{(3)})$$

where $\mathbf{g}^{(3)}$ is specified by $X_{\mathbf{g}^{(3)}} = \psi_3^\top X_{\mathbf{g}''}\psi_3$.

Let $\lambda = \frac{b_4}{b_1 b_2}$ and

$$\psi_4 = \begin{bmatrix} \frac{1}{\sqrt{\lambda}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{bmatrix} \in \mathrm{Aut}_{\langle I,J,K,L\rangle}(\mathfrak{t}).$$

We then have that

$$\psi_4^\top X_{\mathbf{g}^{(3)}}\psi_4 = \begin{bmatrix} \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{bmatrix}.$$

Lemma 4.2.5 implies that

$$(\mathsf{T}, \mathcal{H}_4, \mathbf{g}^{(3)}) \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha})$$

where $\mathbf{h}^{4,\alpha}$ is specified by $X_{\mathbf{h}^{4,\alpha}} = \psi_4^\top X_{\mathbf{g}^{(3)}}\psi_4 = \frac{1}{\lambda} I_4 = \alpha I_4$, where $\alpha = \frac{1}{\lambda} = \frac{b_1 b_2}{b_4} > 0$. Transitivity of the automorphism relation $\cong$ gives the result

$$(\mathsf{T}, \mathcal{H}_4, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha}).$$

We now show that each positive real α determines a unique left-invariant sub-Riemannian structure up to automorphy. That is, given $\alpha, \beta > 0$

$$(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha}) \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\beta})$$

if and only if $\alpha = \beta$. Clearly, if $\alpha = \beta > 0$ then we have that $(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha}) \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\beta})$. Conversely, suppose $\alpha, \beta > 0$ and

$$(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha}) \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\beta}).$$

By Lemma 4.2.5 we have that $X_{\mathbf{h}^{4,\beta}} = \psi^\top X_{\mathbf{h}^{4,\alpha}} \psi$ for some $\psi \in \mathrm{Aut}_{\langle I,J,K,L\rangle}(\mathfrak{t})$. It follows that $X_{\mathbf{h}^{4,\beta}} = \psi^\top X_{\mathbf{h}^{4,\alpha}} \psi$ if and only if $\psi^\top(\alpha I_4)\psi = \beta I_4$.

Let $Eq(i,j)$ be the scalar equation determined by the (i,j)-entry in the matrix equation $\psi^\top(\alpha I_4)\psi = \beta I_4$, where

$$\psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} \\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} \end{bmatrix}.$$

The equations

$$\begin{aligned} Eq(4,1) &: \alpha\psi_{11}\psi_{22}\psi_{41} = 0 \\ Eq(4,2) &: \alpha\psi_{11}\psi_{22}\psi_{42} = 0 \\ Eq(4,3) &: \alpha\psi_{11}\psi_{22}\psi_{43} = 0 \end{aligned}$$

imply that $\psi_{41} = \psi_{42} = \psi_{43} = 0$ or $\psi_{22} = 0$ as $\psi_{11} \neq 0$. However if $\psi_{22} = 0$ we have that $0 = \alpha\psi_{11}^2\psi_{22}^2$, contradicting

$$Eq(4,4) : \alpha\psi_{11}^2\psi_{22}^2 = \beta > 0.$$

Thus $\psi_{41} = \psi_{42} = \psi_{43} = 0$ and $\psi_{22} \neq 0$.

Also,

$$Eq(3,2) : \alpha(\psi_{22}\psi_{23} + \psi_{42}\psi_{43}) = 0.$$

That is, $\alpha(\psi_{22}\psi_{23}) = 0$. As $\psi_{22}, \alpha \neq 0$, we have $\psi_{23} = 0$. Now, $\beta I_4 = \psi^\top(\alpha I_4)\psi$ reduces to

$$\beta I_4 = \begin{bmatrix} \alpha\left(\psi_{11}^2 + \psi_{21}^2 + \psi_{31}^2\right) & \alpha\psi_{21}\psi_{22} & \alpha\psi_{31}\psi_{33} & 0 \\ \alpha\psi_{21}\psi_{22} & \alpha\psi_{22}^2 & 0 & 0 \\ \alpha\psi_{31}\psi_{33} & 0 & \alpha\psi_{33}^2 & 0 \\ 0 & 0 & 0 & \alpha\psi_{11}^2\psi_{22}^2 \end{bmatrix}.$$

As $\alpha \neq 0, \psi_{22} \neq 0$ and $\alpha\psi_{21}\psi_{22} = 0$ we have that $\psi_{21} = 0$. As $\alpha\psi_{33}^2 = \beta \neq 0$, we have $\psi_{33} \neq 0$. It then follow that $\alpha\psi_{31}\psi_{33} = 0$ implies that $\psi_{31} = 0$. Our matrix equation then reduces to

$$\beta I_4 = \begin{bmatrix} \alpha\psi_{11}^2 & 0 & 0 & 0 \\ 0 & \alpha\psi_{22}^2 & 0 & 0 \\ 0 & 0 & \alpha\psi_{33}^2 & 0 \\ 0 & 0 & 0 & \alpha\psi_{11}^2\psi_{22}^2 \end{bmatrix}.$$

From equations $Eq(1,1), Eq(2,2)$ and $Eq(3,3)$ it follows that $\psi_{11} = \psi_{22} = \psi_{33} = \sqrt{\frac{\beta}{\alpha}}$. This reduces $Eq(4,4)$ to $\beta = \frac{\beta^2}{\alpha}$ and thus $\beta = \alpha$ as required. □

Lemma 4.3.3. *Let* $(\mathsf{T}, \mathcal{H}_5, \mathbf{g})$ *be a left-invariant sub-Riemannian structure, then*

$$(\mathsf{T}, \mathcal{H}_5, \mathbf{g}) \cong \left(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)}\right)$$

where the metric $\mathbf{h}^{5,(\alpha,\beta)}$ *is specified by*

$$X_{\mathbf{h}^{5,(\alpha,\beta)}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \beta \end{bmatrix}$$

for some $\alpha, \beta > 0$, *with respect to the ordered basis* (I, J, K, L, M) *of* $\mathcal{H}_5(\mathbf{1}) = \mathfrak{t}$. *Furthermore, every left-invariant sub-Riemannian structure* $(\mathsf{T}, \mathcal{H}_5, \mathbf{g})$ *is automorphic to a unique structure* $(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)})$ *with* $\alpha \geq \beta > 0$.

Proof. Suppose $(\mathsf{T}, \mathcal{H}_5, \mathbf{g})$ is a left-invariant sub-Riemannian structure. Concretely

$$X_{\mathbf{g}} = \begin{bmatrix} h_1 & a_1 & a_2 & a_3 & a_4 \\ a_1 & h_2 & a_5 & a_6 & a_7 \\ a_2 & a_5 & h_3 & a_8 & a_9 \\ a_3 & a_6 & a_8 & h_4 & a_{10} \\ a_4 & a_7 & a_9 & a_{10} & h_5 \end{bmatrix},$$

a positive definite matrix. Let, with the use of Lemma 4.2.7,

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{a_{10}}{h_5} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{a_4}{h_5} & \frac{a_{10}a_9 - a_7 h_5}{{h_5}^2} & -\frac{a_9}{h_5} & -\frac{a_{10}}{h_5} & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}).$$

The entries of ψ_1 are well defined, as $h_5 > 0$ — by Lemma A.1.2. Direct computation gives a positive definite matrix of the form

$$\psi_1^\top X_{\mathbf{g}} \psi_1 = \begin{bmatrix} {h_1}^{(1)} & {a_1}^{(1)} & {a_2}^{(1)} & {a_3}^{(1)} & 0 \\ {a_1}^{(1)} & {h_2}^{(1)} & {a_5}^{(1)} & {a_6}^{(1)} & 0 \\ {a_2}^{(1)} & {a_5}^{(1)} & {h_3}^{(1)} & {a_8}^{(1)} & 0 \\ {a_3}^{(1)} & {a_6}^{(1)} & {a_8}^{(1)} & {h_4}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & {h_5}^{(1)} \end{bmatrix}.$$

Lemma 4.2.5 implies that

$$(\mathsf{T}, \mathcal{H}_5, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_5, \mathbf{g}^{(1)})$$

where $\mathbf{g}^{(1)}$ is specified by $X_{\mathbf{g}^{(1)}} = \psi_1^\top X_{\mathbf{g}} \psi_1$.

Let

$$\psi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\frac{{a_3}^{(1)}}{{h_4}^{(1)}} & -\frac{{a_6}^{(1)}}{{h_4}^{(1)}} & -\frac{{a_8}^{(1)}}{{h_4}^{(1)}} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}).$$

The entries of ψ_2 are well defined, as ${h_4}^{(1)} > 0$ — by Lemma A.1.2. Direct computation give a positive definite matrix of the form

$$\psi_2^\top X_{\mathbf{g}^{(1)}} \psi_2 = \begin{bmatrix} h_1^{(2)} & a_1^{(2)} & a_2^{(2)} & 0 & 0 \\ a_1^{(2)} & h_2^{(2)} & a_5^{(2)} & 0 & 0 \\ a_2^{(2)} & a_5^{(2)} & h_3^{(2)} & 0 & 0 \\ 0 & 0 & 0 & h_4^{(2)} & 0 \\ 0 & 0 & 0 & 0 & h_5^{(2)} \end{bmatrix}.$$

Lemma 4.2.5 implies that

$$(\mathsf{T}, \mathcal{H}_5, \mathbf{g}^{(1)}) \cong (\mathsf{T}, \mathcal{H}_5, \mathbf{g}^{(2)})$$

where $\mathbf{g}^{(2)}$ is specified by $X_{\mathbf{g}^{(2)}} = \psi_2^\top X_{\mathbf{g}^{(1)}} \psi_2$.

Let

$$\psi_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{a_2^{(2)}}{h_3^{(2)}} & -\frac{a_5^{(2)}}{h_3^{(2)}} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{a_5^{(2)}}{h_3^{(2)}} & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}).$$

The entries of ψ_3 are well defined, as $h_3^{(2)} > 0$ — by Lemma A.1.2. Direct computation gives a positive definite matrix of the form

$$\psi_3 X_{\mathbf{g}^{(2)}} \psi_3 = \begin{bmatrix} h_1^{(3)} & a_1^{(3)} & 0 & 0 & 0 \\ a_1^{(3)} & h_2^{(3)} & 0 & 0 & 0 \\ 0 & 0 & h_3^{(3)} & 0 & 0 \\ 0 & 0 & 0 & h_4^{(3)} & 0 \\ 0 & 0 & 0 & 0 & h_5^{(3)} \end{bmatrix}.$$

Lemma 4.2.5 implies that

$$(\mathsf{T}, \mathcal{H}_5, \mathbf{g}^{(2)}) \cong (\mathsf{T}, \mathcal{H}_5, \mathbf{g}^{(3)}),$$

where $\mathbf{g}^{(3)}$ is specified by $X_{\mathbf{g}^{(3)}} = \psi_3^\top X_{\mathbf{g}^{(2)}} \psi_3$.

Let

$$\psi_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{a_1^{(3)}}{h_2^{(3)}} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}).$$

The entries of ψ_4 are well defined, as $h_2^{(3)} > 0$ — by Lemma A.1.2. Direct computation gives a positive definite matrix of the form

$$\psi_4^\top X_{\mathbf{g}^{(3)}} \psi_4 = \begin{bmatrix} b_1 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & 0 & b_5 \end{bmatrix}.$$

Lemma 4.2.5 implies that

$$(\mathsf{T}, \mathcal{H}_5, \mathbf{g}^{(3)}) \cong (\mathsf{T}, \mathcal{H}_5, \mathbf{g}^{(4)})$$

where $\mathbf{g}^{(4)}$ is specified by $X_{\mathbf{g}^{(4)}} = \psi_4^\top X_{\mathbf{g}^{(3)}} \psi_4$.

Let

$$\psi_5 = \begin{bmatrix} \frac{1}{\sqrt{b_1}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{b_2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{b_3}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{b_1}\sqrt{b_2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{b_1}\sqrt{b_3}} \end{bmatrix},$$

Now, ψ_5 is well defined as $b_i > 0$ for $i = 1, 2, ..., 5$ by Lemma A.1.2. Direct computation gives

$$\psi_5^\top X_{\mathbf{g}^{(4)}} \psi_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{b_4}{b_1 b_2} & 0 \\ 0 & 0 & 0 & 0 & \frac{b_5}{b_1 b_3} \end{bmatrix}.$$

Lemma 4.2.5 implies that

$$(\mathsf{T}, \mathcal{H}_5, \mathbf{g}^{(4)}) \cong (\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)})$$

where $\mathbf{h}^{5(\alpha,\beta)}$ is specified by $X_{\mathbf{h}^{5(\alpha,\beta)}} = \psi_5^\top X_{\mathbf{g}^{(4)}} \psi_5$ — taking $\alpha = \frac{b_4}{b_1 b_2}$ and $\beta = \frac{b_5}{b_1 b_3}$.

Transitivity of the automorphism relation $\cong$ gives the result

$$(\mathsf{T}, \mathcal{H}_5, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)}).$$

We determine which left-invariant sub-Riemannian structures of the form $(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)})$ are automorphic. Suppose $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$ and $(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha_1,\beta_1)}) \cong (\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha_2,\beta_2)})$ By Lemma 4.2.5 we have that $X_{\mathbf{h}^{5,(\alpha_1,\beta_1)}} = \psi^\top X_{\mathbf{h}^{5,(\alpha_2,\beta_2)}} \psi$ for some $\psi \in \mathrm{Aut}(\mathfrak{t})$. Let

$$\psi = \begin{bmatrix} \psi_{11} & 0 & 0 & 0 & 0 \\ \psi_{21} & \psi_{22} & \psi_{23} & 0 & 0 \\ \psi_{31} & \psi_{32} & \psi_{33} & 0 & 0 \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{11}\psi_{22} & \psi_{11}\psi_{23} \\ \psi_{51} & \psi_{52} & \psi_{53} & \psi_{11}\psi_{32} & \psi_{11}\psi_{33} \end{bmatrix}.$$

If $\psi_{23} = 0$, then $X_{\mathbf{h}^{5,(\alpha_1,\beta_1)}} = \psi^\top X_{\mathbf{h}^{5,(\alpha_2,\beta_2)}} \psi$ gives the component equations

$$\begin{aligned}
Eq(5,1) &: \beta_2 \psi_{11} \psi_{33} \psi_{51} = 0 \\
Eq(5,2) &: \beta_2 \psi_{11} \psi_{33} \psi_{52} = 0 \\
Eq(5,3) &: \beta_2 \psi_{11} \psi_{33} \psi_{53} = 0 \\
Eq(5,3) &: \beta_2 \psi_{11}^2 \psi_{32} \psi_{33} = 0.
\end{aligned}$$

As $\beta_2 > 0$ and Proposition 3.1.1 implies that $\psi_{11} \neq 0$ and $\psi_{22}\psi_{33} - \psi_{23}\psi_{32} \neq 0$ — thus $\psi_{33} \neq 0$ — we have that $\psi_{51} = \psi_{52} = \psi_{53} = 0$ and $\psi_{32} = 0$.

Taking this into account we have the reduced component equations:

$$\begin{aligned}
Eq(4,1) &: \alpha_2 \psi_{11} \psi_{22} \psi_{41} = 0 \\
Eq(4,2) &: \alpha_2 \psi_{11} \psi_{22} \psi_{42} = 0 \\
Eq(4,3) &: \alpha_2 \psi_{11} \psi_{22} \psi_{43} = 0.
\end{aligned}$$

As $\alpha_2 > 0$ and by Proposition 3.1.1 $\psi_{11} \neq 0$ and $\psi_{22}\psi_{33} - \psi_{23}\psi_{32} \neq 0$ — implying $\psi_{22} \neq 0$ — we have that $\psi_{41} = \psi_{42} = \psi_{43} = 0$.

Component equations $Eq(2,1)$ and $Eq(3,1)$ reduce to

$$\begin{aligned}
Eq(2,1) &: \psi_{21} \psi_{22} = 0 \\
Eq(3,1) &: \psi_{31} \psi_{33} = 0,
\end{aligned}$$

thus we have that $\psi_{21} = \psi_{31} = 0$.

The matrix equation $X_{\mathbf{h}^{5,(\alpha_1,\beta_1)}} = \psi^\top X_{\mathbf{h}^{5,(\alpha_2,\beta_2)}} \psi$ reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 \end{bmatrix} = \begin{bmatrix} \psi_{11}^2 & 0 & 0 & 0 & 0 \\ 0 & \psi_{22}^2 & 0 & 0 & 0 \\ 0 & 0 & \psi_{33}^2 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \psi_{11}^2 \psi_{22}^2 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 \psi_{11}^2 \psi_{33}^2 \end{bmatrix}.$$

Thus, $\psi_{11}^2 = \psi_{22}^2 = \psi_{33}^2 = 1$, and so $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

On the other hand, if $\psi_{23} \neq 0$ then $X_{\mathbf{h}^{5,(\alpha_1,\beta_1)}} = \psi^\top X_{\mathbf{h}^{5,(\alpha_2,\beta_2)}} \psi$ gives the component equation

$$Eq(5,4) : \psi_{11}^2 (\alpha_2 \psi_{22} \psi_{23} + \beta_2 \psi_{32} \psi_{33}) = 0,$$

which gives $\psi_{22} = -\frac{\beta_2 \psi_{32} \psi_{33}}{\alpha_2 \psi_{23}}$. As Proposition 3.1.1 gives $\psi_{22}\psi_{33} - \psi_{23}\psi_{32} \neq 0$, we have that $\left(-\frac{\beta_2 \psi_{32} \psi_{33}}{\alpha_2 \psi_{23}}\right) \psi_{33} - \psi_{23}\psi_{32} \neq 0$. That is, $\psi_{32} \left(\left(-\frac{\beta_2 \psi_{33}}{\alpha_2 \psi_{23}}\right) \psi_{33} - \psi_{23}\right) \neq 0$. Therefore $\psi_{32} \neq 0$.

Solving the component equations

$$\begin{aligned}
Eq(5,1) &: \psi_{11}(\alpha_2 \psi_{23} \psi_{41} + \beta_2 \psi_{33} \psi_{51}) = 0 \\
Eq(5,2) &: \psi_{11}(\alpha_2 \psi_{23} \psi_{42} + \beta_2 \psi_{33} \psi_{52}) = 0 \\
Eq(5,3) &: \psi_{11}(\alpha_2 \psi_{23} \psi_{43} + \beta_2 \psi_{33} \psi_{53}) = 0
\end{aligned}$$

for ψ_{41}, ψ_{42} and ψ_{43}, noting that $\psi_{11} \neq 0$, we have $\psi_{41} = -\frac{\beta_2\psi_{33}\psi_{51}}{\alpha_2\psi_{23}}$, $\psi_{42} = -\frac{\beta_2\psi_{33}\psi_{52}}{\alpha_2\psi_{23}}$, $\psi_{43} = -\frac{\beta_2\psi_{33}\psi_{53}}{\alpha_2\psi_{23}}$. Solving the reduced component equations

$$\begin{aligned} Eq(4,1) &: \beta_2\psi_{11}\psi_{32}\psi_{51}\left(\frac{\alpha_2\beta_2\psi_{33}^2}{\alpha_2^2\psi_{23}^2}+1\right)=0 \\ Eq(4,2) &: \beta_2\psi_{11}\psi_{32}\psi_{52}\left(\frac{\alpha_2\beta_2\psi_{33}^2}{\alpha_2^2\psi_{23}^2}+1\right)=0 \\ Eq(4,3) &: \beta_2\psi_{11}\psi_{32}\psi_{53}\left(\frac{\alpha_2\beta_2\psi_{33}^2}{\alpha_2^2\psi_{23}^2}+1\right)=0 \end{aligned}$$

gives $\psi_{51} = \psi_{52} = \psi_{53} = 0$. This implies that $\psi_{41} = \psi_{42} = \psi_{43} = 0$.

We have the reduced component equation

$$Eq(3,2) : \frac{\psi_{32}\psi_{33}(\alpha_2 - \beta_2)}{\alpha_2} = 0.$$

Thus, as $\psi_{32} \neq 0$, $\alpha_2 = \beta_2$ or $\psi_{33} = 0$. If $\alpha_2 \neq \beta_2$, then $\psi_{33} = 0$. Further, $\psi_{22} = -\frac{\beta_2\psi_{32}\psi_{33}}{\alpha_2\psi_{23}} = -\frac{\beta_2\psi_{32}\cdot 0}{\alpha_2\psi_{23}} = 0$.

This reduces $X_{\mathbf{h}^{5,(\alpha_1,\beta_1)}} = \psi^\top X_{\mathbf{h}^{5,(\alpha_2,\beta_2)}}\psi$ to

$$X_{\mathbf{h}^{5,(\alpha_1,\beta_1)}} = \begin{bmatrix} \psi_{11}^2+\psi_{21}^2+\psi_{31}^2 & \psi_{31}\psi_{32} & \psi_{21}\psi_{23} & 0 & 0 \\ \psi_{31}\psi_{32} & \psi_{32}^2 & 0 & 0 & 0 \\ \psi_{21}\psi_{23} & 0 & \psi_{23}^2 & 0 & 0 \\ 0 & 0 & 0 & \beta_2\psi_{11}^2\psi_{32}^2 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2\psi_{11}^2\psi_{23}^2 \end{bmatrix}.$$

We then have that $\psi_{31}\psi_{32} = 0$ and $\psi_{21}\psi_{23} = 0$ implies $\psi_{31} = \psi_{21} = 0$. Thus $\psi_{11}^2+\psi_{21}^2+\psi_{31}^2 = 1$ gives $\psi_{11}^2 = 1$. We also have $\psi_{32}^2 = \psi_{23}^2 = 1$, thus $\alpha_1 = \beta_2\psi_{11}^2\psi_{32}^2$ and $\beta_1 = \alpha_2\psi_{11}^2\psi_{23}^2$ give $\alpha_1 = \beta_2$ and $\beta_1 = \alpha_2$.

If $\alpha_2 = \beta_2 = \beta$, then $Eq(2,1)$ and $Eq(3,1)$ reduce to

$$\begin{aligned} Eq(2,1) &: \psi_{32}\left(\psi_{31} - \frac{\psi_{21}\psi_{33}}{\psi_{23}}\right) = 0 \\ Eq(3,1) &: \psi_{21}\psi_{23} + \psi_{31}\psi_{33} = 0. \end{aligned}$$

as $\psi_{32} \neq 0$, we have $\psi_{31} = \frac{\psi_{21}\psi_{33}}{\psi_{23}}$. Thus

$$\begin{aligned} \psi_{21}\psi_{23} + \psi_{31}\psi_{33} &= 0 \\ \psi_{21}\psi_{23} + \left(\frac{\psi_{21}\psi_{33}}{\psi_{23}}\right)\psi_{33} &= 0 \\ \psi_{21}\psi_{23}\left(1 + \frac{\psi_{33}^2}{\psi_{23}^2}\right) &= 0. \end{aligned}$$

Thus, as $\psi_{23} \neq 0$, $\psi_{21} = 0$ and $\psi_{31} = \frac{\psi_{21}\psi_{33}}{\psi_{23}} = \frac{0\cdot\psi_{33}}{\psi_{23}} = 0$.

As a result, $Eq(1,1)$ reduces to $\psi_{11}^2 = 1$ and

$$X_{\mathbf{h}^{5,(\alpha_1,\beta_1)}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \psi_{32}^2\left(\frac{\psi_{33}^2}{\psi_{23}^2}+1\right) & 0 & 0 & 0 \\ 0 & 0 & \psi_{23}^2+\psi_{33}^2 & 0 & 0 \\ 0 & 0 & 0 & \frac{\beta\psi_{32}^2(\psi_{23}^2+\psi_{33}^2)}{\psi_{23}^2} & 0 \\ 0 & 0 & 0 & 0 & \beta\left(\psi_{23}^2+\psi_{33}^2\right) \end{bmatrix}.$$

As $\psi_{23}^2 + \psi_{33}^2 = 1$ we have that $\psi_{33} = \sin(x), \psi_{23} = \cos(x)$ for some $x \in [0,\pi]$. $Eq(5,5)$ gives $\beta_1 = \beta$. $Eq(2,2)$ gives $\psi_{32}^2 = \frac{1}{\sec^2(x)}$. Finally, $Eq(4,4)$ simplifies as follows

$$\begin{aligned} \frac{\beta\psi_{32}^2\left(\psi_{23}^2+\psi_{33}^2\right)}{\psi_{23}^2} &= \alpha_1 \\ \beta\cdot\frac{1}{\sec^2(x)}\cdot\frac{1}{\cos^2(x)} &= \alpha_1 \\ \beta &= \alpha_1. \end{aligned}$$

Thus $\alpha_1 = \beta = \alpha_2$ and $\beta_1 = \beta = \beta_2$.

We have shown that the left-invariant sub-Riemannian structure $(\mathsf{T}, \mathcal{H}_5, \mathbf{g})$ is automorphic to some left-invariant sub-Riemannian structure $(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)})$ for $\alpha, \beta > 0$. We have in addition shown that the only left-invariant sub-Riemannian structures of the form $(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\hat\alpha,\hat\beta)})$ with $\hat\alpha, \hat\beta > 0$ that $(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)})$ may be automorphic to are the ones for which $(\hat\alpha,\hat\beta) = (\alpha,\beta)$ (itself) or $(\hat\alpha,\hat\beta) = (\beta,\alpha)$. Next we show that these two structures are always automorphic, which implies that a left-invariant sub-Riemannian structure $(\mathsf{T}, \mathcal{H}_5, \mathbf{g})$ is automorphic to a unique structure $(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\bar\alpha,\bar\beta)})$ with $\bar\alpha \geq \bar\beta > 0$. Indeed,

$$(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)}) \cong (\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\beta,\alpha)})$$

by Lemma 4.2.5 as $X_{\mathbf{h}^{5,(\beta,\alpha)}} = \psi^\top X_{\mathbf{h}^{5,(\alpha,\beta)}}\psi$ for

$$\psi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}).$$

□

Proposition 4.3.4. *For any left-invariant sub-Riemannian structure* $(\mathsf{T}, \mathcal{D}, \mathbf{g})$ *we have that:*

1. *If* $\dim(\mathcal{D}(\mathbf{1})) = 3$*, then*
$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3)$$

2. *If* $\dim(\mathcal{D}(\mathbf{1})) = 4$*, then*
$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha})$$

3. If $\dim(\mathcal{D}(\mathbf{1})) = 5$*, then*

$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)})$$

where $(\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3)$, $(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha})$ *and* $(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)})$ *are as in Lemmas 4.3.1,4.3.2 and 4.3.3.*

Proof. Given a left-invariant sub-Riemannian structure $(\mathsf{T}, \mathcal{D}, \mathbf{g})$, for $\dim(\mathcal{D}) = 3, 4$ and 5 respectively, Corollary 4.2.6 respectively gives

$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_3, \mathbf{g}^3),$$
$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_4, \mathbf{g}^4),$$

and

$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \cong (\mathsf{T}, \mathcal{H}_5, \mathbf{g}^5)$$

for an appropriately defined Riemannian metric $\mathbf{g}^i$, for $i = 3, 4$ and 5.

Respective use of Lemma 4.3.1, Lemma 4.3.2 and Lemma 4.3.3 followed by use of the transitivity of the relation $\cong$ will give the desired results 1. 2. and 3. □

We recall that the isometry of Propositon 4.3.4 is that of Proposition 4.1.7. In particular, isometries ϕ of Proposition 4.1.7 are Lie group automorphisms as defined in Appendix B.2. That is, $\phi \in \mathrm{Aut}(\mathsf{T})$ and preserves the group structure of T in addition to its smooth manifold structure.

In the following we use the classification of Proposition 4.3.4 up to automorphism and the affine nature of isometries between nilpotent Lie groups — as given by Theorem 4.1.9 — to give a classification of the left-invariant sub-Riemannian structures on T up to general isometry (as defined in Definition 4.1.2). For this, we change to the language of equivalence relations, partitions and equivalence classes.

Denote by $\mathcal{U}$ the collection of all left-invariant sub-Riemannian structures on T. Consider the equivalence relations $\cong$ and $\equiv$ on $\mathcal{U}$, where for left-invariant sub-Riemannian structures $(\mathsf{T}, \mathcal{D}, \mathbf{g}), (\mathsf{T}, \mathcal{D}', \mathbf{g}') \in \mathcal{U}$ we have

$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \cong (\mathsf{T}, \mathcal{D}', \mathbf{g}') \Leftrightarrow \exists \phi : (\mathsf{T}, \mathcal{D}, \mathbf{g}) \longrightarrow (\mathsf{T}, \mathcal{D}', \mathbf{g}'), \phi \in \mathrm{Aut}(\mathsf{T})$$

and

$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \equiv (\mathsf{T}, \mathcal{D}', \mathbf{g}') \Leftrightarrow \exists \phi : (\mathsf{T}, \mathcal{D}, \mathbf{g}) \longrightarrow (\mathsf{T}, \mathcal{D}', \mathbf{g}'), \phi \in \mathrm{Diff}(\mathsf{T}),$$

where ϕ is an isometry in both cases.

Proposition 4.3.4 gives the following corollary.

Corollary 4.3.5. *The equivalence relation* $\cong$ *partitions* $\mathcal{U}$ *into the following equivalence classes:*

$$[(\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3)]_{\cong},$$
$$[(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha})]_{\cong} \text{ for } \alpha > 0,$$

and

$$[(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)})]_{\cong} \text{ for } \alpha \geq \beta > 0.$$

As Lie group automorphisms are in particular diffeomorphisms, $\mathrm{Aut}(\mathsf{T}) \subseteq \mathrm{Diff}(\mathsf{T})$ and we have that $[(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\cong} \subseteq [(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv}$ for any structure $(\mathsf{T}, \mathcal{D}, \mathbf{g}) \in \mathcal{U}$. It follows that for any $(\mathsf{T}, \mathcal{D}, \mathbf{g}) \in \mathcal{U}$

$$[(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv} = \bigcup_{(T, \mathcal{D}', \mathbf{g}') \in [(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv}} [(T, \mathcal{D}', \mathbf{g}')]_{\cong}.$$

That is, every $\equiv$-equivalence class is the union of $\cong$-equivalence classes. In fact, as distinct $\cong$-equivalence classes are disjoint, $[(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv}$ can be represented as the disjoint union of $\cong$-equivalence classes

$$[(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv} = \bigsqcup_{r \in \Delta} \left[\left(\mathsf{T}, \mathcal{D}^{(r)}, \mathbf{g}^{(r)}\right)\right]_{\cong},$$

for some index set Δ.

We claim that $[(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv}$ is the disjoint union of exactly one $\cong$-equivalence class.

Proof of Claim: Let $(\mathsf{T}, \mathcal{D}, \mathbf{g}) \in \mathcal{U}$. Now, $[(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\cong} \subseteq [(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv}$. Suppose $[(\mathsf{T}, \mathcal{D}', \mathbf{g}')]_{\cong}$ is another $\cong$-equivalence class contained in $[(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv}$. As $(\mathsf{T}, \mathcal{D}, \mathbf{g}), (\mathsf{T}, \mathcal{D}', \mathbf{g}') \in [(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv}$, there exists an isometry $\phi : (\mathsf{T}, \mathcal{D}, \mathbf{g}) \longrightarrow (\mathsf{T}, \mathcal{D}', \mathbf{g}'), \phi \in \mathrm{Diff}(\mathsf{T})$. By Theorem 4.1.9 we have that $\phi = L_g \circ \psi$ for some $\psi \in \mathrm{Aut}(\mathsf{T})$. Now, $\psi = L_{g^{-1}} \circ \phi : (\mathsf{T}, \mathcal{D}, \mathbf{g}) \longrightarrow (\mathsf{T}, \mathcal{D}', \mathbf{g}')$ is an isometry of left-invariant sub-Riemannian structures on T as it is the composition of the isometries ϕ and $L_{g^{-1}}$. We thus have that

$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \cong (\mathsf{T}, \mathcal{D}', \mathbf{g}')$$

and so $(\mathsf{T}, \mathcal{D}', \mathbf{g}') \in [(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\cong}$. This gives

$$[(\mathsf{T}, \mathcal{D}', \mathbf{g}')]_{\cong} = [(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\cong}.$$

As this holds for all pairs of $\cong$-equivalence classes contained in $[(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv}$ we have that

$$\begin{aligned}[(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv} &= \bigcup_{(T, \mathcal{D}', \mathbf{g}') \in [(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv}} [(T, \mathcal{D}', \mathbf{g}')]_{\cong} \\ &= [(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\cong},\end{aligned}$$

proving the claim.

In light of the above, we have the following result.

Proposition 4.3.6. *The equivalence relation $\equiv$ partitions $\mathcal{U}$ into the following equivalence classes:*

$$[(\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3)]_{\equiv},$$

$$[(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha})]_{\equiv} \text{ for } \alpha > 0,$$

and

$$[(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)})]_{\equiv} \text{ for } \alpha \geq \beta > 0.$$

Proof. This is a direct consequence of the fact that for any left-invariant sub-Riemannian structure $(\mathsf{T}, \mathcal{D}, \mathbf{g})$ on T we have

$$[(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\equiv} = [(\mathsf{T}, \mathcal{D}, \mathbf{g})]_{\cong}$$

and Corollary 4.3.5. □

The classification of left-invariant sub-Riemannian structures on T up to isometries that are Lie automorphisms, given in Proposition 4.3.4, can thus be improved to the following classification up to isometry.

Proposition 4.3.7. *For any left-invariant sub-Riemannian structure* $(\mathsf{T}, \mathcal{D}, \mathbf{g})$ *we have that:*

1. *If* $\dim(\mathcal{D}(\mathbf{1})) = 3$*, then*
$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \equiv (\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3).$$
2. *If* $\dim(\mathcal{D}(\mathbf{1})) = 4$*, then*
$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \equiv (\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha}) \text{ for some } \alpha > 0.$$
3. *If* $\dim(\mathcal{D}(\mathbf{1})) = 5$*, then*
$$(\mathsf{T}, \mathcal{D}, \mathbf{g}) \equiv (\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)}) \text{ for some } \alpha \geq \beta > 0.$$

4.4 Isotropy groups of T

In this section we compute the linearised isotropy groups of the left-invariant sub-Riemannian structures of the Lie algebra T — obtained in Proposition 4.3.7. As a consequence of Proposition 4.1.11, these sufficiently characterize the isometry groups of the left-invariant sub-Riemannian structures $(\mathsf{T}, \mathcal{D}, \mathbf{g})$ on T. The isotropy groups of the left-invariant sub-Riemannian structures $(\mathsf{T}, \mathcal{D}, \mathbf{g})$ on T are merely semidirect products of the group of left-translations by these isotropy groups.

Proposition 4.4.1. *The linearised isotropy groups of the left-invariant sub-Riemannian structures on* T *are given by*

i)
$$\begin{aligned} &d\mathsf{Iso}_{\mathbf{1}}(\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3) \\ &= \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \sigma_2 \sin\theta & 0 & 0 \\ 0 & \sin\theta & -\sigma_2\cos\theta & 0 & 0 \\ 0 & 0 & 0 & \sigma_1\cos\theta & \sigma_1\sigma_2\sin\theta \\ 0 & 0 & 0 & \sigma_1\sin\theta & -\sigma_1\sigma_2\cos\theta \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}) : \sigma_1, \sigma_2 = \pm 1, \theta \in \mathbb{R} \right\} \\ &\cong \mathbb{Z}_2 \times \mathsf{O}(2). \end{aligned}$$

ii)
$$\begin{aligned} d\mathsf{Iso}_{\mathbf{1}}(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha}) &= \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1\sigma_2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1\sigma_3 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}) : \sigma_1, \sigma_2, \sigma_3 = \pm 1 \right\} \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \end{aligned}$$

iii) 1.

$$d\mathsf{Iso}_{\mathbf{1}}(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)}) = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1\sigma_2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1\sigma_3 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}) : \sigma_1, \sigma_2, \sigma_3 = \pm 1 \right\}$$
$$\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

for $\alpha > \beta > 0$.

2.

$$d\mathsf{Iso}_{\mathbf{1}}(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\alpha)})$$
$$= \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \sigma_2 \sin\theta & 0 & 0 \\ 0 & \sin\theta & -\sigma_2\cos\theta & 0 & 0 \\ 0 & 0 & 0 & \sigma_1\cos\theta & \sigma_1\sigma_2\sin\theta \\ 0 & 0 & 0 & \sigma_1\sin\theta & -\sigma_1\sigma_2\cos\theta \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}) : \sigma_1, \sigma_2 = \pm 1, \theta \in \mathbb{R} \right\}$$
$$\cong \mathbb{Z}_2 \times \mathsf{O}(2),$$

were $\alpha > 0$.

Proof. *(i)* Consider the left-invariant sub-Riemannian structure $(\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3)$. Now, if $\tilde{\varphi} \in d\mathsf{Iso}_{\mathbf{1}}(\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3)$, then $\tilde{\varphi} \in \mathrm{Aut}(\mathfrak{t})$ with $\tilde{\varphi} \cdot \mathcal{H}_3(\mathbf{1}) = \mathcal{H}_3(\mathbf{1}) = \langle I, J, K\rangle$, that is $\tilde{\varphi}$ preserves the subspace $\mathcal{H}_3(\mathbf{1})$, then (by Proposition 3.1.1) $\tilde{\varphi}$ has matrix representation

$$\tilde{\varphi} = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & j_3 & k_3 & 0 & 0 \\ 0 & 0 & 0 & i_1 j_2 & i_1 k_2 \\ 0 & 0 & 0 & i_1 j_3 & i_1 k_3 \end{bmatrix}$$

with respect to the ordered basis (I, J, K, L, M) of $\mathfrak{t}$. As $\tilde{\varphi} \cdot \mathcal{H}_3(\mathbf{1}) = \mathcal{H}_3(\mathbf{1}) = \langle I, J, K\rangle$, we may define the map $\varphi : \mathcal{H}_3(\mathbf{1}) \longrightarrow \mathcal{H}_3(\mathbf{1})$ as the restriction of $\tilde{\varphi}$ to the subspace $\mathcal{H}_3(\mathbf{1}) \subseteq \mathfrak{t}$. With respect to the ordered basis (I, J, K) for $\mathcal{H}_3(\mathbf{1})$ we have that φ has matrix representation

$$\varphi = \begin{bmatrix} i_1 & 0 & 0 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{bmatrix}.$$

We also have that φ preserves the inner product $\mathbf{h}^3_{\mathbf{1}}$ on $\langle I, J, K\rangle$. That is, for $A, B \in \mathcal{H}_3(\mathbf{1}) = \langle I, J, K\rangle$

$$\mathbf{h}^3_{\mathbf{1}}(A, B) = \mathbf{h}^3_{\mathbf{1}}(\varphi \cdot A, \varphi \cdot B) \iff A^\top I_3 B = (\varphi \cdot A)^\top I_3 (\varphi \cdot B),$$

where the right-hand side of the bi-conditional is written in coordinates with respect to the ordered basis (I, J, K) of $\mathcal{H}_3(\mathbf{1})$. More explicitly, taking $A = a_1 I + a_2 J + a_3 K$ and

$B = b_1 I + b_2 J + b_3 K$, we have

$$A^\top I_3 B = A^\top \cdot (\varphi^\top I_3 \varphi) \cdot B$$
$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} i_1 & i_2 & i_3 \\ 0 & j_2 & j_3 \\ 0 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} i_1 & 0 & 0 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

As this holds for all $A, B \in \mathcal{H}^3(\mathbf{1})$, we have $I_3 = \varphi^\top \varphi$ when written with respect to the basis (I, J, K) of $\mathcal{H}_3(\mathbf{1})$. That is, the matrix

$$\varphi = \begin{bmatrix} i_1 & 0 & 0 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{bmatrix}$$

is an orthogonal matrix. This implies that the columns and rows of this matrix form an orthonormal basis for $\mathbb{R}^3$. As a result, the first row of the matrix is a unit vector, implying that $i_1 = \sigma_1$, where $\sigma_1 = \pm 1$. As the first column is also a unit vector, we can further deduce that $i_2 = i_3 = 0$. We thus have the matrix

$$[\varphi]_{(I,J,K)} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & j_2 & k_2 \\ 0 & j_3 & k_3 \end{bmatrix}.$$

As $[\varphi]_{(I,J,K)} \in \mathsf{O}(3)$, the submatrix $\begin{bmatrix} j_2 & k_2 \\ j_3 & k_3 \end{bmatrix} \in \mathsf{O}(2)$. That is,

$$\begin{bmatrix} j_2 & k_2 \\ j_3 & k_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sigma_2 \sin\theta \\ \sin\theta & -\sigma_2 \cos\theta \end{bmatrix}$$

for some $\theta \in \mathbb{R}$ and $\sigma_2 = \pm 1$.

We therefore have that

$$\begin{aligned} & d\mathsf{Iso}_{\mathbf{1}}(\mathsf{T}, \mathcal{H}_3, \mathbf{h}^3) \\ &= \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \sigma_2 \sin\theta & 0 & 0 \\ 0 & \sin\theta & -\sigma_2 \cos\theta & 0 & 0 \\ 0 & 0 & 0 & \sigma_1 \cos\theta & \sigma_1 \sigma_2 \sin\theta \\ 0 & 0 & 0 & \sigma_1 \sin\theta & -\sigma_1 \sigma_2 \cos\theta \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}) : \sigma_1, \sigma_2 = \pm 1, \theta \in \mathbb{R} \right\} \\ &\cong \mathbb{Z}_2 \times \mathsf{O}(2). \end{aligned}$$

(ii) Consider the left-invariant sub-Riemannian structure $(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha})$, $\alpha > 0$. Now, for $\tilde{\varphi} \in d\mathsf{Iso}_{\mathbf{1}}(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha})$ we have that $\tilde{\varphi} \in \mathrm{Aut}(\mathfrak{t})$ and $\tilde{\varphi} \cdot \mathcal{H}_4(\mathbf{1}) = \mathcal{H}_4(\mathbf{1}) = \langle I, J, K, L \rangle$. We thus have (by Proposition 3.1.1) that $\tilde{\varphi}$ has the matrix representation

$$\tilde{\varphi} = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & 0 & k_3 & 0 & 0 \\ i_4 & j_4 & k_4 & i_1 j_2 & i_1 k_2 \\ 0 & 0 & 0 & 0 & i_1 k_3 \end{bmatrix}$$

with respect to the ordered basis (I, J, K, L, M) for $\mathfrak{t}$. As $\tilde{\varphi} \cdot \mathcal{H}_4(\mathbf{1}) = \mathcal{H}_4(\mathbf{1}) = \langle I, J, K, L\rangle$, we may define the map $\varphi : \mathcal{H}_4(\mathbf{1}) \longrightarrow \mathcal{H}_4(\mathbf{1})$ as the restriction of $\tilde{\varphi}$ to the subspace $\mathcal{H}_4(\mathbf{1}) \subseteq \mathfrak{t}$. With respect to the ordered basis (I, J, K, L) for $\mathcal{H}_4(\mathbf{1})$ we have that φ has matrix representation

$$\varphi = \begin{bmatrix} i_1 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 \\ i_3 & 0 & k_3 & 0 \\ i_4 & j_4 & k_4 & i_1 j_1 \end{bmatrix}.$$

We also have that φ preserves the inner product $\mathbf{h}^{4,\alpha}$ on $\langle I, J, K, L\rangle$. That is, for $A, B \in \mathcal{H}_4 = \langle I, J, K, L\rangle$

$$\mathbf{h}^{4,\alpha}_{\mathbf{1}}(A, B) = \mathbf{h}^{4,\alpha}_{\mathbf{1}}(\varphi \cdot A, \varphi \cdot B) \iff A^\top(\alpha I_4)B = (\varphi \cdot A)^\top(\alpha I_4)(\varphi \cdot B),$$

where the right-hand side of the bi-conditional is written in coordinates with respect to the ordered basis (I, J, K, L) of $\mathcal{H}_4(\mathbf{1})$. More explicitly, taking $A = a_1 I + a_2 J + a_3 K + a_4 L$ and $B = b_1 I + b_2 J + b_3 K + b_4 L$, we have

$$\alpha(A^\top I_4 B) = \alpha(A^\top \varphi^\top I_4 \varphi B)$$

$$\alpha \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \alpha \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} i_1 & i_2 & i_3 & i_4 \\ 0 & j_2 & 0 & j_4 \\ 0 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & i_1 j_1 \end{bmatrix} \begin{bmatrix} i_1 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 \\ i_3 & 0 & k_3 & 0 \\ i_4 & j_4 & k_4 & i_1 j_1 \end{bmatrix}.$$

As this holds for all $A, B \in \mathcal{H}_4(\mathbf{1})$, we have that $I_4 = \varphi^\top \varphi$ when written with respect to the basis (I, J, K, L) of $\mathcal{H}_4(\mathbf{1})$. That is, the matrix

$$\varphi = \begin{bmatrix} i_1 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 \\ i_3 & 0 & k_3 & 0 \\ i_4 & j_4 & k_4 & i_1 j_1 \end{bmatrix}$$

is an orthogonal matrix. This implies that the columns and row of φ form an orthonormal bases for $\mathbb{R}^4$. As the first row of φ is a unit vector, we have that $i_1 = \sigma_1 = \pm 1$. From this and the fact that the first column of φ is a unit vector, we deduce that $i_2 = i_3 = i_4 = 0$. From $i_3 = 0$ and the fact that the third row of φ is a unit vector, we deduce that $k_3 = \sigma_3 = \pm 1$. From $k_3 = \sigma_3$ and the fact that the third column of φ is a unit vector, we deduce that $k_2 = k_4 = 0$. From the fact that the second row of φ is a unit vector, we deduce $j_2 = \sigma_2 = \pm 1$. Finally, from the fact that the second column of φ is a unit vector and $j_2 = \sigma_2$, we deduce $j_4 = 0$. This results in

$$\varphi = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_1 \sigma_2 \end{bmatrix}.$$

Therefore,

$$d\mathsf{Iso}_{\mathbf{1}}(\mathsf{T}, \mathcal{H}_4, \mathbf{h}^{4,\alpha}) = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1\sigma_2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1\sigma_3 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}) : \sigma_1, \sigma_2, \sigma_3 = \pm 1 \right\}$$
$$\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

(iii) Consider the left-invariant sub-Riemannian structure $(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\beta)})$ with $\alpha \geq \beta > 0$. Now, for $\varphi \in d\mathsf{Iso}_{\mathbf{1}}(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5(\alpha,\beta)})$ we have that $\varphi \in \mathrm{Aut}(\mathfrak{t})$ and $\varphi \cdot \mathcal{H}_5(\mathbf{1}) = \mathcal{H}_5(\mathbf{1}) = \mathfrak{t}$, thus (by Proposition 3.1.1) φ has the matrix representation

$$\varphi = \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & j_3 & k_3 & 0 & 0 \\ i_4 & j_4 & k_4 & i_1 j_2 & i_1 k_2 \\ i_5 & j_5 & k_5 & i_1 j_3 & i_1 k_3 \end{bmatrix}$$

with respect to the ordered basis (I, J, K, L, M) of $\mathfrak{t}$. We also have that φ preserves the inner product $\mathbf{h}_{\mathbf{1}}^{5,(\alpha,\beta)}$ of $\mathfrak{t} = \mathcal{H}_5(\mathbf{1})$. That is, for $A, B \in \mathfrak{t}$,

$$\mathbf{h}_{\mathbf{1}}^{5,(\alpha,\beta)}(A, B) = \mathbf{h}_{\mathbf{1}}^{5,(\alpha,\beta)}(\varphi \cdot A, \varphi \cdot B) \iff A^\top X_{\mathbf{h}^{5,(\alpha,\beta)}} B = (\varphi \cdot A)^\top X_{\mathbf{h}^{5,(\alpha,\beta)}} (\varphi \cdot \beta),$$

were the right-hand side is in coordinates with respect to the basis (I, J, K, L, M) of $\mathcal{H}(\mathbf{1}) = \mathfrak{t}$. In coordinates we have

$$A^\top X_{\mathbf{h}^{5,(\alpha,\beta)}} B = A^\top (\varphi^\top X_{\mathbf{h}^{5,(\alpha,\beta)}} \varphi) B$$

As this holds of all $A, B \in \mathcal{H}_5(\mathbf{1})$, we have that $X_{\mathbf{h}^{5,(\alpha,\beta)}} = \varphi^\top X_{\mathbf{h}^{5,(\alpha,\beta)}} \varphi$. That is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \beta \end{bmatrix}$$
$$= \begin{bmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ 0 & j_2 & j_3 & j_4 & j_5 \\ 0 & k_2 & k_3 & k_4 & k_5 \\ 0 & 0 & 0 & i_1 j_2 & i_1 j_3 \\ 0 & 0 & 0 & i_1 k_2 & i_1 k_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \beta \end{bmatrix} \cdot \begin{bmatrix} i_1 & 0 & 0 & 0 & 0 \\ i_2 & j_2 & k_2 & 0 & 0 \\ i_3 & j_3 & k_3 & 0 & 0 \\ i_4 & j_4 & k_4 & i_1 j_2 & i_1 k_2 \\ i_5 & j_5 & k_5 & i_1 j_3 & i_1 k_3 \end{bmatrix}.$$

The fist row of this matrix equation gives the scalar equations

$$i_1^2 = 1, \;\; i_1 i_2 = 0, \;\; i_1 i_3 = 0, \;\; i_1 i_4 = 0 \;\; \text{and} \;\; i_1 i_5 = 0,$$

giving $i_1 = \sigma_1 = \pm 1$ and $i_2 = i_3 = i_4 = i_5 = 0$.

Therefore, $X_{\mathbf{h}^{5,(\alpha,\beta)}}$ is equal to the matrix

$$\begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 & 0 \\ 0 & {j_2}^2 + {k_2}^2 & j_2j_3 + k_2k_3 & j_2j_4 + k_2k_4 & j_2j_5 + k_2k_5 \\ 0 & j_2j_3 + k_2k_3 & {j_3}^2 + {k_3}^2 & j_3j_4 + k_3k_4 & j_3j_5 + k_3k_5 \\ 0 & j_2j_4 + k_2k_4 & j_3j_4 + k_3k_4 & {j_4}^2 + {k_4}^2 + {j_2}^2\alpha\sigma_1^2 + {k_2}^2\beta\sigma_1^2 & j_2j_3\alpha\sigma_1^2 + k_2k_3\beta\sigma_1^2 + j_4j_5 + k_4k_5 \\ 0 & j_2j_5 + k_2k_5 & j_3j_5 + k_3k_5 & j_2j_3\alpha\sigma_1^2 + k_2k_3\beta\sigma_1^2 + j_4j_5 + k_4k_5 & {j_5}^2 + {k_5}^2 + {j_3}^2\alpha\sigma_1^2 + {k_3}^2\beta\sigma_1^2 \end{bmatrix}.$$

Solving $j_2j_5 + k_2k_5 = 0$ and $j_3j_5 + k_3k_5 = 0$ we have that if $j_2 = 0$ then $k_2 \neq 0$ and $j_3 \neq 0$ as $\varphi \in \mathrm{Aut}(\mathfrak{t})$ (see Proposition 3.1.1) and so $j_2j_5 + k_2k_5 = 0$ gives $k_5 = 0$ while $j_3j_5 + k_3k_5 = 0$ gives $j_5 = 0$. If, on the other hand, $j_2 \neq 0$ then $j_2j_5 + k_2k_5 = 0$ gives $j_5 = -\frac{k_2k_5}{j_2}$ and thus

$$\begin{aligned} j_3j_5 + k_3k_5 &= 0 \\ j_3\left(-\frac{k_2k_5}{j_2}\right) + k_3k_5 &= 0 \\ j_3k_2k_5 &= j_2k_3k_5. \end{aligned}$$

This implies $k_5 = 0$, for if not then $j_3k_2 = j_2k_3$ contradicting $\varphi \in \mathrm{Aut}(\mathfrak{t})$. Now, $k_5 = 0$ gives $j_5 == -\frac{k_2k_5}{j_2} = 0$. Therefore in all cases we have $j_5 = k_5 = 0$.

The above reduction gives that $X_{\mathbf{h}^{5,(\alpha,\beta)}}$ is equal to the matrix

$$\begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 & 0 \\ 0 & {j_2}^2 + {k_2}^2 & j_2j_3 + k_2k_3 & j_2j_4 + k_2k_4 & 0 \\ 0 & j_2j_3 + k_2k_3 & {j_3}^2 + {k_3}^2 & j_3j_4 + k_3k_4 & 0 \\ 0 & j_2j_4 + k_2k_4 & j_3j_4 + k_3k_4 & {j_4}^2 + {k_4}^2 + {j_2}^2\alpha\sigma_1^2 + {k_2}^2\beta\sigma_1^2 & j_2j_3\alpha\sigma_1^2 + k_2k_3\beta\sigma_1^2 \\ 0 & 0 & 0 & j_2j_3\alpha\sigma_1^2 + k_2k_3\beta\sigma_1^2 & {j_3}^2\alpha\sigma_1^2 + {k_3}^2\beta\sigma_1^2 \end{bmatrix}.$$

Similarly, solving $j_2j_4 + k_2k_4 = 0$ and $j_3j_4 + k_3k_4 = 0$ we have that if $j_2 = 0$ then then $k_2 \neq 0$ and $j_3 \neq 0$ as $\varphi \in \mathrm{Aut}(\mathfrak{t})$ and so $j_2j_4 + k_2k_4 = 0$ gives $k_4 = 0$ while $j_3j_4 + k_3k_4 = 0$ gives $j_4 = 0$. If, on the other hand, $j_2 \neq 0$ then $j_2j_4 + k_2k_4 = 0$ gives $j_4 = -\frac{k_2k_4}{j_2}$ and thus

$$\begin{aligned} j_3j_4 + k_3k_4 &= 0 \\ j_3\left(-\frac{k_2k_4}{j_2}\right) + k_3k_4 &= 0 \\ j_3k_2k_4 &= k_3k_4j_2. \end{aligned}$$

This implies that $k_4 = 0$, for if not then $j_3k_2 = k_3j_2$ contradicting $\varphi \in \mathrm{Aut}(\mathfrak{t})$. Now, $k_4 = 0$ gives $j_4 = -\frac{k_2k_4}{j_2} = 0$. Therefore in all cases we have $j_4 = k_4 = 0$.

The above reduces $X_{\mathbf{h}^{5,(\alpha,\beta)}}$ to being equal to

$$\begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 & 0 \\ 0 & {j_2}^2 + {k_2}^2 & j_2j_3 + k_2k_3 & 0 & 0 \\ 0 & j_2j_3 + k_2k_3 & {j_3}^2 + {k_3}^2 & 0 & 0 \\ 0 & 0 & 0 & {j_2}^2\alpha\sigma_1^2 + {k_2}^2\beta\sigma_1^2 & j_2j_3\alpha\sigma_1^2 + k_2k_3\beta\sigma_1^2 \\ 0 & 0 & 0 & j_2j_3\alpha\sigma_1^2 + k_2k_3\beta\sigma_1^2 & {j_3}^2\alpha\sigma_1^2 + {k_3}^2\beta\sigma_1^2 \end{bmatrix}.$$

As $j_2^2 + k_2^2 = 1$ and $j_3^2 + k_3^2 = 1$, we have $j_2 = \cos x, k_2 = \sin x, j_3 = \cos y$ and $k_3 = \sin y$ for some $x, y \in \mathbb{R}$. This gives $X_{\mathbf{h}^{5,(\alpha,\beta)}}$ is equal to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \cos(x-y) & 0 & 0 \\ 0 & \cos(x-y) & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha\cos^2(x) + \beta\sin^2(x) & \begin{smallmatrix}\alpha\cos(x)\cos(y)+ \\ \beta\sin(x)\sin(y)\end{smallmatrix} \\ 0 & 0 & 0 & \begin{smallmatrix}\alpha\cos(x)\cos(y)+ \\ \beta\sin(x)\sin(y)\end{smallmatrix} & \left(\alpha\cos^2(y) + \beta\sin^2(y)\right) \end{bmatrix}.$$

As $\cos(x-y) = 0$ we have that $x - y = \frac{\pi}{2} + n\pi$ for some $n \in \mathbb{Z}$. That is, $x = y + \frac{\pi}{2} + n\pi$ and thus $X_{\mathbf{h}^{5,(\alpha,\beta)}}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\sin(n\pi) & 0 & 0 \\ 0 & -\sin(n\pi) & 1 & 0 & 0 \\ 0 & 0 & 0 & \begin{smallmatrix}\beta\cos^2(\pi n+y)+ \\ \alpha\sin^2(\pi n+y)\end{smallmatrix} & \begin{smallmatrix}\beta\cos(\pi n+y)\sin(y)- \\ \alpha\cos(y)\sin(\pi n+y)\end{smallmatrix} \\ 0 & 0 & 0 & \begin{smallmatrix}\beta\cos(\pi n+y)\sin(y)- \\ \alpha\cos(y)\sin(\pi n+y)\end{smallmatrix} & \left(\alpha\cos^2(y) + \beta\sin^2(y)\right) \end{bmatrix}.$$

Now,

$$\begin{aligned} (\beta\cos(\pi n + y)\sin(y) - \alpha\cos(y)\sin(\pi n + y)) &= 0 \\ (\cos y)(\sin y)(\beta - \alpha) &= 0. \end{aligned}$$

Case 1: Suppose $\alpha \neq \beta$, then $(\cos y) = 0$ or $(\sin y) = 0$ That is, $y = \frac{\pi}{2}n_1$ for some $n_1 \in \mathbb{Z}$. Further,

$$\begin{aligned} \alpha\sin^2(\pi n + y) + \beta\cos^2(\pi n + y) &= \alpha \\ \alpha\sin^2(y) + \beta\cos^2(y) &= \alpha \\ (\alpha - \alpha + \beta)\cos^2(y) + \alpha\sin^2(y) &= \alpha \\ \alpha + (\beta - \alpha)\cos^2(y) &= \alpha \\ (\beta - \alpha)\cos^2(y) &= 0. \end{aligned}$$

As $\beta \neq \alpha$, we have $\cos^2(y) = 0$ - therefore $\cos(y) = 0$ and thus $y = \frac{\pi}{2} + n_2\pi$ for some $n_2 \in \mathbb{Z}$. Now, $\sin^2(y) = \sin^2(\frac{\pi}{2} + n_2\pi) = (\pm 1)^2 = 1$ and in addition $\cos^2(n\pi + y) = 0$ and $\sin^2(n\pi + y) = 1$. This simplifies our matrix equation into the trivial $X_{\mathbf{h}^{5,(\alpha,\beta)}} = X_{\mathbf{h}^{5,(\alpha,\beta)}}$.

Applying the findings,

$$\begin{aligned}
i_1 &= \sigma_1,\\
i_2 &= i_3 = i_4 = i_5 = 0,\\
j_5 &= k_5 = j_4 = k_4 = 0,\\
x &= y + \frac{\pi}{2} + n\pi\\
y &= \frac{\pi}{2} + n_2\pi\\
j_2 &= \cos(x) = \cos\left((n + n_2 + 1)\pi\right) = \sigma_2 = \pm 1,\\
k_2 &= \sin(x) = \sin\left((n + n_2 + 1)\pi\right) = 0,\\
j_3 &= \cos(y) = 0 \text{ and}\\
k_3 &= \sin(y) = \sigma_3 = \pm 1,
\end{aligned}$$

we have

$$\varphi = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0\\ 0 & \sigma_2 & 0 & 0 & 0\\ 0 & 0 & \sigma_3 & 0 & 0\\ 0 & 0 & 0 & \sigma_1\sigma_2 & 0\\ 0 & 0 & 0 & 0 & \sigma_1\sigma_3 \end{bmatrix}.$$

Therefore

$$\begin{aligned}
d\mathsf{Iso}_{\mathbf{1}}(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{4,(\alpha,\beta)}) &= \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0\\ 0 & \sigma_2 & 0 & 0 & 0\\ 0 & 0 & \sigma_3 & 0 & 0\\ 0 & 0 & 0 & \sigma_1\sigma_2 & 0\\ 0 & 0 & 0 & 0 & \sigma_1\sigma_3 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}) : \sigma_1, \sigma_2, \sigma_3 = \pm 1 \right\}\\
&\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.
\end{aligned}$$

Case 2: Suppose $\alpha = \beta$, as $\cos^2\theta + \sin^2\theta = 1$ and $\sin(n\pi) = 0$ for $n \in \mathbb{N}$ the only nontrivial scalar equation in our matrix equation is

$$\beta\cos(\pi n + y)\sin(y) - \alpha\cos(y)\sin(\pi n + y) = 0.$$

Thus we have,

$$\alpha\sin\left(\pi n + y + \frac{\pi}{2}\right)\sin(y) - \alpha\sin\left(y + \frac{\pi}{2}\right)\sin(\pi n + y) = 0,$$

an identity, as $\sin(\theta + \frac{\pi}{2}) = \sin(\theta)$. We are left with no further possible simplification and applying the findings

$$\begin{aligned}
&i_1 = \sigma_1,\\
&i_2 = i_3 = i_4 = i_5 = 0,\\
&j_5 = k_5 = j_4 = k_4 = 0,\\
&x = y + \frac{\pi}{2} + n\pi,\\
&j_2 = \cos(x) = \cos\left(y + \frac{\pi}{2} + n\pi\right),\\
&k_2 = \sin(x) = \sin\left(y + \frac{\pi}{2} + n\pi\right),\\
&j_3 = \cos(y), \text{ and}\\
&k_3 = \sin(y),
\end{aligned}$$

we have

$$\varphi = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \cos\left(y + \frac{\pi}{2} + n\pi\right) & \sin\left(y + \frac{\pi}{2} + n\pi\right) & 0 & 0 \\ 0 & \cos y & \sin y & 0 & 0 \\ 0 & 0 & 0 & \sigma_1 \cos\left(y + \frac{\pi}{2} + n\pi\right) & \sigma_1 \sin\left(y + \frac{\pi}{2} + n\pi\right) \\ 0 & 0 & 0 & \sigma_1 \cos y & \sigma_1 \sin y \end{bmatrix}.$$

Now,

$$\begin{aligned}
\begin{bmatrix} \cos\left(y + \frac{\pi}{2} + n\pi\right) & \sin\left(y + \frac{\pi}{2} + n\pi\right) \\ \cos y & \sin y \end{bmatrix} &= \begin{bmatrix} -\sin\left(y + n\pi\right) & \cos\left(y + n\pi\right) \\ \cos y & \sin y \end{bmatrix} \\
&= \begin{bmatrix} \sigma_2 \sin(y) & -\sigma_2 \cos(y) \\ \cos y & \sin y \end{bmatrix} \in \mathsf{O}(2),
\end{aligned}$$

where $\sigma_2 = \pm 1$.

Therefore,

$$\begin{aligned}
&d\mathsf{Iso}_{\mathbf{1}}(\mathsf{T}, \mathcal{H}_5, \mathbf{h}^{5,(\alpha,\alpha)}) \\
&= \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \sigma_2 \sin\theta & 0 & 0 \\ 0 & \sin\theta & -\sigma_2 \cos\theta & 0 & 0 \\ 0 & 0 & 0 & \sigma_1 \cos\theta & \sigma_1\sigma_2 \sin\theta \\ 0 & 0 & 0 & \sigma_1 \sin\theta & -\sigma_1\sigma_2 \cos\theta \end{bmatrix} \in \mathrm{Aut}(\mathfrak{t}) : \sigma_1, \sigma_2 = \pm 1, \theta \in \mathbb{R} \right\} \\
&\cong \mathbb{Z}_2 \times \mathsf{O}(2).
\end{aligned}$$

□

Chapter 5

Geodesics

Given a left-invariant sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ a curve $\gamma : [0, t_1] \to \mathsf{G}$ is termed admissible if $\dot{\gamma}(t) \in \mathcal{D}(\gamma(t))$ for every $t \in [0, t_1]$. To any admissible curve γ joining two points $\gamma(0) = q_0$ and $\gamma(t_1) = q_1$, we can assign a length

$$\ell(\gamma) = \int_0^{t_1} \sqrt{\mathbf{g}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt.$$

The existence of an admissible curve γ between any two points q_0 and q_1 is guaranteed to us by the bracket generating nature of the distribution $\mathcal{D}$ through the Chow-Rashevsky Theorem (see, e.g.[6]).

The the distance between points q_0 and q_1 on G, given by the Carnot-Carathéordory distance

$$d(q_0, q_1) = \inf \{\ell(\gamma) : \gamma \text{ is an admissible curve joining } q_0 \text{ and } q_1\}$$

An admissible curve, joining q_0 and q_1, that realises this distance is called a *minimizing geodesic*. Such a curve may not necessarily exist or be unique. However, it turns out that for left-invariant sub-Riemannian structures on a Lie group G, any two points in G can be joined by a minimizing geodesic (see e.g.[3], [2])

Necessary conditions for an admissible curve to be a minimising geodesic can be obtained via the Pontryagin Maximum Principle: any minimising geodesic must be the projection of a normal or abnormal extremal curve on the cotangent bundle (see e.g. [4]). For invariant sub-Riemannian structures the normal extremals are integral curves of a single Hamiltonian system on the cotangent bundle (endowed with the canonical symplectic structure). The projections of the normal extremals are called *normal geodesics*. It turns out that sufficiently short subarcs of normal geodesics are in fact minimizing geodesics (see, e.g. [15]).

In this chapter we compute the normal geodesics of the left-invariant sub-Riemannian structures $(\mathsf{T}, \mathcal{D}, \mathbf{g})$ on the Lie group T. (We will not consider the abnormal extremals.) We shall make use of a characterisation of the normal geodesics for sub-Rimannian structures on left-invariant sub-Riemannian structures in terms of Hamilton-Poisson systems on the dual of the Lie algebra. This essentially reduces the problem of finding the normal geodesics to solving some differential equations.

Definition 5.0.1. *Given a left-invariant sub-Riemannian structure* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *with Lie algebra* $\mathfrak{g}$ *of* G*, we denote by* i *the inclusion map* $i : \mathcal{D}(\mathbf{1}) \longrightarrow \mathfrak{g}$ *and by* i^* *its dual* $i^* : \mathfrak{g}^* \longrightarrow \mathcal{D}(\mathbf{1})^*$*. Furthermore, we have the musical isomorphisms* $\flat : \mathcal{D}(\mathbf{1}) \longrightarrow \mathcal{D}(\mathbf{1})^*$*,* $A \mapsto \mathbf{g}(A, \cdot)$ *and* $\sharp = \flat^{-1} : \mathcal{D}(\mathbf{1})^* \longrightarrow \mathcal{D}(\mathbf{1})$*.*

Proposition 5.0.2. *[4, Proposition 4] The normal geodesics* $g(\cdot)$ *of the left-invariant sub-Riemannian structure* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$*, on the matrix Lie group* G *are given by*

$$\begin{cases} \dot{g} & = T_{\mathbf{1}} L_g \cdot (i^* p)^\sharp \\ \dot{p} & = \vec{H}(p) \end{cases}$$

where $H(p) = \frac{1}{2}(i^* p) \cdot (i^* p)^\sharp$*,* $g \in \mathsf{G}$*,* $p \in \mathfrak{g}^*$*.*

Proposition 5.0.3. *Let* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *be a left-invariant sub-Riemannian structure of rank* ℓ *on the* n*-dimensional Lie group* G*. Suppose the following ordered bases are fixed:*

1. $(E_1, E_2 ..., E_n)$ *for the Lie albegra* $\mathfrak{g}$*,*
2. *dual basis* $(E_1^*, E_2^*, ..., E_n^*)$ *for the dual space* $\mathfrak{g}^*$ *and*
3. $(A_1, A_2 ..., A_\ell)$ *for the subspace* $\mathcal{D}(\mathbf{1}) \subseteq \mathfrak{g}$*.*

If $X_{\mathbf{g}}$ *is the positive definite matrix representation of* $\mathbf{g}_{\mathbf{1}}$ *with respect to the ordered basis* $(A_1, A_2, ..., A_\ell)$ *for* $\mathcal{D}(\mathbf{1})$*,* $p \in \mathfrak{g}^*$ *with representation* $\begin{bmatrix} p_1, p_2, ..., p_n \end{bmatrix}$ *with respect to the ordered basis* $(E_1^*, E_2^*, ..., E_n^*)$ *and*

$$B = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \cdots & A_\ell \\ | & | & & | \end{bmatrix}$$

is the $n \times \ell$ *matrix with the vectors* A_i*,* $i = 1, ..., \ell$ *as its columns — where the vectors* A_i *are written with respect to the ordered basis* $(E_1, E_2, ..., E_n)$ *of* $\mathfrak{g}$*, then*

$$H(p) = \frac{1}{2} p B (X_{\mathbf{g}}^{-1})^\top B^\top p^\top .$$

If in addition $g \in \mathsf{G}$ *and* L_g *is the left translation by* g *with tangent map at identity* $T_{\mathbf{1}} L_g$ *represented by an* $n \times n$ *matrix with respect to the basis* $(E_1, E_2, ..., E_n)$ *of* $\mathfrak{g}$*, then*

$$T_{\mathbf{1}} L_g \cdot (i^* p)^\sharp = T_{\mathbf{1}} L_g \cdot B (X_{\mathbf{g}}^{-1})^\top B^\top p^\top .$$

Proof. If $p \in \mathfrak{g}^*$, then pB is a $1 \times \ell$ matrix and thus represent a linear functional on $\mathcal{D}(\mathbf{1})$.

Given any $W \in \mathcal{D}(\mathbf{1})$, where W has coordinates $\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_\ell \end{bmatrix}$ with respect to the ordered basis $(A_1, A_2, ..., A_\ell)$ for $\mathcal{D}(\mathbf{1})$, we have that $B \cdot W$ gives the vector W with respect to the basis $(E_1, E_2, ..., E_n)$ for $\mathfrak{g}$, as B is a change of basis matrix for the subspace $\mathcal{D}(\mathbf{1}) \subseteq \mathfrak{t}$ from the basis $(A_1, A_2, ...A_\ell)$ to the basis $(E_1, E_2, ..., E_n)$.

From the definition of i^*, we have that

$$(i^*p) \cdot W = p \cdot W.$$

In coordinates with respect to the basis $(E_1, E_2, ..., E_n)$ this gives

$$(i^*p) \cdot W = p \cdot B \cdot W.$$

By the definitions of the isomorphisms $\sharp$ and $\flat$, $(i^*p)^\sharp$ is a vector $R \in \mathcal{D}(\mathbf{1})$ such that

$$(i^*p) \cdot Q = \mathbf{g_1}(R, Q), \text{for all } Q \in \mathcal{D}(\mathbf{1}).$$

In coordinates, with respect to the the basis $(A_1, A_2, ..., A_\ell)$ of $\mathcal{D}(\mathbf{1})$, this gives

$$\begin{aligned} (pB) \cdot Q &= R^\top X_{\mathbf{g}} Q \\ (pB) &= R^\top X_{\mathbf{g}}, \end{aligned}$$

where R is given by the $\ell \times 1$ matrix $\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_\ell \end{bmatrix}$ and the second equality follows form the fact that the first holds for every $Q \in \mathcal{D}(\mathbf{1})$. As $X_{\mathbf{g}}$ is positive definite, it is invertible, and thus we have

$$(pB)X_{\mathbf{g}}^{-1} = R^\top$$

and so

$$R = (X_{\mathbf{g}}^{-1})^\top B^\top p^\top.$$

From Proposition 5.0.2 we have that the Hamiltonian H of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is given by

$$\begin{aligned} H(p) &= \frac{1}{2}(i^*p) \cdot (i^*p)^\sharp \\ &= \frac{1}{2} pB \cdot R \\ &= \frac{1}{2} pB(X_{\mathbf{g}}^{-1})^\top B^\top p^\top. \end{aligned}$$

For $g \in \mathsf{G}$, we have that

$$T_{\mathbf{1}}L_g \cdot (i^*p)^\sharp = T_{\mathbf{1}}L_g \cdot \hat{R}, \tag{5.1}$$

where $\hat{R}$ is the vector $R \in \mathcal{D}(\mathbf{1})$ written with respect to the ordered basis $(E_1, E_2, ..., E_n)$. As $R = (X_{\mathbf{g}}^{-1})^\top B^\top p^\top$ with respect to the ordered basis $(A_1, A_2, ..., A_\ell)$ of $\mathcal{D}(\mathbf{1})$, the product BR with the change of basis matrix B gives R with respect to the basis $(E_1, E_2, ..., E_n)$. That is, $\hat{R} = BR$. We thus have that, in coordinates

$$\begin{aligned} T_{\mathbf{1}} L_g \cdot (i^* p)^\sharp &= T_{\mathbf{1}} L_g \cdot BR \\ &= [T_{\mathbf{1}} L_g] B (X_{\mathbf{g}}^{-1})^\top B^\top p^\top . \end{aligned}$$

□

If $g = \begin{bmatrix} 1 & i & l & m \\ 0 & 1 & j & k \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathsf{T}$ then the left translation by g is given by

$$L_g : \begin{bmatrix} 1 & x_1 & x_4 & x_5 \\ 0 & 1 & x_2 & x_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & x_1 + i & x_4 + l + i x_2 & x_5 + m + i x_3 \\ 0 & 1 & x_2 + j & x_3 + k \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

That is, in coordinates,

$$L_g : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + i \\ x_2 + j \\ x_3 + k \\ x_4 + l + i x_2 \\ x_5 + m + i x_3 \end{bmatrix} ,$$

with tangent map at identity given by the matrix

$$T_{\mathbf{1}} L_g = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & i & 0 & 1 & 0 \\ 0 & 0 & i & 0 & 1 \end{bmatrix}$$

with respect to the ordered basis (I, J, K, L, M) of $\mathfrak{t}$.

Consider the rank 3 left-invariant sub-Riemannian structure $(\mathsf{T}, \mathcal{H}_3, \mathbf{h})$ and ordered bases:

1. (I, J, K, L, M) for the Lie algebra $\mathfrak{t}$,
2. dual basis $(I^*, J^*, K^*, L^*, M^*)$ for the dual space $\mathfrak{t}^*$ and
3. ordered basis (I, J, K) for the subspace $\mathcal{H}_3(\mathbf{1}) = \langle I, J, K \rangle \subseteq \mathfrak{t}$.

If $p \in \mathfrak{g}^*$ has representation $\begin{bmatrix} p_i & p_j & p_k & p_l & p_m \end{bmatrix}$ with respect to the ordered basis $(I^*, J^*, K^*, L^*, M^*)$

Proposition 5.0.2 and Proposition 5.0.3 give the Hamiltonian

$$\begin{aligned} H^{(3)}(p) &= \frac{1}{2} pB(X_{\mathbf{h}}^{-1})^\top B^\top p^\top \\ &= \frac{1}{2} pBI_3B^\top p^\top \\ &= \frac{1}{2} \begin{bmatrix} p_i & p_j & p_k & p_l & p_m \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_i \\ p_j \\ p_k \\ p_l \\ p_m \end{bmatrix} \\ \dot{p} = H^{(3)}(p) &= \frac{1}{2} \left(p_i^2 + p_j^2 + p_k^2 \right). \end{aligned}$$

Now, we consider the vector field $\vec{H}^{(3)}$ on $\mathfrak{t}^*$. As $\mathfrak{t}^*$ is a linear space we have that $\left(\frac{\partial}{\partial I^*}|_p, \frac{\partial}{\partial J^*}|_p, \frac{\partial}{\partial K^*}|_p, \frac{\partial}{\partial L^*}|_p, \frac{\partial}{\partial M^*}|_p\right)$ is a basis for the tangent space at $p \in \mathfrak{t}^*$. We let

$$\vec{H}^{(3)}(p) = h_1(p) \frac{\partial}{\partial I^*}\bigg|_p + h_2(p) \frac{\partial}{\partial J^*}\bigg|_p + h_2(p) \frac{\partial}{\partial K^*}\bigg|_p + h_4(p) \frac{\partial}{\partial L^*}\bigg|_p + h_5(p) \frac{\partial}{\partial M^*}\bigg|_p,$$

and we have that the tangent map of the Hamiltonian $H^{(3)}$ at p is given by the matrix representative

$$dH^{(3)}(p) = \begin{bmatrix} p_i & p_j & p_k & 0 & 0 \end{bmatrix}$$

with respect to the dual basis $(I^*, J^*, K^*, L^*, M^*)$ of $\mathfrak{t}^*$.

Suppose $G \in C^\infty(\mathfrak{t}^*)$, then the action of the vector field $\vec{H}^{(3)}$ on the smooth function G is the smooth function $\vec{H}^{(3)}[G](p) = \frac{\partial}{\partial t} \left(G \circ \exp(t\vec{H}^{(3)})p \right)\Big|_{t=0}$. We thus have, in coordinates, that

$$\begin{aligned} \vec{H}^{(3)}[G](p) &= dG \cdot \vec{H}^{(3)}(p) \\ &= \begin{bmatrix} \frac{\partial G}{\partial I^*} & \frac{\partial G}{\partial J^*} & \frac{\partial G}{\partial K^*} & \frac{\partial G}{\partial L^*} & \frac{\partial G}{\partial M^*} \end{bmatrix} \begin{bmatrix} h_1(p) \\ h_2(p) \\ h_3(p) \\ h_4(p) \\ h_5(p) \end{bmatrix} \\ &= h_1(p) \frac{\partial G}{\partial I^*} + h_2(p) \frac{\partial G}{\partial J^*} + h_3(p) \frac{\partial G}{\partial K^*} + h_4(p) \frac{\partial G}{\partial L^*} + h_5(p) \frac{\partial G}{\partial M^*} \end{aligned}$$

with respect to the dual basis $(I^*, J^*, K^*, L^*, M^*)$ for $\mathfrak{t}^*$.

For the smooth functions $\mathrm{Proj}_I : p \mapsto p_i$, $\mathrm{Proj}J : p \mapsto p_j$, $\mathrm{Proj}_K : p \mapsto p_j$, $\mathrm{Proj}_L : p \mapsto p_l$ and $\mathrm{Proj}_M : p \mapsto p_m$ we have that

$$\begin{aligned} d\mathrm{Proj}_I &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ d\mathrm{Proj}_J &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\ d\mathrm{Proj}_K &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\ d\mathrm{Proj}_L &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ d\mathrm{Proj}_M &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

As $\vec{H}^{(3)}[G](p) = -p\left([dG(p), dH(p)]\right)$ [4, Section 2] we have

$$\begin{aligned}\vec{H}^{(3)}[\mathrm{Proj}_I](p) &= -p\left(\left[d\mathrm{Proj}_I(p), dH^{(3)}(p)\right]\right)\\ h_1(p) &= -p\left([I, p_i I + p_j J + p_k K]\right)\\ &= -p\left(p_j L + p_k M\right)\\ &= -p_j p_l - p_k p_m.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}h_2(p) &= \vec{H}^{(3)}[\mathrm{Proj}_J](p) = p_i p_l\\ h_3(p) &= \vec{H}^{(3)}[\mathrm{Proj}_K](p) = p_i p_m\\ h_4(p) &= \vec{H}^{(3)}[\mathrm{Proj}_L](p) = 0\\ h_5(p) &= \vec{H}^{(3)}[\mathrm{Proj}_M](p) = 0.\end{aligned}$$

Now, by Proposition 5.0.2 $\dot{p} = \vec{H}^{(3)}(p)$ and thus we have the system of differential equations

$$\begin{aligned}\begin{bmatrix}\dot{p_i}\\ \dot{p_j}\\ \dot{p_k}\end{bmatrix} &= \begin{bmatrix}0 & -p_l & -p_m\\ p_l & 0 & 0\\ p_m & 0 & 0\end{bmatrix}\begin{bmatrix}p_i\\ p_j\\ p_k\end{bmatrix}\\ \dot{p_l} = \dot{p_m} &= 0.\end{aligned}$$

This has solution

$$\begin{aligned}\begin{bmatrix}p_i\\ p_j\\ p_k\end{bmatrix} &= \frac{1}{\rho^2}\begin{bmatrix}\rho^2\cos(t\rho) & -\rho\sin(t\rho)p_l & -\rho\sin(t\rho)p_m\\ \rho\sin(t\rho)p_l & \cos(t\rho)p_l^2 + p_m^2 & (\cos(t\rho)-1)p_l p_m\\ \rho\sin(t\rho)p_m & (\cos(t\rho)-1)p_l p_m & p_l^2 + \cos(t\rho)p_m^2\end{bmatrix}\begin{bmatrix}p_i(0)\\ p_j(0)\\ p_k(0)\end{bmatrix}\\ p_l &= p_l(0)\\ p_m &= p_m(0)\end{aligned}$$

where $\rho = \sqrt{p_l^2 + p_m^2}$.

Proposition 5.0.2 also give

$$\dot{g} = T_{\mathbf{1}}L_g \cdot (i^* p)^\sharp.$$

Which is given in coordinates as

$$\begin{bmatrix}\dot{i}\\ \dot{j}\\ \dot{k}\\ \dot{l}\\ \dot{m}\end{bmatrix} = [T_{\mathbf{1}}L_g]BX_{\mathbf{h}}B^\top p^\top$$

by Proposition 5.0.3. That is,

$$\begin{bmatrix}\dot{i}\\ \dot{j}\\ \dot{k}\\ \dot{l}\\ \dot{m}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & i & 0 & 1 & 0\\ 0 & 0 & i & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}p_i\\ p_j\\ p_k\\ p_l\\ p_m\end{bmatrix},$$

therefore

$$\begin{bmatrix} \dot{i} \\ \dot{j} \\ \dot{k} \\ \dot{l} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} p_i \\ p_j \\ p_k \\ ip_j \\ ip_k \end{bmatrix}.$$

Integration and the initial condition $g(0) = \mathbf{1}$ — that is, $i(0) = j(0) = k(0) = l(0) = m(0) = 0$ — gives

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = \frac{1}{\rho^2} \begin{bmatrix} \rho \sin(\rho t) & p_l \cos(\rho t) & p_m \cos(\rho t) \\ -p_l \cos(\rho t) & p_m^2 t + \rho^{-1} p_l^2 \sin(\rho t) & p_l p_m (\rho^{-1} \sin(\rho t) - t) \\ -p_m \cos(\rho t) & p_l p_m (\rho^{-1} \sin(\rho t) - t) & \rho^{-1} \sin(\rho t) p_m^2 + p_l^2 t \end{bmatrix} \begin{bmatrix} p_i(0) \\ p_j(0) \\ p_k(0) \end{bmatrix} + \begin{bmatrix} C_i \\ C_j \\ C_k \end{bmatrix}$$

with

$$\begin{aligned} C_i &= i(0) - \frac{1}{\rho^2} \left(p_j(0) p_l + p_k(0) p_m \right) \\ C_j &= j(0) + \frac{p_i(0) p_l}{\rho^2} \\ C_k &= k(0) + \frac{p_i(0) p_m}{\rho^2}. \end{aligned}$$

That is, the components i, j and k have the form

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 \cos(\rho t) + \mathbf{a}_3 \sin(\rho t),$$

where $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ are 3×1 constant matrices determined by the initial conditions of $p(0)$ and $g(0)$. Further,

$$l(t) = a_0 + a_1 t + a_2 \cos(\rho t) + a_3 \sin(\rho t) + a_4 \cos(2\rho t) + a_5 \sin(2\rho t),$$

with

$$\begin{aligned}
a_0 &= -(a_2 + a_4)\\
a_1 &= \frac{1}{2\rho^4}\Big(p_i(0)^2\rho^2 p_l - 2p_j(0)^2 p_l p_m^2 + p_j(0)^2 p_l^3\\
&\qquad + 4p_j(0)p_k(0)p_l^2 p_m - 2p_j(0)p_k(0)p_m^3 + 3p_k(0)^2 p_l p_m^2\Big)\\
a_2 &= \frac{1}{\rho^4}p_i(0)p_l(p_j(0)p_l + p_k(0)p_m) + \frac{1}{\rho^4}p_i(0)p_m(p_k(0)p_l - p_j(0)p_m)\\
a_3 &= -\frac{1}{\rho^5}p_l\left(p_j(0)p_l + p_k(0)p_m\right)^2 + \frac{1}{\rho^5}p_m(p_j(0)p_l + p_k(0)p_m)(p_j(0)p_m - p_k(0)p_l)\\
a_4 &= -\frac{1}{2\rho^4}p_i(0)p_l\left(p_j(0)p_l + p_k(0)p_m\right)\\
a_5 &= \frac{1}{4\rho^5}p_l(p_j(0)p_l + p_k(0)p_m)^2 - \frac{1}{4\rho^3}p_l p_i(0)^2
\end{aligned}$$

and

$$m(t) = b_0 + b_1 t + b_2\cos(\rho t) + b_3\sin(\rho t) + b_4\cos(2\rho t) + b_5\sin(2\rho t),$$

with

$$\begin{aligned}
b_0 &= -(b_2 + b_4)\\
b_1 &= \frac{1}{2\rho^4}\Big(p_i(0)^2\rho^2 p_m + 3p_j(0)^2 p_l^2 p_m - 2p_k(0)^2 p_l^2 p_m\\
&\qquad + 4p_j(0)p_k(0)p_l p_m^2 - 2p_j(0)p_k(0)p_l^3 + p_k(0)^2 p_m^3\Big)\\
b_2 &= \frac{1}{\rho^4}p_i(0)p_m\left(p_j(0)p_l + p_k(0)p_m\right) - \frac{1}{\rho^4}p_i(0)p_l\left(p_k(0)p_l - p_j(0)p_m\right)\\
b_3 &= -\frac{1}{\rho^5}p_m\left(p_j(0)p_l + p_k(0)p_m\right)^2 + \frac{1}{\rho^5}p_l\left(p_j(0)p_l + p_k(0)p_m\right)\left(p_k(0)p_l - p_j(0)p_m\right)\\
b_4 &= -\frac{1}{2\rho^4}p_i(0)p_m\left(p_j(0)p_l + p_k(0)p_m\right)\\
b_5 &= \frac{1}{4\rho^5}p_m(p_j(0)p_l + p_k(0)p_l)^2 - \frac{1}{4\rho^3}p_m p_i(0)^2.
\end{aligned}$$

Normal geodesics thus have the form

$$g(t) = \mathbf{c}_0 + \mathbf{c}_1 t + \mathbf{c}_2\cos(\rho t) + \mathbf{c}_3\sin(\rho t) + \mathbf{c}_4\cos(2\rho t) + \mathbf{c}_5\sin(2\rho t).$$

where $\mathbf{c}_i$, $i = 0, ..., 5$ are 5×1 real column vectors dependent on the initial conditions.

Chapter 6

Conclusion

Our investigation of the five-dimensional, two-step nilpotent Lie group with two-dimensional centre T was initiated in Chapter 2 with the establishment of its basic group properties. The well known approach of studying Lie groups through their associated Lie algebras and inferring results on the Lie group through Lie group-Lie algebra correspondence results led us to the study of the Lie algebra $\mathfrak{t}$.

In Chapter 3, the Lie algebra's automorphism group $\mathrm{Aut}(\mathfrak{t})$ was computed. Relating subspaces by these automorphisms gave an equivalence relation on the collection of subspaces of the Lie algebra $\mathfrak{t}$. The subspace structure of $\mathfrak{t}$ was computed up to this equivalence with the assistance of a full set of scalar invariants determining the structure. This gave three generating subspaces of $\mathfrak{t}$, which give rise to three left-invariant bracket generating distributions on T of ranks 3, 4 and 5 respectively.

The preliminaries to Chapter 4 included a correspondence result between the Lie group automorphisms of a simply connected matrix Lie group G and the Lie algebra automorphisms of its Lie algebra $\mathfrak{g}$. The fact that any diffeomorphism of a metric Lie group decomposes into the composition of a Lie group automorphism and a left translation reduced the classification of the left-invariant sub-Riemannian structures on T up to diffeomorphism into a classification up to Lie group automorphism. As T is a simply connected matrix Lie group, the above mentioned correspondence of automorphism groups lets us use results on the Lie algebra, from Chapter 3, to complete the classification.

All rank 3 structure on T were found to be isometric. The matrix Lie group T was found to have a one-parameter family of rank 4 left-invariant sub-Riemannian structures, parametrized by the positive reals. The rank 5 structures formed a two-parameter family parametrized by positive reals $\alpha \geq \beta > 0$.

Our study ended in Chapter 5 with the computation of the normal geodesics of the rank 3 left-invariant sub-Riemannian structure. There we found that the normal geodesics $g(t)$ of T

consist of linear as well as a trigonometric components with weights and frequencies dependent on initial conditions. That is,

$$g(t) = \mathbf{c}_0 + \mathbf{c}_1 t + \mathbf{c}_2 \cos(\rho t) + \mathbf{c}_3 \sin(\rho t) + \mathbf{c}_4 \cos(2\rho t) + \mathbf{c}_5 \sin(2\rho t),$$

where $\mathbf{c}_i$, $i = 0, ..., 5$ are 5×1 real column vectors and ρ is a constant, determined by the initial conditions $g(0)$ and $\dot{g}(0)$.

Appendix A

Algebra and Topology

Here, for convenient reference, we state standard results that are used throughout the main text. In Appendix A we state standard results of linear algebra, abstract algebra and topology. We refer to the books of D.Poole[18] and C.Meyer[14] for results on linear algebra and those of C. Pinter[17] and J.Munkres[16] for results in abstract algebra and topology respectively.

A.1 Linear Algebra

A.1.1 General Results

Definition A.1.1. *An $n \times n$ symmetric matrix A is* ***positive definite*** *if $\mathbf{x}^\top A\mathbf{x} > 0$ for every non-zero column vector $\mathbf{x} \in \mathbb{R}^{n\times 1}$.*

A **principal submatrix** of an $n \times n$ matrix A is an $r \times r$ submatrix that lies on the same index set of r rows and columns. An $r \times r$ **principal minor** is the determinant of an $r \times r$ principal submatrix. A **leading principal submatrix** of A is a submatrix obtained by taking the first r rows and columns of A. The determinants of these leading principal submatrices are the **leading principal minors** of A.

Lemma A.1.2. *[cf. 14, Section 7.6] Let A be a real $n \times n$ matrix, then the following are equivalent:*

1. *A is positive definite,*
2. *all the leading principal minors of A are positive and*
3. *all the principal minors of A are positive.*

Definition A.1.3. *Given subspaces X and Y of a vector space V, we define the sum of X and Y by*

$$X + Y = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in X, \mathbf{y} \in Y\}.$$

The sum $X + Y$ of subspaces $X, Y \subset V$ is itself a subspace of V and the smallest subspace containing the set $X \cup Y$.

Theorem A.1.4. *[14, Section 4.4] If X and Y are subspaces of a vector space V, then*

$$\dim(X + Y) = \dim(X) + \dim(Y) - \dim(X \cap Y).$$

Corollary A.1.5. *If X and Y are subspaces of a vector space V, then*

$$\dim(X) + \dim(Y) - \dim(V) \leq \dim(X \cap Y) \leq \min(\dim(X), \dim(Y)).$$

Proof. Follows from the fact that $\dim(V) \geq \dim(X + Y)$. □

A.1.2 Orthogonality

Definition A.1.6. *[18, Chapter 5] Vectors $\mathbf{v} = (v_1, v_2, ..., v_n)$ and $\mathbf{w} = (w_1, w_2, ..., w_n)$ in $\mathbb{R}^n$ are said to be **orthogonal** if $\sum_{i=1}^{n} v_i w_i = 0$. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ in $\mathbb{R}^n$ is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. A set of vectors in $\mathbb{R}^n$ is an **orthonormal set** if it is an orthogonal set of unit vectors.*

Definition A.1.7. *[18, Chapter 5] An $n \times n$ matrix Q whose columns form an orthonormal set is called an **orthogonal matrix**.*

Definition A.1.8. *We denote by*

$$\mathsf{O}(n) = \left\{A \in \mathsf{GL}(\mathsf{n}, \mathbb{R}) : AA^\top = I_n\right\}$$

the orthogonal group of invertible orthonormal $n \times n$ real matrices.

As an example we have that the orthogonal group $\mathsf{O}(2)$ is given by

$$\mathsf{O}(2) = \left\{ \begin{bmatrix} \sigma \sin(\theta) & -\sigma \cos(\theta) \\ \cos\theta & \sin\theta \end{bmatrix} : \sigma = \pm 1, \theta \in \mathbb{R} \right\}.$$

Theorem A.1.9. *[18, Theorem 5.7] If Q is an orthogonal matrix, then its rows form an orthonormal set.*

A.2 Abstract algebra

Definition A.2.1. *[19, Chapter2] Given a group* G*. If* $a, b \in \mathsf{G}$*, the* ***commutator*** *of* a *and* b*, denoted by* $[a, b]$*, is*

$$[a, b] = aba^{-1}b^{-1}.$$

The ***commutator subgroup*** *of* G*, denoted by* G'*, is the subgroup of* G *generated by all the commutators.*

Definition A.2.2. *[19, Chapter 5] The* ***lower central series*** *of* G *is the series of subgroups*

$$\mathsf{G} = \gamma_1(\mathsf{G}) \geq \gamma_2(\mathsf{G}) \geq \cdots,$$

where $\gamma_{i+1}(\mathsf{G}) = [\mathsf{G}, \gamma_i(\mathsf{G})]$ *and for subsets* H*,* K *of* G *we define* $[H, K]$ *as the subgroup of* G *generated by commutators of the form* $[h, k] = hkh^{-1}k^{-1}$*,* $h \in H$*,* $k \in K$*.*

Definition A.2.3. *[19, Chapter 5] The* ***higher commutator subgroups*** *of a group* G *are defined inductively by:*

$$\mathsf{G}^{(0)} = \mathsf{G}, \quad \mathsf{G}^{(i+1)} = \mathsf{G}^{(i)\prime}.$$

That is, $\mathsf{G}^{(i+1)}$ *is the commutator subgroup of* $\mathsf{G}^{(i)}$*. The series*

$$\mathsf{G} = \mathsf{G}^{(0)} \geq \mathsf{G}^{(1)} \geq \mathsf{G}^{(2)} \geq \cdots$$

is called the ***derived series*** *of* G*.*

Definition A.2.4. *[19, Chapter 2] A non-trivial group* G *is said to be* ***simple*** *if it has no normal subgroups except* G *and* $\{1_{\mathsf{G}}\}$*.*

Definition A.2.5. *[19, Chapter 5] An* ***automorphism of a group*** G *is an isomorphism* $\varphi : \mathsf{G} \longrightarrow \mathsf{G}$ *that maps* G *to itself. The automorphisms of a group* G *form a group under the operation of composition, called the* ***automorphism group*** *of* G *and denoted by* $\mathrm{Aut}(\mathsf{G})$*.*

Theorem A.2.6. *[17, Chapter 16] Let* $f : \mathsf{G} \longrightarrow \mathsf{H}$ *be a group homomorphism of* G *onto* H*. If* K *is the kernel of* f*, then*

$$\mathsf{H} \cong \mathsf{G}/\mathsf{K}.$$

Definition A.2.7. *[[19, Chapter 7] Let* K *be a (not necessarily normal) subgroup of a group* G*. Then a subgroup* Q *is a* ***complement*** *of* K *in* G *if* $\mathsf{K} \cap \mathsf{Q} = \{\mathbf{1}_{\mathsf{G}}\}$ *and* $\mathsf{KQ} = \mathsf{G}$*. Here,*

$$\mathsf{KQ} = \{kq : k \in \mathsf{K}, q \in \mathsf{Q}\}.$$

Definition A.2.8. *[19, Chapter 7] A group* G *is a* ***semidirect product*** *of subgroup* K *by the subgroup* Q*, denoted by* $\mathsf{G} = \mathsf{K} \rtimes \mathsf{Q}$*, if* $\mathsf{K} \lhd \mathsf{G}$ *—* K *is a normal subgroup of* G *— and* K *has a complement* Q_1 *that is group isomorphic to* Q*.*

Definition A.2.9. *[19, Chapter 7] Let* Q *and* K *be groups, and let* $\theta : \mathsf{Q} \longrightarrow \mathrm{Aut}(\mathsf{K})$ *be a homomorphism. A semidirect product* G *of* K *by* Q ***realizes*** θ *if, for all* $x \in \mathsf{Q}$ *and* $a \in \mathsf{K}$*,*

$$\theta_x(a) = xax^{-1}.$$

Definition A.2.10. *Given groups* Q *and* K *and a homomorphism* $\theta : \mathsf{Q} \longrightarrow \mathrm{Aut}(\mathsf{K})$*, define*

$$\mathsf{G} = \mathsf{K} \rtimes_\theta \mathsf{Q}$$

to be the set of all ordered pairs $(a, x) \in \mathsf{K} \times \mathsf{Q}$ *equipped with the group operation*

$$(a, x)(b, y) = (a\theta_x(b), xy).$$

Theorem A.2.11. *[19, Theorem 7.23] If* G *is a semidirect product of* K *by* Q*, then there exists a homomorphism* $\theta : \mathsf{Q} \longrightarrow \mathrm{Aut}(\mathsf{K})$ *with* $\mathsf{G} \cong \mathsf{K} \rtimes_\theta \mathsf{Q}$*. Namely,*

$$\theta_x(a) = xax^{-1}.$$

Theorem A.2.12. *[19, Chapter 7] The semidirect product* $\mathsf{K} \rtimes_\theta \mathsf{Q}$ *is a direct product* $\mathsf{K} \times \mathsf{Q}$ *if and only if the homomorphism* $\theta : \mathsf{Q} \longrightarrow \mathrm{Aut}(\mathsf{K})$ *is trivial.*

A.3 Topology

Definition A.3.1. *[16, Section 23] Let* X *be a topological space. A* ***separation*** *of* X *is a pair* U, V *of disjoint nonempty open subsets of* X *whose union is* X*. The space* X *is said to be* ***connected*** *if there does not exist a separation of* X*.*

Given points x *and* y *of the topological space* X*, a* ***path*** *in* X *from* x *to* y *is a continuous map* $f : [a, b] \longrightarrow \mathsf{X}$ *of some closed interval* $[a, b]$ *of the real line into* X *such that* $f(a) = x$ *and* $f(b) = y$*. A topological space* X *is said to be* ***path-connected*** *if every pair of points of* X *can be joined by a path in* X*.*

Theorem A.3.2. *A set* $F \subseteq \mathbb{R}^n$ *is closed if and only if for any convergent sequence* $x_n \longrightarrow x$ *in* F*,* $x \in F$*.*

Definition A.3.3. *[1, Definition 0.1.1] Let* G *be a group and at the same time a Hausdorff topological space. Suppose in addition that the group operations*

(1) $(g, h) \mapsto gh$

(2) $g \mapsto g^{-1}$

are continuous, where in (1) *we take the product topology on* $\mathsf{G} \times \mathsf{G}$*. We call* G *a* ***topological group****.*

Equivalent to conditions (1) and (2) above is the condition that the map $(g, h) \mapsto gh^{-1}$ is continuous.

Definition A.3.4. *[1, Chapter 0] Let* G *and* H *be topological groups. A map* $f : \mathsf{G} \longrightarrow \mathsf{H}$ *is called a* ***topological group homomorphism*** *if it is a continuous group homomorphism.*

Appendix B

Lie groups and Manifolds theory

Appendix B presents results on smooth manifolds and Lie groups. It's results drawing mainly from the texts Introduction to Smooth Manifolds by J.Lee[11], Basic Lie Theory by H.Abbaspour and M.Moskowitz[1] and Introduction to Geodesics in Sub-Riemannian Geometry [6]. We assume the reader is familiar with the definition of a smooth manifold, tangent vectors, as well as those of a smooth map and diffeomorphism between manifolds.

B.1 Results on manifolds

Definition B.1.1. *[6, Definition 1.3] A* ***smooth vector field*** *on a smooth manifold M is a smooth map*

$$X : q \mapsto X(q) \in T_q M.$$

Given a complete vector field X on a smooth manifold M, we can consider the family of maps

$$\phi_t : M \longrightarrow M \quad \phi_t(q) = \gamma(t;q) \quad t \in \mathbb{R},$$

where $\gamma(t;q)$ is the integral curve of the vector field X with $\gamma(0) = q$. In other words, $\phi_t(q)$ is the shift for time t along the integral curve of X that starts from q [6].

It follows that $\phi : \mathbb{R} \times M \longrightarrow M, \phi(t,q) = \phi_t(q)$ is smooth in both variables and that $\{\phi_t, t \in \mathbb{R}$ is a one-parameter subgroup of Diff(M). Additionally, by construction, we have

$$\frac{\partial \phi_t(q)}{\partial t} = X(\phi_t(q)), \quad \phi_0(q) = q, \forall q \in M.$$

The family of maps ϕ_t is called the **flow** generated by X. For the flow ϕ_t of the vector field X it is convenient to use the exponential notation $\phi_t := e^{tX}$, for every $t \in \mathbb{R}$[6]. Following the exponential notation, we have

$$\frac{d}{dt}e^{tX} = Xe^{tX}.$$

A smooth vector field X induces an action on the algebra $C^\infty(M)$ of smooth functions on M, defined as follows

$$X : C^\infty(M) \longrightarrow C^\infty(M), \quad a \mapsto Xa,$$

where

$$(Xa)(q) = \left.\frac{d}{dt}\right|_{t=0} a(e^{tX}(q)), \quad q \in M.$$

Definition B.1.2. *[cf. 6, Definition 2.3] Let M be a smooth manifold. A **distribution** $\mathcal{D}$ on M is a family of subspaces*

$$\{\mathcal{D}(q)\}_{q\in M} \text{ where } \mathcal{D}(q) \subseteq T_qM.$$

*The dimension of $\mathcal{D}(q)$ is called the **rank** of the distribution at $q \in M$. The distribution $\mathcal{D}$ is said to have constant rank if $\dim(\mathcal{D}(q_1)) = \dim(\mathcal{D}(q_2))$ for all $q_1, q_2 \in M$.*

Definition B.1.3. *[cf. 13, Section 2.3] (Pull backs and push forwards)*
Let M and N be smooth manifolds with $\phi : M \longrightarrow N$ a diffeomorphism.

(a) ϕ_*X, the push forward of a vector field X by ϕ is the vector field $(\phi_*X)(\phi(z)) = d_z\phi \cdot X(z)$ on N.

(b) ϕ^*Y, the pull back of the vector field Y on N by ϕ is the vector field $\phi^*Y = (\phi^{-1})_*Y$ on M.

(c) $\phi_*(\mathcal{D})$, the push forward of the distribution $\mathcal{D}$ on M by ϕ, is the distribution $\phi_*(\mathcal{D})(\phi(z)) = d_z\phi(\mathcal{D})$ on N.

(d) $\phi^*(\mathcal{D})$, the pull back of the distribution $\mathcal{D}$ on N by ϕ is the distribution $\phi^*(\mathcal{D}) = (\phi^{-1})_*\mathcal{D}$ on M.

(e) $\phi_*(\mathbf{g})$, the push forward of the Riemannian metric $\mathbf{g}$ on M by ϕ is given by the Riemannian metric $\phi_*(\mathbf{g}_{\phi(z)})(d_z\phi(x), d_z\phi(y)) = \mathbf{g}_z(x, y)$ for $z \in M$ and $x, y \in T_zM$ on N.

(f) $\phi^*(\mathbf{g})$, the pull back of the Riemannian metric $\mathbf{g}$ on N by ϕ is given by the Riemannian metric $(\phi^{-1})_*(\mathbf{g})$ on M.

B.2 Lie groups

Definition B.2.1. *[1, Section 0.2] We define a real **Lie group** as a group G which is also a finite-dimensional real smooth manifold whose operations are smooth. That is, the map $\mathsf{G} \times \mathsf{G} \longrightarrow \mathsf{G}$ given by $(g, h) \mapsto (gh^{-1})$ is C^∞ where $\mathsf{G} \times \mathsf{G}$ is endowed with the product manifold structure.*

*A subgroup H of a Lie group G is said to be a **Lie subgroup** if it is a submanifold of the underlying manifold G[7].*

For Lie groups G and H, a **Lie group homomorphism** $f : \mathsf{G} \longrightarrow \mathsf{H}$ is a smooth group homomorphism. A Lie homomorphism $f : \mathsf{G} \longrightarrow \mathsf{H}$ is called a **Lie isomorphism** if there exists a smooth inverse $f^{-1} : \mathsf{H} \longrightarrow \mathsf{G}$. That is, if f is simultaneously an isomorphism of abstract groups and a diffeomorphism of manifolds[7]. We denote by $\mathrm{Aut}(\mathsf{G})$, the group of Lie group automorphisms of G. That is, the groups of bijective Lie group homomorphisms $\phi : \mathsf{G} \longrightarrow \mathsf{G}$ with smooth inverses. Given a Lie group G, the left translations $L_g : \mathsf{G} \longrightarrow \mathsf{G}$, $g \in G$, defined by $L_g(h) = gh$, are global diffeomorphisms of G[1].

Theorem B.2.2. *[see 7, Theorem 3.2] Let N be a normal Lie subgroup of a Lie group G. Then the quotient group G/N is a Lie group.*

Theorem B.2.3. *[see 7, Theorem 3.4] Let $f : \mathsf{G} \longrightarrow \mathsf{H}$ be an epimorphism of Lie groups and let $\mathsf{N} = \ker(f)$. Then the map*

$$\phi : \mathsf{G}/\mathsf{N} \longrightarrow \mathsf{H}, g\mathsf{N} \mapsto f(g)$$

is an isomorphism of Lie groups.

Definition B.2.4. *[20, Section 3.5] The **maximal torus** of a Lie group G is a maximal subgroup of G that is isomorphic to a k-torus*

$$\mathbb{T}^k = \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \mathbb{S}^1 \text{ (a k-fold carteasian product).}$$

Definition B.2.5. *[1, Definition 0.5.1] Let $\mathfrak{g}$ be a finite-dimensional vector space over $\mathbb{R}$. We say that $\mathfrak{g}$ is a (real) **Lie algebra** if it possesses an anti-symmetric bilinear product $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \mapsto \mathfrak{g}$, called the **Lie bracket**, which satisfies the Jacobi identity,*

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = \bar{0}$$

for all $X, Y, Z \in \mathfrak{g}$.

The left-invariant vector fields form a subalgebra of the Lie algebra of all vector fields on a Lie group G. Since left-invariant vector fields act transitively on G it follows that a left-invariant vector field X is fully determined by its value at identity $X(\mathbf{1})$. That is, any $v \in T_{\mathbf{1}}\mathsf{G}$ determines a left-invariant vector field $X(g) = d_{\mathbf{1}}L_g(v) \in T_g\mathsf{G}$.

The linear map $X \mapsto X(\mathbf{1})$ is a vector space isomorphism from the space of left-invariant vector fields to $T_{\mathbf{1}}\mathsf{G}$, which we call the Lie algebra $\mathfrak{g}$ of G. Therefore, $\mathfrak{g}$ is — isomorphic to — a finite dimensional subspace of the vector space of all vector fields on the manifold G[1, Chapter 1].

Definition B.2.6. *An **automorphism** φ of a Lie algebra $\mathfrak{g}$ is an invertible linear map that preserves the Lie bracket. That is,*

$$[\varphi \cdot X, \varphi \cdot Y] = [X,Y] \text{ for every } X, Y \in \mathfrak{g}.$$

*Automorphisms of a Lie algebra $\mathfrak{g}$ form a group under composition called the **automorphism group** of $\mathfrak{g}$, denoted by $\mathrm{Aut}(\mathfrak{g})$.*

Definition B.2.7. *Let $\mathfrak{g}$ be a Lie algebra. The **centre**, $\mathrm{Z}(\mathfrak{g})$, of $\mathfrak{g}$ is the set of all vectors $C \in \mathfrak{g}$ such that $[C,V] = \bar{0}$ for every $V \in \mathfrak{g}$.*

Lemma B.2.8. *If φ is an automorphism of the Lie algebra $\mathfrak{g}$ and C is a central element of $\mathfrak{g}$, then $\varphi \cdot C$ is a central element of $\mathfrak{g}$.*

Proof. Suppose C is a central element of $\mathfrak{g}$ and $V \in \mathfrak{g}$, then as C is central and φ has an inverse, we have

$$\begin{aligned}[\varphi \cdot C, V] &= \left[\varphi \cdot C, \varphi \cdot (\varphi^{-1} \cdot V)\right] \\ &= \varphi \cdot \left[C, \varphi^{-1} \cdot V\right] \\ &= \varphi \cdot \bar{0} \\ &= \bar{0}.\end{aligned}$$

Thus $\varphi \cdot C$ is a central element of $\mathfrak{g}$. □

Definition B.2.9. *[11, Chapter 8] A vector field X on a Lie group G is said to be left-invariant if it is invariant under all left translations. That is,*

$$(L_g)_* X = X \text{ for each } g \in \mathsf{G}.$$

Definition B.2.10. *[11, Chapter 19] A distribution $\mathcal{D}$ on a Lie group G is said to be left-invariant if it is invariant under every left translation. That is,*

$$L_g^*(\mathcal{D}) = \mathcal{D} \text{ for each } g \in \mathsf{G},$$

where L_g is left translation by g.

Theorem B.2.11. *[8, Theorem 5.6] Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and let φ be a Lie algebra homomorphism. If G is simply connected, then there exists a unique Lie group homomorphism $\phi : \mathsf{G} \longrightarrow \mathsf{H}$ such that $\phi(e^X) = e^{\phi(X)}$ for all $X \in \mathfrak{g}$. That is, $\varphi = d_{\mathbf{1}}\phi$.*

Corollary B.2.12. *[8, Corollary 5.6] Suppose G and H are simply connected matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. If $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra isomorphism, then there exists a unique Lie group isomorphism $\phi : \mathsf{G} \longrightarrow \mathsf{H}$, with $d_{\mathbf{1}}\phi = \varphi$.*

Theorem B.2.13. *[8, Theorem 3.28] The differential at identity of a Lie group homomorphism is a Lie algebra homomorphism.*

Definition B.2.14. *[6, Definition 1.21] Let X and Y be smooth vector fields on the smooth manifold M. We define their **Lie bracket** as the vector field*

$$[X, Y] = \left.\frac{\partial}{\partial t}\right|_{t=0} e_*^{-tX} Y.$$

Expanding the above definition, we have that

$$[X, Y]|_q = \left.\frac{\partial}{\partial t}\right|_{t=0} (e_*^{-tX} Y)|_q = \left.\frac{\partial}{\partial t}\right|_{t=0} e_*^{-tX} \left(Y|_{e^{tX(q)}}\right) = \left.\frac{\partial^2}{\partial s \partial t}\right|_{t=s=0} e^{-tX} \circ e^{sY} \circ e^{tX}(q),$$

at a point q on the smooth manifold M.

Definition B.2.15. *[1, Chapter 0] The real general linear group,* $\mathsf{GL}(n,\mathbb{R})$ *is the group of invertible* $n\times n$ *real matrices. Similarly, the complex general linear group* $\mathsf{GL}(n,\mathbb{C})$ *is the group of invertible* $n\times n$ *complex matrices.*

The complex general linear group $\mathsf{GL}(n,\mathbb{C})$ inherits the relative topology when considered as a subset of $\mathsf{M}(n,\mathbb{C})$, where $\mathsf{M}(n,\mathbb{C})$ is identified with $\mathbb{R}^{2n^2}$ equipped with its usual (Euclidean) topology. With this topology the group $\mathsf{M}(n,\mathbb{C})$ is a topological group. With a global smooth chart defined component-wise $\mathsf{M}(n,\mathbb{C})$ is a smooth manifold and in fact a Lie group.

Definition B.2.16. *[20, Chapter 8] With the understanding that the topology of all matrix groups is taken to be relative to* $\mathsf{GL}(n,\mathbb{C})$*, we define a* ***matrix Lie group*** *as a closed subgroup of* $\mathsf{GL}(n,\mathbb{C})$*.*

We present as an example the three-dimensional Heisenberg group, a group which appears in the text and is studied in [4].

Example B.2.17. *By*

$$\mathsf{H}_3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

we denote the three-dimensional Heisenberg group. This is a two-step nilpotent matrix Lie group with one-dimensional centre

$$Z(\mathsf{H}_3) = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{h}_3$ *is given by*

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} = xX + yY + zZ : x, y, z \in \mathbb{R} \right\}$$

with the non-trivial Lie bracket operation $[X, Y] = Z$*.*

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