Open strings and analytic non-integrability

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Submitted in partial fulfilment of the requirements for the degree
Master of Science (Physics)

In the Faculty of Natural & Agricultural Sciences
University of Pretoria
Pretoria

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17 March 2021

Abstract

The AdS/CFT correspondence has proved to be a powerful tool in the analysis of many systems of interest in theoretical physics. Strongly coupled gauge theories that are difficult to solve can be determined by using gravitational theory instead. Based on this concept the integrability, or rather non-integrability of a gauge theory system can be determined using semi-classical string solutions. Integrability has become a coveted property in a system. It indicates that the system can be solved fully. Since there is no systematic method to check for integrability, the analytic non-integrability method was conceived to provide a way to test for non-integrability in string theory by following a set of fixed steps. This method has been able to test for non-integrability in a variety of backgrounds containing closed string solutions. One question that remains is whether the method can be extended to include open strings.

In this dissertation, the analytic non-integrability method is used to test for non-integrability for an open string solution ending on a $Y = 0$, maximal giant graviton. The solution that is used is the Hofman-Maldacena giant magnon. The method is also tested for open strings ending on a D5 and D7 brane. Two variations are used for the metric of the D7 brane. These are the $S^2 \times S^2$ and the nested S^4 metrics of the S^5.

The method was able to reproduce the expected results for the D5 brane and the giant graviton. This is a strong indication that the method can be successfully adapted when checking for non-integrability in open string solutions. There is potential for the method to conclusively prove non-integrability in the D7 brane case if an appropriate open string solution can be found.

Dedication

For my family. Mom for always pushing me to do what I love. Sareesha for being the best big sister and role model I could ever hope for. Ushir for always believing in me and making me laugh. Jay my much better half for love, guidance and unwavering support, my cheerleader. Everything I do is for the four of you.

Declaration

I, **Divania Carelse** declare that the thesis/dissertation, which I hereby submit for the degree **Master of Science(Physics)** at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.
SIGNATURE: ....
DATE: ....

Acknowledgements

This research would not be possible without the funding received from the NRF. I am so grateful for this opportunity.

I would like to thank my supervisor Prof. Konstantinos Zoubos for being a great mentor, keeping my anxiety at bay with his calm and relaxed demeanour. I have learned so much from you. None of this would be possible without your help. Thank you for always taking the time to answer all my questions and concerns whether they were huge problems or little queries.

I would like to thank my husband Jay, for taking care of everything so I have the time and energy to further my education. Thank you for all the lunches made with love, coffee runs and staying up with me if I needed to finish something. You really made this a team effort.

Abbreviations

LIST OF ABBREVIATIONS

AdS	Anti-de Sitter Space
CFT	Conformal Field Theory
SCFT	Super Conformal Field Theory
QFT	Quantum Field Theory
QCD	Quantum Chromo Dynamics
NVE	Normal Variational Equation
SYM	Supersymmetric Yang-Mills

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Chapter 1

Introduction

An important advancement in theoretical physics in recent years is the AdS/CFT correspondence. The correspondence essentially claims that some d dimensional gauge theories can be described by a gravitational theory in $d+1$ dimensions [1]. Strongly coupled gauge theories are usually not easily solved. The correspondence allows for the use of the gravitational theory, that is more easily solved at strong coupling to gain insight into the gauge theory. Certain gauge theories have a property known as integrability [2]. This property allows in principle for a system to be fully solvable at any value of the gauge coupling.

It is not always possible to prove that a system is integrable. No systematic method or algorithm exists to show that a given gauge theory is integrable. Based on the AdS/CFT conjecture, the concept of integrability can be further explored. The analytic non-integrability method provides a way to test if a given theory is non-integrable from the string theory side. The method was first applied in [3] and provides a way to test if a system is non-integrable as long as it has a known AdS/CFT dual. This is important since there isn't a way to test for non-integrability on the gauge theory side.

The method consists of studying the motion of classical strings on the gravitational background. A solution is selected that satisfies the string equations of motion. By linearising around this solution, an attempt is made to reduce the equations of motion to a classical Hamiltonian system. Once this is done, the Kovacic algorithm [4] can be applied to check if the system is non-integrable. If the linearised system is non-integrable then so is the original system. Finding non-integrability on the string theory side translates to a lack of quantum integrability on the gauge theory side.

This method has shown great success in showing non-integrability for closed string solutions. The hope is that it can be extended to the case of open strings. Closed strings satisfy periodic boundary conditions. The open string has either Dirichlet or Neumann boundary conditions depending on the set up being studied. These boundary conditions complicate what solutions are allowed by adding more constraints to the system. The question is, how do these open string boundary conditions affect the way the analytic non-integrability method is applied?

Before the method can be successfully used there are some important concepts that need introducing. Chapter 2 offers an introduction to classical string theory and some important concepts crucial for the study of these string systems. The classical string action and the equations of motion will be discussed. An explanation of the string boundary conditions will also be given. This will be followed by a brief overview of the AdS/CFT correspondence, an important discovery that makes this work possible. A review of some important string solutions will be discussed. The remainder of the chapter will be dedicated to discussing the concept of integrability and the analytic non-integrability method, with some examples of how the method is applied to closed strings.

In chapter 3, the method will be used to test for non-integrability in an open string ending on a giant graviton. The Hofman-Maldacena giant magnon solution will be used as the open string solution [5]. Finally chapters 4 and 5 will consist of testing for non-integrability of an open string ending on a D5 brane and D7 brane respectively.

Appendix A offers a map from the embedding coordinates for the giant graviton to the intrinsic coordinates. Most studies containing giant gravitons use embedding coordinates. However, all the calculations done in this dissertation use the intrinsic coordinate system. It is therefore necessary to have a link between these coordinate systems so that the set-ups and results can be directly compared. Appendix B checks if the Polyakov action for an S^5 can be written as a Principal Chiral Model. Since the Principal Chiral Model is integrable being able to show the equivalence to a Polyakov description means that the Polyakov action is integrable. It is not possible to write a string system on a general background as a Principal Chiral model. However, since S^5 is a symmetric space it can be done. Although this calculation is more suited to a study on integrability as opposed to non-integrability, it was an interesting exercise to gain insight on integrable systems. The calculations in this dissertation were done using Mathematica. Appendix C provides a sample of the code that was used.

The results obtained for the giant graviton and the D5 brane were consistent with expectations from the gauge theory. This provides confidence that the analytic non-integrability method can be consistently expanded to open strings. The method will be a useful in the study of non-integrability in these systems.

Chapter 2

Introduction to String Theory and Analytic Non-Integrability

This chapter contains some of the background knowledge needed to gain a better understanding of open strings and analytic non-integrability. Starting with a brief study of introductory string theory, the Nambu-Goto and Polyakov actions are discussed. Next, an overview of the AdS/CFT correspondence proposed by Maldacena [1] will be given. Finally, there will be a discussion on analytic non-integrability and the Kovacic algorithm.

2.1 Introduction to String Theory

String theory has its origins in the S-matrix theory research direction, started in 1943 by Werner Heisenberg. S-matrix theory was popular from the 1950s into the 60s. Later in 1968, Gabriele Veneziano worked on developing a theory for the nuclear forces that arise from the interactions of hadrons. At the time no one realised that this model had any relevance to string theory. Leonard Susskind, Yoichiro Nambu and Holger Bech Nielsen independently found that they could derive the Veneziano formulation from looking at particles as strings instead of points. Their idea was that quarks are connected by tiny one-dimensional strings [6]. String theory was supposed to describe the strong interactions between quarks but failed at this task. It required twenty six dimensions as well as a particle known as a tachyon. The tachyon is a massless particle that travels faster than the speed of light. Later it was understood that QCD was the correct theory to describe the strong interactions [6]. Although string theory failed in its original purpose it still has merit. While Veneziano's model only consisted of particles acting through the strong force and did not include fermions, Pierre Ramond's reformulation took into account particle spin. This allowed for the inclusion of fermions and bosons. String theory is now able to reproduce Yang Mills gauge theory in the low energy limits, electromagnetism and general relativity [7]. It still gives rise to extra spatial dimensions beyond the usual three observable dimensions. However, it only requires ten instead of the initial twenty six and no longer requires the tachyon. Despite any shortcomings, by thinking of particles as strings of finite length, the re-normalization problem that occurs in general relativity disappears. String theory also provides new results in mathematics, for example mirror symmetry, which describes the interrelation between topologically different Calabi-Yau manifolds [8].

2.1.1 From a Relativistic Point Particle to the Nambu-Goto Action

If a particle is moving in Minkowski space of D- dimensions with the metric $\eta_{\mu\nu} = diag(-1, 1, 1, ..., 1)$ and fixed coordinates $X^\mu = (t, \vec{x})$, the action will be $S = -m \int dt \sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}$ [9]. The particle's motion can be described by giving its position in terms of $D - 1$ functions of time. In order to have re-parametrization invariance, an important property for the point particle action, a new Lagrangian that treats space and time equally will be required. To this end, time is redefined as a dynamical degree of freedom. While the particle may or may not move in space, it must move in time [9]. Introducing a new parameter τ along the world-line allows the particle's motion to be described by D functions $X^\mu(\tau)$. A new action can be formulated using these functions,

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}} \tag{2.1}$$

with $\mu = 0, ..., D - 1$ and $\dot{X}^\mu = \frac{dX^\mu}{d\tau}$. The τ parametrization is arbitrary, picking another parameter $\tilde{\tau}$ will make no difference [8]. After applying an integration by parts, the variation of the new action in (2.1) is given by,

$$\delta S = -m \int d\tau \dot{u}_\mu \delta X^\mu \tag{2.2}$$

with the normalised D-velocity $u^\mu = \frac{\dot{X}^\mu}{\sqrt{-\dot{X}^\nu \dot{X}_\nu}}$ [9]. Taking $u^\mu = 0$ describes the free motion of the particle. There is another action that can be used to describe the relativistic point particle. Introducing another field $e(\tau)$, the action in terms of this field will be,

$$S = \frac{1}{2} \int d\tau (e^{-1} \dot{X}^2 - em^2). \tag{2.3}$$

Here $\dot{X}^2 = \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}$. In this action it appears as though the world line theory is coupled to one dimensional gravity with $e(\tau)$ acting as the einbein. This new action is re-parametrization invariant. Equation(2.3) can be brought to the form

$$S = -\frac{1}{2} \int d\tau \sqrt{-g_{\tau\tau}} (g^{\tau\tau} \dot{X}^2 + m^2), \tag{2.4}$$

by denoting $e(\tau) = \sqrt{-g_{\tau\tau}}$ and where $g_{\tau\tau} = (g^{\tau\tau})^{-1}$ is the world line metric. $e(\tau)$ transforms as a density whilst each X^μ transforms as a scalar on the world line.

Moving on to the case of a string, a worldsheet is a two-dimensional manifold that describes the embedding of the string in spacetime. This worldsheet can be parametrized by a spacelike coordinate σ and a timelike coordinate τ. The two worldsheet coordinates can be combined as $\sigma^\alpha = (\tau, \sigma)$. For a string the action is required to be proportional to the area, A, of the worldsheet. This action needs to describe the string dynamics while being re-parametrization invariant. The metric that will be induced on the worldsheet is the pull-back of the Minkowski flat metric,

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}. \tag{2.5}$$

The action is defined by,

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-det\gamma}, \tag{2.6}$$

where T is the tension of the string. With the definitions $\dot{X}^\mu = \frac{\partial X^\mu}{\partial\tau}$ and $X^{\mu\prime} = \frac{\partial X^\mu}{\partial\sigma}$, the pullback metric can be expressed as,

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix} \tag{2.7}$$

where $\dot{X}^2 = \dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu}$ and $\dot{X}\cdot X' = \dot{X}^\mu X'^\mu\eta_{\mu\nu} = \dot{X}^\mu X'_\mu$. Equation (2.6) can be rewritten in terms of (2.7) as

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-\dot{X}^2X'^2 + (\dot{X}\cdot X')^2}. \tag{2.8}$$

Equation (2.8) is the Nambu-Goto Action for a relativistic string.

The Nambu-Goto action has two types of symmetries. The Poincaré invariance of the spacetime is the first of these symmetries. It is a global symmetry from the worldsheet perspective. The second is reparametrization invariance which is a gauge symmetry.

The momenta (Π) need to be introduced in order to derive the Nambu-Goto string equations of motion. From equation (2.8),

$$\Pi^\tau_\mu = \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} = \frac{-T[(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu]}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} \tag{2.9}$$

$$\Pi^\sigma_\mu = \frac{\partial\mathcal{L}}{\partial X'^\mu} = \frac{-T[(\dot{X}\cdot X')\dot{X}_\mu - (X')^2X'_\mu]}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}. \tag{2.10}$$

The equations of motion in terms of Π are,

$$\frac{\partial\Pi^\tau_\mu}{\partial\tau} + \frac{\partial\Pi^\sigma_\mu}{\partial\sigma} = 0. \tag{2.11}$$

From the action in equation (2.6), along with the pullback metric (2.7) and the equation for the variation of a determinant,

$$\delta\sqrt{-\gamma} = \frac{1}{2}\sqrt{-\gamma}\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta}, \tag{2.12}$$

an alternate form for the equations of motion can be derived. These equations are given by,

$$\partial_\alpha(\sqrt{-det\gamma}\gamma^{\alpha\beta}\partial_\beta X^\mu) = 0. \tag{2.13}$$

The square root in equation (2.8) makes the action difficult to quantize using path integration so a different approach is necessary [8].

2.1.2 The Polyakov Action

There is an alternate form for the string action that is classically equivalent to the Nambu-Goto action. This string action is known as the Polyakov action. This action eliminates the square root in the Nambu-Goto action [6]. However, a new field needs to be introduced,

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \tag{2.14}$$

here $g = detg$. The new field $g_{\alpha\beta}$, is introduced on the worldsheet as a dynamical metric. The Polyakov action is a group of scalar fields X coupled to 2 dimensional gravity from the worldsheet perspective [8]. Based on the Polyakov action the equation of motion for X^μ is,

$$\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta X^\mu) = 0. \tag{2.15}$$

This looks similar to equation (2.13). However, $g_{\alpha\beta}$ is an independent variable determined by its own equation of motion. The $g_{\alpha\beta}$ equation of motion can be derived by varying the action and using the relation in (2.12),

$$\delta S = -\frac{T}{2}\int d^2\sigma\delta g^{\gamma\beta}[\sqrt{-g}\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}\sqrt{-g}g_{\alpha\beta}g^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X^\nu]\eta_{\mu\nu} = 0. \tag{2.16}$$

From (2.16), the worldsheet metric is determined as,

$$g_{\alpha\beta} = 2f(\sigma,\tau)\partial_\alpha X \cdot \partial_\beta X, \tag{2.17}$$

where $f(\sigma,\tau)$ is determined from $f^{-1} = g^{\rho\sigma}\partial_\rho X \cdot \partial_\sigma X$. The metric, $g_{\alpha\beta}$, differs from $\gamma_{\alpha\beta}$ by f, the conformal factor. This does not present a problem because the f drops out of the equations of motion (2.15), since the $\sqrt{-g}$ term is proportional to f [1] and the inverse metric $g^{\alpha\beta}$ is proportional to f^{-1}. This means that the Polyakov and Nambu-Goto actions result in equivalent equations of motion.

The Polyakov action has the same two symmetries as the Nambu-Goto action as well as a third unique symmetry called the Weyl Invariance. The Weyl invariance is the invariance of the theory under a local change of scale that conserves the angles between all lines [8].

The conformal gauge is defined as a gauge in which the worldsheet metric is Minkowski. By selecting the conformal gauge in this case, the theory can be simplified to the flat metric on the worldsheet in Minkowski coordinates. Choosing the flat metric simplifies the Polyakov action to the theory of D scalar fields as well as free fields. The simplified action is given by,

$$S = -\frac{T}{2}\int d^2\sigma\partial_\alpha X \cdot \partial^\alpha X. \tag{2.18}$$

The equations of motion based on this simplified action in (2.18), for X^μ, reduces to the free wave equation,

$$\partial_\alpha\partial^\alpha X^\mu = 0. \tag{2.19}$$

The variation of the action (2.18) with respect to the metric gives rise to the stress-energy tensor $T_{\alpha\beta}$,

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\partial S}{\partial g^{\alpha\beta}}. \tag{2.20}$$

[1] Since g is a tensor density, the $\sqrt{-g}$ is a tensor density with a weight of 1. Since $\sqrt{-g}$ is a tensor density it will transform according to the tensor transformation law [10].

The equations of motion associated with $g_{\alpha\beta}$ occurs when $T_{\alpha\beta} = 0$ or

$$T_{01} = \dot{X} \cdot X' = 0 \tag{2.21}$$

$$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0. \tag{2.22}$$

The string equations of motion are the free wave equations given by (2.19) subject to the two constraints (2.21) and (2.22). These constraints are called the Virasoro constraints. By turning the Nambu-Goto action into the Polyakov action it becomes easier to quantise since the equations of motion are linear.

2.1.3 Open and Closed Strings

A string can take on two forms, open or closed, the classification is determined by the endpoints of the string [6]. Open strings, as the name suggests, are open ended and consist of a worldsheet diffeomorphic to $\mathbb{R} \times [0, \sigma = \pi]$. Closed strings consist of end points that are joined together and a worldsheet diffeomorphic to the cylinder $\mathbb{R} \times S^1$ [6]. Any theory of open strings must also include closed strings. This is because open string endpoints can join together to form a closed string [8]. Open strings require boundary conditions to be imposed on the end points while closed strings do not.

Deriving the open string boundary conditions

To obtain the open string boundary conditions, the first focus is the region of the spacetime where $-\infty \leq \tau \leq \infty$ and $0 \leq \sigma \leq l$, where l is the length of the string. Next the light-cone gauge is selected, in other words the following three conditions must be satisfied [9],

$$X^+ = \tau, \tag{2.23}$$

$$\partial_\sigma \gamma_{\sigma\sigma} = 0 \tag{2.24}$$

and finally,

$$det\gamma_{ab} = 0. \tag{2.25}$$

The timelike coordinate τ must be selected in accordance with equation (2.23), whilst the function $f = \gamma_{\sigma\sigma}(-det\gamma_{ab})^{-\frac{1}{2}}$, transforms as $f'd\sigma' = fd\sigma$ with a fixed τ and a re-parametrization of σ. This allows the invariant length $fd\sigma = dl$ to be defined [9]. If σ is proportional to $\int dl$ from the $\sigma = 0$ endpoint, then the constant of proportionality can be obtained by requiring the $\sigma = l$ endpoint to always be fixed. The $0 \leq \sigma \leq l$ region must remain unchanged so that the τ dependence may be removed. A Weyl transformation must be performed in order for equation (2.25) to be satisfied. Since f is Weyl-invariant $\delta_\sigma f = 0$, this means that equation (2.25) implies (2.24) [9]. The gauge conditions required by (2.25) can be satisfied for $\gamma_{\tau\tau}(\tau, \sigma)$. As a result of the σ independence of $\gamma_{\sigma\sigma}$, the independent degrees of freedom in the metric can be determined by $\gamma_{\sigma\sigma}(\tau)$ and $\gamma_{\sigma\tau}(\tau, \sigma)$. The inverse metric is,

$$\begin{bmatrix} \gamma^{\tau\tau} & \gamma^{\tau\sigma} \\ \gamma^{\sigma\tau} & \gamma^{\sigma\sigma} \end{bmatrix} = \begin{bmatrix} -\gamma_{\sigma\sigma}(\tau) & \gamma_{\tau\sigma}(\tau, \sigma) \\ \gamma_{\tau\sigma}(\tau, \sigma) & \gamma_{\sigma\sigma}^{-1}(\tau)(1 - \gamma_{\tau\sigma}^2(\tau, \sigma)) \end{bmatrix}. \tag{2.26}$$

By separating $X^-(\tau,\sigma)$ into two parts, $x^-(\tau)$ and $Y^-(\tau,\sigma)$ with,

$$x^-(\tau) = \frac{1}{l}\int_0^l d\sigma X^-(\tau,\sigma) \tag{2.27}$$

and

$$Y^-(\tau,\sigma) = X^-(\tau,\sigma) - x^-(\tau), \tag{2.28}$$

the Polyakov Lagrangian is,

$$\begin{aligned}\mathcal{L} = -\frac{T}{2}\int_0^l d\sigma[\gamma_{\sigma\sigma}(2\partial_\tau x^- - \partial_\tau X^i\partial_\tau X^i)\\ - 2\gamma_{\sigma\tau}(\partial_\sigma Y^- - \partial_\tau X^i\partial_\sigma X^i) + \gamma_{\sigma\sigma}^{-1}(1-\gamma_{\tau\sigma}^2)\partial_\sigma X^i\partial_\sigma X^i].\end{aligned} \tag{2.29}$$

In this case $Y^-(\tau,\sigma)$ acts as a Lagrange multiplier [9]. The boundary conditions for the light-cone gauge are therefore,

$$\gamma_{\tau\sigma}\partial_\tau X^\mu - \gamma_{\tau\tau}\partial_\sigma X^\mu = 0, \tag{2.30}$$

at $\sigma = 0, l$.

For $\mu = +$, $\gamma_{\tau\sigma} = 0$ at $\sigma = 0, l$. In fact as a result of $\partial_\sigma^2\gamma_{\tau\sigma}$ being zero, $\gamma_{\tau\sigma}$ is zero everywhere.

For $\mu = i$, the boundary condition is simply,

$$\partial_\sigma X^i = 0, \tag{2.31}$$

at $\sigma = 0, l$.

Classification of Open String Boundary Conditions

The three types of boundary conditions that need to be considered for open strings are:

- Neumann
- Dirichlet
- Mixed.

The Neumann boundary conditions can be thought of as an open string with ends that are free to move in the spacetime [6]. These boundary conditions are,

$$\partial_\sigma X^\mu = 0 \quad \text{at} \quad \sigma = 0, \pi. \tag{2.32}$$

The Dirichlet boundary conditions are given by,

$$\delta X^\mu = 0 \quad \text{at} \quad \sigma = 0, \pi. \tag{2.33}$$

This expression in (2.33) means that the endpoints of the string are fixed to end on a sub-manifold of spacetime. The hypersurface that the brane is connected to when Dirichlet conditions are imposed is called a D-brane. The brane will be located at the fixed position $X^\mu = c^\mu$. More generally there will be a D_p-brane where p is the amount of dimensions. For example a D_0-brane corresponds to a particle and a D_1-brane corresponds to a string [8]. The directions that are parallel to the brane are called Dirichlet while the transverse directions are called Neumann.

The last case consists of mixed boundary conditions where one end of the string is free to move in the spacetime and the other end is fixed to a D-brane. The free end will require Neumann boundary conditions whilst the fixed end requires Dirichlet boundary conditions to be applied.

2.2 AdS/CFT Correspondence

The AdS/CFT correspondence first proposed by Maldacena in [1] claims that there is an equivalence between classical supergravity theory in five dimensional AdS spacetime and strongly coupled four dimensional gauge theory. It is sometimes referred to as holographic theory. By definition, a hologram takes a three dimensional image and encodes it onto a two dimensional surface. Holographic theory encodes a five dimensional theory by a four dimensional one. Following the work first presented in [11] and [12], string theory has a dual holographic description in terms of gauge fields. The correspondence allows fundamental type IIB strings to be identified in a ten dimensional anti-de-Sitter cross sphere background with the dual, maximally supersymmetric Yang-Mills theory with gauge group $SU(N)$ in four dimensions ($\mathcal{N} = 4$ SYM). This spacetime is written as $AdS_5 \times S^5$. The duality was first introduced by 't Hooft [13] who determined that in the large-N limit, the perturbation expansion of $SU(N)$ gauge theories may be interpreted as a genus expansion of two dimensional surfaces constructed using Feynman diagrams. The quantity $\frac{1}{N}$ counts the genus of these diagrams and N also denotes the rank of the gauge group. The 't Hooft coupling λ counts the quantum loops and is given by $\lambda = g_{YM}^2 N$ where g_{YM} is the gauge theory coupling constant. For a review of the correspondence see [14].

The string model has two important parameters. These parameters form the link between the string theory and the gauge theory. The parameters are the coupling constant g_s and the string tension T, which can be expressed as $\frac{R^2}{\alpha'}$, where R denotes the radius of S^5 as well as AdS_5 and α' is the Regge slope. There are also two parameters that govern the gauge theory, λ and N. The correspondence relates these parameters by the following expressions,

$$g_s = \frac{4\pi\lambda}{N} \qquad \text{and} \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}.$$

Another important feature of the conjecture is the AdS/CFT dictionary that relates the excitations of these two theories. In simple terms, the energy eigenvalues of the string denoted by E is hypothesised as being equal to Δ. In this context Δ is the scaling dimension of the gauge theory dual operator. The scaling dimension is calculated from the two point function of the CFT. It is possible to do perturbative calculations on the gauge theory side where $\lambda \ll 1$. However, at a strong coupling where $\lambda \gg 1$, these calculations become difficult. The AdS/CFT duality claims that a gauge theory can be analysed using a weakly curved Anti de Sitter spacetime, or AdS for short, at a strong coupling. The de Sitter spacetime, with a constant positive curvature, provides a solution to the Einstein equation. By extension AdS is a spacetime of constant negative curvature. There is a naturally occurring idea that the AdS spacetime contains a spatial boundary. This boundary is called the AdS boundary and the gauge theory lives on this four dimensional boundary [15]. The result of the correspondence is a duality where the weakly coupled gauge fields are described by highly quantum strings moving in a curved spacetime while the strongly coupled gauge fields are described by classical gravity. There is also a relationship between the symmetries of these two theories. Global symmetries in both the AdS and the CFT correspond to one another. Conformal and R-symmetries on the gauge theory are isometries of the full $AdS_5 \times S^5$ geometry. In particular, the four-dimensional conformal group $SO(4, 2)$ corresponds to isometries of AdS_5. The $SO(6)$ R-symmetry corresponds to the isometry of the five-sphere (S^5).

	String Quantum Number	Gauge Operator
$SO(2,4)$ Labels	E	Δ
	J_1	s_1
	J_2	s_2
S^5 Labels	S_1	R_1
	S_2	R_2
	S_3	R_3

Table 2.1: Table showing which string quantum numbers correspond to the operators on the dual gauge theory.

As in table 2.1, the string is labelled by the energy E and the J_1 and J_2 quantum numbers of $SO(2,4)$. The three spins S_1, S_2 and S_3 are the rotations of the string in the S^5. The scaling dimension Δ of the dual gauge theory must match the energy. The spins s_1 and s_2 corresponds to J_1 and J_2. The R-charges of the scalar fields, R_1, R_2 and R_3 of the $\mathcal{N}=4$ SYM equate to the rotations S_1, S_2 and S_3 respectively. Although there is a great deal of evidence that the AdS/CFT correspondence works, the strong-weak nature of the duality prevents the conjecture from being proven.

While the AdS/CFT correspondence was originally introduced to study the quantum behaviour of gauge invariant theories, it has since been extended to numerous other uses. For example, in non conformal theories it gives rise to an explanation for confinement and Chiral symmetry breaking. Additionally, the correspondence is used to study non-equilibrium phenomena in strongly coupled plasmas. In recent years it has been applied to condensed matter physics. The correspondence verifies the ideas about the behaviour of gauge theories in the large N limit, by naturally implementing the 't Hooft large N expansion [16].

For further reading see [17] and [18] in addition to the references mentioned above.

2.3 Review of String Solutions

As a result of the AdS/CFT correspondence the solution of semi-classical strings should correspond to long gauge invariant operators in the dual theory. This section presents a review of semi-classical spinning string solutions following[14] and [17]. Beginning with the metric of $AdS_5 \times S^5$ spacetime,

$$ds^2_{AdS_5} = d\rho^2 - \cosh^2\rho dt^2 + \sinh^2\rho(d\theta^2 + \cos^2\theta d\varphi_1^2 + \sin^2\theta d\varphi_2^2) \tag{2.34}$$

$$ds^2_{S^5} = d\gamma^2 + \cos^2\gamma d\phi_3^2 + \sin^2\gamma(d\psi^2 + \cos^2\psi d\phi_1^2 + \sin^2\psi d\phi_2^2), \tag{2.35}$$

the Polyakov action is,

$$\begin{aligned} S = -\frac{T}{2}\int d^2\sigma[&\cosh^2\rho(t'^2 - \dot{t}^2) - \dot{\gamma}^2 + \gamma'^2 - \dot{\rho}^2 + \rho'^2 \\ &- \sin^2\gamma(\cos^2\psi(\dot{\phi_1}^2 - \phi_1'^2) + \dot{\psi}^2 - \psi'^2 + \sin^2\psi(\dot{\phi_2}^2 - \phi_2'^2)) \\ - \sinh^2\rho(\dot{\theta}^2 - \theta'^2 &+ \cos^2\theta(\dot{\varphi_1}^2 - \varphi_1'^2) + \sin^2\theta(\dot{\varphi_2}^2 - \varphi_2'^2)) - \cos^2\gamma(\dot{\phi_3}^2 - \phi_3'^2)]. \end{aligned} \tag{2.36}$$

The equations of motion for each of the coordinates are,

$$
\begin{aligned}
\varphi_1 &: 2\sin\theta\sinh\rho(\dot\theta\dot\varphi_1 - \theta'\varphi_1') + 2\cos\theta\cosh\rho(\rho'\varphi_1' - \dot\rho\dot\varphi_1) + \cos\theta\sinh\rho(\varphi_1'' - \ddot\varphi_1) = 0,\\
\varphi_2 &: 2\cos\theta\sinh\rho(\theta'\varphi_2' - \dot\theta\dot\varphi_2) + 2\sin\theta\cosh\rho(\rho'\varphi_2' - \dot\rho\dot\varphi_2) + \sin\theta\sinh\rho(\varphi_2'' - \ddot\varphi_2) = 0,\\
\theta &: 2\sin 2\rho(\theta'\rho' - \dot\theta\dot\rho) + \sin 2\theta\sinh^2\rho(\varphi_1'^2 - \varphi_2'^2 - \dot\varphi_1{}^2 + \dot\varphi_2{}^2) + 2\sinh^2\rho(\theta'' - \ddot\theta) = 0,\\
\rho &: \cosh\rho\sinh\rho(t'^2 - \dot t^2 - \theta'^2 + \dot\theta^2 - \cos^2\theta(\varphi_1'^2 - \dot\varphi_1{}^2) - \sin^2\theta(\varphi_2'^2 - \dot\varphi_2{}^2)) + \rho'' - \ddot\rho = 0,\\
t &: 2\sinh\rho(\dot t\dot\rho - t'\rho') + \cosh\rho(\ddot t - t'') = 0,\\
\phi_1 &: 2\cos\gamma\cos\psi(\gamma'\phi_1' - \dot\gamma\dot\phi_1) - 2\sin\gamma\sin\psi(\psi'\phi_1' - \dot\psi\dot\phi_1) + \cos\psi\sin\gamma(\phi_1'' - \ddot\phi_1) = 0,\\
\phi_2 &: 2\cos\gamma\sin\psi(\gamma'\phi_2' - \dot\gamma\dot\phi_2) + 2\sin\gamma\cos\psi(\psi'\phi_2' - \dot\psi\dot\phi_2) + \sin\psi\sin\gamma(\phi_2'' - \ddot\phi_2) = 0,\\
\phi_3 &: 2\sin\gamma(\dot\gamma\dot\phi_3 - \gamma'\phi_3') + \cos\gamma(\phi_3'' - \ddot\phi_3) = 0,\\
\gamma &: \cos\gamma\sin\gamma(\cos^2\psi(\phi_1'^2 - \dot\phi_1{}^2) + \sin^2\psi(\phi_2'^2 - \dot\phi_2{}^2) - \phi_3'^2 + \dot\phi_3{}^2 + \psi'^2 - \dot\psi^2) + \gamma'' - \ddot\gamma = 0,\\
\psi &: \sin^2\gamma(\sin 2\psi(\dot\phi_1{}^2 - \phi_1'^2 + \phi_2'^2 - \dot\phi_2{}^2) + 2\psi'' - 2\ddot\psi) + 2\sin 2\gamma(\dot\gamma\dot\psi - \gamma'\psi') = 0,
\end{aligned}
$$

where the 'prime' denotes a derivative with respect to σ and a 'dot' is a derivative with respect to τ. The Virasoro constraints are,

$$
\begin{aligned}
&-\cosh^2\rho\dot t t' + \dot\gamma\gamma' + \sinh^2\rho(\dot\theta\theta' + \sin^2\theta\dot\varphi_2\varphi_2' + \cos^2\theta\dot\varphi_1\varphi_1') + \dot\rho\rho'\\
&+ \sin^2\gamma(\cos^2\psi\dot\phi_1\phi_1' + \sin^2\psi\dot\phi_2\phi_2' + \dot\psi\psi') + \cos^2\gamma\dot\phi_3\phi_3' = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\cosh^2\rho(\dot t^2 + t'^2) + \dot\gamma^2 + \gamma'^2 + \sinh^2\rho(\dot\theta^2 + \theta'^2 + \cos^2\theta(\dot\varphi_1{}^2 + \varphi_1'^2 + \sin^2\theta(\dot\varphi_2{}^2 + \varphi_2'^2))\\
&+ \dot\rho^2 + \rho'^2 + \sin^2\gamma(\cos^2\psi(\dot\phi_1{}^2 + \phi_1'^2)\\
&+ \sin^2\psi(\dot\phi_2{}^2 + \phi_2'^2) + \dot\psi^2 + \psi'^2) + \cos^2\gamma(\dot\phi_3{}^2 + \phi_3'^2) = 0.
\end{aligned}
$$

Rotating point particle

The most basic example to consider is the rotating point particle on S^5. This configuration is simply a degenerated string [14]. Consider the ansatz,

$$
t = \kappa\tau, \qquad \rho = 0, \qquad \gamma = \frac{\pi}{2}, \qquad \phi_1 = \kappa\tau, \qquad \phi_2 = \phi_3 = \psi = 0. \tag{2.37}
$$

This particular choice for the coordinates satisfies both the equations of motion and the Virasoro constraints listed above. The cyclic coordinates [2] $(t, \varphi_1, \varphi_2, \phi_1, \phi_2, \phi_3)$ lead to conserved charges $(E, S_1, S_2, J_1, J_2, J_3)$ respectively. These charges correspond to the energy E, spin S and angular momentum J [14]. For the point particle there are only two non zero quantities to be considered, E and J_1,

$$
E = \frac{\partial\mathcal{L}}{\partial\dot t} = \sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\cosh^2\rho\dot t = \sqrt{\lambda}\kappa \tag{2.38}
$$

and

$$
J_1 = -\frac{\partial\mathcal{L}}{\partial\dot\phi_1} = \sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\sin^2\gamma\cos^2\psi\dot\phi_1 = \sqrt{\lambda}\kappa. \tag{2.39}
$$

In the classical limit $E = J$. However, quantum fluctuations may be considered around these solutions.

[2]Cyclic coordinates correspond to $U(1)$ isometries.

Spinning String Solutions

Consider a closed string rotating in an S^3 within an S^5. The string is evolving in time with $J_3 = 0$ and $S_1 = S_2 = 0$. The string is chosen to be at the centre of AdS_5 so $\rho = 0$. With the ansatz,

$$t = \kappa\tau, \quad \rho = 0, \quad \gamma = \frac{\pi}{2}, \quad \phi_1 = \omega_1\tau, \quad \phi_2 = \omega_2\tau, \quad \phi_3 = 0, \quad \psi = \psi(\sigma), \tag{2.40}$$

the choice $\gamma = \frac{\pi}{2}$ corresponds to the equator of the S^5. The action will be defined as,

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau \int_0^{2\pi} d\sigma[\kappa^2 + \psi'^2 - \omega_1^2\cos^2\psi - \omega_2^2\sin^2\psi]. \tag{2.41}$$

The only non-trivial equation of motion is,

$$\psi'' + \sin\psi\cos\psi(\omega_2^2 - \omega_1^2) = 0. \tag{2.42}$$

Defining a new variable $\omega_{21}^2 = \omega_2^2 - \omega_1^2$ and integrating (2.42) yields,

$$\psi' = \omega_{21}\sqrt{q - \sin^2\psi}, \tag{2.43}$$

where q is an integration constant. There will be two distinct cases that arise, one where $q \leq 1$ and $q > 1$. The $q \leq 1$ case results in the well known folded string solution [18]. The string has $q = \sin^2\psi_0$ and extends from $-\psi_0$ to ψ_0 with $\psi' = 0$ at the points where the string turns and folds back onto itself. When $q > 1$, the derivative ψ' is never zero, this results in the circular string that spans a full circle on the S^3. The Virasoro constraints produce an equation for the integration constant,

$$q = \frac{\kappa^2 - \omega_1^2}{\omega_{21}^2} \tag{2.44}$$

where $\omega_{21} \neq 0$. Finally the two angular momenta J_1 and J_2 and the energy E are given by [14],

$$E = \sqrt{\lambda} \tag{2.45}$$

$$J_1 = \sqrt{\lambda}\omega_1\int_0^{2\pi}\frac{d\sigma}{2\pi}\cos^2\psi(\omega) \tag{2.46}$$

$$J_2 = \sqrt{\lambda}\omega_2\int_0^{2\pi}\frac{d\sigma}{2\pi}\sin^2\psi(\omega). \tag{2.47}$$

Solving each of these integrals produces $\frac{1}{2}$. So simplifying equations (2.46) and (2.47) results in,

$$\sqrt{\lambda} = \frac{2J_1}{\omega_1} \quad \text{and} \quad \sqrt{\lambda} = \frac{2J_2}{\omega_2}. \tag{2.48}$$

This leads to an expression for the energy,

$$\sqrt{\lambda} = \frac{J_1}{\omega_1} + \frac{J_2}{\omega_2}. \tag{2.49}$$

The energy of both of these solutions has been determined on the S^3 in the S^5 as a function of J_1 and J_2, the commuting angular momenta. This energy (E) should coincide with the scaling dimension (Δ) of the dual gauge theory. The angular momenta J_1 and J_2 correspond to the spin s_1 and s_2 of the scalar fields of $\mathcal{N} = 4$ SYM.

2.4 What is Integrability?

For a system to be integrable it must exhibit certain properties. The precise definition of integrability is different between different fields. However, the consensus is that an integrable system is fully solvable. In the context of AdS/CFT, integrability provides a link between the strongly coupled and weakly coupled theory. The presence of integrability suggests that physical observables such as dimensions that are functions of the 't Hooft coupling λ should match. These observables calculated from large values of λ in the bulk should correspond to the values obtained from small values of λ on the gauge theory side. The value of λ at intermediate coupling can not always be determined. However, the presence of integrability suggests that it does exists. It is only natural to want to find other integrable structures in a respective theory. The AdS/CFT correspondence allows for determining integrability, or for the purpose of this research, non-integrability on the bulk and relating it to the gauge theory side. As a result only classical integrability for Hamiltonian systems will be briefly discussed in the remainder of this section. The first important concept to address, is what is a Hamiltonian system? From [19], the Hamiltonian H, is a function that consists of integral curves defined by a vector field X_H. These vector fields are known as Hamiltonian vector fields and are defined on the symplectic manifold [3] M. The Hamiltonian is related to X_H by $X_H = \flat^{-1} \cdot dH$, where $\flat$ is a map from the tangent to the cotangent vector bundle $i.e$ $\flat : TM \to T^*M$. The differential equations associated with X_H are called a Hamiltonian system and can be written as,

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = \frac{\partial H}{\partial x_i}. \tag{2.50}$$

To define the Liouville Theorem, consider a Hamiltonian system defined in $2d$-dimensional phase space. The system is described by the canonical variables (x_i, p_i) such that $i = 1, \ldots, d$. Then H will be a function of x_i and p_i. The system requires the Poisson brackets to fulfil the following relations,

$$\{x_i, p_j\} = \delta_{i,j} \qquad \text{and} \qquad \{x_i, x_j\} = \{p_i, p_j\} = 0, \tag{2.51}$$

for all $i, j = 1, \ldots, d$. A system is said to be Liouvillian integrable if there exists independent [4] conserved quantities F_i such that $\{F_i, F_j\} = 0$ with $i, j = 1, \ldots, d$. This conservation requirement of F_i means $\{H, F_i\} = 0$ which in turn implies that H must be a function of F_i. The Liouville theorem states that for a Liouvillian integrable system, the equations of motion of can be solved by straight forward integration. For more information as well as a proof of the Liouville theorem see [20]. There is only one method currently available to determine the integrability of a Hamiltonian system and it is based on the Lax representation. A system is integrable whenever a Lax pair can be determined. A Lax pair is defined as two NxN functions $L(x, p)$ and $M(x, p)$ that are matrix valued and equivalent to the Hamiltonian equations by the following relationship,

$$\dot{L} = [L, M] \leftrightarrow \begin{pmatrix} \dot{x}_i = \{x_i, H\} \\ \dot{p}_i = \{p_i, H\} \end{pmatrix} \tag{2.52}$$

[3] A symplectic manifold is a smooth manifold in differential geometry, that contains a closed non-degenerate two-form ω known as the sympletic form.

[4] In this context independent refers to the set of linearly independent one-forms dF_i.

where $\dot{L} = [L, M]$ is known as the Lax equation. There is no test or systematic approach to check if the Lax pair exists or to find it explicitly [21].

2.5 Analytic Non-integrability and the Kovacic Algorithm

2.5.1 Analytic Non-Integrability Method

The analytic non-integrability method is a definitive method to determine if a given system is non-integrable. Integrability is a coveted property in gauge theory since it implies that the theory is solvable at any value of the gauge coupling. As a result of the AdS/CFT correspondence, if a system is integrable on the string side it directly translates to integrability of the boundary of the gauge theory. Proving that a system is integrable can often be a cumbersome process. It is not always possible to determine the precise Lax connection required to definitively declare a theory as integrable. The method of analytic non-integrability is a more systematic approach. Due to the condition that integrability has to appear everywhere in a given theory, a single instance of non-integrability is enough to state that the entire theory is non-integrable. An important condition for finding non-integrability is that it is still possible to have integrable sub-sectors within the background. As a result of this condition, it is incorrect to infer that a system is integrable if non-integrability is not found. The solution used in the method could simply be contained in the integrable sub-sector of the theory [22].

In order to prove non-integrability in a system of differential equations, the variational equation around a specified solution needs to be analysed. The variational equation refers to the linear system obtained from linearising the vector field around that specific solution. If within a given class of functions, the variational equation does not produce any first integrals, the original non linear system is non-integrable. The Kovacic algorithm [4] provides a test for non-integrability. Once the NVE is in an appropriate form, the Kovacic algorithm can be applied. This algorithm only emits solutions if the system is integrable, thus no solution equates to non integrability.

In order to use the analytic non-integrability method, the system of equations must first be reduced to a two coordinate system, for example $\theta(x)$ and $\phi(x)$, with two non-trivial equations of motion. This is done by selecting trivial solutions for the other coordinates in the system. Select the straight line solution for one of the remaining two coordinates. This is defined by a simple solution, such as $\theta(x) = 0$, that must solve the equation of motion for the selected coordinate. This solution will simplify the equation of motion for ϕ. It is not necessary to determine the solution for ϕ exactly, it can simply be defined as $\bar{\phi}(x)$. Next substitute the $\bar{\phi}$ solution into the θ equation of motion. Consider fluctuations $\eta(x)$ around the straight line solution for θ and the $\bar{\phi}$ solution. The θ equation of motion will become a second order differential equation for η. The $\bar{\phi}$ that is still contained in the η equation of motion will need to be eliminated because the NVE needs to be a second order linear differential equation. This is done by making an appropriate substitution with a new variable, for example z. All the derivatives of η must be determined in terms of this new variable z. After doing this, the NVE will have the form

$f(z)\eta''(z) + g(z)\eta'(z) + h(z)\eta(z) = 0$. Once the NVE has been determined it is entered into the Kovacic algorithm. If no solution is determined by the Kovacic algorithm then the system is non-integrable. If a solution is found, the system may be integrable or the fluctuations considered around the straight line solution that was used to determine the NVE may be contained in an integrable sub-sector of the theory.

The method can be summarised in the following three steps:

- Select the straight line solution for one of the coordinates.
- Compute the NVE.
- Apply the Kovacic algorithm to check the NVE for integrability.

2.5.2 The Kovacic Algorithm

The Kovacic algorithm proposed in 1986 , provides a method to compute a fundamental system of solutions for a second order differential equation as long as that differential equation is integrable. This property makes the algorithm a good check to test for non-integrability in a given system. If the system is non-integrable the algorithm will fail to provide any solution. The Kovacic algorithm is included in symbolic computation software such as Maple. The Maple implementation is used in this dissertation. A comprehensive overview of the algorithm can be found in [19], so only a brief summary will be provided here.

A second order differential equation with the set of coefficients $C(z)$ can be written as an NVE. Within these sets of coefficients there may be a Liouvillian algebra on which the possibility of integrability hinges.

$$\eta'' + g(z)\eta = 0 \tag{2.53}$$

where $g(z) \in C(z)$. Equation (2.53) is Liouvillian integrable if and only if the equation,

$$\frac{d\eta}{dz} = A\eta \tag{2.54}$$

with $A \in Mat(m, K)$ has an algebraic solution. In order for an algebraic solution to exist, the degree n of the associated polynomial $Q(v)$ that is calculated in the algorithm, using differential Galois theory, and has coefficients contained in $C(x)$ must belong to a specific set of possible degrees called L [19]. The algorithm can be broken down into three steps. The first step is to compute the possible n values that make up the set L. If L is the empty set then (2.53) is non-integrable. The values of L determined in step one are fixed throughout steps two and three. Steps two and three attempts to calculate the polynomial $Q(v)$ if it exists, beginning with the first element of L. If the algorithm fails to compute the polynomial then then next value of n is selected and another attempt is made. If the last n in L is selected and no such polynomial exists then (2.54) does not have an algebraic solution and equation (2.53) is non-integrable and by extension so is the original system. If the polynomial is determined then the algebraic solution for (2.54) can be found. The algebraic solution being found means the equation in (2.53) will be integrable.

2.5.3 Testing for Non-integrability in Closed Strings

Following the examples given in [3], this section provides some examples of calculations testing the analytic non-integrability method for closed string solutions.

The S^5 example

The first example is that of the S^5, this particular case has been proven to be integrable ([23] and [24]), by the construction of a Lax pair, so the Kovacic algorithm should produce a solution. The metric of S^5 can be written in terms of the CP^2 metric,

$$ds^2_{CP^2} = d\mu^2 + \sin^2\mu(\sigma_1^2 + \sigma_2^2 + \cos^2\mu\sigma_3^2), \tag{2.55}$$

where,

$$\begin{aligned}
\sigma_1 &= \frac{1}{2}(\cos d\psi d\theta + \sin\psi\sin\theta d\phi),\\
\sigma_2 &= \frac{1}{2}(\sin\psi d\theta - \cos\psi\sin\theta d\phi),\\
\sigma_3 &= \frac{1}{2}(d\psi + \cos\theta d\phi).
\end{aligned}$$

The metric of S^5 is then expressed in the simplified form,

$$d^2_{S^5} = ds^2_{CP^2} + (d\chi + \sin^2\mu\sigma_3)^2. \tag{2.56}$$

The string ansatz that will be used is,

$$\begin{aligned}
&\theta(\tau,\sigma) = \theta(\tau), && \mu(\tau,\sigma) = \mu(\tau), && \chi(\tau,\sigma) = \chi(\tau),\\
&\phi(\tau,\sigma) = \alpha_1\sigma, && \psi(\tau,\sigma) = \alpha_2\sigma.
\end{aligned}$$

Once this ansatz is applied the Lagrangian and non-trivial equations of motion are as follows,

$$\mathcal{L} = -\frac{1}{2\pi\alpha'}\left[\dot{t}^2 - \dot{\mu}^2 - \dot{\chi}^2 - \frac{1}{4}\sin^2\mu(\dot{\theta}^2 - \alpha_1^2\sin^2\theta - (\alpha_2 + \alpha_1\cos\theta)^2)\right], \tag{2.57}$$

$$\ddot{\mu} + \frac{1}{8}\sin(2\mu)\left[\dot{\theta}^2 - 2\alpha_1\alpha_2\cos\theta - \alpha_1^2 - \alpha_2^2\right] = 0, \tag{2.58}$$

$$\ddot{\theta} + 2\dot{\mu}\dot{\theta}\cot\mu + \alpha_1\alpha_2\sin\theta = 0. \tag{2.59}$$

Taking $\mu = \frac{\pi}{2}$ will satisfy the equation of motion given in (2.58) as well as simplify the equation of motion in (2.59) to,

$$\ddot{\theta} + \alpha_1\alpha_2\sin\theta = 0. \tag{2.60}$$

It is possible to find an explicit solution to equation (2.60), however it is not necessary for the analytic non-integrability method. Let $\bar{\theta}$ be the solution to this equation of motion. Substituting $\bar{\theta}$ into equation (2.58) is the first step in finding the NVE,

$$\ddot{\mu} + \frac{1}{8}2\cos\mu\sin\mu\left[\dot{\bar{\theta}}^2 - 2\alpha_1\alpha_2\cos\bar{\theta} - \alpha_1^2 - \alpha_2^2\right] = 0. \tag{2.61}$$

Expanding around the straight line solution $\mu = \frac{\pi}{2}$ such that $\mu = \frac{\pi}{2} + \eta(\tau)$ will result in a equation of motion for η. The new variable η is a small fluctuation around this solution,

$$\ddot{\eta} + \frac{1}{4}\eta\left[\dot{\bar{\theta}}^2 - 2\alpha_1\alpha_2\cos\bar{\theta} - \alpha_1^2 - \alpha_2^2\right] = 0. \tag{2.62}$$

There should not be any variables other than η in the NVE. Normally a change of variables to eliminate $\bar{\theta}$ and change the dependence of η from τ to this new variable will be needed. However, this case is more simple, equation (2.60) is all that is required to eliminate $\bar{\theta}$ from the NVE. With some manipulation equation (2.60) can be written as,

$$\frac{d}{d\tau}(\dot{\bar{\theta}}^2 - 2\alpha_1\alpha_2\cos\theta) = 0. \tag{2.63}$$

In order for (2.63) to be true,

$$\dot{\bar{\theta}}^2 - 2\alpha_1\alpha_2\cos\theta = \text{const.} \tag{2.64}$$

Substituting this into (2.62) gives the final NVE,

$$\ddot{\eta} + \frac{1}{4}\eta\left[\text{const} - \alpha_1^2 - \alpha_2^2\right] = 0. \tag{2.65}$$

All the terms in parenthesis are constants so the NVE is just the equation for a simple harmonic oscillator. An analytical solution for this NVE can easily be obtained and it was integrable when tested with the Kovacic algorithm as expected.

The $T^{p,q}$ example

The next example is in $AdS_5 \times X^5$ space, explicitly the $T^{p,q}$ 5-manifold [3] with non-specified values of p and q. Beginning with the metric,

$$ds^2 = a^2(d\psi + p\cos\theta_1 d\phi_1 + q\cos\theta_2 d\phi_2)^2 + b^2(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + c^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2),$$

with the ansatz:

$$\begin{aligned} \theta_1 &= \theta_1(\tau), & \theta_2 &= \theta_2(\tau), & \psi &= \psi(\tau), \\ t &= t(\tau), & \phi_1 &= \alpha_1\sigma, & \phi_2 &= \alpha_2\sigma. \end{aligned}$$

The Polyakov Lagrangian for the system simplifies to,

$$\begin{aligned} \mathcal{L} = \frac{-1}{2\pi\alpha'}[&\dot{t}^2 - b^2\dot{\theta}_1^2 - c^2\dot{\theta}_2^2 - a^2\dot{\psi}^2 + \alpha_1^2(b^2 - a^2p^2)\sin^2\theta_1 \\ &+ \alpha_2^2(c^2 - a^2q^2)\sin^2\theta_2 + 2\alpha_1\alpha_2 pqa^2\cos\theta_1\cos\theta_2]. \end{aligned} \tag{2.66}$$

The equations of motion can be determined using the Euler-Lagrange equations. For θ_1,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) - \frac{\partial\mathcal{L}}{\partial\theta_1} = 0$$

$$\implies \ddot{\theta}_1 + \frac{\alpha_1}{b^2}\sin\theta_1\left[\alpha_1(b^2 - a^2p^2)\cos\theta_1 - a^2\alpha_2 pq\cos\theta_2\right] = 0. \tag{2.67}$$

For θ_2,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) - \frac{\partial\mathcal{L}}{\partial\theta_2} = 0$$

$$\implies \ddot{\theta}_2 + \frac{\alpha_2}{c^2}\sin\theta_2\left[\alpha_2(c^2-a^2q^2)\cos\theta_2 - a^2\alpha_1 pq\cos\theta_1\right] = 0. \tag{2.68}$$

The equations of motion for t and ψ are both trivial equations of the form, $\ddot{t} = 0$ and $\ddot{\psi} = 0$. Now that the equations of motion are explicitly expressed. The three steps of the method can be applied. First, let the straight line solution be $\theta_2 = \dot{\theta}_2 = 0$. This solves the θ equation of motion (2.68). This solution simplifies (2.67) to,

$$\ddot{\theta}_1 + \frac{\alpha_1}{b^2}\sin\theta_1\left[\alpha_1(b^2-a^2p^2)\cos\theta_1 - a^2\alpha_2 pq\right] = 0. \tag{2.69}$$

An explicit solution can be obtained for θ_1 at this point. However this is not necessary for the analytic non-integrability method. It is sufficient to assume the solution to (2.69) is $\bar{\theta}_1$. Substituting this solution into (2.68) yields,

$$\ddot{\theta}_2 + \frac{\alpha_2}{c^2}\sin\theta_2\left[\alpha_2(c^2-a^2q^2)\cos\theta_2 - a^2\alpha_1 pq\cos\bar{\theta}_1\right] = 0 \tag{2.70}$$

To find the NVE, consider small fluctuations in θ_2 around the solution $\bar{\theta}_1$ such that $\theta_2 = 0 + \eta(\tau)$. Equation (2.70) becomes

$$\ddot{\eta} + \frac{\alpha_2}{c^2}\eta\left[\alpha_2(c^2-a^2q^2) - a^2\alpha_1 pq\cos\bar{\theta}_1\right] = 0. \tag{2.71}$$

All the instances in (2.68) where the terms were zero due to the solution $\theta_2 = 0$ are replaced with the small variation η. To obtain the NVE and to remove any direct dependence on $\bar{\theta}_1$ the substitution $z = \cos\bar{\theta}_1$ is performed and the chain rule is applied,

$$\begin{aligned}
\frac{d\eta}{d\tau} &= \frac{d\eta}{dz}\frac{dz}{d\tau} = -\eta'\dot{\bar{\theta}}_1 sin\bar{\theta}_1\\
\frac{d^2\eta}{d\tau^2} &= \frac{d^2\eta}{dz^2}\left(\frac{dz}{d\tau}\right)^2 + \frac{d\eta}{dz}\left(\frac{d^2z}{d\tau^2}\right)\\
&= \eta'' sin^2\bar{\theta}_1\dot{\bar{\theta}}_1^2 - \eta'(\ddot{\bar{\theta}}_1^2\sin\bar{\theta}_1 + cos\bar{\theta}_1\dot{\bar{\theta}}_1^2),
\end{aligned}$$

all 'primes' denote derivatives with respect to z. Substituting the new derivatives as well as (2.69) into (2.71) gives the NVE,

$$f''(z)\eta''(z) + \frac{1}{2}f'(z)\eta'(z) + \frac{\alpha_2}{c^2}\left[\alpha_2(c^2-a^2q^2) - a^2\alpha_1 pqz\right]\eta(z) = 0, \tag{2.72}$$

where $f(z) = \dot{\bar{\theta}}_1^2\sin^2\bar{\theta}_1 = [6E^2 - \frac{1}{3}(4\alpha_1\alpha_2 z + \alpha_2^2(1-z^2))](1-z^2)$. The NVE in (2.72) is now ready to input into the Kovacic algorithm. In this case the algorithm provides no solution so this NVE and by extension the original system is non-integrable.

In any attempt to expand the method to include open strings, the worldsheet coordinate σ needs to be treated non-trivially. The additional constraints from the open string boundary conditions need to taken into account when selecting the straight line solution and computing the NVE.

2.5.4 Further Reading on Analytic Non-Integrability

In recent years there have been developments where the Analytic Non-Integrability method has been applied as a check for integrability in a variety of set-ups. In this section a brief overview of these papers will be given.

In [25], the method was used to test what effects marginal deformations of $N = 4$ SYM theory and β deformations have on integrability. [26] shows that certain classical string configurations exhibit chaos, resulting in non-integrability. "Analytic Non-Integrability and the $S-$matrix" [27], proposes a connection between integrability and $S-$matrix factorization in theories that violate Lorentz invariance. The reference [28], looks at possible integrability of classical motion in curved p-brane backgrounds.

In [29], it is shown that the string world sheet theory of Gaiotto-Maldacena, holographic duals to $N = 2$ SCFTs fails to be integrable in general. References [30] and [31] both consider six-dimensional $N = (0, 1)$ SCFTs. The authors of [30] show string duals in Massive IIA, specifically the string soliton wrapping and rotating on the Massive IIA background is non-integrable. While [31] explicitly shows the Neveu-Schwarz part of the string sigma model is integrable. "Non integrability of the Ω deformation", [32] tests for the preservation of integrability on the super gravity vacuum dual to the field theory Ω deformation of $N = 4$ SYM theory.

The authors in [33] investigate the relationship between non-integrability and chaos in gravity dual backgrounds of strongly coupled gauge theories with unquenched flavour. Specifically looking at the four dimensional $N = 2$ SYM theory and the three dimensional ABJM theory. Reference [34] classifies D-brane geometries that lead to integrable geodesics. [35] uses the method on string configurations defined over η as well as $\lambda-$deformed backgrounds. In [36], the author classifies generic non-relativistic theories that give rise to non-integrable string solutions. [37] uses the Kovacic algorithm in non-relativistic pulsating strings over torsion Newton-Cartan geometry with topology $R \times S^2$. Finally, [38] checks for classical integrability on two classes of backgrounds in Massive IIA super gravity.

What is missing from this list is a general study of analytic non-integrability for open string solutions. Therefore the following chapters will attempt to expand this method to study various open string solutions. An open string with endpoints on a giant graviton will be analysed and compared to the gauge theory expectations. A configuration of open strings ending on a D5 and D7 brane will also be tested for non-integrability using this method and these results will be compared to the gauge theory counterparts.

Chapter 3

Giant Gravitons

3.1 Introduction

The goal of this chapter is to test the method of analytic non-integrability for an open string ending on a giant graviton. A giant graviton is a wrapped brane that wraps an S^3 in the S^5 [39]. The S^3 wrapped by the brane has a maximum size, this gives rise to the concept of maximal and non-maximal giant gravitons. There is also a giant graviton that wraps an S^3 on the AdS_5. However, these AdS_5 giant gravitons will not be considered in this dissertation. Firstly, the coordinate choices for the giant gravitons are discussed. The Polyakov action and the equations of motion are determined. Lastly, the giant magnon solution proposed by Hofman and Maldacena [5], is used to determine the NVE and check for non-integrability.

The giant magnon solution, first discovered in 2006, is an open string whose endpoints move along the equator of the S^5. It was an attempt to replicate the dispersion relation obtained from the gauge theory by starting from the string theory side. The name giant magnon, comes from making the link from the single magnon states in gauge theory to the giant graviton in the string theory.

3.2 Coordinate Choices

In the literature available on giant gravitons, it is a common occurrence to see references made to $Y = 0$ or $Z = 0$ giant gravitons. The $Y = 0$ and $Z = 0$ giant gravitons permit a BPS ground-state configuration, a state that preserves some of the supersymmetry algebra, [40]. This is associated with the coordinates of embedding an S^5 in $\mathbb{R}^6$.

For the purpose of this study, the embedding coordinates were not used. It is therefore necessary to obtain a transformation between the embedding coordinates and the intrinsic coordinate system. The basic transformation is as follows,

$$Y = \cos\theta\cos\psi e^{i\varphi}, \qquad W = \cos\theta\sin\psi e^{i\eta}, \qquad Z = \sin\theta e^{i\phi}. \tag{3.1}$$

Appendix A shows the full details of this transformation. This creates a correspondence between the work presented here and previous publications on this subject. The distinction of $Y = 0$ and $Z = 0$ giant gravitons simply dictates which sector of the spacetime the system is confined to. By selecting a $Y = 0$ giant graviton, the graviton will be situated at $Y = 0$ and will extend in the other directions.

To obtain a $W = 0$ giant graviton start by making $\psi = 0$. This will result in the embedding coordinates simplifying to, $W = 0$, $Y = \cos\theta e^{i\varphi}$ and $Z = \sin\theta e^{i\phi}$. The way that the embedding coordinates are defined means by shifting ψ by $\frac{\pi}{2}$ radians a $Y = 0$ giant graviton is obtained. The remaining coordinates simplify to $W = \cos\theta e^{i\eta}$ and $Z = \sin\theta e^{i\phi}$. It is clear then, that the $W = 0$ and $Y = 0$ case are the same if the change of coordinates $\eta \to \varphi$ is performed. This change has no physical effect on the system and is simply a convention. The $W = 0$ and $Y = 0$ cases are therefore treated as the same and referred to as the $Y = 0$ giant graviton. The other classification is the $Z = 0$ giant graviton which is obtained by setting $\theta = 0$. The other two coordinates simplify to $Y = \cos\psi e^{i\varphi}$ and $W = \sin\psi e^{i\eta}$.

The reason that the intrinsic coordinates were selected instead of the embedding coordinates is that the giant graviton in this system is not dynamical. It is simply wrapping a surface in the metric and the string endpoints are required to end on it. The string equations of motion use the intrinsic coordinate system so it was the overall better choice.

Another important consideration is how the open string boundary conditions behave when they end on giant gravitons. In the language of the embedding coordinates, according to [40], the $Z = 0$ giant graviton should exhibit Dirichlet boundary conditions. In other words, for the open string boundary at $\sigma = 0$ and $\sigma = \pi$, it is required that $\dot{Z} = 0$ at $Z = 0$. The $Y = 0$ giant graviton on the other hand requires open strings to move on the Z space. This results in the Neumann boundary conditions, $Z' = 0$, since Z is parallel to the world volume of the brane.

3.3 The Giant Magnon

3.3.1 Equations of Motion

Beginning with the $AdS_5 \times S^5$ metric,

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3'^2 + \sin^2\theta d\phi^2 + d\theta^2 + \cos^2\theta d\Omega_3^2, \tag{3.2}$$

with $d\Omega_3^2 = \cos^2\psi d\varphi^2 + d\psi^2 + \sin^2\psi d\eta^2$, the system can be restricted to the $SU(2)$ sector by taking $\rho = 0$ and $\psi = 0$ [41]. The simplified metric for this sector becomes,

$$ds^2 = -dt^2 + \sin^2\theta d\phi^2 + d\theta^2 + \cos^2\theta d\varphi^2. \tag{3.3}$$

The Polyakov action was calculated to be,

$$S = -\frac{T}{2}\int d^2\sigma[t'^2 - \dot{t}^2 - \theta'^2 + \dot{\theta}^2 - \sin^2\theta\phi'^2 + \sin^2\theta\dot{\phi}^2 - \cos^2\theta\varphi'^2 + \cos^2\theta\dot{\varphi}^2]. \tag{3.4}$$

The equations of motion are given by,

$$\cos\theta(\varphi'' - \ddot{\varphi}) - 2\sin\theta(\theta'\varphi' - \dot{\theta}\dot{\varphi}) = 0, \tag{3.5}$$

$$\sin\theta(\phi'' - \ddot{\phi}) + 2\cos\theta(\theta'\phi' - \dot{\theta}\dot{\phi}) = 0, \tag{3.6}$$

$$\theta'' - \ddot{\theta} + \sin\theta\cos\theta(\varphi'^2 - \dot{\varphi}^2 - \phi'^2 + \dot{\phi}^2) = 0. \tag{3.7}$$

This was consistent with the equations obtained following the method given in [41] (see Appendix B), with the Virasoro Constraints,

$$-t'\dot{t} + \theta'\dot{\theta} + \sin^2\theta\phi'\dot{\phi} + \cos^2\theta\varphi'\dot{\varphi} = 0, \tag{3.8}$$

$$t'^2 - \dot{t}^2 + \theta'^2 + \dot{\theta}^2 + \sin^2\theta(\phi'^2 + \dot{\phi}^2) + \cos^2\theta(\varphi'^2 + \dot{\varphi}^2) = 0. \tag{3.9}$$

3.3.2 The Hofman-Maldacena Solution

The Hofman-Maldacena giant magnon [5], is a solution that solves the equations of motion with $\varphi = 0$. In order to understand how this solution was first proposed the authors considered elementary magnons or impurities at large 't Hooft coupling λ. On the string theory side consider a string in flat space. Selecting the light cone gauge with $X^+ = \tau$ and X^- =constant, the string will have a large P^- and $P^+ = 0$. $P^\pm$ is the momentum of the light cone gauge. This corresponds to the string having a light like trajectory. Next two excitations with world sheet momentum p and $-p$ respectively are added to this set up. The idea is to understand the spacetime description of a state that has these excitations on opposite points of a circle on the world sheet. An important concept to keep in mind is that each of the excitations is at a different X^- value. To visualise this set up, think of two particles joined by a string that move along light like trajectories. The string transfers momentum from the leading particle to the trailing particle. Eventually the excitations will pass each other and the leading particle becomes the trailing one. X^- must be periodic for a closed string which leads to the condition that the overall momentum p must be zero. Next expand this concept to an infinite string limit (P^- is infinite), with an infinite string. The string has a single excitation with momentum p. Again there are two light like trajectories that are joined by the string. This time the momentum transfer that occurs will be infinite and never ending since P^- is infinite. The string shape is dependent on the excitations carrying the momentum p.

This is the background information required to formulate the Hofman-Maldacena solution. Keeping this in mind while moving to the AdS_5xS^5 case, with the metric of S^5 the same as (3.2) with $\rho = 0$ and $\psi = 0$, the ϕ coordinate is shifted by J. The string ground state with $E - J = 0$ is a light like trajectory that moves along the ϕ direction with $\phi - t$ =constant. The trajectory sits in the centre of AdS_5 hence the choice $\rho = 0$ and at $\theta = \frac{\pi}{2}$. The authors were searching for the configuration with the lowest possible energy $E - J = \epsilon$. A pair of antipodal points on S^3 that form an S^2 with θ and ϕ were selected. Making the coordinate transformation, $t = \tau$ and $\phi' = \phi - t$, while selecting $\theta(\tau, \sigma) = \theta(\sigma)$, the string action is,

$$S = \frac{\sqrt{\lambda}}{2\pi} \int dt d\phi' \sqrt{\cos^2\theta \theta'^2 + \sin^2\theta}. \tag{3.10}$$

Integrating the equations of motion from this action leads to,

$$\sin\theta = \frac{\sin\theta_0}{cos\phi'}, \tag{3.11}$$

and

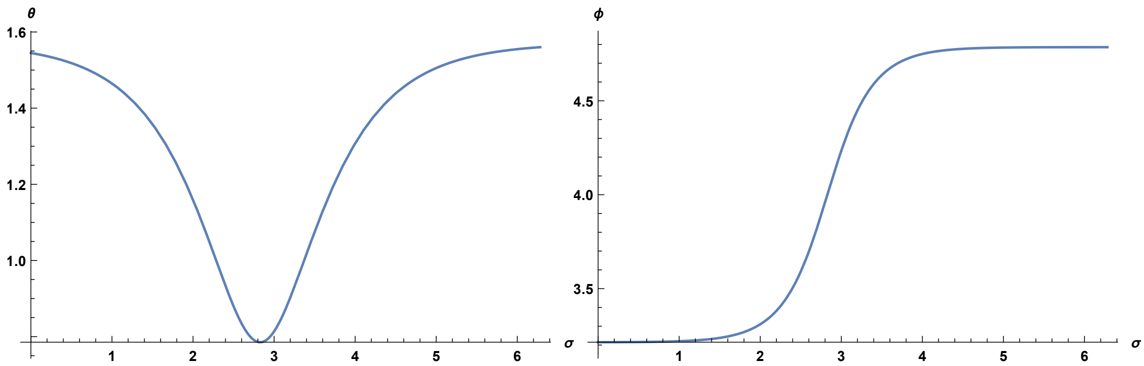
$$-\left(\frac{\pi}{2} - \theta_0\right) \leq \phi' \leq \frac{\pi}{2} - \theta_0, \tag{3.12}$$

where θ_0 is the integration constant with the condition $0 \leq \theta_0 \leq \frac{\pi}{2}$. Although the extent of the string on the world sheet is finite in this coordinate system, around the endpoints they carry an infinite J. The angular difference of the string endpoints is,

$$\Delta\phi' = 2\left(\frac{\pi}{2} - \theta_0\right), \tag{3.13}$$

with energy,

$$E - J = \frac{\sqrt{\lambda}}{\pi}\cos\theta_0 = \frac{\sqrt{\lambda}}{\pi}\sin\frac{\Delta\phi}{2}. \tag{3.14}$$



(a) A plot showing θ as a function of σ as in (3.15) (b) A plot showing ϕ as a function of σ as in (3.16)

Figure 3.1: The Hofman-Maldacena solution for θ and ϕ at time $t = 4$ with $\theta_0 = \frac{\pi}{4}$.

The authors propose that $\Delta\phi = p$. Substituting this proposition in equation (3.14) results in $E - J = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p}{2}\right|$. Negative values are possible since orientation of the string has an influence of the sign of p. Equation (3.14) is a non-relativistic dispersion relation that naturally occurs from the spin chain side due to the periodicity of the lattice. This shows that the authors were successful in replicating the dispersion relation from the string theory perspective with the idea of a giant magnon.

The final step in obtaining the full giant magnon solution is selecting a suitable gauge. The choice of gauge needs to allow the density of J to be constant for the ground state. This condition need only be satisfied away from the excitations. The conformal gauge is selected with $t = \tau$ and σ as the spatial world sheet coordinate. Using the solution in (3.11), a solution for θ and ϕ can be determined in the new gauge. Finally, the giant magnon solution is as follows,

$$\theta = ArcCos\left[\frac{\cos\theta_0}{\cosh\left(\frac{\sigma - \sin\theta_0 t}{\cos\theta_0}\right)}\right] \tag{3.15}$$

and

$$\phi' = ArcTan\left[\cot\theta_0\tanh\left(\frac{\sigma - \sin\theta_0 t}{\cos\theta_0}\right)\right]. \tag{3.16}$$

Figure 3.1 shows the θ and ϕ solution for a given time. This figure 3.1a also shows that the endpoints of the string are fixed at $\frac{\pi}{2}$ in the θ direction. This is consistent for a $Y = 0$ maximal giant graviton. This specific case has been studied in other papers such as [42] so it provides a good way to check the analytic non-integrability method.

3.3.3 The NVE

To obtain the NVE the Hofman-Maldacena giant magnon solution will be used. Usually this solution is considered a closed string solution by patching two of the semi-circular solutions together. However the solution will be used as an open string as it appears in its original form in equations (3.15) and (3.16). The string has its endpoints on a maximal giant graviton and the analytic non-integrability method will be applied to this solution. Since the giant magnon solution holds for $\varphi = 0$, this is selected as the straight line solution to expand around for the NVE. The NVE is a linear order expansion so only terms that are of linear order in φ are considered. Any terms that arise from the equations of motion from a non-zero φ are not of linear order and can therefore be ignored. The solutions given in equations (3.15) and (3.16) can be used without alteration. Equation (3.5) has no explicit ϕ dependence so (3.16) will be ignored for the moment. Let $\varphi = 0 + \eta\left(\cosh\left(\frac{\sigma - \sin\theta_0 t}{\cos\theta_0}\right)\right)$ where $\eta\left(\cosh\left(\frac{\sigma - \sin\theta_0 t}{\cos\theta_0}\right)\right)$ is this small fluctuation around $\varphi = 0$. The variable dependence of η is too complex at this point since it depends on two variables τ and σ, so some simplification will be necessary before finding the NVE. Performing two change of variables

$$y = \sigma - \sin\theta_0 t \qquad\qquad z = \cosh\left(\frac{y}{\cos\theta_0}\right), \tag{3.17}$$

provides a nice simplification such that $\varphi = 0 + \eta(z)$. The equation of motion for φ needs to be modified for this new variable z. Applying the chain rule the first derivative for σ and τ in terms of z is as follows (from now on all 'primes' denote derivatives with respect to z),

$$\frac{d\varphi}{d\sigma} = \frac{d\varphi}{dz}\frac{dz}{dy}\frac{dy}{d\sigma} = \varphi' \sec\theta_0 \sinh\left(\frac{y}{\cos\theta_0}\right), \tag{3.18}$$

$$\frac{d\varphi}{dt} = \frac{d\varphi}{dz}\frac{dz}{dy}\frac{dy}{dt} = -\varphi' \tan\theta_0 \sinh\left(\frac{y}{\cos\theta_0}\right). \tag{3.19}$$

The transformation for the second derivatives is as follows,

$$\begin{aligned}\frac{d^2\varphi}{d\sigma^2} &= \frac{d^2\varphi}{dz^2}\left(\frac{dz}{dy}\right)^2\left(\frac{dy}{d\sigma}\right)^2 + \frac{d^2y}{d\sigma^2}\frac{d\varphi}{dz}\frac{dz}{dy} + \frac{d^2z}{dy^2}\left(\frac{dy}{d\sigma}\right)^2\frac{d\varphi}{dz} \\ &= \varphi'' \sec^2\theta_0 \sinh^2\left(\frac{y}{\cos\theta_0}\right) + \varphi' \sec^2\theta_0 \cosh\left(\frac{y}{\cos\theta_0}\right),\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\frac{d^2\varphi}{dt^2} &= \frac{d^2\varphi}{dz^2}\left(\frac{dz}{dy}\right)^2\left(\frac{dy}{dt}\right)^2 + \frac{d^2y}{dt^2}\frac{d\varphi}{dz}\frac{dz}{dy} + \frac{d^2z}{dy^2}\left(\frac{dy}{dt}\right)^2\frac{d\varphi}{dz} \\ &= \varphi'' \sec^2\theta_0 \sin^2\theta_0 \sinh^2\left(\frac{y}{\cos\theta_0}\right) + \varphi' \sec^2\theta_0 \sin^2\theta_0 \cosh\left(\frac{y}{\cos\theta_0}\right).\end{aligned} \tag{3.21}$$

In the z coordinate the solution in (3.15) simplifies to,

$$\theta = ArcCos\left[\frac{\cos\theta_0}{z}\right].$$

Taking the respective derivatives of new θ expression gives,

$$\frac{d\theta}{d\sigma} = \frac{d\theta}{dz}\frac{dz}{dy}\frac{dy}{d\sigma} = \frac{\sinh\left(\frac{y}{\cos\theta_0}\right)}{z^2\left(1 - \frac{\cos^2\theta_0}{z^2}\right)^{\frac{1}{2}}} \tag{3.22}$$

and

$$\frac{d\theta}{dt} = \frac{d\theta}{dz}\frac{dz}{dy}\frac{dy}{dt} = \frac{-\sin\theta_0 \sinh\left(\frac{y}{\cos\theta_0}\right)}{z^2\left(1 - \frac{\cos^2\theta_0}{z^2}\right)^{\frac{1}{2}}}. \tag{3.23}$$

Lastly, an expression for $\tan\theta$ in terms of z is required,

$$tan\theta = z\left(1 - \frac{\cos^2\theta_0}{z^2}\right)^{\frac{1}{2}} \sec\theta_0. \tag{3.24}$$

To obtain the NVE equations (3.17) to (3.24) needs to be substituted in the equation of motion for φ expressed in (3.5). The resulting NVE is,

$$\frac{z}{2}\cos^2\theta_0\eta'' + \left(\sin^2\theta_0 + \frac{z^2\cos^2\theta_0}{2(z^2-1)} + 1\right)\eta' = 0. \tag{3.25}$$

The NVE in (3.25) can be directly solved. If the solution is a Heun or even a Bessel function with non-integer values of the index, then the NVE is non-integrable. Using the DSolve function in Mathematica leads to a solution for $\eta(z)$,

$$\eta(z) = c_1 + \frac{f(z)}{1 - 2(\sec\theta_0 + \tan^2\theta_0)}, \tag{3.26}$$

where,

$$f(z) = c_2 z^{1-2(\sec\theta_0+\tan^2\theta_0)}$$
$${}_2F_1\left[\frac{1}{2}, \frac{1}{2}(1 - 2(\sec\theta_0 + \tan^2\theta_0)), \frac{3}{2} - \sec\theta_0 - \tan^2\theta_0; z^2\right].$$

Since the parameters are not explicitly integer values depending on the value of θ_0, it is not a very clear indicator for integrability. The Kovacic algorithm should be applied to test this solution for non-integrability.

The algorithm was able to produce a solution. This means that the variation around the Hofman-Maldacena solution with string endpoints on maximal $Y = 0$ giant graviton appears to be integrable.

In obtaining the Hofman-Maldacena giant magnon solutions the string had to be stretch in the spacetime so that the endpoints of the string became $-\infty$ and ∞. Taking the limits of (3.26) as $z \to \infty$ and $z \to -\infty$ will provide some insight on the behaviour of the fluctuations around the endpoints of the string. Despite the solution containing a hyper-geometric function, both of these limits converge to constant values. This means that Neumann boundary conditions imposed on φ are still satisfied. The string is open and has a length in the φ direction. By subtracting the two limits the length of the string opening in the φ direction can be determined.

In figure 3.2, the dependence of the string opening on θ_0 is as expected. The maximum gap length corresponds to the value when $\cos\theta_0$ is a maximum, $\theta_0 = 0$. As the value of $\cos\theta_0$ decreases so does the gap length. Since the expression given in (3.26) contains both a $\tan\theta_0$ and a $\sec\theta_0$, $\theta_0 \neq \pi/2$. In addition to this constraint the gap length between the limit of $\eta(z)$ as $z \to \infty$ and $z \to -\infty$ becomes infinite once $\theta_0 > \frac{\pi}{2}$.

3.4 Discussion

The Hofman-Maldacena solution with string endpoints on maximal $Y = 0$ giant graviton is contained in the integrable subsector. Turning on the φ angles creates

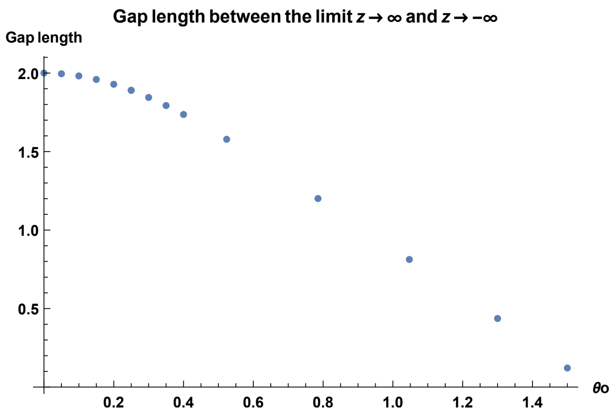


Figure 3.2: The figure above shows the length of the opening of the string in the φ direction, against different values of the angle θ_0. For the purpose of this plot the constants c_1 and c_2 were given the values i and 1 respectively.

a more general open string solution than the giant magnon. From the results of the Kovacic algorithm, perturbing the giant magnon along the φ direction appears compatible with integrability. This is consistent with other studies of this system. This does not mean that all open strings with endpoints on giant gravitons are integrable. Another system where integrability is in question is the $Z = 0$ non-maximal giant graviton. In [42], the author attempts to use the method of images to show that the non-maximal $Z = 0$ giant graviton is integrable in a scattering process. The method of images showed promise for the $Y = 0$ maximal and non-maximal case as well as the $Z = 0$ maximal case. However, it failed to produce any results for the $Z = 0$ non-maximal case since the treatment of the endpoints became too complex. The analytic non-integrability method is an excellent tool to check this case. The method is a more straight forward process and can determine if this case is definitively non-integrable when compared to attempting to prove integrability on the gauge theory side. By producing the expected results for the giant magnon, the method is proving to be a promising way to test for non-integrability in other open string solutions.

Chapter 4

Defects: The D5 Brane

4.1 Introduction

In this chapter open semi-classical strings with endpoints ending on D5 branes [43] are studied. The D5 brane is embedded in $AdS_5 \times S^5$. The idea is to use the set up given in [44] to test for non-integrability. In [44], open spinning strings were studied. In order to have a truly open string solution, it is necessary to have a structure that the end points can be fixed to. This is done by inserting a D5 brane into the bulk of the $AdS_5 \times S^5$. An important consideration is the open string boundary conditions. These boundary conditions, Dirichlet or Neumann, separate the open string from the closed string case by imposing additional constraints.

The first step is to fix the coordinate system by embedding the D5 brane. The authors considered both the $SU(2)$ and $SL(2)$ sectors for finding the open string solutions. The $SU(2)$ sector defines rotation on the S^5, while the $SL(2)$ sector pertains to rotation on the AdS side. For the purpose of this study only the $SU(2)$ sector will be used. The D5 brane is embedded in the $AdS_4 \times S^2$ subspace of $AdS_5 \times S^5$. The D5 brane occupies six of the 10 spacetime coordinates as follows,

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	o	o	o	x	x	x	o	x	x	x
D5	o	o	o	o	o	o	x	x	x	x

Table 4.1: The embeddings of the D3 and D5 branes in $10d$ Minkowski space, with the 'o' showing the coordinates wrapped by each brane. The 'x's' show the coordinates where the branes are pointlike.

The coordinates of greatest interest are those that are occupied by the D5 brane. The four coordinates x_6, x_7, x_8 and x_9 are taken to be zero since they are not occupied by the brane. The coordinates, x_0, x_1, x_2 and x_6 are on the AdS side. The remaining non-zero coordinates that parametrize the 2-sphere are defined by,

$$x_3 = \sin\phi\cos\psi, \qquad x_4 = \sin\phi\sin\psi, \qquad x_5 = \cos\phi.$$

The D5 brane fills the four coordinates of the AdS_4 and wraps an S^2 in the S^5 parametrized by the two coordinates ϕ and ψ. In [44] the solutions for the string sigma model are considered in this particular set up, using the rotating string ansatz. The Elliptical folded is considered in the $SL(2)$ sector. The Circular open string, elliptical folded open strings as well as the Rational Circular open string are considered in the $SU(2)$ sector. The Bethe ansatz [45] was applied on the gauge theory side. This was done to check if the boundary conditions for the Bethe wave function matched with the boundary conditions for open strings on the bulk of $SU(2)$. The open string solutions were related to the close string solutions using the doubling trick. The conclusions that were drawn were as follows. The energy corrections, using the doubling trick, could be determined for all loop levels of the string side. However, on the gauge theory side the Bethe root relation needs to be computed one order in λ at a time, for all values of λ. This calculation was only performed at one-loop level and the system showed signs of integrability. For integrability to be proven it must be done for all loop levels. This means that theoretically integrability is still in question for this problem.

For the remainder of this chapter, the same embedded D5 brane system will be considered. The elliptical folded open string solution will be used to test the system for non-integrability. The solution shows signs of integrability so the Kovacic algorithm is expected to produce a solution when applied to the NVE. It may seem counter intuitive to use a solution that has been shown to be integrable to check for non-integrability. However, the analytic non-integrability method has until now solely been applied to closed string solutions. It is not certain if the method will work when expanded to include open strings. Testing a solution where there are already results available for comparison is important. This will provide a consistency check that the method is being applied correctly. It is precisely for the sake of consistency that a closed string solution will be tested first. The closed strings do not see any effects of the D5 brane so this solution should also be integrable. This consistency check is to determine if the metric for this particular embedding has been correctly defined.

4.2 An Elliptical Folded Open String ending on a D5 brane

4.2.1 Metric and Equations of Motion

The metric in intrinsic coordinates used to determine the equations of motion for all ten of the coordinates is given by,

$$\begin{aligned} ds^2 = d\rho^2 - \cosh^2\rho dt^2 + \sinh^2\rho(d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2) \\ + d\gamma^2 + \cos^2\gamma d\varphi_3^2 + \sin^2\gamma(d\psi^2 + \cos^2\psi d\varphi_1^2 + \sin^2\psi d\varphi_2^2). \end{aligned} \tag{4.1}$$

The equations of motion for each of the coordinates are determined as,

$$\begin{aligned}
&t: 2\sinh\rho(\dot{t}\dot{\rho}-t'\rho')+\cosh\rho(\ddot{t}-t'')=0,\\
&\rho: \cosh\rho\sinh\rho(t'^2-\theta'^2-\dot{t}^2+\dot{\theta}^2-\cos^2\theta(\phi_1'^2-\dot{\phi_1}^2)-\cos^2\theta(\phi_2'^2-\dot{\phi_2}^2))+\rho''-\ddot{\rho}=0,\\
&\theta: 2\sinh 2\rho(\theta'\rho'-\dot{\theta}\dot{\rho})+\sinh^2\rho(\sin 2\theta(\phi_1'^2-\phi_2'^2-\dot{\phi_1}^2+\dot{\phi_2}^2)+2(\theta''-\ddot{\theta}))=0,\\
&\phi_1: \sinh\rho(2\sin\theta(\dot{\theta}\dot{\phi}_1-\theta'\phi_1')+\cos\theta(\phi_1''-\ddot{\phi}_1))+2\cosh\rho\cos\theta(\rho'\phi_1'-\dot{\rho}\dot{\phi}_1)=0,\\
&\phi_2: \sinh\rho(2\cos\theta(\theta'\phi_2'-\dot{\theta}\dot{\phi}_2)+\sin\theta(\phi_2''-\ddot{\phi}_2))+2\cosh\rho\sin\theta(\rho'\phi_2'-\dot{\rho}\dot{\phi}_2)=0,\\
&\varphi_1: 2\cos\gamma\cos\phi(\gamma'\varphi_1'-\dot{\gamma}\dot{\varphi}_1)-2\sin\gamma\sin\psi(\psi'\varphi_1'-\dot{\psi}\dot{\varphi}_1)+\cos\psi\sin\gamma(\varphi_1''-\ddot{\varphi}_1)=0,\\
&\varphi_2: 2\cos\gamma\sin\phi(\gamma'\varphi_2'-\dot{\gamma}\dot{\varphi}_2)+2\sin\gamma\cos\psi(\psi'\varphi_2'-\dot{\psi}\dot{\varphi}_2)+\sin\psi\sin\gamma(\varphi_2''-\ddot{\varphi}_2)=0,\\
&\varphi_3: 2\sin\gamma(\dot{\gamma}\dot{\varphi}_3-\gamma'\varphi_3')+\cos\gamma(\varphi_3''-\ddot{\varphi}_3)=0,\\
&\gamma: \cos\gamma\sin\gamma(\cos^2\psi(\varphi_1'^2-\dot{\varphi_1}^2)+\sin^2\psi(\varphi_2'^2-\dot{\varphi_2}^2)-\varphi_3'^2+\dot{\varphi_3}^2+\psi'^2-\dot{\psi}^2)-\gamma''+\ddot{\gamma}=0,\\
&\psi: \sin^2\gamma(\sin 2\psi(\varphi_1'^2-\dot{\varphi_1}^2-\varphi_2'^2+\dot{\varphi_2}^2)+2(\psi''-\ddot{\psi}))+2\sin 2\gamma(\gamma'\psi'-\dot{\gamma}\dot{\psi})=0.
\end{aligned}$$

The two Virasoro constraints are,

$$\begin{aligned}
&-\cosh^2\rho t'\dot{t}+\gamma'\dot{\gamma}+\sinh^2\rho\theta'\dot{\theta}+\rho'\dot{\rho}+\cos^2\theta\sinh^2\rho\phi_1'\dot{\phi}_1+\cos^2\psi\sin^2\gamma\varphi_1'\dot{\varphi}_1\\
&\quad+\sin^2\theta\sinh^2\rho\phi_2'\dot{\phi}_2+\sin^2\psi\sin^2\gamma\varphi_2'\dot{\varphi}_2+\cos^2\gamma\varphi_3'\dot{\varphi}_3+\sin^2\gamma\psi'\dot{\psi}=0 \qquad (4.2)
\end{aligned}$$

and

$$\begin{aligned}
&-\cosh^2\rho t'^2+\gamma'^2+\sinh^2\rho\theta'^2+\rho'^2+\cos^2\theta\sinh^2\rho\phi_1'^2+\cos^2\psi\sin^2\gamma\varphi'^2_1\\
&\quad+\sin^2\theta\sinh^2\rho\phi_2'^2+\sin^2\psi\sin^2\gamma\varphi_2'^2+\cos^2\gamma\varphi_3'^2+\sin^2\gamma\psi'^2\\
&-\cosh^2\rho\dot{t}^2+\dot{\gamma}^2+\sinh^2\rho\dot{\theta}^2+\dot{\rho}^2+\cos^2\theta\sinh^2\rho\dot{\phi_1}^2+\cos^2\psi\sin^2\gamma\dot{\varphi_1}^2\\
&\quad+\sin^2\theta\sinh^2\rho\dot{\phi_2}^2+\sin^2\psi\sin^2\gamma\dot{\varphi_2}^2+\cos^2\gamma\dot{\varphi_3}^2+\sin^2\gamma\dot{\psi}^2=0. \qquad (4.3)
\end{aligned}$$

Since these are the equations for all ten coordinates of the $AdS_5\times S^5$, they will need to be reduced appropriately to reflect the $AdS_4\times S^2$ embedding of the D5 brane. Choosing an appropriate string ansatz will take care of this simplification. First a closed string solution will be applied to this background to check for non-integrability. Next the elliptical folded open string solution will be tested using the analytic non-integrability method.

4.2.2 A Closed String Solution

As a consistency check on the metric, an analysis must be done for a closed string living in this space time. The ansatz that is used is given by,

$$\begin{aligned}
&\rho(\tau,\sigma)=0, && t=\kappa\tau, && \phi_1=0, && \phi_2=0, && \theta=0,\\
&\psi=\psi(\tau), && \gamma=\gamma(\tau), && \varphi_1=\omega_1\sigma, && \varphi_2=0, && \varphi_3=0. \qquad (4.4)
\end{aligned}$$

This ansatz is the standard type of closed string ansatz such as the one used in [33]. The string is wrapping a $U(1)$ angle φ_1 while some non-$U(1)$ angles ψ and γ are taken to be non-trivial functions of time. The closed string ansatz in (4.4) leaves two non-trivial equations of motion for the γ and ψ coordinates,

$$\gamma: \cos\gamma\sin\gamma(-\omega_1^2\cos^2\psi+\dot{\psi}^2)-\ddot{\gamma}=0, \qquad (4.5)$$

$$\psi: -2\sin 2\gamma\dot{\gamma}\dot{\psi}+\sin^2\gamma(\omega_1^2\sin 2\psi-2\ddot{\psi})=0. \qquad (4.6)$$

Selecting the solution $\gamma = \frac{\pi}{2}$ explicitly solves the equation of motion (4.5). As a result this will be selected as the straight line solution to determine the NVE. This choice for γ simplifies (4.6) to,

$$\omega_1^2 \sin 2\psi - 2\ddot{\psi} = 0, \tag{4.7}$$

and the non-trivial Virasoro constraint (4.3) simplifies to,

$$-\kappa^2 + \omega_1^2 \cos^2\psi + \dot{\psi}^2 = 0. \tag{4.8}$$

The other Virasoro constraint (4.2) is zero. It is not necessary to solve (4.7) explicitly. Simply let $\bar{\psi}$ be the solution to (4.7).

The NVE

Consider small fluctuations around the straight line solution, such that $\gamma = \frac{\pi}{2} + \eta(\tau)$. Expand around this new solution to linear order in η. Substituting this expansion into equation (4.8) gives an equation of motion for η,

$$\eta(\kappa^2 - 2\omega_1^2 \cos^2\bar{\psi}) + \ddot{\eta} = 0. \tag{4.9}$$

The NVE is required to be a single variable function, so the presence of $\bar{\psi}$ is less than ideal. Performing a change of variables, $z = \cos\bar{\psi}$, is the first step in eliminating this dependence. The derivatives of η need to be calculated in terms of the new variable z. The equation of motion for ψ and the non-trivial Virasoro constraint will help get rid of any explicit dependence on $\bar{\psi}$. Applying the chain rule will allow for the derivatives of η to be expressed in terms of the new variable z,

$$\begin{aligned}
\frac{d\eta}{d\tau} &= \frac{d\eta}{dz}\frac{dz}{d\tau} = -\eta'\dot{\bar{\psi}}\sin\bar{\psi} \\
\frac{d^2\eta}{d\tau^2} &= \frac{d^2\eta}{dz^2}\left(\frac{dz}{d\tau}\right)^2 + \frac{d\eta}{dz}\left(\frac{d^2z}{d\tau^2}\right) \\
&= \eta''\sin^2\bar{\psi}\dot{\bar{\psi}}^2 - \eta'(\ddot{\bar{\psi}}^2\sin\bar{\psi} + \cos\bar{\psi}\dot{\bar{\psi}}^2).
\end{aligned}$$

Here all 'primes' denote a derivative with respect to z. To get the NVE, these new derivatives, together with (4.7) and (4.8) substituted in them are entered in equation (4.9). The expression $\bar{\psi} = \arccos(z)$ is also required. The final NVE is,

$$\eta(z)''(z^2-1)(\kappa^2 - \omega_1^2 z^2) + \eta(z)'(z(\omega_1^2(1-2z^2)+\kappa^2)) + \eta(z)(2\omega_1^2 z^2 - \kappa^2) = 0. \tag{4.10}$$

The NVE is only dependent on z and is now ready to be entered in the Kovacic algorithm. When the closed string NVE (4.10) was entered into Kovacic, the algorithm was able to solve the NVE and produce Liouvillian solutions. The presence of these type of solutions indicates that the closed string solutions are integrable. The closed string solutions were expected to be integrable for this set up. This is because closed string should not notice the presence of the D5 brane. Getting a non-integrable result for the closed string solution would mean there is a problem with the selected metric. This result indicates that the metric is consistent and correct. Armed with this knowledge an open string solution can be tested.

4.2.3 An Open String Solution

Now, the elliptical folded open string solution will be applied from [44]. This open string ansatz is defined as follows,

$$\begin{aligned} &\rho(\tau,\sigma)=0, && t=\kappa\tau, && \phi_1=0, && \phi_2=0, && \theta=0,\\ &\psi=\psi(\sigma), && \gamma=\gamma(\sigma), && \varphi_1=\omega_1\tau, && \varphi_2=0, && \varphi_3=0, \end{aligned} \tag{4.11}$$

The open string ansatz in (4.11) differs from the closed string ansatz in (4.4) since this case the non-trivial dependence is on the spatial coordinate σ instead of time. Applying this ansatz to all the equations of motion results in only two non-trivial equations of motion. These equations are for the (γ,ψ)-system,

$$\gamma : \gamma'' + \sin\gamma\cos\gamma(\omega_1^2\cos^2\psi - \psi'^2) = 0, \tag{4.12}$$

$$\psi : \sin\gamma(\psi'' - \omega_1^2\sin\psi\cos\psi) + 2\cos\gamma\gamma'\psi' = 0, \tag{4.13}$$

where 'primes' denote derivatives with respect to σ. Lastly, the non-zero Virasoro constraint is given by,

$$-\kappa^2 + \gamma'^2 + \sin^2\gamma(\psi'^2 + \omega_1^2\cos^2\psi) = 0. \tag{4.14}$$

The Virasoro constraint (4.14) allows a solution for ψ to be obtained, although the explicit solution is not required to compute the NVE. These equations bear a striking resemblance to the closed string case from the previous section. The difference comes in the dependence on the worldsheet coordinate σ. There are angles that are wrapping the worldsheet coordinate τ and the $\gamma - \psi$ system is assumed to be of a more complicated dependence on σ. In the closed string case the angles were wrapping σ and non-trivial in τ. Another very important distinction are the open string boundary conditions. Closed strings do not have boundary conditions that they are required to satisfy.

The boundary conditions are dependent on how the D5 brane is embedded. The D5 brane is located at $\phi_2 = 0$ on the AdS side. For the S^5, the brane sits at $\psi = \varphi_3 = 0$. This means that the brane is wrapping the ψ and φ_3 angles of S^5. The endpoints of the string need to be fixed to the brane, this means that the ϕ_2, ψ, and φ_3 must satisfy Dirichlet boundary conditions. These boundary conditions for the open string are,

$$\psi(0) = \psi(\pi) = 0, \qquad \phi_2(0) = \phi_2(\pi) = 0, \qquad \varphi_3(0) = \varphi_3(\pi) = 0. \tag{4.15}$$

The remainder of the coordinates must satisfy Neumann boundary conditions at the string endpoints. These conditions mean that the first derivatives with respect to σ of these coordinates must be zero at the boundaries where $\sigma = 0$ or $\sigma = \pi$. Most of the boundary conditions are satisfied by the string ansatz. The only two that are in question are for γ and ψ. The elliptical folded string stretches in the ψ direction and is constrained around the equator of the S^5. This equator is defined by $\gamma = \frac{\pi}{2}$. The only remaining boundary condition is,

$$\psi(0) = \psi(\pi) = 0. \tag{4.16}$$

The NVE

To constrain the system to the equator of the S^5, the straight line solution is selected to be $\gamma = \frac{\pi}{2}$. This choice satisfies the condition given by equation (4.12) and equation (4.13) simplifies to,

$$2\psi'' - \omega_1^2 \sin 2\psi = 0, \tag{4.17}$$

and the non-zero Virasoro constraint is reduced to,

$$\psi'^2 = \kappa^2 - \omega_1^2 \cos^2 \psi. \tag{4.18}$$

Once again there is no need to solve for the ψ solution explicitly. If it is assumed that $\bar{\psi}$ is a solution to equation (4.17), the $\bar{\psi}$ solution can be substituted into equation (4.12),

$$\gamma'' + \sin\gamma\cos\gamma(\omega_1^2 \cos^2 \bar{\psi} - \bar{\psi}'^2) = 0. \tag{4.19}$$

Next consider small fluctuations in γ around $\bar{\psi}$ such that $\gamma = \frac{\pi}{2} + \eta(\sigma)$. Equation (4.19) is rewritten as an equation of motion for the fluctuation η,

$$\frac{d^2\eta}{d\sigma^2} + \eta(\omega_1^2 \cos^2 \bar{\psi} - \bar{\psi}'^2) = 0. \tag{4.20}$$

There can only be a one variable dependence in the NVE so the substitution $z = \cos\bar{\psi}^2$ is made. The derivatives of η are needed in terms z so the chain rule is applied. All primes denote a derivative with respect to z for the remainder of this calculation,

$$\begin{aligned}
\frac{d\eta}{d\sigma} &= \frac{d\eta}{dz}\frac{dz}{d\sigma} = -2\eta'\bar{\psi}' \sin\bar{\psi}\cos\bar{\psi} \\
\frac{d^2\eta}{d\sigma^2} &= \frac{d^2\eta}{dz^2}\left(\frac{dz}{d\sigma}\right)^2 + \frac{d\eta}{dz}\left(\frac{dz^2}{d\sigma^2}\right) \\
&= 4\eta'' \cos^2\bar{\psi}\sin^2\bar{\psi}\bar{\psi}'^2 - 2\eta'(\cos^2\bar{\psi}\sin\bar{\psi}\bar{\psi}'' + (\cos^2\bar{\psi} - \sin^2\bar{\psi})\bar{\psi}'^2).
\end{aligned}$$

Using the results of the chain rule and substituting (4.17) and (4.18) into equation (4.20) with $\bar{\psi} = \arccos[\sqrt{z}]$ results in the NVE,

$$\begin{aligned}
&\eta''[4z(z-1)(\omega_1^2 z - \kappa^2)] \\
&\qquad + \eta'[(2-4z)\kappa^2 - 2z\omega_1^2(2-3z)] + \eta[\kappa^2 - 2z\omega_1^2] = 0.
\end{aligned} \tag{4.21}$$

The NVE in (4.21) results in Liouvillian solutions when the Kovacic algorithm is applied. This indicates that this solution may be integrable. This system sometimes called the Karch-Randall D5 brane has been proven to be integrable on the gauge theory side to one-loop in [46] so this result is consistent with the expectations.

By solving (4.21) some insight on the behaviour of the fluctuation η around the endpoints of the string can be gained. The solution to this differential equation seems complex. The relationship between the boundary condition $\psi(0) = \psi(\pi) = 0$ and the new variable z can be helpful since z is a function of $\bar{\psi}$. The ψ boundary condition translates to $z(0) = z(\pi) = 1$. To obtain η at the endpoints of the string the limit as $z \to 1$ of $\eta[z]$ needs to calculated. This limit is calculated to be,

$$lim_{z\to 1}\eta[z] = \frac{2c}{\sqrt{\kappa^2 - \omega_1^2}}, \tag{4.22}$$

where c is a constant of integration. The limit in (4.22) is nothing more than a constant. The boundary condition for γ requires the σ derivatives of γ to vanish and γ is related to η by $\gamma = \frac{\pi}{2} + \eta$. By using the result from the limit in (4.22) it is clear that γ is a constant at the boundaries. This means that the derivative will vanish and that η satisfies the open string boundary condition.

4.2.4 Discussion

Both the closed and open string solutions are integrable for the D5 brane. This is consistent with the results produced on the gauge theory [46]. This is a strong indication that the D5 brane is fully integrable but does not prove integrability. The elliptical folded string uses $\gamma = \frac{\pi}{2}$ so the only angle that is non-trivial in σ is ψ. This solution was enough to test the analytic non-integrability method for open strings on a background containing a defect. The method appears to work and the results are consistent. However, in further research a solution that is a more complex function of σ should be applied. Perhaps this more complex solution will venture out of the integrable subsector and exhibit non-integrability if it is present. Another option is to consider solutions that depend on both σ and τ. The only additional requirement is that any potential solutions that are attempted must satisfy the open string boundary conditions.

Chapter 5

Defects: The D7 Brane

5.1 Introduction

There have been many papers of late on the gauge theory side ([47], [48], [49]) that discuss integrability of open strings ending on the D7 brane. These studies have led to the expectation that this system is non-integrable. In this chapter the hope is to contribute to this ongoing discussion. The method of analytic non-integrability could determine if this sector is non-integrable which would back the current hypothesis. Two different systems will be analysed in a similar manner to chapter 4. The supergravity background is considered to be the geometry of the near horizon limit of the D3 brane. The following table shows the construction of the D7 brane,

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	o	o	o	x	x	x	o	x	x	x
D7	o	o	o	o	o	o	x	o	o	x

Table 5.1: The embedding of the D3 and D7 branes in $10d$ Minkowski space, with the 'o' showing the coordinates wrapped by the brane and the 'x's' denote coordinates where the branes are pointlike.

While the D5 brane was able to preserve supersymmetry, the D7 brane completely loses supersymmetry. The loss of this symmetry becomes evident by the appearance of instabilities. The first of the two systems that will be considered is a string ending on a D7 brane that is wrapping two S^2's in the S^5 of the full $AdS_5 \times S^5$ space. The equations of motion will be determined and an NVE will be computed. The NVE will determine if the system is non-integrable using the Kovacic algorithm. Next, the same calculations will be done for an open string with endpoints on a D7 brane that is wrapping an S^4 in the S^5. In previous literature, the status of integrability in these systems is somewhat open ended. Using the analytic non-integrability method will hopefully bring some clarity to the question of integrability in these sectors.

5.2 The $S^2 \times S^2$ Case

The first step is to determine the equations of motion and find a solution to determine the NVE. The metric for this calculation contains an $S^2 \times S^2$ inside the S^5,

$$ds^2 = d\rho^2 - \cosh^2\rho dt^2 + \sinh^2\rho(d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2) + d\gamma^2 + \cos^2\gamma(d\psi^2 + \sin^2\psi d\varphi^2) + \sin^2\gamma(d\chi^2 + \sin^2\chi d\xi^2). \quad (5.1)$$

The equations of motion are as follows,

$$\begin{aligned}
\theta &: 2\sinh 2\rho(\theta'\rho' - \dot\theta\dot\rho) + \sinh^2\rho(\sin 2\theta(\phi_1'^2 - \dot\phi_1^{\,2} - \phi_2'^2 + \dot\phi_2^{\,2}) - 2\ddot\theta + 2\theta'') = 0,\\
t &: 2\sinh^2\rho(\dot t\dot\rho - t'\rho') + \cosh^2\rho(\ddot t - t'') = 0,\\
\rho &: \cosh\rho\sinh\rho(t'^2 - \dot t^2 - \theta'^2 + \dot\theta^2 - \cos^2\theta(\phi_2'^2 - \dot\phi_1^{\,2}) - \sin^2\theta(\phi_2'^2 - \dot\phi_2^{\,2})) + \rho'' + \ddot\rho = 0,\\
\phi_1 &: \sinh\rho(2\sin\theta(\dot\theta\dot\phi_1 - \theta'\phi_1') + \cos\theta(\phi_1'' - \ddot\phi_1)) + 2\cosh\rho\cos\theta(\rho'\phi_1' - \dot\rho\dot\phi_1) = 0,\\
\phi_2 &: \sinh\rho(2\cos\theta(\theta'\phi_2' - \dot\theta\dot\phi_2) + \sin\theta(\phi_2'' - \ddot\phi_2)) + 2\cosh\rho\sin\theta(\rho'\phi_2' - \dot\rho\dot\phi_2) = 0,\\
\gamma &: \cos\gamma\sin\gamma(\sin^2\chi(\xi'^2 - \dot\xi^2) - \sin^2\psi(\varphi'^2 - \dot\varphi^2) + \chi'^2 - \dot\chi^2 - \psi'^2 + \dot\psi^2) - \ddot\gamma + \gamma'' = 0,\\
\psi &: \cos\gamma(\cos\psi\sin\psi(\dot\varphi^2 - \varphi'^2) + \psi'' - \ddot\psi) - 2\sin\gamma(\gamma'\psi' - \dot\gamma\dot\psi) = 0,\\
\xi &: 2\cos\gamma\sin\chi(\gamma'\xi' - \dot\gamma\dot\xi) + 2\cos\chi\sin\gamma(\xi'\chi' - \dot\xi\dot\chi) + \sin\gamma\sin\chi(\xi'' - \ddot\xi) = 0,\\
\varphi &: 2\sin\gamma\sin\psi(\dot\gamma\dot\varphi - \gamma'\varphi') + 2\cos\psi\cos\gamma(\varphi'\psi' - \dot\varphi\dot\psi) + \cos\gamma\sin\psi(\varphi'' - \ddot\varphi) = 0,\\
\chi &: \sin\gamma(\sin\chi\cos\chi(\dot\xi^2 - \xi'^2) + \chi'' - \ddot\chi) + 2\cos\gamma(\gamma'\chi' - \dot\gamma\dot\chi) = 0.
\end{aligned}$$

The two Virasoro constraints are,

$$\rho'\dot\rho + \cosh^2\rho(-t'\dot t + \theta'\dot\theta + \cos^2\theta\phi_1'\dot\phi_1 + \sin^2\theta\phi_2'\dot\phi_2) + \sin^2\gamma(\sin^2\chi\xi'\dot\xi + \chi'\dot\chi) + \cos^2\gamma(\sin^2\psi\varphi'\dot\varphi + \psi'\dot\psi) = 0, \quad (5.2)$$

and

$$-\cosh^2\rho(t'^2 + \dot t^2) + \sinh^2\rho(\theta'^2 + \dot\theta^2 + \cos^2\theta(\phi_1'^2 + \dot\phi_1^{\,2}) + \sin^2\theta(\phi_2'^2 + \dot\phi_2^{\,2})) + \sin^2\gamma(\chi'^2 + \dot\chi^2 + \sin^2\chi(\xi'^2 + \dot\xi^2)) + \cos^2\gamma(\psi'^2 + \dot\psi^2 + \sin^2\psi(\varphi'^2 + \dot\varphi^2)) + \rho'^2 + \dot\rho^2 = 0. \quad (5.3)$$

5.2.1 A Closed String Solution

Before an open string can be considered the metric must be tested for consistency by calculating an NVE for a closed string. Again the closed string does not feel the effects of the defect and should therefore be integrable. The following ansatz is applied,

$$\begin{aligned}
&t(\tau,\sigma) = \kappa\tau, \quad \theta(\tau,\sigma) = 0, \quad \rho(\tau,\sigma) = 0, \quad \phi_1(\tau,\sigma) = 0, \quad \phi_2(\tau,\sigma) = 0,\\
&\gamma(\tau,\sigma) = \gamma(\tau), \quad \psi(\tau,\sigma) = \psi(\tau), \quad \xi(\tau,\sigma) = \omega_1\sigma, \quad \varphi(\tau,\sigma) = \omega_2\sigma, \quad \chi(\tau,\sigma) = 0.
\end{aligned} \quad (5.4)$$

The string is wrapping the $U(1)$ angles φ and ξ while the non-$U(1)$ angles ψ and γ are taken to be non-trivial functions of time. The remaining non-zero equations of motion are for γ and ψ,

$$\gamma : \sin 2\gamma(\alpha_1^2 \sin^2\psi - \dot{\psi}^2) - 2\ddot{\gamma} = 0, \tag{5.5}$$

$$\psi : -2\sin\gamma\dot{\gamma}\dot{\psi} + \cos\gamma(\alpha_1^2\cos\psi\sin\psi + \ddot{\psi}) = 0. \tag{5.6}$$

Taking $\gamma = 0$ solves equation (5.5) and reduces equation (5.6) to,

$$\ddot{\psi} + \alpha_1^2 \cos\psi \sin\psi = 0. \tag{5.7}$$

The non-trivial Virasoro constraint (5.3) simplifies to,

$$\dot{\psi}^2 = \kappa^2 - \alpha_1^2 \sin^2\psi. \tag{5.8}$$

Let $\bar{\psi}$ be a solution to (5.7). It is not necessary to compute the exact expression for $\bar{\psi}$. However in this case it was fairly straight forward to calculate $\bar{\psi}$,

$$\bar{\psi} = \pm am[\kappa\tau \pm \kappa c, \frac{\alpha_1^2}{\kappa^2}], \tag{5.9}$$

where c is a constant of integration and am is the Jacobi amplitude function [1].

Calculating the NVE

Next consider small fluctuations γ around the solution $\bar{\psi}$. Expanding around $\gamma = 0 + \eta(\tau)$ to first order, results in an equation of motion for η,

$$\eta(\alpha_1^2 \sin^2\bar{\psi} - \dot{\psi}^2) - \ddot{\eta} = 0. \tag{5.10}$$

The NVE cannot contain any τ dependence so a change of coordinates must be implemented. A new coordinate z is selected such that $z = \sin\bar{\psi}$. The derivatives of η need to be determined for the new variable. For the remainder of this calculation all 'primes' will refer to derivatives with respect to z.

$$\begin{aligned}
\frac{d\eta}{d\tau} &= \frac{d\eta}{dz}\frac{dz}{d\tau} \\
&= \eta' \cos\bar{\psi}\dot{\bar{\psi}}, \\
\frac{d^2\eta}{d\tau^2} &= \frac{d^2\eta}{dz^2}\left(\frac{dz}{d\tau}\right)^2 + \frac{d\eta}{dz}\frac{d^2z}{d\tau^2} \\
&= \eta'' \cos^2\bar{\psi}\dot{\bar{\psi}}^2 + \eta'(-\sin\bar{\psi}\dot{\bar{\psi}}^2 + \cos\bar{\psi}\ddot{\bar{\psi}}).
\end{aligned}$$

All that remains is to substitute (5.7) and (5.8) along with the new expressions for the derivatives of η into equation (5.10). With some simplification the NVE turns out to be,

$$\eta''(z^2-1)(\kappa^2 - \alpha_1^2 z^2) + \eta'(z(\alpha_1^2 - 2\alpha_1^2 z^2 + \kappa^2)) + \eta(2\alpha_1^2 z^2 - \kappa^2) = 0. \tag{5.11}$$

When the NVE was checked with the Kovacic algorithm it turned out to be integrable since it admitted Liouvillian solutions. This is consistent with integrability of the full system for closed strings as expected. This is due to the fact that closed strings can not feel the effects of the defects so integrability should be preserved.

[1] For more information on this function see `https://mathworld.wolfram.com/JacobiAmplitude.html`

5.2.2 An Open String Solution

The open string solution differs from the closed string case as a result of the additional constraints that arise from the boundary conditions. In order to calculate the NVE an appropriate ansatz must be applied to the equations of motion. The ansatz will reduce the system of ten equations of motion to just two non-trivial equations. The ansatz is as follows,

$$\begin{array}{lllll} t(\tau,\sigma)=\kappa\tau, & \theta(\tau,\sigma)=0, & \rho(\tau,\sigma)=0, & \phi_1(\tau,\sigma)=0, & \phi_2(\tau,\sigma)=0, \\ \gamma(\tau,\sigma)=\gamma(\sigma), & \psi(\tau,\sigma)=\psi(\sigma), & \xi(\tau,\sigma)=0, & \varphi(\tau,\sigma)=\omega_2\tau, & \chi(\tau,\sigma)=\omega_1\tau. \end{array} \tag{5.12}$$

The non-trivial equations of motion are for the coordinates γ and ψ. Once the ansatz is applied, the $\gamma-\psi$ system reduces to,

$$\gamma : \sin 2\gamma(\psi'^2 - \omega_2^2 \sin^2\psi + \omega_1^2) + 2\gamma'' = 0, \tag{5.13}$$
$$\psi : -2\sin\gamma\gamma'\psi' + \cos\gamma(\omega_2^2\cos\psi\sin\psi + \psi'') = 0. \tag{5.14}$$

The important boundary condition in this case is the boundary condition for the ψ angle. In the D5 brane case, ψ was required to be a Dirichlet direction which means that the function itself had to vanish at the boundary. However in the D7 brane case ψ must satisfy a Neumann boundary condition. The angle γ is also a Neumann direction so the ansatz gives rise to the following boundary conditions,

$$\gamma'(0) = \gamma'(\pi) = 0, \qquad \psi'(0) = \psi'(\pi) = 0. \tag{5.15}$$

The Dirichlet directions are explicitly set to zero so those boundary conditions do not constrain the system based on the ansatz in (5.12).

Calculating the NVE

The first step in calculating the NVE is determining a solution to expand around. In this case the solution $\gamma = 0$ is selected. This solution satisfies the equation in (5.13) and simplifies (5.14) to,

$$\omega_2^2\cos\psi\sin\psi + \psi'' = 0. \tag{5.16}$$

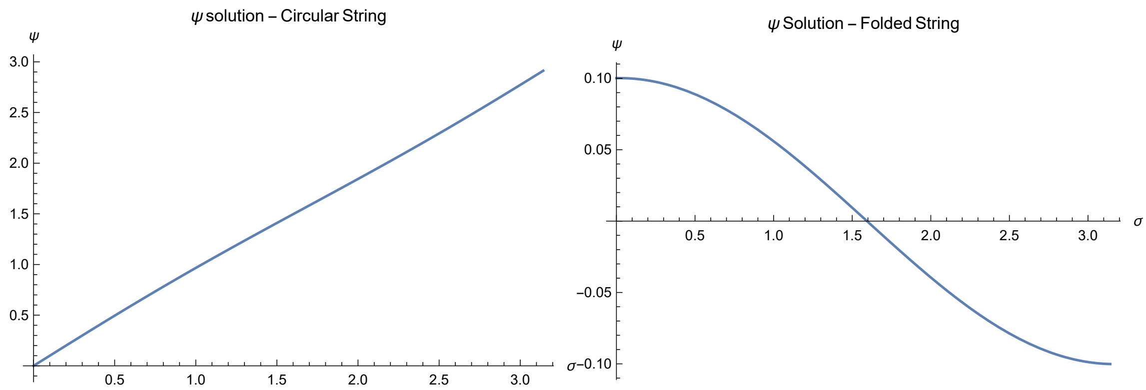
Equation (5.16) gives an expression for ψ'' that will be useful in simplifying the NVE later on. The first Virasoro constraint (5.2) simply reduces to zero once the ansatz is applied. The non-zero Virasoro constraint, (5.3) simplifies to,

$$\psi'^2 + \omega_2^2\sin^2\psi - \kappa^2 = 0. \tag{5.17}$$

This Virasoro constraint provides the required expression for ψ'^2. Solving (5.17) by integration gives the solution of ψ for this system. Of course there is no need to know the exact expression for this solution. The ψ solution turns out to be,

$$\bar{\psi} = \pm am\left[\kappa\sigma \pm \sqrt{2}\kappa c, \frac{\omega_2^2}{\kappa^2}\right], \tag{5.18}$$

where c is a constant of integration.



(a) ψ as a function of σ with $\omega_2 = 0.5$ and $\kappa = 1$.

(b) ψ as a function of σ with $\omega_2 = 1$ and $\kappa = 0.8$.

Figure 5.1: The positive ψ solution as a circular or folded string depending on the choice of the constants κ and ω_2. In 5.1a the solution is periodically identified. Whilst in 5.1b the solution extends between two values of ψ such that $\psi' = 0$ at the boundaries.

Figure 5.1 shows how the solution for ψ changes depending on the choice of the constants. Since the argument of (5.18) contains a ratio of κ and ω_2 depending on when this ratio is greater than or less than 1 results in different behaviour. When the ratio is larger than one, the result is a folded string. The string extends between two values of ψ such that $\psi' = 0$ at the boundaries. This behaviour can be seen in 5.1b. When the ratio is less than 1, the result is a circular string. This is seen in 5.1a where the value of ψ only increases as it goes around the sphere of S^5. Since the string does not turn back or fold onto itself ψ' is never zero. The circular string spans a half circle on the S^3. This can be seen as the graph goes from 0 to π. Since ψ is required to satisfy Neumann boundary conditions, the circular string solution can not be used. The derivative of ψ for the circular string is never zero, so it will not vanish at the boundaries. The folded string satisfies the Neumann boundary conditions as seen in 5.1b, so the folded string is an allowed solution.

Assuming that $\bar{\psi}$ is a solution to (5.16) and using the expression in (5.17), the equation of motion for γ can be expressed in terms of $\bar{\psi}$,

$$\sin 2\gamma(\bar{\psi}'^2 - \omega_2^2 \sin^2 \bar{\psi} + \omega_1^2) + 2\gamma'' = 0. \tag{5.19}$$

Next, consider small fluctuations (η) in γ around the $\bar{\psi}$ solution such that $\gamma = 0 + \eta(\sigma)$. The result is a new equation of motion for $\eta(\sigma)$,

$$\eta'' + \eta(\kappa^2 - 2\omega_2^2 \sin \bar{\psi}^2 + \omega_1^2) = 0. \tag{5.20}$$

In equation (5.20) above, (5.17) has been used for further simplification. The NVE can not have an explicit dependence on σ, so the substitution $z = \sin^2 \bar{\psi}$ is used. The chain rule is required to calculate the derivatives of η with respect to the new variable z. For the remainder of this section all ‘primes’ will refer to derivatives with respect to z.

$$
\begin{aligned}
\frac{d\eta}{d\sigma} &= \frac{d\eta}{dz}\frac{dz}{d\sigma} \\
&= 2\eta' \sin\bar{\psi}\cos\bar{\psi}\bar{\psi}', \\
\frac{d^2\eta}{d\sigma^2} &= \frac{d^2\eta}{dz^2}\left(\frac{dz}{d\sigma}\right)^2 + \frac{d\eta}{dz}\frac{d^2 z}{d\sigma^2} \\
&= 4\eta'' \cos^2\bar{\psi}\sin^2\bar{\psi}\bar{\psi}'^2 + \eta'(2\cos^2\bar{\psi}\bar{\psi}'^2 - 2\sin^2\bar{\psi}\bar{\psi}'^2 + 2\cos\bar{\psi}\sin\bar{\psi}\bar{\psi}'').
\end{aligned}
$$

Substituting these new derivatives along with (5.16) and (5.17) results in the NVE,

$$
\eta'' \sin^2\bar{\psi}(\kappa^2 - \omega_2^2\sin^2\bar{\psi}) + \eta'\left(-\cos\bar{\psi}(\kappa^2 - \omega_2^2\sin^2\bar{\psi}) + \frac{1}{2}\omega_2^2\sin\bar{\psi}\sin 2\bar{\psi}\right) = 0. \tag{5.21}
$$

The final step before the Kovacic algorithm can be applied is to completely get rid of the $\bar{\psi}$ dependence in (5.21). This is done by using $\bar{\psi} = \arccos(\sqrt{z})$. The final NVE is,

$$
\eta''(4z(z-1)(\omega_2^2 z-\kappa^2))+\eta'(\kappa^2(2-4z)-2\omega_2^2 z(2-3z))+\eta(\kappa^2-2\omega_2^2 z+\omega_1^2) = 0. \tag{5.22}
$$

The NVE in (5.22) produced Liouvillian solutions when the Kovacic algorithm was applied. The solution that was used to calculate the NVE had both endpoints on the same S^2. This is therefore a similar set-up as the one proposed for the D5 brane. The key difference lies with the boundary conditions. As a result of ψ having a Neumann boundary condition it is not required for the function to return to zero when $\sigma = \pi$ as it was required for the D5 brane. The Neumann boundary conditions are somewhat less stringent than the Dirichlet conditions and allows more freedom when finding a suitable solution. Since the lesser restriction allows for a wider pool of solutions there is a greater chance that one of these solutions will break integrability. This indicates that there is still potential to show that this system is non-integrable. The gauge theory computation for the $S^2 \times S^2$ defect has led to the expectation of non-integrability [50]. Perhaps a more complicated solution that spans both the S^2's may be required to prove that the system is in fact non-integrable.

5.3 Nested Spherical Metric S^4

In this section an alternate metric to the one in (5.1) will be used. The metric from [51], describes the S^5 as a series of nested spheres. The nested spherical metric is a more natural metric for the $SO(5)$ defect. The $SO(5)$ defect is discussed from the gauge theory side in [52] where the authors found indications of integrability. The metric is defined as,

$$
ds^2 = d\theta_9^2 + \cos^2\theta_9(d\theta_8^2 + \cos^2\theta_8(d\theta_7^2 + \cos^2\theta_7(d\theta_6^2 + \cos^2\theta_6 d\theta_5^2))), \tag{5.23}
$$

5.3.1 Equations of Motion

The equations of motion were computed as,

$$\begin{aligned}
&\theta_5 : 2\cos\theta_7\cos\theta_8\cos\theta_9 sin\theta_6(\dot\theta_5\dot\theta_6-\theta_5'\theta_6')+2\cos\theta_6(\cos\theta_8\cos\theta_9\sin\theta_7(\dot\theta_5\dot\theta_7-\theta_5'\theta_7')\\
&\quad+\cos\theta_7(\cos\theta_9\sin\theta_8(\dot\theta_5\dot\theta_8-\theta_5'\theta_8')+\cos\theta_8\sin\theta_9(\dot\theta_5\dot\theta_9-\theta_5'\theta_9')))\\
&\quad+\cos\theta_6\cos\theta_7\cos\theta_8\cos\theta_9(\theta_5''-\ddot\theta_5)=0,\\
&\theta_6 : \cos\theta_6\cos\theta_7\cos\theta_8\cos\theta_9\sin\theta_6(\theta_5'^2-\dot\theta_5^2)+2\cos\theta_8\cos\theta_9\sin\theta_7(\dot\theta_6\dot\theta_7-\theta_6'\theta_7')\\
&\quad+\cos\theta_7(\cos\theta_9\sin\theta_8(\dot\theta_6\dot\theta_8-\theta_6'\theta_8')+\cos\theta_8\sin\theta_9(\dot\theta_6\dot\theta_9-\theta_6'\theta_9'))+\cos\theta_7\cos\theta_8\cos\theta_9(\theta_6''-\ddot\theta_6)=0,\\
&\theta_7 : \cos\theta_7\cos\theta_8\cos\theta_9\sin\theta_7(\cos^2\theta_6(\theta_5'^2-\dot\theta_5^2)+\theta_6'^2-\dot\theta_6^2)+2\cos\theta_9 sin\theta_8(\dot\theta_7\dot\theta_8-\theta_7'\theta_8')\\
&\quad+2\cos\theta_8\sin\theta_9(\dot\theta_7\dot\theta_9-\theta_7'\theta_9')+\cos\theta_8\cos\theta_9(\theta_7''-\ddot\theta_7)=0,\\
&\theta_8 : \cos\theta_8\cos\theta_9\sin\theta_8(\cos^2\theta_7(\cos^2\theta_6(\theta_5'^2-\dot\theta_5^2)+\theta_6'^2-\dot\theta_6^2)+\theta_7'^2-\dot\theta_7^2)\\
&\quad+2\sin\theta_9(\dot\theta_8\dot\theta_9-\theta_8'\theta_9')+\cos\theta_9(\theta_8''-\ddot\theta_8)=0,\\
&\theta_9 : \cos\theta_9\sin\theta_9(\cos^2\theta_8(\cos^2\theta_7(\cos^2\theta_6(\theta_5'^2-\dot\theta_5^2)+\theta_6'^2-\dot\theta_6^2)+\theta_7'^2-\dot\theta_7^2)+\theta_8'^2-\dot\theta_8^2)+\theta_9''-\ddot\theta_9=0.
\end{aligned}$$

The two Virasoro constraints are as follows,

$$\begin{aligned}
&\cos^2\theta_6\cos^2\theta_8\cos^2\theta_9\cos^2\theta_7\theta_5'\dot\theta_5+\cos^2\theta_8\cos^2\theta_9\cos^2\theta_7\theta_6'\dot\theta_6\\
&\qquad+\cos^2\theta_8\cos^2\theta_9\theta_7'\dot\theta_7+\cos^2\theta_9\theta_8'\dot\theta_8+\theta_9'\dot\theta_9=0
\end{aligned}\tag{5.24}$$

and

$$\begin{aligned}
&\cos^2\theta_6\cos^2\theta_8\cos^2\theta_9\cos^2\theta_7\theta_5'^2+\cos^2\theta_8\cos^2\theta_9\cos^2\theta_7\theta_6'^2+\cos^2\theta_8\cos^2\theta_9\theta_7'^2\\
&\quad+\cos^2\theta_9\theta_8'^2+\theta_9'^2+\cos^2\theta_6\cos^2\theta_8\cos^2\theta_9\cos^2\theta_7\dot\theta_5^2+\cos^2\theta_8\cos^2\theta_9\cos^2\theta_7\dot\theta_6^2\\
&\qquad+\cos^2\theta_8\cos^2\theta_9\dot\theta_7^2+\cos^2\theta_9\dot\theta_8^2+\dot\theta_9^2-\kappa^2=0.
\end{aligned}\tag{5.25}$$

5.3.2 A Closed String Solution

Once again to ensure there are no errors with the metric being used, an NVE is calculated for a simple closed string. The ansatz is given by,

$$\begin{aligned}
\theta_5(\tau,\sigma)&=\omega_1\sigma, & \theta_6(\tau,\sigma)&=0 & \theta_7(\tau,\sigma)&=0,\\
\theta_8(\tau,\sigma)&=\theta_8(\tau), & \theta_9(\tau,\sigma)&=\theta_9(\tau), & t(\tau,\sigma)&=\kappa\tau.
\end{aligned}\tag{5.26}$$

This leaves only the θ_8 and θ_9 coordinates non-trivial for the calculation. The simplified equations of motion are as follows,

$$\theta_8 : 2\sin\theta_9\dot\theta_8\dot\theta_9+\cos\theta_9(\omega_1^2\cos\theta_8\sin\theta_8-\ddot\theta_8)=0,\tag{5.27}$$

$$\theta_9 : \sin 2\theta_9(\omega_1^2\cos^2\theta_8-\dot\theta_8^2)-2\ddot\theta_9=0.\tag{5.28}$$

Selecting the solution $\theta_9=0$ simplifies (5.27) and the Virasoro constraint in (5.25) as follows,

$$\omega_1^2\sin 2\theta_8-2\ddot\theta_8=0,\tag{5.29}$$

and

$$\dot\theta_8^2=\kappa^2-\omega_1^2\cos^2\theta_8.\tag{5.30}$$

Equation (5.29) and (5.30) are compatible by taking a σ derivative of equation (5.30). Let $\bar\theta_8$ be a solution to equation (5.29).

The NVE

Now expanding to linear order around $\theta_9 = 0 + \eta$ results in the following equation for η,

$$\eta(\omega_1^2 \cos^2 \bar{\theta}_8 - \dot{\bar{\theta}}_8^2) - \ddot{\eta} = 0. \tag{5.31}$$

A change of variables is required. Let $z = \cos\bar{\theta}_8$ with 'primes' denoting a derivative with respect to z. Applying the chain rule yields,

$$\begin{aligned}
\frac{d\eta}{d\tau} &= \frac{d\eta}{dz}\frac{dz}{d\tau} = -\eta' \dot{\bar{\theta}}_8 \sin\bar{\theta}_8 \\
\frac{d^2\eta}{d\tau^2} &= \frac{d^2\eta}{dz^2}\left(\frac{dz}{d\tau}\right)^2 + \frac{d\eta}{dz}\left(\frac{d^2 z}{d\tau^2}\right) \\
&= \eta'' \sin^2\bar{\theta}_8 \dot{\bar{\theta}}_8^2 - \eta'(\ddot{\bar{\theta}}_8 \sin\bar{\theta}_8 + \cos\bar{\theta}_8 \dot{\bar{\theta}}_8^2).
\end{aligned}$$

Substituting these new derivatives along with (5.29) and (5.30) into equation (5.31) results in the NVE,

$$\eta''(z^2-1)(\kappa^2 - \omega_1^2 z^2) + \eta'(z(\omega_1^2 - 2\omega_1^2 z^2 + \kappa^2)) + \eta(2\omega_1^2 z^2 - \kappa^2) = 0. \tag{5.32}$$

There is a clear resemblance between (5.11) and (5.32). In fact if $\omega_1 = \alpha_1$ then the two are identical. This is further evidence that both the metrics used in this chapter are correct and consistent. The different embeddings of the D7 brane make no difference to the closed string solutions.

5.3.3 An Open String Solution

The D7 brane wraps an S^4 in this metric so θ_9 is required to be a constant value. As a result, the boundary conditions for an open string are given by,

$$\theta_9 = 0, \qquad \theta_8' = \theta_7' = \theta_6' = \theta_5' = 0 \qquad \text{at} \qquad \sigma = 0, \pi. \tag{5.33}$$

The coordinates $\theta_5, \theta_6, \theta_7$ and θ_8 must satisfy Neumann boundary conditions. The only coordinate that must satisfy a Dirichlet boundary condition is θ_9. The AdS side coordinates apart from the time coordinate are taken to be zero so they are ignored for this calculation. Using the ansatz,

$$\begin{aligned}
\theta_5(\tau,\sigma) &= \omega_1\tau, & \theta_6(\tau,\sigma) &= 0 & \theta_7(\tau,\sigma) &= 0, \\
\theta_8(\tau,\sigma) &= \theta_8(\sigma), & \theta_9(\tau,\sigma) &= \theta_9(\sigma), & t(\tau,\sigma) &= \kappa\tau,
\end{aligned}$$

there are only two non-zero equations of motion,

$$\theta_8 : 2\sin\theta_9 \theta_8' \theta_9' + \cos\theta_9(\omega_1^2 \cos\theta_8 \sin\theta_8 - \theta_8'') = 0, \tag{5.34}$$
$$\theta_9 : \cos\theta_9 \sin\theta_9(\theta_8'^2 - \omega_1^2\cos^2\theta_8) + \theta_9'' = 0. \tag{5.35}$$

The NVE

Expanding around the solution $\theta_9 = 0$ directly satisfies the Dirichlet boundary condition. Equation (5.34) reduces to,

$$\omega_1^2 \cos\theta_8 \sin\theta_8 - \theta_8'' = 0, \tag{5.36}$$

and the non-trivial Virasoro constraint simplifies to,

$$-\kappa^2 + \omega_1^2 \cos^2\theta_8 + \theta_8'^2 = 0, \tag{5.37}$$

The exact solution for θ_8 is not necessary for the NVE. However, in this case it turns out to be a Jacobi amplitude function (am). This suggests that with the appropriate choice of constants the Neumann boundary conditions will be satisfied. Let $\bar{\theta_8}$ be a solution to equation (5.36). Consider small fluctuations in θ_9 around $\bar{\theta_8}$ such that $\theta_9 = 0 + \eta$ leads to,

$$\eta(\kappa^2 - 2\omega_1^2\cos^2\theta_8) + \eta'' = 0. \tag{5.38}$$

Applying the chain rule with the substitution $z = \cos\bar{\theta_8}$ with all 'primes' from this point denoting a derivative with respect to z,

$$\begin{aligned}
\frac{d\eta}{d\sigma} &= \frac{d\eta}{dz}\frac{dz}{d\sigma} = -\eta'\bar{\theta_8}'\sin\bar{\theta_8} \\
\frac{d^2\eta}{d\sigma^2} &= \frac{d^2\eta}{dz^2}\left(\frac{dz}{d\sigma}\right)^2 + \frac{d\eta}{dz}\left(\frac{dz^2}{d\sigma^2}\right) \\
&= \eta''\sin^2\bar{\theta_8}\bar{\theta_8}'^2 - \eta'(\bar{\theta_8}''\sin\bar{\theta_8} + \cos\bar{\theta_8}\bar{\theta_8}'^2) \\
&= \eta''\sin^2\bar{\theta_8}\kappa^2 - \eta'\cos\bar{\theta_8}\kappa^2
\end{aligned}$$

Substituting these results into (5.38) with $\bar{\theta_8} = \arccos(z)$ will produce the NVE,

$$\eta''((z^2-1)(\omega_1^2 z^2 - \kappa^2)) + \eta'(\omega_1^2 z(2z^2-1) - \kappa^2 z) + \eta(\kappa^2 - 2\omega_1^2 z^2) = 0. \tag{5.39}$$

This NVE appears to be integrable when Kovacic is applied so a more complex class of solutions needs to be explored to check for non-integrability.

5.4 Discussion

The solutions used in this chapter are very simple open string solutions. There is still a very good possibility that there are non-integrable solutions in D7 brane backgrounds. Finding a more complex open string solution that satisfies the boundary conditions is not a simple task. In the case of the nested spherical metric, turning on any additional angles results in added constraints that any proposed solution must satisfy. In order to see non-integrability the variation needs to be expanded around a non-trivial function of σ. In the $S^2 \times S^2$ embedding, a good potentially non-integrable solution would be an open string with one end on one of the S^2's and the other endpoint on the other S^2. Another possibility that could be explored is a solution that is non-trivial in both the σ and τ coordinates. Consistent open string solutions of these types that are sufficiently complex have not yet been found and will be the subject of future studies.

Chapter 6

Conclusion

Integrability continues to be an area of interest for many researchers in the string theory field. While there is no simple approach to checking a given theory for integrability, the analytic non-integrability method is a useful tool. The method is simple and provides a step-by-step guide for checking for non-integrability. It consists of studying classical strings moving in the bulk of AdS space. As a result of the AdS/CFT correspondence, and the nature of integrability, showing non-integrability on the bulk directly translates to non-integrability of the gauge theory. In this manner a gauge theory can be shown to be non-integrable without directly doing any gauge theory calculations. This is what makes the method an invaluable tool in the study of integrable systems.

Although the NVE's produced in this dissertation did not show non-integrability, the method consistently produced results that were in line with expectations. This is a good indication that the method can be successfully applied to open string solutions.

The Hofman-Maldacena giant magnon is a well known integrable solution. The method backing up this claim is a good indicator that it can be successfully applied to open strings ending on giant gravitons. The next step is to apply the method to an open string ending on a non-maximal $Z = 0$ giant graviton. This sector is expected to be non-integrable since the gauge theory research has not shown any signs of integrability. The analytic non-integrability method is well suited to make a meaningful contribution to this discussion. By doing a somewhat simple calculation, once an appropriate solution is found, the question of integrability could be settled. If no indication of non-integrability is found it will provide motivation to continue to attempt to look for integrability in this sector.

With respect to the defect theories, the analytic non-integrability method once again produced the expected results. The D5 brane set-up that was used is expected to be integrable. The method did not refute this claim so was once again consistent. The solutions used to study these defects were very simplified open string solutions. The straight line solutions were taken to be constants. In order to find non-integrability a straight line solution that is a more complex function of σ should be tested. For example in the case of the D-7 brane embedded by an $S^2 \times S^2$, a solution where each of the string end points ends on each of the S^2's should be tested. An example of a solution with the required characteristics is a Sigmoid function in the angle γ which controls the radii of the two S^2's. Another important solution to consider is an open string that is non-trivial in both σ and τ. The main constraint within this study is finding an appropriately complex solution that satisfies the string equations of motion, as well as the open string boundary conditions. In practice some defect solutions have fluxes through the spheres that they wrap. These fluxes were not taken into account in this study. It is likely that the fluxes will modify the boundary conditions and thus affect the integrability or not of a given system. This first attempt showed that the method should work once the correct string ansatz is applied. The D7 brane system is expected to be non-integrable and the analytic non-integrability method is the best way at this moment to prove this. There is definitely room to expand this study in the future for a different class of solutions.

Appendix A

A Map from Embedding to Intrinsic Coordinates for S^5

Giant gravitons wrap submanifolds of S^5. In the application of the analytic non-integrability method intrinsic coordinates are used. To understand how the giant gravitons are embedded in the S^5 a map from embedding coordinates to intrinsic coordinates will be needed. The map will also be useful to determine along which coordinate directions the giant graviton extends during the calculations.

The embedding coordinates are given by six real coordinates X_i, $i = 1, \ldots, 6$ or three complex coordinates [42],

$$W = X_1 + iX_2, \qquad Y = X_3 + iX_4, \qquad Z = X_5 + iX_5. \tag{A.1}$$

These coordinates (A.1) must satisfy the identity,

$$|W|^2 + |Y|^2 + |Z|^2 = 1. \tag{A.2}$$

This defines an S^5 embedded in $C^3 = R^6$. Next, the substitution,

$$Y = \cos\theta\cos\psi e^{i\varphi}, \qquad W = \cos\theta\sin\psi e^{i\eta}, \qquad Z = \sin\theta e^{i\phi}, \tag{A.3}$$

is made. As a check, the new coordinates in (A.3) must satisfy, (A.2).

$$|W|^2 + |Y|^2 + |Z|^2 = \cos^2\theta(\sin^2\psi + \cos^2\psi) + \sin^2\theta = 1.$$

To get to the usual metric of S^5 a map for the coordinate transformations needs to be determined. The metric in terms of the new coordinates has the form,

$$ds^2 = dWdW^* + dYdY^* + dZdZ^*. \tag{A.4}$$

Taking the respective derivatives and their complex conjugates results in the following expressions,

$$dY = -\sin\theta\cos\psi e^{i\varphi}d\theta - \cos\theta\sin\psi e^{i\varphi}d\psi + i\cos\theta\sin\psi e^{i\varphi}d\varphi, \tag{A.5}$$
$$dW = -\sin\theta\sin\psi e^{i\eta}d\theta + \cos\theta\cos\psi e^{i\eta}d\psi + i\cos\theta\sin\psi e^{i\eta}d\eta, \tag{A.6}$$
$$dZ = \cos\theta e^{i\phi}d\theta + i\sin\theta e^{i\phi}d\phi. \tag{A.7}$$

The complex conjugates are,

$$dY^* = -\sin\theta\cos\psi e^{-i\varphi}d\theta - \cos\theta\sin\psi e^{-i\varphi}d\psi - i\cos\theta\sin\psi e^{-i\varphi}d\varphi, \tag{A.8}$$
$$dW^* = -\sin\theta\sin\psi e^{-i\eta}d\theta + \cos\theta\cos\psi e^{-i\eta}d\psi - i\cos\theta\sin\psi e^{-i\eta}d\eta, \tag{A.9}$$
$$dZ^* = \cos\theta e^{-i\phi}d\theta - i\sin\theta e^{-i\phi}d\phi. \tag{A.10}$$

Substituting equations (A.5) through (A.10) in (A.4) results in the metric,

$$ds^2 = d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2 + \cos^2\theta\cos^2\psi d\varphi^2 + \cos^2\theta\sin^2\psi d\eta^2. \qquad \text{(A.11)}$$

This is the usual metric in intrinsic coordinates for S^5 so this choice of embedding coordinates is valid.

Appendix B

The Equivalence of Polyakov action and the Principal Chiral Model

Although this section is more suited for a study on integrable systems, it is important to understand the relationship between the Polyakov description of a string system and the Principal Chiral Model (PCM). Since S^5 is a symmetric space, the string action can be written as a PCM. The PCM is integrable since all the Lax connections can be calculated. For integrable systems, there is an equivalence between the Polyakov action and the PCM. It is always possible to write down the Polyakov action for a system. However if it can be shown that the Polyakov action is equivalent to the PCM then the system in question is integrable. Beginning with the string sigma model the equations of motion described by the current, j in [41] will be computed. Then it will be tested if these equations from the PCM turn out to be the same as those obtained by using the Polyakov action. Since S^5 is integrable there should be an equivalence for these two approaches.

For this calculation, the Minkowski metric $\eta = (-1, 1)$ will be used. The Hodge star operator is expressed as,

$$\begin{aligned} *dx^\nu &= \epsilon^\nu_\rho dx^\rho \\ &= \eta^{\nu\lambda}\epsilon_{\lambda\rho}dx^\rho, \end{aligned}$$

where $\epsilon_{\lambda\rho}$ is the usual two dimensional Levi-Civita tensor. For this calculation all that is required is an expression for $*d\tau$ and $*d\sigma$, with this choice of metric, these expression simplify to,

$$*d\tau = -d\sigma \qquad\qquad *d\sigma = -d\tau. \tag{B.1}$$

B.1 Calculating the Action

As in the previous section, the embedded coordinates are defined as $Z = sin\theta e^{i\phi}$ and $Y = \cos\theta e^{i\varphi}$. The $SO(6)$ symmetry can be expressed as an $SO(4)$x$SO(2)$ where the $SO(4)$ can be described as an $SU(2)$x$SU(2)$. Next g, an element of one of these $SU(2)$ sub-sectors of S^5 is introduced,

$$g = \begin{pmatrix} Z & Y \\ -\bar{Y} & \bar{Z} \end{pmatrix} = \begin{pmatrix} \sin\theta e^{i\phi} & \cos\theta e^{i\varphi} \\ -\cos\theta e^{-i\varphi} & \sin\theta e^{-i\phi} \end{pmatrix}, \tag{B.2}$$

The current j is defined as, $j = -g^{-1}dg$. Taking the inverse and the derivative of (B.2) followed by the appropriate matrix multiplication the final expression for the current is,

$$j = \begin{pmatrix} i(\cos^2\theta d\varphi - \sin^2\theta d\phi) & e^{i(\varphi-\phi)}(d\theta - i\sin\theta\cos\theta(d\phi + d\varphi)) \\ -e^{i(\phi-\varphi)}(d\theta + i\sin\theta\cos\theta(d\phi + d\varphi)) & i(\sin^2\theta d\phi - \cos^2\theta d\varphi) \end{pmatrix}. \tag{B.3}$$

According to [41] the action in terms of j is

$$S = -\frac{\lambda}{8\pi}\int\left[\frac{1}{2}Tr(j\wedge *j) + dt\wedge *dt\right]. \tag{B.4}$$

The action only requires the trace of the $j\wedge *j$ matrix so only the first and fourth elements need to be computed. The first element of the matrix is,

$$\begin{aligned} &i(\cos^2\theta d\varphi - \sin^2\theta d\phi)\wedge i(\cos^2\theta * d\varphi - \sin^2\theta * d\phi) \\ &\quad + e^{i(\varphi-\phi)}(d\theta - i\sin\theta\cos\theta(d\phi + d\varphi))\wedge -e^{i(\phi-\varphi)}(*d\theta + i\sin\theta\cos\theta(*d\phi + *d\varphi)) \\ &\qquad = (\cos^4\theta(\dot{\varphi}^2 - \varphi\prime^2) + \sin^4\theta(\dot{\phi}^2 - \phi'^2) + \dot{\theta}^2 - \theta'^2 \\ &\qquad\quad + \sin^2\theta\cos^2\theta(\dot{\phi}^2 - \phi'^2 + \dot{\varphi}^2 - \varphi'^2))d\tau\wedge d\sigma. \end{aligned} \tag{B.5}$$

The fourth element of the matrix is computed to be,

$$\begin{aligned} &-e^{i(\phi-\varphi)}(d\theta + i\sin\theta\cos\theta(d\phi + d\varphi))\wedge e^{i(\varphi-\phi)}(*d\theta - i\sin\theta\cos\theta(*d\phi + *d\varphi)) \\ &\quad + i(\sin^2\theta d\phi - \cos^2\theta d\varphi)\wedge i(\sin^2\theta * d\phi - \cos^2\theta * d\varphi) \\ &\quad = (-\cos^4\theta(-\dot{\varphi}^2 + \varphi'^2) - \sin^4\theta(-\dot{\phi}^2 + \phi'^2) + \dot{\theta}^2 - \theta'^2 \\ &\qquad + \sin^2\theta\cos^2\theta(\dot{\phi}^2 - \phi'^2 + \dot{\varphi}^2 - \varphi'^2))d\tau\wedge d\sigma. \end{aligned} \tag{B.6}$$

The trace of $j\wedge *j$ is then,

$$\begin{aligned} Tr(j\wedge *j) &= (2\dot{\theta}^2 - 2\theta'^2 + 2\cos^2\theta\sin^2\theta(\dot{\varphi}^2 - \varphi'^2 + \dot{\phi}^2 - \phi'^2) \\ &\quad + 2\cos^4\theta(\dot{\varphi}^2 - \varphi'^2) + 2\sin^4\theta(\dot{\phi}^2 - \phi'^2))d\tau\wedge d\sigma \\ &= 2(\dot{\theta}^2 - \theta'^2 + \cos^2\theta(\dot{\varphi}^2 - \varphi'^2) + \sin^2\theta(\dot{\phi}^2 - \phi'^2))d\tau\wedge d\sigma. \end{aligned}$$

Finally, the action is computed by substituting these results into equation (B.4).

$$S = -\frac{\lambda}{8\pi}\int d^2\sigma\left[-\dot{t}^2 + t'^2 + \dot{\theta}^2 - \theta'^2 - \cos^2\theta(\varphi'^2 - \dot{\varphi}^2) - \sin^2\theta(\phi'^2 - \dot{\phi}^2)\right], \tag{B.7}$$

here $d\tau\wedge d\sigma \to d^2\sigma$.

Next, the Polyakov action will be calculated to check if it matches the action obtained from the current. Starting with the AdS_5xS^5 metric with $\rho = 0$,

$$ds^2 = -dt^2 + \sin^2\theta d\phi^2 + d\theta^2 + \cos^2\theta(d\psi^2 + \sin^2\psi d\eta^2 + \cos^2\psi d\varphi^2), \tag{B.8}$$

the Polyakov action is given by the expression,

$$S = \frac{T}{2}\int d^2\sigma\sqrt{-h}h^{ab}g_{\mu\nu}\partial_a X^\mu\partial_b X^\nu, \tag{B.9}$$

where X contains each of the coordinates given in the metric (B.8). Here the space-time can be flattened by fixing to the conformal gauge such that $\sqrt{-h}h^{ab} \to \eta^{ab}$.

$$\begin{aligned} S = \frac{T}{2}\int d^2\sigma - (-\partial_\tau X^t \partial_\tau X^t + \partial_\sigma X^t \partial_\sigma X^t) - \partial_\tau X^\theta \partial_\tau X^\theta + \partial_\sigma X^\theta \partial_\sigma X^\theta \\ + \sin^2\theta(-\partial_\tau X^\phi \partial_\tau X^\phi + \partial_\sigma X^\phi \partial_\sigma X^\phi) + \cos^2\theta(-\partial_\tau X^\psi \partial_\tau X^\psi + \partial_\sigma X^\psi \partial_\sigma X^\psi) \\ + \cos^2\theta\cos^2\psi(-\partial_\tau X^\varphi \partial_\tau X^\varphi + \partial_\sigma X^\varphi \partial_\sigma X^\varphi) \\ + \cos^2\theta\sin^2\psi(-\partial_\tau X^\eta \partial_\tau X^\eta + \partial_\sigma X^\eta \partial_\sigma X^\eta) \end{aligned} \quad \text{(B.10)}$$

By choosing the $SU(2)$ sector, the string is restricted such that $\psi = 0$. The Polyakov action is then reduced to,

$$S = -\frac{T}{2}\int d^2\sigma - \dot{t}^2 + t'^2 + \dot{\theta}^2 - \theta'^2 - \sin^2\theta(\phi'^2 - \dot{\phi}^2) - \cos^2\theta(\varphi'^2 - \dot{\varphi}^2). \quad \text{(B.11)}$$

It is clear from equation (B.7) and (B.11) that using the current definition(B.4) or the Polyakov definition (B.9) results in the same expression for the action.

B.2 The Equations of Motion

The current j must obey the following conditions [41],

$$dj - j \wedge j = 0 \qquad \text{and} \qquad d * j = 0. \quad \text{(B.12)}$$

These expression are the equations of motion that j must satisfy. The equations listed in (B.12) need to be evaluated for (B.3). This will be done element wise. Starting with the derivative of the current matrix (B.3), the first element is evaluated as,

$$dj_{1,1} = -2i\cos\theta\sin\theta(\dot{\theta}\varphi' - \theta'\dot{\varphi} + \dot{\theta}\phi' - \theta'\dot{\phi})d\tau \wedge d\sigma. \quad \text{(B.13)}$$

The second element of the dj matrix is,

$$\begin{aligned} dj_{1,2} = -2ie^{i(\varphi-\phi)}[\dot{\phi}\theta' - \dot{\theta}\phi' + i\cos\theta\sin\theta(\dot{\varphi}\phi' - \dot{\phi}\varphi') \\ + \cos^2\theta(dot\phi\theta' - \dot{\theta}\phi' + dot\varphi\theta' - \dot{\theta}\varphi')]d\tau \wedge d\sigma, \end{aligned} \quad \text{(B.14)}$$

with the third and fourth element given by,

$$\begin{aligned} dj_{2,1} = 2ie^{i(\phi-\varphi)}[\dot{\varphi}\theta' - \dot{\theta}\varphi' - i\cos\theta\sin\theta(\dot{\phi}\varphi' - \dot{\varphi}\phi') \\ + \sin^2\theta(dot\theta\phi' - \dot{\phi}\theta' - dot\varphi\theta' + \dot{\theta}\varphi')]d\tau \wedge d\sigma, \end{aligned} \quad \text{(B.15)}$$

and

$$dj_{2,2} = 2i\sin\theta\cos\theta(\dot{\theta}\phi' - \dot{\phi}\theta' + \dot{\theta}\varphi' - \dot{\varphi}\theta')d\tau \wedge d\sigma, \quad \text{(B.16)}$$

respectively. Next the matrix elements of $j \wedge j$ need to be calculated. The four elements of the matrix are listed in order below.

$$j_{1,1} \wedge j_{1,1} = -2i\cos\theta\sin\theta(\dot{\theta}\varphi' - \theta'\dot{\varphi} + \dot{\theta}\phi' - \theta'\dot{\phi})d\tau \wedge d\sigma. \quad \text{(B.17)}$$

$$\begin{aligned} j_{1,2} \wedge j_{1,2} = -2ie^{i(\varphi-\phi)}[\dot{\phi}\theta' - \dot{\theta}\phi' + i\cos\theta\sin\theta(\dot{\varphi}\phi' - \dot{\phi}\varphi') \\ + \cos^2\theta(dot\phi\theta' - \dot{\theta}\phi' + dot\varphi\theta' - \dot{\theta}\varphi')]d\tau \wedge d\sigma, \end{aligned} \quad \text{(B.18)}$$

$$j_{2,1} \wedge j_{2,1} = 2ie^{i(\phi-\varphi)}[\dot{\varphi}\theta' - \dot{\theta}\varphi' - i\cos\theta\sin\theta(\dot{\phi}\varphi' - \dot{\varphi}\phi') \\ + \sin^2\theta(dot\theta\phi' - \dot{\phi}\theta' - dot\varphi\theta' + \dot{\theta}\varphi')]d\tau \wedge d\sigma, \quad \text{(B.19)}$$

and

$$j_{2,2} \wedge j_{2,2} = 2i\sin\theta\cos\theta(\dot{\theta}\phi' - \dot{\phi}\theta' + \dot{\theta}\varphi' - \dot{\varphi}\theta')d\tau \wedge d\sigma. \quad \text{(B.20)}$$

The elements from each of the two matrices are identical. Once they are subtracted as in the expression $dj - j \wedge j$ it is immediately clear that the equations of motion given by $dj - j \wedge j = 0$ are automatically satisfied. There is nothing additional that needs to be done for this constraint. The non-trivial equations of motion for j will need to come from the set of equations contained in $d * j = 0$. From the first element of (B.3) results in,

$$d[i\cos^2\theta * d\varphi - i\sin^2\theta * d\phi)] = 0 \\ \implies -2\cos\theta\sin\theta(-\dot{\theta}\dot{\varphi} + \theta'\varphi' - \dot{\theta}\dot{\phi} + \theta'\phi') + \cos^2\theta(\ddot{\varphi} - \varphi'') - \sin^2\theta(\ddot{\phi} - \phi'') = 0. \quad \text{(B.21)}$$

The second element produces the equation,

$$d[e^{i(\varphi-\phi)}(*d\theta - i\sin\theta\cos\theta(*d\phi + *d\varphi))] = 0 \\ \implies i(-\dot{\theta}\dot{\varphi} + \theta'\varphi' + \dot{\theta}\dot{\phi} - \theta'\phi') + \sin\theta\cos\theta(-\dot{\varphi}^2 + \varphi'^2 + \dot{\phi}^2 - \phi'^2) + \ddot{\theta} - \theta'' \\ + i(-\dot{\theta}\dot{\varphi} + \theta'\varphi' - \dot{\theta}\dot{\phi} + \theta'\phi')(\sin^2\theta - \cos^2\theta) - i\sin\theta\cos\theta(\ddot{\phi} - \phi'' + \ddot{\varphi} - \varphi'') = 0. \quad \text{(B.22)}$$

The third element yields,

$$d[-e^{i(\phi-\varphi)}(*d\theta + i\sin\theta\cos\theta(*d\phi + *d\varphi))] = 0 \\ \implies i(-\dot{\theta}\dot{\varphi} + \theta'\varphi' + \dot{\theta}\dot{\phi} - \theta'\phi') - \sin\theta\cos\theta(\dot{\varphi}^2 - \varphi'^2 - \dot{\phi}^2 + \phi'^2) + \ddot{\theta} - \theta'' \\ + i(-\dot{\theta}\dot{\varphi} + \theta'\varphi' - \dot{\theta}\dot{\phi} + \theta'\phi')(\cos^2\theta - \sin^2\theta) - i\sin\theta\cos\theta(\ddot{\phi} - \phi'' + \ddot{\varphi} - \varphi'') = 0, \quad \text{(B.23)}$$

and the fourth,

$$d[i\sin^2\theta * d\phi - i\cos^2\theta * d\varphi] = 0 \\ \implies 2\cos\theta\sin\theta(-\dot{\theta}\dot{\varphi} + \theta'\varphi' - \dot{\theta}\dot{\phi} + \theta'\phi') - \cos^2\theta(\ddot{\varphi} - \varphi'') + \sin^2\theta(\ddot{\phi} - \phi'') = 0. \quad \text{(B.24)}$$

Equations (B.21) and (B.24) are identical. This is expected as there are only three independent parameters, θ, φ and ϕ. There should only be three unique equations of motion. These equations are not yet in the correct form and require some manipulation and simplification. To get the equation of motion for θ, add equation (B.22) and (B.23), then simplify,

$$\ddot{\theta} - \theta'' - \sin\theta\cos\theta(\phi'^2 - \dot{\phi}^2 + \dot{\varphi}^2 - \varphi'^2) = 0. \quad \text{(B.25)}$$

Let (B.22)=(B.23) and simplify to obtain an expression for $-\cos^2\theta(\ddot{\varphi} - \varphi'')$. Substituting that expression in equation (B.24), the ϕ equation of motion is,

$$\sin\theta(\ddot{\phi} - \phi'') + 2\cos\theta(\theta'\phi' - \dot{\theta}\dot{\phi}) = 0. \quad \text{(B.26)}$$

Similarly, let (B.22)=(B.23), obtain an expression for $-\sin^2\theta(\ddot{\phi}-\phi'')$ and substitute that expression in equation (B.21). The equation of motion for φ is then,

$$\cos\theta(\ddot{\varphi}-\varphi'')+2\sin\theta(\dot{\theta}\dot{\varphi}-\theta'\varphi')=0. \tag{B.27}$$

The equations of motion given in (B.25), (B.26) and (B.27) exactly match the equations of motion that were produced in the in Chapter 3 using the Polyakov action. This shows the equivalence of the Polyakov and PCM which should indeed be the case for the S^5 metric.

Appendix C

Example of Mathematica Code

The following pages show a sample of the Mathematica code that was used to calculate the NVE for the D7 brane with the $S^2 \times S^2$ metric. The same code was used for all the calculations with only minor adjustments. Mathematica was not able to solve the NVE or directly implement the Kovacic algorithm. The NVE was entered into the Kovacicsols routine in maple and was able to produce a solution indicating that the NVE is integrable.

```
Clear[ds2, n,coordsτ,coords, θ,t,ρ,γ,ϕ1,ϕ2,ψ,ξ,χ,φ];

(*This is where the desired metric is entered and the various angles/variables are
defined. A matrix that will be used for the metric is also created and initialized
to zero.*)

ds2 = FullSimplify[d[ρ]^2-Cosh[ρ]^2 d[t]^2+Sinh[ρ]^2 (  d[θ]^2+ Cos[θ]^2 d[ϕ1]^2
+Sin[θ]^2 d[ϕ2]^2)+ d[γ]^2+Cos[γ]^2 d[ψ]^2+Cos[γ]^2 Sin[ψ]^2 d[φ]^2+Sin[γ]^2 d[χ]^2
+Sin[γ]^2 Sin[χ]^2 d[ξ]^2];
coords = { θ,t,ρ,γ,ϕ1,ϕ2,ψ,ξ,χ,φ};
n = Length[coords];
coordsτ = ConstantArray[0, n] ;
Do[coordsτ[[i]] = coords[[i]][τ, σ], {i,1, n}];
g = ConstantArray[0, {n, n}];

(*Here the metric is written in matrix form so that calculations can be done.*)

Do[g[[i, j]] = 1/2 D[  D[ds2, d[  coords[[i]]   ]   ],  d[  coords[[j]]   ]
]     , {i, 1, n}, {j, 1, n}];
 Do[gX = g/.coords[[i]]->coordsτ[[i]], {i, 1, n}];
 Do[gX = gX/.coords[[i]]->coordsτ[[i]], {i, 1, n}];


(*An array is created for all the angles and they are each given a dependence on τ
and σ*)

X = ConstantArray[0,n];
Do[X[[i]] = coords[[i]][τ, σ], {i, 1, n}];

(*Using the metric the derivatives with respect to τ and σ are calculated*)

dott[a_, b_, met_] := Sum[met[[i, j]]*a[[i]]*b[[j]], {i, 1, n}, {j, 1, n}];
Xtt = Simplify[ dott[D[X, τ], D[X, τ], gX] ];
Xss = Simplify[ dott[D[X, σ], D[X, σ], gX] ] ;
Xst =  Simplify[ dott[D[X, σ], D[X, τ], gX] ];

(*The virasoro constraints and the polyakov action are calculated from these
derivatives. Then the equations of motion are obtained. The superscripts in the
equations of motion denote a derivative with respect to τ or σ as well as if it is
a first derivative or second derivative. For example a (1,0) is a first derivative
with respect to τ and (0,2) is a second derivative with respect to σ*)

lagP =Xtt - Xss;
virasoro1 = Xst;
 virasoro2 = Xtt + Xss;
eqm1 =Simplify[ D[lagP, θ[τ, σ]] - D[D[lagP, (θ^(1,0))[τ,σ]], τ] -  D[D[lagP, (θ^
(0,1))[τ,σ]], σ]  ];
eqm2 =Simplify[ D[lagP, t[τ, σ]] - D[D[lagP, (t^(1,0))[τ,σ]], τ] -  D[D[lagP, (t^
(0,1))[τ,σ]], σ]  ];
eqm3 =Simplify[ D[lagP, ρ[τ, σ]] - D[D[lagP, (ρ^(1,0))[τ,σ]], τ] -  D[D[lagP, (ρ^
(0,1))[τ,σ]], σ]  ];
eqm4 =Simplify[ D[lagP, γ[τ, σ]] - D[D[lagP, (γ^(1,0))[τ,σ]], τ] -  D[D[lagP, (γ^
(0,1))[τ,σ]], σ]  ];
eqm5 =Simplify[ D[lagP, ϕ1[τ, σ]] - D[D[lagP, (ϕ1^(1,0))[τ,σ]], τ] -  D[D[lagP,
(ϕ1^(0,1))[τ,σ]], σ]  ];
eqm6 =Simplify[ D[lagP, ϕ2[τ, σ]] - D[D[lagP, (ϕ2^(1,0))[τ,σ]], τ] -  D[D[lagP,
(ϕ2^(0,1))[τ,σ]], σ]  ];
eqm7 =Simplify[ D[lagP, ψ[τ, σ]] - D[D[lagP, (ψ^(1,0))[τ,σ]], τ] -  D[D[lagP, (ψ^
(0,1))[τ,σ]], σ]  ];
eqm8 =Simplify[ D[lagP, ξ[τ, σ]] - D[D[lagP, (ξ^(1,0))[τ,σ]], τ] -  D[D[lagP, (ξ^
(0,1))[τ,σ]], σ]  ];
eqm9 =Simplify[ D[lagP, χ[τ, σ]] - D[D[lagP, (χ^(1,0))[τ,σ]], τ] -  D[D[lagP, (χ^
(0,1))[τ,σ]], σ]  ];
eqm10=Simplify[ D[lagP, φ[τ, σ]] - D[D[lagP, (φ^(1,0))[τ,σ]], τ] -  D[D[lagP, (φ^
(0,1))[τ,σ]], σ]  ];
```

```
(* In this section the Ansatz is applied to further simplify the 2 non-trivial
equations of motion for γ, χ and ψ *)

eomgamma=Block[{ρ,θ,γ, ψ,t,ξ,χ,φ,ϕ1,ϕ2},t[τ_,σ_]:=κ*τ;θ[τ_,σ_]:=θ[σ];ρ[τ_,σ_]:=0;γ
[τ_,σ_]:=γ[σ];ψ[τ_,σ_]:=ψ[σ];ξ[τ_,σ_]:=0; φ[τ_,σ_]:=ω2*τ; χ[τ_,σ_]:=ω1*τ; ϕ1
[τ_,σ_]:=0; ϕ2[τ_,σ_]:=0;FullSimplify[eqm4]]
eompsi=Block[{ ρ,θ,γ,ψ,t,ξ,χ,φ,ϕ1,ϕ2},t[τ_,σ_]:=κ*τ;θ[τ_,σ_]:=θ[σ];ρ[τ_,σ_]:=0;γ
[τ_,σ_]:=γ[σ];ψ[τ_,σ_]:=ψ[σ];ξ[τ_,σ_]:=0; φ[τ_,σ_]:=ω2*τ; χ[τ_,σ_]:=ω1*τ; ϕ1
[τ_,σ_]:=0; ϕ2[τ_,σ_]:=0;FullSimplify[eqm7]]
eomgamma=Block[{ρ,θ,γ, ψ,t,ξ,χ,φ,ϕ1,ϕ2},t[τ_,σ_]:=κ*τ;θ[τ_,σ_]:=θ[σ];ρ[τ_,σ_]:=0;γ
[τ_,σ_]:=γ[σ];ψ[τ_,σ_]:=ψ[σ];ξ[τ_,σ_]:=0; φ[τ_,σ_]:=ω2*τ; χ[τ_,σ_]:=ω1*τ; ϕ1
[τ_,σ_]:=0; ϕ2[τ_,σ_]:=0;FullSimplify[eqm9]]
eomgamma=Block[{ρ,θ,γ, ψ,t,ξ,χ,φ,ϕ1,ϕ2},t[τ_,σ_]:=κ*τ;θ[τ_,σ_]:=θ[σ];ρ[τ_,σ_]:=0;γ
[τ_,σ_]:=γ[σ];ψ[τ_,σ_]:=ψ[σ];ξ[τ_,σ_]:=0; φ[τ_,σ_]:=ω2*τ; χ[τ_,σ_]:=ω1*τ; ϕ1
[τ_,σ_]:=0; ϕ2[τ_,σ_]:=0;FullSimplify[eqm10]]

(* The outputs were as follows *)

 2 (Cos[γ[σ]] Sin[γ[σ]] (ω1^2-ω2^2 Sin[ψ[σ]]^2+(ψ^′)[σ]^2)+(γ^′′)[σ])
 2 Cos[γ[σ]] (-2 Sin[γ[σ]] (γ^′)[σ] (ψ^′)[σ]+Cos[γ[σ]] (ω2^2 Cos[ψ[σ]] Sin[ψ[σ]]
+(ψ^′′)[σ]))
 0
 0

(* Taking the solution γ=0 satisfies the equation of motion for γ and also reduces
the ψ equation of motion as follows *)

eomgamma0=Block[{ρ,θ,γ, ψ,t,ξ,χ,φ,ϕ1,ϕ2},t[τ_,σ_]:=κ*τ;θ[τ_,σ_]:=θ[σ];ρ[τ_,σ_]:=0;γ
[τ_,σ_]:=0;ψ[τ_,σ_]:=ψ[σ];ξ[τ_,σ_]:=0; φ[τ_,σ_]:=ω2*τ; χ[τ_,σ_]:=ω1*τ; ϕ1
[τ_,σ_]:=0; ϕ2[τ_,σ_]:=0;
FullSimplify[eqm4]]
Output = 0

eompsibar=Block[{ ρ,θ,γ,ψ,t,ξ,χ,φ,ϕ1,ϕ2},t[τ_,σ_]:=κ*τ;θ[τ_,σ_]:=θ[σ];ρ[τ_,σ_]:=0;γ
[τ_,σ_]:=0;ψ[τ_,σ_]:=ψ[σ];ξ[τ_,σ_]:=0; φ[τ_,σ_]:=ω2*τ; χ[τ_,σ_]:=ω1*τ; ϕ1
[τ_,σ_]:=0; ϕ2[τ_,σ_]:=0;FullSimplify[eqm7 ]]
Output= ω2^2 Sin[2 ψ[σ]]+2 (ψ^′′)[σ]

(* Here an expression for (ψ^′′)[σ] is defined for later use to simplify the NVE *)

ψprimeprime=-ω2^2 Sin[2* ψbar[σ]]/2;

(* The Virasoro constraints simplified as follows *)

Vira1 =Block[{ ρ,θ,γ,ψ,t,ξ,χ,φ,ϕ1,ϕ2},t[τ_,σ_]:=κ*τ;θ[τ_,σ_]:=θ[σ];ρ[τ_,σ_]:=0;γ
[τ_,σ_]:=0;ψ[τ_,σ_]:=ψ[σ];ξ[τ_,σ_]:=0; φ[τ_,σ_]:=ω2*τ; χ[τ_,σ_]:=ω1*τ; ϕ1
[τ_,σ_]:=0; ϕ2[τ_,σ_]:=0;FullSimplify[virasoro1]]
Output= 0
Vira2 =Block[{ ρ,θ,γ,ψ,t,ξ,χ,φ,ϕ1,ϕ2},t[τ_,σ_]:=κ*τ;θ[τ_,σ_]:=θ[σ];ρ[τ_,σ_]:=0;γ
[τ_,σ_]:=0;ψ[τ_,σ_]:=ψ[σ];ξ[τ_,σ_]:=0; φ[τ_,σ_]:=ω2*τ; χ[τ_,σ_]:=ω1*τ; ϕ1
[τ_,σ_]:=0; ϕ2[τ_,σ_]:=0;FullSimplify[virasoro2]]
Output= -κ^2+ω2^2 Sin[ψ[σ]]^2+(ψ^′)[σ]^2

(* From virasoro 2 an expression for ψ^′^2 is defined *)

ψprime2=κ^2-ω2^2 Sin[ψbar[σ]]^2;

(* Although not necessary the solution for ψ can be obtained from the expression
for ψ^′^2*)
DSolve[-κ^2+ω2^2 Sin[ψ[σ]]^2+(ψ^′)[σ]^2==0,ψ[σ],σ];

(* Assuming that ψbar is a solution to eompsi and using the above expression for
ψprime2. eomgamma will change as follows *)

eomgamma=Block[{ρ,θ,γ, ψ,t,ξ,χ,φ,ϕ1,ϕ2},t[τ_,σ_]:=κ*τ;θ[τ_,σ_]:=θ[σ];ρ[τ_,σ_]:=0;γ
```

```
[τ_,σ_]:=γ[σ];ψ[τ_,σ_]:=ψbar[σ];ξ[τ_,σ_]:=0; φ[τ_,σ_]:=ω2*τ; χ[τ_,σ_]:=ω1*τ; ϕ1
[τ_,σ_]:=0; ϕ2[τ_,σ_]:=0;FullSimplify[eqm4]]
Output= 2 (Cos[γ[σ]] Sin[γ[σ]] (ω1^2-ω2^2 Sin[ψbar[σ]]^2+(ψbar^′)[σ]^2)+(γ^′′)[σ])

(* Next we consider small fluctuations in γ around the solution ψbar *)

eomgamma = η(ω1^2-ω2^2 Sin[ψbar[σ]]^2+ψprime2)+d2ηdσ2

(* Make the substitution z=Cos[ψbar[σ]] and apply the chain rule to get the
derivatives in terms on the new coordinate z instead of σ *)

z[σ_]:=Sin[ψbar[σ]]^2 ;
dηdσ = ηprime*D[z[σ],σ];
d2ηdσ2 = ηprimeprime*D[z[σ],σ]^2+ηprime*D[D[z[σ],σ],σ];

(* Here the expression for  (ψbar^′′)[σ] from the equation of motion and (ψbar^′)
[σ]^2 from the virasoro constraint is subtituted in *)

d2ηdσ2=4 ηprimeprime Cos[ψbar[σ]]^2 Sin[ψbar[σ]]^2 ψprime2+ηprime (2 Cos[ψbar
[σ]]^2 ψprime2-2 Sin[ψbar[σ]]^2 ψprime2+2 Cos[ψbar[σ]] Sin[ψbar[σ]] ψprimeprime)

(* The last step is to get rid of the explicit ψbar dependence and only have an
NVE in terms of z*)

NVE = Block[{ψbar},ψbar[σ_]:=ArcSin[Sqrt[z]];FullSimplify[eomgamma]]
Output= 2 (ηprime-2 z ηprime-2 (-1+z) z ηprimeprime) κ^2+2 z ((-2+3 z) ηprime+2 (-1
+z) z ηprimeprime) ω2^2+η (κ^2+ω1^2-2 z ω2^2)

(* To check for non-integrability the Kovacic algorithm needs to be applied, this
next step is just to check if mathematica can solve the NVE*)
In[78]:= DSolve[η''[z](4*z(z-1)(z*ω2^2- κ^2))+η'[z](κ^2 (2-4z)-2*ω2^2*z(2-3z))+η[z]
(κ^2+ω1^2-2ω2^2 z)==0,η[z],z]

(*DSolve was not able to solve this NVE*)
```

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