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Convolution algebra for extended Feller convolution

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Abstract We apply the recently introduced framework of admissible homomorphisms in the form of a convolution algebra of $\mathbb{C}^2$-valued admissible homomorphisms to handle two-dimensional uni-directional homogeneous stochastic kernels. The algebra product is a non-commutative extension of the Feller convolution needed for an adequate operator representation of such kernels: a pair of homogeneous transition functions uni-directionally intertwined by the extended Chapman-Kolmogorov equation is a convolution empathy; the associated Fokker-Planck equations are re-written as an implicit Cauchy equation expressed in terms of admissible homomorphisms. The conditions of solvability of such implicit evolution equations follow from the consideration of generators of a convolution empathy.

Keywords Convolution empathy · Feller convolution · extended Chapman-Kolmogorov equation · intertwined homogeneous Markov processes · implicit Fokker-Planck equations · admissible homomorphisms · convolution algebra · two-dimensional uni-directional stochastic kernel

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1 Introduction

The two-space approach[1] of empathy theory to the classical absorbing barrier problem of a stochastic process results in a pair of *non-homogeneous* Markov processes with a pair of distinct finite state spaces [12]. The corresponding pair of transition functions are intertwined by the backward extended Chapman-Kolmogorov equation. In this paper we consider the continuous analogue of this problem in the form of two distinct *continuous* state spaces and two *homogeneous* Markov processes which models random transitions within a continuum of "life" states and from the "life" states to a continuum of "death" states. The absorbing barrier is more realistically modelled as a continuum as "death" states (see [1, §8.1.22] for the case of a single "death" or coffin state).

For a single homogeneous Markov process, the Feller convolution captures the operator representation of the transition function by expressing the Chapman-Kolmogorov equation as a Feller convolution semigroup. Feller's operator representation [6, Ch. VIII.3] was introduced without clear motivation. In §2 we give a precise mathematical interpretation in Palmer's convolution algebra of translation-invariant or admissible linear functionals ([17, Chp. 1.9.7], [8, §19]). Admissible linear functionals replace probability distributions, and the product of admissible linear functionals replaces the Feller convolution of probability distributions. Then a Feller convolution semigroup is equivalent to a star-semigroup (§3 Thm. 3).

In the present problem of two distinct state spaces, there are two types of transitions: (a) from a "life" state to another "life" state and (b) from a "life" state to a "death state". Uni-directional transitions of type (b) cannot be described by Feller's classical notion of a joint conditional distribution (see [6, Chap. V.9, Def. (9.3)]). Thus, we introduce the notion of a uni-directional two-space stochastic kernel (§4, Def. 18).

For an operator representation of a uni-directional two-space stochastic kernel, the Feller convolution is inadequate. The languages of Feller's convolution of distributions and Palmer's convolution algebra are equivalent. This renders Palmer's convolution algebra as inadequate. However we extend Palmer's convolution algebra of admissible linear functionals by a vectorization process that results in the convolution algebra of admissible $\mathbb{C}^2$-valued homomorphisms on a product test space (§5). The product of this extended convolution algebra is an extension of the Feller convolution that gives the required operator representation of a uni-directional two-space stochastic kernel that have probability density functions. The extended Feller convolution expresses the forward extended Chapman-Kolmogorov equation (§6, eqs. (31)) as a convolution empathy or star-empathy (§6, Thm. 6). In this representation, the second evolution family evolves in empathy with the first evolution family, which is an extended Feller convolution semigroup. Thus, the Riesz representation of a uni-directional two-space distribution is a $\mathbb{C}^2$-admissible homomorphism.

[1] The two-space approach to the heat equation, where the boundary of the body is treated as a second distinct body, is presented in Appendix A.

In §7 we consider normed admissible homomorphisms to handle analytic conditions on the behaviour of a Feller convolution semigroup near the time origin. Then such an analytic Feller convolution semigroup can be expressed as a C_0-strongly continuous star-semigroup. When intertwined with another transition distribution function by the extended Chapman-Kolmogorov equation, the resulting pair is a strongly continuous star-empathy that is Laplace transformable (§9 and Lem. 3).

A pair of intertwined Laplace transforms in the form of pseudo-resolvent equations [14, Lem. 2.3] is the starting point of the analysis in the theory of classical strongly continuous empathy [14]. Intertwined Laplace transforms reduces the Fokker-Planck equation of empathy pseudo-Poisson processes [12, eq. (4.7)] into an implicit evolution equation of the form

$$\tfrac{d}{dt}[Bu(t)] = Au(t); \lim_{t\to 0^+}[Bu(t)] = y \in Y, \tag{1}$$

where the "generators" A and B are unbounded linear operators from a common domain in a Banach space X to a Banach space Y (see [12, Cor. 4.2]). Under the invertibility assumption (58), the Laplace transform of the second family is of the form $(\lambda B - A)^{-1}$, where A and B are the pair of generators of an implicit evolution equation (1).

A fully developed Laplace transform theory exists for normed admissible homomorphisms on a product space. Thus we employ a similar approach of intertwined Laplace transforms to construct a Fokker-Planck equation for a strongly continuous Laplace transformable star-empathy. The resulting Fokker-Planck equation takes the analogous form (see (60) and (62))

$$\frac{d}{dt}\langle b' * u'(t), \varphi\rangle = \langle a' * u'(t), \varphi\rangle; \lim_{t\to 0^+}\langle b' * u'(t), \varphi\rangle = \langle \theta'_{(0,0)}, \varphi\rangle, \tag{2}$$

where $*$ is the extended Feller convolution as opposed to the composition of operators on Banach spaces as in (1). Indeed, under the invertibility assumption, the Laplace transform of the second family is of the form $(\lambda b' - a')^{-1}$.

In §12, given only star-empathy pseudo-resolvent as defined in (63), we provide sufficient conditions on the pair of generators $\langle a', b'\rangle$ to construct a star-empathy satisfying a Fokker-Planck equation of the form (2). The classical operator-theoretic version of this "inverse" problem can be solved by an adaptation of Kisyński's algebraic version of the Hille-Yosida theorem to empathy theory. Full details are given in [10, §4]. The approach in §12 is to lift this operator-theoretic version into the framework of admissible homomorphisms. Moreover, under the invertibility assumption, the Laplace transform of the second family is also of the form $(\lambda b' - a')^{-1}$.

2 Probability Distributions as admissible linear functionals

Feller [6] represents a probability distribution as a bounded linear operator on a space of continuous functions by means of the Feller convolution. We now show that Feller's operator representation of a probability distribution has an explicit interpretation in Palmer's convolution algebra.

Let Φ_U be the test space $\mathrm{BUC}(\mathbb{R}, \mathbb{C})$ of bounded uniformly continuous scalar functions defined on a common domain in the form of the additive group $\mathbb{R}$. Given a probability distribution Q, we define the linear functional $Q' : \Phi_U \to \mathbb{C}$ by

$$\langle Q', f \rangle = Q'(f) := \int_{\mathbb{R}} Q\{dy\} f(y) \text{ for all } f \in \Phi_U. \tag{3}$$

We use the notation $\langle Q', f \rangle$ for the action of the linear functional Q' on the test space function $f \in \Phi_U$. Thus $\langle Q', f \rangle$ is the expectation of f, and Q' is the Riesz representation of the distribution $Q\{dy\}$. If we define translation by the parameter $x \in \mathbb{R}$ as the function $f_x(y) = f(y - x)$, then the function

$$x \mapsto [Q \circledast f](x) := \langle Q', f_{-x} \rangle = \int_{\mathbb{R}} Q\{dy\} f(x + y) \tag{4}$$

is well-defined because Φ_U is translation invariant. (We use Bobrowski's alternative format (see [1, Def. 7.5.1]) of the Feller convolution because it is in accordance with the present framework.) In fact, $Q \circledast f \in \Phi_U$ for all $f \in \Phi_U$ [11, Thm. 12].

The constructs (3) and (4) are also well-defined for other test spaces Φ in place of Φ_U. When Q' and Φ have the property

$$Q \circledast f \in \Phi \text{ for all } f \in \Phi, \tag{5}$$

we call Q' a *Φ-admissible linear functional.* The symbol $\mathscr{A}_\Phi$ will denote the set of all Φ-admissible homomorphisms.

Following Feller, we define the bounded linear operator $\mathcal{Q} : \Phi \to \Phi$ by

$$\mathcal{Q} f = Q \circledast f \text{ for all } f \in \Phi. \tag{6}$$

The expressions (3), (4) and (6) illucidate the construction $Q\{dy\} \mapsto Q' \mapsto \mathcal{Q}$ implicit in [6] and made explicit in [17, Chp. 1.9.7] and [8, §19], where $\mathcal{Q}$ is called the *dualism* of Q'. Each admissible linear functional, Q', plays 'dual' roles firstly as a linear functional, and secondly by inducing linear mappings, $\mathcal{Q}$, in the space of test functions, Φ. The mapping $\Gamma : Q \mapsto \mathcal{Q}$ will be called the *dualism mapping.*

The construction (4)–(6) is a special case of the definition of dualism in [17, eq. (17)]. This dualism allows the definition of a product of admissible linear functionals [17, eq. (18)]: if R is another distribution function with associated homomorphism R' and dualism $\mathcal{R}$, the product $Q' * R' \in \mathscr{A}_\Phi$ is defined by

$$\langle Q' * R', f \rangle = \langle Q', \mathcal{R} f \rangle \text{ for all } f \in \Phi. \tag{7}$$

If we define the Feller convolution of distributions by

$$[Q \star R](B) = \int_{\mathbb{R}} Q\{dy\} R(B + y) \tag{8}$$

for all Borel subsets $B \subset \mathbb{R}$, then

$$[Q \star R]' = Q' * R'; \tag{9}$$

$$\Gamma(Q' * R') = \mathcal{Q} \circ \mathcal{R}. \tag{10}$$

At this point we emphasize that Palmer's convolution algebra of admissible linear functionals consists of (i) a test space Φ_U of scalar functions that are defined on a common Abelian group, (ii) a class of linear functionals that are Φ_U-admissible and (iii) an associative product $*$ of such linear functionals.

We will use the following subspaces of Φ_U as test spaces:

(a) $\Phi_0 := C_0(\mathbb{R}, \mathbb{C})$, the space of continuous functions with zero limits at $\pm\infty$;
(b) $\Phi_\infty := C[\mathbb{R}, \mathbb{C}]$, the space of continuous functions with finite limits at $\pm\infty$.

For $\Phi = \Phi_0$ the following result is proved in [8, Lem. 19.5]. Since $\Phi_0 \subset \Phi_\infty \subset \Phi_U$, this result holds for the other spaces as well.

Theorem 1 *Let $\Phi = \Phi_0$, $\Phi = \Phi_\infty$ or $\Phi = \Phi_U$. Then each admissible homomorphism represents a unique distribution, i.e., the mapping $Q \mapsto Q'$ is injective. Moreover, the convolution of distributions lifts as the product of admissible linear functionals, i.e., eq.* (9) *holds.*

Thus, we can work with admissible linear functionals instead of distributions and the corresponding convolution $*$ instead of the Feller convolution $\star$. The versatility of the framework of linear functionals rests on the freedom to change the test space Φ according to the application, with the result such as Theorem 1, remaining valid in the new translation-invariant test space.

3 Feller semigroup as a star semigroup

Let $\mathbf{X} = \{X_t\}_{t>0}$ be a Markov process as defined in [6] with time-homogeneous transition function $\{Q_t(x, B)\}_{t>0}$. We say that $\{Q_t(x, B)\}_{t>0}$ is intertwined by the Chapman-Kolmogorov equation if

$$Q_{t+s}(x, B) = \int_{y \in \mathbb{R}} Q_t(x, \{dy\}) Q_s(y, B) \text{ for all } s, t > 0. \tag{11}$$

Equation (11) expresses the Markov property: given the behaviour of the particle up to time t, the probability to transition in the remaining time s to the Borel set B, i.e. $Q_s(y, B)$, depends only on the intermediary point y reached after time t.

We call $\{Q_t(x, B)\}_{t>0}$ space-homogeneous if

$$Q_t(x, B) = Q_t(x + r, B + r) \text{ for every } r \in \mathbb{R} \text{ and every Borel set } B \subset \mathbb{R}. \tag{12}$$

We call $\mathbf{X}$ a *homogeneous* Markov process if $\{Q_t(x, B)\}_{t>0}$ is both time-homogeneous and space-homogeneous. Then $\{Q_t(x, B)\}_{t>0}$ reduces to a time continuum of distributions $\mathbf{Q} := \{Q_t\{dy\}\}_{t>0} = \{Q_t(0, \{dy\})\}_{t>0}$ since $Q_t\{B - x\} = Q_t(x, B)$, and we call $\mathbf{Q}$ a *distribution transition function.* By (6), $\mathbf{Q}$ has an operator representation $\{\mathcal{Q}_t : \Phi_U \to \Phi_U\}_{t>0}$.

Theorem 2 *Let* $\mathbf{X}$ *be a homogeneous Markov process with transition function* $\{Q_t(x,B)\}_{t>0}$ *intertwined by the Chapman-Kolmogorov equation. Then, in terms of the the Feller convolution* $\star$*, the distribution transition function* $\mathbf{Q}$ *is a Feller convolution semigroup:*

$$Q_{t+s}\{dy\} = Q_t\{dy\} \star Q_s\{dy\} \text{ for all } s,t > 0; \tag{13}$$

$$\mathcal{Q}_{t+s} = \mathcal{Q}_t \circ \mathcal{Q}_s \qquad \text{for all } s,t > 0. \tag{14}$$

Proof Since $\mathbf{X}$ is homogeneous, the Chapman-Kolmogorov equation (11) can be expressed independently of x:

$$Q_{t+s}(0,B) = \int_{y\in\mathbb{R}} Q_t(0,\{dy\})Q_s(0,B-y) \text{ for all } s,t > 0. \tag{15}$$

The right hand side of (15) is precisely the product measure $Q_t\{dy\} \times Q_s\{dy\}$ of the pullback set $B^2 := \{(x,y)|x+y \in B\}$ of B under the sum operation. This proves property (13). The semigroup property (14) then follows.

Without loss of information, we replace the distribution transition function $\mathbf{Q}$ by a time continuum, $\mathfrak{q}' := \{Q'_t\}_{t>0}$, of admissible homomorphisms on Φ_U, which we call an *admissible transition function.* Then the Feller convolution semigroup of Theorem 2 is expressed in the framework of admissible homomorphisms as a star-semigroup. Classical semigroup relations are examples of causal relations built upon composition of evolution operators; a star-semigroup is a similar relation (see (16)) based upon the product $*$.

Theorem 3 *Let* $\mathbf{X}$ *be a homogeneous Markov process intertwined by the Chapman-Kolmogorov equation* (11)*. Then, in terms of the product* $*$*, the admissible transition function* $\mathfrak{q}'$ *is a star-semigroup:*

$$Q'_{t+s} = Q'_t * Q'_s \text{ for all } s,t > 0; \tag{16}$$

$$\mathcal{Q}_{t+s} = \mathcal{Q}_t \circ \mathcal{Q}_s \text{ for all } s,t > 0. \tag{17}$$

Proof Feller's operator representation $\mathcal{Q}_t$ of the distribution $Q_t\{dy\}$ is precisely the dualism of the admissible homomorphism Q'_t. Indeed, the dualism $\mathcal{Q}_t$ backtracks from the space of operators to the space of admissible homomorphism Q'_t by the point evaluation map $\theta'_0 : f \mapsto f(0)$ as follows: $\theta'_0(\mathcal{Q}_t f) = \langle Q'_t, f\rangle$ for each $f \in \Phi_U$.

4 Failure of Riesz representation: Two-space stochastic kernels

In [12] the two-space approach to an absorbing boundary of a Markov process leads to two distinct discrete state spaces: $\mathbb{N}_X := \{1,2,...,m\}$, the set of m "life" states, and $\mathbb{N}_{\bar{Y}} := \{\bar{1},\bar{2},...,\bar{n}\}$, the set of n "death" states. The continuous analogues of $\mathbb{N}_X$ and $\mathbb{N}_{\bar{Y}}$ are the continuums of life and death states, $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$, respectively. We treat the state spaces $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$ as distinct copies of $\mathbb{R}$ (as in the sense of a disjoint union). We write x (or y) for life states, X for the associated random variable, and $B \subset \mathbb{R}_X$ for the Borel sets in which X is manifested. For the death states, the corresponding entities are denoted by $\bar{y}$, $\bar{Y}$ and $\bar{B} \subset \mathbb{R}_{\bar{Y}}$, respectively.

Intertwinement (or interaction) is affected by the possibility of transitioning (a) from a life state to another life state within $\mathbb{R}_X$ and (b) from a life state in $\mathbb{R}_X$ to a death state in $\mathbb{R}_{\bar{Y}}$. The single-state space transition (a) is described by the stochastic kernel $Q(x, B)$, which gives the probability of X transitioning from a point $x \in \mathbb{R}_X$ to a Borel set $B \subset \mathbb{R}_X$. Similarly, the uni-directional transition (b) is described by the *two-space* stochastic kernel $R(x, \bar{B})$, which gives the probability of $\bar{Y}$ transitioning from a point $x \in \mathbb{R}_X$ to a Borel set $\bar{B} \subset \mathbb{R}_{\bar{Y}}$.

Let $Q(x, \{dy\})$ denote the distribution on $\mathbb{R}_X$. The distribution $R(x, \{d\bar{y}\})$ on $\mathbb{R}_{\bar{Y}}$ is defined by the conditional probability

$$R(x, \bar{B}) = P(\bar{Y} \in \bar{B} | X = x) := \lim_{h \to 0} \frac{P(\{\bar{Y} \in \bar{B}\} \text{ and } \{X \in [x, x+h]\})}{F([x, x+h])}, \quad (18)$$

where F is the distribution of X. The two-space stochastic kernel $R(x, \bar{B})$ plays a dual role:

(a) For each $x \in \mathbb{R}_X$, $R(x, \bar{B})$ is the probability distribution $R(x, \{d\bar{y}\})$.
(b) For each Borel set $\bar{B} \subset \mathbb{R}_{\bar{Y}}$, $R(x, \bar{B})$ is the point function that maps $x \in \mathbb{R}_X$ to $P(\bar{Y} \in \bar{B} | X = x)$.

Definition (18) differs from Feller's joint conditional distribution [6, Ch. V.9, Def. (9.3)], which treats both $[x, x+h]$ and $\bar{B}$ as subsets of a single state space $\mathbb{R}$.

Consider a pair of homogeneous Markov processes $(\mathbf{X}, \mathbf{Y})$ intertwined by the backward extended Chapman-Kolmogorov equation introduced in [12]:

$$\begin{cases} Q_{t+s}(x, B) = \displaystyle\int_{y \in \mathbb{R}} Q_t(x, \{dy\}) Q_s(y, B) \text{ for all } s, t > 0; & (19a) \\ R_{t+s}(x, \bar{B}) = \displaystyle\int_{y \in \mathbb{R}} Q_t(x, \{dy\}) R_s(y, \bar{B}) \text{ for all } s, t > 0. & (19b) \end{cases}$$

We call eq. (19b) the *two-space backward (extended Chapman-Kolmogorov) transition equation.* We call the pair $(\mathbf{X}, \mathbf{Y})$ *homogeneous* if both transition functions in the pair $(Q_t(x, B), R_t(x, \bar{B}))_{t>0}$ are homogeneous. In that case it reduces to a pair of distribution transition functions

$$(\mathbf{Q}, \mathbf{R}) := (Q_t\{dy\}, R_t\{d\bar{y}\})_{t>0} = (Q_t(0, \{dy\}), R_t(0, \{d\bar{y}\}))_{t>0}.$$

Then, by arguing as in the proof of Theorem 2, we obtain the following analogue of eq. (15).

Proposition 1 *Let $(\mathbf{X}, \mathbf{Y})$ be a pair of homogeneous Markov processes intertwined by the backward extended Chapman-Kolmogorov equation. The two-space backward transition equation* (19b) *can be expressed in terms of* $(\mathbf{Q}, \mathbf{R})$*:*

$$R_{t+s}\{\bar{B}\} := R_{t+s}(0, \bar{B}) = \int_{y \in \mathbb{R}} Q_t\{dy\} R_s(0, \bar{B} - y) \text{ for all } s, t > 0. \quad (20)$$

Remark 1 Suppose that we do not make a distinction between $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$ and take both $Q_t\{dy\}$ and $R_t\{d\bar{y}\}$ as measures on $\mathbb{R}$. Then the convolution of measures $Q_t\{dy\} \star R_s\{d\bar{y}\}$ is well-defined (see [1, Def. 1.2.16]) and is precisely the right-hand side of eq. (20). Thus, by the commutativity of the Feller convolution [6, Ch. V.4, Thm. 3], eq. (20) has an analogue [2] of eq. (13):

$$R_{t+s}\{d\bar{y}\} = Q_t\{dy\} \star R_s\{d\bar{y}\} = R_s\{d\bar{y}\} \star Q_t\{dy\} \text{ for all } s, t > 0. \tag{21}$$

Now, let $\mathcal{Q}_t$ and $\mathcal{R}_t$ be the operator representations of $Q_t\{dy\}$ and $R_t\{d\bar{y}\}$, respectively. Then it follows from eq. (21) that $\mathcal{R}_{t+s} = \mathcal{Q}_t \circ \mathcal{R}_s = \mathcal{R}_s \circ \mathcal{Q}_t$ for all $s, t > 0$. This commutative relation does not hold in a setting with unidirectional transitions from life states to death states. Moreover, the distribution $R_t\{d\bar{y}\}$ in (20), which involves distinct copies of $\mathbb{R}$, cannot be described by Feller's joint conditional distribution (defined in [6, Ch. V.9, Def. (9.3)]) because eq. (20), unlike eq. (15), cannot be expressed as the Feller convolution of two distributions as in eq. (13) nor as the product of two admissible linear functionals as in eq. (16).

5 Admissible homomorphisms on a product test space

We vectorize Palmer's convolution algebra of admissible linear functionals as a convolution algebra of admissible homomorphisms between a test space of Banach space-valued functions and the Banach space itself. We implement this non-commutative vectorization over three steps.

1. Let $\Phi_P := \mathrm{BUC}(G, Z)$ denote the class of bounded uniformly continuous functions defined on the locally compact Abelian group $(G, +)$; Z is a Banach space. The notation φ shall denote vector-valued functions.

2. Consider (algebraic) homomorphisms $Q'_P : \Phi_P \to Z$. The notation $\langle Q'_P, \varphi \rangle = Q'_P(\varphi)$; $\varphi \in \Phi_P$ — reminiscent of usage for linear functionals — will be used. Such a homomorphism is called *admissible* if the group G-domained function ${}^{Q'_P}\varphi : p \in G \mapsto \langle Q'_P, \varphi_{-p} \rangle$ is back in Φ_P. That is, for a fixed $\varphi \in \Phi_P$, the translations $S^p : \varphi \mapsto \varphi_p$ generate a bundle of "curves" $\{\varphi_p : p \in \mathbb{R}\}$ in Z, which is then mapped by Q'_P to the curve ${}^{Q'_P}\varphi$. Admissibility requires that this curve be in Φ_P. Thus, each admissible homomorphism Q'_P play two roles: the default role of a homomorphism from Φ_P to Z; in the second role they induce operators $\mathcal{Q}$ on Φ_P, called *dualisms*, defined by $\mathcal{Q}f = {}^{Q'_P}\varphi$; $\varphi \in \Phi_P$.

[2] The expression $R_{t+s}\{d\bar{y}\} = R_s\{d\bar{y}\} \star Q_t\{dy\}$ is equivalent to the expression $R_{t+s}(x, \bar{B}) = \int_{y \in \mathbb{R}_X} R_t(x, \{d\bar{y}\}) Q_s(y, \bar{B})$ which has no meaning (compare with (19b)). In particular, for the pair of pseudo-Poisson processes of [12, §2], the integral $\int_{y \in \mathbb{R}_X} R_t(x, \{d\bar{y}\}) Q_s(y, \bar{B})$ reduces to the nonsensical expression $\sum_{k=1}^{|X|} P(Y_{t+s} = \bar{j} | X_t = k) P(Y_t = k | X_0 = i)$ on setting $x = i, y = k$, and $\bar{B} = \bar{j}$.

Point evaluation maps $\theta'_q : \varphi \in \Phi_P \mapsto \varphi(q) \in Z$ are an example of admissible homomorphisms. In fact, ${}^{\theta'_q}\varphi(p) = S^{-q}\varphi(p)$. In particular,

$$\langle Q'_P, \varphi \rangle = \langle \theta'_0, \mathcal{Q}_P \varphi \rangle. \tag{22}$$

The class of admissible homomorphisms is denoted by $\mathscr{A}_{\Phi_P}$.

3. The dual roles of each Φ_P-admissible linear functional R'_P as a homomorphism that is from Φ_P to Z and operators $\mathcal{R}_P : \Phi_P \to \Phi_P$ give an algebraic structure to the class $\mathscr{A}_{\Phi_P}$ by means of an associative product: for Φ_P-admissible homomorphisms Q'_P and R'_P, the product $Q'_P * R'_P$ is given by
$$\langle Q'_P * R'_P, \varphi \rangle = \langle Q'_P, \mathcal{R}_P \varphi \rangle \text{ for all } \varphi \in \Phi_P, \tag{23}$$
where $\mathcal{R}_P = \Gamma(R'_P)$ is the dualism of R'_P. Note that the product $Q'_P * R'_P$ is well-defined even if Q'_P is a homomorphism that is not admissible.

For a judicious choice of Banach space Z and group G, we shall later show that $*$ is a non-commutative extension of the classical commutative Feller convolution $\star$ of Borel measures on $\mathbb{R}$. We therefore call it a *convolution product.* The product $*$ introduced here is related to composition of linear transformations in the following way: If Q'_P and R'_P are in $\mathscr{A}_{\Phi_P}$, then ${}^{(Q'_P * R'_P)}f = \mathcal{Q}_P \mathcal{R}_P \varphi$, where $\mathcal{Q}_P, \mathcal{R}_P$ are the dualisms of Q'_P, R'_P respectively. Thus, the product of homomorphisms corresponds to the composition of their dualisms.

Admissible homomorphisms replace operators in the sense that an operator on the Banach space Z gives rise to an admissible homomorphism: if $A : Z \to Z$ is a bounded linear operator, the mapping $Q'_A : f \mapsto A\theta'_0 f$ is an admissible homomorphism. Indeed, $[\mathcal{Q}_A f](q) = {}^{Q'_A}f(q) = Af(q)$. We call the relation $A \to Q'_A$ the *canonical mapping.*

We consider the converse question: Which linear operators on the test space give rise to admissible homomorphisms from Φ_P to Z. Let Δ be a (vector) subspace of Φ_P. A linear mapping $\mathcal{Q} : \Delta \to \Phi_P$ is called *translatable* if (i) Δ is translation invariant and (ii) for every $p \in G$ and $f \in \Delta$, $S^p \mathcal{Q} f = \mathcal{Q} S^p f$. Note that $\mathcal{Q}$ induces a homomorphism Q' defined by $\langle Q', f \rangle := \langle \theta'_0, \mathcal{Q} f \rangle$, where the induced homomorphism Q' is restricted to the "domain" Δ. We call such homomorphisms, *restricted homomorphisms.* Then we have the following result.

Theorem 4 *A linear mapping $\mathcal{Q} : \Phi_P \to \Phi_P$ is translatable if and only if it is a dualism of a restricted homomorphism. If $\mathcal{Q}$ is a one-to-one dualism and $\Delta := \mathcal{Q}[\Phi_P]$, then $\mathcal{Q}^{-1} : \Delta \to \Phi_P$ is translatable. Sums and compositions of translatable mappings are translatable.*

The framework of admissible homomorphisms is very general. We fix the vectorized test space, Φ_P, and group, G, for the required non-commutative generalization of the Feller convolution: we represent $\bar{Q}_t\{dy\}$ and $\bar{R}_t\{dy\}$ of Remark 1 as admissible homomorphisms on a product test space.

Construction 1 The two-state space approach forces us to consider the product space $\Phi_X \times \Phi_{\bar{Y}}$, where $\Phi_X := BUC(\mathbb{R}_X, \mathbb{C})$ and $\Phi_{\bar{Y}} := BUC(\mathbb{R}_{\bar{Y}}, \mathbb{C})$. We now apply Theorem 1. We replace the intertwined distributions $Q_t\{dy\}$ on $\mathbb{R}_X$ and $R_t\{d\bar{y}\}$ on $\mathbb{R}_{\bar{Y}}$ by the admissible homomorphisms $Q_t'(X)$ on Φ_X and $R_t'(\bar{Y})$ on $\Phi_{\bar{Y}}$, respectively.

We consider the following group G and Banach space Z to fix the vector test space Φ_P:

(a) The diagonal subgroup $G := \{(p,p) | p \in \mathbb{R}\}$, instead of the product group $\mathbb{R}_X \times \mathbb{R}_{\bar{Y}}$, to have one-parameter translations. Then for each $\big(f, \bar{g}\big) \in \Phi_X \times \Phi_{\bar{Y}}$, we define a corresponding function $\varphi : G \to Z := \mathbb{C}^2$ by $\varphi(p,p) = \big(f(p), \bar{g}(\bar{p})\big)$, where p and $\bar{p}$ denote the same numerical value. The product test space Φ_P is defined as the set of all such functions φ.
(b) The Banach space $Z := \mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ with the product norm.
(c) The one-parameter translation operators $R^{(p,p)}$, $p \in G$, are defined by

$$R^{(p,p)}\varphi = \varphi_{(p,p)} : (q,q) \in G \to \varphi(q-p, q-p) = \big(f(q-p), \bar{g}(\bar{q}-\bar{p})\big)\,.$$

We shall henceforth denote (p,p) as p, $R^{(p,p)}$ as R^p and $\varphi_{(p,p)}$ as φ_p.

From this point onwards the Banach space Z, the group G and the test space Φ_P of vector valued test functions will be defined as in Construction 1, unless stated otherwise.

Since $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$ are copies of $\mathbb{R}$, the test spaces Φ_X and $\Phi_{\bar{Y}}$ contain essentially the same functions. Given a test function $f \in \Phi_X$, we define a corresponding test function $\bar{f} \in \Phi_{\bar{Y}}$ by $\bar{f}(\bar{y}) := f(y)$, where y and $\bar{y}$ denote the same numerical value. In order to work with the product test space Φ_P instead of a single test space Φ_X, we define the following mappings to lift classical linear functionals as $\mathbb{C}^2$-linear functionals:

(a) liftings $\ell_1 : \mathbb{C} \to \mathbb{C}^2 : z \mapsto (z, 0)$ and $\ell_2 : \mathbb{C} \to \mathbb{C}^2 : z \mapsto (0, z)$;
(b) liftings $\ell_X : \Phi_X \to \Phi_P : f \mapsto (f, 0_{\bar{Y}})$ and $\ell_{\bar{Y}} : \Phi_{\bar{Y}} \to \Phi_P : f \mapsto (0_X, \bar{f})$, where $0_{\bar{Y}} \in \Phi_{\bar{Y}}$ and $0_X \in \Phi_X$ are the zero functions;
(c) the projection $\pi_X : \Phi_P \to \Phi_X : \varphi = (f, \bar{g}) \mapsto f$.

Lemma 1 (a) *Let Q' be a Φ_X-admissible homomorphism with dualism $\mathcal{Q} = \Gamma(Q') : \Phi_X \to \Phi_X$. Let $Q_P' := \ell_1 \circ Q' \circ \pi_X$, i.e.,*

$$\langle Q_P', \varphi \rangle = \big(\langle Q', f \rangle, 0\big) \in \mathbb{C}^2 \text{ for all } \varphi = (f, \bar{g}) \in \Phi_P. \tag{24}$$

Then Q_P' is a Φ_P-admissible homomorphism and its dualism is $\mathcal{Q}_P = \Gamma(Q_P') = \ell_X \circ \Gamma(Q') \circ \pi_X$, i.e.,

$$\mathcal{Q}_P \varphi = \big(\mathcal{Q} f, 0_{\bar{Y}}\big) \in \Phi_P \text{ for all } \varphi = (f, \bar{g}) \in \Phi_P. \tag{25}$$

(b) *Let R' be a Φ_X-admissible homomorphism with dualism $\mathcal{R} = \Gamma(R') : \Phi_X \to \Phi_X$. Let $R_P' := \ell_2 \circ R' \circ \pi_X$, i.e.,*

$$\langle R_P', \varphi \rangle = \big(0, \langle R', f \rangle\big) \in \mathbb{C}^2 \text{ for all } \varphi = (f, \bar{g}) \in \Phi_P. \tag{26}$$

Then R'_P is a Φ_P-admissible functional and its dualism is $\mathcal{R}_P = \Gamma(R'_P) = \ell_{\bar{Y}} \circ \Gamma(R') \circ \pi_X$, i.e.,

$$\mathcal{R}_P\varphi = \left(0_X, \overline{\mathcal{R}f}\right) \in \Phi_P \text{ for all } \varphi = (f, \bar{g}) \in \Phi_P. \tag{27}$$

Remark 2 Both Q' and R' are "life" homomorphisms on Φ_X. The lifting of Q' as the "life" homomorphism Q'_P is analogous to embedding a real number x as $x + 0i$ on the complex plane. In contrast, the lifting R'_P of the "life" homomorphism R' is analogous to embedding a real number x as a pure imaginary number $0 + xi$. Then R'_P lifts R' as a "death" homomorphism that has a "death" dualism on Φ_P. Note that both types of liftings, Q'_P and R'_P, are very special cases of Φ_P-admissible homomorphisms.

6 Extended Feller convolution empathy

As in [12, §3, Thm 3.1], the operator representation of the uni-directional two-space backward transition equation (19b) is a reverse empathy. In order to apply empathy theory [11], which does not deal with reverse empathies, we introduce the notion of a conjugate stochastic kernel to construct the dual notion of the forward extended Chapman-Kolmogorov equation.

Assume that the stochastic kernels $Q_t(x, B)$ and $R_t(x, \bar{B})$ of eqs. (19) have probability transition density functions $q_t(x, y)$ and $r_t(x, \bar{y})$, respectively. Then for $t > 0$ and Borel sets $B \subset \mathbb{R}_X$, we define the *conjugate kernel* $\bar{Q}_t(y, B)$ by

$$\bar{Q}_t(y, B) := \int_{x \in B} q_t(x, y)dx, \ \ y \in \mathbb{R}_X, \tag{28}$$

and the *conjugate kernel* $\bar{R}_t(\bar{y}, B)$ by

$$\bar{R}_t(\bar{y}, B) := \int_{x \in B} r_t(x, \bar{y})dx, \ \ \bar{y} \in \mathbb{R}_{\bar{Y}}. \tag{29}$$

The symbols $\bar{Q}_t\{dy\} = \bar{Q}_t(0, \{dy\})$ and $\bar{R}_t\{dy\} = \bar{R}_t(\bar{0}, \{dy\})$ will denote the corresponding distributions on $\mathbb{R}_X$.

Note that in the conjugate kernel the roles of the point and the set are interchanged. For example, in (29) the set B of life states is the start of the transition and the death state $\bar{y}$ is the end of the transition. Moreover, the roles of the state spaces $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$ are interchanged: for fixed $\bar{y}$, $\bar{R}_t(\bar{y}, B)$ is a measure in $\mathbb{R}_X$; for fixed x, $R_t(x, \bar{B})$ is a measure in $\mathbb{R}_{\bar{Y}}$. (The adjoint measure $\bar{K}_B(\bar{j}, \{i\})$ defined in [12, p.128] is the discrete version of definition (29).)

Proposition 2 *Let $Q_t(x, B)$ and $R_t(x, \bar{B})$ be the stochastic kernels of eqs.* (19) *with probability transition density functions $q_t(x, y)$ and $r_t(x, \bar{y})$.*

If $Q_t(x, B)$ ($R_t(x, \bar{B})$, resp.) is space-homogeneous, the conjugate kernel $\bar{Q}_t(y, B)$ ($\bar{R}_t(\bar{y}, B)$, resp.) is also space-homogeneous. Moreover, for every Borel set $B \subset \mathbb{R}_X$ and corresponding set $\bar{B} \subset \mathbb{R}_Y$,

$$\bar{Q}_t\{B\} = Q_t\{-B\}, \quad \bar{R}_t\{B\} = R_t\{-\bar{B}\}. \tag{30}$$

Proof Let $x \in \mathbb{R}_X$. For an arbitrary Borel set $\bar{B} \subset \mathbb{R}_{\bar{Y}}$,

$$\int_{\bar{y} \in \bar{B}} r_t(x, \bar{y}) d\bar{y} = R_t(x, \bar{B}) = R_t(x + r, \bar{B} + r) = \int_{\bar{y} \in \bar{B}} r_t(x + r, \bar{y} + r) d\bar{y}.$$

This implies that $r_t(x, \bar{y})$ is space-homogeneous: $r_t(x, \bar{y}) = r_t(x + r, \bar{y} + r)$ for a.e. $\bar{y} \in \mathbb{R}_{\bar{Y}}$ and all $r \in \mathbb{R}_{\bar{Y}}$. By the same argument, $q_t(x, y)$ is space-homogeneous. The assertions follow by integration of these relations.

We will now show that the conjugate transition functions $\bar{Q}_t(y, B)$ and $\bar{R}_t(\bar{y}, B)$ satisfy the *forward extended Chapman-Kolmogorov equation*

$$\begin{cases} \bar{Q}_{t+s}(y, B) = \int_{x \in \mathbb{R}} \bar{Q}_t(y, \{dx\}) \bar{Q}_s(x, B) \text{ for all } s, t > 0; & \text{(31a)} \\ \bar{R}_{t+s}(\bar{y}, B) = \int_{x \in \mathbb{R}} \bar{R}_t(\bar{y}, \{dx\}) \bar{Q}_s(x, B) \text{ for all } s, t > 0. & \text{(31b)} \end{cases}$$

We call eq. (31b) the *conjugate two-space forward (extended Chapman-Kolmogorov) transition equation.*

Theorem 5 *Let* $(\mathbf{X}, \mathbf{Y})$ *be a pair of homogeneous Markov processes intertwined by the backward extended Chapman-Kolmogorov equation* (19) *with a pair of probability transition density functions* $(q_t(x, y), r_t(x, \bar{y}))_{t>0}$. *Then the corresponding pair of conjugate transition functions* $(\bar{Q}_t(y, B), \bar{R}_t(\bar{y}, B))_{t>0}$ *is intertwined by the forward extended Chapman-Kolmogorov equation* (31).

Proof Since $q_t(x, y)$ is the Radon-Nikodym derivative of the measure $Q_t(x, \{dy\})$, it follows from eq. (19) and the Fubini theorem [13, Thm. 4.17] that

$$\begin{cases} q_{t+s}(x, y) = \int_{z \in \mathbb{R}} q_t(x, z) q_s(z, y) dz \text{ for all } s, t > 0; \\ r_{t+s}(x, \bar{y}) = \int_{z \in \mathbb{R}} q_t(x, z) r_s(z, \bar{y}) dz \text{ for all } s, t > 0. \end{cases}$$

By applying the operator $\int_{x \in B} dx$ to these equations and using the Fubini theorem with the fact that $r_s(x, \bar{y})$ is the Radon-Nikodym derivative of the measure $\bar{R}_s(\bar{y}, \{dx\})$, we obtain eq. (31).

The distribution transition functions $\mathbf{Q}$ and $\mathbf{R}$ are defined on distinct spaces, $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$. The conjugation operation produces a corresponding pair of distribution transition functions

$$(\bar{\mathbf{Q}}, \bar{\mathbf{R}}) := (\bar{Q}_t\{dy\}, \bar{R}_t\{dy\})_{t>0} = (\bar{Q}_t(0, \{dy\}), \bar{R}_t(\bar{0}, \{dy\}))_{t>0}$$

on $\mathbb{R}_X$. We now have the following analogue of Proposition 1.

Proposition 3 *Let* $(\mathbf{X}, \mathbf{Y})$ *be a pair of homogeneous Markov processes intertwined by the backward extended Chapman-Kolmogorov equation. The two-space forward transition equation* (31b) *can be expressed in terms of* $(\bar{\mathbf{Q}}, \bar{\mathbf{R}})$*:*

$$\bar{R}_{t+s}\{B\} := \bar{R}_{t+s}(\bar{0}, B) = \int_{x \in \mathbb{R}} \bar{R}_t\{dy\} \bar{Q}_s(0, B - x) \text{ for all } s, t > 0. \tag{33}$$

The distributions $\bar{Q}_t\{dy\}$ and $\bar{R}_t\{dy\}$ associated with (33) commute as in eq. (21) and therefore their operator representations also commute as in Remark 1. However, by the lifting lemma (Lemma 1), we obtain the required non-commutative generalization of the Feller convolution by representing $\bar{Q}_t\{dy\}$ and $\bar{R}_t\{dy\}$ as admissible homomorphisms on a product test space. We replace the intertwined conjugate distributions $\bar{Q}_t\{dy\}$ and $\bar{R}_t\{dy\}$ on $\mathbb{R}_X$ by the admissible homomorphisms $\bar{Q}'_t(X)$ and $\bar{R}'_t(X)$ on Φ_X, respectively. By virtue of Lemma 1 we lift $\bar{Q}'_t(X)$ and $\bar{R}'_t(X)$ as the Φ_P-admissible homomorphisms

$$Q'_P(t) := \ell_1 \circ \bar{Q}'_t(X) \circ \pi_X, \quad R'_P(t) := \ell_2 \circ \bar{R}'_t(X) \circ \pi_X,$$

respectively. This produces a corresponding pair of conjugate Φ_P-admissible transition functions $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P) := (\bar{Q}'_P(t), \bar{R}'_P(t))_{t>0}$.

Theorem 6 *Let* $(\mathbf{X}, \mathbf{Y})$ *be a pair of homogeneous Markov processes intertwined by the backward extended Chapman-Kolmogorov equation* (19)*. Then, in terms of the product* $*$ *defined in* (23)*, the pair of conjugate* Φ_P*-admissible transition functions* $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ *is a star-empathy:*

$$\begin{cases} \bar{Q}'_P(t+s) = \bar{Q}'_P(t) * \bar{Q}'_P(s) \ \textit{for all } s,t>0; & \text{(34a)} \\ \bar{R}'_P(t+s) = \bar{R}'_P(t) * \bar{Q}'_P(s) \ \textit{for all } s,t>0. & \text{(34b)} \end{cases}$$

Moreover, $\bar{Q}'_P(s) * \bar{R}'_P(t)$ *is the zero homomorphism on* Φ_P *for all* $s,t>0$*.*

Proof Equation (34a) follows from eq. (16) by definition (23) and Lemma 1(a).

Since $\bar{Q}_t\{dy\}$ and $\bar{R}_t\{dy\}$ are probability measures on $\mathbb{R}_X$, the right-hand side of eq. (33) is precisely the Feller convolution $\bar{R}_t\{dy\} \star \bar{Q}_t\{dy\}$ (see Remark 1). Thus, by eq. (9),

$$\bar{R}'_{t+s}(X) = \bar{R}'_t(X) * \bar{Q}'_s(X) \text{ for all } s,t>0. \tag{35}$$

Equation (34b) follows from eq. (35) by definition (23) and Lemma 1(b).

In view of eqs. (34) we call $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ the *conjugate extended Riesz representation* on Φ_P of the distribution transition functions $(\mathbf{Q}, \mathbf{R})$ (analogous to def. (3) and eqs. (13)–(14)). In order to exploit the machinery of empathy theory, from this point onward we only deal with the conjugate stochastic kernels where the second conjugate transition family $\bar{\mathfrak{r}}'_P$ evolves in empathy with the first conjugate transition family $\bar{\mathfrak{q}}'_P$ that is an extended Feller convolution semigroup. The evolution family $\bar{\mathfrak{r}}'_P$ is not a semigroup.

7 Normed admissible homomorphisms

With the canonical relationship between distributions and admissible homomorphisms in mind, we have thus far studied time-dependent members of $\mathscr{A}_\Phi$ instead of traditional families of distributions, with the Feller convolution of distributions replaced by the product $*$ of admissible homomorphisms. Thus, in the classical case of a single homogeneous Markov process $\mathbf{X}$, the admissible transition function $\mathfrak{q}' = \{Q'_t\}_{t>0}$ replaces the distribution transition function $\mathbf{Q} = \{Q_t\{dy\}\}_{t>0}$.

Suppose that $\mathbf{Q}$ is an analytic Feller convolution semigroup as defined by Bobrowski [1, Def. 7.6.1]: it satisfies eq. (13) and the initial condition

$$\lim_{t\to 0^+} Q_t\{dy\} = \delta_0. \tag{36}$$

The analytic condition (36) on $\mathbf{Q}$ denotes the weak convergence (or rather weak*) of the distributions $Q_t\{dy\}$ to δ_0, the Dirac measure concentrated at 0, on the space Φ_U of all bounded uniformly continuous functions[3]. This analytic condition (36) can be replaced by the condition

$$\lim_{t\to 0^+} \langle Q'_t, f\rangle = \langle \theta'_0, f\rangle \text{ for all } \mathrm{f} \in \Phi_{\mathrm{U}} \tag{37}$$

in the framework of normed admissible homomorphisms which we now describe.

Under the supremum norm, the test space Φ_U is a Banach test space and we can consider the norms of the bounded linear functional Q' on Φ_U. It follows that every bounded linear functional is Φ-admissible and the map $p \in G \mapsto f_{-p} \in \Phi_U$ is uniformly continuous for each fixed $f \in \Phi_U$. The point evaluation maps $\theta'_q : f \in \Phi_U \mapsto f(q)$ are examples of bounded admissible homomorphisms. The class of bounded admissible homomorphisms is denoted by $\mathscr{A}_B$. The following theorem follows from [11, Thm. 12].

Theorem 7 *The bounded admissible homomorphisms $\mathscr{A}_B$ equipped with the product $*$ is a Banach algebra with unit θ'_0. The dualism mapping $\Gamma : Q' \mapsto \mathcal{Q}$ is an isomorphism into the algebra of linear transformations on Φ_U that preserves norms: $\|Q'\| = \|\mathcal{Q}\|$.*

From this point onward we work in the framework of normed admissible homomorphisms, where we work mostly with bounded admissible homomorphisms. For bounded admissible homomorphisms we get sharper results by studying $\mathfrak{q}' = \{Q'_t \in \mathscr{A}_B\}_{t>0}$ in conjunction with its isometric dualism family $\mathfrak{Q} := \{\mathcal{Q}_t = \Gamma Q'_t\}_{t>0}$. For example, by eq. (37), we have the following result.

Proposition 4 *In the framework of normed admissible homomorphisms, an analytic Feller convolution semigroup is expressed as a C_0-strongly continuous star-semigroup in $\mathscr{A}_{\Phi_U}$ and as an isometric operator C_0-semigroup on Φ_U.*

8 Strongly continuous conjugate convolution semigroup

Our starting point is $(\mathbf{X}, \mathbf{Y})$, a pair of homogeneous Markov processes intertwined by the backward extended Chapman-Kolmogorov equation (19) with a pair of probability transition density functions $(q_t(x, y), r_t(x, \bar{y}))_{t>0}$.

[3] Intuitively, assumption (36) expresses the idea that the transition density function (instead of the actual probabilities) converges to the Dirac delta functional as $t \to 0^+$.

In this section we only consider the first of the pair of distribution functions $(\mathbf{Q}, \mathbf{R}) = (Q_t\{dy\}, R_t\{d\bar{y}\})_{t>0}$ and associated admissible transition function $\mathfrak{q}' = \{Q'_t\}_{t>0}$. We will establish sufficient conditions for the star-semigroup $\mathfrak{q}'$ and its conjugate $\bar{\mathfrak{q}}' = \{\bar{Q}'_t\}_{t>0}$ to be strongly continuous. We will show that the conjugation operation preserves strong continuity. These results will be applied to the conjugate transition family $\bar{\mathfrak{q}}'_P$ in §9.

Consider the classical case of a single homogeneous Markov process $\mathbf{X}$ such that the distribution transition function $\mathbf{Q} = \{Q_t\{dy\}\}_{t>0}$ is an analytic Feller convolution semigroup (see (36)). We henceforth assume that each stochastic kernel $Q_t(x, B)$ has a transition density function $q_t(x, y)$, as in eq. (28). Next we investigate how well the conjugation operation preserves the properties of $\mathbf{Q}$.

Example 1 Let $\mathbf{X}$ be the standard Brownian motion in $\mathbb{R}$. Due to the symmetry of the normal transition density function, the distributions $Q_t\{dy\}$ and $\bar{Q}_t\{dy\}$ are both equal to the normal (Gaussian) distribution with zero mean and variance t. Thus both $\mathbf{Q}$ and $\bar{\mathbf{Q}}$ are convolution semigroups.

Lemma 2 *Let $\mathbf{Q}$ be a homogeneous transition function with a probability transition density function. If $\mathbf{Q}$ is an analytic convolution semigroup then so too is its conjugate $\bar{\mathbf{Q}} = \{\bar{Q}_t\{dy\}\}_{t>0}$.*

Proof For each $f \in \Phi_U$, define $Mf \in \Phi_U$ by $Mf(x) = f(-x)$. Then $\langle \bar{Q}_t\{dy\}, f\rangle = \langle Q_t\{dy\}, Mf\rangle$ by eq. (30).

Theorem 8 *Let $\mathbf{X}$ be a homogeneous Markov process such that $\mathbf{Q}$ is an analytic Feller convolution semigroup. Then $\bar{\mathbf{Q}}$ is an analytic convolution semigroup: $\bar{\mathfrak{q}}'$ is a strongly continuous star-semigroup and the* dualism *transition function $\bar{\mathfrak{Q}} := \{\bar{\mathcal{Q}}_t\}_{t>0}$, where $\bar{\mathcal{Q}}_t = \Gamma(\bar{Q}'_t)$, is an operator C_0-semigroup.*

Proof By Lemma 2 and condition (36), $\langle \bar{Q}'(t), f\rangle \to \langle \theta'_0, f\rangle$ as $t \to 0^+$ for all $f \in \Phi_U$. The strong continuity of $\bar{\mathfrak{q}}'$ follows from the fact that θ'_0 is the identity admissible homomorphism. Thus $\bar{\mathfrak{Q}}$ is an operator C_0-semigroup due to its isometry with $\bar{\mathfrak{q}}'$ [11, §11, Prop. 5].

Theorem 8 holds for any test space that is a closed subspace of Φ_U, in particular the smaller Banach test space Φ_∞ used by Feller [6, §IX.2]. By [9, Thm. 11.5.2], the strongly continuous operator semigroups $\mathfrak{Q}$ and $\bar{\mathfrak{Q}}$ restricted to Φ_∞ have generators defined on dense subspaces of Φ_∞. Feller's definition [6, §IX.2, Def. 4] of a generator differs slightly from that of Hille and Phillips [9]. The Feller generators, A and $\bar{A}$, of $\mathfrak{Q}$ and $\bar{\mathfrak{Q}}$, respectively, are defined on the subspace $C^\infty[\mathbb{R}, \mathbb{C}]$ of infinitely differentiable functions in Φ_∞ with derivatives in Φ_∞.

In the setting of the backward extended Chapman-Kolmogorov equation, the distribution transition function $\mathbf{Q}$ is defective (i.e., $Q_t(x, \mathbb{R}) < 1$ for all $t > 0$ and all $x \in \mathbb{R}$). If $\mathbf{Q}$ is defective, then so too is $\bar{\mathbf{Q}}$, by eq. (30).

Proposition 5 *Let $\mathbf{X}$ be a homogeneous Markov process such that $\mathbf{Q}$ is an analytic Feller convolution semigroup and defective. Then there exist a unique*

$c > 0$ and a unique distribution transition function $\mathbf{P} = \{P_t\{dy\}\}_{t>0}$ that is a convolution semigroup with proper distributions such that

$$Q_t\{dy\} = e^{-ct} P_t\{dy\}. \tag{38}$$

The Feller generator of $\mathbf{Q}$ is $A - cI$, where A is the Feller generator of $\mathbf{P}$.

Proof Let $Q_t\{\mathbb{R}\}$ denote the total mass of the distribution $Q_t\{dy\}$. Then $P_t\{dy\} = p(t)Q_t\{dy\}$, where $p(t) = (Q_t\{\mathbb{R}\})^{-1}$, is a proper distribution. Now $\{P_t\{dy\}\}_{t>0}$ is a convolution semigroup if and only if $p(s+t) = p(s)p(t)$ for all $s, t > 0$ and $p(t) \to 1$ as $t \to 0^+$. Thus, since p is a measurable function, $p(t) = e^{ct}$ for some $c \in \mathbb{R}$, namely $c = -t^{-1} \ln Q_t\{\mathbb{R}\}$. By the defectiveness of $\mathbf{Q}$, $c > 0$.

Let A be the Feller generator of the operator semigroup associated with $\mathbf{P}$. Then $A - cI$ is the Feller generator of the operator semigroup associated with $\mathbf{Q}$ since for small positive t we can approximate e^{-ct} by $1 - ct$.

Example 2 Assume that $\mathbf{X}$ and $\mathbf{Q}$ satisfy the hypotheses of Proposition 5 and that $\mathbf{P}$ is the transition distribution function associated with the standard Brownian motion of Example 1. Then we call $\mathbf{X}$ a defective Brownian motion. In this case the Feller generator $\bar{A}$ of $\bar{\mathbf{Q}}$ is given by

$$\bar{A}f = \Big(\frac{1}{2}\frac{d^2}{dx^2} - c\Big)f \quad \text{for all } f \in C^\infty[\mathbb{R}, \mathbb{C}]. \tag{39}$$

Remark 3 Example 2 is an elementary example of a generator of a conjugate Feller convolution semigroup. From this point onwards we use this elementary example of the generator $\bar{A}$ to simplify the exposition of the second intertwined distribution transition family $\mathbf{R}$.

9 Strongly continuous conjugate convolution empathy

Consider a pair of homogeneous Markov processes $(\mathbf{X}, \mathbf{Y})$ with a pair of distribution transition functions $(\mathbf{Q}, \mathbf{R})$ defined on distinct state spaces $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$ (distinct copies of $\mathbb{R}$) and intertwined by the backward extended Chapman-Kolmogorov equation (19). In §6 the conjugate extended Riesz representation of the distribution transition functions $(\mathbf{Q}, \mathbf{R})$ is the star-empathy $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ of admissible homomorphisms on the test space Φ_P of vector test functions. The framework of normed admissible homomorphisms on the vector test space Φ_P, which is described next, allows us to consider the analytic condition of the strong continuity of $\bar{\mathfrak{q}}'_P$ and $\bar{\mathfrak{r}}'_P$.

We introduce a vector topology on a closed subspace of Φ_P (see Construction 1). The subspace is the product test space associated with $\Phi_\infty(X) \times \Phi_\infty(\bar{Y})$ where,

$$\Phi_\infty(X) := C[\mathbb{R}_X, \mathbb{C}] \subset \Phi_X, \quad \Phi_\infty(\bar{Y}) := C[\mathbb{R}_{\bar{Y}}, \mathbb{C}] \subset \Phi_{\bar{Y}},$$

are two distinct supremum normed Banach spaces (required for joint consideration of the two distinct state spaces $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$). From this point onward, the symbol Φ_P will denote this closed subspace of Φ_P. Equipped with the product norm, Φ_P is a Banach space such that the translation map $p \in G \mapsto R^p \varphi \in \Phi$ is continuous for each fixed $\varphi \in \Phi_P$.

The lifting, $\theta_0'(X)$, of the point evaluation map $\theta_0' : f \in \Phi_\infty(X) \mapsto f(0)$ is an example of a bounded Φ_P-admissible homomorphism. Here $\theta_0'(X) := \ell_1 \circ \theta_0' \circ \pi_X$, i.e.,

$$\langle \theta_0'(X), \varphi \rangle = (\langle \theta_0', f \rangle, 0) = (f(0), 0) \text{ for all } \varphi = (f, \bar{g}) \in \Phi_P.$$

The class of bounded admissible homomorphisms on Φ_P is denoted by $\mathscr{A}_P$.

Theorem 9 [11, Thm.12] *The bounded admissible homomorphisms $\mathscr{A}_P$ equipped with the product $*$ is a Banach algebra with a unit. Every $Q_P' : \Phi_P \to Z$ bounded homomorphism is Φ_P-admissible and the dualism mapping $\Gamma : Q_P' \mapsto \mathcal{Q}_P$ is an isomorphism into the algebra of linear transformations on Φ_U that preserves the norm: $\|Q_P'\| = \|\mathcal{Q}_P\|$.*

Proof The unit of $\mathscr{A}_P$ is the point evaluational functional $\theta'_{(0,0)} : (f, \bar{g}) \mapsto (f(0), \bar{g}(\bar{0}))$ for every $\varphi := (f, \bar{g}) \in \Phi_P$. The inequality $\|Q_P'\| \leqslant \|\mathcal{Q}_P\|$ follows from the analogue of equation (22) and the product norm on Z. The reverse inequality follows form the fact that the translation maps R^p are isometries for all $p \in G$.

The Φ_P-dualism conjugate transition functions are denoted by

$$\bar{\mathfrak{Q}}_P = \{\bar{\mathcal{Q}}_P(t)\}_{t>0}, \quad \bar{\mathfrak{R}}_P := \{\bar{\mathcal{R}}_P(t)\}_{t>0},$$

where $\bar{\mathcal{Q}}_P(t) = \Gamma(\bar{Q}_P'(t))$ and $\bar{\mathcal{R}}_P(t) = \Gamma(\bar{R}_P'(t))$. From Theorem 8 we obtain the following result.

Proposition 6 *If $\mathbf{Q}$ is an analytic Feller convolution semigroup, then $\bar{\mathfrak{q}}_P'$ is strongly continuous, i.e., the mapping $t \mapsto \bar{Q}_P'(t)$ from $(0, \infty)$ to $\mathbb{C}^2$ is continuous, since*

$$\langle \bar{Q}_P'(t), \varphi \rangle \to \langle \theta_0'(X), \varphi \rangle \text{ as } t \to 0^+ \text{ for all } \varphi \in \Phi_P. \tag{40}$$

Moreover, $\bar{\mathfrak{Q}}_P$ is an operator C_0-semigroup on Φ_P.

Corollary 1 *If $\mathbf{Q}$ is an analytic Feller convolution semigroup, then $(\bar{\mathfrak{q}}_P', \bar{\mathfrak{r}}_P')$ is a strongly continuous empathy. The corresponding operator dualism family $(\bar{\mathfrak{Q}}_P, \bar{\mathfrak{R}}_P)$ is also strongly continuous.*

Proof By Theorem 6, $(\bar{\mathfrak{q}}_P', \bar{\mathfrak{r}}_P')$ is a star-empathy. Thus, as in the proof of [16, Thm. 2], the strong continuity of $\bar{\mathfrak{q}}_P'$ ensures the strong continuity of $\bar{\mathfrak{r}}_P'$ on $(0, \infty)$. By the continuity of the dualism mapping Γ, this also implies the strong continuity of the corresponding dualism families, $\bar{\mathfrak{Q}}_P$ and $\bar{\mathfrak{R}}_P$.

10 Intertwined Laplace Transforms

From this point onward we only consider normed admissible homomorphisms on Φ_P. Specifically, we consider time-dependent members, $\mathfrak{x}'_P := \{X'_P(t) \in \mathscr{A}_P\}_{t>0}$, of $\mathscr{A}_P$ that are integrable near the origin, that is, for every $\alpha > 0$, the Bochner integral $\int_{(0,\alpha)} \langle X'_P(t), \varphi \rangle dt$ exists for every $\varphi \in \Phi_P$. We denote this class by $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_P)$. It will be the new algebraic-analytic setting for the study of double families of intertwined admissible transition distribution functions. This vectorization of $\mathcal{L}^1_{loc}((0,\infty), \mathbb{R})$ replaces the algebra $\mathbb{R}$ naturally with another (convolution) algebra $\mathscr{A}_P$. We use the algebra product $*$ of the convolution algebra $\mathscr{A}_P$ to transfer the scalar convolution theorem as opposed to a bounded bilinear form as in well known vector valued convolution theorems of $\mathcal{L}^1_{loc}((0,\infty), X)$ where X is a Banach space. Thus, the framework of *normed* admissible homomorphisms has a full Laplace transform theory, which we will apply to $\bar{\mathfrak{q}}'_P$ and $\bar{\mathfrak{r}}'_P$ (in §11). We give the relevant details in this section. For full details see [11, §5, 11].

Proposition 7 *Let* $(\mathbf{X}, \mathbf{Y})$ *be a pair of homogeneous Markov processes intertwined by the backward extended Chapman-Kolmogorov equation* (19)*. Then the pair of conjugate* Φ_P*-admissible transition functions* $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ *are integrable near the origin.*

Proof Since $\bar{Q}_t\{dy\}$ and $\bar{R}_t\{dy\}$ are probability measures, $\bar{\mathfrak{q}}'_P$ and $\bar{\mathfrak{r}}'_P$ are uniformly bounded on $(0,\infty)$. Furthermore, by Corollary 1, the strong continuity of $\bar{\mathfrak{q}}'_P$ and $\bar{\mathfrak{r}}'_P$ ensures the existence of the Bochner integrals $\int_{(0,\alpha)} \langle \bar{Q}'_P(t), \varphi \rangle dt$ and $\int_{(0,\alpha)} \langle \bar{R}'_P(t), \varphi \rangle dt$ for all $\alpha > 0$.

The following version of a well-known theorem [9, Thm. 3.8.2, p.85] is central to the study of strong integrability of $\mathfrak{x}'_P$.

Theorem 10 [11, Thm.5] *Suppose that for each* $\varphi \in \Phi_P$*, the function* $t \to \langle X'_P(t), \varphi \rangle$; $t \in (0,\infty)$ *are (strongly) Lebesgue measurable in* Z*. If, for an interval* $I \subset (0,\infty)$*, the Bochner integral*

$$\left\langle \int_I \mathfrak{x}'_P, \varphi \right\rangle = \int_I \langle X'_P(t), \varphi \rangle dt, \tag{41}$$

exists for every $\varphi \in \Phi_P$*, the homomorphism* $\int_I \mathfrak{x}'_P : \Phi_P \to Z$ *is bounded and hence* Φ_P*-admissible.*

We now introduce the Laplace transform of the family $\mathfrak{x}'_P$ formally, for $\varphi \in \Phi_P$ and $\lambda > 0$ by the expression

$$\langle \mathfrak{x}'_P(\lambda), \varphi \rangle = \int_{(0,\infty)} e^{-\lambda t} \langle X'_P(t), \varphi \rangle dt = \int_{(0,\infty)} \langle e^{-\lambda t} X'_P(t), \varphi \rangle dt. \tag{42}$$

The class of all functions for which $\mathfrak{x}'_P(\lambda)$ exists will be denoted by $\mathrm{Lap}(\lambda, \mathscr{A}_P)$. The following result is an immediate consequence of Theorem 10.

Theorem 11 *If for some $\lambda > 0$ the Laplace transform $\mathfrak{x}'_P(\lambda)$ exists, it is bounded and therefore in $\mathscr{A}_P$. Moreover,* $\mathrm{Lap}(\lambda, \mathscr{A}_P) \subset \mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_P)$.

Then, by the argument in the proof of Prop. 7, the Laplace transforms of $\bar{\mathfrak{q}}'_P$ and $\bar{\mathfrak{r}}'_P$ exist for all $\lambda > 0$.

Lemma 3 *For all $\varphi \in \Phi_P$ and all $\lambda > 0$, the Laplace transforms of $\bar{\mathfrak{q}}'_P$ and $\bar{\mathfrak{r}}'_P$,*

$$\langle \bar{\mathfrak{q}}'_P(\lambda), \varphi \rangle := \int_{(0,\infty)} e^{-\lambda t} \langle \bar{Q}'_P(t), \varphi \rangle d\,t\,, \ \ \langle \bar{\mathfrak{r}}'_P(\lambda), \varphi \rangle := \int_{(0,\infty)} e^{-\lambda t} \langle \bar{R}'_P(t), \varphi \rangle dt,$$

respectively, exist as Lebesgue integrals.

By Lemma 3 and Theorem 11, for all $\lambda > 0$, the Laplace transforms $\bar{\mathfrak{q}}'_P(\lambda)$ and $\bar{\mathfrak{r}}'_P(\lambda)$ are bounded admissible homomorphisms on Φ_P.

One of the keys to reducing the proof of the classical scalar Convolution Theorem (see [4, Theorem 11.9B]) in $\mathcal{L}^1_{loc}((0,\infty), \mathbb{R})$ to the proof of the Convolution Theorem in $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_P)$ (see [11, Thm.8]) is to combine the notion of closedness with the notion of $\mathscr{A}_P$-Laplace transformability and closedness. We say a family $\mathfrak{x}'_P$ is closed with respect to $\mathfrak{y}'_P \in Lap(\lambda, \mathscr{A}_P)$ if for each fixed $s > 0$,

$$X'_P(s) * \mathfrak{y}'_P(\lambda) = \int_{(0,\infty)} e^{-\lambda t} (X'_P(s) * Y'_P(t))\, dt. \tag{43}$$

It should be noted, however, that the closedness condition (43) is not as elementary at is might seem. In the present it means that for every $\varphi \in \Phi$ the mapping $t \mapsto \langle X'_P(s) * Y'_P(t), \varphi \rangle$ is strongly measurable in Z and the integral $\int_{(0,\infty)} X'_P(s) * e^{-\lambda \cdot} \mathfrak{y}'_P$ is a Φ_P-admissible bounded homomorphism for each $s > 0$. The following result follows from direct calculations based on from eqs. (28)–(29).

Lemma 4 *Let* $(\mathbf{X}, \mathbf{Y})$ *be a pair of homogeneous Markov processes intertwined by the backward extended Chapman-Kolmogorov equation* (19)*. Then $\bar{\mathfrak{q}}'_P$ is Laplace-closed with respect to itself (see [11, eq. (9) & p. 210]) and $\bar{\mathfrak{r}}'_P$ is Laplace-closed with respect to $\bar{\mathfrak{q}}'_P$: for all $s > 0$ and all $\lambda > 0$,*

$$\begin{cases} \bar{Q}'_P(s) * \displaystyle\int_0^\infty e^{-\lambda t} \bar{Q}'_P(t) dt = \int_0^\infty e^{-\lambda t} \bar{Q}'_P(s) * \bar{Q}'_P(t) dt; & \text{(44a)} \\ \bar{R}'_P(s) * \displaystyle\int_0^\infty e^{-\lambda t} \bar{Q}'_P(t) dt = \int_0^\infty e^{-\lambda t} \bar{R}'_P(s) * \bar{Q}'_P(t) dt. & \text{(44b)} \end{cases}$$

The new framework of $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_P)$ has Laplace transform theorems [11, Thm. 9] that are identical in nature to the well-known Laplace transform theorem of $\mathcal{L}^1_{loc}((0,\infty), \mathbb{R})$. These Laplace transform theorems then enable us to capture the different resolvent equations of semigroups of empathy theory, C_0-semigroups, and n-times integrated semigroups in one formulation uniformly. In $\mathcal{L}^1_{loc}((0,\infty), \mathscr{A}_P)$, the convolution of unlike parameters of the Laplace transforms $\mathfrak{x}'_P(\lambda) * \mathfrak{y}'_P(\mu)$ generates the resolvent equations from the causal relations (see [11, eqs.(11)-(13); (14)-(15)]). Similarly by [11, Thm. 11], we have the following intertwined pseudo-resolvent equations for the double family $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ intertwined by the star-empathy causal relation analogous to classical empathy pseudo-resolvent equations [14, Lem. 2.3].

Theorem 12 *Let* $(\mathbf{X}, \mathbf{Y})$ *be a pair of homogeneous Markov processes intertwined by the backward extended Chapman-Kolmogorov equation* (19). *The conjugate extended Riesz representation on* Φ_P *of* $(\mathbf{Q}, \mathbf{R})$ *satisfies the pseudo-resolvent equations*

$$\begin{cases} \bar{\mathfrak{q}}'_P(\lambda) - \bar{\mathfrak{q}}'_P(\mu) = (\mu - \lambda)\bar{\mathfrak{q}}'_P(\lambda) * \bar{\mathfrak{q}}'_P(\mu); & \text{(45a)} \\ \bar{\mathfrak{r}}'_P(\lambda) - \bar{\mathfrak{r}}'_P(\mu) = (\mu - \lambda)\bar{\mathfrak{r}}'_P(\lambda) * \bar{\mathfrak{q}}'_P(\mu) & \text{(45b)} \end{cases}$$

for all $\lambda, \mu > 0$. *In addition, for all* $t > 0$,

$$\bar{\mathfrak{q}}'_P(\lambda) * \bar{Q}'_P(t) = \bar{Q}'_P(t) * \bar{\mathfrak{q}}'_P(\lambda); \tag{46}$$

$$\bar{\mathfrak{r}}'_P(\lambda) * \bar{Q}'_P(t) = \bar{R}'_P(t) * \bar{\mathfrak{q}}'_P(\lambda). \tag{47}$$

We call the pair of intertwined Laplace transform equations (45) the *star-empathy pseudo resolvent equations* and the equations (45)-(47) the *fundamental star-empathy identities.* As in [11], we call $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ an *intertwined* star pseudo-resolvent.

For $\lambda > 0$, let

$$\bar{\mathcal{Q}}_P(\lambda) := \Gamma(\bar{\mathfrak{q}}'_P(\lambda)), \quad \bar{\mathcal{R}}_P(\lambda) := \Gamma(\bar{\mathfrak{r}}'_P(\lambda))$$

be the dualisms of the Laplace transforms. These dualisms may be seen as "induced" Laplace transforms. In fact, $\bar{\mathcal{Q}}_P(\lambda)$ is the λ-potential operator or resolvent operator defined in [3, §1.3, eq. (1.29)]. By Proposition 12, $\bar{\mathcal{Q}}_P(\lambda)$ and $\bar{\mathcal{R}}_P(\lambda)$ are bounded operators on Φ_P.

Example 3 Let $(\mathbf{Q}, \mathbf{R})$ be as in Proposition 6 with $\mathbf{Q}$ as in Example 2. Then the Feller generator of $\bar{\mathfrak{Q}}'_P$ is defined on $\Delta_P := C^\infty[\mathbb{R}_X, \mathbb{C}] \times \Phi_\infty(\bar{Y})$ and is given by

$$\bar{A}_P\varphi = \left(\tfrac{1}{2}f'' - cf, 0_{\bar{Y}}\right) \text{ for all } \varphi := \left(f, \bar{g}\right) \in \Delta_P. \tag{48}$$

Moreover, for all $\varphi := \left(f, \bar{g}\right) \in \Delta_P$,

$$[\bar{\mathcal{Q}}_P(\lambda)\varphi](x, x) = \int_0^\infty e^{-(\lambda+c)t}[p_t * f](x)dt \text{ for all } x \in \mathbb{R}, \tag{49}$$

where $p_t(y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right)$ is the probability density function of the standard Brownian motion (Example 1).

11 Implicit star Fokker-Planck equation

A classical Fokker-Planck equation is a partial differential equation in the smooth probability transition density function instead of the actual Markov process, which can be discontinuous. In [12], implicit Fokker-Planck equations are derived for a pair of discontinuous intertwined counting processes by means of a pair of intertwined Laplace transforms. The unknowns in these Fokker-Planck equations are operator representations of the transition functions that results from a pair of evolution families. One family evolves in empathy with the second family, which is a semigroup. Since these distribution functions are not homogeneous, the Fokker-Planck equations cannot be formulated directly in terms of the distributions. In the present setting, the homogeneity of the transition functions enables us to overcome this limitation by expressing the Fokker-Planck equation as convolution equations. This convolution is the non-commutative extended Feller convolution introduced in §5. We call such an equation an *implicit star Fokker-Planck equation.*

Let $(\mathbf{X}, \mathbf{Y})$, $(\mathbf{Q}, \mathbf{R})$ and $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ be as in §9 and assume that $\mathbf{X}$ is a defective Brownian motion, as in Example 3. We call $(\mathbf{X}, \mathbf{Y})$ an *intertwined Brownian motion* since there is another homogeneous Markov process $\mathbf{Y}$ in empathy with a convolution semigroup $\mathbf{X}$. For each fixed $\varphi = \big(f, \bar{g}\big) \in \varDelta_P$, define a pair of functions u_P and v_P from $(0, \infty) \times \mathbb{R}$ to $\mathbb{C}^2$ by

$$v_P(t, x) = [\bar{\mathcal{Q}}_P(t)\varphi](x, x), \quad u_P(t, x) = [\bar{\mathcal{R}}_P(t)\varphi](x, x). \tag{50}$$

Thus $v_P(t, x) = (v(t, x), 0)$, where $v(t, x) = [\bar{\mathcal{Q}}_t(X)f](x)$.

The analysis of the convolution semigroup $\mathbf{X}$ in the framework of admissible homomorphisms on a product test space is standard. By eq. (48), v_P satisfies the Fokker-Planck equation

$$\frac{\partial v_P}{\partial t} = \bar{A}_P v_P, \tag{51}$$

i.e., v satisfies the scalar Fokker-Planck equation

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} - cv \tag{52}$$

in $(0, \infty) \times \mathbb{R}_X$. In terms of the original homomorphisms $\bar{Q}'_t(X)$, the second component of eq. (51) is simply $0_{\bar{Y}} = 0_{\bar{Y}}$ and the first component is

$$\frac{\partial}{\partial t}\int_{\mathbb{R}_X} \bar{Q}_t\{dy\} f(x + y) = \frac{1}{2}\int_{\mathbb{R}_X} \bar{Q}_t\{dy\}\Big(\frac{\partial^2}{\partial x^2} - 2c\Big) f(x + y). \tag{53}$$

In terms of the associated probability density function

$$q_c(t, y) = \frac{1}{\sqrt{2\pi t}} \exp\Big(-\frac{y^2}{2t} - ct\Big),$$

equation (53) is

$$\frac{\partial}{\partial t}\int_{\mathbb{R}_X} q_c(t,y)f(x+y)dy = \frac{1}{2}\int_{\mathbb{R}_X} q_c(t,y)\Big(\frac{\partial^2}{\partial x^2} - 2c\Big)f(x+y)dy. \tag{54}$$

In contrast, the analysis of the second family $\mathbf{Y}$, which is not a semigroup, is non-standard. We will now use intertwined Laplace transforms to show that $u_P(t,x)$ satisfies a Fokker-Planck equation in the form of an implicit evolution equation formulated in terms of admissible homomorphisms on a product test space. We do not derive the star implicit evolution equation directly from the intertwined pseudo-resolvent $(\bar{\mathfrak{q}}'_P(\lambda), \bar{\mathfrak{r}}'_P(\lambda))_{\lambda>0}$. Instead, we follow the approach of [11, §12]. By applying the dualism mapping Γ to the fundamental star-empathy identities (45)–(47), we obtain the analogous classical empathy pseudo-resolvent equations [14, Lem. 2.3] for the operator-valued dualisms $(\bar{\mathcal{Q}}_P(\lambda), \bar{\mathcal{R}}_P(\lambda))_{\lambda>0}$:

$$\begin{cases} \bar{\mathcal{Q}}_P(\lambda) - \bar{\mathcal{Q}}_P(\mu) = (\mu-\lambda)\bar{\mathcal{Q}}_P(\lambda)\bar{\mathcal{Q}}_P(\mu) = (\mu-\lambda)\bar{\mathcal{Q}}_P(\mu)\bar{\mathcal{Q}}_P(\lambda); & \text{(55a)} \\ \bar{\mathcal{R}}_P(\lambda) - \bar{\mathcal{R}}_P(\mu) = (\mu-\lambda)\bar{\mathcal{R}}_P(\lambda)\bar{\mathcal{Q}}_P(\mu) = (\mu-\lambda)\bar{\mathcal{R}}_P(\mu)\bar{\mathcal{Q}}_P(\lambda) & \text{(55b)} \end{cases}$$

and

$$\bar{\mathcal{Q}}_P(\lambda)\circ\bar{\mathcal{Q}}_P(t) = \bar{\mathcal{Q}}_P(t)\circ\bar{\mathcal{Q}}_P(\lambda); \tag{56}$$

$$\bar{\mathcal{R}}_P(\lambda)\circ\bar{\mathcal{Q}}_P(t) = \bar{\mathcal{R}}_P(t)\circ\bar{\mathcal{Q}}_P(\lambda). \tag{57}$$

Remark 4 Without assuming d-measurability, the dualisms cannot be considered as Laplace transforms. We say that $\bar{\mathfrak{q}}'_P$ is *dualism-measurable* (d-measurable) if for every $\varphi \in \Phi_P$ the mapping $t \mapsto \bar{\mathcal{Q}}_P(t)\varphi$ is measurable in Φ_P. However, the dualisms satisfy the analogous classical empathy pseudo-resolvent equations [14, Lem. 2.3] by the dualism mapping Γ.

We now assume the invertibility assumption of the classical implicit evolution equation (1). In our setting the invertibility assumption is of the form

$$\bar{\mathcal{R}}_P(\xi) \text{ is invertible for some } \xi > 0. \tag{58}$$

We use the invertibility assumption (58) to define the domains

$$\Delta_X := \bar{\mathcal{Q}}_P(\lambda)[\Phi_P], \quad \Delta_{\bar{Y}} := \bar{\mathcal{R}}_P(\lambda)[\Phi_P]$$

for $\lambda > 0$. These subspaces of Φ_P are independent of the choice of λ since $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ is a star-empathy by Theorem 6. Indeed, by [14, Cor. 2.5], $\bar{\mathcal{R}}_P(\lambda)$ is invertible for all $\lambda > 0$.

The pair of generators A and B of (1) from $\Delta_{\bar{Y}}$ to Φ_P are defined by

$$B = \bar{\mathcal{Q}}_P(\lambda)[\bar{\mathcal{R}}_P(\lambda)]^{-1}, \quad A = \lambda B - [\bar{\mathcal{R}}_P(\lambda)]^{-1}.$$

The generators A and B are independent of λ (see [14, §5]). Note that $B[\Delta_{\bar{Y}}] = \Delta_X$.

It now follows from [14, Thm. 2.8(a), Thm. 5.2] that $A = \bar{A}'_P B$, where $\bar{A}'_P$ is the Feller generator of $\bar{\mathfrak{Q}}_P = \{\bar{\mathcal{Q}}_P(t)\}_{t>0}$, and that for each $\varphi = \big(f, 0_{\bar{Y}}\big) \in \Delta_X \cap \Delta_P$,

$$\begin{cases} \dfrac{\partial}{\partial t}(Bu_P) = Au_P; & \text{(59a)} \\ \lim\limits_{t\to 0^+} Bu_P(t,x) = \varphi(x,x),\ x \in \mathbb{R}. & \text{(59b)} \end{cases}$$

The implicit evolution equation (59) can be expressed in terms of the admissible homomorphisms $A' = \theta'_0(X) \circ A$ and $B' = \theta'_0(X) \circ B$:

$$\begin{cases} \dfrac{d}{dt}\langle B' * \bar{R}'_P(t), \varphi\rangle = \langle A' * \bar{R}'_P(t), \varphi\rangle \text{ for a.e. } t > 0; & \text{(60a)} \\ \lim\limits_{t\to 0^+}\langle B' * \bar{R}'_P(t), \varphi\rangle = \langle \theta'_0(X), \varphi\rangle. & \text{(60b)} \end{cases}$$

In terms of the original homomorphisms $\bar{R}'_t(X)$, the second component of eq. (60a) is simply $0_{\bar{Y}} = 0_{\bar{Y}}$ and the first component is

$$\frac{\partial}{\partial t}\langle B' * \bar{R}'_t(X), f_{-x}\rangle = \langle B' * \bar{R}'_t(X), \frac{1}{2}\Big(\frac{\partial^2}{\partial x^2} - 2c\Big) f_{-x}\rangle \text{ for a.e. } t > 0. \quad (61)$$

Note that $\bar{Q}'_t(X) = B' * \bar{R}'_t(X)$ on $\Delta_X \cap \Delta_P$.

Theorem 13 *Let* $(\mathbf{X}, \mathbf{Y})$ *be a pair of homogeneous Markov processes with distribution transition functions* $(\mathbf{Q}, \mathbf{R})$ *intertwined by the backward extended Chapman-Kolmogorov equation* (19)*. If the invertibility assumption* (58) *holds,* $\mathbf{Q}$ *is an analytic Feller convolution semigroup and the transition functions* $\mathbf{Q}$ *and* $\mathbf{R}$ *have probability density functions, then the Fokker-Planck equation is a star-implicit evolution equation* (60) *in the conjugate extended Riesz representation of* $(\mathbf{Q}, \mathbf{R})$*. In fact,* $\bar{\mathfrak{r}}'_P(\lambda) = (\lambda B' - A')^{-1}$ *for* $\lambda > 0$.

12 Solvability criterion for implicit star Cauchy equation

In the setting of $\mathbb{C}^2$-admissible homomorphisms on a product test space Φ_P, the implicit evolution equation (1) takes the more general form

$$\frac{d}{dt}\langle b' * r'(t), \varphi\rangle = \langle a' * r'(t), \varphi\rangle, \lim_{t\to 0^+}\langle b' * r'(t), \varphi\rangle = \langle \theta'_{(0,0)}, \varphi\rangle, \qquad (62)$$

where $\theta'_{(0,0)}$ is the unit of $(\mathscr{A}_P, *)$; $\langle \theta'_{(0,0)}, \varphi\rangle = \varphi(0)$. For the purpose of making the paper self-contained and reducing the level of abstraction of the general setting of an arbitrary Banach space Z-valued admissible homomorphism, we shall tailor the material presented in [11, §12-§14] for the present setting.

Let lower case letters such as r' denote Φ_P-admissible homomorphisms. We indicate the correspondence between homomorphisms r' and dualisms R by changing from upper to lower-case symbols. We call (62) the *implicit star-Cauchy equation.*

Let $\mathfrak{q}'_\lambda := \{q'(\lambda) \in \mathscr{A}_P\}_{\lambda>0}$ and $\mathfrak{r}'_\lambda := \{r'(\lambda) \in \mathscr{A}_P\}_{\lambda>0}$. Then the pair $\langle \mathfrak{q}'_\lambda, \mathfrak{r}'_\lambda \rangle$ is a *star-empathy pseudo-resolvent* if the pair is intertwined by the star empathy pseudo-resolvent equation

$$\begin{cases} \mathfrak{q}'(\lambda) - \mathfrak{q}'(\mu) = (\mu - \lambda)\mathfrak{q}'(\lambda) * \mathfrak{q}'(\mu); & (63a) \\ \mathfrak{r}'(\lambda) - \mathfrak{r}'(\mu) = (\mu - \lambda)\mathfrak{r}'(\lambda) * \mathfrak{q}'(\mu) & (63b) \end{cases}$$

for all $\lambda, \mu > 0$. The problem at hand is: Given a star-empathy pseudo-resolvent $\langle \mathfrak{q}'(\lambda), \mathfrak{r}'(\lambda) \rangle$, construct a star-empathy $\langle \mathfrak{q}', \mathfrak{r}' \rangle$ such that $\mathfrak{q}' := \{q'(t) \in \mathscr{A}_P\}_{t>0}$ and $\mathfrak{r}' := \{r'(t) \in \mathscr{A}_P\}$ satisfy the implicit star-Cauchy equation (62) with $r'(\lambda) = (\lambda b' - a')^{-1}$.

We do not directly solve the problem from the perspective of the given star-empathy pseudo-resolvent $\langle \mathfrak{q}'_\lambda, \mathfrak{r}'_\lambda \rangle$. Instead, we approach the problem from the perspective of their dualisms $\langle \mathfrak{Q}'_\lambda, \mathfrak{R}'_\lambda \rangle$, where $\mathfrak{Q}'_\lambda := \{\mathcal{Q}(\lambda) = \Gamma q'(\lambda)\}_{\lambda>0}$, and $\mathfrak{R}'_\lambda := \{\mathcal{R}(\lambda) = \Gamma r'(\lambda)\}_{\lambda>0}$. Then we backtrack these dualisms as homomorphisms by Theorem 4. We do not assume d-measurabilty. Instead we apply the dualism mapping Γ to the star empathy pseudo-resolvent equations (63) to transfer to the operator-valued empathy pseudo-resolvent equations

$$\begin{cases} \mathcal{Q}(\lambda) - \mathcal{Q}(\mu) = (\mu - \lambda)\mathcal{Q}(\lambda)\mathcal{Q}(\mu), & (64a) \\ \mathcal{R}(\lambda) - \mathcal{R}(\mu) = (\mu - \lambda)\mathcal{R}(\lambda)\mathcal{Q}(\mu). & (64b) \end{cases}$$

We shall follow the approach of [10], which is based upon Kisyński's approach [2] to the Hille-Yosida theorem.

For the first evolution family $\mathfrak{q}'$ that is associated with the generator a' in (62), we adopt Kisyński's approach to the Hille-Yosida theorem. Here, the well-known Banach convolution algebra $\mathbb{L} = L^1_0(\mathbb{R}^+, \bullet)$ of integrable scalar functions defined on the real line with support in $[0, \infty)$ plays a central role; here $\mathbb{R}^+ := (0, \infty)$ and $\bullet$ denotes the classical convolution. The negative exponentials $e_\lambda(x) = \exp\{-\lambda x\}$, $x > 0$, $\lambda > 0$, the characteristic functions $\chi_{(0,t)}$ and the translation operators are fundamental to this approach. It is important to note that the negative exponentials is a fundamental set of $\mathbb{L}$ and a canonical pseudo-resolvent of the convolution algebra $\mathbb{L}$: $e_\lambda - e_\mu = (\mu - \lambda)e_\lambda \bullet e_\mu$ for $\lambda, \mu > 0$.

Let Y be any Banach space. From an arbitrary pseudoresolvent $\mathfrak{Q}_\lambda := \{Q(\lambda) : Y \to Y; \lambda > 0\}$ that satisfies the strong Widder growth condition [18]

$$\sup\{\|[\lambda Q(\lambda)]^k\| : \lambda > 0; k \in \mathbb{N}\} < \infty, \tag{65}$$

a continuous Banach algebra representation T on $\mathbb{L}$ generates the C_0-semigroup $(E(t))_{t\geqslant 0}$ satisfying the abstract Cauchy problem. We establish this over three steps. We define subspaces of Y by (i) $\Delta_E := Q(\lambda)[Y]$ and (ii) $\Delta_K := \{y \in Y : \lim_{\lambda\to\infty} \|\lambda Q(\lambda)y - y\| = 0\}$. Then the following hold:

1. The regularity space, Δ_K, is closed: $\Delta_K = \overline{\Delta_E}$.

2. The map $T : e_\lambda \mapsto Q(\lambda)$ is a unique (bounded) Banach algebra representation $T : \mathbb{L} \to B(Y)$ such that (a) $T(e_\lambda) = Q(\lambda)$ and (b) T reconstructs the space Δ_K in the following algebraic way:

$$\Delta_K = \bigcup_{\phi \in \mathbb{L}} T(\phi)[Y]. \tag{66}$$

We shall refer to Δ_K as the *T-regularity space*.
3. With this reconstructed Δ_K and map T, a semigroup $E(t)$ on Δ_K is constructed from the right-shift operation $R^t f(x) = f(x-t)$ defined for $f \in \mathbb{L}$. The unique C_0-semigroup $\mathcal{E} := \{E(t)) : t \geqslant 0\}$ on Δ_K (see [2]) is defined as follows: for $y = T(\phi)y_\phi \in \Delta_K$,

$$E(t)y := [T(R^t(\phi))]y_\phi. \tag{67}$$

We let A_E denote the generator of $\mathcal{E}$. Furthermore, the pseudo-resolvent $\mathfrak{Q}_\lambda$ becomes a resolvent: $Q(\lambda) = Q(\lambda, A_E) = (\lambda - A_E)^{-1}$ on Δ_K.

Next, for the construction of the admissible homomorphism a', we lift Kisyński's construction into the framework of admissible homomorphisms where the subspaces need to be translation invariant. First note that by (64a), the dualism family $\mathfrak{Q}'_\lambda$ is a pseudo-resolvent on the translation invariant Banach space Φ_P. If we impose the strong Widder condition (65) on $\mathfrak{Q}'_\lambda$ and define the spaces Δ'_K and Δ'_X analogously to Δ_K and Δ_E by

$$\Delta'_K := \{\varphi \in \Phi_P : \lim_{\lambda \to \infty} \|\lambda \mathcal{Q}(\lambda)\varphi - \varphi\| = 0\}, \tag{68}$$

$$\Delta'_X := \mathcal{Q}(\lambda)[\Phi_P], \tag{69}$$

then we have the following lifting of the Kisyński construction.

Theorem 14 *Suppose that $\mathfrak{Q}_\lambda$ satisfies the Widder growth condition* (65)*. Then the following hold:*
(i) The T-regularity space Δ'_K is a translation-invariant closed subspace of Φ_P.
(ii) For each $\lambda > 0$, $\mathcal{Q}(\lambda)[\Delta'_K] \subset \Delta'_K$ and $\mathcal{Q}(\lambda) = (\lambda - A_{\mathcal{Q}})^{-1}$.
(iii) There exists a unique bounded Banach algebra representation T of a commutative algebra of linear operators on Φ_P such that $T(e_\lambda) = \mathcal{Q}(\lambda)$, which algebraically reconstructs the regularity space $\Delta'_K = \overline{\Delta'_X}$ analogous to (66)*.*
(iv) A strongly continuous semigroup of translatable operators $\mathcal{Q}(t) : \Delta'_K \to \Delta'_K$ is constructed by right shift maps on $\mathbb{L}$ analogous to (67)*: for $t \geqslant 0$ and $\phi \in \mathbb{L}$,*

$$\mathcal{Q}(t)[T(\phi)] = [T(R^t \phi)]. \tag{70}$$

The generator $A_{\mathcal{Q}}$ is closed and translatable. Moreover, $D(A_{\mathcal{Q}})$ is dense in Δ'_K.

We can backtrack the operator valued semigroup $(\mathcal{Q}(t))_{t \geqslant 0}$ into the star-semigroup $\mathfrak{q}' := \{q'(t) \in \mathscr{A}_P\}_{t \geqslant 0}$ by setting

$$q'(t) = \theta'_0 \mathcal{Q}(t). \tag{71}$$

The homomorphism $q'(t)$ is admissible since $\mathcal{Q}(t)$ is translatable. Similarly, the homomorphism $a'_{\mathcal{Q}} := \theta'_0 A_{\mathcal{Q}}$ is admissible. We define the first generator a' by

$$a' := a'_{\mathcal{Q}} * b'. \tag{72}$$

Since the dualism mapping is an isometry (Theorem 7), the Widder growth condition (65) can be translated back to the original homomorphisms $\mathfrak{q}'(\lambda)$ if the notation (borrowed from Feller [6]) ${x'}^{n*} := x' * \cdots * x'$ (n-times) is used. The condition then becomes the strong star-Widder growth condition,

$$\sup_{\lambda>0;k\in\mathbb{N}} \{\|[\lambda \mathfrak{q}'(\lambda)]^{k*}\|\} < \infty. \tag{73}$$

We next construct the appropriate second evolution family $\mathfrak{r}'$ that satisfies the implicit star-Cauchy equation (62). Once again we transfer to the corresponding operator-valued dualisms $\langle \mathcal{Q}(\lambda), \mathcal{R}(\lambda)\rangle$ that satisfy the empathy pseudo-resolvent equations (64). The invertibility of $\mathcal{Q}(\lambda)$ when restricted to Δ'_K (Theorem 14(ii)) plays a central role in the construction of $\mathfrak{r}'$. This inverse will be denoted by $\mathcal{Q}^{-1}(\lambda)$.

We now proceed along the lines of the adaptation of Kisyński's version of the Hille-Yosida theorem to empathy theory. Let X and Y denote two Banach spaces. From an empathy pseudo-resolvent $\langle R, P\rangle := \langle \{R_\lambda : Y \to Y\}, \{P_\lambda : Y \to X\}\rangle$, where R satisfies the strong Widder growth condition (65), a homomorphism T^2 on the same algebra $\mathbb{L}$ generates an empathy $\langle S(t), E(t)\rangle_{t>0}$ satisfying the implicit evolution (1). We establish this in three steps. We begin by defining the subspaces (i) $\Delta^2_K = R(\lambda)[\Delta_K]$ and (ii) $\Delta_S := P(\lambda)[Y]$ and the operator $C = P(\lambda)R^{-1}(\lambda)$.

1. The T^2-regularity space Δ^2_K is an isomorphic dense subspace of Δ_K.
2. The map
$$T^2 := CT \tag{74}$$
is a representation of bounded linear operators from Δ^2_K to X by elements of $\mathbb{L}$. However, T^2 is not necessarily an algebra representation or even closed. However the representation map T^2 has algebraic properties in harmony with empathy theory such as $T^2(e_\lambda) = P_\lambda$ and $T^2(e_\lambda * e_\mu) = P_\lambda R_\mu$, the latter expression being a special case of the identity
$$T^2(e_\lambda * \phi) = P_\lambda T(\phi). \tag{75}$$
3. From Δ^2_K and the map T^2, a family of operators $\{S(t) : \Delta^2_K \to \Delta_S\}_{t>0}$ is constructed by
$$S(t)[R_\lambda T(\phi)] = T^2(e_\lambda * R^t \phi), \tag{76}$$
where we use the representation $R_\lambda T(\phi)$ to generate Δ^2_K.

The pair $\langle S(t), E(t)\rangle_{t>0}$ is an empathy defined on Δ_K. It suffices to define each bounded operator $S(t)$ on the dense space Δ^2_K since there is a unique extension to its closure that is a Banach space. The boundedness of $S(t)$ then follows from the boundedness of each $P(\lambda)$. For full details of the above construction, see [10, §4].

Next, for the construction of the admissible homomorphism b', we lift the above adaptation of Kisyński's construction into the framework of admissible homomorphisms where the subspaces need to be translation invariant. We begin by defining the analogous *translation-invariant* subspaces ${\Delta'}^2_K \subset \Delta'_K$, $\Delta'_{\bar{Y}} \subset \Phi_P$ and the operator $C : {\Delta'}^2_K \to \Delta'_{\bar{Y}}$ by

$$ {\Delta'}^2_K = \mathcal{Q}(\lambda)[\Delta'_K], \tag{77} $$
$$ \Delta'_{\bar{Y}} = \mathcal{R}(\lambda)[\Delta'_K], \tag{78} $$
$$ C = \mathcal{R}(\lambda)\mathcal{Q}^{-1}(\lambda). \tag{79} $$

The definitions (77), (78) and (79) are independent of the choice of λ.

Proposition 8 *Let the semigroup $\{\mathcal{Q}(t)\}_{t\geqslant 0}$ on $\Delta'_K \subset \Phi_P$ be defined as in* (70). *Then $\mathcal{Q}(t) : {\Delta'}^2_K \to {\Delta'}^2_K$. Furthermore, for arbitrary $\phi \in \mathbb{L}$, $T(\phi) : {\Delta'}^2_K \to {\Delta'}^2_K$.*

Analogous to (76), by the representation T^2 we may now define a time dependent family $\mathcal{R}(t) : \Delta^2_K \to \Delta'_{\bar{Y}}$ by

$$ \mathcal{R}(t)[\mathcal{Q}(\lambda)T(\phi)] := T^2\,(e_\lambda * R^t\,\phi), \tag{80} $$

Since $\mathcal{Q}(t)$ and $\mathcal{Q}(\lambda)$ commute, the construct (80) can be re-phrased as

$$ \mathcal{R}(t)\varphi = C\mathcal{Q}(t)\varphi \quad \text{on } \Delta^2_K. \tag{81} $$

This leads to the following result.

Theorem 15 *On Δ^2_K the pair of evolution operators $\langle \mathcal{Q}(t), \mathcal{R}(t)\rangle$ is a strongly continuous empathy that satisfies the initial condition*

$$ \lim_{t\to 0^+} \mathcal{R}(t)\varphi = C\varphi \tag{82} $$

for $\varphi \in \Delta^2_K$.

If more structure is added to the empathy pseudo-resolvent $\langle \mathcal{Q}(\lambda), \mathcal{R}(\lambda)\rangle$ in the form of a dualism $B : \mathscr{D} \supset \Delta'_{\bar{Y}} \to \Delta'_K$ such that

$$ \mathcal{Q}(\lambda) = B\mathcal{R}(\lambda) \text{ for all } \lambda > 0, \tag{83} $$

then we use the empathy $\langle \mathcal{Q}(t), \mathcal{R}(t\rangle$ to solve the operator-valued implicit evolution equation (1) on ${\Delta'}^2_K$. For $\varphi \in {\Delta'}^2_K$, $u(t) = \mathcal{R}(t)\varphi$ satisfies the implicit evolution problem

$$ \tfrac{d}{dt}[Bu(t)] = A_{\mathcal{Q}}Bu(t);\ \lim_{t\to 0^+} Bu(t) = \varphi, \tag{84} $$

since $\mathcal{Q}(t) = B\mathcal{R}(t)$ on ${\Delta'}^2_K$ by (81). For this reason, we call the operator pair $\langle A, B\rangle$ where $A = A_{\mathcal{Q}}B$, the *generator* of the empathy $\langle \mathcal{Q}(t), \mathcal{R}(t)\rangle$.

Similarly, we now backtrack the operator-valued family $(\mathcal{R}(t))_{t\geqslant 0}$ into the family $\mathfrak{r}' := \{r'(t) \in \mathscr{A}_P\}_{t\geqslant 0}$ by setting $r'(t) = \theta_0'\mathcal{R}(t)$ where $\langle \mathfrak{q}', \mathfrak{r}' \rangle$ is a strongly continuous star-empathy. The second family $\mathfrak{r}'$ evolves in empathy with the first family $\mathfrak{q}'$ constructed in (71). The second generator is the admissible homomorphism $b' := \theta_0' B$. With this choice of generators $\langle a', b' \rangle$, the family $\mathfrak{r}'$ solves the star-implicit Cauchy problem (62).

Theorem 16 *Consider the star-empathy pseudo-resolvent $\langle \mathfrak{q}'_\lambda, \mathfrak{r}'_\lambda \rangle$, where $\mathfrak{q}'_\lambda$ satisfies the strong star-Widder growth condition* (73)*. Let $\langle \mathfrak{Q}_\lambda, \mathfrak{R}_\lambda \rangle$ denote the operator-valued dualisms. Let B be a dualism defined on $\mathscr{D} \subset \Phi_P$ such that the operators $\mathcal{Q}(\lambda) := B\mathcal{R}(\lambda)$ are bounded.*

Then we can construct an empathy $\langle \mathfrak{q}', \mathfrak{r}' \rangle$ and a pair of generators $\langle a', b' \rangle$ such that for $\varphi \in {\Delta'}^2_K$, the star implicit evolution equation (62) *is satisfied when one sets the generators $b' := \theta_0' B$ and $a' := \theta_0' A$, where $A = A_{\mathcal{Q}} B$.*

For the final requirement of $\mathfrak{r}'(\lambda) = (\lambda b' - a')^{-1}$, more structure is added to the empathy pseudo-resolvent $\langle \mathcal{Q}(\lambda), \mathcal{R}(\lambda) \rangle$ in the form of the invertibility assumption on $\mathfrak{R}_\lambda$ analogous to eq. (58):

$$\mathcal{R}(\xi) \text{ is invertible for some } \xi > 0. \tag{85}$$

Corollary 2 *Let $\langle \mathfrak{q}'_\lambda, \mathfrak{r}'_\lambda \rangle$ be a star-empathy pseudo-resolvent as in the hypothesis of Theorem 16. Under the additional assumption that $\mathfrak{R}_\lambda$ satisfies the invertibility assumption* (85)*, we can construct an empathy $\langle \mathfrak{q}', \mathfrak{r}' \rangle$ and a pair of generators $\langle a', b' \rangle$ such that for $\varphi \in {\Delta'}^2_K$, the star implicit evolution equation* (62) *is satisfied, where $\mathfrak{r}'(\lambda) = (\lambda b' - a')^{-1}$.*

Proof From $\mathcal{R}(\lambda) = (\lambda B - A)^{-1}$, that is, $\mathcal{R}(\lambda)(\lambda B - A) = (\lambda B - A)\mathcal{R}(\lambda) = \mathbf{1}$, it follows that $\mathfrak{r}'(\lambda) = (\lambda b' - a')^{-1}$ since $\theta_0'(X'Y')(\varphi) = \langle \varphi, x' * y' \rangle$.

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A Dynamic boundary condition for heat equation

A dynamic boundary condition for a partial differential equation is a boundary condition in which the time derivative of the unknown appears. For example, such a boundary condition for the heat equation arises if the boundary of the body is considered as a thin conducting film. With its own thermal property, the boundary thermally interacts with the body in a one-way manner like an absorbing-barrier: the body is an external source for the boundary but the boundary is *not* an external source for the body (see Section A.1).

In this appendix, we give a physical motivation of how the implicit evolution equation (1) originates from such a two-space approach to the heat equation. Full details are in [7]. In particular, we give a heuristic formulation of how non-perfect thermal contact between the body and its boundary results in a pair of coupled heat equations that can be formulated as an implicit evolution equation of the form (1). For a mathematically exact treatment please see [5].

A.1 Dynamic boundary condition

For the body material Ω, let $u(x,t), \overrightarrow{\varphi}(x,t)$ and $f(x,t)$ denote the temperature field, the ambient flux and the source at the internal point x of the body Ω at time t, respectively. For the boundary $\Gamma := \partial\Omega$, the corresponding thermal properties at $X \in \Gamma$ at time t are denoted by $U(X,t), \overrightarrow{\Phi}(X,t)$ and $F(X,t)$.

If Ω is three dimensional, then the boundary Γ is a two-dimensional manifold. Then the traditional choice for the ambient flux $\overrightarrow{\varphi}(x,t) = -k\overrightarrow{\nabla}u(x,t)$ is a three-vector, whereas $\overrightarrow{\Phi}(X,t) = -K\overrightarrow{\nabla}_S U(X,t)$ is a two-vector which lives in the tangent plane of Γ at point X; here $\overrightarrow{\nabla}_S$ denotes the surface gradient.

Applying the law of conservation of energy to the body Γ,

$$Q\tfrac{d}{dt}\int_{\mathcal{B}} U(X,t)dX = -\int_{\partial\mathcal{B}} \overrightarrow{\Phi}(X,t)\cdot\widehat{N}(X)ds + \int_{\mathcal{B}} F(X,t) + \overrightarrow{\varphi}(X,t)\cdot\widehat{n}(X)dX \tag{86}$$

where $\mathcal{B}$ denotes the 'volume' patch of the boundary Γ taken as a body and $Q = 1$ without loss of generality. Since $\mathcal{B}$ is a surface patch, the integral on the left hand side of eq. (86) is a surface integral which we denote by $\int_{\mathcal{B}} dX$.

Unlike $F(X,t)$, which is a per-volume per-time quantity, $\overrightarrow{\Phi}(X,t)$ is a per-surface per-time quantity. Thus the first integral on the right hand side of (86) is a surface integral of (the surface) Γ which we denote by $\int_{\partial\mathcal{B}} ds$. Assuming sufficient smoothness on $\mathcal{B}$, with the traditional choices of flux, we have

$$-\int_{\partial\mathcal{B}} \overrightarrow{\Phi}(X,t)\cdot\widehat{N}(X)ds = \int_{\mathcal{B}} K\Delta_S U(X,t)dX, \tag{87}$$

where $\widehat{N}(X)$ is the unit exterior of the body $\Gamma = \partial\Omega$; $\widehat{n}(X)$ denotes the ambient unit exterior of the original body Ω; Δ_S is the surface Laplacian or the Laplace-Beltrami operator. Thus,

$$\tfrac{\partial}{\partial t}U(X,t) = K\Delta_S U(X,t) + F(X,t) + kD_{\widehat{n}}u(X,t) \tag{88}$$

where $D_{\widehat{n}}u(X,t)$ is the directional derivative of $u(X,t)$ in the direction of the ambient unit exterior $\widehat{n}(X)$.

Similarly applying the law of conservation of energy to the body Ω,

$$q\tfrac{d}{dt}\int_{\mathcal{G}} U(x,t)dx = \int_{\mathcal{G}} k\Delta u(x,t)dx + \int_{\mathcal{G}} f(x,t)dx \tag{89}$$

where $\mathcal{G}$ denotes the volume patch of the body Ω and $q = 1$ without loss of generality. The last integral in eq. (89), in contradistinction to the corresponding last integral of eq. (86), lacks an analogous term to the term $\overrightarrow{\varphi}(X,t)\cdot\widehat{n}(X)$, which captures the body Ω acting as a source internal to Γ. The surface flux $\overrightarrow{\Phi}(X,t)$ lies in the tangent plane at X of the boundary surface Γ and therefore has no internal impact on the body Ω. Therefore Γ is an absorbing barrier in the sense that Γ is not a source for Ω, whereas Ω is a source of Γ. The dynamic boundary condition (88) can then be viewed as *one-way* or *one-term coupled*.

Consequently, the law of conservation of energy for the body Ω remains intact as

$$\tfrac{\partial}{\partial t}u(x,t) = k\Delta u(x,t) + f(x,t), \tag{90}$$

so that the pair $\langle u, U\rangle$ is *intertwined* by a pair of heat equations:

$$\begin{cases} \tfrac{\partial}{\partial t}u(x,t) = k\Delta u(x,t) + f(x,t); x \in \Omega \\ \tfrac{\partial}{\partial t}U(X,t) = K\Delta_S U(X,t) + F(X,t) + kD_{\widehat{n}}u(X,t); X \in \Gamma. \end{cases} \tag{91}$$

Now each point $X \in \Gamma$ will be taken as a 'point of contact' between the two bodies. We introduce a pair of trace operators $\langle\gamma_0, \gamma_1\rangle$ to measure the quality of thermal contact between the two bodies

$$\begin{cases} \gamma_0 u(X) := \lim\limits_{x\to X} u(x); x \in \Omega \\ \gamma_1 u(X) := D_{\widehat{n}}u(X); X \in \Gamma. \end{cases} \tag{92}$$

The pair $\langle \gamma_0, \gamma_1 \rangle$ expresses perfect thermal contact and non-perfect thermal contact, respectively, as

$$U(X) = \gamma_0 u(X), \tag{93}$$

$$U(X) = \gamma u(X), \tag{94}$$

where $\gamma = \gamma_0 - C(X)k\gamma_1$ and $C(X)$ is a positive function. Equation (94) is implicit in the *contact constitutive equation* $k\gamma_1 u(X) = \frac{1}{C(X)}(\gamma_0 u(X) - U(X))$, where $C(X)$ measures the quality of thermal contact.

A.2 Trace-induced implicit evolution equation

Consider the first temperature field $u(x,t)$ of the pair $\langle u, U \rangle$ satisfying the intertwined pair of equations (91). We take $u(x,t)$ as a time evolution $u(t) := u(\cdot,t)$ in an appropriate space, X, of functions in $x \in \Omega$. Likewise, the second *dependent* temperature field $U(x,t)$ (see equation (94)) is a time evolution $U(t) := U(\bullet,t)$ in a distinct space of functions in $X \in \Gamma$.

Although we considered the body and boundary as separate distinct bodies in the dynamic boundary formulation, the coupled system of heat equations (91) shows that the body and boundary are thermally inseparable. Therefore we re-write the system of eq. (91) in vector format,

$$\frac{\partial}{\partial t}\begin{pmatrix} u(x,t) \\ U(X,t) \end{pmatrix} = \begin{pmatrix} k\Delta u(x,t) \\ K\Delta_S U(X,t) + kD_{\hat{n}} u(X,t) \end{pmatrix} + \begin{pmatrix} f(x,t) \\ F(X,t) \end{pmatrix}. \tag{95}$$

Thus the solution to the coupled heat equations (91) is the time evolution vector $\begin{pmatrix} u(\cdot,t) \\ U(\bullet,t) \end{pmatrix} \in Y$ evolving from a pair of initial states $y_0 := \begin{pmatrix} u_0 \\ U_0 \end{pmatrix} \in Y$.

The pair of trace operators $\langle \gamma_0, \gamma_1 \rangle$ enables joint consideration of u and U: in the case of perfect thermal contact (see eq. (93)), the trace operator γ_0 is a transition map from the first body Ω into the second body Γ: γ_0 transitions $u(\cdot,t) \in W_2^1(\Omega)$ into $U(\bullet,t) \in L_2(\Gamma)$; thus, the trace operator γ_0 forces the choice of two distinct spaces $W_2^1(\Omega)$ and $L_2(\Gamma)$ corresponding to the two distinct systems Ω and Γ. We use the same choice of spaces for the case for non-perfect thermal contact (94) since γ is a function of γ_0, γ_1 and $C(X)$. The space $W_2^1(\Omega)$ is the golden mean for the classical heat equation which was preserved in the system (91) by virtue of the absorbing-barrier boundary assumption. So a possible choice for the Banach space X is the associated space of definition, $L_2(\Omega)$. Thus the solution space $Y := L_2(\Omega) \times L_2(\Gamma)$.

The vector format of eq. (95) takes the form

$$\frac{d}{dt}Bu(\cdot,t) = Au(\cdot,t) \quad \text{or more concisely,} \quad \frac{d}{dt}Bu(t) = Au(t), \tag{96}$$

if $f(x,t) = 0 = F(X,t)$. Here $B : X \to Y$ denotes the *intertwined trace operator*

$$1 \otimes \gamma : u(\cdot,t) \in X \mapsto \begin{pmatrix} u(\cdot,t) \\ U(\bullet,t) \end{pmatrix} \in Y$$

and $A : X \to Y$ the *intertwined Laplacian operator*, where B only adds the information $U(\bullet,t) = \gamma u(\cdot,t)$:

$$u(\cdot,t) \in X \mapsto \begin{pmatrix} k\Delta u(\cdot,t) \\ K\Delta_S U(\bullet,t) + kD_{\hat{n}} u(\bullet,t) \end{pmatrix}.$$

Remark 5 The example (see **[15, §8]**) shows that the intertwined trace operator B is non-closeable even in an elementary case of perfect thermal contact, $\gamma = \gamma_0$, in a one dimensional body. Therefore we assume B to be non-closeable and so we study eq. (96) as it stands.

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