Comparison theorems for elliptic and parabolic operators in variational form

by

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Declaration

I, Nduduzo Khayelihle Majozi declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Date: 30 November 2020

Abstract

Order properties are normally derived from maximum principles associated with an operator, say P, in classical formulation. This result is often formulated as a comparison theorem for solutions of linear elliptic PDEs as it derives order in the solution space from the order in the space of data. The solutions of the classical formulation in variational form are called weak solutions. The variational formulation and the associated concept of weak solution is widely used in the theory, applications and numerical analysis of elliptic and parabolic PDEs. In practice, often the variational formulation is used in order to accommodate generalized/ weak solutions and also to prepare for the use of numerical schemes such as finite element methods.

The space of weak solutions as well as space of data are Sobolev spaces, which are wider than the respective spaces of solutions and data in the classical formulation. This dissertation proves inverse monotonicity, or equivalently comparison theorems, for this much more general formulation of the operators and respective equations. More precisely, we prove results regarding order/comparison for the solutions of the variational problem through the concept of inverse monotone operators, which put them in a more general framework. We specifically discuss the case of a single equation and the case of systems of PDEs for both elliptic and parabolic equations.

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Chapter 1

Introduction

Mathematical models of real life phenomena are commonly formulated as Partial Differential Equations (PDEs) or systems of PDEs. Let $P : M \mapsto N$, where M and N are partially ordered sets. These type of models can be written in the form

$$P[u] = g, \tag{1.1}$$

where the operator P involves both a differential operator on a function space and boundary condition on u in the study of elliptic PDEs. Additionally the operator P involves the initial condition on u in the case of parabolic PDEs. The operator P is called a *monotone increasing operator* if for every $u, v \in M$ we have

$$u \leq v \implies P[u] \leq P[v].$$

The operator P is called *inverse monotone* if for any $u, v \in M$

$$P[u] \leq P[v] \implies u \leq v. \tag{1.2}$$

We note that if the inverse operator P^{-1} of P exists, then (1.2) implies that P^{-1} is monotone increasing. This motivates the term "inverse monotone", which we use in the sequel. In such settings, statements of the form (1.2) are referred to as comparison theorems. The name reflect the fact that (1.2) provides means of proving order relations between functions in the solution space of equations of the form (1.1). This type of theorem is useful from both theoretical, e.g. proving uniqueness, and practical, e.g. constructing lower/upper bounds, applications, points of view.

Comparison theorems are typically derived from maximum principles associated with the operator P in classical formulation, that is, the domain of P comprises of sufficiently smooth functions so that all derivatives involved in P exist in a classical sense. For this classical setting, there is extensive theory for elliptic and parabolic operators for both one–dimensional (single equation) or multidimensional (systems of PDEs) cases. In practice, very often the operator P is extended to a larger domain in order to accommodate the wider class of physically meaningful solutions. In this dissertation, we follow one of the very popular approaches to formulate the operator P and, respectively, equation (1.1), in variational form. The solutions of (1.1) in variational form are called weak solutions.

Compared to the classical setting, there is little theory related to comparison theorems for operators in variational formulation. Here we need to acknowledge two remarkable results in [6, Chapter XVIII, Section 4, Theorem 2] and [6, Chapter XVIII, Section 4, Theorem 3]. These results provide positivity and maximum principle for the solution of a single parabolic PDE with homogeneous boundary conditions.

In the dissertation we derive comparison theorems for elliptic and parabolic PDEs with non-homogeneous boundary conditions which are further extended to certain classes of systems of PDEs. The approach is similar to the approach in the book of of Walter [22], where the inequalities are not derived for solutions of stated problems, but rather for arbitrary functions in the space of solutions. More precisely, the obtained results derive order in the space of solutions from the order in the space of data. In Walter [22] the obtained results refer to the classical setting of parabolic operators.

The theory presented in the dissertation refers to elliptic and parabolic operators given in variational form. The solutions of the respective equations or systems of equations are called weak solutions of the associated classical problems. The space of weak solutions as well as the space of data are Sobolev spaces, which are wider than the respective spaces in solutions and data in the classical formulation. The main goal of the dissertation is to formulate comparison theorems in this much more general formulation of the operators. Equivalently, this goal can be formulated as deriving order in the domains of the operators under consideration, that is the space of data (forcing term, boundary conditions, initial conditions). The latter property of the operators as defined in (1.2) is called inverse monotonicity.

The method of proof of [6, Chapter XVIII, Section 4, Theorem 2] and preliminary results in [6] are used in deriving the results presented in the dissertation. However, these results are much wider than the mentioned theorems in [6]. The novelty in the dissertation can be summarised as follows:

1) A comparison theorem for the operator associated with an elliptic boundary value problem on $\Omega \subset \mathbb{R}^n$ in variational form. The operator includes the boundary condition and is defined on $H^1(\Omega) \times H^{1/2}(\partial\Omega)$. (Section 3.4)

2) A comparison theorem for the operator in 1) extended to a weakly coupled system of elliptic PDEs. (Section 3.6)

3) A comparison theorem for the operator associated with a parabolic initial–boundary value problem on $[0, \theta] \times \Omega$. The domain of the operator is $L^2(0, \theta; H^1(\Omega), H^{-1}(\Omega)) \times L^2(0, \theta; H^{1/2}(\partial\Omega)) \times L^2(\Omega)$. (Section 4.3)

4) A comparison theorem for the operator in 3) extended to systems of weakly coupled parabolic PDEs. (Section 4.5)

5) All results in 1)–4) are restated as inverse monotonicity of the respective operator, thus providing a different point of view of the comparison theorems, namely as a "pull–back" mechanism of the order in the target space of the operator's domain. (Sections 3.5, 3.6, 4.4, 4.5)

The dissertation is structured as follows: in the next chapter we we provide mathematical preliminaries which give basic results regarding the spaces in which weak solutions lie. We further state the comparison theorem for systems of Ordinary Differential Equations (ODEs). In Chapter 3, we recall the comparison theorem derived from the maximum principle for elliptic operators. We also deal with one and multidimensional operators in variational form related to a single elliptic PDE or a weakly coupled system of elliptic PDEs. In Chapter 4, we extend the concept studied in the elliptic case. To be more precise, we study the operators in variational form related to parabolic PDEs. Different assumptions on the differential operator are explored and the appropriate comparison theorems are formulated and proved. In Chapter 5 we give some concluding remarks as well as a discussion of possible future work.

Chapter 2

Mathematical preliminaries

In this chapter we provide mathematical preliminaries as a toolbox needed for the theory of PDEs. The theory, that is, preliminary results given here, include L^p, Sobolev and dual spaces. We also introduce Comparison Principles for systems of ODEs as they have a significant role in the theory of parabolic PDEs.

2.1 L^p Spaces

In this section we introduce function spaces which are commonly used in the theory of PDEs. The presentation of the results here follow the book of Quarteroni [17, Chapter 1.2]. Let Ω be an open subset of $\mathbb{R}^n$ and in Ω we consider the Lebesgue measure. The Lebesgue measure gives a way of describing the size of some subsets of $\mathbb{R}^n$. The term "almost everywhere in Ω", abbreviated a.e., means everywhere in Ω except on a subset of Ω with Lebesgue measure zero. We look at L^p spaces which belong to the family of Banach spaces. Let $p \in [1, \infty)$ and

$$L^p(\Omega) = \{f : \Omega \mapsto \mathbb{R} |\ f \text{ is measurable and } |f|^p \text{ is measurable and integrable}\}$$

and the associated norm is given by

$$||f||_{L^p(\Omega)} = \left(\int_\Omega |f|^p dx \right)^{1/p}.$$

We also have

$$L^p_{loc} = \{f : \Omega \mapsto \mathbb{R} |\ f \in L^p(V) \text{ for each } V \subset\subset \Omega\}.$$

Moreover, in this dissertation our main focus is on the space $L^2(\Omega)$, which is a Hilbert space with a scalar/ inner product given by

$$(f, g) = \int_\Omega fg\ dx \tag{2.1}$$

and we see that

$$||f||_{L^2(\Omega)} = \left(\int_\Omega |f|^2 dx \right)^{1/2}.$$

Theorem 2.1 (Hölder's Inequality)**.** *[9, Appendix B2] Assume* $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then if* $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, *we have*

$$\int_\Omega |uv|\ dx \le ||u||_{L^p(\Omega)}\ ||v||_{L^q(\Omega)}. \tag{2.2}$$

2.2 Sobolev spaces

Sobolev spaces are named after a Russian mathematician Sergei Sobolev. The development of Sobolev spaces is driven by the need to accommodate weak solutions of PDEs. In this dissertation we study precisely solutions that belong to Sobolev spaces. The theory discussed in this section mostly follows the book of Evans [9, Chapter 5]. Let Ω denote an open, connected and bounded subset of $\mathbb{R}^n$. Let C_c^∞ denote the space of infinitely differentiable functions $\phi : \Omega \mapsto \mathbb{R}$, with compact support in Ω. A function φ belonging to $C_c^\infty(\Omega)$ is called a test function which we use in the text. Let k be a positive integer and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be a multi–index of order $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n = k$. Then we state the definition of a weak derivative as follows:

Definition 2.1 (Weak derivative)**.** *Let $u, v \in L^1_{loc}(\Omega)$, and α be a multiindex. We say that v is the α^{th}- weak partial derivative of u, written*

$$D^\alpha u = v,$$

provided

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v \varphi dx$$

for all test functions $\varphi \in C_c^\infty(\Omega)$.

The general definition of a Sobolev space is

Definition 2.2 (Sobolev space)**.** *The Sobolev space $W^{k,p}(\Omega)$ consists of all summable functions $u : \Omega \mapsto \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and $D^\alpha u \in L^p(\Omega)$.*

With regards to this dissertation we consider the case where $k = 1$ and $p = 2$, this space is denoted as $H^1(\Omega)$. To be more precise,

$$H^1(\Omega) = W^{1,2}(\Omega) = \left\{ u \in L^2(\Omega) \;\middle|\; \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, 2, ..., n \right\}. \tag{2.3}$$

The inner product on $H^1(\Omega)$ is defined as

$$(f, g)_{H^1(\Omega)} = \int_\Omega fg \; dx + \int_\Omega \nabla f \cdot \nabla g \; dx,$$

with the associated norm

$$||u||_{H^1(\Omega)} = \left(||u||^2_{L^2(\Omega)} + ||\nabla u||^2_{L^2(\Omega)} \right)^{1/2}.$$

Theorem 2.2. *The Sobolev space $H^1(\Omega)$ is a Hilbert space.*

Let us recall the concept of trace as given in the following theorem.

Theorem 2.3 (Trace Theorem). *[17, Theorem 1.3.1] Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$.*

(a) There exists a unique linear continuous map $T : H^1(\Omega) \mapsto H^{1/2}(\partial\Omega)$ such that $Tu = u|_{\partial\Omega}$ for each $u \in H^1(\Omega) \cap C(\bar{\Omega})$.

(b) There exists a linear continuous map $T^{-1} : H^{1/2}(\partial\Omega) \mapsto H^1(\Omega)$ such that $TT^{-1}(\varphi) = \varphi$ for each $\varphi \in H^{1/2}(\partial\Omega)$.

Using the concept of trace given in (a) we denote

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : \; Tu = 0\}.$$

Definition 2.3 (Convergence). *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $H^1(\Omega)$. We say (u_n) converges to u in $H^1(\Omega)$, that is*

$$u_n \to u \text{ in } H^1(\Omega)$$

provided that

$$\lim_{n\to\infty} ||u_n - u||_{H^1(\Omega)} = 0.$$

Remark 2.1. *The space $C^\infty(\Omega)$ is dense in $H^1(\Omega)$ [15].*

Remark 2.2. *The space $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ with respect to the norm in $H^1(\Omega)$.*

The Sobolev space is a space of real valued functions and the following theorem gives an assertion to that.

Theorem 2.4 (Sobolev space as function space). *The Sobolev space $H^1(\Omega)$ is a Banach space.*

Theorem 2.5 (Poincaré's Inequality). *Assume Ω is a bounded, open subset of $\mathbb{R}^n$ and $u \in H_0^1(\Omega)$. Then we have the estimate*

$$||u||_{L^2(\Omega)} \leq C||\nabla u||_{L^2(\Omega)} \tag{2.4}$$

for the constant C depending on Ω.

Lemma 2.1. *[5, Chapter IV, Section 7, Lemma 1] Let $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$. Then for $u \in H^1(\Omega)$, with Ω an arbitrary open set in $\mathbb{R}^n$, we have:*

$$f \circ u \in H^1(\Omega).$$

Furthermore,

$$grad(f \circ u) = (f' \circ u) grad\ u.$$

2.3 Dual spaces

We briefly discuss the theory of dual spaces as given in the book of Quarteroni [17, Chapter 1.2]. Let $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ be normed spaces. We denote the set of linear continuous functions from V into W by $\mathcal{L}(V;W)$.

Definition 2.4. *Let $L \in \mathcal{L}(V;W)$ then the norm for L is defined as*

$$||L||_{\mathcal{L}(V;W)} := \sup_{v\in V\setminus\{0\}} \frac{||Lv||_W}{||v||_V}. \tag{2.5}$$

The expression $||\cdot||_{\mathcal{L}(V;W)}$ defines a norm on $\mathcal{L}(V;W)$ so that this space is also a normed space. If W is a Banach space, it follows that $\mathcal{L}(V;W)$ is a Banach space too.

Definition 2.5. *If $W = \mathbb{R}$, then the space $\mathcal{L}(V;\mathbb{R})$ equipped with the norm (2.5) is called the dual space of V is denoted by V'.*

Definition 2.6 (Duality Pairing)**.** *The bilinear form $\langle\cdot,\cdot\rangle : V' \times V \mapsto \mathbb{R}$ defined by $\langle L, \varphi\rangle := L(\varphi)$ is called the duality pairing between V' and V.*

As stated before, our theory focuses on weak solutions, that is, solutions in the Sobolev space. Hence we investigate duality of Sobolev spaces. The dual space to $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ [9, Definition 1, Chapter 5.9.1]. Evans [9, Definition 2, Chapter 5.9.1] defines the norm of $H^{-1}(\Omega)$ for a function $f \in H^{-1}(\Omega)$, as

$$||f||_{H^{-1}(\Omega)} = \sup\left\{ \frac{\langle f, u\rangle}{||u||_{H_0^1(\Omega)}} \;\middle|\; u \in H_0^1(\Omega), ||u||_{H_0^1(\Omega)} \neq 0 \right\}.$$

If $f \in L^2(\Omega)$ then

$$\varphi \mapsto (f, \varphi)$$

is a bounded linear functional on $H_0^1(\Omega)$. In this case,

$$\langle f, \varphi\rangle = (f, \varphi)$$

and

$$||f||_{H^{-1}(\Omega)} = ||f||_{L^2(\Omega)}.$$

Theorem 2.6 (Characterisation of $H^{-1}(\Omega)$)**.** *[9, Theorem 1, Chapter 5.9.1] Let $\varphi \in H_0^1(\Omega)$ and assume $f \in H^{-1}(\Omega)$. Then there exist functions $f^0, f^1, ..., f^n$ in $L^2(\Omega)$ such that*

$$\langle f, \varphi\rangle = \int_\Omega f^0\varphi + \sum_{i=1}^n f^i \frac{\partial \varphi}{\partial x_i}\, dx.$$

Remark 2.3. *If $f \in L^2(\Omega)$ then $f^0 = f$ and $f^i = 0$ for $i = 1, ..., n$. This representation does not contradict the fact that $L^2(\Omega)$ is a subspace of $H^{-1}(\Omega)$.*

Showalter [18, Chapter 3, Section 2] gives the following result which we later make use of in obtaining the existence and uniqueness of weak solutions of parabolic PDEs:

Theorem 2.7. *Let V be a separable Hilbert space with dual V', then $\mathcal{V} \equiv L^2(0, \theta, V)$ is a Hilbert space with dual $\mathcal{V}' \equiv L^2(0, \theta, V')$. Assume for each $t \in [0, \theta]$ we are given a continuous bilinear form $B(t; \cdot, \cdot)$ on V such that for each pair $u, \varphi \in V$ the function $B(\cdot, u, \varphi)$ is in $L^\infty(0, \theta, \mathbb{R})$. Then*

(a) by the uniform boundedness principle we have an $M > 0$ for which

$$|B(t; u, \varphi)| \leq M||u|| \ ||\varphi|| \text{ for } u, \varphi \in V \text{ and } t \in [0, \theta];$$

(b) if $u \in \mathcal{V}$ and $\varphi \in V$ we have that $t \mapsto B(t; u, \varphi)$ is measurable.

2.4 Comparison theorems for systems of ODEs

In this section our aim is to provide the comparison theorem for systems of ODES as given in the book of Walter [22]. We consider a system of non-autonomous ordinary differential equations given by:

$$\frac{du}{dt} = g(t, u) \tag{2.6}$$

where $g : \mathscr{U}(g) \mapsto \mathbb{R}^n$, $\mathscr{U}(g) \subset [0, \infty) \times \mathcal{U}$ and $\mathcal{U}$ is an open subset of $\mathbb{R}^n$.

Definition 2.7 (Class $Z(g)$)**.** *The function class $Z(g)$ contains all functions $u : [0, \infty) \mapsto \mathbb{R}^n$ which are continuous in $[0, \infty)$ and differentiable in $(0, \infty)$ and satisfy $(t, u(t)) \in \mathscr{U}(g)$ for $t \in (0, \infty)$.*

Definition 2.8 (Defect Q)**.** *The defect $Q[u]$ of a function $u(t)$ with respect to equation* (2.6) *is only defined for $u \in Z(g)$. and given by*

$$Q[u](t) = \frac{du(t)}{dt} - g(t, u(t)), \quad t \in (0, \infty).$$

Note: The defect operator is applicable to all functions in $Z(g)$.

Definition 2.9. *A function $\phi : \mathcal{U} \mapsto \mathbb{R}^n$ is called one-sided Lipschitz on $\mathcal{U} \subset \mathbb{R}^n$ if*

$$\langle \phi(u) - \phi(v), u - v \rangle \;\; \leq \;\; L||u - v||^2$$

for some positive real constant L and all $u, v \in \mathcal{U}$ where $\langle \cdot, \cdot \rangle$ is a dot product on $\mathbb{R}^n$.

Theorem 2.8 (Forward Uniqueness)**.** *If g is one-sided Lipschitz with respect to $u \in \mathcal{U}$, then any solution u of* (2.6) *is unique on its domain.*

In Walter [22] a more general case of functions $\omega(t, z)$ for which uniqueness follows from a one-sided condition is given as

$$f(t, z) - f(t, \bar{z}) \leq \omega(t, z - \bar{z}) \text{ for } z \geq \bar{z} \text{ and } (t, z), (t, \bar{z}) \in \mathcal{U}. \tag{2.7}$$

We show that (2.7) implies the one-sided Lipschitz condition here below. Suppose f is Lipschitz and pick $\omega(t, z) = \frac{L}{n\sqrt{n}}||z||$. Also note that $y \leq x$ then

$$\begin{aligned}
\langle f(x) - f(y), x - y \rangle &= \sum_{i=1}^{n} (f(x) - f(y)) \cdot (x_i - y_i) \\
&\leq n \frac{L}{n\sqrt{n}} ||x - y||_2 \; \sum_{i=1}^{n} (x_i - y_i) \\
&\leq \frac{L}{\sqrt{n}} ||x - y||_2 \; \sum_{i=1}^{n} ||x_i - y_i|| \\
&= \frac{L}{\sqrt{n}} ||x - y||_2 \; ||x - y||_1 \\
&\leq \frac{L}{\sqrt{n}} ||x - y||_2 \; (\sqrt{n} ||x - y||_2) \\
&= \frac{L}{\sqrt{n}} \sqrt{n} ||x - y||_2^2 \\
&= L ||x - y||_2^2
\end{aligned}$$

Definition 2.10 (Quasi-monotone function). *A vector function $\phi(z) = (\phi_1(z), \phi_2(z), ..., \phi_n(z))$ depending on $z = (z_1, ..., z_n) \in \mathbb{R}^n$ is said to be quasi-monotone increasing if for $i = 1, .., n$*

$$\phi_i(z) \leq \phi_i(\bar{z}) \text{ for } z \leq \bar{z}, z_i = \bar{z}_i.$$

Theorem 2.9 (Comparison Theorem). *Suppose the function g is quasi-monotone increasing and satisfies the forward uniqueness condition with respect to u. Then for any $u, v \in Z(g)$ we have*

$$\begin{pmatrix} u(0) \leq v(0) \\ Q[u] \leq Q[v] \end{pmatrix} \implies u \leq v.$$

As discussed in the Introduction, Theorem 2.9 can be equivalently represented by inverse monotonicity. In our case, the operator P is defined as

$$P[u] = \begin{pmatrix} u(0) \\ Q[u] \end{pmatrix} \in Z(g).$$

<u>Application to Monotone Dynamical Systems</u>
We consider a system of ordinary differential equations given by:

$$\frac{du}{dt} = f(u) \tag{2.8}$$

where $f : \mathcal{U} \mapsto \mathbb{R}^n$ and $\mathcal{U}$ is an open subset of $\mathbb{R}^n$.

Definition 2.11. *Equation* (2.8) *defines a <u>forward dynamical system</u> on $\mathcal{U}$ if for every $u^0 \in \mathcal{U}$ there exists a unique solution of* (2.8) *defined on* $[0, \infty)$ *such that* $u(0) = u^0$*. Further, if for any two solutions u and v of* (2.8) *we have* $u(0) \leq v(0) \implies u(t) \leq v(t)$ *for all* $t \geq 0$*, then* (2.8) *is called <u>monotone</u>.*

Theorem 2.10 (Application of the Comparison Theorem). *Let* (2.8) *define a forward dynamical system on $\mathcal{U}$. If f is quasi-monotone and one-sided Lipschitz on $\mathcal{U}$ then this dynamical system is monotone.*

If f is smooth then $f(u)$ is quasi-monotone on $u \in \mathcal{U}$ if and only if the Jacobian of f, (J_f), has non-negative off-diagonal entries. J_f is a Metzler matrix, that is, there exists λ such that $J_f + \lambda I \geq 0$ [14].
It is important to remark that the theorem of monotone dynamical systems provides order between solutions of the respective system. This is more restrictive in application compared to comparison principles as the latter one applies to arbitrary functions in $Z(g)$ and not only solutions. Specifically, one can use comparison theorems to derive lower and upper approximations of solutions of ODEs and PDEs.

Chapter 3

Elliptic PDEs

3.1 Classical formulation

Let us recall that an operator P from a partially ordered set M to a partially ordered set N is called monotone if it preserves order in M. Further, the operator P is said to be inverse monotone if the order of N is preserved in the mapping as stated in (1.2). In this chapter we are going to consider an operator which involves a second-order differential operator on $C^2(\Omega)$ given by:

$$L[u] \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu, \tag{3.1}$$

where $a_{ij}, b_i, c \in L^\infty(\Omega)$ are functions of $x = (x_1, x_2, ..., x_n) \in \Omega$ with $c \geq 0$ and Ω an open bounded subset of $\mathbb{R}^n$. We assume the symmetry condition, that is $a_{ij} = a_{ji}$ for $i, j = 1, 2, ..., n$. Hence, the $n \times n$ matrix

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & ... & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & ... & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & ... & a_{nn}(x) \end{pmatrix} \tag{3.2}$$

is symmetric.

Definition 3.1. *[16, Definition 1, Chapter 2.2] The operator L is said to be elliptic if for every $x \in \Omega$, and every $\xi \in \mathbb{R}^n \setminus \{0\}$ we have*

$$\xi \cdot A(x)\xi > 0.$$

We say that L is uniformly elliptic if there exists a constant $\mu_0 > 0$ such that

$$\xi \cdot A(x)\xi \geq \mu_0 |\xi|^2$$

for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

Remark 3.1. *Since A is symmetric, all its eigenvalues are real numbers. Definition 3.1 implies that L is elliptic if all the eigenvalues of $A(x)$ are positive and that L is uniformly elliptic if the eigenvalues of $A(x)$ are bounded below on Ω by a positive constant μ_0.*

For the comparison theorem of a uniformly elliptic operator we require the Maximum Principle, which is stated here below:

Theorem 3.1 (Maximum Principle). *[9, Theorem 1, Section 6.4.1] Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $c \equiv 0$ in Ω. If*

$$L[u] \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} \leq 0 \text{ in } \Omega, \tag{3.3}$$

then

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x).$$

The maximum principle holds not only for solutions of PDEs but for all functions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that satisfy the differential inequality (3.3). A consequence of the maximum principle is the Comparison Theorem stated here below:

Theorem 3.2 (Comparison Theorem). *Let L be uniformly elliptic on $C^2(\Omega)$. Then for every $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ we have $L[u] \leq L[v], u|_{\partial\Omega} \leq v|_{\partial\Omega} \implies u \leq v$ on Ω.*

Proof. Let $w = u - v$. Since L is a linear operator we have that $L[w] = L[u] - L[v]$. Since $L[u] \leq L[v]$, it follows that $L[w] \leq 0$. Using Theorem 3.1 we have that:

$$\begin{aligned} \max_{x\in\overline{\Omega}} w(x) &= \max_{x\in\partial\Omega} w(x) \\ &= \max_{x\in\partial\Omega}(u(x) - v(x)) \\ &\leq 0. \end{aligned} \tag{3.4}$$

From(3.4) we have that

$$u(x) - v(x) = w(x) \leq \max_{x\in\partial\Omega} w(x) \leq 0.$$

It follows that $u(x) - v(x) \leq 0$ on Ω, hence $u(x) \leq v(x)$ on Ω. □

As discussed in the Introduction, the Comparison Theorem can be represented in terms of inverse monotonicity of an appropriate operator. And in this case we have $P : C^2(\Omega) \cap C(\overline{\Omega}) \mapsto C(\Omega) \times C(\partial\Omega)$ defined as

$$P[u] = \begin{pmatrix} L[u] \\ u|_{\partial\Omega} \end{pmatrix}. \tag{3.5}$$

This operator L is related to problems of PDEs with Dirichlet boundary conditions, homogeneous or non-homogeneous, such as:

$$\begin{cases} L[u] = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where L is given by (3.1) and $f \in L^2(\Omega)$ and $g \in C(\partial\Omega)$. Then (3.6) can be equivalently written as

$$P[u] = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{3.7}$$

Using the concept of inverse monotonicity we can reformulate the Comparison Theorem 3.2 as follows:

Theorem 3.3. *If L is uniformly elliptic on $C^2(\Omega)$, the operator $P[u] = \begin{pmatrix} L[u] \\ u|_{\partial\Omega} \end{pmatrix}$ is inverse monotone.*

Proof. $P[u] \leq P[v] \iff \begin{pmatrix} L[u] \\ u|_{\partial\Omega} \end{pmatrix} \leq \begin{pmatrix} L[v] \\ v|_{\partial\Omega} \end{pmatrix} \iff (L[u] \leq L[v],\, u|_{\partial\Omega} \leq v|_{\partial\Omega})$. □

Existence and uniqueness of the solution of (3.6) implies that
$P^{-1} : C(\Omega) \times C(\partial\Omega) \mapsto C^2(\Omega) \cap C(\overline{\Omega})$ exists. In fact,

$$u = P^{-1} \begin{pmatrix} f \\ g \end{pmatrix}$$

is the solution of (3.7).

3.2 Variational formulation in the case of homogeneous Dirichlet boundary conditions

Let Ω be an open, bounded subset of $\mathbb{R}^n$. Assume that $u \in C^2(\Omega)$. We consider the following homogeneous Dirichlet boundary value problem:

$$\begin{cases} L[u] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

where L is given by (3.1) and $f \in L^2(\Omega)$.
It is usually useful to write the (3.1) in the following vector form:

$$L[u] = -\nabla \cdot (A\nabla u) + \vec{b} \cdot \nabla u + cu \tag{3.9}$$

where A is defined as in equation (3.2), $\vec{b} = (b_1, b_2, ..., b_n)'$. Using the vector form (3.9), we multiply the first equation in (3.8) by $\varphi \in C_c^\infty(\Omega)$ and integrate

$$\begin{aligned} \int_\Omega (L[u]\ \varphi)\ dx &= \int_\Omega f\varphi\ dx \\ \int_\Omega (-\nabla \cdot (A\nabla u)\varphi + \vec{b} \cdot \nabla u\varphi + cu\varphi)\ dx &= \int_\Omega f\varphi\ dx. \end{aligned} \tag{3.10}$$

We focus on the left hand side of (3.10) and using Green's formula we obtain:

$$\int_\Omega (L[u]\varphi)\ dx = \int_\Omega \nabla \cdot (A\nabla u)\varphi\ dx + \int_\Omega \vec{b} \cdot \nabla u\varphi\ dx + \int_\Omega cu\varphi\ dx - \int_{\partial\Omega} A\nabla u\varphi \cdot \vec{n}\ ds.$$

Since $u = 0$ on $\partial\Omega$ and $\varphi \in C_c^\infty(\Omega)$ we have that

$$\begin{aligned} \int_\Omega (L[u]\varphi)\ dx &= \int_\Omega \nabla \cdot (A\nabla u)\varphi\ dx + \int_\Omega \vec{b} \cdot \nabla u\varphi\ dx + \int_\Omega cu\varphi\ dx \\ &= \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\ dx + \int_\Omega \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} \varphi\ dx + \int_\Omega cu\varphi\ dx \end{aligned}$$

Let

$$B(u, \varphi) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\ dx + \int_\Omega \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} \varphi\ dx + \int_\Omega cu\varphi\ dx \tag{3.11}$$

where $u \in C^2(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$. Then every solution u of (3.8) satisfies

$$B(u, \varphi) = (f, \varphi) \text{ for all } \varphi \in C_c^\infty(\Omega), \tag{3.12}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$.
Recall the Sobolev space

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) \middle| \ \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ i = 1, 2, ..., n \right\}$$

as defined in (2.3). We also remember from Remark 2.2 that the space $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ with respect to the norm in $H^1(\Omega)$ then (3.12) implies

$$B(u, \varphi) = (f, \varphi) \text{ for all } \varphi \in H_0^1(\Omega). \tag{3.13}$$

The formulation of (3.13) does not require $u \in C^2(\Omega)$ but only $u \in H_0^1(\Omega)$.

Definition 3.2. *[9] A function $u \in H_0^1(\Omega)$ is called a weak solution of the boundary value problem* (3.8) *if*

$$B(u, \varphi) = (f, \varphi) \text{ for all } \varphi \in H_0^1(\Omega).$$

It follows from Definition 3.2 that the homogeneous Dirichlet boundary value problem can be reformulated as:

$$\text{Given } f \in L^2(\Omega), \text{ find } u \in H_0^1(\Omega) \text{ such that (3.13) holds.} \tag{3.14}$$

This reformulation is called the variational formulation. When we reach this stage, one may ask whether or not the solution exists for this problem that we have transformed so much. We then use the Lax-Milgram Theorem to answer this question. It is not easy to find an explicit formula for the solution of any given problem, therefore we need to show existence implicitly using the Lax-Milgram theorem stated here below:

Theorem 3.4 (Lax-Milgram)**.** *[9, Theorem 1, Section 6.2.1] Let H be a Hilbert space. Assume that $B : H \times H \mapsto \mathbb{R}$ is a bilinear mapping, for which there exist constants α and $\beta \geq 0$ such that*

(a) $$|B(u, \varphi)| \leq \alpha ||u|| \; ||\varphi|| \tag{3.15}$$

(b) $$B(u, u) \geq \beta ||u||^2 \tag{3.16}$$

where $|| \cdot ||$ is a norm in H. Let $f : H \mapsto \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique element $u \in H$ such that

$$B(u, \varphi) = (f, \varphi)$$

for all $\varphi \in H$.

For us to prove that our problem (3.8) has a unique solution, we are required to prove that the bilinear form given in (3.11) satisfies the conditions of the Lax-Milgram Theorem, that is, (3.15) and (3.16). We prove that these conditions are satisfied in the lemmas given below

Lemma 3.1. *If the operator L is uniformly elliptic, then the bilinear form* (3.11) *is bounded, that is, there exists a constant $\alpha \geq 0$ so that*

$$|B(u, \varphi)| \leq \alpha ||u|| \; ||\varphi||$$

for all $u, \varphi \in H_0^1(\Omega)$

Proof.

$$\begin{aligned}
|B(u, \varphi)| &= \left| \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \, dx + \int_\Omega \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} \varphi \, dx + \int_\Omega c u \varphi \, dx \right| \\
&\leq \int_\Omega \left| \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} \varphi + c u \varphi \right| dx \\
&\leq \int_\Omega \sum_{i,j=1}^n ||a_{ij}||_{L^\infty(\Omega)} \left| \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right| \, dx + \int_\Omega \sum_{i=1}^n ||b_i||_{L^\infty(\Omega)} \left| \frac{\partial u}{\partial x_i} \varphi \right| \, dx + \int_\Omega ||c||_{L^\infty(\Omega)} |u\varphi| \, dx
\end{aligned}$$

From Hölders inequality (2.2) we have that

$$\begin{aligned}|B(u,\varphi)| &\leq \sum_{i,j=1}^{n} ||a_{ij}||_{L^\infty(\Omega)}\Bigg(\bigg|\bigg|\frac{\partial u}{\partial x_i}\bigg|\bigg|_{L^2(\Omega)}\bigg|\bigg|\frac{\partial \varphi}{\partial x_j}\bigg|\bigg|_{L^2(\Omega)}\Bigg) + \sum_{i=1}^{n} ||b_i||_{L^\infty(\Omega)}\Bigg(\bigg|\bigg|\frac{\partial u}{\partial x_i}\bigg|\bigg|_{L^2(\Omega)}||\varphi||_{L^2(\Omega)}\Bigg)\\ &+ ||c||_{L^\infty(\Omega)}||u||_{L^2(\Omega)}||\varphi||_{L^2(\Omega)}\\ &= \hat{a}||\nabla u||_{L^2(\Omega)}||\nabla\varphi||_{L^2(\Omega)} + \hat{b}||\nabla u||_{L^2(\Omega)}||\varphi||_{L^2(\Omega)} + \hat{c}||u||_{L^2(\Omega)}||\varphi||_{L^2(\Omega)}\end{aligned}$$

where $\hat{a} = \sum_{i,j=1}^{n} ||a_{ij}||_{L^\infty(\Omega)}$, $\hat{b} = \sum_{i=1}^{n} ||b_i||_{L^\infty(\Omega)}$ and $\hat{c} = ||c||_{L^\infty(\Omega)}$. Applying the Poincaré inequality (2.4) we obtain

$$\begin{aligned}|B(u,\varphi)| &\leq \hat{a}||\nabla u||_{L^2(\Omega)}||\nabla\varphi||_{L^2(\Omega)} + \hat{b}C||\nabla u||_{L^2(\Omega)}||\nabla\varphi||_{L^2(\Omega)} + \hat{c}\tilde{C}||\nabla u||_{L^2(\Omega)}||\nabla\varphi||_{L^2(\Omega)}\\ &= M||\nabla u||_{L^2(\Omega)}||\nabla\varphi||_{L^2(\Omega)}\\ &\leq M(||\nabla u||_{L^2(\Omega)}||\nabla\varphi||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}||\varphi||_{L^2(\Omega)})\\ &\leq M||u||_{H^1(\Omega)}||\varphi||_{H^1(\Omega)}\end{aligned}$$

□

Lemma 3.2. *If the operator L is uniformly elliptic and* $\operatorname{ess\,inf}(c - \nabla\cdot\vec{b}) \geq 0$ *then $B(\cdot,\cdot)$ is coercive on $H_0^1(\Omega)$, that is, there exists a $\beta > 0$ such that for every $u \in H_0^1(\Omega)$ we have*

$$B(u,u) \geq \beta||u||^2_{H_0^1(\Omega)} = \beta\Bigg(\int_\Omega u^2 dx + \int_\Omega \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2 dx\Bigg).$$

Proof. Consider the bilinear form

$$B(u,u) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dx + \int_\Omega \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} u\, dx + \int_\Omega cu^2\, dx.$$

Since L is uniformly elliptic, it follows from Remark 3.1 that

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \geq \mu_0|\nabla u|^2 = \mu_0\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2.$$

Consider any $\varepsilon > 0$. Then Hence

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dx \geq \mu_0\int_\Omega |\nabla u|^2 dx = \mu_0||\nabla u||^2_{L^2(\Omega)}. \tag{3.17}$$

$$\begin{aligned}\mu_0||\nabla u||^2_{L^2(\Omega)} &= (\mu_0 - \varepsilon + \varepsilon)||\nabla u||^2_{L^2(\Omega)}\\ &= (\mu_0 - \varepsilon)||\nabla u||^2_{L^2(\Omega)} + \varepsilon||\nabla u||^2_{L^2(\Omega)}.\end{aligned}$$

From Poincaré's inequality (2.4) we have that

$$\mu_0||\nabla u||^2_{L^2(\Omega)} \geq (\mu_0 - \varepsilon)||\nabla u||^2_{L^2(\Omega)} + \frac{\varepsilon}{C^2}||u||^2_{L^2(\Omega)}.$$

Choose $\varepsilon = \dfrac{\mu_0 C^2}{1+C^2}$. It follows that

$$\begin{aligned}
\mu_0 ||\nabla u||^2_{L^2(\Omega)} &\geq \left(\mu_0 - \frac{\mu_0 C^2}{1+C^2}\right)||\nabla u||^2_{L^2(\Omega)} + \left(\frac{\mu_0 C^2}{C^2(1+C^2)}\right)||u||^2_{L^2(\Omega)} \\
&= \left(\frac{\mu_0 + \mu_0 C^2 - \mu_0 C^2}{1+C^2}\right)||\nabla u||^2_{L^2(\Omega)} + \left(\frac{\mu_0}{1+C^2}\right)||u||^2_{L^2(\Omega)} \\
&= \frac{\mu_0}{1+C^2}\left(||\nabla u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}\right) \\
&= \frac{\mu_0}{1+C^2}||u||^2_{H^1_0(\Omega)}.
\end{aligned}$$

Taking the square root on both sides of the inequality we obtain

$$\sqrt{\mu_0}||u||_{L^2(\Omega)} \geq \sqrt{\frac{\mu_0}{1+C^2}}||u||_{H^1_0(\Omega)} = \beta||u||_{H^1_0(\Omega)} \tag{3.18}$$

where $\beta = \sqrt{\dfrac{\mu_0}{1+C^2}}$. Combining (3.17) and (3.18) we get

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dx \geq \beta||u||_{H^1_0(\Omega)}. \tag{3.19}$$

Next we consider

$$\begin{aligned}
\int_\Omega \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} u\, dx + \int_\Omega c u^2\, dx &= \int_\Omega \vec{b}\cdot(\nabla u) u\, dx + \int_\Omega c u^2\, dx \\
&= \frac{1}{2}\int_\Omega \vec{b}\cdot\nabla(u^2)\, dx + \int_\Omega c u^2\, dx \\
&= -\frac{1}{2}\int_\Omega u^2(\nabla\cdot\vec{b})\, dx + \frac{1}{2}\int_{\partial\Omega} u^2(\vec{b}\cdot\vec{n})\, ds + \int_\Omega c u^2\, dx \\
&= \int_\Omega \left(c - \frac{1}{2}\nabla\cdot\vec{b}\right)u^2\, dx \\
&\geq \int_\Omega (c - \nabla\cdot\vec{b})u^2\, dx \\
&\geq ess\inf(c - \nabla\cdot\vec{b})\int_\Omega u^2\, dx \\
&\geq 0.
\end{aligned} \tag{3.20}$$

From equations (3.19) and (3.20) it follows that

$$B(u,u) \geq \beta||u||^2_{H^1_0(\Omega)}. \tag{3.21}$$

$\square$

Remark 3.2. *Let us note that for the purpose of this proof it is enough to have* $B(u,u) \geq \gamma||u||_{L^2(\Omega)}$ *with* $\gamma = \frac{\mu_0}{C^2}$*, which follows from* (3.17),(3.20) *and Poincaré inequality.*

Remark 3.3. *The assumption ess* $\inf(c - \nabla \cdot \vec{b}) \geq 0$ *looks a bit strange within what looks to be a quite elegant theory, particularly when compared to the simple condition* $c \geq 0$ *in the classical case. When* b *is a constant vector, there is not a problem as* $\nabla \cdot b = 0$*. To simplify the condition ess* $\inf(c - \nabla \cdot \vec{b}) \geq 0$ *when* b *is a function of* $x \in \Omega$ *we can re-write the operator as follows*

$$\begin{aligned} L[u] &= \nabla \cdot (A\nabla u) + b \cdot \nabla u + cu \\ &= \nabla \cdot (A\nabla u) + \frac{1}{2} b \cdot \nabla u + \frac{1}{2} \nabla \cdot (bu) + \left(c - \frac{1}{2} \nabla \cdot b\right) u. \end{aligned}$$

Then we have

$$L[u] = \nabla \cdot (A\nabla u) + \tilde{b} \cdot \nabla u + \nabla(\tilde{b}u) + \tilde{c}u, \tag{3.22}$$

where $\tilde{b} = \frac{1}{2} b$ *and* $\tilde{c} = c - \frac{1}{2} \nabla \cdot b$*. The bilinear form associated with (3.22) is*

$$\tilde{B}(u, \varphi) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \int_\Omega \sum_{i=1}^n \tilde{b}_i \left(\frac{\partial u}{\partial x_i} \varphi - u \frac{\partial \varphi}{\partial x_i} \right) + \int_\Omega \tilde{c} u \varphi. \tag{3.23}$$

Then

$$\tilde{B}(u, u) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \int_\Omega \tilde{c} u^2$$

does not depend on b *at all and it is easy to see that coercivity follows from the uniform ellipticity of* L *and* $\tilde{c} \geq 0$.

3.3 Variational formulation in the case of non-homogeneous Dirichlet boundary conditions

Let Ω be an open bounded subset of $\mathbb{R}^n$. We consider the following non-homogeneous Dirichlet boundary value problem:

$$\begin{cases} L[u] = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma := \partial\Omega, \end{cases} \tag{3.24}$$

where L is given by (3.1). It follows from (b) in Theorem 2.3 that the function g can be extended on the whole Ω in such a way that the extension is in $H^1(\Omega)$. More precisely, such extension is given by $\tilde{g} = T^{-1}(g)$. We consider the following homogeneous boundary value problem

$$\begin{cases} L[\tilde{u}] = f - L[\tilde{g}] & \text{in } \Omega \\ \tilde{u} = 0 & \text{on } \Gamma. \end{cases} \tag{3.25}$$

Let $\tilde{f} = f - L[\tilde{g}]$, see [9, Remark, page 297], then (3.25) becomes

$$\begin{cases} L[\tilde{u}] = \tilde{f} & \text{in } \Omega \\ \tilde{u} = 0 & \text{on } \Gamma. \end{cases} \tag{3.26}$$

Theorem 3.5. *A function $u \in C^2(\Omega)$ is a solution of the non-homogeneous boundary value problem* (3.24) *if and only if $\tilde{u} = u - \tilde{g}$ is a solution of the homogeneous boundary value problem* (3.26).

Similarly, for the variational formulation we have:

Theorem 3.6. *A function $u \in H^1(\Omega)$ is a solution of the variational problem*

$$B(u, \varphi) = (f, \varphi) \text{ for all } \varphi \in H_0^1(\Omega) \tag{3.27}$$

$$Tu = g, \tag{3.28}$$

if and only if

(1) $\tilde{u} = u - \tilde{g} \in H_0^1(\Omega)$ and

(2) $\tilde{u}$ is a solution of (3.27).

Corollary 3.1. *The variational problem* (3.27)–(3.28) *has a unique solution under the assumptions of Theorem 3.4.*

Remark 3.4. *We can multiply the first equation of* (3.26) *by $\varphi \in H_0^1(\Omega)$, do integration by parts and consider the boundary conditions to obtain a bilinear form that is equivalent to the bilinear form defined in equation* (3.11) *where u will be replaced by $\tilde{u}$ and f by $\tilde{f}$.*

Remark 3.5. *The function $\tilde{u}$ defined in Theorem 3.6 is a solution for Problem* (3.26) *and a weak solution for the homogeneous Problem* (3.24) *[9].*

To summarize, from Theorem 3.6 it follows that the non-homogeneous problem (3.24) can be written in the following form:
Find $u \in H^1(\Omega)$ such that

$$\begin{cases} B(u,\varphi) & = (f,\varphi) \text{ for all } \varphi \in H_0^1(\Omega) \\ Tu & = g. \end{cases} \tag{3.29}$$

3.4 Positivity and comparison theorems

In the previous sections we discussed classical solutions and solutions of the problem in variational form. In this section we have results derived for the case where order preservation is the property we are interested in. This means that the concepts of monotonicity and inverse monotonicity come in. The result that we derive for the weak formulation of the differential operator (3.1) use the important property of $H^1(\Omega)$ being a lattice. Consider a function $w \in H^1(\Omega)$. The functions w^+ and w^- are defined as follows

$$w^+(x) = \begin{cases} w(x) & \text{if } w(x) > 0 \\ 0 & \text{if } w(x) \leq 0 \end{cases} \tag{3.30}$$

$$w^-(x) = \begin{cases} 0 & \text{if } w(x) \geq 0 \\ -w(x) & \text{if } w(x) < 0. \end{cases} \tag{3.31}$$

We have that $w = w^+ - w^-$. We have that $H^1(\Omega)$ is a lattice, hence for $w \in H^1(\Omega)$ it follows that $w^+, w^- \in H^1(\Omega)$. Also if $w \in H_0^1(\Omega)$ then $w^+, w^- \in H_0^1(\Omega)$. It is also shown in the proof of [5, Chapter 4.7, Propostion 6] that for any $j = 1, 2, ...n$

$$\frac{\partial}{\partial x_j} w^+(x) = \begin{cases} \dfrac{\partial}{\partial x_j} w(x) & \text{if } w(x) > 0 \\ 0 & \text{if } w(x) \leq 0 \end{cases} \tag{3.32}$$

and

$$\frac{\partial}{\partial x_j} w^-(x) = \begin{cases} 0 & \text{if } w(x) \geq 0 \\ -\dfrac{\partial}{\partial x_j} w(x) & \text{if } w(x) < 0. \end{cases} \tag{3.33}$$

Theorem 3.7 (Positivity). *Let $u \in H^1(\Omega)$ and assume that the bilinear form B is coercive. If*

(a) $B(u, \varphi) \geq 0$ for all $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$ and

(b) $Tu \geq 0$

then $u \geq 0$ a.e. in Ω.

Proof. Let $u \in H^1(\Omega)$ satisfy (a) and (b). Let $u = u^+ - u^-$, where u^+ and u^- are given in (3.30) and (3.31) respectively. By the assumption of $u \in H^1(\Omega)$, we have that $u^+,\ u^- \in H^1(\Omega)$. Using (3.30), (3.31), (3.32) and (3.33) we obtain

$$B(u^+, u^-) = \int_\Omega \sum_{i=1}^{n} a_{ij} \frac{\partial u^+}{\partial x_i} \frac{\partial u^-}{\partial x_j} + \int_\Omega \sum_{i=1}^{n} b_i \frac{\partial u^+}{\partial x_i} u^- + \int_\Omega c u^+ u^- = 0$$

since in all products, the factors have disjoint support. Let $u \in H^1(\Omega)$ satisfy (a) and (b). Then $Tu \geq 0$ implies $Tu = Tu^+$ and $Tu^- = 0$. Therefore $u^- \in H_0^1(\Omega)$. Taking $\varphi = u^-$ in (a) we have

$$\begin{aligned} B(u^+ - u^-, u^-) &\geq 0, \\ B(u^+, u^-) - B(u^-, u^-) &\geq 0, \\ B(u^-, u^-) &\leq 0. \end{aligned}$$

Using the coercivity of B we have

$$\beta||u^-||_{H_0^1(\Omega)} \leq B(u^-, u^-) \leq 0.$$

Therefore $u^- = 0$ a.e. in Ω, which implies that $u = u^+ \geq 0$ a.e. in Ω. □

Theorem 3.8 (Comparison Theorem)**.** *Let the bilinear form B be coercive on $H_0^1(\Omega)$. Then, if for some $u, v \in H^1(\Omega)$ we have*

(a) $B(u, \varphi) \geq B(v, \varphi)$ for all $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$ and

(b) $Tu \geq Tv$

then $u \geq v$ a.e. in Ω.

Proof. Let $z = u - v$, then

$$\begin{aligned} B(z, \varphi) &= B(u - v, \varphi) \\ &= B(u, \varphi) - B(v, \varphi). \end{aligned}$$

Since $B(u, \varphi) \geq B(v, \varphi)$ it follows that $B(z, \varphi) \geq 0$ and

$$\begin{aligned} Tz &= T(u - v) \\ &= Tu - Tv \\ &\geq 0 \end{aligned}$$

since $Tu \geq Tv$. The function $z \in H^1(\Omega)$ satisfies the conditions of Theorem 3.7. Hence $z \geq 0$ a.e. in Ω. This implies that

$$\begin{aligned} u - v &\geq 0 \text{ a.e. in } \Omega \\ u &\geq v \text{ a.e. in } \Omega. \end{aligned}$$

□

3.5 Extension of the operator P in (3.5) to $H^1(\Omega)$

Let us take note that the bilinear form is defined on a much wider domain than in (3.11). To be more precise, it is defined for all $u, \varphi \in H^1(\Omega)$. We note that for any $u \in H^1(\Omega)$, $B(u, \cdot)$ is a linear functional on $H^1_0(\Omega)$ defined via

$$\varphi \to B(u, \varphi),\ \ \varphi \in H^1_0(\Omega).$$

Let B satisfy the boundedness condition (3.15). To be more precise,

$$\begin{aligned} B(u, \varphi) &\leq \alpha ||u||_{H^1(\Omega)}\ ||\varphi||_{H^1(\Omega)} \\ \frac{B(u, \varphi)}{||\varphi||_{H^1(\Omega)}} &\leq \alpha ||u||_{H^1(\Omega)} \end{aligned}$$

Therefore, $B(u, \cdot)$ is bounded and we have $B(u, \cdot) \in H^{-1}(\Omega)$ (see section 2.4). The existence and uniqueness theory is derived for an even more general problem than (3.14) given as:

$$\begin{aligned} &\text{Given } f \in H^{-1}(\Omega), \text{ find } u \in H^1_0(\Omega) \text{ such that} \\ &B(u, \varphi) = \langle f, \varphi \rangle \text{ for all } \varphi \in H^1_0(\Omega). \end{aligned} \tag{3.34}$$

The main result is that (3.34) has a unique solution provided B satisfies the conditions of the Lax–Milgram theorem. The theory is given in detail in many books, e.g. [9]. Further, this means that the non-homogeneous problem (3.24) can be generalized to the following variational formulation:

$$\begin{aligned} &\text{Given } f \in H^{-1}(\Omega) \text{ and } g \in H^{1/2}(\partial\Omega), \text{ find } u \in H^1(\Omega) \text{ such that} \\ &B(u, \varphi) = \langle f, \varphi \rangle \text{ for all } \varphi \in H^1_0(\Omega) \text{ and} \\ &Tu = g, \end{aligned} \tag{3.35}$$

where a unique solution exists under the same conditions for B on $H^1_0(\Omega)$. Considering problem (3.35), the operator P in (3.5) can be extended to $P : H^1(\Omega) \to H^{-1}(\Omega) \times H^{1/2}(\partial\Omega)$ as

$$P[u] = \begin{pmatrix} B(u, \cdot) \\ Tu \end{pmatrix}. \tag{3.36}$$

Using this notation, problem (3.35) can be written in the form (3.7), where both the data and the solution belong to much larger spaces than in the classical case, specifically $f \in H^{-1}(\Omega)$, $g \in H^{1/2}(\partial\Omega)$ and $u \in H^1(\Omega)$. The existence and uniqueness of solution of the equation (3.36) implies that P is actually a bijection. Our primary interest is in the preservation of order. The goal here is to show that the operator P in (3.36) preserves order in $H^{-1}(\Omega)$ which is induced by order in $H^1_0(\Omega)$.

Theorem 3.9. *Let the bilinear form B be coercive on $H^1_0(\Omega)$. Then the operator P defined in* (3.36) *is inverse monotone.*

Proof. Let $P[u] = \begin{pmatrix} B(u, \cdot) \\ Tu \end{pmatrix}$ and $P[v] = \begin{pmatrix} B(v, \cdot) \\ Tv \end{pmatrix}$.

We have $P[u] \leq P[v] \implies \begin{pmatrix} B(u, \cdot) \\ Tu \end{pmatrix} \leq \begin{pmatrix} B(v, \cdot) \\ Tv \end{pmatrix}$.

This implies that

$$\begin{aligned} B(u,\varphi) &\le B(v,\varphi) \text{ for all } \varphi \in H^1_+(\Omega) := \{\varphi \in H^1_0(\Omega) : \varphi \ge 0\} \\ Tu &\le Tv \end{aligned}$$

Hence by the Comparison Theorem 3.8 we have that $u \le v$ a.e. in Ω. □

Corollary 3.2. *The boundary value problem* (3.27)– (3.28) *is equivalent to*

$$\begin{aligned} B(u,\varphi) &= \langle f,\varphi\rangle \textit{ for all } \varphi \in H^1_+(\Omega) \\ Tu &= g. \end{aligned}$$

Proof. Let $\varphi \in H^1_0(\Omega)$. We consider the functions:

$$\varphi^+(x) = \begin{cases} \varphi(x) & \text{if } \varphi(x) > 0 \\ 0 & \text{if } \varphi(x) \le 0 \end{cases}$$

$$\varphi^-(x) = \begin{cases} 0 & \text{if } \varphi(x) \ge 0 \\ -\varphi(x) & \text{if } \varphi(x) < 0. \end{cases}$$

Hence $\varphi = \varphi^+ - \varphi^-$, where $\varphi^+, \varphi^- \in H^1_+(\Omega)$ [5, Chapter 4.7, Propostion 6]. Then

$$\begin{aligned} B(u,\varphi) &= B(u,\varphi^+ - \varphi^-) \\ &= B(u,\varphi^+) - B(u,\varphi^-) \\ &= (f,\varphi^+) - (f,\varphi^-) \\ &= (f,\varphi^+ - \varphi^-) \\ &= (f,\varphi). \end{aligned}$$

□

Remark 3.6. *It is sufficient for the variational problem* (3.24) *to be satisfied for any* $\varphi \in H^1_+(\Omega)$.

3.6 Systems of elliptic PDEs

We consider in this section operators associated with weakly coupled systems of elliptic PDEs. In the classical form such a system is formulated for $u \in (C^2(\Omega) \cap C(\overline{\Omega}))^m$ as

$$\begin{cases} L_k[u] &= f_k \text{ in } \Omega, \\ u_k &= g_k \text{ on } \partial\Omega, \end{cases} \tag{3.37}$$

where

$$L_k[u] \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}^{(k)} \frac{\partial u_k}{\partial x_i}\right) + \sum_{i=1}^{n} b_i^{(k)} \frac{\partial u_k}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial u_k}{\partial x_i}(b_i^{(k)} u_k) + \sum_{\ell=1}^{m} c_{k,\ell} u_\ell, \tag{3.38}$$

for $k = 1, ..., m$. Weakly coupling refers to the fact that the equations are not coupled in the differential part of the operators L_k. If we denote $A^{(k)} = (a_{ij}^{(k)})_{i,j=1}^n$ and $b^{(k)} = (b_1^{(k)}, ..., b_n^{(k)})'$, then the differential operator (3.38) can be written in the following vector form

$$L_k[u] = -\nabla \cdot (A^{(k)} \nabla u_k) + b^{(k)} \cdot \nabla u_k + \nabla \cdot (b^{(k)} u_k) + \sum_{\ell=1}^{m} c_{k,\ell} u_\ell. \tag{3.39}$$

We assume that the uniform ellipticity condition is satisfied, namely there exists a $\mu_0 > 0$ such that for every $k = 1, ..., m$ we have

$$\xi \cdot A^{(k)}(x)\xi \geq \mu_0 |\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. We introduce vector notation by letting

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad L = \begin{pmatrix} L_1[u] \\ \vdots \\ L_m[u] \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \text{ and } g = \begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix}.$$

As a result, the system (3.37) can be written as a single vector equation and boundary condition as

$$\begin{cases} L[u] &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{cases} \tag{3.40}$$

Furthermore, the operator admits the following convenient vector representation

$$L[u] \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(A_{ij} \frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} B_i \frac{\partial u_k}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial u_k}{\partial x_i}(B_i u) + Cu.$$

We assume that A_{ij} and B_i are diagonal $m \times m$ matrices, that is, $A_{ij} = \text{diag}(a_{ij}^{(1)}, ..., a_{ij}^{(m)})$ and $B_i = \text{diag}(b_i^{(1)}, ..., b_i^{(m)})$. We further make assumptions on matrix C as follows:

$$(i) \quad c_{k\ell} \leq 0 \text{ for } k \neq \ell, \tag{3.41}$$

$$(ii) \quad \text{the matrix } C + C' \text{ is positive semi-definite for any } x \in \Omega. \tag{3.42}$$

We note that the assumption (3.41) implies that matrix C is an $M-$matrix. In other words, $-C$ is a Metzler matrix. This assumption is essential for deriving the intended monotonicity.

A similar assumption is made in [22] for obtaining a comparison theorem in the space of classical solutions. The assumption (3.42) generalizes the condition $c \geq 0$ in the one dimensional case. From this assumption we obtain that for any $x \in \Omega$ and $\eta \in \mathbb{R}^m$ we have

$$(C(x)\eta) \cdot \eta = \frac{1}{2}((C(x) + C'(x))\eta) \cdot \eta \geq 0. \tag{3.43}$$

In order to derive a variational formulation we assume that all the coefficients in L are measurable and uniformly bounded. Let $f \in (L^2(\Omega))^m$. Multiply the first equation of the boundary value problem (3.40) by a test function $\varphi \in (C_c^\infty(\Omega))^m$, integrate over Ω and apply Green's formula. The resulting bilinear form is given in equation (1) in the research paper of O'Connor [13] for all $\varphi \in (H_0^1(\Omega))^m$ as

$$B(u,\varphi) = \sum_{i,j=1}^{n} \int_\Omega \left(A_{ij}\frac{\partial u}{\partial x_i}\right)\cdot\frac{\partial \varphi}{\partial x_i}\,dx + \sum_{i=1}^{n} \int_\Omega \left(B_i\frac{\partial u}{\partial x_i}\right)\cdot\varphi\,dx + \sum_{i=1}^{n} \int_\Omega (B_i)\cdot\frac{\partial \varphi}{\partial x_i}\,dx + \int_\Omega (Cu)\cdot\varphi\,dx. \tag{3.44}$$

We note that bilinear form (3.44) has a mapping $B : (H_0^1(\Omega))^m \times (H_0^1(\Omega))^m \mapsto \mathbb{R}$. Similar to the one dimensional case we generalize (3.37) to the problem

$$\begin{cases} \text{Given } f \in (H^{-1}(\Omega))^m \text{ and } g \in (H^{1/2}(\partial\Omega))^m, \\ \text{find } u \in (H^1(\Omega))^m \text{ such that} \\ B(u,\varphi) = \langle f, \varphi\rangle \text{ for all } \varphi \in (H_0^1(\Omega))^m \text{ and} \\ Tu = g. \end{cases} \tag{3.45}$$

Assumption (3.42) is a quadratic form, which is relaxed compared to the symmetry condition of C especially in application. With this inequality we can obtain the coercivity of the bilinear form on $(H_0^1(\Omega))^m$. Indeed, for any $u \in (H_0^1(\Omega))^m$ we have

$$\begin{aligned} B(u,u) &= \sum_{k=1}^{m} \int_\Omega \left((A^{(k)}\nabla u_k) \cdot \nabla u_k\right) + \int_\Omega (Cu) \cdot u \\ &\geq \mu_1 \left(\sum_{k=1}^{m} ||\nabla u_k||^2\right). \end{aligned}$$

Then, following the standard approach of using the Poincaré inequality as in Lemma 3.2 we obtain that there exists a constant μ_2 such that

$$B(u,u) \geq \mu_2 \left(\sum_{k=1}^{m} ||u_k||^2_{H^1(\Omega)}\right) = \mu_2 ||u||^2_{(H^1(\Omega))^m}. \tag{3.46}$$

The boundedness of $B(u,v)$ is obtained in a similar way to the one dimensional case. Then, the existence and uniqueness of solution of (3.45) follows from the Lax-Milgram Theorem - first for the homogeneous problem and then extended to the non-homogeneous one. More detailed existence and uniqueness theory of variational formulation of general systems of elliptic PDEs is given in [13].

Our main interest in this section is the order properties of general elliptic systems of PDEs. All product spaces in this section are considered with the associated coordinate-wise partial order.

Theorem 3.10 (Positivity Theorem). *Let $u \in (H^1(\Omega))^m$ and assume that the bilinear form B is coercive on $(H_0^1(\Omega))^m$. If*

(a) $B(u,\varphi) \geq 0$ for all $\varphi \in (H_+^1(\Omega))^m$ and

(b) $Tu \geq 0$

then $u \geq 0$ a.e. in Ω.

Proof. Let $u = u^+ - u^-$, where u^+ and u^- are given in (3.30) and (3.31) respectively. Let us note that $Tu \geq 0$ implies that $Tu^- = 0$, so that $u^- \in (H_0^1(\Omega))^m$. By the assumption of $u \in (H_0^1(\Omega))^m$, we have that $u^+,\ u^- \in (H_0^1(\Omega))^m$ by the lattice property of $H_0^1(\Omega)$. Using also (3.32), (3.33) and (3.41) we obtain

$$\begin{aligned}
B(u^+,u^-) &= \sum_{k=1}^{m}\left(\int_\Omega \sum_{i,j=1}^{n}\left(a_{ij}^{(k)}\frac{\partial u_k^+}{\partial x_j}\right)\cdot\frac{\partial u_k^-}{\partial x_i} + \int_\Omega \sum_{i=1}^{n} b_i^{(k)}\frac{\partial u_k^+}{\partial x_i}u_k^-\right) + \sum_{\ell=1}^{m}\int_\Omega c_{kk}u_k^+u_k^- \\
&\quad + \sum_{k\neq\ell}\int_\Omega c_{k\ell}u_k^+u_\ell^- \\
&= \int_\Omega \sum_{k=1}^{m}\sum_{i,j=1}^{n} a_{ij}^{(k)}\frac{\partial u_k^+}{\partial x_j}\cdot\frac{\partial u_k^-}{\partial x_i} + \int_\Omega \sum_{k=1}^{m}\sum_{i=1}^{n} b_i^{(k)}\frac{\partial u_k^+}{\partial x_i}u_k^- + \int_\Omega \sum_{\ell=1}^{m} c_{kk}u_k^+u_k^- \\
&\quad + \int_\Omega \sum_{k\neq\ell} c_{k\ell}u_k^+u_\ell^- \\
&= \int_\Omega \sum_{i,j=1}^{n}\sum_{k=1}^{m} a_{ij}^{(k)}\frac{\partial u_k^+}{\partial x_j}\cdot\frac{\partial u_k^-}{\partial x_i} + \int_\Omega \sum_{i=1}^{n}\sum_{k=1}^{m} b_i^{(k)}\frac{\partial u_k^+}{\partial x_i}u_k^- + \int_\Omega \sum_{\ell=1}^{m} c_{kk}u_k^+u_k^- \\
&\quad + \int_\Omega \sum_{k\neq\ell} c_{k\ell}u_k^+u_\ell^- \\
&= \int_\Omega \sum_{i,j=1}^{n} 0 + \int_\Omega \sum_{i=1}^{n} 0 + \int_\Omega \sum_{\ell=1}^{m} 0 + \int_\Omega \sum_{k\neq\ell} c_{k\ell}u_k^+u_\ell^- \\
&= \int_\Omega \sum_{k\neq\ell} c_{k\ell}u_k^+u_\ell^- \\
&\leq 0
\end{aligned}$$

Using (a) with $\varphi = u^-$ we obtain

$$\begin{aligned}
B(u^+ - u^-, u^-) &\geq 0 \\
B(u^+,u^-) - B(u^-,u^-) &\geq 0 \\
B(u^-,u^-) &\leq 0.
\end{aligned}$$

By the coercivity (3.46) of B it follows that

$$\mu_2||u^-||^2_{(H^1(\Omega))^m} \leq B(u^-,u^-) \leq 0.$$

Hence, $||u^-||^2_{(H^1(\Omega))^m} = 0$, or equivalently $u^- = 0$ a.e. on Ω. Then, $u = u^+ \geq 0$ a.e. on Ω. $\square$

Theorem 3.11 (Comparison Theorem of an Elliptic System of PDEs)**.** *Let the bilinear form B be coercive on $(H_0^1(\Omega))^m$. Then if for some $u, v \in (H^1(\Omega))^m$*

(a) $B(u,\varphi) \geq B(v,\varphi)$ for all $\varphi \in (H_+^1(\Omega))^m$ and

(b) $Tu \geq Tv$

then $u \geq v$ a.e in Ω.

Proof. Let $z \in (H^1(\Omega))^m$ be the function defined by $z := u - v$, then

$$\begin{aligned} B(z,\varphi) &= B(u-v,\varphi) \\ &= B(u,\varphi) - B(v,\varphi). \end{aligned}$$

Since $B(u,\varphi) \geq B(v,\varphi)$ it follows that $B(z,\varphi) \geq 0$ and

$$\begin{aligned} Tz &= T(u-v) \\ &= Tu - Tv \\ &\geq 0 \end{aligned}$$

since $Tu \geq Tv$. The function z satisfies the conditions of Theorem 3.10, hence $z \geq 0$ a.e. in Ω. This implies that

$$\begin{aligned} u - v &\geq 0 \text{ a.e. in } \Omega \\ u &\geq v \text{ a.e. in } \Omega. \end{aligned}$$

□

We can associate with the problem (3.45) the operator
$P : (H^1(\Omega))^m \mapsto (H^{-1}(\Omega))^m \times (H^{1/2}(\partial\Omega))^m$ defined through

$$P[u] = \begin{pmatrix} B(u,\cdot) \\ Tu \end{pmatrix}. \tag{3.47}$$

Theorem 3.12. *Let C satisfy* (3.41)–(3.42)*. The the operator P defined in (3.47) is inverse monotone, that is,*

$$P[u] \leq P[v] \implies u \leq v \textit{ for all } u, v \in (H^1(\Omega))^m.$$

Proof. Let K be the positive cone of $(H_0^1(\Omega))^m$, that is

$$K = \{\varphi \in (H_0^1(\Omega))^m : \varphi_k(x) \geq 0, \text{ for a.e. } x \in \Omega, k = 1, ..., m\}.$$

Then the positive cone in $(H^{-1}(\Omega))^m$ is defined by

$$K^* = \{f \in (H^{-1}(\Omega))^m : \langle f, \varphi\rangle \geq 0 \text{ for all } \varphi \in K\}.$$

Equivalently, this means that for any $f_1, f_2 \in (H^{-1}(\Omega))^m$

$$f_1 \leq f_2 \iff \langle f_1, \varphi\rangle \leq \langle f_2, \varphi\rangle \text{ for all } \varphi \in K.$$

Specifically with reference to the operator P in (3.47), we have

$$B(u,\cdot) \leq B(v,\cdot) \iff B(u,\varphi) \leq B(v,\varphi) \text{ for all } \varphi \in (H_+^1(\Omega))^m.$$

Then the proof follows directly from Theorem 3.11 □

Chapter 4

Parabolic PDEs

4.1 Variational formulation in the case of Dirichlet boundary conditions

Let Ω be an open, bounded subset of $\mathbb{R}^n$ and $\Omega_\theta = \Omega \times (0, \theta]$ where $\theta > 0$ is some fixed time. We study partial differential equations of the form

$$u_t + L[u] = f \text{ in } \Omega_\theta, \tag{4.1}$$

where $f : \Omega_\theta \mapsto \mathbb{R}$ and $u : \overline{\Omega}_\theta \mapsto \mathbb{R}$ is the unknown that we seek. Let $u \equiv u(x,t)$. In (4.1), the letter L denotes the second order differential operator

$$L[u] \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u \tag{4.2}$$

and we assume that $a_{ij}, b_i, c \in L^\infty(\Omega_\theta)$. The operator $\frac{\partial}{\partial t} + L$ is said to be a uniformly parabolic operator in Ω_θ if there exists a positive constant μ_0 such that

$$\xi \cdot A(x,t)\xi \geq \mu_0 |\xi|^2 \tag{4.3}$$

for all $(x,t) \in \Omega_\theta$ and all $\xi \in \mathbb{R}^n$. Note that for each fixed time $t \in [0, \theta]$ we assume that the operator L is uniformly elliptic in the spatial variable x [9, Remark, Chapter 7.1.1]. To be more explicit we state the following definition

Definition 4.1. *[17, Definition 11.1.1] The operator*

$$\frac{\partial u}{\partial t} + L[u],$$

defined for all real functions on Ω_θ which are $C^1((0,\theta])$ with respect to t and $C^2(\Omega)$ with respect to x, is said to be parabolic if L is uniformly elliptic on Ω_θ.

Unlike in Chapter 3, we have here the presence of the time derivative, hence an initial condition has to be imposed in the problem formulation. We impose certain boundary conditions to (4.1) in order to complete the formulation of the problem. Parabolic PDEs are well known in Mathematical Physics as they describe the time evolution of spatial states, for example, the distribution of heat. In more recent times parabolic PDEs and systems of parabolic PDEs are applied to biological models, such as population dynamics. The biological meaning of the homogeneous Dirichlet and Neumann boundary conditions are as follows:

1. Dirichlet: $u(x,t) = 0,\ x \in \partial\Omega$, describes a domain out of which individuals cannot survive
2. Neumann: $\frac{\partial}{\partial n_a} u(t,x) = 0\ x \in \partial\Omega$ describes an isolated domain with no movement of individuals in and out of the domain.

In the heat equation, the homogeneous Dirichlet boundary conditions mean that the temperature is kept zero at the end points. Neumann boundary conditions are used to describe the lateral surface of the bar that is perfectly insulated, that is, no heat can come in or go out of the surface. We focus on the following initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + L[u] & = f \text{ in } \Omega_\theta, \\ u(\cdot,0) & = u_0(\cdot) \text{ on } \Omega, \\ u & = 0 \text{ on } \partial\Omega \times (0,\theta), \end{cases} \tag{4.4}$$

where $f \in L^2(\Omega_\theta)$ and $u_0 \in L^2(\Omega)$. As in the elliptic case studied before, we derive a weak formulation for (4.4). For equation (4.4), we expect a solution $u(t,x)$ to lie in $C^1((0,\theta])$ with respect to t and in $C^2(\Omega)$ with respect to x and we call such a solution a classical solution. The derivation of a weak solution can be obtained by multiplying the first equation of (4.4) by $\varphi \in C_c^\infty(\Omega)$ and integrate:

$$\int_\Omega \frac{\partial u}{\partial t}\varphi\ dx + \int_\Omega L[u]\varphi\ dx = \int_\Omega f\varphi\ dx. \tag{4.5}$$

After using integration by parts, Green's formula and the boundary conditions we obtain

$$\int_\Omega \frac{\partial u}{\partial t}\varphi\ dx + \int_\Omega \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j}\ dx + \int_\Omega \sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i}\varphi\ dx + \int_\Omega c(x,t)u\varphi\ dx = \int_\Omega f\varphi\ dx.$$

For each $t \in [0,\theta]$ we let

$$B(t;u,\varphi) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j}\ dx + \int_\Omega \sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i}\varphi\ dx + \int_\Omega c(x,t)u\varphi\ dx. \tag{4.6}$$

Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ with respect to the norm in $H^1(\Omega)$, we have that every solution u of (4.4) satisfies

$$\left(\frac{\partial u}{\partial t}, \varphi\right) + B(t;u,\varphi) = (f,\varphi) \text{ for all } \varphi \in H_0^1(\Omega), \tag{4.7}$$

where $(\cdot,\cdot)$ is an inner product in $L^2(\Omega)$. Unlike (4.1) the operator on the left hand side of (4.7) is defined on a much wider space of functions than the space considered for the solutions of (4.1). Specifically, the bilinear form is defined if for any fixed $t \in (0,\theta]$ we have $u(t,\cdot) \in H^1(\Omega)$. Our initial-boundary value problem (4.4) has homogeneous Dirichlet boundary conditions, that is $Tu = 0$ on $\partial\Omega \times (0,\theta)$. As a result, we expect for any fixed t the solution to lie on $H_0^1(\Omega)$. Moreover, the integral $\displaystyle\int_\Omega \frac{\partial u}{\partial t}\varphi\ dx$ is defined on the dual pairing of

$H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Therefore it is well-defined for any t if $\dfrac{\partial u}{\partial t} \in H^{-1}(\Omega)$. From [9, Chapter 7.1b] we let $\boldsymbol{u}$ be a mapping

$$\boldsymbol{u} : [0, \theta] \mapsto H_0^1(\Omega)$$

defined by

$$[\boldsymbol{u}(t)](x) := u(x,t) \text{ for } x \in \Omega, t \in [0, \theta].$$

This means that we consider $\boldsymbol{u}$ as a mapping of t into the space $H_0^1(\Omega)$. Similarly we define

$$\boldsymbol{f} : [0, \theta] \mapsto L^2(\Omega)$$

by

$$[\boldsymbol{f}(t)](x) := f(x,t) \text{ for } x \in \Omega, t \in [0, \theta].$$

As a result, the problem of this form reduces to a system of ODEs with initial condition $\boldsymbol{u}(0) = u_0$. We denote by $L^2(0, \theta; H_0^1(\Omega))$ the space

$$L^2(0, \theta; H_0^1(\Omega)) := \left\{ u : (0, \theta) \mapsto H^1(\Omega) | \; ||u(\cdot)||_{H_0^1(\Omega)} \text{ is measurable and } \int_0^\theta ||u(t)||^2_{H_0^1(\Omega)} dt < \infty \right\}.$$

The following property is useful in getting to the variational formulation of problem (4.4):

Proposition 4.1. *[6, Chapter XVIII, Section 1, Proposition 7] Let V be a Hilbert space and let V' be the dual space of V. For $\boldsymbol{u} \in L^2(0, \theta; V)$ with $\boldsymbol{u}' \in L^2(0, \theta; V')$ and $\varphi \in V$ we have*

$$\langle \boldsymbol{u}'(\cdot), \varphi \rangle = \frac{d}{dt}(u(\cdot), \varphi) \text{ in } [0, \theta].$$

It is important to note that the distributional derivative $\dfrac{d\boldsymbol{u}}{dt}$ generalizes the derivative $\dfrac{\partial u}{\partial t}$ of u in Ω_θ. Proposition 4.1 allows us to rewrite the first term of (4.7) as $\langle \dfrac{d\boldsymbol{u}}{dt}(t), \varphi \rangle$. Hence the variational formulation is

$$\begin{cases} \text{Given } \boldsymbol{f} \in L^2(0, \theta; H^{-1}(\Omega)) \text{ and } u_0 \in L^2(\Omega) \\ \text{Find } \boldsymbol{u} \in L^2(0, \theta; H_0^1(\Omega)) \text{ with } \boldsymbol{u}' \in L^2(0, \theta; H^{-1}(\Omega)) \text{ such that} \\ \left\langle \dfrac{d\boldsymbol{u}}{dt}(t), \varphi \right\rangle + B(t; \boldsymbol{u}(t), \varphi) = \langle \boldsymbol{f}(t), \varphi \rangle \text{ for all } \varphi \in H_0^1(\Omega) \\ \boldsymbol{u}(0) = u_0 \end{cases} \tag{4.8}$$

for each $t \in [0, \theta]$.

Definition 4.2. *A function satisfying* (4.8) *is called a weak solution of* (4.4)*.*

We introduce short-hand notation for the space defined where the solution lies, motivated by Definition 4.2. Define

$$\mathcal{W}(H_0^1(\Omega), H^{-1}(\Omega)) := \{ \boldsymbol{u} \in L^2(0, \theta; H_0^1(\Omega)) \mid \boldsymbol{u}' \in L^2(0, \theta; H^{-1}(\Omega)) \}.$$

Properties of the bilinear form:

Let $\theta > 0$ be fixed. For *a.e.* $t \in [0, \theta]$ the bilinear form $B(t; \boldsymbol{u}, \varphi) : H_0^1(\Omega) \times H_0^1(\Omega) \mapsto \mathbb{R}$ satisfies the following properties for all $\boldsymbol{u} \in \mathcal{W}(H_0^1(\Omega), H^{-1}(\Omega))$ and $\varphi \in H_0^1(\Omega)$:

(P1) the function $t \mapsto B(t; \boldsymbol{u}(t), \varphi)$ is measurable,

(P2) $|B(t; \boldsymbol{u}(t), \varphi)| \leq M ||\boldsymbol{u}(t)||_{H_0^1(\Omega)} \, ||\varphi||_{H_0^1(\Omega)}$ for a.e. $t \in [0, \theta]$ and

(P3) $B(t; \boldsymbol{u}(t), \boldsymbol{u}(t)) \geq \alpha ||\boldsymbol{u}(t)||^2_{H_0^1(\Omega)} - \beta ||\boldsymbol{u}(t)||^2_{L^2(\Omega)}$ for *a.e.* $t \in [0, \theta]$,

where $\alpha, \beta, M > 0$.
Note that (P2) means that the bilinear form is continuous (P3) is the definition of L^2 coercivity. The approach of J.-L. Lions is well known in the theory of PDEs as it allows us to prove the existence and uniqueness of a weak solution for parabolic initial-boundary value problems. J.-L Lions' theorem can be seen as an equivalent of the Lax-Milgram theorem for parabolic equations. We state the theorem here below

Theorem 4.1 (J.-L. Lions)**.** *[3, Theorem 10.9] Let V and H be Hilbert spaces and assume B satisfies (P1)–(P3). Given $\boldsymbol{f} \in L^2(0, T; V')$ and $u_0 \in H$, there exists a unique function $\boldsymbol{u}$ satisfying*

$$\boldsymbol{u} \in L^2(0, \theta; V) \cap C([0, \theta]; V), \ \boldsymbol{u}'(t) \in L^2(0, \theta; V')$$

$$\langle \boldsymbol{u}'(t), \varphi \rangle + B(t; \boldsymbol{u}(t), \varphi) = \langle \boldsymbol{f}(t), \varphi \rangle \text{ for a.e. } t \in (0, \theta) \text{ for all } \varphi \in V$$

and

$$\boldsymbol{u}(0) = u_0.$$

For our initial-boundary value problem (4.4), Brezis [3] states that in Theorem 4.1 we can let $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Since the dual space to $H_0^1(\Omega)$ is given by $H^{-1}(\Omega)$, we let $V' = H^{-1}(\Omega)$. In order to apply Theorem 4.1 to problem (4.8) we need to show that the bilinear form (4.6) satisfies (P1)–(P3).

Proof. (P1)–(P2): The proofs of (P1) and (P2) follow directly from Theorem 2.7 by setting $V = H_0^1(\Omega)$, then $V' = H^{-1}(\Omega)$, $\mathcal{V} = L^2(0, \theta; H_0^1(\Omega))$ and $\mathcal{V}' = L^2(0, \theta; H^{-1}(\Omega))$.

(P3): Consider the bilinear form

$$B(t; u, u) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx + \int_\Omega \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} u \, dx + \int_\Omega c u^2 \, dx.$$

Since L is uniformly elliptic, it follows that

$$\sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \geq \mu_0 |\nabla u|^2 = \mu_0 \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2.$$

Hence

$$\int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx \geq \mu_0 \int_\Omega |\nabla u|^2 dx = \mu_0 ||\nabla u||^2_{L^2(\Omega)}. \tag{4.9}$$

Let $\varepsilon > 0$. Then

$$\begin{aligned} \mu_0 ||\nabla u||^2_{L^2(\Omega)} &= (\mu_0 - \varepsilon + \varepsilon) ||\nabla u||^2_{L^2(\Omega)} \\ &= (\mu_0 - \varepsilon) ||\nabla u||^2_{L^2(\Omega)} + \varepsilon ||\nabla u||^2_{L^2(\Omega)}. \end{aligned}$$

From Poincaré's inequality (2.4) we have that

$$\mu_0||\nabla u||^2_{L^2(\Omega)} \geq (\mu_0 - \varepsilon)||\nabla u||^2_{L^2(\Omega)} + \frac{\varepsilon}{C^2}||u||^2_{L^2(\Omega)}.$$

Choose $\varepsilon = \dfrac{\mu_0 C^2}{1+C^2}$, it follows that

$$\begin{aligned}
\mu_0||\nabla u||^2_{L^2(\Omega)} &\geq \left(\mu_0 - \frac{\mu_0 C^2}{1+C^2}\right)||\nabla u||^2_{L^2(\Omega)} + \left(\frac{\mu_0 C^2}{C^2(1+C^2)}\right)||u||^2_{L^2(\Omega)} \\
&= \left(\frac{\mu_0 + \mu_0 C^2 - \mu_0 C^2}{1+C^2}\right)||\nabla u||^2_{L^2(\Omega)} + \left(\frac{\mu_0}{1+C^2}\right)||u||^2_{L^2(\Omega)} \\
&= \frac{\mu_0}{1+C^2}\left(||\nabla u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}\right) \\
&= \frac{\mu_0}{1+C^2}||u||^2_{H^1_0(\Omega)}.
\end{aligned}$$

Taking the square root on both sides of the inequality we obtain

$$\sqrt{\mu_0}||u||_{L^2(\Omega)} \geq \sqrt{\frac{\mu_0}{1+C^2}}||u||_{H^1_0(\Omega)} = \alpha||u||_{H^1_0(\Omega)} \tag{4.10}$$

where $\alpha = \sqrt{\dfrac{\mu_0}{1+C^2}}$. Combining (4.9) and (4.10) we get

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\;dx \geq \alpha||u||_{H^1_0(\Omega)}. \tag{4.11}$$

Next we consider

$$\begin{aligned}
\int_\Omega \sum_{i=1}^n b_i\frac{\partial u}{\partial x_i}u\;dx + \int_\Omega cu^2\;dx &= \int_\Omega \vec{b}\cdot(\nabla u)u\;dx + \int_\Omega cu^2\;dx \\
&= \frac{1}{2}\int_\Omega \vec{b}\cdot\nabla(u^2)\;dx + \int_\Omega cu^2\;dx \\
&= -\frac{1}{2}\int_\Omega u^2(\nabla\cdot\vec{b})\;dx + \frac{1}{2}\int_{\partial\Omega} u^2(\vec{b}\cdot\vec{n})\;ds + \int_\Omega cu^2\;dx \\
&= \int_\Omega \left(c - \frac{1}{2}\nabla\cdot\vec{b}\right)u^2\;dx \\
&\geq -\beta\int_\Omega u^2\;dx \\
&= -\beta||u||^2_{L^2(\Omega)}
\end{aligned} \tag{4.12}$$

where $\beta = \max\{\max(\frac{1}{2}\nabla\cdot\vec{b} - c), 0\}$. From equations (4.11) and (4.12) it follows that $B(t;u,u) \geq \alpha||u||^2_{H^1_0(\Omega)} - \beta||u||_{L^2(\Omega)}$. □

4.2 Variational formulation in the case of non-homogeneous Dirichlet boundary conditions

Let Ω be an open, bounded subset of $\mathbb{R}^n$ and $\Omega_\theta = \Omega \times (0, \theta]$ where $\theta > 0$ is some fixed time. We consider the following non-homogeneous Dirichlet initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + L[u] & = f \text{ in } \Omega_\theta \\ u(\cdot, 0) & = u_0(\cdot) \text{ on } \Omega \\ u & = g \text{ on } \partial\Omega \times (0, \theta) \end{cases} \tag{4.13}$$

where L is given by (4.2) and $g \in H^{1/2}(\partial\Omega)$. We recall that a non-homogeneous Dirichlet boundary value problem can be transformed to a homogeneous Dirichlet boundary value problem using the fact that g can be extended on the whole Ω in such a way that the extension is in $H^1(\Omega)$. It follows form (b) in Theorem 2.3 that this extension is given by a sufficiently smooth function $\tilde{g} = T^{-1}(g)$. As a result, we obtain the following homogeneous initial-boundary value problem

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} + L[\tilde{u}] & = f - L[\tilde{g}] \text{ in } \Omega_\theta \\ \tilde{u}(\cdot, 0) & = \tilde{u}_0(\cdot) \text{ on } \Omega \\ \tilde{u} & = 0 \text{ on } \partial\Omega \times (0, \theta). \end{cases} \tag{4.14}$$

Letting $\tilde{f} = f - L[\tilde{g}]$ we have that (4.14) becomes

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} + L[\tilde{u}] & = \tilde{f} \text{ in } \Omega_\theta \\ \tilde{u}(\cdot, 0) & = \tilde{u}_0(\cdot) \text{ on } \Omega \\ \tilde{u} & = 0 \text{ on } \partial\Omega \times (0, \theta). \end{cases} \tag{4.15}$$

To obtain the variational formulation, we multiply the first equation of (4.15) by $\varphi \in H_0^1(\Omega)$, do integration by parts and take into consideration the boundary conditions to obtain a bilinear form equivalent to the bilinear form given in (4.6) where u will be replaced by $\tilde{u}$ and f replaced by $\tilde{f}$.

Theorem 4.2. *A function u that lies in $C^1((0, \theta])$ with respect to t and $C^2(\Omega)$ with respect to x is a solution of the non-homogeneous initial-boundary value problem* (4.13) *if and only if $\tilde{u} = u - \tilde{g}$ is a solution of the homogeneous boundary value problem* (4.15).

We can state the variational formulation for this problem similar to the previous section as:

$$\begin{cases} \text{Given } \boldsymbol{f} \in L^2(0, \theta; H^{-1}(\Omega)),\ g \in L^2(0, \theta; H^{1/2}(\partial\Omega)) \text{ and } u_0 \in L^2(\Omega), \\ \text{find } \boldsymbol{u} \in L^2(0, \theta; H^1(\Omega)) \text{ with } \boldsymbol{u}' \in L^2(0, \theta; H^{-1}(\Omega)) \text{ such that} \\ \langle \boldsymbol{u}'(t), \varphi \rangle + B(t; \boldsymbol{u}(t), \varphi) = \langle \boldsymbol{f}(t), \varphi \rangle \text{ for all } \varphi \in H_0^1(\Omega) \\ \boldsymbol{u}(0) = u_0 \\ T(\boldsymbol{u}(t)) = g \end{cases} \tag{4.16}$$

for each $t \in [0, \theta]$. We can make use of short–hand notation for the space on which the solution lies as

$$\mathcal{W}(H^1(\Omega), H^{-1}(\Omega)) := \{\boldsymbol{u} \in L^2(0, \theta; H^1(\Omega))|\ \boldsymbol{u}' \in L^2(0, \theta; H^{-1}(\Omega))\}.$$

We make use of the following Lemma for functions $\boldsymbol{u}$ which belong to the space $\mathcal{W}(H^1(\Omega), H^{-1}(\Omega))$.

Lemma 4.1. *Let $f \in C^1(\mathbb{R})$ and $f' \in L^\infty(\mathbb{R})$. If $\boldsymbol{u} \in \mathcal{W}(H^1(\Omega), H^{-1}(\Omega))$ then $f(\boldsymbol{u}) = f(\boldsymbol{u}(t)) = f \circ (\boldsymbol{u}(t)) \in \mathcal{W}(H^1(\Omega), H^{-1}(\Omega))$ for each $t \in [0, \theta)$ and $\dfrac{df(\boldsymbol{u})}{dt} = f'(\boldsymbol{u}).\dfrac{d\boldsymbol{u}}{dt}$.*

Proof. Since $f \in C^1(\mathbb{R})$ and $f' \in L^\infty(\mathbb{R})$, it then follows from Lemma 2.1 that for $\boldsymbol{u} \in \mathcal{W}(H^1(\Omega), H^{-1}(\Omega)$ we have $f(\boldsymbol{u}(t)) \in H^1(\Omega)$ for all $t \in [0, \theta]$.
First we show that $f'(\boldsymbol{u}).\dfrac{d\boldsymbol{u}}{dt} \in H^{-1}(\Omega)$:
Consider $\phi_k \in L^2(\Omega)$ for $k = 0, 1, ..., n$ and let $\dfrac{d\boldsymbol{u}}{dt} = (\phi_0, \phi_1, ..., \phi_n)$ by Proposition 4.1, that is

$$\left\langle \frac{d\boldsymbol{u}}{dt}, \varphi \right\rangle = \int_\Omega \left(\phi_0 \varphi + \sum_{k=1}^{n} \phi_k \frac{\partial \varphi}{\partial x_k} \right) \text{ for } \varphi \in H_0^1(\Omega).$$

$$f'(\boldsymbol{u})\frac{d\boldsymbol{u}}{dt} = (f'(\boldsymbol{u})\phi_0, f'(\boldsymbol{u})\phi_1, ..., f'(\boldsymbol{u})\phi_n) \in H^{-1}(\Omega).$$

Using the method of proof of the second part of Lemma 2.1, one can prove that $\dfrac{d}{dt} f(\boldsymbol{u}) = f'(\boldsymbol{u})\dfrac{d\boldsymbol{u}}{dt}$. □

4.3 Positivity and comparison theorems on $\mathcal{W}(H^1(\Omega), H^{-1}(\Omega))$

Positivity of solutions of the variational problem (4.8) has been published previously but there hasn't been results that can be directly applied to our variational problem. These results appear in publications such as [8, Proposition 6.11] and [12, Theorem 1]. We state a more general result for functions in $\mathcal{W}(H^1(\Omega), H^{-1}(\Omega))$ and provide a detailed proof. Let us first recall the definition of the space

$$H^1_+(\Omega) = \{\varphi \in H^1_0(\Omega) : \varphi \geq 0\}.$$

We need the following result to derive the positivity theorem.

Proposition 4.2.

$$\langle (\boldsymbol{u}^+(t))', \boldsymbol{u}^-(t)\rangle = 0. \tag{4.17}$$

Proof. We consider for all $\varepsilon > 0$ the function $f_\varepsilon : \mathbb{R} \mapsto \mathbb{R}$ given by

$$f_\varepsilon(\xi) = \begin{cases} \sqrt{\xi^2+\varepsilon^2} - \varepsilon & \text{if } \xi > 0 \\ 0 & \text{if } \xi \leq 0. \end{cases}$$

For all $\varepsilon > 0$, $f_\varepsilon \in C^1(\mathbb{R})$. Next we consider

$$f'_\varepsilon(\xi) = \begin{cases} \dfrac{\xi}{\sqrt{\xi^2+\varepsilon^2}} & \text{if } \xi > 0 \\ 0 & \text{if } \xi \leq 0 \end{cases} \quad \text{and} \quad \lim_{\varepsilon\to 0} f_\varepsilon(\xi) = \begin{cases} 1 & \text{if } \xi > 0 \\ 0 & \text{if } \xi \leq 0. \end{cases}$$

We see that f'_ε is continuous and bounded, that is, $|f'_\varepsilon(\xi)| \leq 1$. By Lemma 2.1 we have that for every $t \in (0,\theta)$ $f_\varepsilon \circ \boldsymbol{u}(t) = f_\varepsilon(\boldsymbol{u}(t)) \in H^1(\Omega)$. Further from Proposition 4.1 we have that

$$\frac{d}{dt}(f_\varepsilon(\boldsymbol{u}(t)), \varphi) = \left\langle \frac{d}{dt} f_\varepsilon(\boldsymbol{u}(t)), \varphi \right\rangle.$$

Fix $\boldsymbol{u} \in \mathcal{W}(H^1(\Omega), H^{-1}(\Omega))$, $\varphi \in H^1_0(\Omega)$. Let $\Phi_\varepsilon(t) = (f_\varepsilon(\boldsymbol{u}), \varphi) \in L^2(0,\theta)$.

$$\frac{d}{dt}\Phi_\varepsilon(t) = \left\langle \frac{d}{dt} f_\varepsilon(\boldsymbol{u}), \varphi \right\rangle = \left\langle f'_\varepsilon(\boldsymbol{u})\frac{d\boldsymbol{u}}{dt}, \varphi \right\rangle \in L^2(0,\theta).$$

Therefore $\Phi_\varepsilon(t) \in H^1(0,\theta)$. Fix $t \in (0,\theta)$. We have $f_\varepsilon(u(x,t)) \longrightarrow u^+(x,t)$ pointwise for all $x \in \Omega$. Using that $(\cdot,\cdot)$ is an integral, and using the dominated convergence theorem we obtain convergence of the respective integral, that is

$$\Phi_\varepsilon(t) = (f_\varepsilon(\boldsymbol{u}), \varphi) = \int_\Omega f_\varepsilon(u(x,t))\varphi(x)dx \longrightarrow \int_\Omega u^+(x,t)\varphi(x)dx = (\boldsymbol{u}^+(t), \varphi).$$

Similarly,

$$f'_\varepsilon(u(x,t)) \xrightarrow[0]{\varepsilon} \zeta(x,t) = \begin{cases} 1 & \text{if } u(t,x) > 0 \\ 0 & \text{if } u(t,x) \leq 0 \end{cases} \tag{4.18}$$

for all $x \in \Omega$. The duality pairing $\langle \cdot, \cdot \rangle$ has an integral presentation as given in Proposition 4.1. Therefore, from the dominated convergence theorem we obtain

$$\frac{d}{dt}\Phi_\varepsilon(t) = \left\langle f'_\varepsilon(\boldsymbol{u})\frac{d\boldsymbol{u}}{dt}, \varphi \right\rangle \xrightarrow[\infty]{\varepsilon} \left\langle \zeta \frac{d\boldsymbol{u}}{dt}, \varphi \right\rangle.$$

The convergence of Φ_ε and $\frac{d}{dt}\Phi_\varepsilon$ is pointwise for $t \in [0, \theta]$. Since the functions are also bounded we have convergence in the $L^2(0, \theta)$ norm as well, therefore

$$\frac{d}{dt}\langle \boldsymbol{u}^+, \varphi \rangle = \left\langle \zeta \frac{d\boldsymbol{u}}{dt}, \varphi \right\rangle.$$

Let $t \in [0, \theta]$ and take $\varphi = \boldsymbol{u}^-(t)$. Then

$$\begin{aligned} \frac{d}{dt}\langle \boldsymbol{u}^+(t), \boldsymbol{u}^-(t) \rangle &= \int_\Omega \zeta(x,t)\left(\phi_0(x)u^-(x,t) + \sum_{k=1}^m \phi_k(x)\frac{\partial}{\partial x_k}u^-(x,t) \right) \\ &= \int_\Omega \left(\phi_0(x)\zeta(x)u^-(x,t) + \sum_{k=1}^m \phi_k(x)\zeta(x)\frac{\partial}{\partial x_k}u^-(x,t) \right). \end{aligned}$$

Using (3.33) and (4.18) we have $\zeta(x)u^-(x,t) = 0$ and $\zeta(x)\frac{\partial}{\partial x_k}u^-(x,t) = 0$ for all $x \in \Omega$, then $\frac{d}{dt}\langle \boldsymbol{u}^+(t), \boldsymbol{u}^-(t) \rangle = 0$. Therefore

$$\left\langle \frac{d\boldsymbol{u}(t)}{dt}, \boldsymbol{u}^-(t) \right\rangle = \frac{d}{dt}(\boldsymbol{u}^+(t) - \boldsymbol{u}^-(t), \boldsymbol{u}^-(t)) = \frac{d}{dt}(\boldsymbol{u}^+(t), \boldsymbol{u}^-(t)) - \frac{d}{dt}(\boldsymbol{u}^-(t), \boldsymbol{u}^-(t)) = -\frac{d}{dt}||\boldsymbol{u}^-(t)||^2_{L^2(0,\theta)}.$$

$\square$

Since our main interest is to prove order properties in this section we make use of the following lemma and theorem:

Lemma 4.2 (Gronwall). *[8, Lemma 6.9] Let $\beta \in \mathbb{R}$, $\rho \in C^1([0,\theta];\mathbb{R})$ and $f \in C^0([0,\theta];\mathbb{R})$ such that*

$$\frac{d\rho}{dt} \leq \beta\rho + f.$$

Then, for all $t \in [0, \theta]$ we have

$$\rho(t) \leq e^{\beta t}\rho(0) + \int_0^\theta e^{\beta(t-\tau)}f(t)d\tau.$$

Theorem 4.3. *[9, Theorem 3, Section 5.9] Suppose $\boldsymbol{u} \in \mathcal{W}(H_0^1(\Omega), H^{-1}(\Omega))$.*

(a) Then $\boldsymbol{u} \in C([0,\theta]; L^2(\Omega))$.

(b) The mapping $t \mapsto ||\boldsymbol{u}(t)||^2_{L^2(\Omega)}$ is absolutely continuous, with

$$\frac{d}{dt}||\boldsymbol{u}(t)||^2_{L^2(\Omega)} = 2\langle \boldsymbol{u}'(t), \boldsymbol{u}(t) \rangle$$

for a.e. $t \in [0, \theta]$.

Theorem 4.4 (Positivity). *Let the bilinear form $B(t;\boldsymbol{u},\varphi)$ be L^2-coercive on $H_0^1(\Omega)$. If $\boldsymbol{u} \in \mathcal{W}(H^1(\Omega), H^{-1}(\Omega))$ is such that:*

a) $\langle \boldsymbol{u}'(t), \varphi\rangle + B(t;\boldsymbol{u}(t),\varphi) \geq 0$ *for all* $t \in [0,\theta]$ *and for all* $\varphi \in H_+^1(\Omega)$*,*

b) $T(\boldsymbol{u}(t)) \geq 0$*,*

c) $\boldsymbol{u}(0) \geq 0$*,*

then $\boldsymbol{u}(t) \geq 0$ *for a.e.* $t \in [0,\theta]$*.*

Proof. Let $\boldsymbol{u} \in \mathcal{W}(H^1(\Omega), H^{-1}(\Omega))$ satisfy conditions (a)–(c). For every $t \in [0,\theta]$ we have $\boldsymbol{u}(t) = \boldsymbol{u}^+(t) - \boldsymbol{u}^-(t)$ and using the lattice property of $H^1(\Omega)$ it follows that $\boldsymbol{u}^+(t), \boldsymbol{u}^-(t) \in H^1(\Omega)$. These functions are defined as in (3.30) and (3.31). Using also (3.32) and (3.33) we obtain

$$B(t;\boldsymbol{u}^+,\boldsymbol{u}^-) = \int_\Omega \sum_{i,j=1}^n a_{ij}\frac{\partial \boldsymbol{u}^+}{\partial x_i}\frac{\partial \boldsymbol{u}^-}{\partial x_j} + \int_\Omega \sum_{i=1}^n b_i \frac{\partial \boldsymbol{u}^+}{\partial x_i}\boldsymbol{u}^- + \int_\Omega c\boldsymbol{u}^+\boldsymbol{u}^- = 0 \tag{4.19}$$

since in all products the factors have disjoint support. We note that $T(\boldsymbol{u}(t)) \geq 0$ implies that $T(\boldsymbol{u}^-(t)) = 0$ so that $\boldsymbol{u}^- \in H_0^1(\Omega)$. Using (a) with $\varphi = \boldsymbol{u}^-$ as well as (4.17) and (4.19) we obtain

$$\begin{aligned} \langle (\boldsymbol{u}^+)' - (\boldsymbol{u}^-)', \boldsymbol{u}^-\rangle + B(t;\boldsymbol{u}^+ - \boldsymbol{u}^-, \boldsymbol{u}^-) &\geq 0 \\ \langle (\boldsymbol{u}^+)', \boldsymbol{u}^-\rangle - B(t;\boldsymbol{u}^+,\boldsymbol{u}^-) - \langle (\boldsymbol{u}^-)', \boldsymbol{u}^-\rangle - B(t;\boldsymbol{u}^-,\boldsymbol{u}^-) &\geq 0 \\ \langle (\boldsymbol{u}^-)', \boldsymbol{u}^-\rangle + B(t;\boldsymbol{u}^-,\boldsymbol{u}^-) &\leq 0 \\ \langle (\boldsymbol{u}^-)', \boldsymbol{u}^-\rangle &\leq B(t;\boldsymbol{u}^-,\boldsymbol{u}^-). \end{aligned} \tag{4.20}$$

If follows from the L^2-coercivity of B and Theorem 4.3 that

$$\begin{aligned} \langle (\boldsymbol{u}^-)', \boldsymbol{u}^-\rangle &\leq B(t;\boldsymbol{u}^-,\boldsymbol{u}^-) \\ \frac{1}{2}\frac{d}{dt}||(\boldsymbol{u}^-)||^2_{L^2(\Omega)} &\leq -\alpha||\boldsymbol{u}^-||^2_{H_0^1(\Omega)} + \beta||\boldsymbol{u}^-||^2_{L^2(\Omega)} \\ \frac{1}{2}\frac{d}{dt}||(\boldsymbol{u}^-)||^2_{L^2(\Omega)} &\leq \beta||\boldsymbol{u}^-||^2_{L^2(\Omega)} \\ \frac{d}{dt}||(\boldsymbol{u}^-)||^2_{L^2(\Omega)} &\leq 2\beta||\boldsymbol{u}^-||^2_{L^2(\Omega)} \end{aligned}$$

From Gronwall's Lemma 4.2 we have

$$\frac{d}{dt}||\boldsymbol{u}^-(t)||^2_{L^2(\Omega)} \leq ||\boldsymbol{u}^-(t)||^2_{L^2(\Omega)} \leq ||\boldsymbol{u}^-(0)||^2_{L^2(\Omega)}e^{2\beta t}.$$

Since $\boldsymbol{u}^-(0) = 0$ we have that $||\boldsymbol{u}^-(0)||_{L^2(\Omega)} = 0$ this implies that $||\boldsymbol{u}^-(t)||_{L^2(\Omega)} = 0$. Therefore $\boldsymbol{u}^-(t) = 0$ in $(0,\theta]$, hence $\boldsymbol{u}(t) = \boldsymbol{u}^+(t) \geq 0$ a.e. $t \in [0,\theta]$. □

We use this result to prove the comparison theorem for weak solutions of parabolic PDEs given as follows:

Theorem 4.5 (Comparison Theorem)**.** *Let the bilinear form $B(t;\boldsymbol{u},\varphi)$ be L^2-coercive on $H_0^1(\Omega)$. Let $\boldsymbol{u},\boldsymbol{v} \in \mathcal{W}(H^1(\Omega), H^{-1}(\Omega))$. If*

a) $\langle \boldsymbol{u}'(t),\varphi\rangle + B(t;\boldsymbol{u}(t),\varphi) \geq \langle \boldsymbol{v}'(t),\varphi\rangle + B(t;\boldsymbol{v}(t),\varphi)\ \forall\ t\in[0,\theta]$ *and for all* $\varphi \in H_+^1(\Omega)$,

b) $T(\boldsymbol{u}(t)) \geq T(\boldsymbol{v}(t))$ *and*

c) $\boldsymbol{u}(0) \geq \boldsymbol{v}(0)$

then $\boldsymbol{u}(t) \geq \boldsymbol{v}(t)$ *a.e.* $t \in [0,\theta]$.

Proof. Let $\boldsymbol{z}(t) = \boldsymbol{u}(t) - \boldsymbol{v}(t)$ and $\boldsymbol{z}'(t) = \boldsymbol{u}'(t) - \boldsymbol{v}'(t)$, then

$$\begin{aligned}\langle \boldsymbol{z}'(t),\varphi\rangle + B(t;\boldsymbol{z}(t),\varphi) &= \langle \boldsymbol{u}'(t)-\boldsymbol{v}'(t),\varphi\rangle + B(t;\boldsymbol{u}(t)-\boldsymbol{v}(t),\varphi)\\ &= \langle \boldsymbol{u}'(t),\varphi\rangle + B(t;\boldsymbol{u}(t),\varphi) - \langle \boldsymbol{v}'(t),\varphi\rangle - B(t;\boldsymbol{v}(t),\varphi)\\ &\geq 0\end{aligned}$$

$$\begin{aligned}T(\boldsymbol{z}(t)) &= T(\boldsymbol{u}(t)-\boldsymbol{v}(t))\\ &= T(\boldsymbol{u}(t)) - T(\boldsymbol{v}(t))\\ &\geq 0\\ \boldsymbol{z}(0) &= \boldsymbol{u}(0)-\boldsymbol{v}(0) \geq 0\end{aligned}$$

It follows from Theorem 4.4 that $\boldsymbol{z}(t) \geq 0$ a.e. for $t\in[0,\theta]$. This implies that
$\boldsymbol{u}(t)-\boldsymbol{v}(t) \geq 0 \implies \boldsymbol{u}(t) \geq \boldsymbol{v}(t)$ a.e. $t\in[0,\theta]$. □

4.4 Inverse monotonicity of an operator on $\mathcal{W}(H^1(\Omega), H^{-1}(\Omega))$

In this section we consider problem (4.16) and define the operator $P : \mathcal{W}(H^1(\Omega), H^{-1}(\Omega)) \mapsto L^2(0, \theta; H^{-1}(\Omega)) \times L^2(0, \theta; H^{1/2}(\partial\Omega)) \times L^2(\Omega)$ as

$$P[\boldsymbol{u}] = \begin{pmatrix} B(t; \boldsymbol{u}, \cdot) \\ T(\boldsymbol{u}(t)) \\ u_0 \end{pmatrix}. \tag{4.21}$$

Theorem 4.6. *If the bilinear form in* (4.21) *satisfies the properties (P1)–(P3) then the operator P is inverse monotone, that is*

$$P[\boldsymbol{u}] \le P[\boldsymbol{v}] \implies \boldsymbol{u} \le \boldsymbol{v} \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in \mathcal{W}(H^1(\Omega), H^{-1}(\Omega)).$$

Proof. Let

$$P[\boldsymbol{u}] = \begin{pmatrix} B(t; \boldsymbol{u}, \cdot) \\ T(\boldsymbol{u}(t)) \\ u_0 \end{pmatrix} \text{ and } P[\boldsymbol{v}] = \begin{pmatrix} B(t; \boldsymbol{v}, \cdot) \\ T(\boldsymbol{v}(t)) \\ v_0 \end{pmatrix}.$$

Then $P[\boldsymbol{u}] \le P[\boldsymbol{v}] \implies \begin{pmatrix} B(t; \boldsymbol{u}, \cdot) \\ T(\boldsymbol{u}(t)) \\ u_0 \end{pmatrix} \le \begin{pmatrix} B(t; \boldsymbol{v}, \cdot) \\ T(\boldsymbol{v}(t)) \\ v_0 \end{pmatrix}.$

It follows from the Comparison Theorem 4.5 that $\boldsymbol{u}(t) \le \boldsymbol{v}(t)$. □

4.5 Systems of parabolic equations with Metzler matrix in the reaction term

In this section we consider a system of PDEs of the form

$$u_{t,k}(t,x) + L_k[u(t,\cdot)] = f_k(t,x) \text{ in } \Omega_\theta \tag{4.22}$$

where

$$L_k[u(t,\cdot)] = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}^{(k)}(t,x)\frac{\partial u_k}{\partial x_i}\right) + \sum_{i=1}^{n} b_i^{(k)}(t,x)\frac{\partial u_k}{\partial x_i} + \sum_{\ell=1}^{m} c_{k\ell}(t,x)u_\ell \tag{4.23}$$

for $k = 1, ..., m$. If we denote $A^{(k)} = (a^{(k)})_{i,j=1}^n$ and $\vec{b}^{(k)} = (b_1^{(k)}, ..., b_n^{(k)})'$, then (4.23) can be written in the following more compact form

$$L_k[u(t,\cdot)] = -\nabla\cdot(A^{(k)}\nabla u) + \vec{b}^{(k)}\cdot\nabla u + \sum_{\ell=1}^{m} c_{k\ell}(t,x)u_\ell. \tag{4.24}$$

We assume that all operators $L_1, ..., L_m$ satisfy the uniform ellipticity condition, that is, there exists a $\mu_0 > 0$ such that for every $k = 1, ..., m$ we have

$$\xi\cdot A^{(k)}(t,x)\xi \geq \mu_0|\xi|^2$$

for all (t,x) in Ω_θ and $\xi\in\mathbb{R}^n$. Using vector notation

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix},\ L[u(t,\cdot)] = \begin{pmatrix} L_1[u(t,\cdot)] \\ \vdots \\ L_m[u(t,\cdot)] \end{pmatrix} \text{ and } f = \begin{pmatrix} f_1(t,x) \\ \vdots \\ f_m(t,x) \end{pmatrix}$$

the system (4.22) can be represented as a single vector equation as

$$u_t(t,x) + L[u(t,\cdot)] = f(t,x) \text{ in } \Omega_\theta. \tag{4.25}$$

Furthermore, the operator can be written in the following convenient vector representation

$$L[u(t,\cdot)] = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(A_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} B_i\frac{\partial u}{\partial x_i} + Cu \tag{4.26}$$

where the partial derivatives are implemented coordinate-wise, $A_{ij},\ i,j = 1, ..., n$ and $B_i,\ i = 1, ..., n$ are $m\times m$ matrices and $C = (c_{k\ell})_{k,\ell=1}^m$. In order to derive a variational formulation we assume that all coefficients in L are measurable and uniformly bounded. We multiply (4.25) by $\varphi\in(C_c^\infty(\Omega))^m$ and integrate

$$\int_\Omega u_t(t,x)\varphi\ dx + \int_\Omega L[u(t,\cdot)]\varphi\ dx = \int_\Omega f(t,x)\varphi\ dx. \tag{4.27}$$

For $\int_\Omega L[u(t,\cdot)]\varphi\ dx$ we use Green's formula to obtain

$$\int_\Omega L[u(t,\cdot)]\varphi\, dx = \int_\Omega\left(\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial u}{\partial x_j}\right)\cdot\frac{\partial\varphi}{\partial x_i} + \sum_{i=1}^{n} B_i\frac{\partial u}{\partial x_i}\varphi + (Cu)\cdot\varphi\right)dx - \int_{\partial\Omega}\sum_{i,j=1}^{n}\varphi\cdot A_{ij}\left(\frac{\partial u}{\partial x_j}\right)n_i ds$$

The bilinear form is

$$B(t;u,\varphi) := \int_\Omega \sum_{i,j=1}^{n} \left(A_{ij}\frac{\partial u}{\partial x_j}\right)\cdot\frac{\partial \varphi}{\partial x_i} + \int_\Omega \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i}\varphi + \int_\Omega (Cu)\cdot\varphi - \int_{\partial\Omega} \sum_{i,j=1}^{n} \varphi\cdot A_{ij}\left(\frac{\partial u}{\partial x_j}\right) n_i ds. \tag{4.28}$$

Taking note that $\varphi \in (C_c^\infty(\Omega))^m$ the bilinear form (4.28) becomes

$$B(t;u,\varphi) = \int_\Omega \sum_{i,j=1}^{n} \left(A_{ij}\frac{\partial u}{\partial x_j}\right)\cdot\frac{\partial \varphi}{\partial x_i}\, dx + \int_\Omega \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i}\varphi\, dx + \int_\Omega (Cu)\cdot\varphi\, dx \tag{4.29}$$

We have that any solution of (4.25) satisfies

$$(u_t,\varphi) + B(t;u,\varphi) = (f,\varphi) \text{ for all } \varphi \in (C_c^\infty(\Omega))^m \tag{4.30}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega_\theta)$. Since the space $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ with respect to the norm in $H^1(\Omega)$, we have that (4.30) implies

$$(u_t,\varphi) + B(t;u,\varphi) = (f,\varphi) \text{ for all } \varphi \in (H_0^1(\Omega))^m.$$

This variational formulation is associated with systems of parabolic PDEs with homogeneous Dirichlet boundary conditions. It can be extended to systems with non-homogeneous Dirichlet boundary conditions in a similar was as section 4.2. Generalizations for other types of boundary conditions are also possible but in order to stay close to the main ideas we consider only Dirichlet boundary conditions. Let us recall the mappings $\boldsymbol{u} : [0,\theta] \mapsto H_0^1(\Omega)$ and $\boldsymbol{f} : [0,\theta] \mapsto L^2(\Omega)$ where $[\boldsymbol{u}(t)](x) := u(x,t)$ and $[\boldsymbol{f}(t)](x) := f(x,t)$ for $x \in \Omega,\ t \in [0,\theta]$. Using the fact that $L^2(\Omega) \subset H^{-1}(\Omega)$ [3, Remark, Section 8.3] and that the dual space to $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ we get that the variational formulation for problem (4.25) with homogeneous Dirichlet boundary conditions can be presented in the following very general form:

$$\begin{cases} \text{Given } \boldsymbol{f} \in (L^2(0,\theta;H^{-1}(\Omega))^m \text{ and } u_0 \in (L^2(\Omega))^m, \\ \text{find } \boldsymbol{u} \in (\mathcal{W}(H_0^1(\Omega),H^{-1}(\Omega)))^m \text{ such that} \\ \langle \boldsymbol{u}'(t),\varphi\rangle + B(t;\boldsymbol{u}(t),\varphi) = \langle \boldsymbol{f}(t),\varphi\rangle \text{ for all } \varphi \in (H_0^1(\Omega))^m \\ \boldsymbol{u}(0) = u_0 \end{cases}$$

for each $t \in [0,\theta]$.

Our main interest is to prove the order properties of a system of PDEs of the form (4.25). We have seen in the previous sections that such properties are formulated immediately after imposing non-homogeneous Dirichlet boundary conditions to the problem. Hence we consider the non-homogeneous Dirichlet boundary conditions to (4.25), that is,

$$T(u_k(t)) = g_k(t) \text{ for } t \in (0,\theta) \text{ and } k = 1,...,m$$

where $g_k \in L^2(0,\theta;H^{1/2}(\partial\Omega))$. The variational formulation of (4.25) with these boundary conditions is as follows:

$$\begin{cases} \text{Given } \boldsymbol{f} \in (L^2(0,\theta;H^{-1}(\Omega)))^m,\ g \in (L^2(0,\theta;H^{1/2}(\partial\Omega)))^m \text{ and } u_0 \in (L^2(\Omega))^m, \\ \text{find } \boldsymbol{u} \in (\mathcal{W}(H^1(\Omega),H^{-1}(\Omega)))^m \text{ such that} \\ \langle \boldsymbol{u}'(t),\varphi\rangle + B(t;\boldsymbol{u}(t),\varphi) = \langle \boldsymbol{f}(t),\varphi\rangle \text{ for all } \varphi \in (H_0^1(\Omega))^m \\ \boldsymbol{u}(0) = u_0 \\ T(\boldsymbol{u}(t)) = g \end{cases} \tag{4.31}$$

for each $t \in [0,\theta]$ where $g = (g_1(t), ..., g_m(t))'$.

The comparison theorems are obtained under the assumption for reactions which are similar to systems of ODEs. More precisely, it is assumed that the interaction between any two different species has negative impact on both of them. This means that

$$c_{k\ell} \leq 0 \text{ for } k \neq \ell, \tag{4.32}$$

that is, $-C$ is a Metzler matrix.
Since the partial derivatives are implemented coordinate-wise, it follows that Theorem 4.3 holds for $\boldsymbol{u} \in (\mathcal{W}(H^1(\Omega), H^{-1}(\Omega)))^m$. For positivity of solutions we formulate the following theorem

Theorem 4.7 (Positivity). *Let $B(t;\boldsymbol{u},\varphi)$ be L^2-coercive on $(H_0^1(\Omega))^m$. If $\boldsymbol{u} \in (\mathcal{W}(H^1(\Omega), H^{-1}(\Omega)))^m$ is such that*

(a) $\langle \boldsymbol{u}'(t), \varphi\rangle + B(t;\boldsymbol{u},\varphi) \geq 0$ for a.e $t \in (0,\theta]$, for all $\varphi \in (H_+^1(\Omega))^m$

(b) $T(\boldsymbol{u}(t)) \geq 0$ and

(c) $\boldsymbol{u}(0) \geq 0$

then $\boldsymbol{u}(t) \geq 0$ for a.e. $t \in [0,\theta]$.

Proof. Let $\boldsymbol{u} \in (\mathcal{W}(H^1(\Omega), H^{-1}(\Omega)))^m$ satisfy conditions (a)–(c). For every $t \in [0,\theta]$ we have that $\boldsymbol{u}(t) = \boldsymbol{u}^+(t) - \boldsymbol{u}^-(t)$ where $\boldsymbol{u}^+(t), \boldsymbol{u}^-(t) \in (H_0^1(\Omega))^m$. Using (3.30), (3.31), (3.32), (3.33) and (4.32), we obtain

$$\begin{aligned}
B(t;\boldsymbol{u}^+,\boldsymbol{u}^-) &= \sum_{k=1}^m \left(\int_\Omega \sum_{i,j=1}^n \left(a_{ij}^{(k)} \frac{\partial u_k^+}{\partial x_j}\right) \cdot \frac{\partial u_k^-}{\partial x_i} + \int_\Omega \sum_{i=1}^n b_i^{(k)} \frac{\partial u_k^+}{\partial x_i} u_k^- \right) + \sum_{\ell=1}^m \int_\Omega c_{kk} u_k^+ u_k^- \\
&\quad + \sum_{k\neq\ell} \int_\Omega c_{k\ell} u_k^+ u_\ell^- \\
&= \int_\Omega \sum_{k=1}^m \sum_{i,j=1}^n a_{ij}^{(k)} \frac{\partial u_k^+}{\partial x_j} \cdot \frac{\partial u_k^-}{\partial x_i} + \int_\Omega \sum_{k=1}^m \sum_{i=1}^n b_i^{(k)} \frac{\partial u_k^+}{\partial x_i} u_k^- + \int_\Omega \sum_{\ell=1}^m c_{kk} u_k^+ u_k^- \\
&\quad + \int_\Omega \sum_{k\neq\ell} c_{k\ell} u_k^+ u_\ell^- \\
&= \int_\Omega \sum_{i,j=1}^n \sum_{k=1}^m a_{ij}^{(k)} \frac{\partial u_k^+}{\partial x_j} \cdot \frac{\partial u_k^-}{\partial x_i} + \int_\Omega \sum_{i=1}^n \sum_{k=1}^m b_i^{(k)} \frac{\partial u_k^+}{\partial x_i} u_k^- + \int_\Omega \sum_{\ell=1}^m c_{kk} u_k^+ u_k^- \\
&\quad + \int_\Omega \sum_{k\neq\ell} c_{k\ell} u_k^+ u_\ell^- \\
&= \int_\Omega \sum_{i,j=1}^n 0 + \int_\Omega \sum_{i=1}^n 0 + \int_\Omega \sum_{\ell=1}^m 0 + \int_\Omega \sum_{k\neq\ell} c_{k\ell} u_k^+ u_\ell^- \\
&= \int_\Omega \sum_{k\neq\ell} c_{k\ell} u_k^+ u_\ell^- \\
&\leq 0
\end{aligned}$$

since in all products the factors have disjoint support. From (b) we have that $T\boldsymbol{u} \geq 0 \implies T\boldsymbol{u} = T\boldsymbol{u}^+,\ T\boldsymbol{u}^- = 0$. Taking $\varphi = \boldsymbol{u}^-$ in (a) and using (4.17) we obtain

$$\begin{aligned}
0 &\leq \langle \boldsymbol{u}'(t), \boldsymbol{u}^- \rangle + B(t; \boldsymbol{u}, \boldsymbol{u}^-) \\
&= \langle (\boldsymbol{u}^+)'(t) - (\boldsymbol{u}^-)'(t), \boldsymbol{u}^- \rangle + B(t; \boldsymbol{u}^+(t) - \boldsymbol{u}^-(t), \boldsymbol{u}^-(t)) \\
&= \langle (\boldsymbol{u}^+)'(t), \boldsymbol{u}^-(t) \rangle - \langle (\boldsymbol{u}^-)'(t), \boldsymbol{u}^- \rangle + B(t; \boldsymbol{u}^+(t), \boldsymbol{u}^-(t)) - B(t; \boldsymbol{u}^-(t), \boldsymbol{u}^-(t)) \\
&\leq -\langle (\boldsymbol{u}^-)'(t), \boldsymbol{u}^-(t) \rangle - B(t; \boldsymbol{u}^-(t), \boldsymbol{u}^-(t))
\end{aligned}$$

This implies that $\langle \boldsymbol{u}'(t), \boldsymbol{u}^-(t) \rangle + B(t; \boldsymbol{u}^-(t), \boldsymbol{u}^-(t)) \leq 0$. Using the fact that the bilinear form is L^2–coercive and Theorem 4.3 we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\boldsymbol{u}^-(t)||^2_{(L^2(\Omega))^m} &\leq -B(t; \boldsymbol{u}^-(t), \boldsymbol{u}^-(t)) \\
\frac{1}{2}\frac{d}{dt}||\boldsymbol{u}^-(t)||^2_{(L^2(\Omega))^m} &\leq -\alpha||\boldsymbol{u}^-(t)||^2_{(H^1_0(\Omega))^m} + \beta||\boldsymbol{u}^-(t)||^2_{(L^2(\Omega))^m} \\
\frac{1}{2}\frac{d}{dt}||\boldsymbol{u}^-(t)||^2_{(L^2(\Omega))^m} &\leq \beta||\boldsymbol{u}^-(t)||^2_{(L^2(\Omega))^m} \\
\frac{d}{dt}||\boldsymbol{u}^-(t)||^2_{(L^2(\Omega))^m} &\leq 2\beta||\boldsymbol{u}^-(t)||^2_{(L^2(\Omega))^m}
\end{aligned}$$

From Gronwall's Lemma 4.2 we have

$$\frac{d}{dt}||\boldsymbol{u}^-(t)||^2_{(L^2(\Omega))^m} \leq ||\boldsymbol{u}^-(t)||^2_{(L^2(\Omega))^m} \leq e^{2\beta t}||\boldsymbol{u}^-(0)||^2_{(L^2(\Omega))^m}.$$

Since $u^-(0) = 0$ we have that $||\boldsymbol{u}^-(0)||_{(L^2(\Omega))^m} = 0$ this implies that $||\boldsymbol{u}^-(t)||_{(L^2(\Omega))^m} = 0$. Therefore $\boldsymbol{u}^-(t) = 0$ in $(0, \theta]$, hence $\boldsymbol{u}(t) = \boldsymbol{u}^+(t) \geq 0$ a.e. $t \in [0, \theta]$.

□

Remark 4.1. *Take note that the zero in Theorem 4.7 is not a number but a zero function in the space* $H^1_0(\Omega)$.

Theorem 4.8 (Comparison Theorem)**.** *Let* $B(t; \boldsymbol{u}, \varphi)$ *be* L^2-*coercive on* $(H^1_0(\Omega))^m$. *Let* $\boldsymbol{u}, \boldsymbol{v} \in (\mathcal{W}(H^1(\Omega), H^{-1}(\Omega)))^m$. *If*

a) $\langle \boldsymbol{u}'(t), \varphi \rangle + B(t; \boldsymbol{u}, \varphi) \geq \langle \boldsymbol{v}'(t), \varphi \rangle + B(t; \boldsymbol{v}, \varphi)$ *for a.e* $t \in (0, \theta]$, *for all* $\varphi \in H^1_+(\Omega)$,

b) $T(\boldsymbol{u}(t)) \geq T(\boldsymbol{v}(t))$ *and*

c) $\boldsymbol{u}(0) \geq \boldsymbol{v}(0)$

then $\boldsymbol{u}(t) \geq \boldsymbol{v}(t)$ *for a.e.* $t \in [0, \theta]$.

Proof. Let $\boldsymbol{z}(t) = \boldsymbol{u}(t) - \boldsymbol{v}(t)$ and $\boldsymbol{z}'(t) = \boldsymbol{u}'(t) - \boldsymbol{v}'(t)$.

$$\begin{aligned}
\langle \boldsymbol{z}'(t), \varphi \rangle + B(t; \boldsymbol{z}, \varphi) &= \langle \boldsymbol{u}'(t) - \boldsymbol{v}'(t), \varphi \rangle + B(t; \boldsymbol{u} - \boldsymbol{v}, \varphi) \\
&= \langle \boldsymbol{u}'(t), \varphi \rangle + B(t; \boldsymbol{u}, \varphi) - \langle \boldsymbol{v}'(t), \varphi \rangle - B(t; \boldsymbol{v}, \varphi) \\
&\geq 0
\end{aligned}$$

$$\begin{aligned}
T(\boldsymbol{z}(t)) &= T(\boldsymbol{u}(t) - \boldsymbol{v}(t)) \\
&= T(\boldsymbol{u}(t)) - T(\boldsymbol{v}(t)) \\
&\geq 0
\end{aligned}$$

$$\boldsymbol{z}(0) = \boldsymbol{u}(0) - \boldsymbol{v}(0) \geq 0$$

It follows from Theorem 4.7 that $\boldsymbol{z}(t) \geq 0$ a.e. $t \in [0, \theta]$. This implies that
$\boldsymbol{u}(t) - \boldsymbol{v}(t) \geq 0 \implies \boldsymbol{u}(t) \geq \boldsymbol{v}(t)$ a.e. $t \in [0, \theta]$. $\square$

We can associate with the problem (4.31) the operator
$P : (\mathcal{W}(H^1(\Omega), H^{-1}(\Omega)))^m \mapsto (L^2(0, \theta; H^{-1}(\Omega)))^m \times (L^2(0, \theta; H^{1/2}(\partial\Omega)))^m \times (L^2(\Omega))^m$

$$P[\boldsymbol{u}] = \begin{pmatrix} B(t; \boldsymbol{u}, \cdot) \\ T(\boldsymbol{u}(t)) \\ u_0 \end{pmatrix}. \tag{4.33}$$

Theorem 4.9. *If the bilinear form in* (4.33) *satisfies the properties (P1)–(P3) then the operator P is inverse monotone, that is*

$$P[\boldsymbol{u}] \leq P[\boldsymbol{v}] \implies \boldsymbol{u} \leq \boldsymbol{v} \textit{ for all } \boldsymbol{u}, \boldsymbol{v} \in (\mathcal{W}(H^1(\Omega), H^{-1}(\Omega)))^m.$$

Proof. Let

$$P[\boldsymbol{u}] = \begin{pmatrix} B(t; \boldsymbol{u}, \cdot) \\ T(\boldsymbol{u}(t)) \\ u_0 \end{pmatrix} \text{ and } P[\boldsymbol{v}] = \begin{pmatrix} B(t; \boldsymbol{v}, \cdot) \\ T(\boldsymbol{v}(t)) \\ v_0 \end{pmatrix}.$$

Then $P[\boldsymbol{u}] \leq P[\boldsymbol{v}] \implies \begin{pmatrix} B(t; \boldsymbol{u}, \cdot) \\ T(\boldsymbol{u}(t)) \\ u_0 \end{pmatrix} \leq \begin{pmatrix} B(t; \boldsymbol{v}, \cdot) \\ T(\boldsymbol{v}(t)) \\ v_0 \end{pmatrix}$.

It follows from the Comparison Theorem 4.8 that $\boldsymbol{u}(t) \leq \boldsymbol{v}(t)$. $\square$

Chapter 5

Conclusion

This dissertation presents results related to the order properties of the operators associated with elliptic and parabolic PDEs. This means that in the cases of both classical and variational formulation, we have shown that the comparison theorems can be recast in a general framework, namely in terms of inverse monotone operators, see (1.2).

In Chapter 3, we have considered one–dimensional and multidimensional elliptic PDEs. The results in this chapter concern preserving the order in the target space as given by the concept of inverse monotonicity. In the classical solution space, this property is derived from the maximum principle. We have multiplied the one–dimensional and multidimensional equation by $\varphi \in C_c^\infty(\Omega)$ and $(\varphi \in C_c^\infty(\Omega))^m$ respectively. Different boundary conditions were taken into account and this led to variational formulations. In the dissertation, we provide inequalities which can be derived directly in the space of weak solutions.

We have considered weakly coupled systems of elliptic PDEs. This is because the maximum principle has been extended to such systems in the classical case [7], [23]. We note that the conditions on the matrix C in [7] and [23] are very similar to the conditions (3.41)–(3.42) assumed in the text. The maximum principle does not have a natural extension to strongly coupled systems. The results in the dissertation extend the classical order properties of the operators in the mentioned elliptic problems, namely single PDEs or weakly coupled systems, to a wider space of weak solutions, namely $H^{-1}(\Omega)$. We have shown indirectly that the theory of elliptic PDEs has a natural extension to parabolic PDEs and systems of parabolic PDEs.

In Chapter 4 we have considered a single parabolic PDE and a system of parabolic equations with a Metlzer matrix on the reaction term. We saw that the space of solutions and the space of data are Sobolev spaces which are wider that the respective spaces considered in classical formulation. Unlike in Chapter 3, here we observed that the bilinear form should satisfy three properties, namely measurability, boundedness and L^2 coercivity. We were able to prove that the bilinear form satisfies all these properties. As a result, Lions' theorem could be applied, meaning that existence and uniqueness of a weak solution was guaranteed. We were not able to find the theory for the case of non-homogeneous Dirichlet boundary conditions in parabolic equations. However, since parabolic equations are a natural extension of elliptic equations, we were able to transform the initial-boundary value problem with non-homogeneous boundary conditions to an initial–boundary value problem with homogeneous boundary conditions. In this case we required for any fixed $t \in (0, \theta]$, $u(t, \cdot) \in H^1(\Omega)$ and $g(t) \in H^{1/2}(\partial\Omega)$. This meant that we required the space of solutions to be $\mathcal{W}(H^1(\Omega), H^{-1}(\Omega))$.

One may observe that the positivity theorems followed immediately after the case of non–homogeneous Dirichlet boundary conditions, reason being, we need the space of data, that is forcing term, boundary conditions and initial conditions. We have followed the method of proof of [6, Chapter XVIII, Section 4, Theorem 2] to prove the positivity theorems. The proofs of the comparison theorems simply required us to introduce a function $\boldsymbol{z}$ which was a difference of the two functions being compared. As a consequence, the function $\boldsymbol{z}$ satisfied the positivity theorems and the desired result was obtained. In both the cases of one–dimensional and multidimensional parabolic equations, the comparison theorems have been recast in terms of inverse monotone operators. This motivates our future work where we intend to consider monotone properties of non–linear elliptic and parabolic equations and systems of such equations which often occur in application in Biosciences, see [1] and [4].

The biggest challenge is that there is very much limited theory available for the analysis of systems arising in practical applications which are modelled by systems of elliptic and parabolic PDEs. Our goal will be to establish means of obtaining asymptotic properties of general dynamical systems as they arise in application by using the theory of monotone dynamical systems. More precisely, this goal can be formulated as deriving comparison theorem which provide for upper and lower approximations of a dynamical system via monotone dynamical systems.

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