

Fejér monotonicity and fixed point theorems with applications to a nonlinear integral equation in complex valued Banach spaces

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ABSTRACT

It is our purpose in this paper to prove some fixed point results and Fejér monotonicity of some faster fixed point iterative sequences generated by some nonlinear operators satisfying rational inequality in complex valued Banach spaces. We prove that results in complex valued Banach spaces are valid in cone metric spaces with Banach algebras. Furthermore, we apply our results in solving certain mixed type Volterra-Fredholm functional nonlinear integral equation in complex valued Banach spaces.

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1. INTRODUCTION

Fixed point theory, which is famous in sciences and engineering due to its applications in solving several nonlinear problems in these fields of study became one of the most interesting area of research in the last sixty years. For example,

it has shown the importance of theoretical subjects, which are directly applicable in different applied fields of science. Other areas of applications includes optimization problems, control theory, economics and a host of others. In particular, it plays an important role in the investigation of existence of solutions to differential and integral equations, which direct the behaviour of several real life problems for which the existence of solution is critical (see, e.g. [25], [42]). In 1922, Banach [12] provided a general iterative method to construct a fixed point result and proved its uniqueness under linear contraction in complete metric spaces. This famous results of Banach have been generalized in several directions by many researchers. These generalization were made either by using the contractive condition or by imposing some additional conditions on the ambient space. Some of these generalizations of metric spaces includes: rectangular metric spaces, pseudo metric spaces, D-metric spaces, partial metric spaces, G-metric spaces and cone metric spaces (see, e.g. [1], [22], [23]).

The notion of complex valued metric spaces was introduced by Azam *et al.* [11] in 2011. They established some fixed point theorems for a pair of mappings satisfying rational inequality. Their results is intended to define rational expressions which are meaningless in cone metric spaces. Although complex valued metric spaces form a special class of cone metric spaces (see, e.g. [2], [6]), yet the definition of cone metric spaces rely on the underlying Banach space which is not a division ring. Consequently, rational expressions are not meaningful in cone metric spaces, this means that results involving mappings satisfying rational expressions cannot be generalized to cone metric spaces. In view of this deficiency, Azam *et al.* [11] introduced the concept of complex valued metric spaces. It is known that in cone metric spaces the underlying metric assumes values in linear spaces where the linear space may be even infinite dimensional, whereas in the case of complex valued metric spaces the metric values belong to the set of complex numbers which is one dimensional vector space over the complex field. This instance is the major motivation for the consideration of complex valued metric spaces independently (see, [6]). Hence, results in this direction cannot be generalized to cone metric spaces, but to complex valued metric spaces. It is known that complex valued metric space is useful in many branches of Mathematics, including number theory, algebraic geometry, applied Mathematics as well as in physics including hydrodynamics, mechanical engineering, thermodynamics and electrical engineering (see, e.g. [41]). Several authors have obtained interesting and applicable results in complex valued metric spaces (see, e.g. [2], [3], [5], [8], [6], [11], [25], [36], [40], [41], [42]).

It is known that there is a close relationship between the problem of solving a nonlinear equation and that of approximating fixed points of a corresponding contractive type operator (see, e.g. [14], [15], [32]). Hence, there is a practical and theoretical interests in approximating fixed points of several contractive type operators. Since, the introduction of the notion of complex valued metric spaces by Azam *et al.* [11] in 2011, most results obtained in literature by many

authors are existential in nature (see, e.g. [8], [11], [36], [41], [42]). Consequently, there is a gap in literature with respect to the approximation of the fixed point of several nonlinear mappings in this type of space. Recently, Okeke [32] exploited the idea of complex valued metric spaces to define the concept of complex valued Banach spaces and then initiated the idea of approximating the fixed point of nonlinear mappings in complex valued Banach spaces.

The theory of integral and differential equations is an important aspect of nonlinear analysis and the most applied tool for proving the existence of the solutions of such equations is the fixed point technique (see, e.g. [12], [18], [19], [33]). One of the most frequent and difficult problems faced by scientists in mathematical sciences is nonlinear problems. This is because nature is intrinsically nonlinear (see, e.g. [19]). Solving nonlinear equations is cumbersome but important to mathematicians and applied mathematicians such as engineers and physicist. Some authors have used the fixed point iterative methods in solving such equations (see, e.g. [18], [19], [33]). In this paper, we apply our results in solving certain mixed type Volterra-Fredholm functional nonlinear integral equation in complex valued Banach spaces.

It is our purpose in this paper to prove some fixed point results and Fejér monotonicity of some faster fixed point iterative sequences generated by some nonlinear operators satisfying rational inequality in complex valued Banach spaces. We prove that results in complex valued Banach spaces are valid in cone metric spaces with Banach algebras. Our results validates the fact that fixed point theorems in the setting of cone metric spaces with Banach algebras are more useful than the standard results in cone metric spaces and that results in cone metric spaces with Banach algebras cannot be reduced to corresponding results in cone metric spaces. Furthermore, we apply our results in solving certain mixed type Volterra-Fredholm functional nonlinear integral equation in complex valued Banach spaces. Our results unify, generalize and extend several known results to complex valued Banach spaces, including the results of [4], [9], [10], [18], [19], [28], [33]) among others.

2. PRELIMINARIES

The following symbols, notations and definitions which can be found in [11] will be useful in this study. Let \mathbb{C} be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order \lesssim on \mathbb{C} as follows:

$$z_1 \lesssim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), \quad Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \lesssim z_2$ if one of the following conditions is satisfied:

- (i) $Re(z_1) = Re(z_2), \quad Im(z_1) < Im(z_2)$,
- (ii) $Re(z_1) < Re(z_2), \quad Im(z_1) = Im(z_2)$,
- (iii) $Re(z_1) < Re(z_2), \quad Im(z_1) < Im(z_2)$,
- (iv) $Re(z_1) = Re(z_2), \quad Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \not\lesssim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied.

Note that

- (a) $a, b \in \mathbb{R}$ and $a \leq b \implies az \preceq bz$ for all $z \in \mathbb{C}$;
- (b) $0 \preceq z_1 \preceq z_2 \implies |z_1| < |z_2|$;
- (c) $z_1 \preceq z_2$ and $z_2 \prec z_3 \implies z_1 \prec z_3$.

Definition 2.1 ([11]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow \mathbb{C}$, satisfies:

- 1. $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- 2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
- 3. $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X , and (X, d) is called a complex valued metric space.

Recently, Okeke [32] defined a complex valued Banach space and proved some interesting fixed point theorems in the framework of complex valued Banach spaces.

Definition 2.2 ([32]). Let E be a linear space over a field \mathbb{K} , where $\mathbb{K} = \mathbb{R}$ (the set of real numbers) or \mathbb{C} (the set of complex numbers). A complex valued norm on E is a complex valued function $\|\cdot\| : E \rightarrow \mathbb{C}$ satisfying the following conditions:

- 1. $\|x\| = 0$ if and only if $x = 0$, $x \in E$;
- 2. $\|kx\| = |k| \cdot \|x\|$ for all $k \in \mathbb{K}$, $x \in E$;
- 3. $\|x + y\| \preceq \|x\| + \|y\|$ for all $x, y \in E$.

A linear space with a complex valued norm defined on it is called a *complex valued normed linear space*, denoted by $(E, \|\cdot\|)$. A point $x \in E$ is called an *interior point* of a set $A \subseteq E$ if there exist $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in E : \|x - y\| \prec r\} \subseteq A.$$

A point $x \in E$ is called a limit point of the set A whenever for each $0 \prec r \in \mathbb{C}$, we have

$$B(x, r) \cap (AnE) \neq \emptyset.$$

The set A is said to be open if each element of A is an interior point of A . A subset $B \subseteq E$ is said to be closed if it contains each of its limit point. The family

$$F = \{B(x, r) : x \in E, 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ on E .

Suppose x_n is a sequence in E and $x \in E$. If for all $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\|x_n - x_{n+m}\| \prec c$, then $\{x_n\}$ is called a Cauchy sequence in $(E, \|\cdot\|)$. If every Cauchy sequence is convergent in $(E, \|\cdot\|)$, then $(E, \|\cdot\|)$ is called a complex valued Banach space.

Example 2.3 ([32]). Let $E = \mathbb{C}$ be the set of complex numbers. Define $\|\cdot\| : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ by

$$\|z_1 - z_2\| = |x_1 - x_2| + i|y_1 - y_2| \quad \forall z_1, z_2 \in \mathbb{C},$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Clearly, $(\mathbb{C}, \|\cdot\|)$ is a complex valued normed linear space.

Example 2.4 ([32]). Let $E = \mathbb{C}$ be the set of complex numbers. Define a mapping $\|\cdot\| : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ by

$$\|z_1 - z_2\| = e^{ik}|z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{C}, \quad \text{where } k \in [0, \frac{\pi}{2}], \quad z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2.$$

Then $(\mathbb{C}, \|\cdot\|)$ is a complex valued normed linear space.

Example 2.5 ([32]). Let $(C[a, b], \|\cdot\|_\infty)$ be the space of all continuous complex valued functions on a closed interval $[a, b]$, endowed with the Chebyshev norm

$$\|x - y\|_\infty = \max_{t \in [a, b]} |x(t) - y(t)|e^{ik}, \quad x, y \in C[a, b], \quad k \in [0, \frac{\pi}{2}].$$

Then $(C[a, b], \|\cdot\|_\infty)$ is a complex valued Banach space, since the elements of $C[a, b]$ are continuous functions, and convergence with respect to the Chebyshev norm $\|\cdot\|_\infty$ corresponds to uniform convergence. We can easily show that every Cauchy sequence of continuous functions converges to a continuous function, i.e. an element of the space $C[a, b]$.

In 1975, Dass and Gupta [21] extended the Banach contraction mapping principle by proving the following theorem for mappings satisfying contractive condition of the rational type in the framework of complete metric spaces.

Theorem 2.6 ([21]). *Let (X, d) be a complete metric space and let T be a mapping on X . Assume that there exist $\alpha, \beta \in (0, 1)$ satisfying $\alpha + \beta < 1$ and*

$$d(Tx, Ty) \leq \alpha d(y, Ty) \frac{1 + d(x, Tx)}{1 + d(x, y)} + \beta d(x, y) \tag{2.1}$$

for all $x, y \in X$. Then T has a unique fixed point z . Moreover $\{T^n x\}$ converges to z for all $x \in X$.

The following theorem for a Meir-Keeler contraction of the rational type was proved in 2013 by Samet *et al.* [39] in the framework of complete metric spaces.

Theorem 2.7 ([39]). *Let (X, d) be a complete metric space and T be a mapping from X into itself. We assume that the following hypothesis holds: given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$2\varepsilon \leq d(y, Ty) \frac{1 + d(x, Tx)}{1 + d(x, y)} + d(x, y) < 2\varepsilon + \delta(\varepsilon) \implies d(Tx, Ty) < \varepsilon. \tag{2.2}$$

Then T has a unique fixed point $\zeta \in X$. Moreover, for any $x \in X$, the sequence $\{T^n x\}$ converges to ζ .

In 2007, Agarwal *et al.* [7] introduced the S iteration process as follows:

$$\begin{cases} x_0 \in D, \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n Ty_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.3}$$

In 2014, Gürsoy and Karakaya [20] introduced the Picard-S iterative process as follows:

$$\begin{cases} x_0 \in D, \\ z_n = (1 - \beta_n)x_n + \beta_nTx_n, \\ y_n = (1 - \alpha_n)Tx_n + \alpha_nTz_n \\ x_{n+1} = Ty_n. \end{cases} \quad (2.4)$$

In 2015, Thakur *et al.* [44] introduced the following iterative process:

$$\begin{cases} x_0 \in D, \\ z_n = (1 - \beta_n)x_n + \beta_nTx_n, \\ y_n = T((1 - \alpha_n)x_n + \alpha_nz_n) \\ x_{n+1} = Ty_n. \end{cases} \quad (2.5)$$

The authors in [44] proved that the Thakur iterative process (2.5) converges faster than Picard, Mann [30], Ishikawa [26], S [7], Noor [31] and Abbas [4] iteration processes for Suzuki's generalized nonexpansive mappings. Recently, Ullah and Arshad [45] introduced the M-iteration. They proved that this iterative process converges faster than all of S [7], Picard-S [19], Picard, Mann [30], Ishikawa [26], Noor [31], SP [35], CR [17], S^* [27], Abbas [4] and Normal-S [38] iteration processes. The following is the M-iteration process introduced by Ullah and Arshad [45] in 2018.

$$\begin{cases} x_0 \in D, \\ z_n = (1 - \alpha_n)x_n + \alpha_nTx_n, \\ y_n = Tz_n \\ x_{n+1} = Ty_n. \end{cases} \quad (2.6)$$

In 2013, Khan [28] introduced the Picard-Mann hybrid iterative process. The iterative process for one mapping case is given by the sequence $\{m_n\}_{n=1}^{\infty}$.

$$\begin{cases} m_1 = m \in D, \\ m_{n+1} = Tz_n, \\ z_n = (1 - \alpha_n)m_n + \alpha_nTm_n, \quad n \in \mathbb{N}, \end{cases} \quad (2.7)$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is in $(0, 1)$. Khan [28] proved that this iterative process converges faster than all of Picard, Mann and Ishikawa iterative processes in the sense of Berinde [15] for contractive mappings.

Recently, Okeke and Abbas [33] introduced the Picard-Krasnoselskii hybrid iterative process defined by the sequence $\{x_n\}_{n=1}^{\infty}$ as follows:

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = Ty_n, \\ y_n = (1 - \lambda)x_n + \lambdaTx_n, \quad n \in \mathbb{N}, \end{cases} \quad (2.8)$$

where $\lambda \in (0, 1)$. The authors proved that this new hybrid iteration process converges faster than all of Picard, Mann, Krasnoselskii and Ishikawa iterative processes in the sense of Berinde [15]. They also used this iterative process to find the solution of delay differential equations.

Definition 2.8 ([15]). Let $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty$ be two sequences of positive numbers that converge to a , respectively b . Assume there exists

$$l = \lim_{n \rightarrow \infty} \frac{|a_n - a|}{|b_n - b|}. \tag{2.9}$$

1. If $l = 0$, then it is said that the sequence $\{a_n\}_{n=0}^\infty$ converges to a faster than the sequence $\{b_n\}_{n=0}^\infty$ to b ;
2. If $0 < l < \infty$, then we say that the sequences $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ have the same rate of convergence.

Definition 2.9. Let D be a nonempty subset of a complex valued Banach space $(E, \|\cdot\|)$. The diameter of D is

$$\text{diam}_D = \sup_{(x,y) \in D \times D} \|x - y\|. \tag{2.10}$$

The distance to D is the function

$$\|\cdot\|_D : E \rightarrow \mathbb{C} : x \rightarrow \inf \|x - D\|. \tag{2.11}$$

The following lemma will be useful in this study.

Lemma 2.10 ([32]). *Let $(E, \|\cdot\|)$ be a complex valued Banach space and let $\{x_n\}$ be a sequence in E . Then $\{x_n\}$ converges to x if and only if $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 2.11 ([32]). *Let $(E, \|\cdot\|)$ be a complex valued Banach space and let $\{x_n\}$ be a sequence in E . Then $\{x_n\}$ is a Cauchy sequence if and only if $\|x_n - x_{n+m}\| \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 2.12 ([43]). *Let $\{\beta_n\}_{n=0}^\infty$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ one has satisfied the inequality*

$$\beta_{n+1} \leq (1 - \mu_n)\beta_n + \mu_n\gamma_n,$$

where $\mu_n \in (0, 1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^\infty \mu_n = \infty$ and $\gamma_n \geq 0, \forall \mathbb{N}$. Then the following inequality holds

$$0 \leq \limsup_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \gamma_n.$$

3. FEJÉR MONOTONICITY AND FIXED POINT THEOREMS IN COMPLEX VALUED BANACH SPACES

In this section, we prove some Fejér monotonicity and fixed point results in the framework of complex valued Banach spaces. Our results improves and extend some known results in the framework of complex valued Banach spaces, including the results of Bauschke and Combettes [13], Cegielski [16] and Dass and Gupta [21] among others. We begin this section by defining the concept of Fejér monotonicity in the framework of complex valued Banach spaces and also provide some examples.

Definition 3.1. Let D be a nonempty subset of a complex valued Banach space $(E, \|\cdot\|)$ and let $\{x_n\}$ be a sequence in E . Then $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone with respect to D if for each $x \in D$ and each $n \in \mathbb{N}$,

$$\|x_{n+1} - x\| \lesssim \|x_n - x\|. \tag{3.1}$$

Example 3.2. Suppose $\{x_n\}_{n \in \mathbb{N}}$ is a bounded sequence in \mathbb{C} that is increasing (respectively decreasing). Then the sequence $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone with respect to $[\sup\{x_n\}_{n \in \mathbb{N}}, +\infty)$ (respectively $(-\infty, \inf\{x_n\}_{n \in \mathbb{N}}]$).

Example 3.3. Let D be a nonempty subset of a complex valued Banach space $(E, \|\cdot\|)$ and $T : D \rightarrow D$ be a mapping on D with $F(T) := \{x \in D : Tx = x\} \neq \emptyset$. Assume that there exist $\alpha, \beta \in (0, 1)$ satisfying $\alpha + \beta < 1$ and

$$\|Tx - Ty\| \lesssim \alpha\|y - Ty\| \frac{1 + \|x - Tx\|}{1 + \|x - y\|} + \beta\|x - y\| \tag{3.2}$$

for all $y \in F(T)$. Let $x_0 \in D$ and set $x_{n+1} = Tx_n, \forall n \in \mathbb{N}$. Then $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone with respect to $F(T)$.

Now, using relation (3.2) together with the facts that $\alpha, \beta \in (0, 1)$ and $y \in F(T)$, we have

$$\begin{aligned} \|x_{n+1} - y\| &\lesssim \alpha\|y - Ty\| \frac{1 + \|x_n - Tx_n\|}{1 + \|x_n - y\|} + \beta\|x_n - y\| \\ &= \alpha \cdot 0 \left(\frac{1 + \|x_n - Tx_n\|}{1 + \|x_n - y\|} \right) + \beta\|x_n - y\| \\ &= \beta\|x_n - y\| \\ &\lesssim \|x_n - y\|. \end{aligned} \tag{3.3}$$

Hence, $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone with respect to $F(T)$.

Proposition 3.4. Let D be a nonempty subset of a complex valued Banach space $(E, \|\cdot\|)$ and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in E . Suppose that $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone with respect to D . Then the following hold:

- (i) $\{x_n\}_{n \in \mathbb{N}}$ is bounded.
- (ii) For every $x \in D$, $(\|x_n - x\|)_{n \in \mathbb{N}}$ converges.
- (iii) $\{\|\cdot\|_D(x_n)\}_{n \in \mathbb{N}}$ is decreasing and converges.
- (iv) Let $m \in \mathbb{N}$ and let $n \in \mathbb{N}$. Then

$$\|x_{n+m} - x_n\| \leq 2\|\cdot\|_D(x_n). \tag{3.4}$$

Proof. (i) Suppose $x \in D$. It follows from (3.1) that $\{x_n\}_{n \in \mathbb{N}}$ lies in $B(x, \|x_0 - x\|)$. Hence, $\{x_n\}_{n \in \mathbb{N}}$ is bounded.

(ii) By (3.1), we have

$$\|x_{n+1} - x\| \leq \|x_n - x\| \longrightarrow 0 \text{ as } n \rightarrow \infty. \tag{3.5}$$

Hence, by Lemma 2.10, we have that $\{x_n\}_{n \in \mathbb{N}} \longrightarrow x$ as $n \rightarrow \infty$.

(iii) Suppose $x \in D$, since $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone, it follows that

$$\{\|\cdot\|_D(x_{n+1})\}_{n \in \mathbb{N}} = \inf \|x_{n+1} - x\| \leq \inf \|x_n - x\| = \{\|\cdot\|_D(x_n)\}_{n \in \mathbb{N}} \tag{3.6}$$

Hence, by Lemma 2.10, we obtain $\{\|\cdot\|_D(x_n)\}_{n \in \mathbb{N}} \rightarrow 0$ as $n \rightarrow \infty$.

(iv) Since $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone, then by (3.1), we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - x\| + \|x_n - x\| \\ &\leq 2\|x_n - x\|. \end{aligned} \tag{3.7}$$

By taking infimum over $x \in D$ in (3.7), we have the desired result. The proof of Proposition 3.4 is completed. \square

Proposition 3.5. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E, \|\cdot\|)$ and $T : D \rightarrow D$ be a mapping on D with $F(T) := \{x \in D : Tx = x\} \neq \emptyset$. Assume that there exist $\lambda, \beta \in (0, 1)$ satisfying $\lambda + \beta < 1$ and*

$$\|Tx - Ty\| \lesssim \lambda \|y - Ty\| \frac{1 + \|x - Tx\|}{1 + \|x - y\|} + \beta \|x - y\| \tag{3.8}$$

for all $x, y \in D$. For arbitrary chosen $x_0 \in D$, let the sequence $\{x_n\}$ be generated by the M -iteration process (2.6), where $\alpha_n \in (0, 1)$ for each $n \in \mathbb{N}$, then $\{x_n\}$ is Fejér monotone with respect to $F(T)$.

Proof. Suppose $p \in F(T)$, then by (2.6), (3.8) and the facts that $\lambda, \beta \in (0, 1)$ and $\alpha_n \in (0, 1)$ for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| \\ &\lesssim \lambda \|p - Tp\| \frac{1 + \|y_n - Ty_n\|}{1 + \|y_n - p\|} + \beta \|y_n - p\| \\ &= \lambda \cdot 0 \left(\frac{1 + \|y_n - Ty_n\|}{1 + \|y_n - p\|} \right) + \beta \|y_n - p\| \\ &= \beta \|y_n - p\| \\ &= \beta \|Tz_n - p\| \\ &\lesssim \beta \left[\lambda \|p - Tp\| \frac{1 + \|z_n - Tz_n\|}{1 + \|z_n - p\|} + \beta \|z_n - p\| \right] \\ &\lesssim \lambda \cdot 0 \left(\frac{1 + \|z_n - Tz_n\|}{1 + \|z_n - p\|} \right) + \beta \|z_n - p\| \\ &= \beta \|z_n - p\| \\ &= \beta \|(1 - \alpha_n)x_n + \alpha_n Tx_n - p\| \\ &\lesssim (1 - \alpha_n)\|x_n - p\| + \alpha_n \|Tx_n - p\| \\ &\lesssim (1 - \alpha_n)\|x_n - p\| + \alpha_n \left[\lambda \|p - Tp\| \frac{1 + \|x_n - Tx_n\|}{1 + \|x_n - p\|} + \beta \|x_n - p\| \right] \\ &= (1 - \alpha_n)\|x_n - p\| + \alpha_n \left[\lambda \cdot 0 \left(\frac{1 + \|x_n - Tx_n\|}{1 + \|x_n - p\|} \right) + \beta \|x_n - p\| \right] \\ &= (1 - \alpha_n)\|x_n - p\| + \alpha_n \beta \|x_n - p\| \\ &\lesssim (1 - \alpha_n)\|x_n - p\| + \alpha_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.9}$$

This means that $\|x_{n+1} - p\| \lesssim \|x_n - p\|$ as desired. Therefore, $\{x_n\}$ is Fejér monotone. The proof of Proposition 3.5 is completed. \square

Theorem 3.6. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E, \|\cdot\|)$ and $T : D \rightarrow D$ be a mapping on D . Assume that there exist $\lambda, \beta \in (0, 1)$ satisfying $\lambda + \beta < 1$ and*

$$\|Tx - Ty\| \lesssim \lambda \|y - Ty\| \frac{1 + \|x - Tx\|}{1 + \|x - y\|} + \beta \|x - y\| \tag{3.10}$$

for all $x, y \in D$. For arbitrary chosen $x_0 \in D$, let the sequence $\{x_n\}$ be generated by the M -iteration process (2.6), where $\alpha_n \in (0, 1)$ for each $n \in \mathbb{N}$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a unique fixed point p of T .

Proof. We want to show that $x_n \rightarrow p$ as $n \rightarrow \infty$. Now, using relation (2.6) and (3.10), we obtain:

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - Tp\| \\ &\lesssim \lambda \|p - Tp\| \frac{1 + \|y_n - Ty_n\|}{1 + \|y_n - p\|} + \beta \|y_n - p\| \\ &= \lambda \cdot 0 \left(\frac{1 + \|y_n - Ty_n\|}{1 + \|y_n - p\|} \right) + \beta \|y_n - p\| \\ &= \beta \|y_n - p\|. \end{aligned} \tag{3.11}$$

Next, we obtain the following estimate:

$$\begin{aligned} \|y_n - p\| &= \|Tz_n - Tp\| \\ &\lesssim \lambda \|p - Tp\| \frac{1 + \|z_n - Tz_n\|}{1 + \|z_n - p\|} + \beta \|z_n - p\| \\ &= \lambda \cdot 0 \left(\frac{1 + \|z_n - Tz_n\|}{1 + \|z_n - p\|} \right) + \beta \|z_n - p\| \\ &= \beta \|z_n - p\| \\ &= \beta \|(1 - \alpha_n)x_n + \alpha_n Tx_n - p\| \\ &\lesssim \beta(1 - \alpha_n)\|x_n - p\| + \beta \alpha_n \|Tx_n - Tp\| \\ &\lesssim \beta(1 - \alpha_n)\|x_n - p\| + \beta \alpha_n \left[\lambda \|p - Tp\| \frac{1 + \|x_n - Tx_n\|}{1 + \|x_n - p\|} + \beta \|x_n - p\| \right] \\ &= \beta(1 - \alpha_n)\|x_n - p\| + \beta^2 \alpha_n \|x_n - p\| \\ &= \beta(1 - \alpha_n(1 - \beta))\|x_n - p\|. \end{aligned} \tag{3.12}$$

Using (3.12) in (3.11), we have

$$\begin{aligned} \|x_{n+1} - p\| &\lesssim \beta \|y_n - p\| \\ &\lesssim \beta^2(1 - \alpha_n(1 - \beta))\|x_n - p\|. \end{aligned} \tag{3.13}$$

Continuing this process gives the following relations

$$\left\{ \begin{aligned} \|x_{n+1} - p\| &\lesssim \beta^2(1 - \alpha_n(1 - \beta))\|x_n - p\| \\ \|x_n - p\| &\lesssim \beta^2(1 - \alpha_{n-1}(1 - \beta))\|x_{n-1} - p\| \\ \|x_{n-1} - p\| &\lesssim \beta^2(1 - \alpha_{n-2}(1 - \beta))\|x_{n-2} - p\| \\ &\vdots \\ \|x_1 - p\| &\lesssim \beta^2(1 - \alpha_0(1 - \beta))\|x_0 - p\|. \end{aligned} \right. \tag{3.14}$$

From relation (3.14), we obtain the followings:

$$\|x_{n+1} - p\| \lesssim \|x_0 - p\| \beta^{2(n+1)} \prod_{k=0}^n (1 - \alpha_k(1 - \beta)). \tag{3.15}$$

Using the fact that $\beta \in (0, 1)$ and $\alpha_n \in (0, 1)$ for each $n \in \mathbb{N}$, we have

$$(1 - \alpha_n(1 - \beta)) < 1. \tag{3.16}$$

In classical analysis, it is known that $1 - x \leq e^{-x}$ for all $x \in [0, 1]$. Now using these facts together with relation (3.15), we obtain

$$\|x_{n+1} - p\| \lesssim \|x_0 - p\| \beta^{2(n+1)} e^{-(1-\beta) \sum_{k=0}^n \alpha_k}. \tag{3.17}$$

From (3.17), it follows that

$$\|x_{n+1} - p\| \leq \|x_0 - p\| \beta^{2(n+1)} e^{-(1-\beta) \sum_{k=0}^n \alpha_k} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.18)$$

Hence, by Lemma 2.10, it follows that $\{x_n\} \rightarrow p$ as $n \rightarrow \infty$.

Next, we show that the fixed point p of T is unique. Now suppose that p^* is another fixed point of T , then by (3.10), we have

$$\begin{aligned} \|p - p^*\| &\lesssim \lambda \|p^* - Tp^*\| \frac{1 + \|p - Tp\|}{1 + \|p - p^*\|} + \beta \|p - p^*\| \\ &= \lambda \cdot 0 \left(\frac{1}{1 + \|p - p^*\|} \right) + \beta \|p - p^*\| \\ &= \beta \|p - p^*\|. \end{aligned} \quad (3.19)$$

Relation (3.19) implies that

$$\|p - p^*\| \leq \beta \|p - p^*\|. \quad (3.20)$$

Which is a contradiction, since $\beta \in (0, 1)$. Hence, $p = p^*$ as desired. The proof of Theorem 3.6 is completed. \square

Lemma 3.7. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E, \|\cdot\|)$ and $T : D \rightarrow D$ be a mapping on D with $F(T) \neq \emptyset$. Assume that there exist $\lambda, \beta \in (0, 1)$ satisfying $\lambda + \beta < 1$ and*

$$\|Tx - Ty\| \lesssim \lambda \|y - Ty\| \frac{1 + \|x - Tx\|}{1 + \|x - y\|} + \beta \|x - y\| \quad (3.21)$$

for all $x, y \in D$. For arbitrary chosen $x_0 \in D$, let the sequence $\{x_n\}$ be generated by the M -iteration process (2.6), then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for any $p \in F(T)$.

Proof. From relation (3.9), it follows that

$$\|x_{n+1} - p\| \leq \|x_n - p\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.22)$$

Hence, by Lemma 2.10 we see that $\{\|x_n - p\|\}$ is bounded and non-increasing for each $p \in F(T)$. Therefore, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists as desired. The proof of Lemma 3.7 is completed. \square

Lemma 3.8. *Let D be a nonempty subset of a complex valued Banach space $(E, \|\cdot\|)$. Let the sequence $\{x_n\} \subseteq E$ be Fejér monotone with respect to D . If at least one cluster point x^* of $\{x_n\}$ belongs to D , then $x_n \rightarrow x^*$.*

Proof. Since every Fejér monotone sequence is bounded, it follows that $\{x_n\}$ has a weak cluster point x^* . Let a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ converge to $x^* \in D$. We now prove that $\{x_n\}$ converges to x^* . Suppose $x' \in E$, $x' \neq x^*$ is another cluster point of $\{x_n\}$ such that a subsequence $\{x_{m_n}\}$ converges to x' . Suppose $\varepsilon := \frac{1}{2} \|x' - x^*\| > 0$, let $n_0 \in \mathbb{N}$ be such that $\|x_{m_n} - x'\| < \varepsilon$ and $\|x_{k_n} - x^*\| < \varepsilon$, for all $n \geq n_0$ and let $m_n > k_{n_0}$. Using the triangle inequality and Fejér monotonicity of $\{x_n\}$ with respect to D , we have

$$2\varepsilon = \|x' - x^*\| \lesssim \|x' - x_{m_n}\| + \|x_{m_n} - x^*\| < 2\varepsilon. \quad (3.23)$$

This means that

$$2\varepsilon = \|x' - x^*\| \leq \|x' - x_{m_n}\| + \|x_{m_n} - x^*\| < 2\varepsilon, \quad (3.24)$$

which is a contradiction. Therefore, by Lemma 2.10, it follows that $x_n \rightarrow x^*$. The proof of Lemma 3.8 is completed. \square

4. CONE METRIC SPACES WITH BANACH ALGEBRAS

Let \mathcal{A} denote a real Banach algebra. This means that \mathcal{A} is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for each $x, y, z \in \mathcal{A}$, $\alpha \in \mathbb{R}$):

- (i) $(xy)z = x(yz)$;
- (ii) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
- (iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
- (iv) $\|xy\| \leq \|x\|\|y\|$.

In this paper, we assume that \mathcal{A} has a unit; i.e. a multiplicative identity e such that $ex = xe = x$ for each $x \in \mathcal{A}$. The inverse of x is denoted by x^{-1} (see, e.g. Rudin [37]).

In 2012, Öztürk and Başarir [34] generalized the concept of cone metric spaces introduced by Huang and Zhang [23] by replacing a Banach space with a Banach algebra \mathcal{A} in cone metric spaces. They called this new concept BA-cone metric spaces. Abbas *et al.* [2] proved that complex valued metric spaces introduced in [11] is a BA-cone metric space, that is a cone metric space over a solid cone in commutative division Banach algebra \mathcal{A} (see, [2], [34]).

Perhaps unaware of the work of Öztürk and Başarir [34], in 2013 Liu and Xu [29] introduced the concept of cone metric spaces with Banach algebras, by replacing Banach spaces with Banach algebras as the underlying space of cone metric spaces. They proved that fixed point theorems in the setting of cone metric spaces with Banach algebras are more useful than the standard results in cone metric spaces and that results in cone metric spaces with Banach algebras cannot be reduced to corresponding results in cone metric spaces. .

Example 4.1 ([29]). Let $\mathcal{A} = M_n(\mathbb{R}) = \{a = (a_{ij})_{n \times n} | a_{ij} \in \mathbb{R} \text{ for all } 1 \leq i, j \leq n\}$ be the algebra of all n -square real matrices, and define the norm

$$\|a\| = \sum_{1 \leq i, j \leq n} |a_{ij}|.$$

Then \mathcal{A} is a real Banach algebra with the unit e , the identity matrix.

Let $\mathcal{P} = \{a \in \mathcal{A} | a_{ij} \geq 0 \text{ for all } 1 \leq i, j \leq n\}$. Then $\mathcal{P} \subset \mathcal{A}$ is a normal cone with normal constant $M = 1$.

Let $X = M_n(\mathbb{R})$, and define the metric $d : X \times X \rightarrow \mathcal{A}$ by

$$d(x, y) = d((x_{ij})_{n \times n}, (y_{ij})_{n \times n}) = (|x_{ij} - y_{ij}|)_{n \times n} \in \mathcal{A}.$$

Then (X, d) is a cone metric space with a Banach algebra \mathcal{A} .

Example 4.2 ([29]). Let \mathcal{A} be the Banach space $\mathcal{C}(\mathcal{K})$ of all continuous real-valued functions on a compact Hausdorff topological space \mathcal{K} , with multiplication defined pointwise. Then \mathcal{A} is a Banach algebra, and the constant function $f(t) = 1$ is the unit of \mathcal{A} .

Let $\mathcal{P} = \{f \in \mathcal{A} \mid f(t) \geq 0 \text{ for all } t \in \mathcal{K}\}$. Then $\mathcal{P} \subset \mathcal{A}$ is a normal cone with a normal constant $M = 1$.

Let $X = \mathcal{C}(\mathcal{K})$ with the metric mapping $d : X \times X \rightarrow \mathcal{A}$ defined by

$$d(f, g) = |f(t) - g(t)|, \quad \text{where } t \in \mathcal{K}.$$

Then (X, d) is a cone metric space with a Banach algebra \mathcal{A} .

Example 4.3 ([29]). Let $\mathcal{A} = \ell^1 = \{a = (a_n)_{n \geq 0} \mid \sum_{n=0}^{\infty} |a_n| < \infty\}$ with convolution as multiplication:

$$ab = (a_n)_{n \geq 0} (b_n)_{n \geq 0} = \left(\sum_{i+j=n} a_i b_j \right)_{n \geq 0}.$$

Thus \mathcal{A} is a Banach algebra. The unit e is $(1, 0, 0, \dots)$.

Let $\mathcal{P} = \{a = (a_n)_{n \geq 0} \in \mathcal{A} \mid a_n \geq 0 \text{ for all } n \geq 0\}$, which is a normal cone in \mathcal{A} . And let $X = \ell^1$ with the metric $d : X \times X \rightarrow \mathcal{A}$ defined by

$$d(x, y) = d((x_n)_{n \geq 0}, (y_n)_{n \geq 0}) = (|x_n - y_n|)_{n \geq 0}.$$

Then (X, d) is a cone metric space with \mathcal{A} .

Motivated by the results above, we now prove that results in complex valued Banach spaces (see, e.g. Okeke [32]) are true in the context of cone metric spaces with Banach algebras. Moreover, we show that our results cannot be deduced in cone metric spaces.

Theorem 4.4. *Let D be a nonempty closed convex subset of a complete cone metric space with Banach algebras $(\mathcal{A}, \|\cdot\|)$ and $T : D \rightarrow D$ be a contraction mapping satisfying the following contractive condition*

$$\|Tx - Ty\| \lesssim \frac{\varphi(\|x - Tx\|) + a\|x - y\|}{e + M\|x - Tx\|}, \quad \forall x, y \in D, \quad a \in [0, 1), \quad M \geq 0, \quad (4.1)$$

where $\varphi : \mathbb{C}_+ \rightarrow \mathbb{C}_+$ is a monotone increasing function such that $\varphi(0) = 0$. Let $\{m_n\}$ be an iterative sequence generated by the Picard-Mann hybrid iterative process (2.7) with real sequence $\{\alpha_n\}_{n=0}^{\infty}$ in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{m_n\}$ converges strongly to a unique fixed point p of T .

Proof. We now show that $x_n \rightarrow p$ as $n \rightarrow \infty$. Using (2.7) and (4.1), we obtain:

$$\begin{aligned} \|m_{n+1} - p\| &= \|Tz_n - p\| \\ &\lesssim \frac{\varphi(\|p - Tp\|) + a\|z_n - p\|}{e + M\|p - Tp\|} \\ &= \frac{\varphi(\|0\|) + a\|z_n - p\|}{e + M\|0\|} \\ &= a\|(1 - \alpha_n)m_n + \alpha_n Tm_n - p\| \\ &\lesssim a(1 - \alpha_n)\|m_n - p\| + a\alpha_n\|Tm_n - p\| \\ &\lesssim a(1 - \alpha_n)\|m_n - p\| + a\alpha_n \left[\frac{\varphi(\|p - Tp\|) + a\|m_n - p\|}{e + M\|p - Tp\|} \right] \\ &= a(1 - \alpha_n)\|m_n - p\| + a\alpha_n \left[\frac{\varphi(\|0\|) + a\|m_n - p\|}{e + M\|0\|} \right] \\ &= a(1 - \alpha_n)\|m_n - p\| + a^2\alpha_n\|m_n - p\| \\ &= a(1 - \alpha_n(1 - a))\|m_n - p\|. \end{aligned} \tag{4.2}$$

Using the fact that $(1 - \alpha_n(1 - a)) < 1$ and $a \in [0, 1)$, we obtain the following inequalities from (4.2).

$$\begin{cases} \|m_{n+1} - p\| \lesssim a(1 - \alpha_n(1 - a))\|m_n - p\| \\ \|m_n - p\| \lesssim a(1 - \alpha_{n-1}(1 - a))\|m_{n-1} - p\| \\ \|m_{n-1} - p\| \lesssim a(1 - \alpha_{n-2}(1 - a))\|m_{n-2} - p\| \\ \vdots \\ \|m_2 - p\| \lesssim a(1 - \alpha_1(1 - a))\|m_1 - p\|. \end{cases} \tag{4.3}$$

From relation (4.3), we derive

$$\|m_{n+1} - p\| \lesssim \|m_1 - p\| a^{n+1} \prod_{k=1}^n (1 - \alpha_k(1 - a)), \tag{4.4}$$

where $(1 - \alpha_k(1 - a)) \in (0, 1)$, since $a \in [0, 1)$ and $\alpha_k \in [0, 1]$ for all $k \in \mathbb{N}$. It is well-known in classical analysis that $1 - x \leq e^{-x}$ for all $x \in [0, 1]$. Using these facts together with relation (4.4), we have

$$\|m_{n+1} - p\| \lesssim \frac{\|m_1 - p\| a^{n+1}}{e^{(1-a)\sum_{k=1}^n \alpha_k}}. \tag{4.5}$$

Therefore,

$$\lim_{n \rightarrow \infty} \|m_{n+1} - p\| \leq \left\{ \frac{\|m_1 - p\| a^{n+1}}{e^{(1-a)\sum_{k=1}^n \alpha_k}} \right\} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{4.6}$$

Therefore by Lemma 2.1 we have that $\lim_{n \rightarrow \infty} \|m_n - p\| = 0$. This means that $m_n \rightarrow p$ as $n \rightarrow \infty$ as desired.

Next, we show that T has a unique fixed point $p \in F(T) := \{p \in D : Tp = p\}$. Assume that p^* is another fixed point of T , then we have

$$\begin{aligned} \|p - p^*\| &= \|Tp - Tp^*\| \\ &\lesssim \frac{\varphi(\|p - Tp\|) + a\|p - p^*\|}{e + M\|p - Tp\|} \\ &= \frac{\varphi(\|0\|) + a\|p - p^*\|}{e + M\|0\|} \\ &= a\|p - p^*\|. \end{aligned} \tag{4.7}$$

This implies that

$$\|p - p^*\| \leq \|p - p^*\|. \tag{4.8}$$

Hence, by Lemma 2.10 we have that $p = p^*$. The proof of Theorem 4.4 is completed. \square

Proposition 4.5. *Let D be a nonempty closed convex subset of a complete cone metric space with Banach algebras $(\mathcal{A}, \|\cdot\|)$ and let $T : D \rightarrow D$ be a mapping defined as follows*

$$\|Tx - Ty\| \lesssim \frac{\varphi(\|x - Tx\|) + a\|x - y\|}{e + M\|x - Tx\|}, \quad \forall x, y \in D, \quad a \in [0, 1), \quad M \geq 0, \tag{4.9}$$

where $\varphi : \mathbb{C}_+ \rightarrow \mathbb{C}_+$ is a monotone increasing function such that $\varphi(0) = 0$. Suppose that each of the iterative processes (2.7) and (2.8) converges to the same fixed point p of T where $\{\alpha_n\}_{n=0}^\infty$ and λ are such that $0 < \alpha \leq \lambda, \alpha_n < 1$

for all $n \in \mathbb{N}$ and for some α . Then the sequence $\{x_n\}$ generated by the Picard-Krasnoselskii hybrid iterative process (2.8) have the same rate of convergence as the sequence $\{m_n\}$ generated by the Picard-Mann hybrid iterative process (2.7).

Proof. The proof of Proposition 4.5 follows similar lines as in the proofs of ([32], Proposition 2.2) and Theorem 4.4. \square

Remark 4.6. Observe that the results in Theorem 4.4 and Proposition 4.5 was proved for mappings satisfying rational inequality, which is meaningless in cone metric spaces. This means that these results cannot be reduced to some corresponding results in cone metric spaces.

5. APPLICATIONS TO A NONLINEAR INTEGRAL EQUATION

It is our purpose in this section to show that the M- iterative process (2.6) converges strongly to the solution of a mixed type Volterra-Fredholm functional nonlinear integral equation in complex valued Banach spaces. Our results generalize and extend some known results to complex valued Banach spaces, including the results of Crăciun and Şerban [18], Gürsoy [19] among others.

In 2011, Crăciun and Şerban [18] considered the following mixed type Volterra-Fredholm functional nonlinear integral equation:

$$x(t) = F \left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t, s, x(s)) ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t, s, x(s)) ds \right), \tag{5.1}$$

where $[a_1; b_1] \times \cdots \times [a_m; b_m]$ be an interval in \mathbb{R}^m , $K, H : [a_1; b_1] \times \cdots \times [a_m; b_m] \times [a_1; b_1] \times \cdots \times [a_m; b_m] \times \mathbb{R} \rightarrow \mathbb{R}$ continuous functions and $F : [a_1; b_1] \times \cdots \times [a_m; b_m] \times \mathbb{R}^3 \rightarrow \mathbb{R}$. They established the following results.

Theorem 5.1 ([18]). *We assume that:*

- (i) $K, H \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R})$;
- (ii) $F \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R}^3)$;
- (iii) *there exist α, β, γ nonnegative constants such that:*

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \leq \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$;

- (iv) *there exist L_K and L_H nonnegative constants such that:*

$$|K(t, s, u) - K(t, s, v)| \leq L_K|u - v|,$$

$$|H(t, s, u) - H(t, s, v)| \leq L_H|u - v|,$$

for all $t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, $u, v \in \mathbb{R}$;

- (v) $\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \cdots (b_m - a_m) < 1$.

Then, the equation (5.1) has a unique solution $x^* \in C([a_1, b_1] \times \cdots \times [a_m, b_m])$.

Remark 5.2 ([18]). Let $(\mathbb{B}, |\cdot|)$ be a Banach space. Then Theorem 5.1 remains also true if we consider the mixed type Volterra-Fredholm functional nonlinear integral equation (5.1) in the Banach space \mathbb{B} instead of Banach space \mathbb{R} .

Let D be a nonempty subset of a complex valued Banach space $(E, \|\cdot\|)$ and let $\{m_n\}$ be an iterative process defined by the M -iteration associated with F , which is generated as follows:

$$\begin{cases} m_0 \in D \\ z_n = (1 - \alpha_n)m_n + \alpha_n Tm_n \\ g_n = Tz_n \\ m_{n+1} = Tg_n, \end{cases} \tag{5.2}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$. Consequently, we now obtain the following analogue of Theorem 5.1 in complex valued Banach spaces.

Theorem 5.3. *We consider the complex valued Banach space $\mathcal{B}_{\mathbb{C}} = C([a_1, b_1] \times \dots \times [a_m, b_m], \|\cdot\|_{\mathbb{C}})$, where $\|\cdot\|_{\mathbb{C}}$ is the Chebyshev's norm defined by $\|x - y\|_{\mathbb{C}} = |x - y|i, \forall x, y \in \mathcal{B}_{\mathbb{C}}$. We assume that:*

- (i) $K, H \in C([a_1, b_1] \times \dots \times [a_m, b_m] \times [a_1, b_1] \times \dots \times [a_m, b_m] \times \mathbb{R})$;
- (ii) $F \in C([a_1, b_1] \times \dots \times [a_m, b_m] \times \mathbb{R}^3)$;
- (iii) there exist α, β, γ nonnegative constants such that:

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \leq \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a_1, b_1] \times \dots \times [a_m, b_m], u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$;

- (iv) there exist L_K and L_H nonnegative constants such that:

$$|K(t, s, u) - K(t, s, v)| \leq L_K|u - v|,$$

$$|H(t, s, u) - H(t, s, v)| \leq L_H|u - v|,$$

for all $t, s \in [a_1, b_1] \times \dots \times [a_m, b_m], u, v \in \mathbb{R}$;

- (v) $\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \dots (b_m - a_m) < 1$.

Then, the mixed type Volterra-Fredholm functional integral equation (5.1) has a unique solution $p \in C([a_1; b_1] \times \dots \times [a_m; b_m])$.

Proof. Since our analysis is in the complex valued Banach space $\mathcal{B}_{\mathbb{C}} = C([a_1, b_1] \times \dots \times [a_m, b_m], \|\cdot\|_{\mathbb{C}})$, where $\|\cdot\|_{\mathbb{C}}$ is the Chebyshev's norm defined by $\|x - y\|_{\mathbb{C}} = |x - y|i, \forall x, y \in \mathcal{B}_{\mathbb{C}}$, and the operator

$$A : \mathcal{B}_{\mathbb{C}} \rightarrow \mathcal{B}_{\mathbb{C}},$$

defined by

$$A(x)(t) = F \left(t, x(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, x(s)) ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, x(s)) ds \right) \tag{5.3}$$

Using conditions (iii) and (iv), we have

$$\begin{aligned}
 |A(u)(t) - A(v)(t)| &\lesssim \alpha|u(t) - v(t)| + \beta \left| \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} (K(t, s, u(s)) - K(t, s, v(s))) ds \right| + \\
 &\quad \gamma \left| \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} (H(t, s, u(s)) - H(t, s, v(s))) ds \right| \\
 &\lesssim \alpha|u(t) - v(t)| + \beta \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} L_K |u(s) - v(s)| ds + \\
 &\quad \gamma \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} L_H |u(s) - v(s)| ds \\
 &\lesssim [\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \cdots (b_m - a_m)] \|u - v\|_{\mathbb{C}} \\
 &= [\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \cdots (b_m - a_m)] |u - v|_i.
 \end{aligned}
 \tag{5.4}$$

It follows from relation (5.4) that

$$\begin{aligned}
 \|A(u)(t) - A(v)(t)\|_{\mathbb{C}} &\leq [\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \cdots (b_m - a_m)] |u - v|_i \\
 &= [\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \cdots (b_m - a_m)] |u - v|.
 \end{aligned}
 \tag{5.5}$$

Using Lemma 2.10 in (5.5) together with condition (v), we see that operator A is a contraction, so that by the Banach contraction mapping principle, we have that operator A has a unique fixed point $F(A) = \{p\}$. This means that our equation (5.1) has a unique solution $p \in C([a_1; b_1] \times \cdots \times [a_m; b_m])$. The proof of Theorem 5.3 is completed. \square

Theorem 5.4. *Suppose that all the conditions (i) - (v) in Theorem 4.2 are satisfied. Let the sequence $\{m_n\}$ be generated by the M-iteration process (5.2), where $\{\alpha_n\} \subset (0, 1)$ is a real sequence satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the mixed type Volterra-Fredholm functional integral equation (5.1) has a unique solution, say $p \in C([a_1; b_1] \times \cdots \times [a_m; b_m])$ and the sequence $\{m_n\}$ converges to p .*

Proof. We consider the complex valued Banach space $\mathcal{B}_{\mathbb{C}} = C([a_1, b_1] \times \cdots \times [a_m, b_m], \|\cdot\|_{\mathbb{C}})$, where $\|\cdot\|_{\mathbb{C}}$ is the Chebyshev's norm defined by $\|x - y\|_{\mathbb{C}} = |x - y|_i, \forall x, y \in \mathcal{B}_{\mathbb{C}}$. Let $\{m_n\}$ be the sequence generated by the M-iteration process (5.2) for the operator $A : \mathcal{B}_{\mathbb{C}} \rightarrow \mathcal{B}_{\mathbb{C}}$ defined by

$$A(x)(t) = F \left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t, s, x(s)) ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t, s, x(s)) ds \right).
 \tag{5.6}$$

We want to show that $m_n \rightarrow p$ as $n \rightarrow \infty$. Using (5.2), (5.1) and assumptions (i) - (v) we obtain

$$\begin{aligned}
 \|m_{n+1} - p\|_{\mathbb{C}} &= |A(g_n)(t) - A(p)(t)| \\
 &= \left| F \left(t, g_n(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, g_n(s)) ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, g_n(s)) ds \right) - \right. \\
 &\quad \left. F \left(t, p(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, p(s)) ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, p(s)) ds \right) \right| \\
 &\lesssim \alpha |g_n(t) - p(t)| + \beta \left| \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, g_n(s)) ds - \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, p(s)) ds \right| + \\
 &\quad \gamma \left| \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, g_n(s)) ds - \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, p(s)) ds \right| \\
 &\lesssim \alpha |g_n(t) - p(t)| + \beta \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} |K(t, s, g_n(s)) - K(t, s, p(s))| ds + \\
 &\quad \gamma \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} |H(t, s, g_n(s)) - H(t, s, p(s))| ds \\
 &\lesssim \alpha |g_n(t) - p(t)| + \beta \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} L_K |g_n(s) - p(s)| ds + \\
 &\quad \gamma \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} L_H |g_n(s) - p(s)| ds \\
 &\lesssim [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)] \|g_n - p\|_{\mathbb{C}}.
 \end{aligned} \tag{5.7}$$

Next, we have the following estimate

$$\begin{aligned}
 \|g_n - p\|_{\mathbb{C}} &= |A(z_n)(t) - A(p)(t)| \\
 &= \left| F \left(t, z_n(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, z_n(s)) ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, z_n(s)) ds \right) - \right. \\
 &\quad \left. F \left(t, p(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, p(s)) ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, p(s)) ds \right) \right| \\
 &\lesssim \alpha |z_n(t) - p(t)| + \beta \left| \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, z_n(s)) ds - \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, p(s)) ds \right| + \\
 &\quad \gamma \left| \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, z_n(s)) ds - \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, p(s)) ds \right| \\
 &\lesssim \alpha |z_n(t) - p(t)| + \beta \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} |K(t, s, z_n(s)) - K(t, s, p(s))| ds + \\
 &\quad \gamma \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} |H(t, s, z_n(s)) - H(t, s, p(s))| ds \\
 &\lesssim \alpha |z_n(t) - p(t)| + \beta \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} L_K |z_n(s) - p(s)| ds + \\
 &\quad \gamma \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} L_H |z_n(s) - p(s)| ds \\
 &\lesssim [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)] \|z_n - p\|_{\mathbb{C}}.
 \end{aligned} \tag{5.8}$$

$$\begin{aligned}
 \|z_n - p\|_{\mathbb{C}} &\lesssim (1 - \alpha_n) |m_n(t) - p(t)| + \alpha_n |A(m_n)(t) - A(p)(t)| \\
 &= (1 - \alpha_n) |m_n(t) - p(t)| + \\
 &\quad \alpha_n \left| F \left(t, m_n(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, m_n(s)) ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, m_n(s)) ds \right) - \right. \\
 &\quad \left. F \left(t, p(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, p(s)) ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, p(s)) ds \right) \right| \\
 &\lesssim (1 - \alpha_n) |m_n(t) - p(t)| + \alpha_n \alpha |m_n(t) - p(t)| + \alpha_n \beta \int_{a_1}^{t_1} \dots \\
 &\quad \int_{a_m}^{t_m} L_K |m_n(s) - p(s)| ds \\
 &\quad + \alpha_n \gamma \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} L_H |m_n(s) - p(s)| ds \\
 &\lesssim \{1 - \alpha_n (1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)])\} \|m_n - p\|_{\mathbb{C}}.
 \end{aligned} \tag{5.9}$$

Using (5.8) and (5.9) in (5.7), together with the fact that $[\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)] < 1$ in assumption (v) we obtain

$$\begin{aligned} \|m_{n+1} - p\|_{\mathbb{C}} &\lesssim [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]^2 \times \\ &\quad \{1 - \alpha_n (1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)])\} \|m_n - p\|_{\mathbb{C}} \\ &\lesssim \{1 - \alpha_n (1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)])\} \|m_n - p\|_{\mathbb{C}}. \end{aligned} \tag{5.10}$$

Hence, by induction (5.10) becomes

$$\|m_{n+1} - p\|_{\mathbb{C}} \lesssim \prod_{k=0}^n \left\{ 1 - \alpha_k \left(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)] \right) \right\} \|m_0 - p\|_{\mathbb{C}}. \tag{5.11}$$

From the fact that $\alpha_k \in (0, 1)$ for each $k \in \mathbb{N}$, together with assumption (v), we have

$$\left\{ 1 - \alpha_k \left(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)] \right) \right\} < 1. \tag{5.12}$$

It is known in analysis that $e^x \geq 1 - x$ for all $x \in [0, 1]$. Therefore (5.11) becomes

$$\begin{aligned} \|m_{n+1} - p\|_{\mathbb{C}} &\lesssim \|m_0 - p\|_{\mathbb{C}} e^{-(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]) \sum_{k=0}^n \alpha_k} \\ &= |m_0 - p| e^{-(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]) \sum_{k=0}^n \alpha_k}. \end{aligned} \tag{5.13}$$

This means that

$$\begin{aligned} \|\|m_{n+1} - p\|_{\mathbb{C}}\| &\lesssim \|\|m_0 - p\|_{\mathbb{C}} e^{-(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]) \sum_{k=0}^n \alpha_k}\| \\ &= |m_0 - p| e^{-(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]) \sum_{k=0}^n \alpha_k} \longrightarrow 0 \end{aligned} \tag{5.14}$$

as $k \rightarrow \infty$. Therefore, by Lemma 2.10, we have $x_n \rightarrow p$ as $n \rightarrow \infty$ as desired. The proof of Theorem 5.4 is completed. \square

We now turn our attention to proving the data dependence of the solution for the integral equation (5.1) via the M-iterative process (5.2).

Suppose $\mathcal{B}_{\mathbb{C}}$ is as in Theorem 5.3 and the operators $T, \tilde{T} : \mathcal{B}_{\mathbb{C}} \rightarrow \mathcal{B}_{\mathbb{C}}$ are defined by

$$T(x)(t) = F \left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t, s, x(s)) ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t, s, x(s)) ds \right) \tag{5.15}$$

$$\tilde{T}(x)(t) = F \left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} \tilde{K}(t, s, x(s)) ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \tilde{H}(t, s, x(s)) ds \right), \tag{5.16}$$

where $K, \tilde{K}, H, \tilde{H} \in C([a_1; b_1] \times \cdots \times [a_m; b_m] \times [a_1; b_1] \times \cdots \times [a_m; b_m] \times \mathbb{R})$.

Theorem 5.5. *Let F, K and H be defined as in Theorem 5.3 and let $\{m_n\}$ be the iterative sequence generated by the M-iteration process (5.2) associated with T . Let $\{\tilde{m}_n\}$ be an iterative sequence generated by*

$$\begin{cases} \tilde{m}_0 \in D, \\ \tilde{z}_n = (1 - \alpha_n)\tilde{m}_n + \alpha_n\tilde{T}\tilde{m}_n, \\ \tilde{g}_n = \tilde{T}\tilde{z}_n \\ \tilde{m}_{n+1} = \tilde{T}\tilde{g}_n, \end{cases} \quad (5.17)$$

where $\mathcal{B}_{\mathbb{C}}$ is as defined in Theorem 5.2 and $\{\alpha_n\}$ is a real sequence in $(0, 1)$ satisfying

(a) $\frac{1}{2} \leq \alpha_n$ for each $n \in \mathbb{N}$, and

(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Furthermore, suppose

(c) there exist nonnegative constants λ_1 and λ_2 such that $|K(t, s, u) - \tilde{K}(t, s, u)| \leq \lambda_1$ and $|H(t, s, u) - \tilde{H}(t, s, u)| \leq \lambda_2$, for all $u \in \mathbb{R}$ and for all $t, s \in [a_1; b_1] \times \dots \times [a_m; b_m]$.

If p and \tilde{p} are solutions of corresponding nonlinear equations (5.15) and (5.16) respectively, then we have

$$\|p - \tilde{p}\| \leq \frac{4(\beta\lambda_1 + \gamma\lambda_2) \prod_{i=1}^m (b_i - a_i)}{1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]}. \quad (5.18)$$

Proof. We consider the complex valued Banach space $\mathcal{B}_{\mathbb{C}} = C([a_1, b_1] \times \dots \times [a_m, b_m], \|\cdot\|_{\mathbb{C}})$, where $\|\cdot\|_{\mathbb{C}}$ is the Chebyshev's norm defined by $\|x - y\|_{\mathbb{C}} = |x - y|_i, \forall x, y \in \mathcal{B}_{\mathbb{C}}$.

Now using (5.1), (5.2), (5.15), (5.16), (5.17) and assumptions (i) - (v) together with conditions (a) - (c), we have

$$\begin{aligned} \|m_{n+1} - \tilde{m}_{n+1}\|_{\mathbb{C}} &= \|Tg_n - \tilde{T}\tilde{g}_n\|_{\mathbb{C}} \\ &= \left| F\left(t, g_n(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, g_n(s)) ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, g_n(s)) ds\right) - \right. \\ &\quad \left. F\left(t, \tilde{g}_n(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} \tilde{K}(t, s, \tilde{g}_n(s)) ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} \tilde{H}(t, s, \tilde{g}_n(s)) ds\right) \right| \\ &\lesssim \alpha |g_n(t) - \tilde{g}_n(t)| + \beta \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} |K(t, s, g_n(s)) - \tilde{K}(t, s, \tilde{g}_n(s))| ds + \\ &\quad \gamma \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} |H(t, s, g_n(s)) - \tilde{H}(t, s, \tilde{g}_n(s))| ds \\ &\lesssim \alpha |g_n(t) - \tilde{g}_n(t)| + \\ &\quad \beta \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} (|K(t, s, g_n(s)) - K(t, s, \tilde{g}_n(s))| + \\ &\quad |K(t, s, \tilde{g}_n(s)) - \tilde{K}(t, s, \tilde{g}_n(s))|) ds + \\ &\quad \gamma \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} (|H(t, s, g_n(s)) - H(t, s, \tilde{g}_n(s))| + \\ &\quad |H(t, s, \tilde{g}_n(s)) - \tilde{H}(t, s, \tilde{g}_n(s))|) ds \\ &\lesssim \alpha |g_n(t) - \tilde{g}_n(t)| + \beta \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} (L_K |g_n(s) - \tilde{g}_n(s)| + \lambda_1) ds + \\ &\quad \gamma \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} (L_H |g_n(s) - \tilde{g}_n(s)| + \lambda_2) ds \\ &\lesssim \alpha \|g_n - \tilde{g}_n\|_{\mathbb{C}} + \beta (L_K \|g_n - \tilde{g}_n\|_{\mathbb{C}} + \lambda_1) \prod_{i=1}^m (b_i - a_i) + \\ &\quad \gamma (L_H \|g_n - \tilde{g}_n\|_{\mathbb{C}} + \lambda_2) \prod_{i=1}^m (b_i - a_i) \\ &\lesssim [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)] \|g_n - \tilde{g}_n\|_{\mathbb{C}} + \\ &\quad (\beta\lambda_1 + \gamma\lambda_2) \prod_{i=1}^m (b_i - a_i). \end{aligned} \quad (5.19)$$

$$\begin{aligned}
 \|g_n - \tilde{g}_n\|_{\mathbb{C}} &= \|Tz_n - \tilde{T}\tilde{z}_n\|_{\mathbb{C}} \\
 &= \left| F\left(t, z_n(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K(t, s, z_n(s)) ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H(t, s, z_n(s)) ds\right) - \right. \\
 &\quad \left. F\left(t, \tilde{z}_n(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} \tilde{K}(t, s, \tilde{z}_n(s)) ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} \tilde{H}(t, s, \tilde{z}_n(s)) ds\right) \right| \\
 &\lesssim \alpha |z_n(t) - \tilde{z}_n(t)| + \\
 &\quad \beta \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} (|K(t, s, z_n(s)) - K(t, s, \tilde{z}_n(s))| + \\
 &\quad |K(t, s, \tilde{z}_n(s)) - \tilde{K}(t, s, \tilde{z}_n(s))|) ds + \\
 &\quad \gamma \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} (|H(t, s, z_n(s)) - H(t, s, \tilde{z}_n(s))| + \\
 &\quad |H(t, s, \tilde{z}_n(s)) - \tilde{H}(t, s, \tilde{z}_n(s))|) ds \\
 &\lesssim \alpha |z_n(t) - \tilde{z}_n(t)| + \beta \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} (L_K |z_n(s) - \tilde{z}_n(s)| + \lambda_1) ds + \\
 &\quad \gamma \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} (L_H |z_n(s) - \tilde{z}_n(s)| + \lambda_2) ds \\
 &\lesssim \alpha \|z_n - \tilde{z}_n\|_{\mathbb{C}} + \beta (L_K \|z_n - \tilde{z}_n\|_{\mathbb{C}} + \lambda_1) \prod_{i=1}^m (b_i - a_i) + \\
 &\quad \gamma (L_H \|z_n - \tilde{z}_n\|_{\mathbb{C}} + \lambda_2) \prod_{i=1}^m (b_i - a_i) \\
 &\lesssim [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)] \|z_n - \tilde{z}_n\|_{\mathbb{C}} + \\
 &\quad (\beta \lambda_1 + \gamma \lambda_2) \prod_{i=1}^m (b_i - a_i).
 \end{aligned} \tag{5.20}$$

$$\begin{aligned}
 \|z_n - \tilde{z}_n\|_{\mathbb{C}} &\lesssim (1 - \alpha_n) |m_n(t) - \tilde{m}_n(t)| + \alpha_n |T(m_n)(t) - \tilde{T}(\tilde{m}_n)(t)| \\
 &\lesssim (1 - \alpha_n) |m_n(t) - \tilde{m}_n(t)| + \alpha_n \{ \alpha |m_n(t) - \tilde{m}_n(t)| + \\
 &\quad \beta \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} (L_K |m_n(s) - \tilde{m}_n(s)| + \lambda_1) ds + \\
 &\quad \gamma \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} (L_H |m_n(s) - \tilde{m}_n(s)| + \lambda_2) ds \} \\
 &\lesssim \{ 1 - \alpha_n (1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]) \} \|m_n - \tilde{m}_n\|_{\mathbb{C}} + \\
 &\quad \alpha_n (\beta \lambda_1 + \gamma \lambda_2) \prod_{i=1}^m (b_i - a_i).
 \end{aligned} \tag{5.21}$$

Using (5.21) in (5.20), together with assumption (v), we have:

$$\|g_n - \tilde{g}_n\|_{\mathbb{C}} \lesssim \{ 1 - \alpha_n (1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]) \} \|m_n - \tilde{m}_n\|_{\mathbb{C}} + \\
 \alpha_n (\beta \lambda_1 + \gamma \lambda_2) \prod_{i=1}^m (b_i - a_i) + (\beta \lambda_1 + \gamma \lambda_2) \prod_{i=1}^m (b_i - a_i). \tag{5.22}$$

Using (5.22) in (5.19) together with assumption (v), we have

$$\begin{aligned}
 \|m_{n+1} - \tilde{m}_{n+1}\|_{\mathbb{C}} &\lesssim \{ 1 - \alpha_n (1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]) \} \times \\
 &\quad \|m_n - \tilde{m}_n\|_{\mathbb{C}} + \\
 &\quad \alpha_n (\beta \lambda_1 + \gamma \lambda_2) \prod_{i=1}^m (b_i - a_i) + \\
 &\quad (\beta \lambda_1 + \gamma \lambda_2) \prod_{i=1}^m (b_i - a_i) + \\
 &\quad (\beta \lambda_1 + \gamma \lambda_2) \prod_{i=1}^m (b_i - a_i) \\
 &\lesssim \{ 1 - \alpha_n (1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]) \} \times \\
 &\quad \|m_n - \tilde{m}_n\|_{\mathbb{C}} + \\
 &\quad \alpha_n (1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]) \times \\
 &\quad \left(\frac{4(\beta \lambda_1 + \gamma \lambda_2) \prod_{i=1}^m (b_i - a_i)}{(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)])} \right).
 \end{aligned} \tag{5.23}$$

From relation (5.23), we choose the sequences β_n , μ_n and γ_n as follows:

$$\begin{cases} \beta_n = \|m_n - \tilde{m}_n\|_{\mathbb{C}}, \\ \mu_n = \alpha_n(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)]) \in (0, 1), \\ \gamma_n = \frac{4(\beta\lambda_1 + \gamma\lambda_2) \prod_{i=1}^m (b_i - a_i)}{(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)])}. \end{cases} \quad (5.24)$$

Therefore, from relation (5.23), we see that all the conditions of Lemma 2.3 are satisfied. Hence, we have

$$\|p - \tilde{p}\|_{\mathbb{C}} \lesssim \frac{4(\beta\lambda_1 + \gamma\lambda_2) \prod_{i=1}^m (b_i - a_i)}{(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)])}. \quad (5.25)$$

This implies that

$$\|p - \tilde{p}\|_{\mathbb{C}} \leq \frac{4(\beta\lambda_1 + \gamma\lambda_2) \prod_{i=1}^m (b_i - a_i)}{(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^m (b_i - a_i)])}. \quad (5.26)$$

The proof of Theorem 5.5 is completed. \square

Remark 5.6. Theorem 5.4 and Theorem 5.5 generalize, unify and extend several known results from real Banach spaces to complex valued Banach spaces, including the results of Gürsoy [19] among others.

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