VARIETIES OF DE MORGAN MONOIDS: COVERS OF ATOMS

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Abstract. The variety DMM of De Morgan monoids has just four minimal subvarieties. The join-irreducible covers of these atoms in the subvariety lattice of DMM are investigated. One of the two atoms consisting of idempotent algebras has no such cover; the other has just one. The remaining two atoms lack nontrivial idempotent members. They are generated, respectively, by 4–element De Morgan monoids $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$, where $\boldsymbol{C}_4$ is the only nontrivial 0–generated algebra onto which finitely subdirectly irreducible De Morgan monoids may be mapped by non-injective homomorphisms. The homomorphic *pre-images* of $\boldsymbol{C}_4$ within DMM (together with the trivial De Morgan monoids) constitute a proper quasivariety, which is shown to have a largest subvariety U. The covers of the variety $\mathbb{V}(\boldsymbol{C}_4)$ within U are revealed here. There are just ten of them (all finitely generated). In exactly six of these ten varieties, all nontrivial members have $\boldsymbol{C}_4$ as a *retract*. In the varietal join of those six classes, every subquasivariety is a variety—in fact, every finite subdirectly irreducible algebra is projective. Beyond U, all covers of $\mathbb{V}(\boldsymbol{C}_4)$ [or of $\mathbb{V}(\boldsymbol{D}_4)$] within DMM are discriminator varieties. Of these, we identify infinitely many that are finitely generated, and some that are not. We also prove that there are just 68 minimal quasivarieties of De Morgan monoids.

1. Introduction

De Morgan monoids, introduced by Dunn [7, 22], are involutive distributive residuated lattices satisfying $x \leqslant x^2$. The theory of residuated lattices descends from the study of ideal multiplication in rings, and from the calculus of binary relations, but the algebras also model substructural logics; see [11]. In particular, the relevance logic $\mathbf{R^t}$ of Anderson and Belnap [1] is

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algebraized by the variety DMM of De Morgan monoids, provided that the monoid identity e is distinguished in the algebraic signature. From the general theory of algebraization [5], it follows that the axiomatic extensions of $\mathbf{R^t}$ and the subvarieties of DMM form anti-isomorphic lattices, and the latter are susceptible to the methods of universal algebra.

Accordingly, in [23], we initiated a study of the lattice of varieties of De Morgan monoids. Among other results, we proved that this lattice has just four atoms. The idempotent De Morgan monoids (a.k.a. Sugihara monoids) are very well understood and encompass two of the minimal varieties, viz. the class BA of Boolean algebras (whose nontrivial members satisfy $\neg e < e$) and the variety $\mathbb{V}(\boldsymbol{S}_3)$ generated by the 3–element Sugihara monoid (in which $\neg e = e$). The remaining two are generated, respectively, by two 4–element algebras $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$, where $\boldsymbol{C}_4$ is totally ordered (with $e < \neg e$), while e and $\neg e$ are incomparable in $\boldsymbol{D}_4$. We established in [23] that a subvariety of DMM omits $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$ iff it consists of Sugihara monoids.

The present paper is primarily an investigation of the covers of these four atoms within DMM. It suffices to consider the join-irreducible covers, as the subvariety lattice of DMM is distributive. We show that BA has no join-irreducible cover within DMM, and that $\mathbb{V}(\boldsymbol{S}_3)$ has just one; the situation for $\mathbb{V}(\boldsymbol{C}_4)$ and $\mathbb{V}(\boldsymbol{D}_4)$ is much more complex (see Theorem 7.2).

The covers of $\mathbb{V}(\boldsymbol{C}_4)$ are distinctive, in view of a result of Slaney [26]: $\boldsymbol{C}_4$ is the only 0–generated nontrivial algebra onto which finitely subdirectly irreducible De Morgan monoids may be mapped by non-injective homomorphisms. We demonstrate that there is a largest variety U of De Morgan monoids consisting of homomorphic *pre-images* of $\boldsymbol{C}_4$ (along with trivial algebras), as well as a largest subvariety M of DMM such that $\boldsymbol{C}_4$ is a *retract* of every nontrivial member of M. Thus, $\mathbb{V}(\boldsymbol{C}_4) \subseteq \mathsf{M} \subseteq \mathsf{U}$. We furnish U and M with finite equational axiomatizations; each has an undecidable equational theory and uncountably many subvarieties (see Sections 4 and 6). We also provide representation theorems for the members of U and M (Corollaries 5.6 and 5.8), involving a 'skew reflection' construction of Slaney [27].

With the help of these representations, we identify all of the covers of $\mathbb{V}(\boldsymbol{C}_4)$ within U. There are just ten, of which exactly six fall within M (Theorem 8.10, Corollary 8.14). All ten of these varieties are finitely generated.

Within DMM, every cover of $\mathbb{V}(\boldsymbol{D}_4)$ is semisimple—in fact, a discriminator variety. The same applies to the covers of $\mathbb{V}(\boldsymbol{C}_4)$ that are not contained in U. In both cases, we identify infinitely many such covers that are finitely generated, and some that are not even generated by their finite members (see Sections 9 and 10).

In the literature of substructural logics, subvariety lattices are more prominent than subquasivariety lattices, because they mirror the extensions of a logic by new axioms, as opposed to new inference rules. Nevertheless, some natural logical problems call for a consideration of quasivarieties if they are to be approached algebraically, e.g., the identification of the structurally complete axiomatic extensions of $\mathbf{R^t}$. Although this particular question is deferred to a subsequent paper, we throw some fresh light here on the subquasivariety lattice of DMM.

Specifically, each of the four minimal varieties of De Morgan monoids is also minimal as a quasivariety, but they are not alone in this. Indeed, we prove that DMM has just 68 minimal subquasivarieties (Corollary 3.6, Remark 5.9). The proof exploits Slaney's description of the free 0–generated De Morgan monoid in [25]. We show, moreover, that in the varietal join J of the six covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M, every finite subdirectly algebra is projective. It follows that every subquasivariety of J is a variety. (See Theorems 8.12 and 8.13.)

2. Residuated Structures and De Morgan Monoids

Some key definitions and results are recalled briefly below. Unproved assertions in this section were either referenced or proved in [23], where additional citations and/or attributions can be found. Familiarity with [23] is not presupposed, however.

An *involutive (commutative) residuated lattice*, or briefly, an *IRL*, is an algebra $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, e\rangle$ comprising a commutative monoid $\langle A; \cdot, e\rangle$, a lattice $\langle A; \wedge, \vee\rangle$ and a function $\neg\colon A \longrightarrow A$, called an *involution*, such that $\boldsymbol{A}$ satisfies $\neg\neg x = x$ and

$$(1) \qquad x \cdot y \leqslant z \iff \neg z \cdot y \leqslant \neg x.$$

Here, $\leqslant$ denotes the lattice order and $\neg$ binds more strongly than any other operation; we refer to $\cdot$ as *fusion*. It follows that $\neg$ is an anti-automorphism of $\langle A; \wedge, \vee\rangle$, and if we define $x \to y := \neg(x \cdot \neg y)$ and $f := \neg e$, then $\boldsymbol{A}$ satisfies

$$(2) \qquad x \cdot y \leqslant z \iff y \leqslant x \to z \quad \text{(the law of residuation)},$$

$$(3) \qquad \neg x = x \to f, \quad \text{hence} \quad x \cdot \neg x \leqslant f,$$

$$(4) \qquad x \to y = \neg y \to \neg x \quad \text{and} \quad x \cdot y = \neg(x \to \neg y),$$

$$(5) \qquad e \leqslant x = x \cdot x \iff x \cdot \neg x = \neg x \iff x = x \to x.$$

An algebra $\boldsymbol{A} = \langle A; \cdot, \to, \wedge, \vee, e\rangle$ is called a *(commutative) residuated lattice*—or an *RL*—if $\langle A; \cdot, e\rangle$ is a commutative monoid, $\langle A; \wedge, \vee\rangle$ is a lattice and $\to$ is a binary operation—called *residuation*—such that $\boldsymbol{A}$ satisfies (2).

Every RL satisfies the following well known laws. Here and subsequently, $x \leftrightarrow y$ abbreviates $(x \to y) \wedge (y \to x)$.

$$(6) \qquad x \cdot (x \to y) \leqslant y \quad \text{and} \quad x \leqslant (x \to y) \to y$$

$$(7) \qquad x \leqslant y \to z \iff y \leqslant x \to z$$

$$(8) \qquad x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$$

$$(9) \qquad x \to (y \wedge z) = (x \to y) \wedge (x \to z)$$

$$(10) \qquad (x \vee y) \to z = (x \to z) \wedge (y \to z)$$

$$(11) \qquad x \leqslant y \implies \begin{cases} x \cdot z \leqslant y \cdot z \text{ and} \\ z \to x \leqslant z \to y \text{ and } y \to z \leqslant x \to z \end{cases}$$

$$(12) \qquad x \leqslant y \iff e \leqslant x \to y$$

$$x = y \iff e \leqslant x \leftrightarrow y \tag{13}$$

$$e \leqslant x \to x \;\text{ and }\; e \to x = x. \tag{14}$$

By (13), an RL $\boldsymbol{A}$ is nontrivial (i.e., $|A| > 1$) iff e is not its least element. A class of algebras is said to be *nontrivial* if it has a nontrivial member.

An RL $\boldsymbol{A}$ is said to be *bounded* if there are *extrema* $\bot, \top \in A$, by which we mean that $\bot \leqslant a \leqslant \top$ for all $a \in A$. In this case, for each $a \in A$,

$$a \cdot \bot = \bot = \top \to \bot \;\text{ and }\; \bot \to a = \top = a \to \top = \top \cdot \top. \tag{15}$$

If, moreover, $\top \cdot a = \top$ for all $a \in A \backslash \{\bot\}$, then $\boldsymbol{A}$ is said to be *rigorously compact* [21], in which case $a \to \bot = \bot = \top \to b$ for all $a \in A \backslash \{\bot\}$ and $b \in A \backslash \{\top\}$. The extrema of a bounded [I]RL are not distinguished in the algebra's signature, so they are not always retained in subalgebras.

Lemma 2.1. *Let $\boldsymbol{A}$ be a rigorously compact RL, with extrema $\bot, \top$, and let $h \colon \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism that is not a constant function. Then*

(i) $h^{-1}[\{h(\bot)\}] = \{\bot\}$ *and* $h^{-1}[\{h(\top)\}] = \{\top\}$.

(ii) *If $h(\bot)$ is meet-irreducible in $\boldsymbol{B}$, then $\bot$ is meet-irreducible in $\boldsymbol{A}$. Likewise, $\top$ is join-irreducible if $h(\top)$ is.*

(iii) *If $\boldsymbol{B}$ is totally ordered (as a lattice), then $\bot$ is meet-irreducible and $\top$ join-irreducible in $\boldsymbol{A}$.*

Proof. (i) If $\bot < a \in A$, with $h(a) = h(\bot)$, then $\top \cdot a = \top$, by rigorous compactness, so $h(\top) = h(\top) \cdot h(a) = h(\top) \cdot h(\bot) = h(\top \cdot \bot) = h(\bot)$. Similarly, if $\top > b \in A$, with $h(b) = h(\top)$, then $h(\top) = h(\bot)$, because $\top \to b = \bot$. As h is isotone, we conclude in both cases that $|h[A]| = 1$, contradicting the fact that h is not constant.

(ii) follows easily from (i), and (iii) from (ii). ☐

In an RL $\boldsymbol{A}$, we define $x^0 := e$ and $x^{n+1} := x^n \cdot x$ for $n \in \omega$. We say that $\boldsymbol{A}$ is *square-increasing* if it satisfies $x \leqslant x^2$. Every [square-increasing] RL can be embedded into a [square-increasing] IRL, and every finitely generated square-increasing IRL is bounded. The following laws obtain in all square-increasing IRLs:

$$x \wedge y \leqslant x \cdot y \tag{16}$$

$$x, y \leqslant e \implies x \cdot y = x \wedge y \tag{17}$$

$$x \to (x \to y) \leqslant x \to y \tag{18}$$

$$e \leqslant x \vee \neg x \tag{19}$$

$$f \leqslant x \implies x^3 = x^2 \quad \text{(in particular, } f^3 = f^2\text{)}. \tag{20}$$

An RL $\boldsymbol{A}$ is said to be *distributive* [resp. *modular*] if its reduct $\langle A; \wedge, \vee \rangle$ is a distributive [resp. modular] lattice.

Recall that a [quasi]variety is the model class of a set of [quasi-]equations in an algebraic signature. (Quasi-equations have the form

$$(\alpha_1 = \beta_1 \;\&\; \dots \;\&\; \alpha_n = \beta_n) \implies \alpha = \beta,$$

where $n \in \omega$.) The class of all RLs and that of all IRLs are finitely axiomatized varieties. They are congruence distributive, congruence permutable and have the congruence extension property (CEP); see [11] for instance.

In the lemma below, the acronym [F]SI abbreviates '[finitely] subdirectly irreducible'. (In any given algebraic signature, the direct product of an empty family is a trivial algebra, hence SI algebras are nontrivial, as are simple algebras.) Every variety is generated by its SI finitely generated members, as Birkhoff's Subdirect Decomposition Theorem [2, Thm. 3.24] says that every algebra is isomorphic to a subdirect product of SI homomorphic images of itself (and since an equation involves only finitely many variables).

Lemma 2.2. *Let $\boldsymbol{A}$ be a (possibly involutive) RL.*

(i) *$\boldsymbol{A}$ is FSI iff e is join-irreducible in $\langle A; \wedge, \vee\rangle$. In this case, therefore, the subalgebras of $\boldsymbol{A}$ are also FSI.*
(ii) *When $\boldsymbol{A}$ is distributive, it is FSI iff e is join-prime (i.e., whenever $a, b \in A$ with $e \leqslant a \vee b$, then $e \leqslant a$ or $e \leqslant b$).*
(iii) *If there is a largest element strictly below e, then $\boldsymbol{A}$ is SI. The converse holds if $\boldsymbol{A}$ is square-increasing.*
(iv) *If e has just one strict lower bound, then $\boldsymbol{A}$ is simple. The converse holds when $\boldsymbol{A}$ is square-increasing.*

The class of all [F]SI members of a class L of algebras shall be denoted by $\mathsf{L}_{[\mathrm{F}]\mathrm{SI}}$. The class operator symbols $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\mathbb{S}}$ and $\mathbb{P}_{\mathbb{U}}$ stand, respectively, for closure under isomorphic and homomorphic images, subalgebras, direct and subdirect products, and ultraproducts, while $\mathbb{V}$ and $\mathbb{Q}$ denote varietal and quasivarietal generation, i.e., $\mathbb{V} = \mathbb{HSP}$ and $\mathbb{Q} = \mathbb{ISPP}_{\mathbb{U}} = \mathbb{IP}_{\mathbb{S}}\mathbb{SP}_{\mathbb{U}}$. For each class operator $\mathbb{O}$, we abbreviate $\mathbb{O}(\{\boldsymbol{A}_1, \dots, \boldsymbol{A}_n\})$ as $\mathbb{O}(\boldsymbol{A}_1, \dots, \boldsymbol{A}_n)$. Recall that $\mathbb{P}_{\mathbb{U}}(\mathsf{L}) \subseteq \mathbb{I}(\mathsf{L})$ for any finite set L of finite similar algebras [6, Lem. IV.6.5].

Jónsson's Theorem [16, 18] asserts that, for any subclass L of a congruence distributive variety, $\mathbb{V}(\mathsf{L})_{\mathrm{FSI}} \subseteq \mathbb{HSP}_{\mathbb{U}}(\mathsf{L})$. In particular, if L consists of finitely many finite similar algebras, then $\mathbb{V}(\mathsf{L})_{\mathrm{FSI}} \subseteq \mathbb{HS}(\mathsf{L})$, provided that $\mathbb{V}(\mathsf{L})$ is congruence distributive. Note also that $\mathbb{HS}(\mathsf{L}) = \mathbb{SH}(\mathsf{L})$ for any class L of [I]RLs, owing to the CEP.

Corollary 2.3. *Let K be any class of simple square-increasing [I]RLs. Then the variety $\mathbb{V}(\mathsf{K})$ is* semisimple, *i.e., its SI members are simple algebras. In fact, its SI members are just the nontrivial algebras in $\mathbb{ISP}_{\mathbb{U}}(\mathsf{K})$.* [1]

Proof. By Jónsson's Theorem, the SI members of $\mathbb{V}(\mathsf{K})$ belong to $\mathbb{HSP}_{\mathbb{U}}(\mathsf{K})$, but the criterion for simplicity in Lemma 2.2(iv) is first order-definable and therefore persists in ultraproducts (by Łos' Theorem [6, Thm. V.2.9]), while the CEP ensures that nontrivial subalgebras of simple algebras are simple. $\square$

[1] Actually, $\mathbb{V}(\mathsf{K})$ is a discriminator variety, so it consists of Boolean products of simple algebras, in the sense of [6, Sec. IV.8–9]. This stronger conclusion will not be needed here, but it follows because the discriminator varieties are just the congruence permutable semisimple varieties with equationally definable principal congruences (EDPC) [4, 9], and because square-increasing [I]RLs have EDPC [11, Thm. 3.55].

An element a of an [I]RL $\boldsymbol{A}$ is said to be *idempotent* if $a^2 = a$. We say that $\boldsymbol{A}$ is *idempotent* if all of its elements are.

An IRL is said to be *anti-idempotent* if it is square-increasing and satisfies $x \leqslant f^2$ (or equivalently, $\neg(f^2) \leqslant x$). This terminology is justified by Theorem 2.4(iii), which implies that a square-increasing IRL $\boldsymbol{A}$ is anti-idempotent iff $\mathbb{V}(\boldsymbol{A})$ has no nontrivial idempotent member.

Theorem 2.4.

(i) *A square-increasing IRL is idempotent iff it satisfies $f \leqslant e$, iff it satisfies $f^2 = f$. Consequently:*
(ii) *A square-increasing non-idempotent IRL has no idempotent subalgebra (and in particular, no trivial subalgebra).*
(iii) *A variety of square-increasing IRLs has no nontrivial idempotent member iff it satisfies $x \leqslant f^2$ (i.e., it consists of anti-idempotent algebras).*
(iv) *In a simple anti-idempotent IRL $\boldsymbol{A}$, if $e < a \in A$, then $a \cdot f = f^2$.*

Proof. (i)–(iii) were proved in [23, Thm. 3.3 and Cor. 3.6].

(iv) Let $e < a \in A$. By (11), $f = e \cdot f \leqslant a \cdot f$, but by (1), $a \cdot f \nleqslant f$ (since $a \cdot e \nleqslant e$), so $f < a \cdot f$. As $\boldsymbol{A}$ is simple and square-increasing, Lemma 2.2(iv) and involution properties show that f has just one strict upper bound in $\boldsymbol{A}$, which must be f^2, by anti-idempotence. Thus, $a \cdot f = f^2$. □

Definition 2.5. A *De Morgan monoid* is a distributive square-increasing IRL. The variety of De Morgan monoids shall be denoted by DMM.

A De Morgan monoid satisfies $x \leqslant e$ iff it is a Boolean algebra (in which the operation $\wedge$ is duplicated by fusion).

In a partially ordered set $\langle P; \leqslant \rangle$, we denote by $[a)$ the set of all upper bounds of an element a (including a itself), and by $(a]$ the set of all lower bounds. If $a \leqslant b \in P$, we use $[a, b]$ to denote the interval $\{c \in P : a \leqslant c \leqslant b\}$. If $a < b$ and $[a, b] = \{a, b\}$, we say that b *covers* (or is a *cover* of) a.

Theorem 2.6. *Let $\boldsymbol{A}$ be a De Morgan monoid that is FSI. Then*

(i) $A = [e) \cup (f\,]$;
(ii) *if $\boldsymbol{A}$ is bounded, then it is rigorously compact. Consequently,*
(iii) *every finitely generated subalgebra of $\boldsymbol{A}$ is rigorously compact.*

A *Sugihara monoid* is an idempotent De Morgan monoid; see [1, 8, 12, 13, 24]. The variety SM of all Sugihara monoids coincides with $\mathbb{V}(\boldsymbol{S}^*)$ for the algebra

$$\boldsymbol{S}^* = \langle \{a : 0 \neq a \in \mathbb{Z}\}; \cdot, \wedge, \vee, \neg, 1 \rangle$$

on the set of all nonzero integers, where the lattice order is the usual total order, the involution is the usual additive inversion and

$$a \cdot b \;=\; \begin{cases} \text{the element of } \{a, b\} \text{ with the greater absolute value, if } |a| \neq |b|; \\ a \wedge b \;\text{ if } |a| = |b|. \end{cases}$$

An IRL is said to be *odd* if it satisfies $f = e$. By Theorem 2.4(i), every odd De Morgan monoid is a Sugihara monoid. In the odd Sugihara monoid

$\boldsymbol{S} = \langle \mathbb{Z}; \cdot, \wedge, \vee, -, 0\rangle$ on the set of *all* integers, the operations are defined like those of $\boldsymbol{S}^*$, except that 0 takes over from 1 as the neutral element for $\cdot$. The variety of all odd Sugihara monoids is $\mathbb{Q}(\boldsymbol{S})$, whereas $\mathsf{SM} = \mathbb{Q}(\boldsymbol{S}, \boldsymbol{S}^*)$.

For each positive integer n, let $\boldsymbol{S}_{2n}$ denote the subalgebra of $\boldsymbol{S}^*$ with universe $\{-n, \ldots, -1, 1, \ldots, n\}$ and, for $n \in \omega$, let $\boldsymbol{S}_{2n+1}$ be the subalgebra of $\boldsymbol{S}$ with universe $\{-n, \ldots, -1, 0, 1, \ldots, n\}$. Note that $\boldsymbol{S}_2$ is a Boolean algebra. Up to isomorphism, the algebras $\boldsymbol{S}_n$ $(1 < n \in \omega)$ are precisely the finitely generated SI Sugihara monoids, whence the algebras $\boldsymbol{S}_{2n+1}$ $(0 < n \in \omega)$ are just the finitely generated SI odd Sugihara monoids. The algebra $\boldsymbol{S}_3$ is a homomorphic image of $\boldsymbol{S}_n$ for all integers $n \geq 3$. Thus, every nontrivial variety of Sugihara monoids includes $\boldsymbol{S}_2$ or $\boldsymbol{S}_3$.

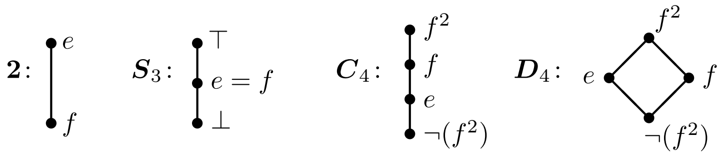
Theorem 2.7.

(i) ([24, 12]) *Every quasivariety of odd Sugihara monoids is a variety.*

(ii) *The lattice of varieties of odd Sugihara monoids is the chain*

$$\mathbb{V}(\boldsymbol{S}_1) \subsetneq \mathbb{V}(\boldsymbol{S}_3) \subsetneq \mathbb{V}(\boldsymbol{S}_5) \subsetneq \ldots \subsetneq \mathbb{V}(\boldsymbol{S}_{2n+1}) \subsetneq \ldots \subsetneq \mathbb{V}(\boldsymbol{S}).$$

An algebra is said to be *n–generated* (where n is a cardinal) if it has a generating subset with at most n elements. Thus, an IRL is 0–generated iff it has no proper subalgebra. We depict below the two-element Boolean algebra $\mathbf{2}$ $(= \boldsymbol{S}_2)$, the three-element Sugihara monoid $\boldsymbol{S}_3$, and two four-element De Morgan monoids, $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$. In each case, the labeled Hasse diagram determines the structure.



Theorem 2.8. *A De Morgan monoid is simple and 0–generated iff it is isomorphic to* $\mathbf{2}$ *or to* $\boldsymbol{C}_4$ *or to* $\boldsymbol{D}_4$.

Lemma 2.9. *Let* $\boldsymbol{A}$ *be a nontrivial square-increasing IRL, and* K *a variety of square-increasing IRLs.*

(i) *If* $\boldsymbol{A}$ *is anti-idempotent, with* $e \leqslant f$, *then* $e < f$.

(ii) *If* $e < f$ *in* $\boldsymbol{A}$, *then* $\boldsymbol{C}_4$ *can be embedded into* $\boldsymbol{A}$.

(iii) *If* $\boldsymbol{A}$ *is simple and* $\boldsymbol{C}_4$ *or* $\boldsymbol{D}_4$ *can be embedded into* $\boldsymbol{A}$, *then* $\boldsymbol{A}$ *is anti-idempotent.*

(iv) *If* $\boldsymbol{C}_4$ *can be embedded into every SI member of* K, *then* K *consists of anti-idempotent algebras and satisfies* $e \leqslant f$.

(v) *If* $\boldsymbol{D}_4$ *can be embedded into every SI member of* K, *then* K *consists of anti-idempotent algebras.*

Proof. (i) and (ii) were proved in [23, Sec. 5].

(iii) follows from Lemma 2.2(iv), because $\neg(f^2) < e$ in $\boldsymbol{C}_4$ and in $\boldsymbol{D}_4$.

(iv) Suppose $\boldsymbol{C}_4$ embeds into every SI member of K. Then K satisfies $e \leqslant f$, as $\boldsymbol{C}_4$ does. Now let $\boldsymbol{B} \in \mathsf{K}$ be nontrivial. Then $\boldsymbol{B} \in \mathbb{IP}_{\mathbb{S}}\{\boldsymbol{B}_i : i \in I\}$

for suitable SI algebras $\boldsymbol{B}_i \in \mathsf{K}$, by the Subdirect Decomposition Theorem. As $\boldsymbol{C}_4$ embeds into each $\boldsymbol{B}_i$, it embeds diagonally into $\prod_{i \in I} \boldsymbol{B}_i$, and therefore into $\boldsymbol{B}$, because it is 0–generated. Thus, no nontrivial $\boldsymbol{B} \in \mathsf{K}$ is idempotent, and so K satisfies $x \leqslant f^2$, by Theorem 2.4(iii).

The proof of (v) is similar. □

The next lemma generalizes [26, Thms. 2, 3] (where it was confined to FSI De Morgan monoids).

Lemma 2.10. *Let $\boldsymbol{A}$ be a rigorously compact IRL.*

(i) *There is at most one homomorphism from $\boldsymbol{A}$ into $\boldsymbol{C}_4$.*
(ii) *If there is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}_4$, then $\neg(f^2) \leqslant a \leqslant f^2$ for all $a \in A$.*

Proof. Let $\bot, \top$ be the extrema of $\boldsymbol{A}$. Suppose $h_1, h_2 \colon \boldsymbol{A} \to \boldsymbol{C}_4$ are homomorphisms, and note that they are surjective, because $\boldsymbol{C}_4$ is 0–generated. For each $i \in \{1, 2\}$, as h_i is isotone and preserves $\cdot, \neg, e$, we have

$$h_i(f^2) = f^2 = h_i(\top) \text{ and } h_i(\neg(f^2)) = \neg(f^2) = h_i(\bot),$$

so by Lemma 2.1(i), $f^2 = \top$ and $\neg(f^2) = \bot$ (proving (ii)) and

$$h_i^{-1}[\{f^2\}] = \{f^2\} \text{ and } h_i^{-1}[\{\neg(f^2)\}] = \{\neg(f^2)\}. \tag{21}$$

Therefore, if $h_1 \neq h_2$, then $h_1(a) = e$ and $h_2(a) = f$ for some $a \in A$. In that case, $h_2(a^2) = f^2$, so $a^2 = f^2$ (by (21)), whence $h_1(a^2) = f^2$, contradicting the fact that $h_1(a^2) = (h_1(a))^2 = e^2 = e$. Thus, $h_1 = h_2$, proving (i). □

Theorem 2.11. (Slaney [26, Thm. 1]) *Let $h \colon \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism, where $\boldsymbol{A}$ is an FSI De Morgan monoid, and $\boldsymbol{B}$ is nontrivial and 0–generated. Then h is an isomorphism or $\boldsymbol{B} \cong \boldsymbol{C}_4$.*

3. Minimality

The following general result will be needed in our study of the subvariety lattice of DMM.

Theorem 3.1. ([17, Cor. 4.1.13]) *If a nontrivial algebra of finite type is finitely generated, then it has a simple homomorphic image.*

A quasivariety is said to be *minimal* if it is nontrivial and has no nontrivial proper subquasivariety. If we say that a variety is *minimal* (without further qualification), we mean that it is nontrivial and has no nontrivial proper subvariety. When we mean instead that it is *minimal as a quasivariety*, we shall say so explicitly, thereby avoiding ambiguity.

Recall that $\mathbb{V}(\mathbf{2})$ is the class of all Boolean algebras.

Theorem 3.2. ([23, Thm. 6.1]) *The distinct classes $\mathbb{V}(\mathbf{2})$, $\mathbb{V}(\boldsymbol{S}_3)$, $\mathbb{V}(\boldsymbol{C}_4)$ and $\mathbb{V}(\boldsymbol{D}_4)$ are precisely the minimal varieties of De Morgan monoids.*

A variety K is said to be *finitely generated* if $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$ for some finite algebra $\boldsymbol{A}$ (or equivalently, $\mathsf{K} = \mathbb{V}(\mathsf{L})$ for some finite set L of finite algebras). Every finitely generated variety is *locally finite*, i.e., its finitely generated members are finite algebras [6, Thm. II.10.16]. Bergman and McKenzie [3] showed that every locally finite congruence modular minimal variety is also minimal as a quasivariety, so by Theorem 3.2, $\mathbb{V}(\mathbf{2})$, $\mathbb{V}(\boldsymbol{S}_3)$, $\mathbb{V}(\boldsymbol{C}_4)$ and $\mathbb{V}(\boldsymbol{D}_4)$ are minimal as quasivarieties. We proceed to show that the total number of minimal subquasivarieties of DMM is still finite, but much greater than four.

Lemma 3.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be nontrivial algebras, where $\boldsymbol{A}$ is 0–generated.*

(i) *If $\boldsymbol{B} \in \mathbb{Q}(\boldsymbol{A})$, then $\boldsymbol{A}$ can be embedded into $\boldsymbol{B}$, whence $\mathbb{Q}(\boldsymbol{A}) = \mathbb{Q}(\boldsymbol{B})$.*
(ii) *$\mathbb{Q}(\boldsymbol{A})$ is a minimal quasivariety.*
(iii) *If $\boldsymbol{B} \in \mathbb{Q}(\boldsymbol{A})$ and $\boldsymbol{B}$ is 0–generated, then $\boldsymbol{A} \cong \boldsymbol{B}$.*
(iv) *If $\boldsymbol{A}$ has finite type and $\mathbb{Q}(\boldsymbol{A})$ is a variety, then $\boldsymbol{A}$ is simple.*

Proof. (i) Let $\boldsymbol{B} \in \mathbb{Q}(\boldsymbol{A}) = \mathbb{ISPP}_{\mathbb{U}}(\boldsymbol{A})$. Then $\boldsymbol{B}$ embeds into a direct product $\boldsymbol{D}$ of ultrapowers of $\boldsymbol{A}$, where the index set of the direct product is not empty (because $\boldsymbol{B}$ is nontrivial). Clearly, if a variable-free equation ε is true in $\boldsymbol{A}$, then it is true in $\boldsymbol{B}$. Conversely, if ε is true in $\boldsymbol{B}$, then it is true in $\boldsymbol{D}$, as variable-free equations persist in extensions (i.e., super-algebras). In that case, since ε persists in homomorphic images, it is true in an ultrapower $\boldsymbol{U}$ of $\boldsymbol{A}$, whence it is true in $\boldsymbol{A}$, because all first order sentences persist in ultraroots. There is therefore a well defined injection $k\colon A \longrightarrow B$, given by

$$\alpha^{\boldsymbol{A}}(c_1^{\boldsymbol{A}}, c_2^{\boldsymbol{A}}, \dots) \;\mapsto\; \alpha^{\boldsymbol{B}}(c_1^{\boldsymbol{B}}, c_2^{\boldsymbol{B}}, \dots),$$

where $c_1, c_2, \dots$ are the nullary operation symbols of the signature and α is any term. Clearly, k is a homomorphism from $\boldsymbol{A}$ into $\boldsymbol{B}$, so $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$.

(ii) follows immediately from (i).

(iii) In the proof of (i), the image of the embedding k is a subalgebra of $\boldsymbol{B}$. So, if $\boldsymbol{B}$ is 0–generated, then k is surjective, i.e., $k\colon \boldsymbol{A} \cong \boldsymbol{B}$.

(iv) Suppose $\boldsymbol{A}$ has finite type and is not simple. As $\boldsymbol{A}$ is 0–generated and nontrivial, it has a simple homomorphic image $\boldsymbol{C}$, by Theorem 3.1, and $\boldsymbol{C}$ is still 0–generated. If $\boldsymbol{C} \in \mathbb{Q}(\boldsymbol{A})$, then $\boldsymbol{A} \cong \boldsymbol{C}$, by (iii), contradicting the non-simplicity of $\boldsymbol{A}$. So, $\boldsymbol{C} \notin \mathbb{Q}(\boldsymbol{A})$, whence $\mathbb{Q}(\boldsymbol{A})$ is not a variety. □

Theorem 3.4. *A quasivariety of De Morgan monoids is minimal iff it is $\mathbb{V}(\boldsymbol{S}_3)$ or $\mathbb{Q}(\boldsymbol{A})$ for some nontrivial 0–generated De Morgan monoid $\boldsymbol{A}$.*

Proof. Sufficiency follows from Lemma 3.3(ii) and previous remarks about $\mathbb{V}(\boldsymbol{S}_3)$. Conversely, let K be a minimal subquasivariety of DMM. Being minimal, K is $\mathbb{Q}(\boldsymbol{A})$ for some nontrivial De Morgan monoid $\boldsymbol{A}$. Let $\boldsymbol{B}$ be the smallest subalgebra of $\boldsymbol{A}$. If $\boldsymbol{B}$ is trivial, then $\boldsymbol{A}$ satisfies $e = f$, so K is a variety, by Theorems 2.4(i) and 2.7(i). In this case, as K is a minimal variety of odd Sugihara monoids, it is $\mathbb{V}(\boldsymbol{S}_3)$, by Theorem 2.7(ii). On the other hand, if $\boldsymbol{B}$ is nontrivial, then $\mathsf{K} = \mathbb{Q}(\boldsymbol{B})$ (again by the minimality of K), and this completes the proof, because $\boldsymbol{B}$ is 0–generated. □

A *deductive filter* of a (possibly involutive) RL $\boldsymbol{A}$ is a lattice filter G of $\langle A; \wedge, \vee\rangle$ that is also a submonoid of $\langle A; \cdot, e\rangle$. Thus, $[e)$ is the smallest deductive filter of $\boldsymbol{A}$. The lattice of deductive filters of $\boldsymbol{A}$ and the congruence lattice $\boldsymbol{Con}\,\boldsymbol{A}$ of $\boldsymbol{A}$ are isomorphic. The isomorphism and its inverse are given by

$$G \mapsto \boldsymbol{\Omega} G := \{\langle a, b\rangle \in A^2 \colon a \leftrightarrow b \in G\};$$
$$\theta \mapsto \{a \in A \colon \langle a \wedge e, e\rangle \in \theta\}.$$

For a deductive filter G of $\boldsymbol{A}$ and $a, b \in A$, we often abbreviate $\boldsymbol{A}/\boldsymbol{\Omega} G$ as $\boldsymbol{A}/G$, and $a/\boldsymbol{\Omega} G$ as a/G, noting that $a \to b \in G$ iff $a/G \leqslant b/G$ in $\boldsymbol{A}/G$. In the square-increasing case, the deductive filters of $\boldsymbol{A}$ are just the lattice filters of $\langle A; \wedge, \vee\rangle$ that contain e, by (16), so $[b)$ is a deductive filter whenever $e \geqslant b \in A$, and if $\boldsymbol{A}$ is finite, then all of its deductive filters have this form.

For any quasivariety K and any cardinal m, the free m–generated algebra in K shall be denoted by $\boldsymbol{F}_{\mathsf{K}}(m)$ if it exists (i.e., if $m > 0$ or the signature of K includes a constant symbol).

Theorem 3.5. *The minimal subquasivarieties of* DMM *form a finite set, whose cardinality is the number of lower bounds of* e *in* $\boldsymbol{F}_{\mathsf{DMM}}(0)$.

Proof. Let $\boldsymbol{F} = \boldsymbol{F}_{\mathsf{DMM}}(0)$. Slaney [25] proved that $\boldsymbol{F}$ has just 3088 elements; its bottom element is $e^{\boldsymbol{F}} \leftrightarrow f^{\boldsymbol{F}}$ (see [23, Thm. 3.2]). By the Homomorphism Theorem, every 0–generated De Morgan monoid is isomorphic to a factor algebra of $\boldsymbol{F}$, so DMM has only finitely many minimal subquasivarieties, by Theorem 3.4.

Now consider a factor algebra $\boldsymbol{F}/G$, where G is a deductive filter of $\boldsymbol{F}$. As $\boldsymbol{F}$ is finite, $G = [\alpha^{\boldsymbol{F}})$ for some nullary term α in the language of IRLs, where $\alpha^{\boldsymbol{F}} \leqslant e^{\boldsymbol{F}}$. If $\boldsymbol{F}/G$ is nontrivial, i.e., $\alpha^{\boldsymbol{F}} \nleqslant e^{\boldsymbol{F}} \leftrightarrow f^{\boldsymbol{F}}$, then $\boldsymbol{F}/G$ is not odd (by (13)), whence $\mathbb{Q}(\boldsymbol{F}/G) \neq \mathbb{V}(\boldsymbol{S}_3)$. The function $\alpha^{\boldsymbol{F}} \mapsto \mathbb{Q}(\boldsymbol{F}/[\alpha^{\boldsymbol{F}}))$ is therefore a well defined surjection from the lower bounds of $e^{\boldsymbol{F}}$ in $\boldsymbol{F}$ to the set consisting of the trivial subvariety (corresponding to the bottom element of $\boldsymbol{F}$) and the minimal subquasivarieties of DMM, *other* than $\mathbb{V}(\boldsymbol{S}_3)$. It remains only to show that this map is injective. To that end, suppose $\boldsymbol{F}/[\alpha^{\boldsymbol{F}})$ and $\boldsymbol{F}/[\beta^{\boldsymbol{F}})$ generate the same quasivariety, where $\alpha^{\boldsymbol{F}}, \beta^{\boldsymbol{F}} \leqslant e^{\boldsymbol{F}}$. Then there is an isomorphism $g \colon \boldsymbol{F}/[\alpha^{\boldsymbol{F}}) \cong \boldsymbol{F}/[\beta^{\boldsymbol{F}})$, by Lemma 3.3(iii). As $\beta^{\boldsymbol{F}} \leqslant e^{\boldsymbol{F}}$, we have $\beta^{\boldsymbol{F}} \leftrightarrow e^{\boldsymbol{F}} = \beta^{\boldsymbol{F}} \in [\beta^{\boldsymbol{F}})$, by (12) and (14), so $\beta^{\boldsymbol{F}/[\beta^{\boldsymbol{F}})} = e^{\boldsymbol{F}/[\beta^{\boldsymbol{F}})}$. Now

$$g(\beta^{\boldsymbol{F}}/[\alpha^{\boldsymbol{F}})) = g(\beta^{\boldsymbol{F}/[\alpha^{\boldsymbol{F}})}) = \beta^{\boldsymbol{F}/[\beta^{\boldsymbol{F}})} = e^{\boldsymbol{F}/[\beta^{\boldsymbol{F}})} = g(e^{\boldsymbol{F}}/[\alpha^{\boldsymbol{F}})),$$

but g is injective, so $\beta^{\boldsymbol{F}}/[\alpha^{\boldsymbol{F}}) = e^{\boldsymbol{F}}/[\alpha^{\boldsymbol{F}})$, i.e., $\beta^{\boldsymbol{F}} = \beta^{\boldsymbol{F}} \leftrightarrow e^{\boldsymbol{F}} \in [\alpha^{\boldsymbol{F}})$. This means that $\alpha^{\boldsymbol{F}} \leqslant \beta^{\boldsymbol{F}}$ and, by symmetry, $\alpha^{\boldsymbol{F}} = \beta^{\boldsymbol{F}}$, completing the proof. ☐

Corollary 3.6. *There are exactly* 68 *minimal quasivarieties of De Morgan monoids.*

Proof. By Theorem 3.5, we need to show that e has just 68 lower bounds in $\boldsymbol{F}_{\mathsf{DMM}}(0)$. The argument will be given in Remark 5.9, after the notion of a 'skew reflection' has been defined. ☐

4. Crystalline Varieties

We begin this section with some general observations about retracts, that will be needed later.

Recall that an algebra $\boldsymbol{A}$ is said to be a *retract* of an algebra $\boldsymbol{B}$ if there are homomorphisms $g\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ and $h\colon \boldsymbol{B} \longrightarrow \boldsymbol{A}$ such that $h \circ g$ is the identity function id_A on A. This forces g to be injective and h surjective; we refer to h as a *retraction* (of $\boldsymbol{B}$ onto $\boldsymbol{A}$). The composite of two retractions, when defined, is clearly still a retraction.

Remark 4.1. Given similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, the first canonical projection $\pi_1\colon \boldsymbol{A} \times \boldsymbol{B} \longrightarrow \boldsymbol{A}$ is a retraction iff there exists a homomorphism $f\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$. (Sufficiency: as id_A and f are homomorphisms, so is the function g from $\boldsymbol{A}$ to $\boldsymbol{A} \times \boldsymbol{B}$ defined by $a \mapsto \langle a, f(a)\rangle$, and clearly $\pi_1 \circ g = \mathrm{id}_A$.) Consequently, if an algebra $\boldsymbol{C}$ is a retract of every member of a class K, then $\boldsymbol{D}$ is a retract of $\boldsymbol{D} \times \boldsymbol{E}$ for all $\boldsymbol{D}, \boldsymbol{E} \in \mathsf{K}$, because there is always a composite homomorphism from $\boldsymbol{D}$ to $\boldsymbol{E}$ (whose image is isomorphic to $\boldsymbol{C}$).

Remark 4.2. A 0–generated algebra $\boldsymbol{A}$ is a retract of an algebra $\boldsymbol{B}$ if there exist homomorphisms $g\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ and $h\colon \boldsymbol{B} \longrightarrow \boldsymbol{A}$. For in this case, every element of A has the form $\alpha^{\boldsymbol{A}}(c_1, \dots, c_n)$ for some term α and some *distinguished* elements $c_i \in A$, whence $h \circ g = \mathrm{id}_A$, because homomorphisms preserve distinguished elements (and respect terms).

Lemma 4.3. *Let K be a variety of finite type, and let $\boldsymbol{A} \in \mathsf{K}$ be finite, simple and 0–generated. Then the following conditions are equivalent.*

(i) *$\boldsymbol{A}$ is a retract of every nontrivial member of K.*
(ii) *Every simple algebra in K is isomorphic to $\boldsymbol{A}$ and embeds into every nontrivial member of K.*

Proof. (i) $\Rightarrow$ (ii): For each simple $\boldsymbol{C} \in \mathsf{K}$, there is a homomorphism h from $\boldsymbol{C}$ onto $\boldsymbol{A}$, by (i), and h must be an isomorphism (as $\boldsymbol{A}$ is nontrivial and $\boldsymbol{C}$ is simple). Thus, the embedding claim also follows from (i).

(ii) $\Rightarrow$ (i): By (ii) and Theorem 3.1, $\boldsymbol{A}$ is a homomorphic image of every finitely generated nontrivial member of K. Consider an arbitrary nontrivial algebra $\boldsymbol{B} \in \mathsf{K}$. By (ii), $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$. Like any nontrivial algebra, $\boldsymbol{B}$ embeds into an ultraproduct $\boldsymbol{U}$ of finitely generated nontrivial subalgebras $\boldsymbol{B}_i$ of $\boldsymbol{B}$ (cf. [6, Thm. V.2.14]). As $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B}_i)$ for all i, and as $\mathbb{P}_{\mathbb{U}}\mathbb{H}(\mathsf{L}) \subseteq \mathbb{HP}_{\mathbb{U}}(\mathsf{L})$ for any class L of similar algebras, there is a homomorphism h from $\boldsymbol{U}$ onto an ultrapower of $\boldsymbol{A}$. But $\boldsymbol{A}$, being finite, is isomorphic to all of its ultrapowers, so h restricts to a homomorphism from $\boldsymbol{B}$ into $\boldsymbol{A}$. Therefore, $\boldsymbol{A}$ is a retract of $\boldsymbol{B}$, by Remark 4.2. □

Generalizing the usage of [26], we say that an IRL $\boldsymbol{A}$ is *crystalline* if there is a homomorphism $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{C}_4$ (in which case h is surjective).[2] Theorem 2.11 motivates the following definitions.

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[2] For the sake of Theorem 4.5, we have dropped the requirement in [26] that crystalline algebras be FSI.

Definition 4.4.

(i) $\mathsf{W} := \{\boldsymbol{A} \in \mathsf{DMM} : |A| = 1 \text{ or } \boldsymbol{A} \text{ is crystalline}\}$;

(ii) $\mathsf{N} := \{\boldsymbol{A} \in \mathsf{DMM} : |A| = 1 \text{ or } \boldsymbol{C}_4 \text{ is a retract of } \boldsymbol{A}\} \subseteq \mathsf{W}$.

By Lemma 2.10(ii), the rigorously compact algebras in W are anti-idempotent. Also, $\boldsymbol{A}$ is a retract of $\boldsymbol{A} \times \boldsymbol{B}$ for all nontrivial $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{N}$, by Remark 4.1.

Theorem 4.5. W *and* N *are quasivarieties.*

Proof. As W and N are isomorphically closed, we must show that they are closed under $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_{\mathbb{U}}$, bearing Remark 4.2 in mind. If $\boldsymbol{B} \in \mathbb{S}(\boldsymbol{A})$ and $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{C}_4$ is a homomorphism, then so is $h|_B\colon \boldsymbol{B} \longrightarrow \boldsymbol{C}_4$, while any embedding $\boldsymbol{C}_4 \longrightarrow \boldsymbol{A}$ maps into $\boldsymbol{B}$, as $\boldsymbol{C}_4$ is 0–generated. Thus, W and N are closed under $\mathbb{S}$. Let $\{\boldsymbol{A}_i \colon i \in I\}$ be a subfamily of W, where, without loss of generality, $I \neq \emptyset$. For any $j \in I$, the projection $\prod_{i \in I} \boldsymbol{A}_i \longrightarrow \boldsymbol{A}_j$ can be composed with a homomorphism $\boldsymbol{A}_j \longrightarrow \boldsymbol{C}_4$, so $\prod_{i \in I} \boldsymbol{A}_i \in \mathsf{W}$. If, moreover, $\boldsymbol{A}_i \in \mathsf{N}$ for all i, then $\boldsymbol{C}_4$ embeds diagonally into $\prod_{i \in I} \boldsymbol{A}_i$, whence $\prod_{i \in I} \boldsymbol{A}_i \in \mathsf{N}$. Every ultraproduct of $\{\boldsymbol{A}_i \colon i \in I\}$ can be mapped into $\boldsymbol{C}_4$, as in the proof of Lemma 4.3 ((ii) $\Rightarrow$ (i)). Also, as $\boldsymbol{C}_4$ is finite and of finite type, the property of having a subalgebra isomorphic to $\boldsymbol{C}_4$ is first order-definable and therefore persists in ultraproducts. Thus, W and N are closed under $\mathbb{P}$ and $\mathbb{P}_{\mathbb{U}}$. $\square$

Nevertheless, W and N are not varieties, i.e., they are not closed under $\mathbb{H}$. To see this, consider any simple De Morgan monoid $\boldsymbol{A}$ of which $\boldsymbol{C}_4$ is a proper subalgebra, and let $\boldsymbol{B} = \boldsymbol{C}_4 \times \boldsymbol{A}$. Then $\boldsymbol{B} \in \mathsf{N}$, by Remark 4.1. Now $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B})$ but $\boldsymbol{A} \notin \mathsf{W}$, because $\boldsymbol{A}$ is simple and not isomorphic to $\boldsymbol{C}_4$. Concrete examples of finite simple 1–generated De Morgan monoids having $\boldsymbol{C}_4$ as sole proper subalgebra are given in Section 9.

An [I]RL is said to be *semilinear* if it is a subdirect product of totally ordered algebras. The semilinear De Morgan monoids are axiomatized, relative to DMM, by $e \leqslant (x \to y) \vee (y \to x)$ [15]. The examples in Section 9 show that even the semilinear anti-idempotent algebras in W or N do not form a variety. Note that N contains (semilinear) algebras that are not anti-idempotent. For instance, $\boldsymbol{C}_4 \times \boldsymbol{C}_4^{\#} \in \mathsf{N}$ does not satisfy $x \leqslant f^2$, where $\boldsymbol{C}_4^{\#}$ denotes the rigorously compact extension of $\boldsymbol{C}_4$ by new extrema $\bot, \top$.

As W and N are not varieties, it is not obvious that either of them possesses a largest subvariety, but we shall show that both do. Purely equational axioms will be needed in the proof, and the opaque postulate (24), which abbreviates an equation, is introduced below for that reason. The following convention helps to eliminate some burdensome notation.

Convention 4.6. In an anti-idempotent IRL, we define

$$\mathit{1} := f^2 \quad \text{and} \quad \mathit{0} := \neg \mathit{1} = \neg(f^2).$$

(These abbreviations will be used when they enhance readability, rather than always. The typeface distinguishes them from standard uses of $0, 1$.)

Definition 4.7. We denote by U the variety of De Morgan monoids satisfying

(22) $x^2 \vee (\neg x)^2 = \mathit{1}$

(23) $\mathit{1} \to (x \vee y) \leqslant (\mathit{1} \to x) \vee (\mathit{1} \to y)$

(24) $\mathit{1} \cdot x \cdot y \cdot q(x) \cdot q(y) \leqslant q(x \cdot y) \wedge q(x \vee y) \wedge q(x \to y) \wedge (\mathit{1} \cdot (x \to y))$,

where $q(x) := \mathit{1} \to (\neg x)^2$. (Note that U consists of anti-idempotent algebras, by (22), so our use of the symbol $\mathit{1}$ in this definition is justified.)

Lemma 4.8. *Every rigorously compact member of* W *belongs to* U.

Proof. Let $\boldsymbol{A} \in \mathsf{W}$ be rigorously compact. We may assume that $\boldsymbol{A}$ is nontrivial, so there is a (surjective) homomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}_4$. Because $\boldsymbol{C}_4$ satisfies (22),

$$[\mathit{1} \to (x \vee y)] \to [(\mathit{1} \to x) \vee (\mathit{1} \to y)] = \mathit{1} \quad \text{and}$$
$$[\mathit{1} \cdot x \cdot y \cdot q(x) \cdot q(y)] \to [q(x \cdot y) \wedge q(x \vee y) \wedge q(x \to y) \wedge (\mathit{1} \cdot (x \to y))] = \mathit{1},$$

it follows from Lemma 2.1(i) that $\boldsymbol{A}$ satisfies the same laws. Then $\boldsymbol{A}$ satisfies (23) and (24), by (12). Thus, $\boldsymbol{A} \in \mathsf{U}$. □

Theorem 4.9. U *is the largest subvariety of* W, *i.e.,* U *is the largest variety of crystalline (or trivial) De Morgan monoids.*

Proof. To see that $\mathsf{U} \subseteq \mathsf{W}$, let $\boldsymbol{A} \in \mathsf{U}$ be SI. It suffices to show that $\boldsymbol{A} \in \mathsf{W}$, because W, like any quasivariety, is closed under $\mathbb{IP}_{\mathbb{S}}$. Now $\boldsymbol{A}$ is nontrivial and bounded by $\mathit{0}, \mathit{1}$, so $\mathit{0} < e \leqslant \mathit{1}$ and $\boldsymbol{A}$ is rigorously compact, by Theorem 2.6(ii). It follows from (23), Lemma 2.2(ii) and (12) that $\mathit{1}$ is join-irreducible (whence $\mathit{0}$ is meet-irreducible) in $\boldsymbol{A}$. Let

$$B = \{a \in A : a \neq \mathit{0} \text{ and } (\neg a)^2 = \mathit{1}\} \text{ and } B' = \{a \in A : \neg a \in B\}.$$

Then $e \in B$ (by definition of $\mathit{1}$) and $\mathit{1} \notin B$ (as $\mathit{0}^2 = \mathit{0} \neq \mathit{1}$), so $e < \mathit{1}$.

We claim that B is closed under the operations $\cdot, \to, \wedge, \vee$ of $\boldsymbol{A}$. Indeed, let $b, c \in B$, so $\mathit{0} < b, c \in A$ and $(\neg b)^2 = \mathit{1} = (\neg c)^2$, i.e., $q(b) = \mathit{1} = q(c)$. Then $b \wedge c \neq \mathit{0}$ and $(\neg(b \wedge c))^2 = \mathit{1}$ (because $(\neg(b \wedge c))^2 \geqslant (\neg b)^2$), so $b \wedge c \in B$. Clearly, $b \vee c \neq \mathit{0}$. Also, $b \cdot c \neq \mathit{0}$, by (16), and $\mathit{1} \cdot b \cdot c \cdot q(b) \cdot q(c) = \mathit{1}$, by rigorous compactness. Then, by (24), each of $q(b \cdot c), q(b \vee c), q(b \to c)$ and $\mathit{1} \cdot (b \to c)$ is $\mathit{1}$. Thus, $\mathit{1} = (\neg(b \cdot c))^2 = (\neg(b \vee c))^2 = (\neg(b \to c))^2$, again by rigorous compactness, and $b \to c \neq \mathit{0}$. This shows that $b \cdot c$, $b \vee c$, $b \to c \in B$, as claimed.

Let $a \in A \backslash \{\mathit{0}, \mathit{1}\}$. Since $\mathit{1}$ is join-irreducible, (22) shows that $a \in B$ or $\neg a \in B$, i.e., $a \in B \cup B'$. Suppose $a \in B \cap B'$, i.e., $a, \neg a \in B$. Then $\neg a \to a = \neg((\neg a)^2) = \neg \mathit{1}$ (as $a \in B$) $= \mathit{0}$, so $(\neg a \to a)^2 = \mathit{0} \neq \mathit{1}$, so $\neg a \to a \notin B$, contradicting the fact that B is closed under $\to$. Therefore, A is the disjoint union of B, B', $\{\mathit{0}\}$ and $\{\mathit{1}\}$.

Suppose $b, c \in B$, with $\neg c \leqslant b$. Then $b \neq \mathit{1}$, so $\neg b \neq \mathit{0}$ and $b^2 \geqslant (\neg c)^2 = \mathit{1}$, so $b^2 = \mathit{1}$, hence $\neg b \in B$, i.e., $b \in B \cap B' = \emptyset$, a contradiction. Thus, no element of B has a lower bound in B'. This, together with the meet- [resp.

join-] irreducibility of 0 [resp. 1], shows that $b \wedge d \in B$ and $b \vee d \in B'$ for all $b \in B$ and $d \in B'$.

Let $h\colon A \longrightarrow C_4$ be the function such that $h(0) = 0$ and $h(1) = 1$ and $h(b) = e$ and $h(\neg b) = f$ for all $b \in B$. It follows readily from the above conclusions that h is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}_4$, so $\boldsymbol{A} \in \mathsf{W}$, as required.

Finally, let K be a subvariety of W. The finitely generated SI algebras in K are rigorously compact, by Theorem 2.6(iii), so they belong to U, by Lemma 4.8. Thus, $\mathsf{K} \subseteq \mathsf{U}$. □

Remark 4.10. In $\boldsymbol{C}_4$, we have $f \to a = 0$ iff $a \in \{0, e\}$, while $a \to e = 0$ iff $a \in \{f, 1\}$. Therefore, $\boldsymbol{C}_4$ satisfies $(f \to x) \vee (x \to e) \neq 0$, and hence also

$$((f \to x) \vee (x \to e)) \to 0 = 0. \tag{25}$$

So, because every SI homomorphic image of a member of U is rigorously compact and crystalline, it follows from Lemma 2.1(i) that U satisfies (25). Note that N and W do not satisfy (25), as (25) fails in the algebra $\boldsymbol{C}_4 \times \boldsymbol{C}_4^{\#}$ mentioned before Convention 4.6.

Definition 4.11. We denote by M the variety of anti-idempotent De Morgan monoids satisfying $e \leqslant f$ and (25).

Lemma 4.12. *$\boldsymbol{C}_4$ is a retract of every nontrivial member of* M.

Proof. Because M satisfies $e \leqslant f$, it also satisfies

$$x \leqslant f \cdot x, \tag{26}$$

and therefore

$$e \leqslant x \implies f \vee x \leqslant f \cdot x. \tag{27}$$

As M satisfies (25) and $0 \to 0 = 1$, its nontrivial members satisfy

$$(f \to x) \vee (x \to e) \neq 0, \quad \text{i.e.,} \quad \neg(f \cdot \neg x) \vee \neg(f \cdot x) \neq 0,$$

or equivalently (by De Morgan's laws),

$$(f \cdot x) \wedge (f \cdot \neg x) \neq 1. \tag{28}$$

By Lemma 2.9(i),(ii), every nontrivial member of M satisfies $e < f$ and has a subalgebra isomorphic to $\boldsymbol{C}_4$. So, by Lemma 4.3, it suffices to show that every simple member of M is isomorphic to $\boldsymbol{C}_4$. Suppose $\boldsymbol{A} \in \mathsf{M}$ is simple. We may assume that $\boldsymbol{C}_4 \in \mathbb{S}(\boldsymbol{A})$.

We claim that the intervals $[0, e]$, $[e, f]$ and $[f, 1]$ of $\boldsymbol{A}$ are doubletons, i.e.,

$$[0, e] = \{0, e\} \text{ and } [e, f] = \{e, f\} \text{ and } [f, 1] = \{f, 1\}. \tag{29}$$

The first and third assertions in (29) follow from Lemma 2.2(iv) and involution properties. To prove the middle equation, suppose $a \in A$ with $e < a < f$. As $f = \neg e$, it follows that $e < \neg a < f$ and, by (27), $f = f \vee a \leqslant f \cdot a$. As $e \cdot a \nleqslant e$, we have $f \cdot a \nleqslant f$ (by (1)), so $f < f \cdot a$. Then $f \cdot a = 1$, as $[f, 1] = \{f, 1\}$. By symmetry, $f \cdot \neg a = 1$, so $(f \cdot a) \wedge (f \cdot \neg a) = 1$, contradicting (28). Therefore, $[e, f] = \{e, f\}$, as claimed.

To complete the proof, it suffices to show that every element of $\boldsymbol{A}$ is comparable with e, as that will imply, by involution properties, that every element is comparable with f, forcing $A = \{0, e, f, 1\} = C_4$.

Suppose, on the contrary, that $a \in A$ is incomparable with e, i.e., $\neg a$ is incomparable with f. As $a \nleqslant e$ and $\neg a \nleqslant f$, we have $e < e \vee a$ and $f < f \vee \neg a$, as well as $e \leqslant \neg a$ (by Theorem 2.6(i)), i.e., $a \leqslant f$. So, by (27), $f \vee \neg a \leqslant f \cdot \neg a$, hence $f < f \cdot \neg a$, and so $f \cdot \neg a = 1$, because $[f, 1] = \{f, 1\}$.

Again, as $e \cdot a \nleqslant e$, we have $f \cdot a \nleqslant f$, so $e \leqslant f \cdot a$, by Theorem 2.6(i). This, with $e < f$, gives $e \leqslant f \wedge (f \cdot a)$. Also, $a \leqslant f \cdot a$, by (26), so $a \leqslant f \wedge (f \cdot a)$. Therefore, $e \vee a \leqslant f \wedge (f \cdot a)$.

If we can argue that $f \wedge (f \cdot a) < f$, then $e < e \vee a < f$, contradicting the fact that $[e, f] = \{e, f\}$. So, to finish the proof, it suffices to show that f is incomparable with $f \cdot a$, and we have already shown that $f \cdot a \nleqslant f$. If $f < f \cdot a$, then $f \cdot a = 1$, as $[f, 1] = \{f, 1\}$, but since $f \cdot \neg a = 1$, this yields $(f \cdot a) \wedge (f \cdot \neg a) = 1$, contradicting (28). Therefore, f and $f \cdot a$ are indeed incomparable, as required. □

Theorem 4.13. M *is the largest subvariety of* N.

Proof. By Lemma 4.12, M is a subvariety of N. Let K be any subvariety of N. Clearly, K satisfies $e \leqslant f$ and, by Lemma 2.9(iv), its members are anti-idempotent. Now K is a subvariety of U, by Theorem 4.9, because $\mathsf{N} \subseteq \mathsf{W}$. By Remark 4.10, (25) is satisfied by U, so it holds in K. Thus, $\mathsf{K} \subseteq \mathsf{M}$. □

Corollary 4.14. M *is the class of all algebras in* U *satisfying* $e \leqslant f$. *In particular,* M *satisfies* (22), (23) *and* (24).

Corollary 4.15. *Every rigorously compact algebra in* N *belongs to* M.

Proof. This follows from Lemma 4.8 and Corollary 4.14. □

At this point in our account, N and M are organizational tools, suggested by Theorem 2.11. They will assume an additional significance when we discuss structural completeness in a subsequent paper. (The structurally complete varieties of De Morgan monoids fall into two classes—a denumerable family that is fully understood, and a more opaque family of subvarieties of M.)

5. Skew Reflections and U

We are going to provide a representation theorem for algebras in U, using ideas of Slaney [27].[3]

Definition 5.1. Let $\boldsymbol{B} = \langle B; \cdot^{\boldsymbol{B}}, \to^{\boldsymbol{B}}, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, e\rangle$ be a square-increasing RL, with lattice order $\leqslant^{\boldsymbol{B}}$. Let $B' = \{b' : b \in B\}$ be a disjoint copy of the set B, let $0, 1$ be distinct non-elements of $B \cup B'$, and let $S = B \cup B' \cup \{0, 1\}$. Let $\leqslant$ be a binary relation on S such that

[3] The nomenclature of [27] is untypical. There, 'De Morgan monoids' were not required to be distributive, and likewise the 'Dunn monoids' of Definition 5.5.

(i) $\leqslant$ is a lattice order whose restriction to B^2 is $\leqslant^{\boldsymbol{B}}$

(the meet and join operations of $\langle S; \leqslant \rangle$ being denoted by $\wedge$ and $\vee$, respectively), and for all $b, c \in B$,

(ii) $b' \leqslant c'$ iff $c \leqslant b$,
(iii) $b \leqslant c'$ iff $e \leqslant (b \cdot^{\boldsymbol{B}} c)'$,
(iv) $b' \nleqslant c$,
(v) $\mathit{0} \leqslant b \leqslant \mathit{1}$ and $\mathit{0} \leqslant b' \leqslant \mathit{1}$.

The *skew $\leqslant$–reflection* $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ *of* $\boldsymbol{B}$ is the algebra $\langle S; \cdot, \wedge, \vee, \neg, e \rangle$ such that

(vi) $\cdot$ is a commutative binary operation on S, extending $\cdot^{\boldsymbol{B}}$,
(vii) $a \cdot \mathit{0} = \mathit{0}$ for all $a \in S$, and if $\mathit{0} \neq a \in S$, then $a \cdot \mathit{1} = \mathit{1}$,
(viii) $b \cdot c' = (b \to^{\boldsymbol{B}} c)'$ and $b' \cdot c' = \mathit{1}$ for all $b, c \in B$,
(ix) $\neg \mathit{0} = \mathit{1}$ and $\neg \mathit{1} = \mathit{0}$ and $\neg b = b'$ and $\neg(b') = b$ for all $b \in B$.

A *skew reflection of* $\boldsymbol{B}$ is any algebra of the form $\mathrm{S}^{\leqslant}(\boldsymbol{B})$, where $\leqslant$ is a binary relation on S satisfying (i)–(v). (Some examples are pictured before Lemma 8.6.)

Definition 5.1 is essentially due to Slaney [27]. (In [27], (iii) is formulated in an ostensibly more general manner, as

$$\text{for all } a, b, c \in B, \text{ we have } a \cdot^{\boldsymbol{B}} b \leqslant c' \text{ iff } a \leqslant (b \cdot^{\boldsymbol{B}} c)'.$$

This follows from (iii), however. Indeed, for $a, b, c \in B$,

$$a \cdot^{\boldsymbol{B}} b \leqslant c' \text{ iff } e \leqslant ((a \cdot^{\boldsymbol{B}} b) \cdot^{\boldsymbol{B}} c)' = (a \cdot^{\boldsymbol{B}} (b \cdot^{\boldsymbol{B}} c))' \text{ iff } a \leqslant (b \cdot^{\boldsymbol{B}} c)'.)$$

By an *RL–subreduct* of an IRL $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, e \rangle$, we mean a subalgebra of the RL–reduct $\langle A; \cdot, \to, \wedge, \vee, e \rangle$ of $\boldsymbol{A}$.

Theorem 5.2. ([27, Fact 1]) *A skew reflection* $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ *of a square-increasing RL* $\boldsymbol{B}$ *is a square-increasing IRL, and* $\boldsymbol{B}$ *is an RL–subreduct of* $\mathrm{S}^{\leqslant}(\boldsymbol{B})$.

Remark 5.3. In a skew reflection $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ of a square-increasing RL $\boldsymbol{B}$, we have $f = e'$, hence $f^2 = \mathit{1}$, so $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ is anti-idempotent and our use of $\mathit{0}, \mathit{1}$ in Definition 5.1 is consistent with Convention 4.6. By definition, $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ is rigorously compact. Because it has $\boldsymbol{B}$ as an RL–subreduct, $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ satisfies $(f \to x) \vee (x \to e) \neq \mathit{0}$, and hence also (25). It satisfies (22) and (24) as well. (In verifying (24), we may assume that its left-hand side is not $\mathit{0}$, whence $x, y, q(x), q(y) \neq \mathit{0}$. This forces $x, y \in B$, whence each conjunct of the right-hand side is $\mathit{1}$.) The fact that elements of B lack lower bounds in B' has two easy but important consequences. First,

$$\mathrm{S}^{\leqslant}(\boldsymbol{B}) \text{ is simple iff } \boldsymbol{B} \text{ is trivial (i.e., } e \text{ is the least element of } \boldsymbol{B}),$$

in view of Lemma 2.2(iv). Secondly, by Lemma 2.2(iii),

$$\mathrm{S}^{\leqslant}(\boldsymbol{B}) \text{ is SI iff } \boldsymbol{B} \text{ is SI or trivial.}$$

Specifically, when $\boldsymbol{B}$ is not trivial, an element of $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ is the greatest strict lower bound of e in $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ iff it is the greatest strict lower bound of e in $\boldsymbol{B}$.

Elements of B might lack upper bounds in B', e.g., $\boldsymbol{D}_4$ arises in this way from a trivial RL. Such cases are eliminated in the next theorem, however.

Theorem 5.4. *The following two conditions on a square-increasing IRL $\boldsymbol{A}$ are equivalent.*

(i) *There is a homomorphism $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{C}_4$ and $\boldsymbol{A}$ is rigorously compact.*
(ii) *$\boldsymbol{A}$ is a skew reflection of a square-increasing RL $\boldsymbol{B}$, and $\mathit{0}$ is meet-irreducible in $\boldsymbol{A}$.*

In this case, in the notation of Definition 5.1,

(iii) *h is unique and surjective, and $\mathit{1}$ is join-irreducible in $\boldsymbol{A}$;*
(iv) *$b \wedge c' \in B$ and $b \vee c' \in B'$ for all $b, c \in B$, so each element of B has an upper bound in B', and elements of B' have lower bounds in B;*
(v) *if $\boldsymbol{B}$ is distributive and $\boldsymbol{A}$ is modular, then $\boldsymbol{A}$ is distributive and therefore a De Morgan monoid, belonging to U.*

Proof. Note first that, in (iii), the uniqueness of h follows from Lemma 2.10(i) (and its surjectivity from the fact that $\boldsymbol{C}_4$ is 0–generated).

(i) ⇒ (ii): Being crystalline, $\boldsymbol{A}$ is nontrivial. The set $B := h^{-1}[\{e\}]$ is the universe of an RL–subreduct $\boldsymbol{B}$ of $\boldsymbol{A}$, which inherits the square-increasing law, and $b \mapsto b' := \neg b$ defines an antitone bijection from B onto $B' := h^{-1}[\{f\}]$. Clearly, $B \cap B' = \emptyset$ and no element of B' is a lower bound of an element of B, because h is isotone and $e < f$ in $\boldsymbol{C}_4$. As h fixes $\mathit{0}$ and $\mathit{1}$, Lemma 2.1 shows that $\boldsymbol{A}$ is anti-idempotent, with $h^{-1}[\{\mathit{0}\}] = \{\mathit{0}\}$ and $h^{-1}[\{\mathit{1}\}] = \{\mathit{1}\}$, and that $\mathit{0}$ [resp. $\mathit{1}$] is meet- [resp. join-] irreducible in $\boldsymbol{A}$, finishing the proof of (iii). In particular, $A = B \cup B' \cup \{\mathit{0}\} \cup \{\mathit{1}\}$ (disjointly).

We verify that $\boldsymbol{A}$ satisfies conditions (iii) and (viii) of Definition 5.1. Let $b, c \in B$. Because B is closed under the operation $\cdot$ of $\boldsymbol{A}$, we have

$$b \leqslant c' \text{ iff } b \cdot e \leqslant \neg c \text{ iff } b \cdot c \leqslant f \text{ (by (1), deployed in } \boldsymbol{A}\text{), iff } e \leqslant (b \cdot c)'.$$

Clearly, $b \cdot c' = (b \to c)'$ and $h(b' \cdot c') = \neg h(b) \cdot \neg h(c) = f^2 = \mathit{1}$, so $b' \cdot c' = \mathit{1}$. This completes the proof that $\boldsymbol{A} = \mathrm{S}^{\leqslant}(\boldsymbol{B})$, where $\leqslant$ is the lattice order of $\boldsymbol{A}$.

(ii) ⇒ (i): Rigorous compactness was noted in Remark 5.3. Definition 5.1 shows that $\cdot, \neg$ and e are preserved by the function $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{C}_4$ such that $h(\mathit{0}) = \mathit{0}$, $h(\mathit{1}) = \mathit{1}$, $h(b) = e$ and $h(b') = f$ for all $b \in B$. As $\mathit{0}$ is meet-irreducible (whence $\mathit{1}$ is join-irreducible) in $\boldsymbol{A}$, the map h preserves $\wedge, \vee$ too. Indeed, if $b, c \in B$, then $b \geqslant b \wedge c' \neq \mathit{0}$ and b has no lower bound in B', so $b \wedge c' \in B$ and, by involution properties, $b \vee c' \in B'$. This proves (i) and (iv).

By (iv), when $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ is modular, it will be distributive iff the five-element lattice with three atoms doesn't embed into the sublattice $B \cup B'$ of $\mathrm{S}^{\leqslant}(\boldsymbol{B})$; see [6, Thms. I.3.5, I.3.6]. That is true if $\boldsymbol{B}$ is distributive, as B and B' are then distributive sublattices of $B \cup B'$. This, with Lemma 4.8, proves (v). □

Definition 5.5. A *Dunn monoid* is a square-increasing distributive RL.

Dunn monoids originate in [7] and acquired their name in [19].

Corollary 5.6. *A De Morgan monoid belongs to U iff it is isomorphic to a subdirect product of skew reflections of Dunn monoids, where $\mathit{0}$ is meet-irreducible in each subdirect factor.*

Proof. The forward implication follows from Theorem 5.4 and the Subdirect Decomposition Theorem, because the SI homomorphic images of members of U are bounded by $0, 1$, are rigorously compact (Theorem 2.6(ii)) and are still crystalline (U being a variety), and because RL–subreducts of De Morgan monoids inherit distributivity. Conversely, by Remark 5.3, skew reflections of Dunn monoids satisfy the defining postulates of U, except possibly for (23) and distributivity (which are effectively given here), and U, like any quasivariety, is closed under $\mathbb{IP}_{\mathbb{S}}$. □

Lemma 5.7. *Let $\boldsymbol{A} = \mathrm{S}^{\leqslant}(\boldsymbol{B})$ be a skew reflection of a square-increasing RL $\boldsymbol{B}$, where $\boldsymbol{A}$ satisfies $e \leqslant f$. Then, in the notation of Definition* 5.1,

(i) $b \leqslant (b \to e)'$ *for all* $b \in B$, *and*
(ii) 0 *is meet-irreducible and* 1 *is join irreducible in* $\boldsymbol{A}$.

Proof. (i) Let $b \in B$. By (6), $b \cdot (b \to e) \leqslant e$, so $e \leqslant f \leqslant (b \cdot (b \to e))'$. Then $b \leqslant (b \to e)'$, by Definition 5.1(iii).

(ii) Let $b, c \in B$. By (i), $c \leqslant (c \to e)'$, i.e., $c \to e \leqslant c'$. Because $\boldsymbol{B}$ is an RL–subreduct of $\boldsymbol{A}$ and $0 \notin B$, we have $b \wedge c' \geqslant b \wedge (c \to e) \in B$, so $b \wedge c' \neq 0$. As B and B' are both sublattices of $\boldsymbol{A}$, this shows that 0 is meet-irreducible (whence 1 is join-irreducible) in $\boldsymbol{A}$. □

Corollary 5.8. *A De Morgan monoid belongs to M iff it satisfies $e \leqslant f$ and is isomorphic to a subdirect product of skew reflections of Dunn monoids.*

Proof. This follows from Lemma 5.7(ii) and Corollaries 4.14 and 5.6. □

Remark 5.9. We can now complete the proof of Corollary 3.6. In [25], Slaney showed that the free 0–generated De Morgan monoid $\boldsymbol{F}$ is $\mathbf{2} \times \boldsymbol{D}_4 \times \boldsymbol{A}$, where $\boldsymbol{A}$ is a skew reflection of the direct product of four Dunn monoids, called 'the α segments of CA6, CA10a, CA10b and CA14'. In each of $\mathbf{2}$, $\boldsymbol{D}_4$ and the four 'α segments', e has just one strict lower bound. So, from the structure of skew reflections, it follows that the number of lower bounds of $e^{\boldsymbol{F}}$ in $\boldsymbol{F}$ (including $e^{\boldsymbol{F}}$ itself) is $2 \times 2 \times ((2 \times 2 \times 2 \times 2) + 1) = 68$.

6. Reflections and M

Definition 6.1. Let $\boldsymbol{B}$ be a square-increasing RL, with lattice order $\leqslant^{\boldsymbol{B}}$, and let $S = B \cup B' \cup \{0, 1\}$, where $B' = \{b' : b \in B\}$ is a disjoint copy of B and $0, 1$ are distinct non-elements of $B \cup B'$. Let $\leqslant$ be the unique partial order of S whose restriction to B^2 is $\leqslant^{\boldsymbol{B}}$, such that

$$b \leqslant c' \text{ for all } b, c \in B$$

and conditions (ii), (iv) and (v) of Definition 5.1 hold. As (i) and (iii) obviously hold too, we may define the *reflection* $\mathrm{R}(\boldsymbol{B})$ of $\boldsymbol{B}$ to be the resulting skew reflection $\mathrm{S}^{\leqslant}(\boldsymbol{B})$. This definition is essentially due to Meyer; see [20] or [1, pp. 371–373].

By Theorem 5.2, every Dunn monoid $\boldsymbol{B}$ is an RL–subreduct of its reflection $\mathrm{R}(\boldsymbol{B})$, and $\mathrm{R}(\boldsymbol{B})$ satisfies $e \leqslant f$ (by definition) and is distributive (as $\boldsymbol{B}$ is), so $\mathrm{R}(\boldsymbol{B}) \in \mathsf{M}$, by Corollary 5.8. Conversely, the RL–reduct of an algebra from M is of course a Dunn monoid, whence so are its subalgebras. This justifies a variant of the 'Crystallization Fact' of [26, p. 124]:

Theorem 6.2. *The variety of Dunn monoids coincides with the class of all RL–subreducts of members of* M.

Corollary 6.3. *The equational theory of* M *is undecidable.*

Proof. This follows from Theorem 6.2, because Urquhart [28, p. 1070] proved that the equational theory of Dunn monoids is undecidable. □

Corollary 6.4. M *is not generated (as a variety) by its finite members.*

Proof. This follows from Corollary 6.3, as M is finitely axiomatized. □

Clearly, in the statements of Theorem 6.2 and Corollary 6.3, we may replace M by any variety K such that $\mathsf{M} \subseteq \mathsf{K} \subseteq \mathsf{DMM}$. The same applies to Corollary 6.4 if K is also finitely axiomatized. In particular, the variety U is not generated by its finite members.

The notational conventions of Definition 5.1 are assumed in the next lemma.

Lemma 6.5. *Let* $\boldsymbol{B}$ *be a Dunn monoid.*

(i) *If* $\boldsymbol{C}$ *is a subalgebra of* $\boldsymbol{B}$, *then* $C \cup \{c' \colon c \in C\} \cup \{\mathit{0}, \mathit{1}\}$ *is the universe of a subalgebra of* $\mathrm{R}(\boldsymbol{B})$ *that is isomorphic to* $\mathrm{R}(\boldsymbol{C})$, *and every subalgebra of* $\mathrm{R}(\boldsymbol{B})$ *arises in this way from a subalgebra of* $\boldsymbol{B}$.

(ii) *If* θ *is a congruence of* $\boldsymbol{B}$, *then*

$$\mathrm{R}(\theta) := \theta \cup \{\langle a', b'\rangle \colon \langle a, b\rangle \in \theta\} \cup \{\langle \mathit{0}, \mathit{0}\rangle, \langle \mathit{1}, \mathit{1}\rangle\}$$

is a congruence of $\mathrm{R}(\boldsymbol{B})$, *and* $\mathrm{R}(\boldsymbol{B})/\mathrm{R}(\theta) \cong \mathrm{R}(\boldsymbol{B}/\theta)$. *Also, every proper congruence of* $\mathrm{R}(\boldsymbol{B})$ *has the form* $\mathrm{R}(\theta)$ *for some* $\theta \in \mathit{Con}\,\boldsymbol{B}$.

(iii) *If* $\{\boldsymbol{B}_i \colon i \in I\}$ *is a family of Dunn monoids and* $\mathcal{U}$ *is an ultrafilter over* I, *then* $\prod_{i \in I} \mathrm{R}(\boldsymbol{B}_i)/\mathcal{U} \cong \mathrm{R}\big(\prod_{i \in I} \boldsymbol{B}_i/\mathcal{U}\big)$.

Proof. The first assertions in (i) and (ii) are straightforward. For the final assertions, one shows that if $\boldsymbol{D}$ is a subalgebra and φ a proper congruence of $\mathrm{R}(\boldsymbol{B})$, then $\boldsymbol{D}$ is the reflection of the subalgebra of $\boldsymbol{B}$ on $D \cap B$, while $\varphi = \mathrm{R}(B^2 \cap \varphi)$. To see that $\varphi \subseteq \mathrm{R}(B^2 \cap \varphi)$, observe that if φ identifies a with b' $(a, b \in B)$, and therefore a' with b, it must identify $\mathit{1} = a' \cdot b'$ with $b \cdot a \in B$. But this contradicts Lemma 2.1(i), because $\mathrm{R}(\boldsymbol{B})$ is rigorously compact.

(iii) For each $i \in I$, let $\mathit{0}_i$ and $\mathit{1}_i$ denote the extrema of $\mathrm{R}(\boldsymbol{B}_i)$ and, for convenience, define $\overline{\mathit{0}}_i = \{\mathit{0}_i\}$ and $\overline{\mathit{1}}_i = \{\mathit{1}_i\}$ and $(B')_i = B_i'$. By $\mathit{0}, \mathit{1}$, we mean (for the moment) the extrema of $\mathrm{R}\big(\prod_{i \in I} \boldsymbol{B}_i/\mathcal{U}\big)$. Consider $x \in \prod_{i \in I} \mathrm{R}(B_i)$. As $\mathcal{U}$ is an ultrafilter, there is a unique $F(x) \in \{B, B', \overline{\mathit{0}}, \overline{\mathit{1}}\}$ such that

$$\{i \in I : \, x(i) \in F(x)_i\} \in \mathcal{U}$$

(see [6, Cor. IV.3.13(a)]). If $F(x)$ is $\overline{0}$ [resp. $\overline{1}$], define $h(x)$ to be 0 [resp. 1]. If $F(x) = B$, define $h(x) = z/\mathcal{U}$, where $z \in \prod_{i \in I} B_i$ and, for each $i \in I$,

$$z(i) \;=\; \begin{cases} x(i) & \text{if } x(i) \in B_i; \\ e^{\boldsymbol{B}_i} & \text{otherwise.} \end{cases}$$

If $F(x) = B'$, define $h(x) = (z/\mathcal{U})'$, where $z \in \prod_{i \in I} B_i$ and, for each $i \in I$,

$$z(i) \;=\; \begin{cases} \text{the unique } b \in B_i \text{ such that } x(i) = b', \text{ if this exists;} \\ e^{\boldsymbol{B}_i}, \text{ otherwise.} \end{cases}$$

Then h is a homomorphism from $\prod_{i \in I} \mathrm{R}(\boldsymbol{B}_i)$ onto $\mathrm{R}\big(\prod_{i \in I} \boldsymbol{B}_i/\mathcal{U}\big)$, whose kernel is $\{\langle x, y\rangle \in \big(\prod_{i \in I} \mathrm{R}(B_i)\big)^2 : \{i \in I : x(i) = y(i)\} \in \mathcal{U}\}$, so the result follows from the Homomorphism Theorem. ☐

Definition 6.6. Given a variety K of Dunn monoids, the *reflection* $\mathbb{R}(\mathsf{K})$ of K is the subvariety $\mathbb{V}\{\mathrm{R}(\boldsymbol{B}) : \boldsymbol{B} \in \mathsf{K}\}$ of M.

As a function from the lattice of varieties of Dunn monoids to the subvariety lattice of M, the operator $\mathbb{R}$ is obviously isotone.

Lemma 6.7. $\mathbb{R}$ *is order-reflecting and therefore injective.*

Proof. Let $\mathbb{R}(\mathsf{K}) \subseteq \mathbb{R}(\mathsf{L})$, where K and L are varieties of Dunn monoids. We must show that $\mathsf{K} \subseteq \mathsf{L}$. Let $\boldsymbol{A} \in \mathsf{K}$ be SI. It suffices to show that $\boldsymbol{A} \in \mathsf{L}$. By assumption, $\mathrm{R}(\boldsymbol{A}) \in \mathbb{R}(\mathsf{L})$. Also, $\mathrm{R}(\boldsymbol{A})$ is SI (because $\boldsymbol{A}$ is), so by Jónsson's Theorem, $\mathrm{R}(\boldsymbol{A}) \in \mathbb{HSP}_{\mathbb{U}}\{\mathrm{R}(\boldsymbol{B}) : \boldsymbol{B} \in \mathsf{L}\}$. Because L is closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}_{\mathbb{U}}$, it follows from Lemma 6.5 that $\mathrm{R}(\boldsymbol{A}) \cong \mathrm{R}(\boldsymbol{B})$ for some $\boldsymbol{B} \in \mathsf{L}$, whence $\boldsymbol{A} \cong \boldsymbol{B}$, and so $\boldsymbol{A} \in \mathsf{L}$. ☐

A *Brouwerian algebra* is an RL satisfying $x \cdot y = x \wedge y$, or equivalently, a Dunn monoid satisfying $x \leqslant e$. Every variety of countable type has at most $2^{\aleph_0}$ subvarieties, and it is known that there are $2^{\aleph_0}$ distinct varieties of Brouwerian algebras [29]. So, the injectivity of $\mathbb{R}$ in Lemma 6.7 yields the following conclusion.

Theorem 6.8. *The variety* M *has* $2^{\aleph_0}$ *distinct subvarieties.*

7. Covers of Atoms

When a lattice $\boldsymbol{L}$ has a least element $\bot$, its *atoms* are the covers of $\bot$. Provided that $\boldsymbol{L}$ is modular, the join of any two distinct atoms covers each join-and, so a cover c of an atom is interesting when it is *not* the join of two atoms. If $\boldsymbol{L}$ is distributive, that is equivalent to the ostensibly stronger demand that c be join-irreducible.

The lattice of subvarieties of a congruence distributive variety E is itself distributive [16, Cor. 4.2]. Therefore, once the atoms of this lattice have been determined, the immediate concern is to identify the join-irreducible covers of each atom E'; we refer to these as covers of E' *within* E. In particular, it behoves us to investigate the join-irreducible covers, within DMM, of the four varieties in Theorem 3.2.

By Theorem 2.7(ii), $\mathbb{V}(\boldsymbol{S}_5)$ is a join-irreducible cover of $\mathbb{V}(\boldsymbol{S}_3)$ within DMM.

For each $\boldsymbol{X} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$ and each variety K of De Morgan monoids, if $\boldsymbol{A} \in (\mathsf{K} \backslash \mathbb{I}(\boldsymbol{X}))_{\mathrm{FSI}}$ is nontrivial, then $\boldsymbol{A} \notin \mathbb{V}(\boldsymbol{X})$, by Jónsson's Theorem, because the nontrivial members of $\mathbb{HS}(\boldsymbol{X})$ belong to $\mathbb{I}(\boldsymbol{X})$. In this case, if K covers $\mathbb{V}(\boldsymbol{X})$, then $\mathsf{K} = \mathbb{V}(\boldsymbol{A}, \boldsymbol{X})$, so if K is also join-irreducible, it coincides with $\mathbb{V}(\boldsymbol{A})$. In other words:

Fact 7.1. *If $\boldsymbol{X} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$, then every join-irreducible cover of $\mathbb{V}(\boldsymbol{X})$ within DMM is generated by each of its nontrivial FSI members, other than the isomorphic copies of $\boldsymbol{X}$.*

In the subvariety lattice of U, the only atom is $\mathbb{V}(\boldsymbol{C}_4)$ (as $\mathsf{U} \subseteq \mathsf{W}$), so every cover of $\mathbb{V}(\boldsymbol{C}_4)$ within U is join-irreducible.

Up to isomorphism, there are just seven nontrivial 0–generated [F]SI De Morgan monoids, all of which are finite. They were identified by Slaney [25], and are denoted by $\boldsymbol{C}_2, \boldsymbol{C}_3, \ldots, \boldsymbol{C}_8$ in [26].

Slaney's $\boldsymbol{C}_2$, $\boldsymbol{C}_3$ and $\boldsymbol{C}_4$ are our $\mathbf{2}$, $\boldsymbol{D}_4$ and $\boldsymbol{C}_4$, respectively. His $\boldsymbol{C}_5, \ldots, \boldsymbol{C}_8$ will not be defined in full here, but their significant additional properties, for present purposes, are as follows. For $n \in \{5, 6, 7, 8\}$, $\boldsymbol{C}_n$ is anti-idempotent and not totally ordered (in fact, e and f are incomparable), with $|(e]| = 3$, so $\boldsymbol{C}_n$ is not simple. It is therefore a homomorphic pre-image of $\boldsymbol{C}_4$, by Theorems 3.1 and 2.11, i.e., $\boldsymbol{C}_n \in \mathsf{W}$. But, because $\boldsymbol{C}_n$ is rigorously compact (Theorem 2.6(ii)) and violates $e \leqslant f$, Lemma 4.8 shows that $\boldsymbol{C}_n \in \mathsf{U} \backslash \mathsf{M}$, whence $\mathbb{V}(\boldsymbol{C}_n) \subseteq \mathsf{U}$. Moreover, as $|(e]| = 3$, $\boldsymbol{C}_n$ has just three deductive filters, and hence just three factor algebras. The class of nontrivial members of $\mathbb{HS}(\boldsymbol{C}_n)$ is therefore $\mathbb{I}(\boldsymbol{C}_n, \boldsymbol{C}_4)$, because $\boldsymbol{C}_n$ is 0–generated. Thus, $\mathbb{V}(\boldsymbol{C}_n)$ is a (join-irreducible) cover of $\mathbb{V}(\boldsymbol{C}_4)$ within U, by Jónsson's Theorem.

Theorem 7.2.

(i) $\mathbb{V}(\mathbf{2})$ *has no join-irreducible cover within* DMM.

(ii) $\mathbb{V}(\boldsymbol{S}_5)$ *is the only join-irreducible cover of* $\mathbb{V}(\boldsymbol{S}_3)$ *within* DMM.

(iii) *If* K *is a join-irreducible cover of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* DMM, *then* K *consists of anti-idempotent algebras and exactly one of the following holds.*

  (1) $\mathsf{K} \subseteq \mathsf{M}$.

  (2) $\mathsf{K} = \mathbb{V}(\boldsymbol{C}_n)$ *for some* $n \in \{5, 6, 7, 8\}$.

  (3) $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$ *for some simple* 1*–generated De Morgan monoid* $\boldsymbol{A}$, *where* $\boldsymbol{C}_4$ *is a proper subalgebra of* $\boldsymbol{A}$.

(iv) *If* K *is a join-irreducible cover of* $\mathbb{V}(\boldsymbol{D}_4)$ *within* DMM, *then* $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$ *for some simple* 1*–generated De Morgan monoid* $\boldsymbol{A}$, *where* $\boldsymbol{D}_4$ *is a proper subalgebra of* $\boldsymbol{A}$. *In this case,* K *consists of anti-idempotent algebras.*

Proof. Let $\boldsymbol{X} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$, and let K be a join-irreducible cover of $\mathbb{V}(\boldsymbol{X})$ within DMM. As $\mathbb{V}(\boldsymbol{X}) \subsetneq \mathsf{K}$, there exists a finitely generated SI algebra $\boldsymbol{A} \in \mathsf{K} \backslash \mathbb{V}(\boldsymbol{X})$. Then $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$, by Fact 7.1. Note that $\boldsymbol{A}$ is rigorously compact, by Theorem 2.6(iii). Let $\boldsymbol{B}$ be the 0–generated subalgebra of $\boldsymbol{A}$, so

$\boldsymbol{B}$ is FSI, by Lemma 2.2(i). Now $\boldsymbol{B}$ is finite, by the aforementioned result of Slaney, so $\boldsymbol{B}$ is SI or trivial.

If $\boldsymbol{B}$ is trivial, then $\boldsymbol{A}$ is an odd Sugihara monoid (by Theorem 2.4(i)), whence K consists of odd Sugihara monoids, forcing $\boldsymbol{X} = \boldsymbol{S}_3$ and $\mathsf{K} = \mathbb{V}(\boldsymbol{S}_5)$ (by Theorem 2.7(ii)), as K covers $\mathbb{V}(\boldsymbol{X})$.

We may therefore assume that $\boldsymbol{B}$ is nontrivial, in view of the present theorem's statement. By Theorems 3.1 and 2.11, $\boldsymbol{B}$ is simple or crystalline, so by Theorem 2.8, we may assume that $\boldsymbol{B} \in \{\mathbf{2}, \boldsymbol{D}_4\}$ or $\boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{B})$.

If $\boldsymbol{B} = \mathbf{2}$, then $\boldsymbol{A}$ is idempotent (by Theorem 2.4(ii)). In this case, if $\boldsymbol{X} \neq \mathbf{2}$, then $\mathsf{K} = \mathbb{V}(\boldsymbol{X}, \mathbf{2})$, while if $\boldsymbol{X} = \mathbf{2}$, then $\boldsymbol{A} \ncong \mathbf{2}$ (as $\boldsymbol{A} \notin \mathbb{V}(\boldsymbol{X})$), so $\boldsymbol{S}_3 \in \mathbb{H}(\boldsymbol{A})$ (by the remark preceding Theorem 2.7), whereupon $\mathsf{K} = \mathbb{V}(\mathbf{2}, \boldsymbol{S}_3)$. Either way, this contradicts the join-irreducibility of K, so $\boldsymbol{B} \neq \mathbf{2}$, whence $\boldsymbol{D}_4 = \boldsymbol{B}$ or $\boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{B})$.

For the same reason, the cases $\boldsymbol{X} \neq \boldsymbol{D}_4 = \boldsymbol{B}$ and $\boldsymbol{X} \neq \boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{B})$ are ruled out, as K would be $\mathbb{V}(\boldsymbol{X}, \boldsymbol{D}_4)$ in the first of these, and $\mathbb{V}(\boldsymbol{X}, \boldsymbol{C}_4)$ in the second. If $\boldsymbol{X} = \boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{B}) \backslash \mathbb{I}(\boldsymbol{B})$, then $\mathsf{K} = \mathbb{V}(\boldsymbol{B})$, instantiating (iii)(2), in view of Slaney's findings. The assertion '$\boldsymbol{X} = \boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{B}) \backslash \mathbb{I}(\boldsymbol{B})$' may therefore be assumed false. (The exclusivity claim in (iii) will be proved separately below.)

It follows that $\boldsymbol{B} \cong \boldsymbol{X} \in \{\boldsymbol{C}_4, \boldsymbol{D}_4\}$. We identify $\boldsymbol{B}$ with $\boldsymbol{X}$ and refer henceforth only to the latter. Thus, $\boldsymbol{X}$ is a subalgebra of $\boldsymbol{A}$, and $\boldsymbol{X} \neq \boldsymbol{A}$ (as $\boldsymbol{A} \notin \mathbb{V}(\boldsymbol{X})$), so $\boldsymbol{A}$ is not 0–generated. Also, K has no nontrivial idempotent member (otherwise K would be $\mathbb{V}(\boldsymbol{X}, \mathbf{2})$ or $\mathbb{V}(\boldsymbol{X}, \boldsymbol{S}_3)$), so K consists of anti-idempotent algebras, by Theorem 2.4(iii).

By Theorem 3.1, there is a surjective homomorphism $h \colon \boldsymbol{A} \longrightarrow \boldsymbol{E}$ for some simple $\boldsymbol{E} \in \mathsf{K}$. Now $\boldsymbol{E} \ncong \boldsymbol{D}_4$, by Theorem 2.11, because $\boldsymbol{A}$ is not 0–generated.

If $\boldsymbol{X} = \boldsymbol{C}_4$, then $\boldsymbol{C}_4 \in \mathbb{S}(\boldsymbol{A})$. If, moreover, $\boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{A})$, then $\boldsymbol{A} \in \mathsf{N}$, by Remark 4.2, so $\boldsymbol{A} \in \mathsf{M}$, by Corollary 4.15, because $\boldsymbol{A}$ is rigorously compact. In this case, $\mathsf{K} \subseteq \mathsf{M}$, because $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$.

We may therefore assume that $\boldsymbol{X} = \boldsymbol{D}_4$ or $\boldsymbol{X} = \boldsymbol{C}_4 \notin \mathbb{H}(\boldsymbol{A})$. In both cases, $\boldsymbol{E} \ncong \boldsymbol{X}$. As $\boldsymbol{E}$ is a nontrivial member of K, it is not idempotent, so the subalgebra $h[\boldsymbol{X}]$ of $\boldsymbol{E}$ cannot be trivial (by Theorem 2.4(ii)). Therefore, $h|_X$ embeds $\boldsymbol{X}$ into $\boldsymbol{E}$, because $\boldsymbol{X}$ is simple. Since $\boldsymbol{X}$ is 0–generated and finite, it is isomorphic to a proper subalgebra of a 1–generated subalgebra $\boldsymbol{E}'$ of $\boldsymbol{E}$. As $\boldsymbol{E}$ is simple, so is $\boldsymbol{E}'$, by the CEP for IRLs. Thus, because $\boldsymbol{X} \ncong \boldsymbol{E}' \in \mathsf{K}$, Fact 7.1 gives $\mathsf{K} = \mathbb{V}(\boldsymbol{E}')$, witnessing (iii)(3) or (iv).

For the mutual exclusivity claim in (iii), note that (1) precludes (2) (as $\boldsymbol{C}_n \notin \mathsf{M}$) and (3) (as $\boldsymbol{C}_4 \notin \mathbb{H}(\boldsymbol{A})$ for the simple generator $\boldsymbol{A}$ of K in (3)). Also, (2) precludes (3), by Corollary 2.3, because $\boldsymbol{C}_n$ is SI but not simple. □

If K and $\boldsymbol{A}$ are as in Theorem 7.2(iii)(3) [resp. 7.2(iv)], then K is semisimple, by Corollary 2.3. If, moreover, $\boldsymbol{A}$ is finite, then the class of simple members of K is $\mathbb{I}(\boldsymbol{C}_4, \boldsymbol{A})$ [resp. $\mathbb{I}(\boldsymbol{D}_4, \boldsymbol{A})$], by Jónsson's Theorem and the CEP. The options for $\boldsymbol{A}$ are discussed in Sections 9 and 10.

An immediate consequence of Theorem 7.2(iii) is the following.

Corollary 7.3. *The varieties* $\mathbb{V}(\boldsymbol{C}_5)$, $\mathbb{V}(\boldsymbol{C}_6)$, $\mathbb{V}(\boldsymbol{C}_7)$ *and* $\mathbb{V}(\boldsymbol{C}_8)$ *are exactly the covers of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* U *that are not within* M.

In the next section, we shall show that $\mathbb{V}(\boldsymbol{C}_4)$ has just six covers within M. Some preparatory results will be required. The subalgebra of an algebra $\boldsymbol{A}$ generated by a subset X of A shall be denoted by $\boldsymbol{Sg}^{\boldsymbol{A}}X$.

Lemma 7.4. *Let* K *be a cover of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* U. *Then* $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$ *for some skew reflection* $\boldsymbol{A}$ *of an SI Dunn monoid* $\boldsymbol{B}$, *where* 0 *is meet-irreducible in* $\boldsymbol{A}$, *and* $\boldsymbol{A}$ *is generated by the greatest strict lower bound of* e *in* $\boldsymbol{B}$.

Proof. By assumption, there is an SI algebra $\boldsymbol{G} \in \mathsf{K}\backslash\mathbb{V}(\boldsymbol{C}_4)$, and K is join-irreducible in the subvariety lattice of DMM. As $\boldsymbol{G} \in \mathsf{U}$, Corollary 5.6 shows that $\boldsymbol{G}$ is a skew reflection of a Dunn monoid $\boldsymbol{H}$, and 0 is meet-irreducible in $\boldsymbol{G}$. Now $\boldsymbol{H}$ is nontrivial, because $\boldsymbol{G} \ncong \boldsymbol{C}_4$, so Remark 5.3 shows that $\boldsymbol{H}$ is SI, and that $\boldsymbol{H}$ includes the greatest strict lower bound of e in $\boldsymbol{G}$, which we denote by c. Then $\boldsymbol{A} := \boldsymbol{Sg}^{\boldsymbol{G}}\{c\} \in \mathsf{K}$ is SI, by Lemma 2.2(iii), and $\boldsymbol{A} \ncong \boldsymbol{C}_4$, as $0 < c < e$. Consequently, $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$, by Fact 7.1. Clearly, $\boldsymbol{A}$ is the skew reflection of the SI Dunn monoid $\boldsymbol{B} := \boldsymbol{Sg}^{\boldsymbol{A}}(H \cap A)$, with respect to the restricted order of $\boldsymbol{A}$, and 0 is meet-irreducible in $\boldsymbol{A}$. □

A partial converse of Lemma 7.4 is supplied below. It extends the claim about $\boldsymbol{C}_5, \ldots, \boldsymbol{C}_8$ preceding Theorem 7.2.

Lemma 7.5. *If a skew reflection* $\boldsymbol{A} \in \mathsf{U}$ *of a finite simple Dunn monoid* $\boldsymbol{B}$ *is generated by the least element of* $\boldsymbol{B}$, *then* $\mathbb{V}(\boldsymbol{A})$ *is a (join-irreducible) cover of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* U.

Proof. Let $\bot$ be the least element of $\boldsymbol{B}$. By Lemma 2.2(iv), $\bot$ is the only strict lower bound of e in $\boldsymbol{B}$. The lower bounds of e in $\boldsymbol{A}$ therefore form the chain $0 < \bot < e$, so $\boldsymbol{A} \ncong \boldsymbol{C}_4$, but $\boldsymbol{A}$ is SI, by Remark 5.3. Therefore, $\boldsymbol{A} \notin \mathbb{V}(\boldsymbol{C}_4)$, by Jónsson's Theorem, and so $\mathbb{V}(\boldsymbol{C}_4) \subsetneq \mathbb{V}(\boldsymbol{A}) \subseteq \mathsf{U}$. To see that $\mathbb{V}(\boldsymbol{A})$ covers $\mathbb{V}(\boldsymbol{C}_4)$, let $\boldsymbol{E} \in \mathbb{V}(\boldsymbol{A})\backslash\mathbb{V}(\boldsymbol{C}_4)$ be SI. We must show that $\boldsymbol{A} \in \mathbb{V}(\boldsymbol{E})$. Since $\boldsymbol{A}$ is finite, Jónsson's Theorem gives $\boldsymbol{E} \in \mathbb{HS}(\boldsymbol{A})$. Any subalgebra $\boldsymbol{D}$ of $\boldsymbol{A}$ is nontrivial, so e has a strict lower bound in $\boldsymbol{D}$, by (13). If 0 is the only strict lower bound of e in $\boldsymbol{D}$, then $\boldsymbol{D}$ is a simple member of U (by Lemma 2.2(iv)), whence $\boldsymbol{D} \cong \boldsymbol{C}_4$ (as $\mathsf{U} \subseteq \mathsf{W}$). Otherwise, $\bot \in D$, in which case $\boldsymbol{D} = \boldsymbol{A}$, as $\boldsymbol{A}$ is generated by $\bot$. Thus, $\mathbb{S}(\boldsymbol{A}) \subseteq \{\boldsymbol{C}_4, \boldsymbol{A}\}$, and so $\boldsymbol{E}$ is a homomorphic image of $\boldsymbol{A}$ (as $\boldsymbol{C}_4$ is simple). Now $\boldsymbol{A}$ has only three deductive filters (because $|(e]| = 3$ in $\boldsymbol{A}$), so $\boldsymbol{A}$ has just three factor algebras, of which $\boldsymbol{A}$ and a trivial algebra are two. The other is isomorphic to $\boldsymbol{C}_4$, as $\boldsymbol{A} \in \mathsf{U} \subseteq \mathsf{W}$. Therefore, $\boldsymbol{E} \cong \boldsymbol{A}$, whence $\boldsymbol{A} \in \mathbb{V}(\boldsymbol{E})$, as required. □

The RL–reducts of $\mathbf{2}$, $\boldsymbol{S}_3$ and $\boldsymbol{C}_4$ shall be denoted by $\mathbf{2}^+$, $\boldsymbol{S}_3^+$ and $\boldsymbol{C}_4^+$, respectively. (In fact, $\boldsymbol{S}_3$ and $\boldsymbol{S}_3^+$ are termwise equivalent, because $\neg x$ is definable as $x \to e$ in $\boldsymbol{S}_3^+$.) The following result will be needed later.

Theorem 7.6. *Let* $\boldsymbol{B}$ *be a square-increasing RL that is SI. Let* c *be the greatest strict lower bound of* e *in* $\boldsymbol{B}$ *(which exists, by Lemma* 2.2(iii)*).*

If $c \to e = e$, *then* $Sg^{\boldsymbol{B}}\{c\} = \{c, e\}$ *and* $\boldsymbol{Sg}^{\boldsymbol{B}}\{c\} \cong \boldsymbol{2}^+$.

If $c \to e \neq e$, *then* $\boldsymbol{Sg}^{\boldsymbol{B}}\{c\} \cong \boldsymbol{S}_3^+$, *its lattice reduct being* $c < e < c \to e$.

Proof. As $c < e$, we have $c^2 = c$, by (17), and $e \leqslant c \to e$, by (12). Then $c \to c = c \to e$, because (11), (14) and (18) yield

$$c \to c \leqslant c \to e \leqslant c \to (c \to c) \leqslant c \to c.$$

Therefore, in view of (14), if $c \to e = e$, then $\{c, e\}$ is the universe of a subalgebra of $\boldsymbol{B}$, isomorphic to $\boldsymbol{2}^+$.

We may now assume that $e < c \to e$. Then $(c \to e) \to e \leqslant e \to e = e$, by (11), whereas $e \nleqslant (c \to e) \to e$, by (12), so $(c \to e) \to e < e$. Then $(c \to e) \to e \leqslant c$, by definition of c, so $(c \to e) \to e = c$, by (6). It suffices, therefore, to show that the chain $(c \to e) \to e < e < c \to e$ constitutes a subalgebra of $\boldsymbol{B}$, isomorphic to $\boldsymbol{S}_3^+$, but this was already proved by Galatos [10, Thm. 5.7]. Although its statement in [10] assumes idempotence (and a weak form of commutativity) for fusion, all appeals to idempotence in the proof require only the square-increasing law. □

8. Covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M

If K is a cover of $\mathbb{V}(\boldsymbol{C}_4)$ within M, then by Lemma 7.4 and Remark 5.3, there exist $\boldsymbol{A}$, $\boldsymbol{B}$ and $\bot$ such that $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$,

- $\boldsymbol{B}$ is an SI Dunn monoid, $\boldsymbol{A}$ is a skew reflection of $\boldsymbol{B}$ in which $e < f$, and $\boldsymbol{A} = \boldsymbol{Sg}^{\boldsymbol{A}}\{\bot\}$, where $\bot \in B$ is the greatest strict lower bound of e in $\boldsymbol{A}$.

The displayed properties of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\bot$ will now be *assumed*, until the 'conclusions' after Lemma 8.9. By Lemma 5.7(ii), they imply that *0* is meet-irreducible (and *1* join-irreducible) in $\boldsymbol{A}$. We shall prove that they also force $\boldsymbol{A}$ to be finite and $\boldsymbol{B}$ simple, with $|A| \leq 14$ (i.e., $|B| \leq 6$).

We define $\top = \bot \to e$, so $\top \in B$. By Theorem 7.6, $\boldsymbol{Sg}^{\boldsymbol{B}}\{\bot\}$ consists of $\bot, e, \top$ and is isomorphic to $\boldsymbol{2}^+$ (with $e = \top$) or to $\boldsymbol{S}_3^+$ (with $e < \top$). The respective tables for $\cdot$ and $\to$ in $\boldsymbol{Sg}^{\boldsymbol{B}}\{\bot\}$ are recalled below. (There is no guarantee that $\boldsymbol{Sg}^{\boldsymbol{B}}\{\bot\}$ exhausts $\boldsymbol{B}$.)

$\cdot$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\top$

$\to$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$

or

$\cdot$	$\bot$	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
e	$\bot$	e	$\top$
$\top$	$\bot$	$\top$	$\top$

$\to$	$\bot$	e	$\top$
$\bot$	$\top$	$\top$	$\top$
e	$\bot$	e	$\top$
$\top$	$\bot$	$\bot$	$\top$

Warning. Although $\bot, \top$ will turn out to be extrema for $\boldsymbol{B}$, that fact will emerge only after Lemma 8.9. Until then, our use of these symbols should not be taken to justify claims like '$b \cdot \bot = \bot$ for all $b \in B$' on the basis of (15) alone. Such claims will be justified directly when needed.

All this notation will remain fixed until the 'conclusions' after Lemma 8.9. We use freely the notation from Definition 5.1 as well, e.g., $\bot' = \neg^{\boldsymbol{A}} \bot \in B'$ and $\top' = \neg^{\boldsymbol{A}} \top \in B'$. The superscript $\boldsymbol{A}$ will normally be omitted.

Theorem 8.1. *The algebra* $\boldsymbol{A}$ *is the reflection of* $\boldsymbol{B}$ *iff* e *and* $\top'$ *are comparable. In this case,* $e < \top'$ *and* $\boldsymbol{B} = \boldsymbol{Sg}^{\boldsymbol{B}}\{\bot\}$*, so* $\boldsymbol{B}$ *consists of* $\bot, e, \top$ *only, and*

(i) $e = \top$ *iff* $\boldsymbol{B} \cong \boldsymbol{2}^+$*, iff* $\boldsymbol{A} \cong \mathrm{R}(\boldsymbol{2}^+)$;
(ii) $e \neq \top$ *iff* $\boldsymbol{B} \cong \boldsymbol{S}_3^+$*, iff* $\boldsymbol{A} \cong \mathrm{R}(\boldsymbol{S}_3^+)$.

Proof. In the first assertion, necessity follows from the definition of reflection. Conversely, suppose e and $\top'$ are comparable, and let h be the unique homomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}_4$. As h is isotone and $h(e) = e < f = h(\top')$, we can't have $\top' \leqslant e$, so $e < \top' = (\top \cdot \top)'$. Then, by Definition 5.1(iii), $\top \leqslant \top'$, so

$$0 < \bot < e \leqslant \top < \top' \leqslant f < \bot' < 1, \tag{30}$$

where $e = \top$ iff $\top' = f$. The elements $\bot, e, \top \in B$ are closed under $\cdot, \to$, so items (vi)–(ix) of Definition 5.1 ensure that the elements listed in (30) are closed under $\cdot, \wedge, \vee, \neg$. They include e, so they constitute a subalgebra of $\boldsymbol{A}$. As they also include $\bot$, which generates $\boldsymbol{A}$, they exhaust $\boldsymbol{A}$. Consequently, $\boldsymbol{A} = \mathrm{R}(\boldsymbol{B})$, where $\boldsymbol{B}$ consists of $\bot, e, \top$ only, and is therefore generated by $\bot$. Then (i) and (ii) follow from Theorem 7.6. □

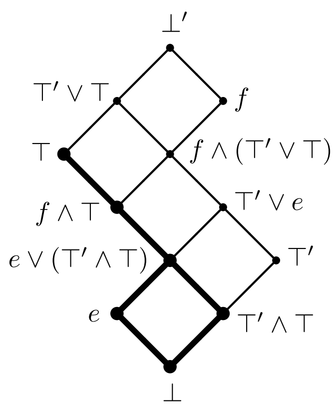
Corollary 8.2. *If* $\boldsymbol{Sg}^{\boldsymbol{B}}\{\bot\} \ncong \boldsymbol{S}_3^+$*, then* $\boldsymbol{A} \cong \mathrm{R}(\boldsymbol{2}^+)$.

Proof. In this case, $\boldsymbol{Sg}^{\boldsymbol{B}}\{\bot\} \cong \boldsymbol{2}^+$, by Theorem 7.6, so $e = \top$. As $\boldsymbol{A} \in \mathsf{M}$, we have $e < f = e' = \top'$, so the result follows from Theorem 8.1. □

By Lemma 7.5, $\mathrm{R}(\boldsymbol{2}^+)$ and $\mathrm{R}(\boldsymbol{S}_3^+)$ generate covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M, as each is generated by its own unique atom. Because our aim is now to isolate the *other* covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M, the previous two results allow us to assume, until further notice, that

- $\boldsymbol{Sg}^{\boldsymbol{B}}\{\bot\}$ is isomorphic to $\boldsymbol{S}_3^+$ and has universe $\bot < e < \top$, and
- $\boldsymbol{A}$ is *not* the reflection of $\boldsymbol{B}$, i.e., e and $\top'$ are incomparable.

Consider the formal diagram below.



By Theorem 5.4(iv), the labels on the thicker points all identify elements of B, but we do *not* claim that the twelve depicted elements are distinct in $\boldsymbol{A}$. (It will turn out that they exhaust $\boldsymbol{A}$, but that is not yet obvious.)

Lemma 8.3. *The subset of A comprising the elements depicted above is closed under the operations $\wedge, \vee$ and $\neg$ of $\boldsymbol{A}$.*

Moreover, the label on the diagrammatic join of any two elements is the actual join in $\boldsymbol{A}$ of the labels on those elements, and similarly for meets.

Proof. Note first that the diagram order is sound, in the sense that wherever x is depicted as a lower bound of y, then $x \leqslant y$ in $\boldsymbol{A}$. This is easy to see, except perhaps for the $\boldsymbol{A}$–inequalities $\bot \leqslant \top'$ and $\top' \wedge \top \leqslant f$ $(= e')$. The first of these follows from Lemma 5.7(i), as $\top = \bot \to e$, and the second from Definition 5.1(iii), because $e \leqslant \top \leqslant \top \vee \top' = (\top' \wedge \top)' = ((\top' \wedge \top) \cdot e)'$ in $\boldsymbol{A}$.

Closure under $\neg$ follows from the identity $\neg\neg x = x$ and De Morgan's laws.

Let x, y be expressions from the diagram. We shall show that $x \vee y$ (computed in $\boldsymbol{A}$) is equal (in $\boldsymbol{A}$) to the label on the diagrammatic join of x, y. In view of the chosen labels and De Morgan's laws, the same will then follow for meets. We go through the possible values of x. Because the diagram order is sound, we can eliminate cases where y is comparable (according to the diagram) with x. This eliminates $\bot$ and $\bot'$ as values for x and for y.

If x is e, then the uneliminated values of y are $\top'$ and $\top' \wedge \top$. These cases are disposed of by noting that $e \vee \top'$ and $e \vee (\top' \wedge \top)$ appear (up to commutativity of $\vee$) in the diagram, and that the diagram order makes them the least upper bounds, respectively, of $e, \top'$ and of $e, \top' \wedge \top$.

When x is $\top' \wedge \top$, the only uneliminated value of y is e, which we have just considered.

When x is $e \vee (\top' \wedge \top)$, the uneliminated possibilities for y are $e, \top' \wedge \top, \top'$, of which all but $\top'$ have been considered. As $e \vee (\top' \wedge \top) \vee \top' = \top' \vee e$, which appears (in the correct place) in the diagram, we are done with this case.

When x is $\top'$, the only uneliminated choices for y, not already considered, are $\top$ and $f \wedge \top$. Now $\top' \vee \top$ is well-placed in the diagram, and in $\boldsymbol{A}$, we have $\top' \vee (f \wedge \top) = (\top' \vee f) \wedge (\top' \vee \top) = f \wedge (\top' \vee \top)$, which is also well-placed.

When x is $\top' \vee e$, the only interesting possibilities for y are $\top$ and $f \wedge \top$. And in $\boldsymbol{A}$, we have well-placed values $(\top' \vee e) \vee \top = \top' \vee \top$ and

$$(\top' \vee e) \vee (f \wedge \top) = (\top' \vee e \vee f) \wedge (\top' \vee e \vee \top) = f \wedge (\top' \vee \top).$$

From cases already considered, it follows that $(f \wedge \top) \vee y$ is well-placed in the diagram, for every y.

When x is $f \wedge (\top' \vee \top)$, the only interesting y is $\top$. In $\boldsymbol{A}$, we have

$$(f \wedge (\top' \vee \top)) \vee \top = (f \vee \top) \wedge ((\top' \vee \top) \vee \top) = (f \vee \top) \wedge (\top' \vee \top).$$

This expression will simplify to the well-placed $\top' \vee \top$, provided that $f \vee \top = \bot'$, or equivalently, $e \wedge \top' = \bot$ (in $\boldsymbol{A}$), which we now show. We have already verified that $\bot \leqslant \top'$, so $\bot \leqslant e \wedge \top'$. As $e \not\leqslant \top'$, we have $e \wedge \top' < e$, whence $e \wedge \top' \leqslant \bot$ (by definition of $\bot$), and so $e \wedge \top' = \bot$, as required.

When x is $\top$, the only new y to consider is f, but we have just shown that $\top \vee f = \bot'$, which is well-placed.

When x is f, the only new y is $\top' \vee \top$, and $\top \leqslant \top' \vee \top \leqslant \bot'$. In $\boldsymbol{A}$, we have seen that $f \vee \top = \bot' = f \vee \bot'$, so $f \vee (\top' \vee \top) = \bot'$, which is well-placed.

When x is $\top' \vee \top$, there is no longer any unconsidered option for y. □

Lemma 8.4. *Fusion in* $\boldsymbol{A}$ *behaves as in the following table.*

$\cdot$	$\bot$	e	$\top' \wedge \top$	$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
e		e	$\top' \wedge \top$	$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$\top' \wedge \top$			$\top' \wedge \top$	$\top' \wedge \top$	$\top' \wedge \top$	$\top' \wedge \top$
$e \vee (\top' \wedge \top)$				$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$f \wedge \top$						$\top$
$\top$						$\top$

Proof. The submatrix involving only $\bot, e, \top$ is justified, because $\boldsymbol{Sg}^{\boldsymbol{B}}\{\bot\}$ is an RL–subreduct of $\boldsymbol{A}$. If $x \in \{\top' \wedge \top,\, e \vee (\top' \wedge \top),\, f \wedge \top\}$, then $\bot \leqslant x \leqslant \top$, so $\bot \cdot x = \bot$, by (11), since $\bot^2 = \bot = \bot \cdot \top$. This justifies the first row; the second records the neutrality of e in $\boldsymbol{A}$.

To see that $(\top' \wedge \top) \cdot \top = \top' \wedge \top$, note the following consequences of (11), Definition 5.1(viii) and the tables for $\boldsymbol{Sg}^{\boldsymbol{B}}\{\bot\}$:

$$\begin{aligned}(\top' \wedge \top) \cdot \top &\leqslant (\top' \cdot \top) \wedge \top^2 = (\top \to \top)' \wedge \top \\ &= \top' \wedge \top = (\top' \wedge \top) \cdot e \leqslant (\top' \wedge \top) \cdot \top.\end{aligned}$$

For any $x \in \{\top' \wedge \top,\ e \vee (\top' \wedge \top),\ f \wedge \top\}$, we now have

$$\top' \wedge \top \leqslant (\top' \wedge \top)^2 \leqslant (\top' \wedge \top) \cdot x \leqslant (\top' \wedge \top) \cdot \top = \top' \wedge \top,$$

so $(\top' \wedge \top) \cdot x = \top' \wedge \top$. If $y \in \{e \vee (\top' \wedge \top),\ f \wedge \top,\ \top\}$, then

$$\top = e \cdot \top \leqslant y \cdot \top \leqslant \top^2 = \top,$$

whence $y \cdot \top = \top$. By (8) and the idempotence of $\top' \wedge \top$,

$$(e \vee (\top' \wedge \top))^2 = e \vee (\top' \wedge \top) \vee (\top' \wedge \top)^2 = e \vee (\top' \wedge \top).$$

Finally, by (8) and since $\top' \leqslant f$, we have

$$\begin{aligned}(e \vee (\top' \wedge \top)) \cdot (f \wedge \top) &= (e \cdot (f \wedge \top)) \vee ((\top' \wedge \top) \cdot (f \wedge \top)) \\ &= (f \wedge \top) \vee (\top' \wedge \top) = f \wedge \top.\end{aligned}$$

□

Lemma 8.5. *Residuation in* $\boldsymbol{A}$ *behaves as in the following table.*

$\to$	$\bot$	e	$\top' \wedge \top$	$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
e	$\bot$	e	$\top' \wedge \top$	$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$\top' \wedge \top$			$\top$	$\top$	$\top$	$\top$
$e \vee (\top' \wedge \top)$	$\bot$		$\top' \wedge \top$	$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$f \wedge \top$	$\bot$		$\top' \wedge \top$		$e \vee (\top' \wedge \top)$	$\top$
$\top$	$\bot$	$\bot$	$\top' \wedge \top$	$\top' \wedge \top$	$\top' \wedge \top$	$\top$

Proof. All elements in the table lie between $\bot$ and $\top$. The submatrix involving only $\bot, e, \top$ documents residuation in $\boldsymbol{Sg}^{\boldsymbol{B}}\{\bot\}$, so the first row and last column follow from (11), because $\top = \bot \to \bot = \bot \to \top = \top \to \top$. The second row is justified by (14). Then $(\top' \wedge \top) \to \top' = \neg((\top' \wedge \top) \cdot \top) = (\top' \wedge \top)'$ (Lemma 8.4) $= \top' \vee \top$, so by (9),

$$(\top' \wedge \top) \to (\top' \wedge \top) = ((\top' \wedge \top) \to \top') \wedge ((\top' \wedge \top) \to \top) = (\top' \vee \top) \wedge \top = \top.$$

Now, for each $x \in \{e \vee (\top' \wedge \top),\, f \wedge \top\}$,

$$\top = (\top' \wedge \top) \to (\top' \wedge \top) \leqslant (\top' \wedge \top) \to x \leqslant (\top' \wedge \top) \to \top = \top,$$

by (11), whence $(\top' \wedge \top) \to x = \top$.

Clearly, $e \vee (\top' \wedge \top) \leqslant \top = \bot \to \bot$, so (7) and (11) yield

$$\bot \leqslant (e \vee (\top' \wedge \top)) \to \bot \leqslant e \to \bot = \bot,$$

whence $(e \vee (\top' \wedge \top)) \to \bot = \bot$. By Lemma 8.4, $e \vee (\top' \wedge \top)$ is an idempotent upper bound of e, so $(e \vee (\top' \wedge \top)) \to (e \vee (\top' \wedge \top)) = e \vee (\top' \wedge \top)$, by (5). Also, by (10),

$$\begin{aligned}(e \vee (\top' \wedge \top)) \to (\top' \wedge \top) &= (e \to (\top' \wedge \top)) \wedge ((\top' \wedge \top) \to (\top' \wedge \top)) \\ &= (\top' \wedge \top) \wedge \top = \top' \wedge \top;\end{aligned}$$

$$\begin{aligned}(e \vee (\top' \wedge \top)) \to (f \wedge \top) &= (e \to (f \wedge \top)) \wedge ((\top' \wedge \top) \to (f \wedge \top)) \\ &= (f \wedge \top) \wedge \top = f \wedge \top.\end{aligned}$$

Similarly, from $f \wedge \top \leqslant \top = \bot \to \bot$ and $e \leqslant f \wedge \top$ and (7), (11), we obtain $\bot \leqslant (f \wedge \top) \to \bot \leqslant e \to \bot = \bot$, hence $(f \wedge \top) \to \bot = \bot$. Also, by (9),

$$\begin{aligned}(f \wedge \top) \to (\top' \wedge \top) &= ((f \wedge \top) \to \top') \wedge ((f \wedge \top) \to \top) \\ &= \neg((f \wedge \top) \cdot \top) \wedge \top = \top' \wedge \top \ \text{(by Lemma 8.4)};\end{aligned}$$

$$\begin{aligned}(f \wedge \top) \to (f \wedge \top) &= ((f \wedge \top) \to f) \wedge ((f \wedge \top) \to \top) = \neg(f \wedge \top) \wedge \top \\ &= (e \vee \top') \wedge \top = e \vee (\top' \wedge \top) \ \text{(by distributivity, since } e \leqslant \top);\end{aligned}$$

$$\begin{aligned}\top \to (\top' \wedge \top) &= (\top \to \top') \wedge (\top \to \top) = \neg(\top \cdot \top) \wedge \top = \top' \wedge \top; \\ \top \to (f \wedge \top) &= (\top \to f) \wedge (\top \to \top) = \neg\top \wedge \top = \top' \wedge \top,\end{aligned}$$

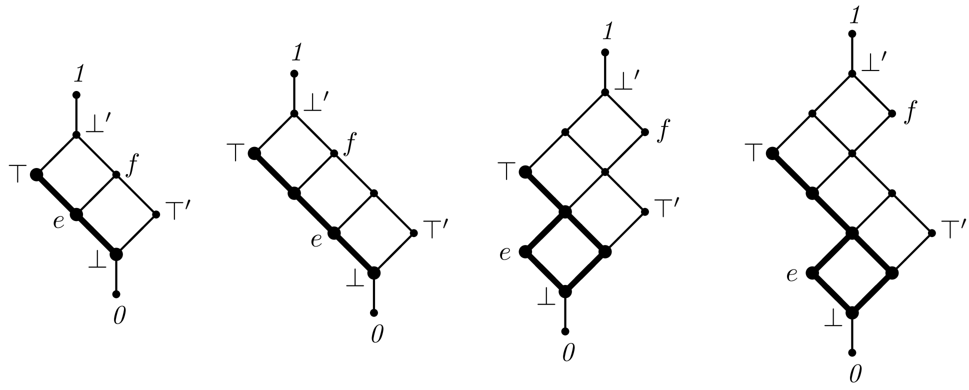
so, because $\top' \wedge \top \leqslant e \vee (\top' \wedge \top) \leqslant f \wedge \top$, (11) yields

$$\top \to (e \vee (\top' \wedge \top)) = \top' \wedge \top. \qquad \square$$

Recall that $\boldsymbol{A}$ has the following properties (under present assumptions):

- $[\top') \cap (\top] = \emptyset$, by the definition of a skew reflection,
- $\bot < e < \top$ and $\top' < f < \bot'$, as $\boldsymbol{Sg}^{\boldsymbol{B}}\{\bot\} \cong \boldsymbol{S}_3^+$, and
- e and $\top'$ are incomparable, by Theorem 8.1, as $\boldsymbol{A} \neq \mathrm{R}(\boldsymbol{B})$.

There are only four ways to identify elements from the diagram preceding Lemma 8.3 while respecting these rules. Thus, $\boldsymbol{A}$ must have one of the four Hasse diagrams below, where the thicker points denote the elements of $\boldsymbol{B}$.



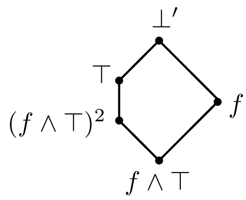
Case I Case II Case III Case IV

Lemma 8.6. *In Cases II, III and IV, we have* $(f \wedge \top) \to e = \bot$.

Proof. From $f \wedge \top \leqslant \top = \bot \to e$ and (7) we infer $\bot \leqslant (f \wedge \top) \to e$. As $e \leqslant f \wedge \top$, (11) shows that $(f \wedge \top) \to e \leqslant e \to e = e$. In Cases II, III and IV, $f \wedge \top \nleqslant e$, so (12) shows that $(f \wedge \top) \to e < e$. Therefore, $(f \wedge \top) \to e \leqslant \bot$ (by definition of $\bot$), hence the result. □

Lemma 8.7. *In Cases II and IV, we have* $(f \wedge \top)^2 = \top$.

Proof. By (11), $(f \wedge \top)^2 \leqslant \top^2 = \top$. By Lemma 8.5, $(f \wedge \top) \to (f \wedge \top) = e \vee (\top' \wedge \top)$. In Cases II and IV, therefore, $(f \wedge \top) \to (f \wedge \top) \neq f \wedge \top \geqslant e$, so $f \wedge \top$ is not idempotent (by (5)), i.e., $f \wedge \top < (f \wedge \top)^2$. Suppose, with a view to contradiction, that $(f \wedge \top)^2 < \top$. Then the diagram below depicts a five-element sub-poset of $\langle A; \leqslant \rangle$, which we claim is a sublattice of $\langle A; \wedge, \vee \rangle$. That will contradict the distributivity of $\boldsymbol{A}$, finishing the proof.



The claim amounts to the assertion that $f \vee (f \wedge \top)^2 = \bot'$. As f and $(f \wedge \top)^2$ are incomparable, we have $f < f \vee (f \wedge \top)^2 \leqslant \bot'$. In $\boldsymbol{A}$, however, $\bot'$ is the smallest strict upper bound of f (because $\bot$ is the greatest strict lower bound of e). Therefore, $f \vee (f \wedge \top)^2 = \bot'$. □

Lemma 8.8. *In Cases III and IV,* $\bot$ *is the value of all three of*

$$(\top' \wedge \top) \to e, \ \ (\top' \wedge \top) \to \bot \ \ \textit{and} \ \ (e \vee (\top' \wedge \top)) \to e.$$

Proof. As $\top' \wedge \top \leqslant \top = \bot \to \bot$ and $\bot \leqslant e$, we have

$$\bot \leqslant (\top' \wedge \top) \to \bot \leqslant (\top' \wedge \top) \to e,$$

by (7) and (11). Thus, the first claim subsumes the second. Suppose, with a view to contradiction, that

$$\bot < (\top' \wedge \top) \to e. \tag{31}$$

Since $e \leqslant \top$, it follows from (11) that

$$(\top' \wedge \top) \to e \leqslant \top \cdot ((\top' \wedge \top) \to e). \tag{32}$$

On the other hand, $(\top' \wedge \top) \cdot \top = \top' \wedge \top$, by Lemma 8.4, so

$$(\top' \wedge \top) \cdot \top \cdot ((\top' \wedge \top) \to e) = (\top' \wedge \top) \cdot ((\top' \wedge \top) \to e) \leqslant e,$$

by (6), whence $\top \cdot ((\top' \wedge \top) \to e) \leqslant (\top' \wedge \top) \to e$, by (2). Then, by (32),

$$\top \cdot ((\top' \wedge \top) \to e) = (\top' \wedge \top) \to e = d, \text{ say.} \tag{33}$$

In Cases III and IV, $\top' \wedge \top \nleqslant e$, so $e \nleqslant d$, by (12). Consequently, $d \leqslant f$, by Theorem 2.6(i), because $\boldsymbol{A}$ is SI. Therefore, $e \leqslant \neg d = \neg(d \cdot \top)$ (by (33)) $= d \to \top'$, so $d \leqslant \top'$, i.e., $(\top' \wedge \top) \to e \leqslant \top'$. Also, $\bot \leqslant \top' \wedge \top$, so by (11), $(\top' \wedge \top) \to e \leqslant \bot \to e = \top$. Therefore,

$$(\top' \wedge \top) \to e \leqslant \top' \wedge \top. \tag{34}$$

Now, by (2),

$$\begin{aligned} e \geqslant (\top' \wedge \top) \cdot ((\top' \wedge \top) \to e) &\geqslant ((\top' \wedge \top) \to e)^2 \text{ (by (34))} \\ &\geqslant (\top' \wedge \top) \to e > \bot \text{ (by (31)).} \end{aligned}$$

This forces

$$(\top' \wedge \top) \cdot ((\top' \wedge \top) \to e) = e, \tag{35}$$

by definition of $\bot$. Then, by Lemma 8.4,

$$\top' \wedge \top = (\top' \wedge \top)^2 \geqslant (\top' \wedge \top) \cdot ((\top' \wedge \top) \to e) \text{ (by (34))} = e \text{ (by (35))},$$

contradicting the diagrams for Cases III and IV. Thus, $(\top' \wedge \top) \to e = \bot$.

Finally, by (10) and the claim just proved,

$$(e \vee (\top' \wedge \top)) \to e = (e \to e) \wedge ((\top' \wedge \top) \to e) = e \wedge \bot = \bot. \qquad \square$$

The next lemma applies to Case IV. Its statement remains true in Case II, but is redundant there, as $\top' \wedge \top = \bot$ and $e \vee (\top' \wedge \top) = e$ in Case II (cf. Lemma 8.6).

Lemma 8.9. *In Case IV, we have* $(f \wedge \top) \to (e \vee (\top' \wedge \top)) = \top' \wedge \top$.

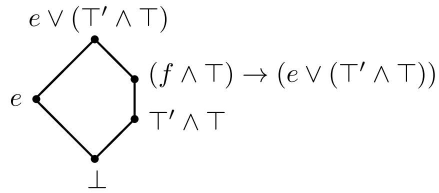
Proof. Observe that

$$\begin{aligned} \top' \wedge \top &= (f \wedge \top) \to (\top' \wedge \top) \text{ (by Lemma 8.5)} \\ &\leqslant (f \wedge \top) \to (e \vee (\top' \wedge \top)) \text{ (by (11))} \\ &\leqslant e \to (e \vee (\top' \wedge \top)) \text{ (by (11), as } e \leqslant f \wedge \top) \\ &= e \vee (\top' \wedge \top) \text{ (by (14)).} \end{aligned}$$

In Case IV, $f \wedge \top \nleqslant e \vee (\top' \wedge \top)$, so $e \nleqslant (f \wedge \top) \to (e \vee (\top' \wedge \top))$, by (12). Thus, $(f \wedge \top) \to (e \vee (\top' \wedge \top)) \neq e \vee (\top' \wedge \top)$. Suppose, with a view to contradiction, that $(f \wedge \top) \to (e \vee (\top' \wedge \top)) \neq \top' \wedge \top$. Then

$$\top' \wedge \top \;<\; (f \wedge \top) \to (e \vee (\top' \wedge \top)) \;<\; e \vee (\top' \wedge \top),$$

so the Hasse diagram below depicts a five-element sub-poset of $\langle A; \leqslant \rangle$.



Using the fact that $\bot$ is the greatest strict lower bound of e in $\boldsymbol{A}$, we obtain

$$e \wedge ((f \wedge \top) \to (e \vee (\top' \wedge \top))) \leqslant \bot$$

(cf. the proof of Lemma 8.7). On the other hand, by Lemma 8.4,

$$(f \wedge \top) \cdot \bot = \bot \leqslant e \vee (\top' \wedge \top),$$

so by (2), $\bot \leqslant (f \wedge \top) \to (e \vee (\top' \wedge \top))$. Also, $\bot \leqslant e$, so

$$\bot \leqslant e \wedge ((f \wedge \top) \to (e \vee (\top' \wedge \top))).$$

Therefore, $e \wedge ((f \wedge \top) \to (e \vee (\top' \wedge \top))) = \bot$, whence the elements depicted above form a sublattice of $\langle A; \wedge, \vee \rangle$, contradicting the distributivity of $\boldsymbol{A}$. $\square$

This completes the tables from Lemmas 8.4 and 8.5 in all cases.

Conclusions. The above arguments put constraints on $\boldsymbol{B}$ and on the order $\leqslant$ if $\boldsymbol{A} = \mathrm{S}^{\leqslant}(\boldsymbol{B})$ is to generate a cover of $\mathbb{V}(\boldsymbol{C}_4)$ within M. In particular, $\boldsymbol{B}$ must be finite and simple, with $|B| \leq 6$ (i.e., $|A| \leq 14$), and in each of Cases I–IV, there is at most one way to choose $\leqslant$ and the operations $\cdot, \to$ on $\boldsymbol{B}$ if this is to happen, in view of Lemmas 8.3–8.9. It remains, however, to check that in each case, $\boldsymbol{B}$ really is a Dunn monoid for which $\boldsymbol{Sg}^{\boldsymbol{A}}\{\bot\} = \boldsymbol{A}$. If so, then since $\boldsymbol{B}$ is finite and simple, $\mathbb{V}(\boldsymbol{A})$ will indeed be a cover of $\mathbb{V}(\boldsymbol{C}_4)$ within M, by Lemma 7.5, and the resulting varieties will be the only covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M, apart from $\mathrm{R}(\mathbf{2}^+)$ and $\mathrm{R}(\boldsymbol{S}_3^+)$.

In Case I, the intended $\boldsymbol{B}$ is clearly the Dunn monoid $\boldsymbol{S}_3^+$, which is generated by $\bot$, so $\boldsymbol{Sg}^{\boldsymbol{A}}\{\bot\} = \boldsymbol{A}$.

In Case II, $\boldsymbol{B} \cong \boldsymbol{C}_4^+$, so $\boldsymbol{B}$ is a Dunn monoid. Its elements form the chain

$$\bot < e < f \wedge \top < \top.$$

As the co-atom of $\boldsymbol{B}$ is $f \wedge \top$, it is clear that $\bot$ generates the skew reflection $\boldsymbol{A}$ of $\boldsymbol{B}$ shown in the diagram for Case II.

In Case III, the intended elements of $\boldsymbol{B}$ are

$$\bot,\ e,\ \top' \wedge \top,\ \top \ \text{ and } \ f \wedge \top = e \vee (\top' \wedge \top).$$

That the operations in the lemmas turn this into a Dunn monoid (actually, an idempotent one) with neutral element e can be verified mechanically, the only real issues being the associativity of fusion and the law of residuation; we omit the details.

We shall call this Dunn monoid $\boldsymbol{T}_5$. It is clear from the above description of its elements that its skew reflection $\boldsymbol{A}$, in the diagram for Case III, is generated by $\bot$.

Finally, in Case IV, the intended elements of $\boldsymbol{B}$ are

$$\bot,\ e,\ \top' \wedge \top,\ e \vee (\top' \wedge \top),\ f \wedge \top \text{ and } \top.$$

We suppress the mechanical verification that this becomes a Dunn monoid, with neutral element e, when equipped with the operations in the lemmas.

We denote this Dunn monoid by $\boldsymbol{T}_6$. Again, the above description of its elements shows that its skew reflection $\boldsymbol{A}$, in the diagram for Case IV, is generated by $\bot$.

We have now proved the following.

Theorem 8.10. *The covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M are just*

$$\mathbb{V}(\mathrm{R}(\mathbf{2}^+)),\ \mathbb{V}(\mathrm{R}(\boldsymbol{S}_3^+)),\ \mathbb{V}(\mathrm{S}^{\leqslant}(\boldsymbol{S}_3^+)),\ \mathbb{V}(\mathrm{S}^{\leqslant}(\boldsymbol{C}_4^+)),\ \mathbb{V}(\mathrm{S}^{\leqslant}(\boldsymbol{T}_5)) \text{ and } \mathbb{V}(\mathrm{S}^{\leqslant}(\boldsymbol{T}_6)),$$

for the last four of which $\leqslant$ is as in the respective diagrams of Cases I–IV.

The claim that these varieties cover $\mathbb{V}(\boldsymbol{C}_4)$ will be strengthened in Theorem 8.13. To facilitate the proof, let $\boldsymbol{G}_1, \dots, \boldsymbol{G}_6$ abbreviate the six algebras mentioned in Theorem 8.10, so that $\mathbb{V}(\boldsymbol{G}_i)$, $i = 1, \dots, 6$, are the covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M. Their varietal join $\mathbb{V}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6)$ is locally finite, like any finitely generated variety. For each $i \in \{1, \dots, 6\}$, recall that $\boldsymbol{G}_i$ and $\boldsymbol{C}_4$ are the only subalgebras of $\boldsymbol{G}_i$ and are also, up to isomorphism, the only nontrivial homomorphic images of $\boldsymbol{G}_i$ (because $|(e]| = 3$ in $\boldsymbol{G}_i$). By Jónsson's Theorem, therefore,

$$\text{if } \emptyset \neq \mathsf{X} \subseteq \{\boldsymbol{G}_1, \dots, \boldsymbol{G}_6, \boldsymbol{C}_4\}, \text{ then } \mathbb{V}(\mathsf{X})_{\mathrm{SI}} = \mathbb{I}(\mathsf{X} \cup \{\boldsymbol{C}_4\}). \tag{36}$$

Lemma 8.11. *Let $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n \in \{\boldsymbol{G}_1, \dots, \boldsymbol{G}_6, \boldsymbol{C}_4\}$, where $0 < n \in \omega$. Then $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n$ are retracts of each algebra that embeds subdirectly into $\prod_{i=1}^n \boldsymbol{Z}_i$.*

Proof. The proof is by induction on n. The case $n = 1$ is trivial, so let $n > 1$. For $\boldsymbol{Z} := \prod_{i=1}^n \boldsymbol{Z}_i$, suppose that $\boldsymbol{A} \in \mathbb{S}(\boldsymbol{Z})$ and that $\pi_i[A] = Z_i$ for each canonical projection $\pi_i \colon \boldsymbol{Z} \to \boldsymbol{Z}_i$.

Let $\boldsymbol{B} = \pi[\boldsymbol{A}]$, where $\pi \colon \boldsymbol{Z} \to \prod_{i=1}^{n-1} \boldsymbol{Z}_i$ is the homomorphism

$$\langle z_1, \dots, z_{n-1}, z_n \rangle \mapsto \langle z_1, \dots, z_{n-1} \rangle.$$

Then $\boldsymbol{B}$ embeds subdirectly into $\prod_{i=1}^{n-1} \boldsymbol{Z}_i$, so by the induction hypothesis,

$$\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_{n-1} \text{ are retracts of } \boldsymbol{B}. \tag{37}$$

Also, $\boldsymbol{A}$ embeds subdirectly into $\boldsymbol{B} \times \boldsymbol{Z}_n$ and shall be identified here with the image of the obvious embedding. *Fleischer's Lemma* [2, Thm. 6.2] applies to any algebra that embeds subdirectly into the direct product of a pair of algebras from a congruence permutable variety—in particular, from any variety of IRLs. According to this lemma, there exist an algebra $\boldsymbol{C}$ and surjective homomorphisms $g\colon \boldsymbol{B} \longrightarrow \boldsymbol{C}$ and $h\colon \boldsymbol{Z}_n \longrightarrow \boldsymbol{C}$ such that

$$A = \{\langle x, y\rangle \in B \times Z_n : g(x) = h(y)\}. \tag{38}$$

As $\boldsymbol{C} \in \mathbb{H}(\boldsymbol{Z}_n)$, we may assume that $\boldsymbol{C}$ is $\boldsymbol{Z}_n$ or $\boldsymbol{C}_4$ or a trivial algebra.

If $\boldsymbol{C}$ is trivial, then $\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{Z}_n$, by (38), so the retracts of $\boldsymbol{A}$ include $\boldsymbol{B}$ and $\boldsymbol{Z}_n$ (as noted before Theorem 4.5) and hence all of $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n$, by (37).

If $\boldsymbol{C} = \boldsymbol{Z}_n \ncong \boldsymbol{C}_4$, then h is an isomorphism and

$$\boldsymbol{Z}_n \in \mathbb{H}(\boldsymbol{B}) \subseteq \mathbb{HP}_{\mathbb{S}}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_{n-1}) \subseteq \mathbb{V}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_{n-1}),$$

but $\boldsymbol{Z}_n$ is SI, so $\boldsymbol{Z}_n \in \mathbb{I}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_{n-1})$, by (36), whence $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n$ are retracts of $\boldsymbol{B}$, by (37). In this case, therefore, it suffices to show that $\boldsymbol{B}$ is a retract of $\boldsymbol{A}$. As $\mathrm{id}_B\colon \boldsymbol{B} \longrightarrow \boldsymbol{B}$ and $h^{-1} \circ g\colon \boldsymbol{B} \longrightarrow \boldsymbol{Z}_n$ are homomorphisms, so is the function $k\colon \boldsymbol{B} \longrightarrow \boldsymbol{B} \times \boldsymbol{Z}_n$ defined by $x \mapsto \langle x, h^{-1}g(x)\rangle$, and $k[B] \subseteq A$, by (38). Obviously, $\pi \circ k = \mathrm{id}_B$, so $\pi|_A$ is the desired retraction.

We may therefore assume, for the remainder of the proof, that $\boldsymbol{C} = \boldsymbol{C}_4$.

First, let $i \in \{1, \dots, n-1\}$. By (37), there are homomorphisms $r\colon \boldsymbol{Z}_i \longrightarrow \boldsymbol{B}$ and $s\colon \boldsymbol{B} \longrightarrow \boldsymbol{Z}_i$ (and hence $s \circ \pi|_A\colon \boldsymbol{A} \longrightarrow \boldsymbol{Z}_i$) with $s \circ r = \mathrm{id}_{Z_i}$. Because $g \circ r\colon \boldsymbol{Z}_i \longrightarrow \boldsymbol{C}_4 \in \mathbb{S}(\boldsymbol{Z}_n)$ is a homomorphism, so is the map $p\colon \boldsymbol{Z}_i \longrightarrow \boldsymbol{B} \times \boldsymbol{Z}_n$ defined by $x \mapsto \langle r(x), gr(x)\rangle$. Now $h|_{C_4}$ is an endomorphism of $\boldsymbol{C}_4$, which can only be id_{C_4}, so $gr(x) = hgr(x)$ for all $x \in \boldsymbol{Z}_i$, whence $p[Z_i] \subseteq A$, by (38). Clearly, $s \circ \pi|_A \circ p = \mathrm{id}_{Z_i}$, so $\boldsymbol{Z}_i$ is a retract of $\boldsymbol{A}$.

It remains to show that $\boldsymbol{Z}_n$ is a retract of $\boldsymbol{A}$. As $h\colon \boldsymbol{Z}_n \longrightarrow \boldsymbol{C}_4 \in \mathbb{S}(\boldsymbol{B})$ is a homomorphism, so is the function $t\colon \boldsymbol{Z}_n \longrightarrow \boldsymbol{B} \times \boldsymbol{Z}_n$ given by $x \mapsto \langle h(x), x\rangle$. Since the endomorphism $g|_{C_4}$ of $\boldsymbol{C}_4$ is the identity map, we have $gh(x) = h(x)$ for all $x \in Z_n$. Therefore, $t[Z_n] \subseteq A$, by (38), while $\pi_n|_A \circ t = \mathrm{id}_{Z_n}$. □

Theorem 8.12. *In the variety* $\mathbb{V}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6)$*, every finite subdirectly irreducible algebra* $\boldsymbol{E}$ *is projective.*

Proof. Recall that a [finitely generated] member of a variety K is projective in K iff it is a retract of each of its [finitely generated] homomorphic pre-images in K. Let $\boldsymbol{A} \in \mathsf{J} := \mathbb{V}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6)$ be a finitely generated homomorphic pre-image of $\boldsymbol{E}$. Then $\boldsymbol{A}$ is finite (as J is locally finite) and nontrivial. Also, $\mathsf{J}_{\mathrm{SI}} = \mathbb{I}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6, \boldsymbol{C}_4)$, by (36), and there are only finitely many maps from $\boldsymbol{A}$ to members of $\{\boldsymbol{G}_1, \dots, \boldsymbol{G}_6, \boldsymbol{C}_4\}$. Therefore, by the Subdirect Decomposition Theorem, there exist an integer $n > 0$ and (not necessarily distinct) algebras $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n \in \{\boldsymbol{G}_1, \dots, \boldsymbol{G}_6, \boldsymbol{C}_4\}$ such that $\boldsymbol{A}$ embeds subdirectly into $\prod_{i=1}^n \boldsymbol{Z}_i$. By Lemma 8.11, $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n$ are retracts of $\boldsymbol{A}$. Now $\boldsymbol{E}$ is an SI member of $\mathbb{HP}_{\mathbb{S}}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n) \subseteq \mathbb{V}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n)$, so $\boldsymbol{E} \in \mathbb{I}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n, \boldsymbol{C}_4)$, by (36). As $\boldsymbol{C}_4$ is a retract of each nontrivial member of M, this show that $\boldsymbol{E}$ is a retract of $\boldsymbol{A}$, and hence that $\boldsymbol{E}$ is projective in J. □

Theorem 8.13. *Every subquasivariety of* $\mathbb{V}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6)$ *is a variety.*

Proof. Gorbunov [14] proved that a locally finite variety K has no subquasivariety other than its subvarieties iff every finite SI member of K embeds into each of its homomorphic pre-images in K (cf. [24, Sec. 9]). This, with Theorem 8.12, delivers the result, because $\mathbb{V}(\boldsymbol{G}_1,\dots,\boldsymbol{G}_6)$ is locally finite. □

Combining Theorem 8.10 and Corollary 7.3, we obtain the following.

Corollary 8.14. *There are just ten covers of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* U, *viz. the six listed in Theorem* 8.10 *and* $\mathbb{V}(\boldsymbol{C}_5),\dots,\mathbb{V}(\boldsymbol{C}_8)$.

In contrast with Theorem 8.13, for each $n \in \{5,6,7,8\}$, the quasivariety $\mathbb{Q}(\boldsymbol{C}_n)$ omits $\boldsymbol{C}_4$, and is therefore strictly smaller than $\mathbb{V}(\boldsymbol{C}_n)$. Indeed, the quasi-equation $e = e \wedge f \implies x = y$ holds in $\boldsymbol{C}_n$ but not in $\boldsymbol{C}_4$.

By Theorem 7.2 and Corollaries 2.3 and 8.14, the non-semisimple covers of atoms in the subvariety lattice of DMM (regardless of join-irreducibility) are just $\mathbb{V}(\boldsymbol{S}_5)$ and the ones contained in U. All of these are finitely generated varieties. Example 9.5 will show, however, that $\mathbb{V}(\boldsymbol{C}_4)$ has at least one join-irreducible cover within DMM that is not finitely generated.

9. Other Covers of $\mathbb{V}(\boldsymbol{C}_4)$

We have seen that each cover of $\mathbb{V}(\boldsymbol{C}_4)$ within U is generated by a finite non-simple algebra. By Lemma 2.9(iii), a *simple* De Morgan monoid $\boldsymbol{A}$ is anti-idempotent if it has $\boldsymbol{C}_4$ as a subalgebra (cf. Theorem 7.2(iii)(3)). If $\boldsymbol{A}$ is finite as well, then it generates a cover of $\mathbb{V}(\boldsymbol{C}_4)$ exactly when $\boldsymbol{C}_4$ is its *only* proper subalgebra, by Jónsson's Theorem and the CEP. In that case, by the same arguments, $\mathbb{V}(\boldsymbol{A})$ is join-irreducible in the subvariety lattice of DMM.

In fact, $\mathbb{V}(\boldsymbol{C}_4)$ has infinitely many finitely generated covers within DMM witnessing Theorem 7.2(iii)(3), as the next example shows.

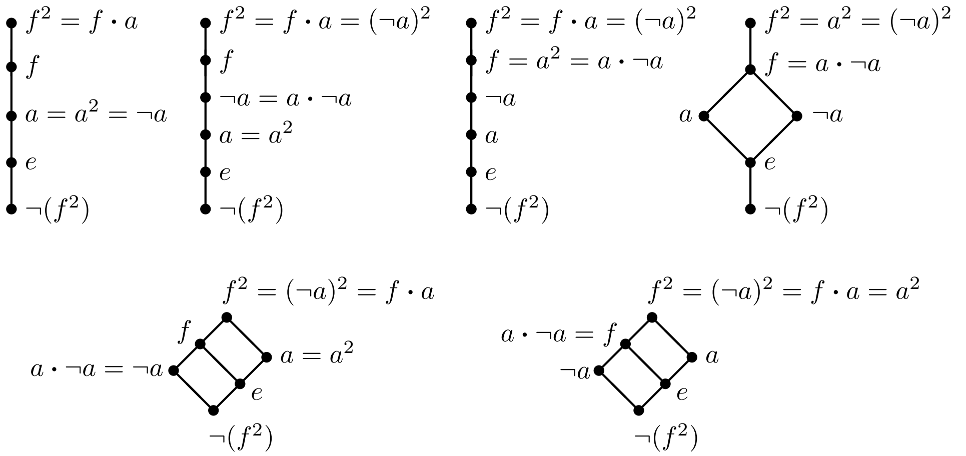
Example 9.1. For each positive integer p, let $\boldsymbol{A}_p$ be the rigorously compact De Morgan monoid on the chain $0 < 1 < 2 < 4 < 8 < \ldots < 2^p < 2^{p+1}$, where fusion is multiplication, truncated at 2^{p+1}. Thus, $|A_p| = p+3$ and e is the integer 1, while $f = 2^p$ and $\neg(2^k) = 2^{p-k}$ for all $k \in \{0,1,\dots,p\}$. Clearly, $\boldsymbol{A}_p$ is simple and generated by 2, and we may identify $\boldsymbol{C}_4$ with the subalgebra of $\boldsymbol{A}_p$ on $\{0,1,2^p,2^{p+1}\}$.

Now suppose p is prime. We claim that $\boldsymbol{C}_4$ is the only proper subalgebra of $\boldsymbol{A}_p$.

It suffices to show that, whenever $k \in \{1,2,\dots,p-1\}$, then $2 \in Sg^{\boldsymbol{A}_p}\{2^k\}$. The proof is by induction on k and the base case is trivial, so let $k > 1$. As p is prime, it is not divisible by k, whence there is a positive integer n such that $kn \in \{p-1, p-2, \dots, p-(k-1)\}$, so $\neg(2^{kn}) \in \{2,4,\dots,2^{k-1}\} \cap Sg^{\boldsymbol{A}_p}\{2^k\}$. Because $\neg(2^{kn}) = 2^r$, where $1 \leq r < k$, the induction hypothesis implies that $2 \in Sg^{\boldsymbol{A}_p}\{\neg(2^{kn})\} \subseteq Sg^{\boldsymbol{A}_p}\{2^k\}$, as required.

Thus, $\mathbb{V}(\boldsymbol{A}_p)$ is a (join-irreducible) cover of $\mathbb{V}(\boldsymbol{C}_4)$ within DMM. And by Jónsson's Theorem, $\mathbb{V}(\boldsymbol{A}_p) \neq \mathbb{V}(\boldsymbol{A}_q)$ for distinct primes p, q, vindicating the claim preceding this example.

The $\wedge, \vee$ reduct of a simple De Morgan monoid is a self-dual distributive lattice in which e is an atom and f a co-atom. It is therefore not difficult to verify that, up to isomorphism, there are just eight simple De Morgan monoids $\boldsymbol{A}$ on at most 6 elements (and none on 7 elements) such that $\boldsymbol{C}_4$ is the only proper subalgebra of $\boldsymbol{A}$. Six such algebras are depicted below; the other two are $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ from Example 9.1. Each of these eight De Morgan monoids is 1–generated and generates a (join-irreducible) cover of $\mathbb{V}(\boldsymbol{C}_4)$ exemplifying Theorem 7.2(iii)(3).



The exhaustiveness of this eight-item list will not be proved here, as we shall not rely on it below.[4] Some features of the covers of $\mathbb{V}(\boldsymbol{C}_4)$ consisting of semilinear algebras deserve to be established, however. We consider first the case where $\boldsymbol{A}$ has an idempotent element outside $\boldsymbol{C}_4$.

Theorem 9.2. *Let $\boldsymbol{A}$ be a simple totally ordered De Morgan monoid, having $\boldsymbol{C}_4$ as a proper subalgebra, and suppose $a^2 = a \in A \backslash C_4$. Then a generates a subalgebra of $\boldsymbol{A}$ isomorphic to one of the first two algebras pictured above.*

Proof. By Lemmas 2.2(iv) and 2.9(iii), $\boldsymbol{A}$ is anti-idempotent, with $e < a < f$ and $e < \neg a < f$. Now $a \leqslant \neg a$, by (1), as $a^2 \leqslant f$. Also, $a \cdot \neg a = \neg a$, by (5), and $f \cdot a = f^2 = f \cdot \neg a$, by Theorem 2.4(iv). If $\neg a = a$, then $\boldsymbol{Sg}^{\boldsymbol{A}}\{a\}$ matches the first of the two pictured algebras. If $\neg a \nleqslant a$ then $(\neg a)^2 \nleqslant f$, by (1), whence $(\neg a)^2 = f^2$ and $\boldsymbol{Sg}^{\boldsymbol{A}}\{a\}$ matches the second pictured algebra. $\square$

Now we consider the case where $\boldsymbol{A}$ has no idempotent element outside $\boldsymbol{C}_4$, assuming that $\boldsymbol{A}$ is finite.

[4] Readers wanting to confirm it should note that all self-dual distributive lattices on 5, 6 or 7 elements are pictured above, except for the 7–element chain (ruled out by Theorem 9.4) and the 7–element lattice that stacks one 4–element diamond on another, gluing them at the juncture. The latter supports several simple De Morgan monoids $\boldsymbol{A}$ that extend $\boldsymbol{C}_4$, but in each case, the vertical 'midpoint' a of $\boldsymbol{A}$ is a fixed point of $\neg$, and $a \cdot f = f^2$, by Theorem 2.4(iv) and Lemma 2.9(iii), while $a \leqslant a^2 = a \cdot \neg a \leqslant f$, so $a^2 \in [a, f] = \{a, f\}$. Thus, $\boldsymbol{Sg}^{\boldsymbol{A}}\{a\}$ is a proper subalgebra of $\boldsymbol{A}$, strictly containing $\boldsymbol{C}_4$, so $\mathbb{V}(\boldsymbol{A})$ does not cover $\mathbb{V}(\boldsymbol{C}_4)$. The arguments for the lattices depicted above are no more difficult.

Theorem 9.3. *Let $\boldsymbol{A}$ be a finite simple totally ordered De Morgan monoid, having $\boldsymbol{C}_4$ as a proper subalgebra, where no element of $A \backslash C_4$ is idempotent.*

Let c be the cover of e in $\boldsymbol{A}$, and n the smallest positive integer such that $c^{n+1} = c^{n+2}$. Then

(i) *$c^n = f$ and $c^{n+1} = f^2$;*
(ii) *$\neg(c^{m+1}) < c^{n-m} \leqslant \neg(c^m)$ for each positive integer $m < n$;*
(iii) *$b \cdot \neg b = f$ for all $b \in A \backslash \{f^2, \neg(f^2)\}$.*

If, moreover, $|A|$ is odd, then $\boldsymbol{Sg}^{\boldsymbol{A}}\{a\} \cong \boldsymbol{A}_2$ (as defined in Example 9.1), where a is the fixed point of $\neg$ in $\boldsymbol{A}$.

Proof. Again, recall that $\boldsymbol{A}$ is anti-idempotent, with $A = \{\neg(f^2), f^2\} \cup [e, f]$, and note that f covers $\neg c$, by definition of c.

(i) As c^{n+1} is idempotent, we have $e < c^{n+1} \in C_4$ (by assumption), whence $c^{n+1} = f^2$. As c^n is not idempotent, $c^n < f^2$, i.e., $c^n \leqslant f$. But $c^n \nleqslant \neg c$ (by (1), since $c^{n+1} \nleqslant f$), so $c^n = f$, because f covers $\neg c$.

(ii) Consider a positive integer $m < n$. We cannot have $c^{n-m} \leqslant \neg(c^{m+1})$, otherwise $c^{n+1} = c^{m+1} \cdot c^{n-m} \leqslant c^{m+1} \cdot \neg(c^{m+1}) \leqslant f$ (by (3)), a contradiction. Thus, $\neg(c^{m+1}) < c^{n-m}$. By (2), $c^{n-m} \leqslant c^m \to c^n = c^m \to f = \neg(c^m)$.

(iii) Let $b \in A \backslash \{f^2, \neg(f^2)\}$. Then $b \cdot \neg b \leqslant f$, by (3). Since $b \cdot \neg b = f$ for $b \in \{e, f\}$, we may assume that $e < b < f$, i.e., $c \leqslant b \leqslant \neg c$. Suppose $b \cdot \neg b < f$, i.e., $b \cdot \neg b \leqslant \neg c$. Then $b \cdot c \leqslant b$, by (1). As $c \leqslant b < f = c^n$, we have $c^p \leqslant b < c^{p+1}$ for some positive integer $p < n$. Then $c^{p+1} \leqslant b \cdot c \leqslant b < c^{p+1}$, a contradiction. Thus, $b \cdot \neg b = f$.

Finally, let $|A|$ be odd, so $\neg a = a$ for some (unique) $a \in A$, as $\neg$ is a bijection. Then $a \notin C_4$, so $a^2 = a \cdot \neg a = f$, by (iii), whence $Sg^{\boldsymbol{A}}\{a\} = C_4 \cup \{a\}$ and $\boldsymbol{Sg}^{\boldsymbol{A}}\{a\} \cong \boldsymbol{A}_2$. □

The third example pictured above shows that, in Theorem 9.3, when $\boldsymbol{A}$ has even cardinality, it need not have a subalgebra of the form $\boldsymbol{A}_p$ for $p > 1$.

Theorem 9.4. *If $\mathbb{V}(\boldsymbol{A})$ is a cover of $\mathbb{V}(\boldsymbol{C}_4)$ within* DMM, *where $\boldsymbol{A}$ is finite, simple and totally ordered, then $|A|$ is 5 or an even number.*

Proof. The hypothesis implies that $\boldsymbol{C}_4$ is the only proper subalgebra of $\boldsymbol{A}$, as noted earlier. If $|A|$ is odd, then $|A| \neq 6$, so by Theorems 9.2 and 9.3, $\boldsymbol{A}$ has a 5–element subalgebra, which cannot be proper, so $|A| = 5$. □

In the statement of Theorem 7.2(iii)(3), the algebra $\boldsymbol{A}$ cannot always be chosen finite, in view of the following example.

Example 9.5. The set $B = \{0\} \cup \{2^n : n \in \omega\} \cup \{\infty\}$ is the universe of a rigorously compact Dunn monoid $\boldsymbol{B}$ whose lattice order is the conventional total order, and whose fusion is ordinary multiplication on the finite elements of B (hence $e = 1$). For finite nonzero $b, c \in B$, the value of $b \to c$ is c/b if b divides c; otherwise it is 0. It is well known that there is a unique totally ordered De Morgan monoid $\boldsymbol{A}_\infty$, having $\boldsymbol{B}$ as an RL–subreduct and having exactly the additional elements indicated and ordered below:

$$0 < 1 < 2 < 4 < 8 < 16 < \ldots < \neg 16 < \neg 8 < \neg 4 < \neg 2 < \neg 1 < \infty.$$

Here, $b \cdot \neg c = \neg(b \to c)$ and $\neg b \cdot \neg c = \infty$ for all finite nonzero $b, c \in B$. Note that $\boldsymbol{A}_\infty$ is generated by 2. The subalgebra of $\boldsymbol{A}_\infty$ on $\{0, 1, \neg 1, \infty\}$ is isomorphic to $\boldsymbol{C}_4$. Clearly, $\boldsymbol{A}_\infty$ is simple, so $\boldsymbol{A}_\infty \notin \mathsf{W}$, whence $\boldsymbol{A}_\infty$ is not the reflection of a Dunn monoid.

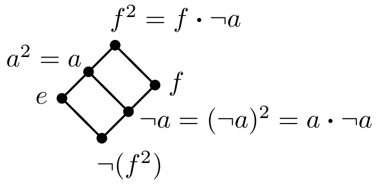
By Corollary 2.3, every SI algebra $\boldsymbol{C} \in \mathbb{V}(\boldsymbol{A}_\infty)$ embeds into an ultrapower of $\boldsymbol{A}_\infty$, and it is easily deduced that $\boldsymbol{C}$ contains an isomorphic copy of $\boldsymbol{A}_\infty$, unless $\boldsymbol{C} \cong \boldsymbol{C}_4$. In particular, $\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{D}_4 \notin \mathbb{V}(\boldsymbol{A}_\infty)$, and $\mathbb{V}(\boldsymbol{A}_\infty)$ is not generated by its finite members. This establishes that $\mathbb{V}(\boldsymbol{A}_\infty)$ is a join-irreducible cover of $\mathbb{V}(\boldsymbol{C}_4)$ within DMM, exemplifying Theorem 7.2(iii)(3), and that $\mathbb{V}(\boldsymbol{A}_\infty)$ is not finitely generated.

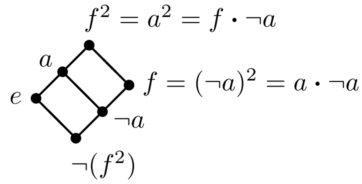
Actually, $\boldsymbol{A}_\infty$ embeds naturally into an ultraproduct of the algebras $\boldsymbol{A}_p$ (p a positive prime) from Example 9.1, so $\mathbb{V}(\boldsymbol{A}_\infty)$ is contained in the varietal join of the $\mathbb{V}(\boldsymbol{A}_p)$.

10. Covers of $\mathbb{V}(\boldsymbol{D}_4)$

Suppose $\boldsymbol{D}_4$ is a subalgebra of an FSI De Morgan monoid $\boldsymbol{A}$. Then A is the *disjoint* union of the anti-isomorphic sublattices $[e)$ and $(f]$ of $\langle A; \wedge, \vee \rangle$, by Theorem 2.6(i). Consequently, if $\boldsymbol{A}$ is finite, then $|A|$ is even. Also, if $\boldsymbol{A}$ is simple (cf. Theorem 7.2(iv)), then it is anti-idempotent, by Lemma 2.9(iii). When $\boldsymbol{A}$ is both finite and simple, then $\boldsymbol{D}_4$ is the sole proper subalgebra of $\boldsymbol{A}$ iff $\mathbb{V}(\boldsymbol{A})$ is a cover of $\mathbb{V}(\boldsymbol{D}_4)$ within DMM, in which case $\mathbb{V}(\boldsymbol{A})$ is join-irreducible (the arguments being just as for $\boldsymbol{C}_4$).

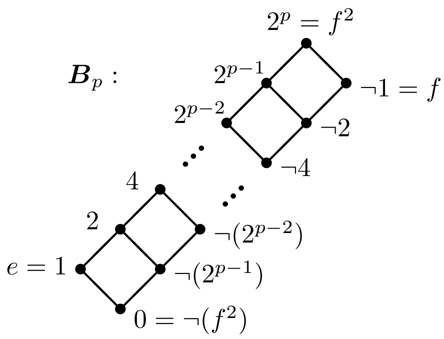
Example 10.1. Each of the simple De Morgan monoids depicted below generates a cover of $\mathbb{V}(\boldsymbol{D}_4)$ within DMM, witnessing Theorem 7.2(iv). Up to isomorphism, they are the only two such algebras on 6 elements. (The one on the left is a subalgebra of the eight-element 'Belnap lattice'; it is obtained by removing from that structure the elements labeled 2 and -2 in [1, p. 252]. The one on the right is isomorphic to the algebra $\boldsymbol{B}_2$ in Example 10.2 below.)





In analogy with the case of $\boldsymbol{C}_4$, there are infinitely many finitely generated covers of $\mathbb{V}(\boldsymbol{D}_4)$ within DMM, as well as a cover that is not finitely generated (nor even generated by its finite members). This is shown by the next two examples.

Example 10.2. For each positive integer p, it can be checked that there is a unique rigorously compact (simple) De Morgan monoid $\boldsymbol{B}_p$ having the labeled Hasse diagram and fusion indicated below, where it is understood that $m, n, k, \ell \in \omega$ with $m, n \leq p$ and $k, \ell < p$.



$$2^m \cdot 2^n = 2^{\min\{m+n,p\}}$$

$$2^m \cdot \neg(2^\ell) = \begin{cases} \neg(2^{\ell-m}) & \text{if } \ell \geq m; \\ 2^p & \text{if } \ell < m. \end{cases}$$

$$\neg(2^k) \cdot \neg(2^\ell) = \begin{cases} \neg(2^{k+\ell-p}) & \text{if } k+\ell \geq p; \\ 2^p & \text{if } k+\ell < p. \end{cases}$$

The subalgebra of $\boldsymbol{B}_p$ on $\{0, 1, \neg 1, 2^p\}$ may be identified with $\boldsymbol{D}_4$.

Now suppose p is prime. We claim that $\boldsymbol{B}_p$ has no proper subalgebra other than $\boldsymbol{D}_4$. It suffices (by involution properties) to show, by induction on k, that $2 \in Y := Sg^{\boldsymbol{B}_p}\{2^k\}$ for each positive integer $k < p$. The base case is trivial, so let $k > 1$. As k does not divide p, we have $r := p - kn \in \{1, 2, \ldots, k-1\}$ for some positive integer n, so $\neg(2^{p-r}) = \neg(2^{kn}) \in Y$, whence $2^r = e \vee \neg(2^{p-r}) \in Y$. By the induction hypothesis, $2 \in Sg^{\boldsymbol{B}_p}\{2^r\}$, so $2 \in Y$, completing the proof. Thus, $\mathbb{V}(\boldsymbol{B}_p)$ is a cover of $\mathbb{V}(\boldsymbol{D}_4)$ within DMM and, by Jónsson's Theorem, $\mathbb{V}(\boldsymbol{B}_p) \neq \mathbb{V}(\boldsymbol{B}_q)$ for distinct primes p, q.

Example 10.3. In each $\boldsymbol{B}_p$ above, the element e has a unique cover. That is a first order property, so it persists in the rigorously compact simple ultraproduct $\prod_p \boldsymbol{B}_p/\mathcal{F}$, for each nonprincipal ultrafilter $\mathcal{F}$ over the set of positive primes. By similar applications of Łos' Theorem, in any such ultraproduct, the rigorously compact simple subalgebra $\boldsymbol{B}_\infty$ generated by the cover of e (still denoted by 2) has the infinite lattice reduct shown in the next diagram, and its fusion is determined by the following additional information, where m, n are positive integers:

$$f \cdot x = f^2 \text{ whenever } x \in B_\infty \backslash \{0, e\}$$

$$2^m \cdot 2^n = 2^{m+n}$$

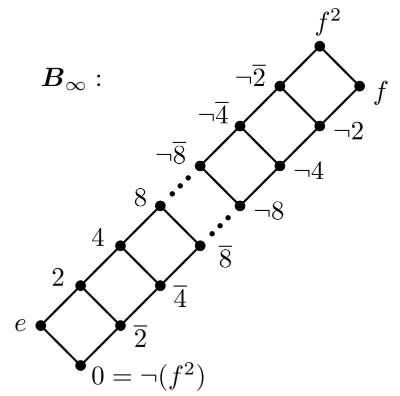
$$2^m \cdot \overline{2^n} = \overline{2^{m+n}} = \overline{2^m} \cdot \overline{2^n}$$

$$\neg(2^m) \cdot \neg\overline{2^n} = f^2 = \neg\overline{2^m} \cdot \neg\overline{2^n} = \neg(2^m) \cdot \neg(2^n)$$

$$2^m \cdot \neg\overline{2^n} = \left\{ \begin{array}{ll} \neg\overline{2^{n-m}} & \text{if } m \leq n \\ f^2 & \text{if } m > n \end{array} \right\} = 2^m \cdot \neg(2^n) = \overline{2^m} \cdot \neg(2^n) = \overline{2^m} \cdot \neg\overline{2^n}.$$

We claim that $\mathbb{V}(\boldsymbol{B}_\infty)$ is a join-irreducible cover of $\mathbb{V}(\boldsymbol{D}_4)$ within DMM, not generated by its finite members. For this, it suffices, as in Example 9.5, to establish the following.

Fact 10.4. *Let $\boldsymbol{D}$ be a subalgebra of an ultrapower of $\boldsymbol{B}_\infty$, where $\boldsymbol{D} \ncong \boldsymbol{D}_4$. Then $\boldsymbol{B}_\infty$ can be embedded into $\boldsymbol{D}$.*



Proof. (Sketch) Identifying $\boldsymbol{D}_4$ with the 0–generated subalgebra of the ultrapower $\boldsymbol{U}$ (and hence of $\boldsymbol{D}$), we see that $\boldsymbol{D}$ is anti-idempotent, rigorously compact and simple, and that D is the disjoint union of its subsets $[e)$ and $(f]$. We may choose $a \in D \backslash D_4$, because $\boldsymbol{D} \not\cong \boldsymbol{D}_4$ and $\boldsymbol{D}_4$ is finite. Membership and non-membership of $\boldsymbol{D}_4$ are first order properties, because e is distinguished. We can arrange that

$$e < a < a^2 < f^2,$$

because the following properties of $\boldsymbol{B}_\infty$ are expressible as universal first order sentences (which therefore persist in both $\boldsymbol{U}$ and $\boldsymbol{D}$):

$$x = x^2 \implies x \in D_4;$$
$$(x \notin D_4 \;\;\&\;\; x^2 = f^2) \implies e < (e \vee \neg x) < (e \vee \neg x)^2 < f^2.$$

(So, we may replace a by $e \vee \neg a \in D$ if $a^2 = f^2$, and by $e \vee a \in D$ if $e \not< a$ with $a^2 < f^2$.) As 2 generates $\boldsymbol{B}_\infty$, there is at most one homomorphism $h \colon \boldsymbol{B}_\infty \to \boldsymbol{D}$ sending 2 to a. To see that h is well defined and injective, it suffices (by (13)) to show that for any unary term α in $\cdot, \wedge, \neg, e$, we have

$$e \leqslant \alpha^{\boldsymbol{B}_\infty}(2) \;\text{ iff }\; e \leqslant \alpha^{\boldsymbol{D}}(a).$$

This can be shown by induction on the complexity of α. At the inductive step, the case of $\wedge$ is trivial, while $\neg$ is straightforward, because D is the disjoint union of $[e)$ and $(f]$. Fusion requires an examination of subcases, which is aided by noting that $\boldsymbol{B}_\infty$ (and hence $\boldsymbol{D}$) has properties of the following kind, where n, m, p are any *fixed* positive integers with $p \geq m$:

$$e < x < x^2 < f^2 \implies (x^m \cdot x^n = x^{m+n} \;\;\&\;\; x^m \cdot (e \vee \neg(x^p)) = e \vee \neg(x^{p-m})). \quad \square$$

References

[1] A.R. Anderson, N.D. Belnap, Jnr., 'Entailment: The Logic of Relevance and Necessity, Vol. 1', Princeton University Press, 1975.

[2] C. Bergman, 'Universal Algebra. Fundamentals and Selected Topics', CRC Press, Taylor & Francis, 2012.

[3] C. Bergman, R. McKenzie, *Minimal varieties and quasivarieties*, J. Austral. Math. Soc. Ser. A **48** (1990), 133–147.

[4] W.J. Blok, P. Köhler, D. Pigozzi, *On the structure of varieties with equationally definable principal congruences II*, Algebra Universalis **18** (1984), 334–379.
[5] W.J. Blok, D. Pigozzi, 'Algebraizable Logics', Memoirs of the American Mathematical Society 396, Amer. Math. Soc., Providence, 1989.
[6] S. Burris, H.P. Sankappanavar, 'A Course in Universal Algebra', Graduate Texts in Mathematics, Springer-Verlag, New York, 1981.
[7] J.M. Dunn, 'The Algebra of Intensional Logics', PhD thesis, University of Pittsburgh, 1966.
[8] J.M. Dunn, *Algebraic completeness results for R-mingle and its extensions*, J. Symbolic Logic **35** (1970), 1–13.
[9] E. Fried, E.W. Kiss, *Connection between congruence-lattices and polynomial properties*, Algebra Universalis **17** (1983), 227–262.
[10] N. Galatos, *Minimal varieties of residuated lattices*, Algebra Universalis **52** (2005), 215–239.
[11] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, 'Residuated Lattices. An Algebraic Glimpse at Substructural Logics', Studies in Logic and the Foundations of Mathematics 151, Elsevier, 2007.
[12] N. Galatos, J.G. Raftery, *A category equivalence for odd Sugihara monoids and its applications*, J. Pure Appl. Algebra **216** (2012), 2177–2192.
[13] N. Galatos, J.G. Raftery, *Idempotent residuated structures: some category equivalences and their applications*, Trans. Amer. Math. Soc. **367** (2015), 3189–3223.
[14] V.A. Gorbunov, *Lattices of quasivarieties*, Algebra and Logic **15** (1976), 275–288.
[15] J. Hart, L. Rafter, C. Tsinakis, *The structure of commutative residuated lattices*, Internat. J. Algebra Comput. **12** (2002), 509–524.
[16] B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. **21** (1967), 110–121.
[17] B. Jońsson, 'Topics in universal algebra', Lecture Notes in Math., Vol. 250, Springer-Verlag, Berlin and New York, 1972.
[18] B. Jónsson, *Congruence distributive varieties*, Math. Japonica **42** (1995), 353–401.
[19] R.K. Meyer, *Conservative extension in relevant implication*, Studia Logica **31** (1972), 39–46.
[20] R.K. Meyer, *On conserving positive logics*, Notre Dame J. Formal Logic **14** (1973), 224–236.
[21] R.K. Meyer, *Sentential constants in R and* $R^{\neg}$, Studia Logica **45** (1986), 301–327.
[22] R.K. Meyer, J.M. Dunn, H. Leblanc, *Completeness of relevant quantification theories*, Notre Dame J. Formal Logic **15** (1974), 97–121.
[23] T. Moraschini, J.G. Raftery, J.J. Wannenburg, *Varieties of De Morgan monoids: minimality and irreducible algebras*, manuscript, 2017. Available at `http://uivty.cs.cas.cz/ moraschini/files/submitted/VarietiesofDMMs1.pdf`
[24] J.S. Olson, J.G. Raftery, *Positive Sugihara monoids*, Algebra Universalis **57** (2007), 75–99.
[25] J.K. Slaney, *3088 varieties: a solution to the Ackermann constant problem*, J. Symbolic Logic **50** (1985), 487–501.
[26] J.K. Slaney, *On the structure of De Morgan monoids with corollaries on relevant logic and theories*, Notre Dame J. Formal Logic **30** (1989), 117–129.
[27] J.K. Slaney, *Sentential constants in systems near R*, Studia Logica **52** (1993), 443–455.
[28] A. Urquhart, *The undecidability of entailment and relevant implication*, J. Symbolic Logic **49** (1984), 1059–1073.
[29] A. Wroński, *The degree of completeness of some fragments of the intuitionistic propositional logic*, Rep. Math. Logic **2** (1974), 55–62.

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