The implications of teachers' understanding of learner errors in mathematics

by

Cebisa Faith Mtumtum

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Declaration

I declare that "The implications of teachers' understanding of learner errors in mathematics" is my own work and that all the sources used or quoted herein, have been indicated and acknowledged by means of complete references. I further declare that this work or part of it has not been previously submitted for examination or another qualification at this or any other higher education institution.

Cebisa Faith Mtumtum

30 April 2020

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Ethics statement

The author, whose name appears on the tittle page of this dissertation, has obtained for the research described in this work, the applicable research ethics approval. The author declares that she has observed the ethical research standards required in terms of the University of Pretoria's *Code of ethics for the researchers and the Policy Guidelines for responsible research.*

Dedication

I dedicate this research

to

my mother, Mrs PN Mtumtum who has supported me in all the goals I have set to achieve

and

my daughter, Nasiphi and son, Lethukuthula who sacrificed a lot during this study and encouraged me throughout to reach my dreams.

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Abstract

Low levels of learner performance in Mathematics in the Senior Phase (Grades 7-9) in South Africa is often attributed to insufficient mathematics content knowledge among teachers. Although this view might be justifiable, it is often incorrect to assume that content knowledge alone will solve the problem of low performance in mathematics. This study, therefore, argues that understanding learner misconceptions and/or errors and their underlying intricacies could provide the basis for instructional decision making, subsequently improved performance in mathematics. The purpose of the study was to explore the implications of teachers' understanding of learner errors for mathematics learning. The study was guided by qualitative methods using a case study design which involved data collection from two schools, followed by in-depth data analysis. Two theoretical lenses, namely, Cognitively Guided Instruction (CGI) and Constructivist theory were used to explore the main research question: *What are the implications of the teachers' understanding of learner errors on the learning of school mathematics in the Senior Phase (specifically Grade 9)?* Data was collected through lesson observations, analysis of learners' test responses and interviews. The findings revealed that teachers' understanding of learner errors from written responses differed notably from intricacies of same errors emanating from interviewing the learners as well as the same errors analysed by the researcher. The implications of these findings suggest the likelihood of a mismatch between teachers' instructional decision making and learner misconception/errors and this may hamper effective learning of mathematics.

Key words: Error analysis, misconceptions, conceptual understanding, procedural fluency, Cognitively Guided Instruction, conceptual errors, procedural errors, symbolic errors.

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To whom it may concern,
I hereby confirm that I undertook the language editing for the article,

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by

Cebisa Faith Mtumtum

Petrus Johannes Cillié Swart BA (Harvard) MBA (Kuehne)

Editor
Tel: +27 (0)73 612 0278
pjcswart@yahoo.com

List of abbreviations

ACE	Advanced Certificate in Education
ANA	Annual National Assessments
CAPS	Curriculum and Assessment Policy Statement
CGI	Cognitively Guided Instruction
DBE	Department of Basic Education
DoE	Department of Education
EC	Eastern Cape
EFAL	English First Additional Language
ECDoE	Eastern Cape Department of Education
FET	Further Education and Training
GET	General Education and Training
LoLT	Language of Learning and Teaching
NCS	National Curriculum Statements
PCK	Pedagogical Content Knowledge
SACMEQ	Southern and Eastern Africa Consortium for Monitoring Education Quality
SE	Systemic Evaluation
SG	Superintendent General
SP	Senior Phase
STD	Secondary Teachers Diploma
TIMSS	Trends in International Mathematics and Science Study

WNB	Whole-Number Bias
ZPD	Zone of Proximal Development

Table of Contents

List of Annexures

List of Figures

List of Tables

1 CHAPTER ONE: GENERAL ORIENTATION OF THE STUDY

1.1. Introduction

Various studies have reported about limitations in the teaching and learning of mathematics and science in South Africa (Howie, 2003; Reddy, 2004). Numerous factors are cited for the poor performance in Mathematics in the Senior Phase (Grades 7-9) in South Africa. Poor performance is often attributed to insufficient content knowledge among teachers (Pournara, Hodgen, Adler & Pillay, 2015). Although this assertion might carry an element of truth in it and might be justifiable, in my opinion, one of the key neglected areas that could be explored and could contribute meaningfully towards effective teaching and learning of mathematics is teachers' understanding of the errors that learners commit when solving mathematical problems. These errors result from employing mathematically unsound or incorrect procedures, or the lack of conceptual understanding of mathematics. If error analysis could be institutionalised in such a way that it forms the basis of classroom practice, and not be considered only after external assessment is administered, teachers could have a better understanding of learners' knowledge deficiencies. Understanding learners' misconceptions and errors could form the basis of teachers' self-reflection and instructional decision making, and subsequently improve learner proficiency in mathematics.

1.2. Problem statement

Despite several professional development initiatives to improve the quality of mathematics teaching and learning, there appears to be no significant change in mathematics learner performance as evidenced in, *inter alia,* Annual National Assessment (ANA) (DBE, 2015). One of the major causes to which poor performance can be attributed is the disregard for the errors that learners make and not utilising them to improve the quality of teaching. The national assessments such as the 2005 Systemic Evaluation (SE), 2014 Annual National Assessment (ANA) and the international studies such as the 2017 Southern and Eastern Africa Consortium for

Monitoring Educational Quality (SACMEQ IV) as well as 2015 Trends in Mathematics and Science Study (TIMSS) provide sufficient empirical evidence that learners in South Africa perform poorly in Mathematics compared to their regional and international counterparts (DBE, 2005; DBE, 2015; DBE, 2017; Zuze, Reddy, Visser, Winnaar & Govender, 2017). Given this scenario, the unacceptably low levels of learner performance in mathematics in the lower grades, especially in the Senior Phase (Grades 7-9), have been a major problem for South Africa in recent years. This has contributed to the low levels of performance in National Senior Certificate (NSC) examination.

Literature on, and theories about, teaching and learning emphasises the importance of focusing on how learners construct knowledge (Guerrero, 2014; Mohyuddin, 2014). However, little research has been done on how teachers' understanding of learners' mathematical errors can be used to inform instructional decision making and, by implication, enhance learner performance. One of the most critical aspects of effective mathematics teaching entails teachers' ability to understand and explore learners' mathematical thinking. Several researchers (Herholdt & Sapire, 2014; Brown, Skow & the IRIS Center, 2016; Riccomini, 2016; Resnick, Jordan, Hansen, Rajan, Rodrigues, Siegler & Fuchs, 2016) refer to this skill as error analysis. Merely understanding learners' errors is not the same as utilising them (errors) for instruction. Researchers such as Freeman, Eddy, McDonough, Smith, Okoroafor, Jordt, & Wenderoth, (2014) demonstrated that to improve learner achievement in mathematics, teachers must focus on remedial interventions that target common errors and misconceptions. Many teachers understand most of the learners' errors, but it is not always clear whether teachers understand the implications of learner errors on learning. One of the neglected areas is the utilisation of learner errors to inform instruction.

This study argues that the lack of attention to learners' thinking processes in relation to conceptual and procedural knowledge and the utilisation thereof to inform learning lies at the heart of the problem. In instances where teachers do indeed pay attention to learners' thinking processes, little has been explored and documented as regards the implications thereof on learning.

1.3. Rationale for the study

After having taught Grade 10 to Grade 12 mathematics at a high school for eleven years, since 2008 I worked as a mathematics curriculum advisor for Grade 4 to Grade 9. During this period, I found that students hold certain misconceptions about basic mathematics concepts that are needed to study Algebra at Further Education and Training (FET) level (Grade 10 – 12). My experience as a subject advisor taught me that even though error analysis was conducted at the end of every year in the Eastern Cape Province, it had no implications for learning. The same errors emerge year in and year out. I realised that there might be factors that contribute to these similar errors amongst learners of different grade cohorts. One of these factors could be the implications of a teacher's understanding of learner errors for teaching and learning of mathematics. Many studies have been conducted on errors and error analysis, but a limited number of studies focused on the above-mentioned factor. Given the issues articulated above, there was a need to explore the implications of teacher's understanding of learner errors for mathematics learning in South Africa. This motivated me to conduct this study and attempt to fill this gap.

1.4. Purpose and significance of the study

Essentially, there is a need for a better understanding of the importance of learners' errors in teaching and learning. Primarily, the purpose of the study was to explore the implications of teacher's understanding of learner errors for mathematics learning. In other words, the study aimed at exploring how teachers conduct error analysis and utilise their findings to inform effective learning. Although the study was about learning, the teaching could not be divorced from learning (Brew, 2003). Specifically, the objective of the study was to explore the implications of teachers' understanding of learners' errors on learning and instructional decision-making in mathematics.

The assumption was that, understanding learners' thinking process through error analysis can help improve mathematics learning. The end goal of the study was to contribute towards improving the quality of teaching and learning in South African mathematics classes in the Senior Phase, specifically Grade 9. The findings of this study would inform teachers about some of the factors that could impact on learner performance in mathematics. Conducting the study at Grade 9 level was salient since this is the exit grade from Senior Phase (SP) to FET, where learners are required to decide whether or not to choose mathematics as a subject in the FET and pursue mathematical-related careers.

The study also reported on misconceptions and errors that learners hold or commit in mathematics. This was important since teachers would understand the implications of learner errors in the learning of mathematics and could assist them to prepare lessons that would inform conceptual understanding. Clearing misconceptions and correcting learner errors at this level would afford learners more confidence in learning new mathematics concepts. Effectively, this study will contribute towards addressing the poor performance of learners in the SP (Grade 7 – 9) by exploring the teaching and learning opportunity emanating from mathematics error analysis and help develop a formalised way of ensuring that learners' errors are not perceived as '**bad'** and/or '**wrong'.** Instead**,** errors could be utilised meaningfully to benefit teachers (teaching process) and learners (learning process).

This study does not provide or offer solutions for low learner performance in mathematics but attempts to provide insight into how learners' errors can be viewed for targeted instruction.

1.5. Literature Review and the framework

As mentioned earlier, the general aim of this study was to explore the implications of teachers' understanding of learner errors for mathematics learning. I examined literature focusing on error analysis and instruction to get a deeper understanding of what the literature says. Although there are numerous debates about errors, their origin and the mitigation of errors in mathematics, the description of errors by most researchers mainly revolves around anything that results in an incorrect solution (Mohyuddin & Khalil, 2016; Godden, Mbekwa & Julie, 2013; Luneta & Makonye, 2010; Zienkiewics, Taylor & Zhu, 2005) and can either be systematic or unsystematic (Muthukrishnan, Kee & Sidhu, 2019). Errors are caused by misconceptions or carelessness, and these errors contribute negatively to learners' conceptual, and subsequently procedural, knowledge. Conceptual errors, procedural errors, factual errors, careless errors and encoding errors are some of the types of errors that have been identified by numerous researchers (Dlamini, 2017; Brown, Skow & the IRIS Center, 2016; Riccomini, 2016; Khalo and Bayaga, 2015 Elbrink, 2008) and used in this study. These are further grouped into systematic and unsystematic (Muthukrishnan et al., 2019; Ricommini, 2005). These types of errors will be discussed in detail in Chapter Two.

In addition to errors and error analysis, I also reviewed literature on learning styles and teaching approaches. I believed that learners learn differently, and teaching approaches should accommodate such diversity. I therefore employed two theoretical lenses for teaching and learning, namely Cognitively Guided Instruction (CGI) and Constructivism. CGI is mainly concerned with how teachers must facilitate learning and the types of environments that yield the desired learning. Constructivism is mainly concerned about how learners construct knowledge.

1.6. Research questions under investigation

To explore the phenomenon of the implications of teachers' understanding of learner errors for mathematics learning, the primary research question in this study was:

What are the implications of teachers' understanding of errors on learning of school mathematics in the Senior Phase (specifically Grade 9)?

The following secondary research questions guided this study:

(a) How do teachers analyse learners' errors?
(b) What instructional decisions emanate from teachers' understanding of learners' errors?
(c) How do teachers' understanding of learners' errors inform learners' conceptual understanding in mathematics?
(d) How do teachers' understanding of learners' errors inform learners' procedural knowledge in mathematics?

These research questions helped me explore and analyse teachers' understanding of learner errors and the utilisation of these errors to inform instruction.

1.7. Clarification of terms

Due to disparate definitions of the concepts in literature, it is necessary to clarify the key concepts as discussed in the literature review. This would help in comprehending any conceptual differences that may exist among these terms and how they could be used in the context of this study.

1.7.1 Errors

An error is regarded as having occurred when an actual result differs from the objective or from what is expected, or when the procedures or techniques used differ from the accepted procedures (Mohyuddin & Khalil, 2016; Godden, Mbekwa & Julie, 2013). In the context of this study, errors are to be regarded as an incorrect mathematical procedure or an inappropriate application of a procedure and they are classified as conceptual, procedural or symbolic.

1.7.2 Misconceptions

Errors caused by a lack of conceptual knowledge are called misconceptions (Mohyuddin, 2014). Mohyuddin, (2014) states that misconceptions "interfere with learning and because learners are emotionally and intellectually attached to them (misconceptions)" (p. 1- 16), it is often difficult to clear them up. In the context of this study, misconceptions will be regarded as errors due to a lack of meaningful connections between the concepts.

1.7.3 Error analysis

Herholdt and Sapire (2014) define error analysis as follows:

> "…the study of errors in learners' work with a view to looking for possible explanations for these errors. It is a multifaceted activity involving analysis

of correct, partially correct and incorrect processes and thinking about possible remediating strategies” (p. 43).

In this study, error analysis determines whether the errors result from existing knowledge that interferes with the new knowledge, or a lack of understanding of procedures to be followed.

1.7.4 Conceptual understanding

Samuelsson (2010) describes conceptual understanding as “comprehension of mathematical concepts, operations and relationships” (p. 62). In this study, a learner would have demonstrated conceptual understanding if they are able to express mathematical ideas in an interrelated manner and apply these ideas in different contexts.

1.7.5 Procedural fluency

Procedural fluency refers to the ability to employ procedures in a flexible, correct and proper manner, and perform basic calculation processes efficiently (Samuelsson, 2010). In this study, procedural fluency was extended to any correct procedure that resulted in a correct solution.

1.7.6 Quality teaching and learning

Although quality is mainly concerned with value for money, in this study quality means fitness for purpose (Harvey & Green, 2006). Therefore, for the purpose of this study, quality teaching and learning will refer to the sharing (between teachers and learners) of purposeful knowledge and the acquisition thereof, which leads to a deep conceptual understanding of mathematical concepts and procedural fluency when solving mathematical problems.

1.7.7 Learner performance

Although learner performance is mainly concerned with quantifiable scores and grades presented in, for example, percentages, quality was the main characteristic of learner performance in this study. In mathematics, the main traits of learner performance are, therefore, conceptual understanding and procedural knowledge.

1.8. Methodological considerations

This study is underpinned by the interpretivist paradigm, a research paradigm which states that knowledge construction is not only based on prior knowledge, but also on cultural or social context, and is therefore co-constructed (Krauss, 2005; Creswell & Clark, 2018). The ontological and epistemological foundations of this qualitative research were related to the interpretivist paradigm where the world is perceived as existing based on an individual's perception and therefore it is subjective, since it can be interpreted differently depending on individuals' perspectives, experiences and positions (Cohen, Manion & Morrison, 2018; Elshafie, 2013; Nieuwenhuis (2007).

The study adopted a qualitative approach with a case study design, where participants were studied within their environment, since it aimed at understanding how participants construct knowledge (Yin, 2017; Harling, 2012; Daymon & Holloway, 2010). Qualitative research produces descriptive data based on the experiences and perceptions of the participants (Lewis, 2015; Brynard, Hanekom, & Brynard, 2014)

This study used a multiple case study design, with two schools that differ according to their location, namely a rural and an urban school. The objective was to understand the intricacies and essences of the topic under investigation from different contexts.

The population of this study consisted of all the Grade 9 learners from one district of the Eastern Cape Province of South Africa. The sample was made up of two public ordinary Senior Phase (Grade 7 – 9) schools offering Grade 9 in one education district. The sample was conveniently selected according to the schools' proximity to my workplace, to minimise time constraints and costs. They were purposively selected based on the language of Teaching and Learning (LoLT). Both teachers were suitably qualified to teach Mathematics.

Data collection was done through self-developed tests, interviews, and lesson observation. The three data collection techniques enabled me to acquire an in-depth understanding of the implications of teachers' understanding of learner errors for mathematics learning.

Two semi-structured interviews were conducted per teacher. One interview took place after the observation of the first lesson and the second interview took place after the learner interviews. Learner interviews took place after the second test was administered, marked and analysed. To further explore their thinking processes, an interview protocol was conducted for face-to-face interviews with learners (*see* **Annexure A**) on selected items where most of them (learners) had committed common errors in the tests. The interviews were recorded to save time and avoid missing some information when writing and listening to the respondents. The recorded responses were later transcribed.

Two teachers were observed for two lessons each. Interview schedules and observation schedules were used. For the purpose of the lesson observation, the lesson observation sheet (*see* **Annexure B**) was developed to gather data on how the lesson addressed the misconceptions identified from the learner responses (from the test) and the extent to which the emerging (unexpected) errors were addressed.

Two tests (diagnostic test and summative test) were conducted to ensure that the relevant research questions were appropriately addressed. The tests were administered in Term 1 of 2019, at the end of February and March, respectively. To ensure fairness, validity and reliability, the tests were moderated by the Mathematics Curriculum Advisor from the Free State and North West provinces.

To ensure appropriateness and fairness, the tests were piloted on a group of Grade 9 learners from another school that was not one of the sampled schools (Kinchin, Ismail, & Edwards, 2018). The learners' scripts were kept as documentary sources. Mathematical errors that emerged from the test responses were documented as a guide on which to structure the interview questions.

1.9. Data analysis

The teachers were required to conduct error analysis on learners' responses to explore their understanding thereof. In addition, I conducted error analysis on the same learners' responses to, firstly, inform the teachers' interviews and secondly, explore and get a first-hand understanding of learners' procedural and conceptual understanding. The errors found were tagged according to different categories from literature for representing the data and developing questions that were used during learners' interviews. Error analysis was conducted to check the extent to which inappropriate and incorrect procedures led to errors.

The analysis of data focused on, *inter alia*, the following categories: the extent to which the errors identified before were addressed by the teacher, the promptness and the extent to which the emerging (unexpected) errors were addressed by the teacher, the classroom context, and the extent to which error analysis affects learning. In other words, the implications of teachers' understanding of learner errors as identified through his or her analysis of the learners' responses from the test were explored in-depth using the aforementioned categories.

With teachers, data from the interviews was analysed according to teachers' *awareness* of the error, and *procedural* and *conceptual* explanations in relation to the error (Sapire, Shalem, Wilson-Thompson & Paulsen, 2016). Learners' responses were analysed to gather explanation relating to the committed errors, either for corroboration or to gain in-depth understanding thereof.

1.10. Ethical considerations

Certain practices were observed to ensure that the study was carried out in an ethical manner. According to Shenton (2004), issues of credibility (truthfulness of the data), confirmability (correctness of the data), dependability (consistency of the findings) and transferability (the applicability of findings) should continue to ensure the trustworthiness of the study. All four aspects of trustworthiness discussed under *quality criteria* in Chapter 3 were observed.

1.11. Chapter outline

Chapter One: General orientation of the study

This chapter presents an overview of the whole study and includes the background; rationale; problem statement; purpose and significance of the study; research questions; theoretical framework and literature overview; and the methodology and ethical considerations. Concepts used in the study are also clarified.

Chapter Two: Literature review

This chapter provides a review of some of the literature associated with the study to gain insights into issues pertaining to mathematical errors or misconceptions and how they can be used to inform instruction. In addition, the theoretical framework that underpins the study is explained.

Chapter Three: Research methodology and design

This chapter examines the research methodology and design applied in the study. The relevant ethical issues are also expounded.

Chapter Four: Presentation of research findings

In this chapter, research findings from data collection are presented. Although the study is qualitative, graphs will be used in some instances to give an overview of the findings that emerged from the tests.

Chapter Five: Discussion of the findings, conclusions, and recommendations

In this chapter the research findings are discussed to address the research questions. The utility of the theoretical framework is also reflected upon. The chapter concludes with the limitations of the study, as well as recommendations for implementation, and future research.

2 CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

This chapter firstly explores literature on the definition of errors and error analysis. Secondly, it also addresses the role of error analysis for the learning and the teaching of school mathematics. This is supported by teaching approaches and learning styles associated with doing mathematics through error analysis. Finally, the theoretical framework which underpins this study is presented.

Although research has shown much interest in learners' errors over the past decades (Pournara, Hodgen, Sanders & Adler, 2016), South Africa has recently seen an increase in research on learners' errors and error analysis in mathematics (Herholdt & Sapire, 2014). Researchers like Pournara et al. (2016), Machaba (2016); McNamara and Shaughnessy (2011); Essien and Setati (2006); Resnick, Jordan, Hansen, Rajan, Rodrigues, Siegler and Fuchs (2016); Knuth, Stephens, McNeil and Alibali (2006) conducted valuable research on errors and error analysis, but there is limited research on the implications of teachers' understanding of learners' errors for mathematics learning. This study therefore aims to address such gaps.

2.2 Definition of mathematical errors and error analysis

Mohyuddin (2014) posits that "learners' learning difficulties can be presented in the form of errors" (p. 21). An error is regarded as having occurred when an actual result differs with the objective or with what was expected, or when the procedures or techniques used differ from the accepted procedures (Mohyuddin & Khalil, 2016; Godden, Mbekwa & Julie, 2013; Luneta & Makonye, 2010; Zienkiewics, Taylor & Zhu, 2005). Underlying causes of errors include a lack of conceptual understanding or misconceptions, mathematical generalisation, use of incorrect procedures, and misapplication of a rule (Makhubele, Nkhoma & Luneta, 2015; Mohyuddin, 2014; Godden et al., 2013).

Brown, Skow and the IRIS Center (2016) and Riccomini (2016) identify three types of errors:

(a) **factual errors** where a learner shows a lack of basic mathematical facts or misidentification of operational signs or digits, or a lack of knowledge of a formula;

(b) **procedural errors** which result from application of incorrect steps, or missing steps to complete the procedure or incorrect use of rules or algorithm when solving mathematical problems; and

(c) **conceptual errors** which result from misconceptions that learners hold about a concept or a misunderstanding of the principles surrounding the solving of a problem. Muthukrishnan, Kee and Sidhu (2019) posit that all these errors are a result of knowledge deficiencies or no understanding at all.

Elbrink (2008) categorises errors as follows:

(a) **calculation errors** which are a result of careless mistakes that occur when one pays too little attention to four basic operation signs, namely addition, subtraction, multiplication, and division,

(b) **procedural errors** occurring due to application of incorrect procedures, and

(c) **symbolic errors** which result from incorrect association of mathematical problems that use identical symbols.

In addition to the types of errors identified, Riccomini (2016), Dlamini (2017), Godden, Mbekwa and Julie (2013) identify **careless errors,** where a learner becomes easily distracted or that are caused by carelessness due to a lack of concentration as well as other factors apart from knowledge and skills (Muthukrishnan et al., 2019). They are also referred to as **unsystematic**. Whilst Riccomini (2016) and Brown et al. (2016) share the same sentiments with regards to factual, procedural and conceptual errors, Dlamini (2017) and Godden, Mbekwa and Julie (2013) identify **an application error** as when a learner shows understanding of a procedure and applies it correctly but to an incorrect situation.

Amongst other types of errors, Khalo and Bayaga (2015) identify **encoding errors,** which are where a learner goes through the correct procedures but fails to come to a correct solution.

Although similar in characteristics, there appear to be many variations in the way researchers interpret different types of errors. The calculation errors identified by Elbrink (2008) are similar to careless errors identified by Mbekwa and Julie (2013), Dlamini (2017), Riccomini (2016), Mohyuddin (2014) and Godden, Mbekwa and Julie (2013), and so the term *careless errors* was used in the context of this study. Factual errors identified by Brown et al. (2016) and Riccomini (2016) are similar to symbolic errors identified by Elbrink (2008), and in the context of this study the term symbolic errors were used. The procedural errors identified by Brown et al. (2016) and Riccomini (2016) are similar to the calculation errors identified by Dlamini (2017) and Godden, Mbekwa and Julie (2013). The term *procedural errors* was used in this study. Since each pair of errors above are similar, the characteristics of each pair will be incorporated to describe the preferred type of error in this study. Therefore, for the purpose of this study, errors were categorised into conceptual, procedural, application, careless, symbolic and encoding errors.

All types of errors may result in incorrect solutions to mathematical problems and lead to poor performance. All the errors, except for careless errors, are *systematic* (Muthukrishnan et al., 2019; Ricommini, 2005) since they indicate an unfounded concept. Learners are sometimes not convinced that certain procedure are carried out only with certain problems and not all (Elbrink, 2008). In the context of this study, all the systematic errors will be viewed as misconception and errors caused by lack of conceptual understanding. Table 1 will be used to inform categorisation of errors in the context of this study:

Table 1: Categories of misconceptions and errors

Category	Type of error	Description
Systematic	**Factual or symbolic errors**	• lack of basic mathematical facts • misidentification of operational signs or digits, • lack of knowledge of formula
	Procedural or application errors	• application of incorrect steps • missing steps to complete the procedure • incorrect use of rules or algorithms • correct procedure applied in incorrect situation
	Conceptual errors	• misconceptions that learners hold about a concept • misunderstanding of the principles of solving a problem • procedure used in a different concept
	Encoding errors	• Correct carrying out of procedures but fail to come to a correct solution.
Unsystematic	**Calculation or careless errors**	• Paying too little attention to the four basic operation signs. • Incorrect transcription

Although the descriptions of conceptual error and procedural error (Riccomini, 2016) are very similar, in the context of this study procedural errors will be regarded as errors where learners carry out procedures incorrectly and correctly in an inappropriate situation (Dlamini, 2017; Riccomini, 2016; Elbrink, 2008). However, if the cause of the procedural error is a result of not understanding a concept, it will be referred to as conceptual error (Riccomini; 2016; Mohyuddin, 2014). Thus, an error can be categorised as both conceptual and procedural, which means that if the procedure is carried out inappropriately, it would be categorised as a conceptual error. A conceptual error can also result in a procedural error.

Brown et al. (2016) used the following practical examples involving fractions (Table 2) to illustrate the procedural errors likely to be committed by learners:

Table 2: Examples of errors committed by learners (Brown et al., 2016)

Example 1	Example 2
$\frac{3}{4}+\frac{1}{3}=\frac{4}{7}$	$\frac{1}{2}\div 2=\frac{1}{2}\times 2=\frac{2}{2}=1$

Evidently, example 1 shows that the learner has treated fractions as whole numbers. The learner added "the numerators together as well as the denominators" (Resnick, Jordan, Hansen, Rajan, Rodrigues, Siegler & Fuchs, 2016, p.747). Adding numerators and denominators, as is normally done with whole numbers, show a lack of understanding of the concept of fractions, which means there is no understanding of what the difference between a whole number and a fraction is. Key to mathematics learning is definition of concepts. The learner does not understand that a fraction is a part of a whole (Cortina & Visnovska, 2015). Therefore, because the learner applied a correct procedure for adding whole numbers in a concept of fractions, which differs from whole numbers, I argue that, contrary to Brown et al. (2016), the error is a conceptual error. Conceptual error is more linked to the understanding of the concept being dealt with. Because the learner carried out a procedure incorrectly, the error in example 2, is regarded as a procedural error. If a learner converted both fractions in example 2, the error would have been regarded as a conceptual error in the context of this study.

Analysing learners' errors may therefore reveal the actual problem that led to the wrong solutions. Error analysis provides information on how to correct these misconceptions and may lead to an improved conceptual understanding. The inability to address learners' errors may lead to poor mathematics proficiency, which is a potential threat to numeracy skills.

Brown et al. (2016) define error analysis as "a type of diagnostic assessment that can help a teacher determine what types of errors a student is making and why" (p. 1). For instance, assuming that a learner did not invert the second fraction without actually understanding why the learner did not do so, may lead to designing the instruction in a way that is not compatible with the error. Teachers need to understand the reason behind the errors because errors may vary based on how knowledge was constructed. Error analysis involves identifying and understanding learners' errors and whether these errors occur persistently. Siegler (2009) says that "understanding what children know before they enter school is critical both for identifying what they still need to be taught and also identifying strengths on which further instruction can be based" (p. 219). Muthukrishnan, Kee and Sidhu (2019) define error analysis as "a process of reviewing the errors with an objective to provide feedback and remediation instructions to improve the learning and performance" (p. 116). This definition implies that instructional decisions should be based on an analysis of errors to help provide feedback and demystify misconceptions, which generally seems not to be the case.

Below are some examples of errors committed by learners as extracted from Riccomini (2016), Brown et al. (2016) and Schumacher and Malone (2017), followed by an interpretation thereof to share insights into the essence of this study.

Table 3: Different examples of learner errors

Learner 1	Learner 2	Learner 3
$\frac{1}{5} \div \frac{4}{5} = \frac{5}{1} \times \frac{4}{5} = \frac{20}{5} = 4$	$\frac{1}{3} \times \frac{2}{3} = \frac{2}{3}$ and $\frac{2}{6} \times \frac{7}{8} = \frac{14}{48}$	$\frac{1}{2} + \frac{3}{8} = \frac{4}{10}$
(Riccomini, 2016, p.18)	(Brown, Skow and the IRIS center, 2016, p.12)	(Schumacher and Malone, 2017, p.114)

Learner 1 consistently inverted the first fraction instead of the second one, even with other examples of the same kind. The learner inverted the first fraction $\frac{1}{5}$ to get $\frac{5}{1}$,and correctly changed the division sign to a multiplication sign. Thus, the learner committed a procedural error (Riccomini, 2016) as inverting the first fraction is not a correct procedure for dividing fractions. However, after writing $\frac{1}{5} \div \frac{4}{5}$ as $\frac{5}{1} \times \frac{4}{5}$, the learner showed correct application of procedure of multiplying fractions and simplifying fractions to get $\frac{20}{5}$, subsequently 4. Even though the learner displayed procedural fluency in the latter part of the calculations, where the procedure of multiplying and simplifying fractions were carried out correctly, this does not negate the fact that the learner committed a procedural error. Similarly, Learner 2 committed a procedural error by applying the incorrect procedure to multiply fractions. Evidently, the learner is challenged by multiplication of common fractions with equal denominators as the learner can correctly multiply fractions with different denominators. Instead of applying the correct procedure to get $\frac{2}{9}$ in $\frac{1}{3} \times \frac{2}{3}$, the learner was influenced by the rule for adding common fractions (Ojose, 2015), hence $\frac{2}{3}$. Bilalić, McLeod and Gobet (2008) refer to the phenomenon where learners apply a correct mathematical procedure in an inappropriate context because of their fixation on the previously introduced procedure as the *Einstellung effect*. This phenomenon accounts for a significant number of learners' errors and misconceptions of learners in school mathematics. The learner understands the procedure but not the concept. Thus, Muthukrishnan et al. (2019) aver that procedural knowledge is not conceptual knowledge but conceptual knowledge can lead to correct procedures.

When solving $\frac{1}{2} + \frac{3}{8}$, Learner 3 wrote the answer as $\frac{4}{10}$. It is evident that the learner added $1 + 3$ to get 4 and $2 + 8$ to get 10. According to Kallai and Tzelgov (2009) and Schumacher and Malone (2017), this type of calculation is due to approaching fractions with *whole-number bias (WNB),* where the learner views the numerators and denominators as whole numbers.

Even though the learner showed a lack of understanding of the procedure for adding common fractions, this error is due to a lack of understanding of the concept of fractions. Contrary to Brown et al. (2016) who identified the error as procedural error, in this study, it is considered a conceptual error. This conceptual error resulted from not understanding the magnitude of a fraction compared to the whole number. Thus, mathematics involves the precise definition of terms (Milgram, 2007; Wu, 2008), which is lacking with this learner. Not understanding the meaning of fractions leads to difficulty in applying any fractional calculations (Schumacher & Malone, 2017). Defining a fraction as numbers between a set of two consecutive integers can lead to understanding a proper fraction as a fraction whose numerator is smaller than the denominator and an improper fraction as a fraction whose numerator is "equal to or greater than the denominator" (Resnick et al., 2015: p. 747). Resnick et al. (2016) provided only positive examples of these fractions. In the context of this study, since a fraction is a rational number which is expressed as a quotient of integers, such that the denominator integer is a non-zero (Wu, 2008), a proper fraction will be defined as fraction whose absolute value is less than 1 and an improper fraction will be defined as a fraction whose absolute value is greater than or equal to 1.

The above explanations on errors committed by learners in Table 3 are typical examples of error analysis and is essential in understanding errors. Therefore, the classification of learners' errors should not be shallow (Leu & Wu, 2005), but instead be extended to analysing the source of the error by further asking what might have led to the solution (Panadero & Alonso-Tapia, 2013), which is key to effective teaching (Ball, Hill and Bass, 2005). Learners' errors may be effectively utilised for teaching and learning (Herholdt & Sapire, 2014). Error analysis is an important skill (Riccomini, 2005; Khalo & Bayaga, 2015) for all mathematics teachers.

2.3 The role of error analysis for instructional decision-making

The main objective of error analysis is to understand the reason behind the learners' errors (Chan & Yeung, 2001) as a way of informing instruction. Riccomini (2005) posits that the possibility of learners making errors is high during the instruction process, and it is through error analysis that one can develop informed decisions about how to teach the concept for improved understanding, and thereafter procedural fluency. Therefore, error analysis necessitates teachers to have, amongst other things, a sound knowledge and skill of how to interpret learners' errors (Muthukrishnan, Kee & Sidhu, 2019). Chan and Yeung (2001) distinguish the following reasons as being behind learners' errors: (a) distorted definition where a learner apply rules in incorrect situation, (b) misused data which involves ignoring specific datum, (c) logically invalid inference where a learner draws invalid information from a piece of information and (d) technical error which involves computational errors such as carelessness.

Muthukrishnan et al. (2019) argue that conceptual knowledge depends on how one meaningfully connects the concepts. This may involve the relationship between previous knowledge and the new concept. Previous knowledge to solve new problems with similar features does not always yield effective learning (Chrysikou, Motyka, Nigro, Yang & Thompson-Schill, 2016). The negative effect is called *functional fixedness* (Munoz-Rubke, Olson, Will & James, 2018).

Thus, error analysis assists with a deeper understanding of how the concepts that resulted in errors were connected. In the context of mathematics, the teacher should allow learners to explain how they came up with their solutions. Detailed error analysis when done well through asking probing questions based on learners' answers reduces assumptions as to the causes of errors. Error analysis provides teachers with clear and focused strategies for correcting the learners' misconceptions. However, the learner must first be given a chance to analyse his or her response and correct the mistake themselves (Rushton, 2018). Thus, the teacher undoubtedly helps the learner better understand the concept (Adom & Ankrah, 2016), when engaging the learner on his or her response.

An, Kulm and Wu (2004) assert that there are two processes of teaching, namely convergent and divergent processes. According to An et al. (2004), a convergent process of teaching is grounded on understanding and promoting learners' thinking process, building on previous knowledge, correcting misconceptions and involving them (learners) in mathematics learning. Contrary to convergent process of teaching, a divergent process of teaching focuses on content and curriculum knowledge without considering how a student thinks mathematically (An et al., 2004). Any teacher must view teaching as a convergent process for effective learning to take place. Viewing teaching as a convergent process (An et al., 2004) allows the teacher to modify teaching strategies based on learners' ability and learning styles (Makonye, 2016; Kim, 2005), and is in line with error analysis as alluded in Section 2.2.

Cognitively Guided Instruction (CGI), which will later be discussed under Theoretical Framework, also provides guidelines on how learning occurs. Carpenter, Fennema, Franke, Levi and Empson (2015) assert that a CGI teacher helps students make sense of the problem by eliciting students' thinking through asking questions about what children understand about a problem and how they got to the solution. A teacher who uses CGI avoids giving learners algorithms, and instead uses learners' errors that emanated from learners' discoveries to make instructional decisions (Guerrero, 2014; Mohyuddin, 2014). CGI can lead to active learning which is beneficial to learners since it can improve on conceptual understanding (Freeman, Eddy, McDonough, Smith, Okoroafor, Jordt, & Wenderoth, 2014).

Based on this argument, the teacher must do detailed error analysis to identify error patterns and accordingly plan the lesson for remediation. In this study, and in the context of mathematics, it is envisaged that error analysis will assist teachers in understanding deficiencies in learners' conceptual knowledge. This will further assist teachers in adjusting their instructional practices, including assessment practices for addressing learners' difficulties in mathematical concepts or targeted instruction.

2.4 The role of error analysis for the learning of school mathematics

Tarlow (2014) argues that learners' individual explorations, together with social interaction with other learners, support the construction of new knowledge. It is during this knowledge construction and solving problems that learners commit errors through interaction with others when learning school mathematics.

Learning depends on background knowledge and involves making connections between information (Brieger, Arghode & McLean, 2020). According to Bhattacharjee (2015), learners construct new knowledge during learning. In addition, learners learn effectively when creating their own knowledge, and when learning from one another (Bhattacharjee, 2015). Based on the above argument, learners are likely to unknowingly make mathematical errors or build misconceptions, primarily because of an incorrect understanding of mathematical concepts and procedures. These mathematical errors can be a hindrance to learning (Knuth, Stephens, McNeil & Alibali, 2006). When errors are analysed well, one can view them as an opportunity to learn (Makhubele et al., 2015) and they can help improve instruction.

Muthukrishnan et al. (2019) posit that the failure to analyse learners' errors poses a potential threat to acquiring mathematics skills and leads to difficulties in learning mathematics.

In mathematics, error analysis should not be viewed as punitive (Zamora, Suárez & Ardura, 2018), but as assisting the teachers in identifying misconceptions during construction of new knowledge at an early stage. This will help in designing informed instruction to correct these misconceptions, hence leading to effective learning.

2.5 The relationship between learning styles and error analysis

Learning styles also play a role in learning. Researchers such as Fleming and Baume (2006) describe learning style as a package that includes one's preferred way of learning. For the purpose of this study, learning styles will be defined as "a particular way in which an individual learns" (Pritchard, 2017, p. 41-56) and differs from one individual to another. It is therefore referred to as one's learning style preference. Different learning styles must be catered for as they can be accompanied by emotions, and according to Trigwell (2012) and Corcoran and Tormey (2012), emotions have an impact on teaching and learning.

Learners achieve more in learning environments that meet their learning styles (Othman & Amiruddin, 2010; Pritchard, 2017). Such a learning environment is designed and managed in a way that helps learners freely express their views without fear of disapproval by anybody, and neither dismisses the impression that there is one solution to a problem nor one way to come to a solution (Mahmood & Gondal, 2017; Vakalisa, 2016).

Pritchard (2017) describes three learning styles, namely visual (seeing), auditory (listening) and kinaesthetic (doing), as identified by Neuro-Linguistic Programming (NLP), which is concerned with how communication affects learning. Othman and Amiruddin (2010) posit that humans use all the senses when learning. In the context of mathematics, this implies that, concrete objects or modelling, group discussions and being actively engaged in class can clear up the misconceptions (Kim, 2005) that learners hold about certain concepts.

According to Pritchard (2017) and Jones et al. (2003), Kolb's Learning Style Model categorises individuals over two dimensions, each with different modes. The first dimension, which has preference for the concrete experience mode, or the abstract conceptualisation mode, is concerned with how the learner takes in information. The second dimension which has a preference for active experimentation mode or the reflective observation mode is concerned with how the learner internalises information. Kolb and Kolb (2005) contend that even though everyone makes use of all learning modes to some extent, everyone has their own preferred learning style. Understanding learning style preferences of learners assists teachers in planning tasks that will help individual learners achieve their learning goals (Pritchard, 2017).

Based on the information above, teachers should not apply a blanket approach when teaching (Jones et al., 2003) and teach according to their own learning preferences, disregarding the individual preferences for a learner. In the context of error analysis, the learners' learning style should be understood well by the teachers so that they can develop a focused remediation strategy that is suitable for a specific learner.

2.6 The relationship between error analysis and teaching approaches

There are two categories of approaches to teaching, namely teacher-focused, which is mainly concerned with the transmission of knowledge, and student-focused, where the teacher is a facilitator of knowledge (Trigwell, 2012). Most teachers are still using the traditional approach where teaching of mathematics is initiated in a symbolic or abstract manner and leads to memorisation which has little benefit for learners. Cognitive development is not enhanced (Zakaria & Syamaun, 2017). There seems to be an agreement that the traditional approach, which is a common practice that is teacher-centred and involves telling or recitation (Alsup, 2004; Saadati, Cerda, Giaconi, Reyes & Felmer, 2019), must be replaced by an approach that will arouse learners' interests (Abrie et al., 2016). This occurs in a participative environment which seeks to collaborate human knowledge and experience (Abrie, Blom & Fraser, 2016). Dewey views a school as "life and not preparation for life" (Abrie et al., 2016, p. 1-37), which implies that learners must take responsibility for their learning by being actively and not passively involved.

Vakalisa (2016) states that participative teaching is a process of engaging learners actively where its possibility depends on a sound content knowledge, employing different teaching strategies to accommodate different learning styles and maintaining conducing learning environment through effective management by teachers. Vakalisa (2016) further contends that the dynamics of participative teaching includes teacher guided activities that are based on teachers' understanding of how individual learners learn. Teachers must therefore move away from being mere knowledge dispensers (Vakalisa, 2016) and take learners by hand, to achieve conceptual understanding. Based on the above, investigative approaches and problem-based learning are some of the teaching approaches that can assist learners to correct their mistakes on their own during the process of knowledge construction.

2.6.1 Investigative approach

Investigative approach is a learner centred approach in which curiosity and exploration is fostered. It differs from rehearsed algorithms (Quinnell, 2010) which are examples of rote learning with little understanding involved (Pritchard,2017), in the sense that it gives the learner an opportunity to formulate the algorithm and make conjectures (Van de Walle, Karp and Bay-Williams, 2015). Van de Walle et al. (2015) refers to this approach as teaching through problem solving, which is opposite to the traditional approach, where according to Aljaberi and Gheith (2018) a teacher sees his or her role as passing information. In addition, this approach involves teaching for conceptual understanding and reduces the amount of meaningless standardised algorithms (Tripet & Chapman, 2019). According to Grouws and Cebulla (2000), research has shown that learners who spend more time on discovering and inventing mathematical ideas, show good conceptual understanding, and thereafter procedural knowledge. The investigative approach is thus strong in discovering and inventing mathematics ideas and can be regarded as teaching for meaning and understanding, which is an important practice in the teaching of mathematics. However, this approach does not dismiss the role of a teacher in guiding learners as regards what needs to be discovered. One may refer to this approach as Partially Guided Discovery Learning, as it increases scaffolding (Smagorinsky, 2018) through teacher's guidance. The objective is to come up with a scientific algorithm, unlike in unguided discovery learning where learners choose their own tasks (Baroody, Clements, & Sarama, 2019). Concrete materials, pictures and symbols can strengthen this approach. The method of using, concrete materials, pictures and symbols is called the Concrete Pictorial and Abstract (CPA) approach. The CPA approach is one of the methods of teaching that is based on Bruner's learning theory and consists of 3 stages of learning, namely (1) concrete (real objects); (2) pictorial/representational (diagrams); and (3) abstract (symbols) (Purwadi, Sudiarta & Suparta, 2019). According to Purwadi et al. (2019), CPA approach can improve students' conceptual understanding.

Investigative approach is beneficial in the sense that it gives learners a better opportunity to apply knowledge that they have developed on their own when solving mathematical problems, as opposed to simply copying a correctly worked example of the teacher (Van de Walle et al., 2015; Rushton, 2018). In an investigative classroom, the teacher reinforces learners' effective thinking skills (Adom & Ankrah, 2016), through creating a less formal environment that encourages discussions and debates (Quinnell, 2010).

Through investigations, both learners and teachers may get to understand how others think and reason (Pritchard, 2017) Misconceptions that learners might have, can therefore be cleared up with the help of the teacher or peers. Therefore, investigations should mostly be done in class and findings must be reported for everyone to interact with, so that learners can correct their own mistakes. In the context of mathematics and error analysis, learners should be given many examples to explore so that they can formulate conjectures. They should not be given readily made rules or algorithms (Quinnell, 2010) to use in calculations.

2.6.2 Problem-based learning (PBL)

Mathematics is, among other things, characterised by the solving of well-structured problems (Milgram, 2007). PBL is an approach where learning is problem driven. Learners are confronted with a problem to be solved whereby they "interpret the problem, gather needed information, identify possible solutions, evaluate options, and present conclusions" (Roh, 2003, p.1; Padmavathy & Mareesh, 2013, p.47). This statement attests that mathematical reasoning and problem solving should not be treated in isolation as different entities (Milgram, 2007). One of the reasons for failing to achieve the outcomes for PBL is that it is sometimes confused with problem solving and inappropriate assessment methods are therefore used (Savery, 2006).

Hmelo-Silver (2004) posits two issues that are at the heart of PBL: (a) collaborative active construction of knowledge by learners and (b) teacher being the facilitator of collaborative learning and not the source of knowledge. PBL should aim at a deep understanding of mathematics and the nature of mathematics. This can be achieved when a teacher knows when to ask questions and follow-up questions (Pritchard, 2017), and is able to employ a variety of strategies at different stages of PBL. Hmelo-Silver (2004) regards this as the qualities of a PBL teacher.

Through PBL, learners flexibly construct their knowledge, become effective problem solvers, develop lifelong self-directed learning skills and are able to work as a team (Padmavathy & Mareesh, 2013), and this leads to intrinsic motivation (Hmelo-Silver, 2004). In the context of this study, PBL can assist learners in learning from one another. During teamwork, they (learners) can clear up all the misconceptions before presenting their solution to the problem that was given.

Most teaching approaches are not mutually exclusive but complementary (Grouws & Cebulla, 2000), and the teacher must choose a suitable approach for conceptual understanding. All the teaching approaches lead to Opportunity to Learn (OTL). According to Grouws and Cebulla (2000), OTL comprises of what is to be taught, how to teach it, and the relationship between what learners must learn and what they already know. OTL therefore equips learners with the necessary mathematics skills for the 21st century, which allows learners to perform well globally in a technologically inclined society (Grouws & Cebulla, 2000).

With these approaches, the teacher should facilitate active learning to diagnose misconceptions (Freeman et al., 2014) by asking learners to justify their answers, whether correct or incorrect (Rushton, 2018). Asking learners to justify their solutions is beneficial in that: (1) with a correct solution given, a teacher can verify whether the learner understood the concept clearly and did not come up with a solution by luck and (2) with an incorrect solution given, a teacher can determine the actual problem that led to the incorrect solution. The latter also helps the learner correct his or her own mistakes through follow up questions that may be asked by a teacher, and a learner can then internalise information (Pritchard, 2017). For an example, Learner 2 in Table 3 must be asked to explain how he or she came up with the solutions in order to understand the inconsistency that has occurred in the calculations. Dismissing or ignoring learners' errors is a recipe for reinforcing the misconception (Setiawan & Koimah, 2019). Maximum class management and efficacy relies on active participation and genuine support of, and by, students (Petress, 2006). Whilst learners actively participate in learning, they "can develop independence, responsibility, and accountability only if the teacher gives them opportunities to do so" (Hoosain & Chance, 2004, p. 476). For effective learning to take place best, learners must be allowed to voice their ideas followed by feedback which not only provides for correct solutions but are also suggestive and analytical (Setiawan & Koimah, 2019).

One of the key aspects that can be linked to these teaching approaches is engaging in reflective teaching. Reflective teaching is a process where a teacher examines his or her own teaching attitudes and values in teaching with an intention to improve on them (Ratminingsih, Artini & Padmadewi, 2017; Jacobs, 2016). Thus, a reflective teacher acknowledges that he or she holds misconceptions or does not understand a certain concept (Isiksa and Cakiroglu, 2011). Reflective teaching can be aligned to lesson study, a Japanese initiative where one allows colleagues to observe his or her lessons and gives feedback through open discussions and analysis. A reflective teacher is open-minded and interprets self-evaluation through reflection from colleagues as an opportunity for growth. In the context of mathematics, reflective teaching can help in determining whether the misconceptions that the learners hold were transferred to them by the teacher through the method used when teaching and whether or not the methodology used allows for conceptual understanding.

2.7 The role of error analysis in doing mathematics

According to Van de Walle et al. (2015), doing mathematics means devising a plan to solve a problem, carrying out the plan, and checking the reasonableness of the solution. These are George Polya's four steps for teaching about problem solving. Problem solving is not only key to doing mathematics but is also one of the 21st century skills required to address real-life situations. This approach is also referred to as teaching about problem solving (Van de Walle et al., 2015). On the other hand, the traditional method of teaching requires students to follow a routine procedure (Baroody, Clements, & Sarama, 2019) by listening, copying, and memorising what the teacher says, which sometimes does not make sense and can lead to low-level thinking (Van de Walle et al., 2015). Whilst contrary to low-level thinking which does not equip learners with the skills for doing mathematics, higher-level thinking helps learners make sense of what they are doing by figuring out how to reach a solution (Van de Walle et al., 2015).

When doing mathematics, the teacher presents problems which will require the use of the verbs such as "describe, justify, predict, compare, explain, conjecture, explore, formulate, investigate" (Van de Walle et al., 2015). In a class where the teacher asks learners different questions using the above verbs, it assists learners to reason, which is one of the key strands in teaching mathematics for understanding (Kilpatrick, 2011). Whilst doing so, learners are checking if their solutions make sense. In addition, learners get the understanding of where they might have missed the point and are able to correct themselves. This does not happen automatically, but at first with the aid of the teacher asking probing questions. Later learners can do error analysis on their own.

2.8 THEORETICAL FRAMEWORK

This section presents the theoretical framework upon which the study is framed. The theoretical framework is a basis on which to build and support the research, and is derived from generally accepted and tested existing theories (Grant & Osanloo, 2014). The study is mainly underpinned by two theoretical lenses, namely Cognitively Guided Instruction (CGI), and Constructivist theory. Each makes a unique contribution to the study to explore and gain insight into the topic. In this section, the two theoretical lenses are explicated and their presumed relevance and utility (fit-for-purpose) for the study are presented.

2.8.1 Cognitively Guided Instruction

Cognitively Guided Instruction (CGI) aimed to understand learners' thinking as a guide to inform instruction and was first practiced at the University of Wisconsin and with the Foundation Phase learners (Hoosain & Chance, 2004;). However, the approach can be effectively practiced at any level.

Many researchers have been objective about the efficacy of using this approach throughout the phases and assert that it should form the basis of a mathematics curriculum, due to its insistence on individual's reasoning comprehension and critical thinking (Hoosain & Chance, 2004). The focus in the CGI classroom is on the processes involved in obtaining the answer rather than the answer itself (Guerrero, 2014; Hoosain & Chance, 2004).

CGI also provides guidelines on how learning occurs. Carpenter, Fennema, Franke, Levi and Empson (2015) and Hoosain and Chance (2004) state that CGI places a prominence on individual attention to track if the concept is well grounded and learner's critical thinking is developed by eliciting students to think through asking a series of questions on what children understand about a problem and how they got the solution. Understanding and building up on learners' mathematical thinking leads to changes in teachers' practice (Glennan, 2004).

Since children come to school with mathematical knowledge, the duty of the teacher in a CGI classroom is to reveal and strengthen that knowledge through formative assessment (Hoosain & Chance, 2004). Thus, a CGI teacher considers learners' previous knowledge as a powerful tool which enables them to solve problems on their own (Carpenter et al., 2015). In the CGI classroom, learners decide to use any material available to them as they wish in solving the problems, using different strategies that they have chosen (Hoosain & Chance, 2004; Munday, 2016). This may lead to learners' discovery of meaning on their own by moving from concrete to pictures, then abstract, which supports the natural brain functioning (Guerrero, 2014). Thus, learners do not have to be given procedures or strategies on how to solve problems beforehand.

Since learners in the CGI classroom spend most of their time solving problems on their own (Guerrero, 2014), they definitely come up with different ways to the solution (Carpenter et al., 2015). This is in line with Polya's problem-solving techniques where "success in problem solving does not solely depend on the acquisition of concepts but also depend on the choice of the relevant problem-solving technique" (Khalo & Bayaga, 2015, p.102). Thus, there is no one way to the solution, and the logic in processes involved when solving the problem is imperative. To strengthen learners' cognitive demand, activities should be carefully chosen and be purposeful. The role of a teacher in determining classroom activities is key (Aljaberi & Gheith, 2018; Ngoaka, 2018). Thus, each learner reports the solution to the group and questions may be asked by peers (Guerrero, 2014) and the teacher in trying to understand how the problem was solved (Carpenter et al., 2015). One may regard self and peer assessment as beneficial for learning (Ratminingsih, Artini & Padmadewi, 2017) in the CGI classroom.

According to Carpenter, Fennema and Franke (1996), the CGI model focuses not only on teachers' knowledge of learners' thinking, but also on understanding the ways to explain the concepts such that they (concepts) are intelligible. The ability to explain concepts to learners in a comprehensible manner is a primary component of teachers' pedagogical content knowledge (PCK). In addition, there is a shift of focus from presenting knowledge by teachers to own construction of knowledge by learners (Carpenter et al., 1996).

Different reports to solutions give a teacher a deeper insight in terms of how guidance should be tailored to suit each child. In the context of mathematics, the implications of CGI are that learners can solve problems on their own using the previously constructed knowledge. Based on the previous statement, I therefore argue that as learners solve problems, they are susceptible to making errors and misconceptions are likely to develop. Thus, the teacher must allow learners to attempt problems and thereafter analyse the errors committed. The analysis of errors could be strengthened by allowing learners to explain their answers in trying to gain more insights in their thinking process and their procedural knowledge. Learners must therefore be asked follow-up questions based on individuals' responses to questions so that their (learners') thinking processes can be explored further.

Given the salient characteristics of CGI, its relevance to this study is that it is mainly concerned with what effective teaching and learning consists of. It is also concerned with utilisation of learner errors to inform instruction to clear student misconceptions. The study is aimed at exploring the teacher's understanding of learner errors to inform instruction, and CGI advocates for the utilisation of learner errors through eliciting learners' thinking processes.

2.8.2 Constructivism

According to Bada and Olusegun (2015) constructivism is defined as follows:

> *an approach to teaching and learning based on the premise that cognition (learning) is the result of 'mental construction'. In other words, students learn by fitting new information together with what they already know (p. 66).*

Thus, a teacher needs to afford the learners an opportunity to learn new concepts by building on previous knowledge through scaffolding. Meaning learning cannot take place in isolation.

Machaba (2016) contends that constructivism is a study about how people individually learn and most importantly this learning process occurs actively through the construction of new ideas depending on new or existing knowledge. Construction of new knowledge, as outlined in the constructivist theory involves two processes: accommodation and assimilation (Bada & Olusegun, 2015). Assimilation is the incorporation of new ideas into the current structure without changing that structure, whereas accommodation is restructuring or modification of what is already known to incorporate new information or a “process of reframing one's mental representation of the external world to fit new experiences” (Bhattacharjee, 2015, p.66; Bada & Olusegun, 2015; Adom, Yeboah, & Ankrah, 2016, p. 3). Thus, one’s mental presentation is being reorganised or restructured to house new information.

Bada and Olusegun (2015) further contend that constructivism helps learners actively participate in their learning and make significant interrelations between prior knowledge, new information and the leaning processes. Thus, by implication, key to the principles of constructivism is that learning is an “active process” (Bada and Olusegun, 2015, p. 69) which involves a teacher and a learner. This implies that the learner must be actively involved in the learning process and take charge of his or her own learning. Bada and Olusegun (2015) add that through constructivism, social and communication skills are promoted by designing a classroom environment that encourages collaboration and the exchange of ideas.

Kim (2005) avers that constructivist teaching assists learners to “internalise and transform new information” where transformation ensues when there is formation of new comprehension from emerging mental processes (p. 10). By implication, a constructivist teacher helps the learner absorb and change the information with the purpose of improving on it or adding some features or characteristics or properties without changing its value.

Wachira and Mburu (2019) posit that the focal point of a cognitive constructivists is an individual learner whilst social constructivists considers knowledge to be constructed through social interaction, hence co-constructed. Thus, peer assessment is core to the construction of knowledge (Ratminingsih et al., 2017).

According to Bhattacharjee (2015), constructivist teaching is based on the assertion that learning involves the active participation of learners in a process of meaning and knowledge construction as opposed to peaceably receiving information. Thus, learner passivity cannot yield the desired outcome (Aljaberi & Gheith, 2018) during learning. Constructivist teaching promotes higher order thinking, and produces inspired and self-driven learners (Pritchard, 2017; Bhattacharjee, 2015).

Kim (2005) describes constructivist teaching as comprising of the following three fundamental aspects: (1) Learning involves knowledge construction and not knowledge acquisition. (2) Teaching supports the construction of knowledge and does not involve spoon feeding the learner. (3) Teaching is learner-centred and not teacher-centred. In the context of mathematics, a constructivist teacher must stimulate learners' thinking (Yusmarni, Fauzan, Amanda & Musdi, 2019) and learning through experiments and the use of real-world problems that lead to critical thinking (Bada & Olusegun, 2015). Setiawan and Koimah (2019) posit that learners come to school with correct or incorrect information about almost every topic to be learnt, and in the process of understanding what must be learnt, the new information may be distorted or rejected completely.

Whilst taking into cognisance that "learners should not be regarded as empty vessels that have to be filled with knowledge by the teachers" (Makonye, 2016, p.291), constructivism does not therefore dismiss the role of a teacher in class. Instead, it allows the teacher to become dominant in probing learners' critical thinking to uncover what they (learners) already know (Makhubele et al., 2015). This implies that learners have some information about any concept to be learnt, and it is the duty of the teacher to assist the learner to learn the new concept by building on what the learner already knows.

In constructivism, classrooms become diverse, with various activities at different levels occurring concurrently and teachers acting as facilitators, supports, guides and models of learning (Cohen, Manion & Morrison, 2004; Slavin, 2019) and not transmitters of information (Russo, Bobis, Sullivan, Downton, Livy, McCormick & Hughes, 2020). Thus, instruction should be built around more complex problems

(Slavin, 2019), and not only problems with clear answers to stimulate higher order thinking. Based on the previous statement, mathematical errors will be analysed differently according to different activities, levels of complexity and different learners' responses. For this to be successful, there should be a dialogue stimulated by the teacher until it gains its impetus. This should be done in a sequential manner which Pritchard (2008) and Smagorinsky (2018) refer to as "scaffolding" (p.70). Scaffolding can be viewed as a transitional process of teaching where a teacher uses simple examples that learners understand as a way of linking them (examples) with the new concept. As an example, teaching addition of common fractions can first be taught through modelling where learners could understand that adding and subtracting parts of the same whole with equal parts would give an answer where the denominator is the same as that of the parts that were added or subtracted.

Learners would then formulate a conjecture that the denominators of the parts of a whole are neither added nor subtracted. After learners have understood how to add fraction parts of a whole with equal parts, they could be given two *wholes* with different fraction parts, where denominators are multiples of each other to add, and they would apply equivalence to make the denominators the same. This could be extended to fractions with different denominators. Although sociocultural theory is not the main theory framing this study, Pritchard (2008) posits that the concept of scaffolding can be understood better with the idea of Vygotsky's concept of Zone of Proximal Development (ZPD). Vygotsky's concept of ZPD emphasises the importance of the role of the teacher in guiding the learners as development cannot take place if the content can be achieved without the guidance of a teacher (Abrie et al., 2016). Thus, the teacher needs to provide guidance on mathematics tasks and activities that the learners cannot complete without assistance (Slavin, 2019) to narrow the ZPD. Essentially, the defining characteristic of ZPD is the difference between mathematics tasks and activities that the learner can complete dependently and independently (Abrie et al., 2016). Learners' ZPD are not the same (Pritchard, 2008; Smagorinsky, 2018) due to culture, society and experience (Kim, 2005). In the context of mathematics, this implies that the construction of knowledge in learners cannot be the same and teachers need to cater for learners' different ZPD. Error analysis can provide an opportunity for teachers to understand learners' mathematical errors and plan lessons that will assist the learners to navigate the ZPD. This will later increase the possibility for learners to solve problems independently.

Piaget believes that knowledge construction is related to interaction with the physical and social environment and defines categories of knowledge as schemas (Abrie et al., 2016). Through error analysis, the teacher will assist the learners to assimilate and accommodate the information into the existing schemas or transform information. Bada and Olusegun (2015) in their research state that constructivism restricts knowledge transmission by teachers, a practice evident in traditional classrooms. Table 4 illustrates the differences between traditional classroom and constructive classroom as extracted from Bada and Olusegun (2015) and Bhattacharjee (2015).

Table 4: Differences between a traditional and a constructivist classroom (Bada & Olusegun, 2015; Bhattacharjee, 2015)

Traditional Classroom	**Constructivist Classroom**
Primarily use textbooks and workbooks.	Resources include manipulatives.
Learning is based on repetition.	Learning is interactive, building on what the student already knows.
Teachers transmit information to learners; learners are recipients of knowledge.	Teachers facilitate knowledge, assisting learners to construct their own knowledge.
Teacher's role is directive, rooted in authority.	Teacher's role is interactive, rooted in negotiation.
Assessment is through testing and focuses on correct answers.	Assessment includes observations, and discussions. The process is as important as the answer.
Knowledge is seen as stagnant.	Knowledge is seen as dynamic, ever changing with experiences.
Students work primarily as individuals.	Students work primarily in groups.

Constructivism advocates for a learner-centred approach where a teacher facilitates knowledge by assisting learners to construct knowledge based on what they already know. In addition, in a constructivist classroom, the teacher is interactive whilst promoting discussion among learners. In the context of this study, teachers would be expected to understand and utilise learners' errors to clear up misconceptions or strengthen procedural fluency by engaging learners through seeking for explanations on errors committed.

3 CHAPTER THREE: METHODOLODY

3.1 Introduction

This chapter presents a description of the methodology used in this study. To understand the case being investigated, firstly the research paradigm and its philosophical assumptions are discussed, followed by the research approach and design that underpin the study. The sample selection, and data collection strategies are elucidated. Lastly, the data analysis strategies used are spelled out, as are the quality criteria in terms of trustworthiness, validity and reliability and ethical considerations related to the study.

3.2 Research paradigm

According to Wisniewski (2010), the term *paradigm* was discovered by historian and scientist Thomas Kuhn in 1962 who defines a paradigm as “beliefs, values and techniques” shared by societal members and as such they (beliefs, values and techniques) cannot be linear but the old are challenged to form the new (p. 55). In the context of research, a paradigm is defined as follows:

> *A set of assumptions or beliefs about fundamental aspects of reality which gives rise to a particular worldview – it addresses fundamental assumptions taken on faith, such as beliefs about the nature of reality (ontology), the relationship between knower and known (epistemology) and assumptions about methodologies* (Nieuwenhuis, 2007, p. 47).

Various paradigms differ according to the reality, knowledge and action plan which informs the choice of method(s) to gather the reality about the knowledge of different research approaches. The first step towards research is a paradigm (Elshafie, 2013) and is regarded as setting the purpose for the research (Mackenzie and Knipe, 2006). The three constituents of paradigm, namely ontology, epistemology and methodology (Shah& Al-Bargi, 2013; Daymon & Holloway, 2010), will be discussed under philosophical assumptions.

Interpretivist paradigm, in which this study is embedded, perceives the world as existing on individual's perception and therefore it is subjective since it can be interpreted differently depending on the perspectives, experiences and positions of individuals (Cohen, Manion & Morrison, 2018; Elshafie, 2013). Therefore, it involves the idea of multiple reality. Contrary to nomothetic which is mainly concerned with application to many people generally, interpretivists are interested in the unique individual, which is known as idiographic and takes cognisance of understanding over scientific explanation (Daymon & Holloway, 2010; Nieuwenhuis, 2007; Ponterotto, 2005).

According to Cohen, Manion and Morrison (2018), the interpretivist strives to understand how individuals construct knowledge through reaching out to them to focus on their behaviour. In the context of the study, lesson observations and interviews were conducted to observe how learners construct knowledge during learning and seek for explanations behind learner errors that emerged from diagnostic tests. Cohen et al. (2018) refer to the approach of understanding and seeking explanations of learners' errors to discover meaning as '*verstehen'* and *hermeneutic*, respectively. Lastly the study adopted an interpretivist paradigm, since the experience that I gained as a curriculum advisor assisted me with interpretation of data to understand the underlying causes of learners' errors in the context of mathematics.

There are two dominant philosophical schools of thought or assumptions in research, namely positivism which underpins quantitative methodology, and interpretivism, which underpins qualitative methodology (Adom, Yeboah & Ankrah, 2016; Tuli, 2010; Daymon & Holloway, 2010).

This investigation followed a qualitative approach, as discussed in paragraph 3.4 later in this chapter. The research focused on exploring the teacher's understanding of learners' errors for mathematics learning. This was done to understand how teachers analyse learners' errors; practically, and hence the investigation was in the context of individual classroom communities. In the context of mathematics, different learners construct different knowledge and hence multiple realities. Corresponding to the argument in the preceding statement, this study is located within the interpretivist paradigm.

Although some researchers such as Creswell and Clark (2018), Tuli (2010), Bahari (2010), Mackenzie and Knipe (2006), Ponterotto (2005) and Krauss (2005) use constructivist, interpretivist, subjectivism and naturalistic paradigms interchangeably, the interpretivist paradigm will be used for this study.

Even though mathematics discipline is objective in the sense that it is based on measurable facts, learner error during construction of knowledge is based on interpretations, beliefs and assumptions, and is thus subjective. Based on the above statement, this study is aligned with the interpretivist paradigm since its focus is on learner errors and the explanation thereof.

3.3 Philosophical assumptions underpinning Interpretivist paradigm

Ontology, epistemology, and methodology are some of the philosophical assumptions that characterise the research paradigm (Bunniss & Kelly, 2010). In the next sub-sections, the implications of interpretivist paradigm for the ontological, epistemological and methodological perspectives will be explained as they pertain to this study.

3.3.1 Ontology

According to Daymon and Holloway (2010), ontology refers the study that involves different human understanding of how the social world is, in reality.

Qualitative research methodology is grounded on interpretivist ontology (Tuli, 2010) according to which reality exists as multiple realities since it is created by individuals in groups, and not as a single reality (Nguyen, Gardner & Sheridan, 2019; Creswell & Clark, 2018; Krauss, 2005). As alluded to in the previous section, this study seeks to explore the *implications of teachers' understanding of learner errors for mathematics learning in the Senior Phase (specifically Grade 9*). This was done through studying teachers and learners in their environments to understand how they construct knowledge on the same mathematical concept differently, which may lead to different errors.

In this study, the ontological assumption was that different leaners construct different knowledge on the same mathematical concept, hence interpretivist. Corresponding to the ontological belief of the interpretivist, there are multiple realities and truths which are subject to change due to the social world even though they are individually and uniquely constructed (Daymon & Holloway, 2010). The analysis of errors committed by different learners during knowledge construction on mathematical concepts would assist in understanding what instructional decisions emanate from teachers' understanding of learners' errors.

3.3.2 Epistemology

Daymon and Holloway (2010) define epistemology as follows:

> *The philosophical study or theory of knowledge and determines what counts for valid knowledge. The key questions in the field of epistemology are 'What is knowledge, and how is it acquired?' or, put another way, 'How do I know the world?' It also asks, 'What is the relationship between the enquirer and the known?* (p. 100).

Understanding how individuals constructed knowledge that they have is therefore paramount in this study as it would clear up some assumptions that one may hold.

Tuli (2010) describe epistemology as posing the following questions: "What is the relationship between the knower and what is known? How do we know what we know? What counts as knowledge?" (p. 99). Since no single reality is created, these multiple realities need to be interpreted with the view of understanding the meaning of ideas (Nguyen, Gardner & Sheridan, 2019; Creswell & Clark, 2018).

In the context of this study and in line with the prescripts of the interpretivist paradigm, the epistemological assumption is that learners construct new knowledge daily, depending on different mathematical concepts that they learn. To understand the deeper meaning of the knowledge constructed by learners, there was a need to interact with individual teachers (Ponterotto, 2005) during the teaching and learning process with the aim of exploring the implications of their (teachers') understanding of learners' errors for mathematics learning. Therefore, it was necessary to engage with teachers and learners to subjectively interpret learners' errors with the aim of gaining insight into the teachers' understanding of learner errors in mathematics learning. The interpretation of learners' errors may be used to develop meaningful strategies to deconstruct incorrect information and improve learner proficiency in mathematics.

3.3.3 Methodology

Daymon and Holloway (2010) argue that methodology is sometimes confused with methods, and hence describe methodology as how knowledge is acquired and methods as techniques used to collect data during this process.

According to Crotty (1998) and Troudi (2010) as cited in Elshafie (2013), methodology is the action plan which informs the choice of method(s) used in the research, whilst methods are strategies or instruments used to collect data, respectively.

Kothari (2004), describes research methodology as an approach to solve the research problem and may be perceived as an investigation of reviewing how research is systematically conducted. Nieuwenhuis, (2016) alludes that methodology comprises of steps that are undertaken by a researcher to collect, analyse, describe data and understand a concept that is being studied. Based on the descriptions above, this study adopted a qualitative approach through case study design where participants were studied within their environment, which is in line with interpretivist paradigm. In the context of error analysis, the participants were observed and interviewed to explore how they construct new knowledge after errors have been analysed.

3.4 Research methodology and design

This section entails the research methodology and research design used in conducting the study.

3.4.1 Research methodology

The two main and most common forms of research are qualitative and quantitative methods (Tuli, 2010). In previous sections, it was indicated that the study was based on the interpretive paradigm and the qualitative research method was adopted.

Edmonds and Kennedy (2017) describe qualitative method as a method usually used to delve into the how and why of systems and human behaviour, with the aim of divulging and perceiving phenomena within a particular context without trying to deduce causality. Qualitative research produces descriptive data based on the experiences and perceptions of the participants (Lewis, 2015; Brynard et al., 2014). Tuli (2010) contends that qualitative methodology aims at understanding the intricacy of the world through participants' experiences and how participants communicate meaning is an ongoing process, which may change daily due to social interaction. This approach seeks to produce data based on participants' experiences with the aim of understanding how they view certain mathematical concepts.

According to Daymon and Holloway (2010), qualitative researchers use an inductive approach that does not seek to hypothesise any theories but works from the premise of understanding how individuals construct knowledge in their context through interactions.

This study adopted the qualitative research approach where I:

- observed teaching in two different Grade 9 schools since the "best way to understand any phenomenon is to view it in its context" (Krauss, 2005, p. 759). The study focused on individual teachers in their classrooms, considered as their habitat, and the intention was not to control the social settings, but rather to enable conditions to develop normally.

- observed how the teachers interacted with learners and how learners interacted with one another during teaching. This was done to understand how individual learners and group of learners make sense of the mathematics concepts studied (Daymon & Holloway, 2010).
- did not interfere with the lessons observed but reflected critically on her role as a research tool/instrument, hence it was reflexive (Daymon & Holloway, 2010; Nieuwenhuis, 2007) in exploring the implications of the teachers' understanding of learner errors for mathematics learning. The learners' behaviour towards learning and teacher's response towards the learner's behaviour could be understood in a specific setting, in this case, a classroom. My presence was therefore necessary in the classroom as an observer during the teaching and learning.
- interviewed the learners after analysing the errors that emerged from the tests. This was done to gain an in-depth understanding of reasons behind those errors (Vogt, Gardner & Haeffele, 2012; Moriarty, 2011).
- interviewed the teachers after error analysis. This was done to check the awareness and their opinion/understanding of learners' errors during error analysis (Vogt et al., 2012).

3.4.2 Research design

The study was guided by qualitative method with a case study design since it was aimed at exploring the *implications of teachers' understanding of learner errors for mathematics learning in the Senior Phase (specifically Grade 9*).

Gerring (2004) asserts that the use of the case study is one of the unique features of the qualitative research paradigm. A case study is described as a method that studies humans, their interdependence and their context, where one can only find out the reality and truth about humans when interacting with them in their contextual environment (Yin, 2017; Harling, 2012; Daymon & Holloway, 2010; Baxter & Jack, 2008; Zainal, 2007). Contrary to quantitative research which first generalises to predict observations through a top-down approach, the qualitative research paradigm uses bottom-up movement by exploring the specific features and conditions of a case, which Creswell and Poth (2016) refer to as "theoretical lens".

This study can be described as multiple case study where data is collected from different sources (Harling, 2012; Daymon & Holloway, 2010; Baxter & Jack, 2008; Zainal, 2007), namely the individual teachers and the learners, from two Grade 9 schools that differ according to their location, rural and urban. The idea was to explore how teachers in different environments analyse learners' errors, how learners in different environments perceive certain mathematical concepts, and the implications teachers' understanding of learners' errors have on learning. Based on the assertion above, the study is exploratory and explanatory.

A multiple case study design allowed me to compare the cases and is reliable (Yin, 2017; Baxter & Jack, 2008). Thus, this study could not be confined to a single case design which according to Zainal (2007) is conducted when there is no possibility of duplication. My assertion linked to multiple case study was that where there is knowledge construction, errors are prone to occur regardless of the environment.

Even though the study is a multiple case study, generalisation cannot be made to a bigger population, but rather to theory at acceptable levels (Baskarada, 2014; Daymon & Holloway, 2010), should there be similarities between the two cases. This is contrary to Noor (2008) and Zainal (2007) where case studies may generally lead to the generalisation of findings to a bigger population. The benefit of this design is that there is enough room to conduct an in depth investigation; however, Yin (2017) and Crowe, Cresswell, Robertson, Huby, Avery and Sheikh (2011) posit that some of the limitations of the case study are collecting large unnecessary volumes of data or too little significant data and that findings cannot be generalised to a bigger population. In trying to mitigate these limitations which may be possible in the case of multiple case studies, the focus was only on data that was in line with research questions (Crowe et al., 2011). Subsequently, in-depth exploration of the cases is crucial, thus the study was not aimed at generalising to a bigger population.

The study followed some of the defining characteristics of the case study, namely holistic, empirical, interpretive (Yazan, 2015). Its *holistic* nature focuses on how teachers and learners relate to errors, it is *empirical* as it relies on my observations, and it is *interpretive* as it is based on my constructivist epistemology. In addition, the study was in accordance with the distinguishing features of the case study, namely particularistic, descriptive, and heuristic (Yazan, 2015). It is *particularistic* as it centred on the teacher’s connections with learners' mathematical errors as a concept to be understood, in a Grade 9 mathematics classroom to be relevant, where the unit of analysis was the awareness and understanding of the learners’ errors by the teacher and the explanation of the errors by the learners. The narratives from classroom observations and interviews, contextual descriptions and learners’ vignettes yield a strong and valuable *description* of the study. The *heuristic* property of this case study lies in the connection between the literature review and the theoretical framework and helps the reader argue and interpret the information to make conclusions.

3.5 Sampling

This section elucidates the selection of the sample and the context of the research sites.

3.5.1 Selection of the sample

Sampling is a procedure employed to select a small group (the sample) with the prospect to discover the characteristics of a large group (the population) (Brynard et al., 2014).

Grade 9 learners of two ordinary public schools from the population of all the public ordinary schools in the General Education and Training (GET) band (Grade R-9) in one of the districts of the Eastern Cape Province, constituted the sample of this study. My choice to concentrate on the Grade 9 class was motivated by the coherent presumption that since Grade 9 is the exit grade in the Senior Phase and Mathematics and is mandatory for all learners in the GET band (Grades 1-9) in South Africa, most concepts and skills prescribed for the phase would have been mastered and that greater variation in learners’ mathematical reasoning would be more persuasive than that of the Grade 8 learners. This would give a broad understanding of the implications of the teachers’ understanding of learner errors for mathematics learning in the Senior Phase. The sampled schools followed the school mathematics curriculum as suggested in the CAPS document and utilise different accredited textbooks and educational materials.

Two different schools were conveniently and purposively selected. The participating schools were chosen through convenience sampling, which is easy, convenient, quick, and cheap (Etikan, Musa & Alkassim, 2016; Fink, 2010). Thus, the sampled schools are in proximity to my work area, and time constraints, and costs would be minimised due to easy accessibility. Given the traits of a multiple case study mentioned in Section 3.4.2, it became paramount to purposively select schools that exhibited similar characteristics to link the design and sampling to avoid a disconnect between the two.

In the rural school, all learners belonged to the same cultural group (Africans) with similar practice (Hodge & Cobb, 2019), which is also referred to as homogenous (Etikan, Musa & Alkassim, 2016). The learners from the urban school were from heterogeneous cultures, comprising of Africans, Coloureds, Indians and other foreign nationals. The sample was therefore independent of race and gender. In South Africa, the general trend is that schools are categorised into primary (Grades R-7) and secondary schools (Grades 8-12). However, there are some schools which have Grade 8 and 9 in primary schools. The Grade 8 and 9 learners of the two sampled schools are in the primary school. Since the sampled schools start from Grade R to Grade 9, the assumption is that there is continuity in the teaching and learning and learners have adapted to the environment.

My choice of a rural and an urban school was based on the desire to study the same phenomenon through different contextual lenses and motivated by the assumption that regardless of the environment, where there is knowledge construction, errors are bound to occur, and that qualified teachers have to have the capacity to deal with learners' errors. This assumption was in line with the conclusion by Riccomini (2005) that the possibility for learners to commit errors is high during the instruction process.

The two criteria for consideration of the two participating teachers were the profile of the teacher in terms of qualifications and mathematics teaching experience, and the language of instruction. Both teachers in the two schools were qualified to teach mathematics. The difference in years of experience in the teaching of mathematics at Grade 9 level for these teachers was 15 years. The sampling did not consider gender and race. Both teachers had two English classes each.

The inclusion and exclusion criteria for selection in this study are listed in Table 5.

Table 5: The inclusion and the exclusion criteria

	Inclusion criteria	Exclusion criteria
School	• English medium public ordinary schools. • Grade 9 learners in Primary schools.	• Private/Independent schools • Grade 9 learners in high schools.
Teachers	Experience of teaching Grade 9 mathematics	Unqualified mathematics teachers
Classes	• Maximum of two Grade 9 classes • Grade 9 classes taught by same teacher. • All Grade 9 learners in the school	• More than two Grade 9 classes • Grade 9 classes taught by different teachers.

3.5.2 Research sites

This section presents the research sites individually. Each presentation is classified according to the profile of the school which includes the school location, the number of teachers and learners, the number of Grade 9 Mathematics classes, and the number of grade 9 mathematics learners per class. Generally, both schools are co-educational public ordinary primary schools from Grade R-9, following the South African school curriculum. The textbooks used in both schools were approved textbooks from the national catalogue, including a state-owned textbook and workbooks.

School A (Urban school)

The school was an urban school located 200m away from the local town. It had an enrolment of 1 173 learners with a total number of 29 teachers including the school principal. Although the school was a multi-cultural school with racial integration of Indians, Coloureds and Africans including Africans from foreign countries, the medium of instruction was English. Even though the school was situated in a Coloured location, more than three quarters of the learners were Africans. The majority of the learners used scholar transport. The school had two Grade 9 classes with an average of 29 learners in each class.

School B (***Rural school)***

This was a rural school situated close to other public primary schools. It had an enrolment of 941 learners with a total number of 24 teachers including the school principal. Although this school was an African school with two home languages, isiXhosa and Sesotho, the medium of instruction in the Intermediate and Senior Phase was English. The school had two Grade 9 classes, each with an average of 36 learners. Very few of the learners from far-away villages used scholar transport.

3.6 Data collection and documentation

The types of data collection techniques for case study design include interviews, observations and documents (Yin, 2017; Crowe, Cresswell, Robertson, Huby, Avery, & Sheikh, 2011; Baxter, & Jack, 2008; Noor, 2008). The data collection procedures that were employed in this study were two lesson observations per teacher, semi-structured interviews for teachers and learners, and two tests of which one was a diagnostic assessment and the other a summative assessment. Both of these tests were analysed qualitatively, however. Before presenting detailed discussions of the data collection instruments, a summary and presentation of the alignment between these instruments, the theoretical lenses, and the research questions are presented in Table 6.

Table 6: Alignment between research questions, data collection instruments and theoretical lenses used in the study

Research questions	Data collection instruments	Theoretical lenses
1. How do teachers analyse learners' errors?	• Documentary sources (learner responses from tests) • Interviews • Observations	Cognitively Guided Instruction
2. What instructional decisions emanate from teachers' understanding of learners' errors?	• Documentary sources (lesson plans) • Interviews • Observations	Cognitively Guided Instruction
3. How do teachers' understanding of learners' errors inform learner conceptual understanding in mathematics?	• Documentary sources (lesson plans) • Interviews • Observations	Constructivism
4. How do teachers' understanding of learners' errors inform learners' procedural knowledge in mathematics?	• Documentary sources (lesson plans) • Interviews • Observations	Constructivism

Given the information in Table 6, it should be understood that although CGI is mainly relevant for Questions 1 and 2, it does not exclude some traits of constructivism. This is similarly applicable to Questions 3 and 4 as it relates to constructivism. The three data collection techniques were perceived as most relevant in enabling me to acquire an in-depth understanding of *the implications of teachers' understanding of learners' errors for mathematics learning in the Senior Phase (specifically Grade 9*). The data collection took place during the first semester (February to May) of 2019.

I played a paramount and key role in the research processes of data collection and data analysis. In this study, knowledge construction concerning the realities of the participants through understanding their (participants) communication and actions, was subjectively involved (Maree & Van der Westhuizen, 2007) and meaning to the data was inductively internalised with evidence (Blaikie & Priest, 2019).

The process of data collection as illustrated in Section 1.8 of Chapter One is elucidated below.

3.6.1 Documentary sources

According to Nieuwenhuis (2007), documentary sources include any document that is linked to the investigation and may shed light on the phenomenon to be investigated. Documents serve to confirm information obtained from interviews and observation (Noor, 2008). In this study, the documents that were used were self-developed tests, lesson plans used during instruction, and learners' responses to the tests in the form of answer sheets or scripts. The aforementioned documents served as the primary source of data due to their originality (Nieuwenhuis, 2007). The lesson plans were used to inform whether errors that emerged from the diagnostic test were built into them (lesson plans) to inform instruction.

Self-developed tests (diagnostic and summative tests) were used to collect qualitative data. These tests were used to answer the first, third and fourth research sub-questions about how teachers analyse learners' errors and how their (teachers') understanding of learners' errors inform learners' conceptual understanding and procedural knowledge in mathematics, respectively.

The objective of the two self-developed tests was to ensure that all the relevant research sub-questions which were meant to address the main research question, were appropriately addressed. The tests were administered in Term One of 2019. There was one test at the end February, which lasted 45 minutes. The other test was during March and was one hour and 30 minutes long. The first test (written on the 28th February in school B and 1st March in school A) was a diagnostic assessment (**Annexure C1**) that tested prior knowledge on Common Fractions which is required to learn Fractional Algebraic Fractions following the Annual Teaching Plan (ATP) as presented in the Grade 9 Term One of the South African Curriculum (CAPS). Test Two (**Annexure C2**) was a summative assessment which included all the topics learnt in Grade 9 Term One as prescribed by the CAPS and included problems involving fractions. The diagnostic assessment was 45 minutes long whilst the summative assessment was 60 minutes long.

Test Two was set up in a way that it could be utilised for progression and promotion requirements for Term one, according to CAPS requirements. This was done to avoid administering two tests in March, one for the research study and the other for progression and promotion requirements. Both tests were moderated before being administered by two curriculum officials, one from Free State Province and the other from North West Province, to ensure their validity, reliability and fairness (DBE, 2011), The tests were piloted to minimise and rectify any unseen errors that might affect the sample (Kinchin, Ismail, & Edwards, 2018).

The tests were invigilated and marked by the participating teachers. I moderated all the learners' scripts for both schools. In test two, the focus was on questions that involved fractions to establish the implications of the teacher's understanding of learners' errors for mathematics learning.

Groth (2017) posits that the effect of instruction cannot be achieved from learners' scores but the way they think and their misconceptions. Thus, clues about, and insights into, learners' mathematical errors and misconceptions cannot be gathered through learner's quantifiable scores, but rather qualitatively. Therefore, the focus of the study was not on learners' achievement, and hence learners' scores were not recorded. The relevance and prominence of their performance revealed the problem in question, however, and will be discussed in the chapter that follows.

The concept and skill assessed in the diagnostic test was calculations with fractions which include: (a) subtraction of common fractions with denominators that are not multiples of each other, (b) addition of squares and square roots of common fractions, (c) subtraction of cubes and cube roots of common fractions, (d) finding fractions of a whole, (e) multiplication of common fractions with same denominator, (f) multiplication of common fractions by a mixed number, (g) division of a whole number by a common fraction, (h) division of common fractions and (i) division of a mixed number by a common fraction. The concepts and skills that were assessed in the summative test were: (a) Solving non-routine unseen problems involving fractions (b) solving algebraic equations involving fractions and (c) solving fractional equations involving exponents (DBE,2011).

3.6.2 Lesson observations

According to Siabe (2012) observation is a data collection method that depends on the data collector's senses without interacting with the participants. It allows for direct exploration of phenomenon in natural contexts and provides information about participants' behavior in natural contexts (Morgan, Pullon, Macdonald, McKinlay, & Gray, 2017). It records data as it occurs from the participants' natural environment (MacMillan & Schumacher, 2010; Noor, 2008).

Exploring the implications of the teacher's understanding of learners' errors for mathematics learning was done through observing the instructional practices of these teachers in their natural context and examining how they dealt with learners' errors. An audio-recorder was used to capture data during lesson observation. The audio-recorded lessons were transcribed to assist in the analysis of data.

As a curriculum advisor and a researcher, I was conscious of the potential pressure this double character could make, "which might result in situations where the researcher has to choose one identity and its associated obligations over the other, in the best interests of the study participant, and perhaps to the detriment of the study itself" (Bloomer et al., 2012, p. 27). I had to choose the identity of a researcher to ensure that the quality of the study was not compromised, by not participating in the classroom activities, as I would normally do under normal circumstances where I visit the school to observe lessons as a curriculum advisor.

The data collection first focused on describing the classroom context as it appeared at the time of observation. I observed the lesson presentations without interfering in the process of lesson presentation. For the purpose of the lesson observation, the lesson observation sheet (*see* **Annexure B**) was developed to gather data on how the teacher addressed the misconceptions identified from the learner responses during the process of knowledge construction, and the extent to which the emerging (unexpected) errors were addressed. This was done to address the second research sub-question, namely: What instructional decisions emanate from teachers' understanding of learners' errors?

The subjective realities of the participants and the individual explanations assigned to mathematics, the teaching and learning of mathematics, and learners' errors were recorded during lesson observations in the format of a questionnaire. Subjective information was recorded in participants' own words without selections and assumptions of importance (Blaikie & Priest, 2019).

3.6.3 Interviews

Easwaramoorthy and Zarinpoush (2006) define interviews as "a conversation for gathering information… involves an interviewer who coordinates the process of the conversation and asks questions, and interviewee (respondent), who responds to the questions" (p. 1). Interviews can be categorised into three types: structured, semi-structured and unstructured. *Semi-structured interviews* are used to collect in-depth information from interviewees in an orderly manner and uses a set of predetermined questions where respondents answer in their own words when clarification has been sought (Doody, & Noonan, 2013; Easwaramoorthy and Zarinpoush, 2006).

The semi-structured interview was most appropriate in this study since it allowed me to collect data on the same phenomena to be studied whilst using a different questioning approach for the different respondents (Noor, 2008) to seek clarity (Doody, & Noonan, 2013). The semi-structured interviews allowed me to interact with respondents differently based on their unique responses and their understanding of mathematical concepts that resulted in errors. One interview conducted was about the general profile of the teachers and their understanding about learner errors and error analysis in general. The second interview was conducted at the end of data collection process to understand the awareness of the errors that learners committed in both tests. However, teachers were aware that the research is centred around the learners' errors in mathematics. The interview schedule, questions, and sequence of questions was similar in both schools.

An interview protocol was conducted for face-to-face interviews with learners (*see* **Annexure A**) on selected items where the majority of them (learners) have committed common errors. In addition, the interviews were conducted to gather an in-depth understanding of how learners construct knowledge meaningfully (Hox & Boeije, 2005). Since the learners' interview questions emanated from the analysis of their (learners') responses as per the test and could not be pre-empted, questions varied from learner to learner.

With learners, interviews sought to gather explanations of their specific errors. With teachers, the interviews (*see* **Annexure D**) sought to gather teachers' understanding of the process of error analysis as well as "teachers' *procedural* explanation in relation to the error, teachers' *conceptual* explanation in relation to the error, teachers' *awareness* of the error" (Sapire, Shalem, Wilson-Thompson & Paulsen, 2016, p. 4). In addition, interviews were conducted to answer the third and fourth research sub-questions, namely, *How do teachers' understanding of learners' errors inform learners' conceptual understanding in mathematics*, and, *How do teachers' understanding of learners' errors inform learners' procedural knowledge in mathematics*, respectively. Interviews were recorded to save time and avoid missing some information when writing for the purpose of analysis whilst listening to the respondents.

Similarly, as in lesson observation, the subjective realities of the participants and the individual explanations assigned to mathematics, the teaching and learning of mathematics, and learners' errors, were recorded during lesson interviews, in the format of a questionnaire. Subjective information was recorded in participants' own words without selections and assumptions of importance (Blaikie & Priest, 2019).

3.7 Data analysis and documentation

Muyeghu (2008) views data analysis as a stage of describing data meaningfully. This stage of research requires one to be open to both positive and negative explanations (Muyeghu, 2008) because with the interpretivist paradigm the experiences of the participants are paramount. I had to understand their (participants) experiences and interpretations thereof through their eyes and voices. Data analysis gives the significant meaning of the results of the research (Flick, 2013). Data analysis in this study involved three stages, namely (a) organising and understanding, (b) reducing data through coding, and (c) interpreting and representing data (Maxwell, 2012).

Stage 1: Organising and understanding

In accordance with learners' name in the mark sheets, each learner was designated by a code**.** For instance, learners from the rural school were marked using code R1 to R72 which translated to R for rural and 1 for learner entry in the mark sheet. Similarly, learners from the urban school were designated using codes U1 to U58. The copies from the diagnostic test and summative test for each learner were stapled together according to the same codes. The teachers were assigned pseudonyms, namely Tozy and Nozy, for the rural and urban schools, respectively.

For the learners who were interviewed, the recordings were also labelled according to the code of the learner, and so were the teachers' recordings.

Stage 2: Reducing data through coding

According to Nieuwenhuis (2007) coding is "marking of the segments of data with symbols, or descriptive words or unique identifying names" (p. 105).

The learner responses were firstly analysed and categorised as correct (all steps calculated correctly), partially correct (some steps not correctly calculated), incorrect (all steps calculated incorrectly) and did not write (question not attempted). The learner responses with partially correct and incorrect responses were further analysed as they were the ones form where errors or misconceptions emerged.

The students' misconceptions and errors were coded using priori coding according to types of errors that were identified in the literature, namely conceptual, procedural or application, careless or calculation, factual or symbolic and encoding errors (Dlamini, 2017; Brown et al., 2016; Riccomini, 2016; Khalo & Bayaga, 2015; Mohyuddin, 2014; Elbrink, 2008) were grouped into systematic and unsystematic (Muthukrishnan et al., 2019; Ricommini, 2005).

Inductive analysis involved categorising the learners' errors according to the themes that emerged from different categories. Coding the data assisted in working with the voluminous amount of data.

Stage 3: Representing and Interpreting data

Interpretation and representation involved reporting on the findings of the data collected according to categories that were used and themes that emerged. The research questions were answered during the interpretation of data.

3.7.1 Analysis of data collected through tests

The teachers conducted error analysis on learner responses to explore their understanding thereof. I also conducted error analysis on the same learner responses in order to, firstly, decide on the teachers' and learners' interview questions, and secondly, explore and get a first-hand understanding of learners' procedural and conceptual understanding. The data analysis was aimed at addressing the sub-questions as stated earlier on, in data collection and to explore the implications of the teacher's understanding of learner's errors for mathematics learning. The errors found were tagged according to different categories for representing the data and for the purposes of developing questions that would be used during learners' interviews. Error analysis was conducted to check the extent to which incorrect procedures and inappropriate procedures were used. Even though the summative test covered all the concepts and skills that were supposed to be covered in Term 1 and was meant to contribute towards the School Based Assessment (SBA), the focus of the data was only on questions that involved fractions, specifically addition of fractions involving square roots and solving equations involving fractions. Learner interviews were conducted to confirm whether the errors arose from the deficiencies mentioned in thee previous statement and to gain an in-depth understanding of those errors.

3.7.2 Analysis of data collected through observation

The analysis focused on, *inter alia*, the following categories: the extent to which the errors identified before were addressed by the teacher, the promptness and the extent to which the emerging (unexpected) errors were addressed by the teacher, the classroom context, and the extent to which error analysis permeated learning. In other words, the implications of teachers' understanding of learners' errors as identified through his or her analysis of the learner responses from the test was explored in-depth using the aforementioned categories.

3.7.3 Analysis of data collected through the interviews

Data from the teacher interviews was analysed according to teachers' *awareness* of the error and *procedural* and *conceptual* explanations in relation to the error (Sapire et al., 2016). In other words, the implications of teachers' understanding of learners' errors as identified through their (teachers) analysis of the learner responses from the test were explored in-depth during the interviews. This was done to understand teachers' procedural and conceptual explanation of learners' errors after they (learners' errors) have been identified.

Data from learner interviews were analysed to gather elucidation relating to the submitted errors, either for corroboration, or to gain an in-depth understanding thereof. In other words, the learner interviews were conducted to confirm the reasons behind the conceptual and procedural errors that the they (learners) committed.

3.8 Quality criteria

Qualitative research is based on trustworthiness. According Shenton (2004), the issues of credibility (truthfulness of the data), confirmability (correctness of the data), dependability (consistency of the findings) and transferability (the applicability of findings) should be the main concerns in conducting trustworthy qualitative research.

Credibility

According to Nieuwenhuis (2016), credibility can be ensured through the use of well-established research methods, research designs that are relevant to answer the research questions, and the theoretical foundation related to the study. The study used multiple-case design where the research venture was strengthened using triangulation through a self-developed test (and learner responses), lesson observations, and semi-structured interviews, to ensure congruency of the findings with reality.

Confirmability

Cope (2014) describes confirmability as the ability of the researcher to prove that there were no signs of bias or misrepresentation of data due to his or her own experience when representing data. Akaranga and Makau, (2016) and Resnik (2011) posit that truthful acquisition of knowledge can be promoted by avoiding possible errors arising due to providing false information, fabrications, and misinterpretation of information. To ensure the correct interpretation of data, the data collected was verified with the participants (Nieuwenhuis, 2016, Cope, 2014). In reporting qualitative research, this can be exhibited by providing rich quotes from the participants that depict each emerging theme. To exhibit the findings of the data, quotes from the participants that illustrate emerging themes were provided where necessary, without compromising confidentiality and anonymity.

Transferability

Transferability refers to meaningful association of the results of the study to individuals who were not involved in the research (Cope, 2014). To ensure transferability of results, the results were verified with other teachers who were not involved in the study, to establish the similarity of their experience as regards understanding learners' errors compared to that of the data collected (Nieuwenhuis, 2016).

Dependability

Shenton (2004) describes dependability as a process of providing a detailed methodological description of the study to permit future researchers to repeat the investigation. To ensure dependability, a qualitative case study was conducted through a multiple case design with two Grade 9 schools within the same district from the Eastern Cape province. Sampling was convenient and purposive. Since the study aimed at exploring the implications of teachers' understanding of learner errors in mathematics learning, the sources of data that were used in this study were self-developed test (and learner responses), lesson observation, and semi-structured interviews. Lastly, data analysis was conducted according to the categories from the literature.

3.9 Ethical considerations

Ethical consideration in research involves treating the participants and their data with respect (Resnik, 2011; Hofman et al., 2018). The following ethical practices were considered:

Permission was obtained from the Ethics Committee at the University of Pretoria as well as the Eastern Cape Department of Education (ECDoE) and the district director from where the study was conducted, as well as the principals of the two schools, and the participants, who were Grade 9 mathematics teachers and learners. For the learners, permission was granted by the parents, since the learners were minors. An application and request to conduct research were submitted after the proposal was successfully defended at faculty level and before the data was collected (*see* **Annexures E1 to E5).** The schools were first approached telephonically to conduct research. Grade 9 teachers were also approached to discuss the nature of the study and to ask for permission for the learners to participate in the research.

After the letter of permission was received from the ECDoE (**see Annexure F**), meetings with individual schools were held to explain the purpose and the nature of the research. Also, the meetings were held with participants to establish trust and confidence in the study (Getz, 2002) by explaining the role of each participant.

According to Getz (2002), informed consent is a decision-making process on any investment (beneficial) programme made by a client after having a clear appreciation and understanding of the facts and consequences of the programme.

Participants were presented with letters of informed consent to sign after asking questions for clarity (see **Annexures G1 to G2**). The letters explained the research and highlighted the fact that their involvement was voluntary. Parents gave consent (*see* **Annexure H)** on behalf of their children and the learners consented to participate in the study.

Participants were assured that they would remain **anonymous** and the information would be kept **confidential**, and that they would not be exposed to harm. For confidentiality and anonymity, pseudonyms were assigned to each participant (teachers and learners) and schools. Even though it was impossible for the participants to remain anonymous to me (Saunders, Kitzinger & Kitzinger, 2015), participants' privacy was not compromised. As the audio recordings were not to be accessed by the public, they were encrypted to ensure anonymity. The data remained the institutional property of the University of Pretoria and was handed over for safe-keeping.

Notwithstanding that, the educators reiterated to the learners my presence in the classrooms and assured the learners of the protection of the confidentiality of any information they share. Participants were constantly reminded of the essence of optional involvement and that they could decide to withdraw from the study at any time, if they wanted to (Konza & Cowan, 2012).

4 CHAPTER 4: PRESENTATION OF RESEARCH FINDINGS

4.1 Introduction

The purpose of this chapter is to present the findings that emerged from the data collected from documentary sources, lesson observations and interviews. The findings revealed that learners solved the questions incorrectly and partially. Their incorrect and partially correct solutions further revealed that most learners committed conceptual and procedural errors. The findings also revealed that the way teachers taught did not seem to address the errors that were committed by learners in the diagnostic test. The instruction was thus not in line with learners' errors.

The first part of this chapter presents the findings from the data collected from teachers through observation, interviews and the error analysis that they (teachers) conducted. This includes the summary of teachers' profile and the actual classroom practice gleaned through lesson observations. The second part presents the findings emanating from the data collected from the learners. This includes data gleaned through the tests and interview and will be presented in an integrated manner. Even though the study is qualitative in nature, there are instances where I have used graphs specifically to capture the overview of the findings emanating from the analysis of the tests and not for statistical purpose as is the case with quantitative data.

4.2 Presentation of findings of data collected from teachers

The data collected from the teachers through observations and interviews focused on the teachers' background and classroom practice in general and what transpired from the interviews about errors and error analysis. In other words, the implications of teachers' understanding of learners' errors as identified through his or her analysis of the learner responses from the test was explored in-depth.

4.2.1 Synopsis of the profiles of the two participant teachers

In this section, a resume of the participant teachers in terms of their qualifications, experience and language is presented. Understanding the teachers' background and profile is essential as it will assist in understanding the findings from a particular context.

Both teachers were sufficiently qualified to teach Grade 9 Mathematics. Tozy, a Sesotho speaking teacher had a Secondary Teachers Diploma (STD) and was teaching in a rural school (school B) with learners from African culture. Nozy, an isiXhosa speaking teacher had an Advanced Certificate in Education (ACE) and was teaching in a multicultural urban school (School A). Sesotho and isiXhosa are two of twelve official languages (including sign language) in South Africa. The Language of Teaching and learning (LoLT) in both schools was officially English First Additional Language (EFAL). Since the home language for Tozy was Sesotho, code switching to any home language was not a challenge. All learners understood both isiXhosa and Sesotho. Code switching was not an option for Tozy since not all learners understood isiXhosa or Sesotho. Both teachers were female. Although both teachers are experienced teachers, Tozy had more teaching experience (32 years) than Nozy (17 years).

4.2.2 Teachers' practice gleaned from lesson observation

This category focused on data collected from lesson observations. The data analysis focused on the following categories: the classroom context as it appeared during observations, the extent to which the previously identified errors were addressed by the teacher, and the promptness and extent to which the emerging (unexpected) errors were addressed by the teacher.

Classroom context

The learners in both schools were seated in pairs and in columns, and they would work as individuals, in pairs and in groups when given work to do. Classroom management was satisfactory for both teachers as learners were disciplined. In both schools, teachers were innovative to ensure that learners had the necessary material for mathematics learning to take place. For instance, learners from Tozy's school shared the textbook in pairs whilst Nozy made copies for each learner on each activity to be done per day, since there were not enough textbooks for sharing.

The textbook played an elementary role in both schools as the mathematical problems that were solved by learners were mainly chosen from the textbooks. Tozy referred learners to the textbooks whilst Nozy referred them to the copies she had made. Tozy used more than three approved SA textbooks for reference, and some of the examples of classwork or homework given to learners would be taken from those books, including the state-owned textbook, but for classroom activities, learners commonly used the textbook which they were sharing. Since Nozy was making copies for each learner, the work to be done by learners per day was taken from more than eight approved textbooks that she used for reference. The textbooks that were used in both schools adhered to the requirements of the–Curriculum and Assessment Policy Statement (CAPS) for Grade 7 - 9, a policy that outlines the content to be taught in SA.

The learning processes and lesson preparation

Both teachers did not administer baseline tests on topics to be taught. Instead, they used oral questions based on previous knowledge of fractions to introduce the new topics. During the fieldwork at School A, Nozy's school, the topics under discussion were functions and relationships and fractions, whilst at School B, Tozy's school, the topics under discussion were common fractions and algebraic expressions.

In both schools, the duration of mathematics periods were four hours and thirty minutes per week, as prescribed in the CAPS.A degree of equivalence was noted in the segmentation of the mathematics period. The segmentation entailed the introduction of the lesson and teaching of the subsequent topic, including the written classwork, followed by the homework. Nozy also included a consolidation segment in her classes through questioning, and as a result, unlike Tozy, she rarely marked learners' classwork books. Tozy allowed learners to write their solutions on the board whilst Nozy would write the solutions herself as she was giving feedback to the learners with their involvement as a group.

Both teachers prepared lesson plans. Nozy planned her lessons and Tozy used the lesson plans provided by the Department of Education. However, the lesson plans did not indicate remediation of the errors that learners committed as gleaned from the diagnostic test. The diagnostic test in this study was administered to all the grade 9 learners in both schools to gauge their conceptual understanding on fractional computations. The diagnostic test focused on computations with fractions because fractions can be integrated with almost all the topics to be studied in the Senior Phase (Grade 7 – 9) Mathematics curriculum in South Africa (SA).

In both schools, the findings during lesson observation revealed that a new topic was introduced by stating the appropriate heading and was recorded on the board for learners to do the same in both schools. Regarding the previous knowledge, Tozy believed that the previous knowledge that learners needed to learn algebraic expressions were integers and common fractions as they were done in Grade 8. Therefore, the lessons she used from the Department of Education considered these concepts. Similarly, Nozy used common fractions as previous knowledge to teach functions and relationships.

Teachers' interaction with learner errors

Interesting trends were observed in terms of the teachers' questioning. Both participants asked questions aimed at recalling knowledge and applying knowledge, but adequate evidence of probing learners' thinking was not perceived to be evident. The participating teachers assessed learners informally on a regular basis. However, asking questions to obtain clarity about learners' conceptions and understanding was occasionally observed. When the solution was incorrect, both teachers would occasionally ask follow-up questions from the learners, and in most cases, teachers would ask other learners to give their alternative responses. Learners raised their hands to indicate that they had different solutions. In a case where learners would be observed raising hands even after the correct solution had been given, Nozy would give those learners a chance to give their opinion, which was followed by questions to the respondent until an agreement was reached. Tozy would proceed to the next question by appraising the responded with words like "*good or correct*". Figure 1 presents a problem that was solved by learners in Tozy's class. One group, the Snakes, presented their solution to $\frac{9}{10} \div \frac{3}{50}$ on the board. Snakes correctly made the denominators to be 50 by multiplying $\frac{9}{10}$ by $\frac{5}{5}$ and they correctly got $\frac{45}{50} \div \frac{3}{50}$. However, the final answer was erroneously written as $\frac{15}{50}$. The group correctly divided the numerators but did not divide the denominators.

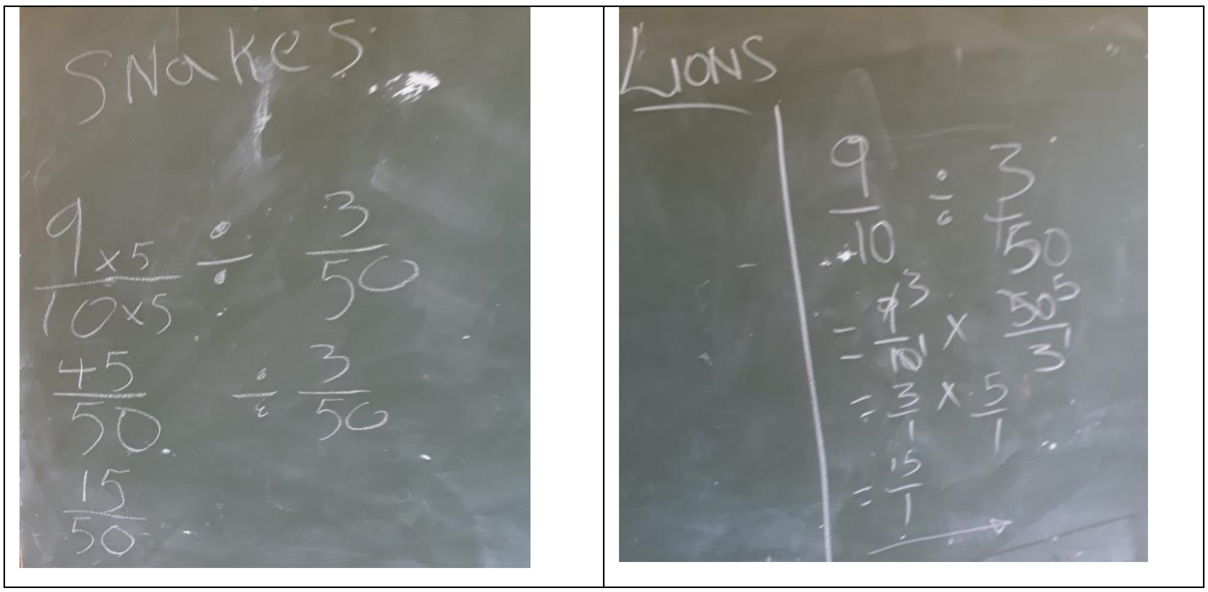

Figure 1: Learners' different methods of calculation

Evidently, these learners were influenced by the rule of adding or subtracting the fractions and hence got $\frac{15}{50}$. These learners were not engaged to explain their thinking, and instead another group, the Lions, was asked to present their solution. The Lions correctly carried out the procedure of dividing fractions by changing the division sign to multiplication and then inverting the second fraction to get 15 as a solution.

Since both teachers rarely asked learners to explain their solutions, they (teachers) constantly explained the concepts by writing corrections on the board after learners failed to respond to questions.

The findings revealed that both teachers were aware of the challenge that almost all learners faced when it came to fractions in general. During the lesson observations on functions and relationships and algebraic fractions, the teachers introduced the lessons with problems involving calculations with fractions. However, as indicated earlier on in this section, lesson plans did not address the errors, and both teachers ran through the problems without actually addressing the errors. Teachers asked questions from learners, and if the learner answered incorrectly, another learner would be asked to respond. Teachers wrote solutions on the board according to correct learners' responses or allowed learners to write solutions on the board.

There were similarities in the disposition towards learners' mathematical contributions, including learners' suggested alternatives to solving problems from the participating teachers. Similarly, the responsibility for the mathematical evaluation of learners' contributions seemed to mostly reside with the teacher, as learners' contributions were conditionally accommodated. A dissimilarity observed was that in Tozy's class, learners were not free to respond to questions, but in Nozy's class learners were once observed cross questioning one another.

One of the findings promoting interaction among learners and between learners and teacher Nozy, was a Grade 9 WhatsApp group, where learners assisted one another with homework from 18:00 to 21:00 from Monday to Friday. Nozy was observed constantly reminding learners to discuss mathematically related issues in the WhatsApp group.

4.2.3 Presentation of findings collected from teacher interviews

Data from the teacher interviews was analysed according to teachers' *awareness* of the error, as well as their *procedural* and *conceptual* explanation in relation to the error (Sapire et al., 2016).

During interviews, both teachers seemed to understand the importance of error analysis for planning, teaching and learning, and assessment.

Nozy associated the learners' errors with misconceptions, whilst Tozy associated them with carelessness as given in her interview transcript below:

> *Some of them are careless mistakes that they themselves are able to correct. For instance, the learner who read the question to be 4+6=, instead of 4×6=, but still managed to write the answer to be 24, and then on the next question instead of writing 4+6=10 his answer is 4 + 6= 24.*

Based on Nozy's response that learners' errors were caused by misconceptions, she further said that when learners seemed not to understand a concept, she would re-teach the concept. Tozy's response was that learners needed to practice a lot. Nozy said that she did not always have time to address learners' errors, since she had to complete the syllabus, whereas Tozy announced that she always did revision with learners. Below is the interview transcript that transpired from the interviews with Nozy:

Teacher:	Yeah and another thing about error analysis, sometimes you… you get the errors, then it shows you that oh haye that I have…my children are having errors on this thing, that means I…I need to...to go back and teach this, then you don't have time for that. You need to go along with the syllabus and the time, what is it called…the…?
Researcher:	Annual Teaching Plan
Teacher:	Yes, the plan, you have to go with the plan because now it's taking you back, maybe to previous grades work, you see? and you don't have time for that, that's another thing about error analysis.

Nozy seemed to have noticed all the errors that emerged from the learners during the test and said that she would correct them if she understood them. Nozy confessed that sometimes she found it difficult to explain a concept that seemed to be a misconception to learners. She gave an example of an error where the learner inverted the first fraction instead of the second fraction. She alluded that she did not know how to explain to the learner why it had to be the second fraction that must be inverted and not the first one.

Nozy's extract from the interviews below supports the previous statement (words in brackets are my translation into English):

> *When dividing fractions, others do not understand the concept of reciprocal, they… they don't understand which one should they change, others they change the first fraction not the one… in fact I don't even know how to.. to. to… (teacher sighs). I know that when you change the first fraction you get an incorrect answer, but I don't know why it is like that. Mhmm (thinking), then I can't explain to them why. I know that it's going to be wrong, but I don't know why it is wrong, yabona? (you see?) so I can't explain to them why you have to change the second one, you see?*

The findings revealed that both teachers marked the learners incorrectly in Figure 2 as the learners did not use the strategy of inverting the second fraction to divide fractions.

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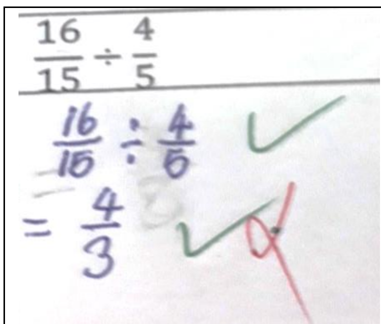
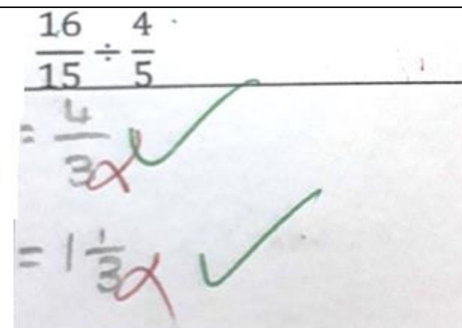
<tr><td>
</td><td>
</td></tr>
<tr><td>Nozy (Learner U54)</td><td>Tozy (Learner R58)</td></tr>
</table>

Figure 2: Learner answers from different schools

It also transpired from the interviews that teachers were not exposed to some strategies of solving problems. One learner was marked incorrectly because the learner did not use the strategy of changing the division sign to a multiplication sign and inverting the second fraction to get the answer when solving $\frac{16}{15} \div \frac{4}{5}$, i.e. $\frac{16}{15} \times \frac{5}{4} = \frac{4}{3}$. Instead, the learner showed the knowledge of dividing fractions from previous grades where the learner divided 16 by 4 and 15 by 5 to get $\frac{4}{3}$ as shown in Figure 2.

Researcher: There is one of the questions where the learner…. I don't even know if it's U22, where the learner divided the fractions without changing anything.

Teacher: But sometimes, they divide…they divide the numerators and divide the denominators

Researcher: What's your take on that?

Teacher: I still think it's the same thing that they…they… I think the concept…hayi (no), I don't know, because I can't say that the concept which they know is multiplication, it means they don't know it kwa (for) the fact that they use that strategy on all of the fractions even if they add, they add the numerators and the denominators together. If they multiply, they multiply the denominators and the numerators. If they divide, they also do the same thing. So, I don't know whether I would say they got the concept or not if they're doing it like that.

Researcher: But some, you find that when they…they are dividing, let's make an example. I'm not seeing this learner now. Let's make an example where the learner divides, let's say $\frac{16}{15} \div \frac{4}{5}$ then they divide both those… (Thinking)

Teacher: But it works for that one

Researcher: It works fine.

Teacher: It works for some.

Researcher: It works for some not for all? So, would you say that strategy is not correct to be used?

Teacher: It's not correct because it's not working for all of them. So, it's only working for some of the fractions. There may be an investigation…an investigation that can check where does that approach work, might help.

Another finding was that full marks were often not awarded to the learner. This could be attributed to the fact that the learner used a strategy different to the one that was taught, namely setting the denominators the same to eliminate them (denominators). Instead, the learner used a different strategy to solve the problem. Figure 3 shows how the learner solved the problem.

Figure 3: An example of how a learner solved the problem

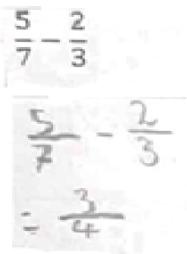
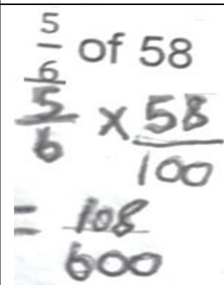
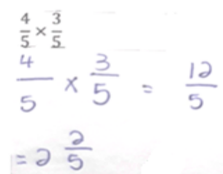
During the interviews, both teachers stated that when they noticed that the learner held a misconception about a concept, they would assist the learner. This was sometimes, but not always, during the classroom observations where they would ask the learner a follow up question.

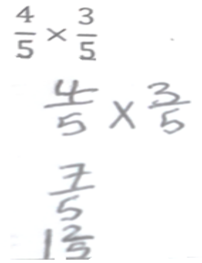
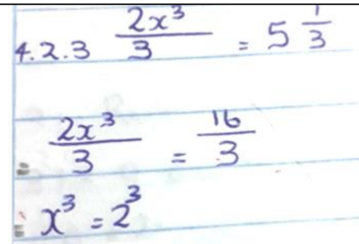
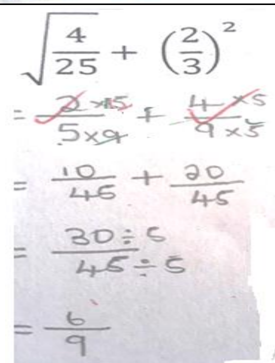
4.3 Presentation of the findings collected from the learners

The data was collected from learners through the diagnostic test and the summative test (both analysed diagnostically), and the interviews. As mentioned earlier, in Section 4.2.2, the diagnostic test only focused on calculations with fractions and was administered at the end of February. Even though the summative test covered all the concepts and skills that were supposed to be covered in Term 1 and was meant to contribute towards the School Based Assessment (SBA), the focus of the data was only on questions that involved fractions, specifically the addition of fractions involving square roots and solving equations, including exponential equations, involving fractions. The reason for such focus was to check the extent to which the errors that emerged from the diagnostic test were addressed, as well as the effect of error analysis on learning. Findings that emerged from the tests and interviews will be presented in an integrated manner as the learner interviews were conducted to gather an explanation relating to the emerged errors, for corroboration, or to gain in-depth understanding thereof.

As presented in Chapter 3 of Section 3.6.1, the tests mainly addressed different aspects of common fractions, especially operations with fractions, including squares, cubes, square roots and cube roots of fractions, and solving algebraic equations involving fractions. The learners' misconceptions in the tests were grouped into the following categories and themes:

Table 7: Categories of learner misconceptions and errors

	Category	Theme	Example	Description and researcher's comment
Systematic	Conceptual errors	Treating fractions as whole numbers		Fraction parts were treated as whole numbers ***Example***: To subtract $\frac{5}{7}-\frac{2}{3}$, the learner subtracted 2 from 5 to get 3 and 3 from 7 to get 4, and then presented the answer as $\frac{3}{4}$.
		Misunderstanding of the principles of the concept		These involve complete lack of understanding of the concepts and principles required to solve a problem. ***Example***: To solve $\frac{5}{6}$ of 58, learner wrote 58 as $\frac{58}{100}$. From there, the learner treated 5 as t50, then added 50 and 58 to get 58 and multiplied the denominators to get 600, thus wrote the answer as $\frac{108}{600}$.
	Procedural or application error	Misunderstanding and misapplication of procedures		This involves partial application of the correct rule to a question. These errors are serious but indicate that the student possess some of the knowledge required ***Example***: When solving $\frac{4}{5}\times\frac{3}{5}$. The learner correctly multiplied the numerators to get 12 and wrote the common denominators.

	Symbolic or factual errors	Misidentification of sign or lack of basic facts		This involves learners who misidentifies signs due to associating it with what it implies. Also, this category includes learners who lack basic facts. ***Example***: When solving $\frac{4}{5} \times \frac{3}{5}$. the learner added 4 and 3 to get 7 and wrote 5 as a common denominator. The learner was influenced by the understanding that multiplication means repeated addition.
	Encoding error	Apply correct procedure but fail to come to a solution		This involves learners who correctly apply the procedure but do not understand how to continue to reach the solution ***Example***: The learner correctly changed $5\frac{1}{3}$, to $\frac{16}{.3}$.In addition the learner correctly applied the multiplicative inverse to get $2x^3 = 2^3$ but failed to come up with a solution.
Unsystematic	Careless or calculation error	Slips or careless mistakes		This involves careless mistakes or slips that are sometimes related to lack of paying attention to basic facts. ***Example*** When solving $\frac{16}{15} \div \frac{4}{5}$, the learner carelessly wrote $\frac{5}{3}$ instead of $\frac{4}{3}$

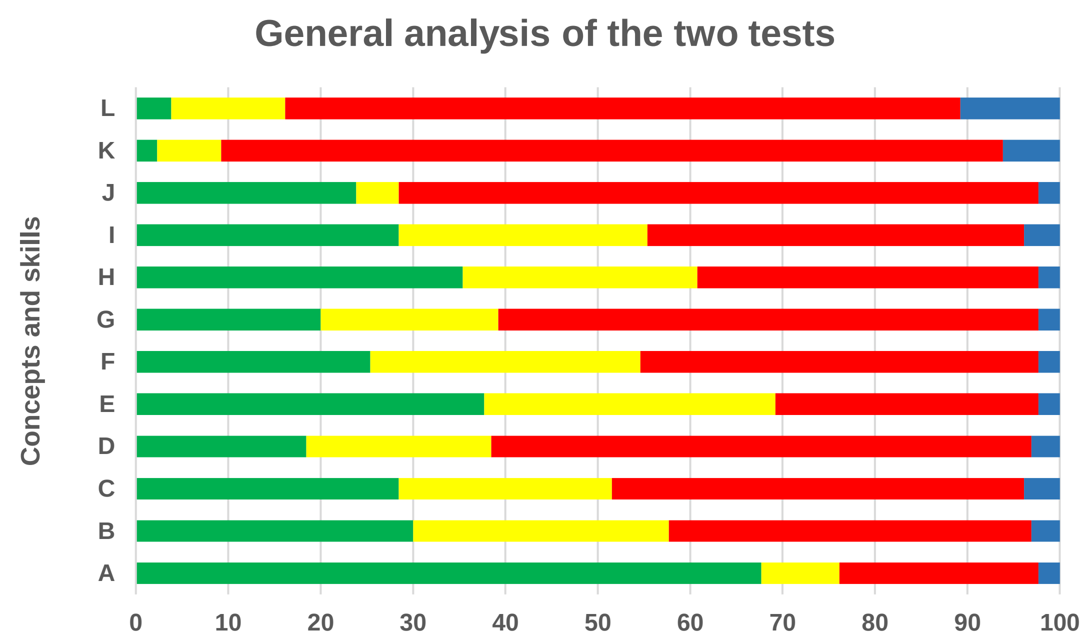
As mentioned earlier in Section 3.7, the learners' scripts were initially analysed as correct (C), partially correct (PC), incorrect (I) and did not write (DNW). After this initial analysis, responses that were correct (C) and those where the learner did not write anything (DNW) were excluded. The focus was on partially correct (PC) and incorrect (I) responses. The responses were then analysed and categorised according to the categories in Table 7.

The concepts and skills assessed were assigned different letters of the alphabet for easy interpretation of the graph as indicated in Table 8. Each letter of the alphabet represents the corresponding concept and skill that featured in the tests.

Table 8: Table indicating the concepts and skills that were assessed in both tests

TEST	LETTER	CONCEPT AND SKILL
DIAGNOSTIC	A	subtraction common fractions with denominators that are not multiples of each other
	B	Addition of squares and square roots of common fractions
	C	Subtraction of cubes and cube roots of common fractions
	D	Finding fractions of a whole
	E	Multiplication of common fractions with same denominator
	F	Multiplication of common fractions by a mixed number
	G	Division of a whole number by a common fraction
	H	Division of common fractions
	I	Division of a mixed number by a common fraction
SUMMA TIVE	J	Solving non-routine unseen problems involving fractions
	K	Solving algebraic equations involving fractions
	L	Solving fractional equations involving exponents

As mentioned in Section 4.1, the findings of the data will also be presented graphically to give a clear picture of the overview of the findings that emerged from the analysis of the tests. Figure 4 shows the findings from general analysis in terms of number of learners who did not write (DNW) the question and number of learners who responded to the questions correctly (C), partially correctly (PC) and incorrectly (I), in both tests.



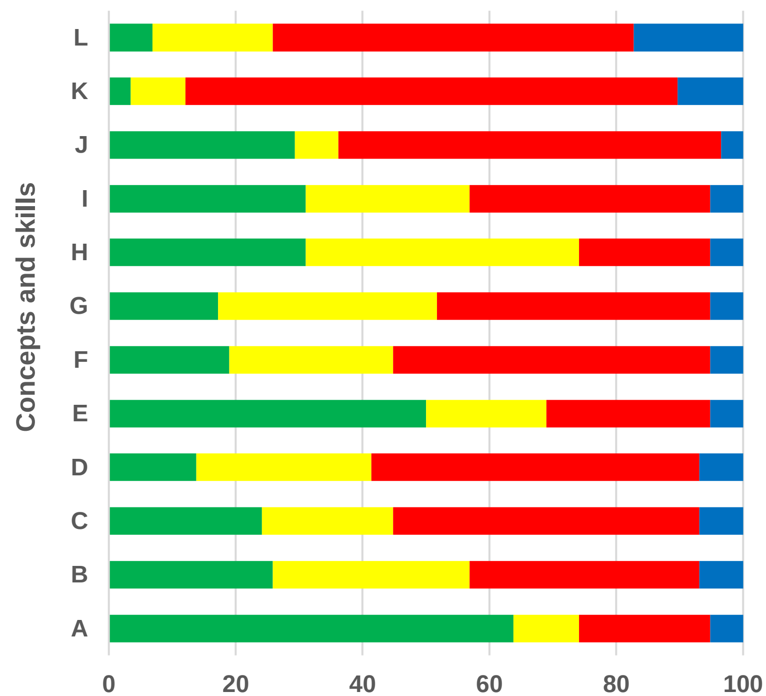
	A	B	C	D	E	F	G	H	I	J	K	L
C- %	68	30	28	18	38	25	20	35	28	24	2	4
PC- %	8	28	23	20	32	29	19	25	27	5	7	12
I- %	22	39	45	58	28	43	58	37	41	69	85	73
DNW- %	2	3	4	3	2	2	2	2	4	2	6	11

Figure 4: Graph showing general analysis of the two tests

Figure 4 shows that most learners solved the problems incorrectly (I) and partially correct (PC), especially when solving algebraic equations involving fractions in the summative tests. The findings revealed that in almost all the questions, there are learners who did not write (DNW) the question. Very few learners answered the questions correctly (C) without errors, especially the questions involving algebraic equations.

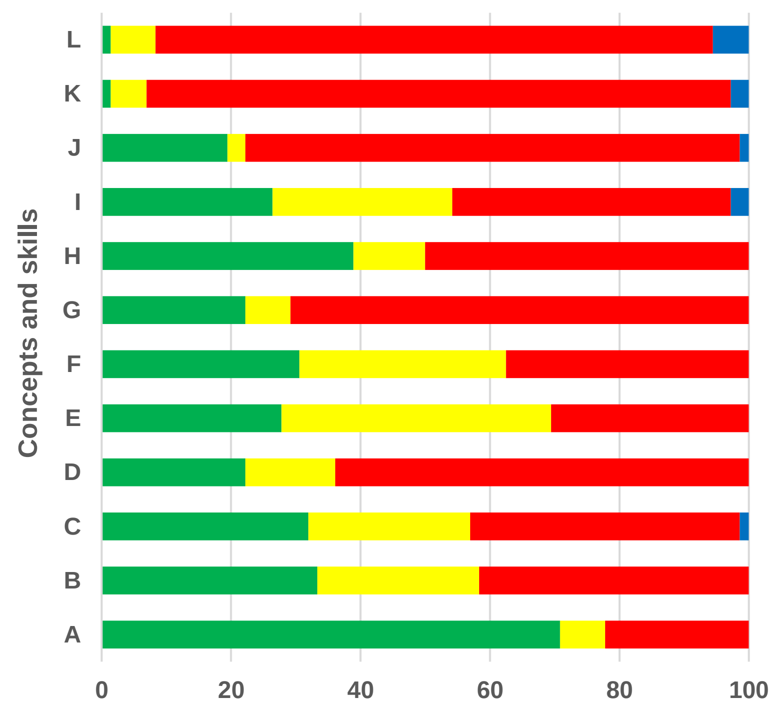
Whilst the data findings from the analysis of learners' scripts from both schools show no notable difference in performance when comparing schools by location, the findings revealed that, unlike at the rural school, not all learners attempted all the questions at the urban school. Even though, as indicated earlier in Section 4.2.2, learners from the urban school were free to ask questions in class, the findings revealed that these learners committed more errors in the diagnostic test than the learners from the rural school. However, in the summative test, the opposite was the case. Figure 5 illustrates the data presented in Figure 4 by location.

General analysis of the tests - urban school



	A	B	C	D	E	F	G	H	I	J	K	L
C- %	64	26	24	14	50	19	17	31	31	29	3	7
PC- %	10	31	21	28	19	26	34	43	26	7	9	19
I- %	21	36	48	52	26	50	43	21	38	60	78	57
DNW- %	5	7	7	7	5	5	5	5	5	3	10	17

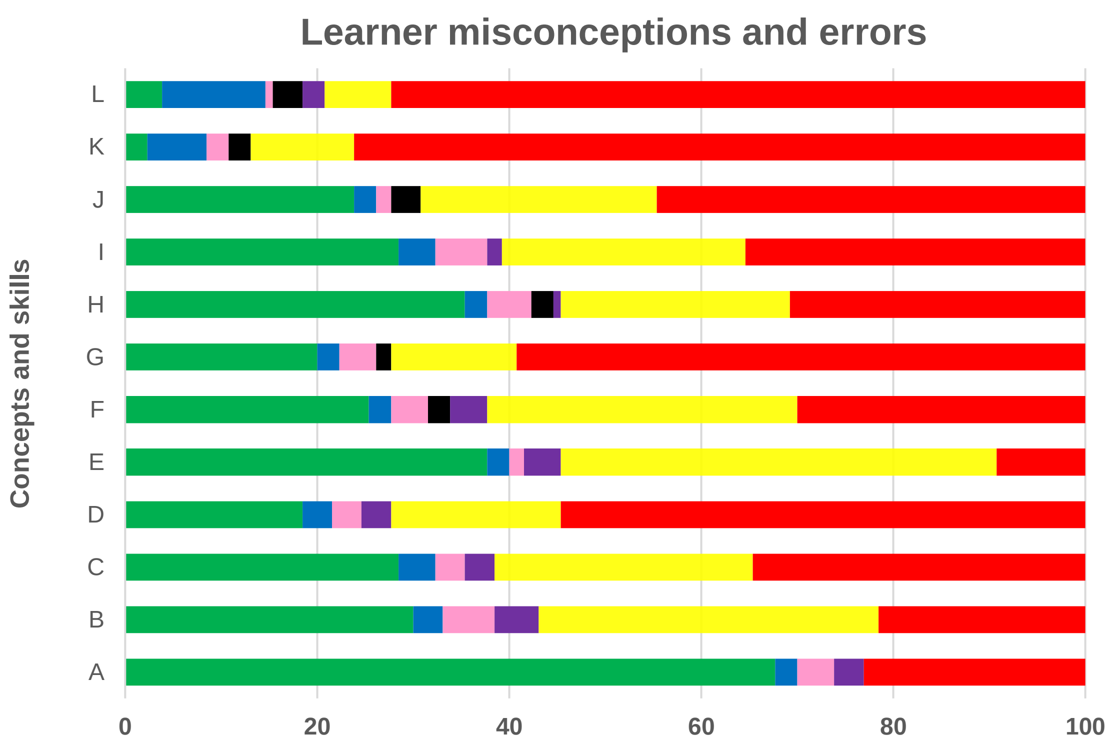
General analysis of the tests - rural school



	A	B	C	D	E	F	G	H	I	J	K	L
C- %	71	33	32	22	28	31	22	39	26	19	1	1
PC- %	7	25	25	14	42	32	7	11	28	3	6	7
I- %	22	42	42	64	31	38	71	50	43	76	90	86
DNW- %	0	0	1	0	0	0	0	0	3	1	3	6

Figure 5: Graph showing general analysis of the tests by location

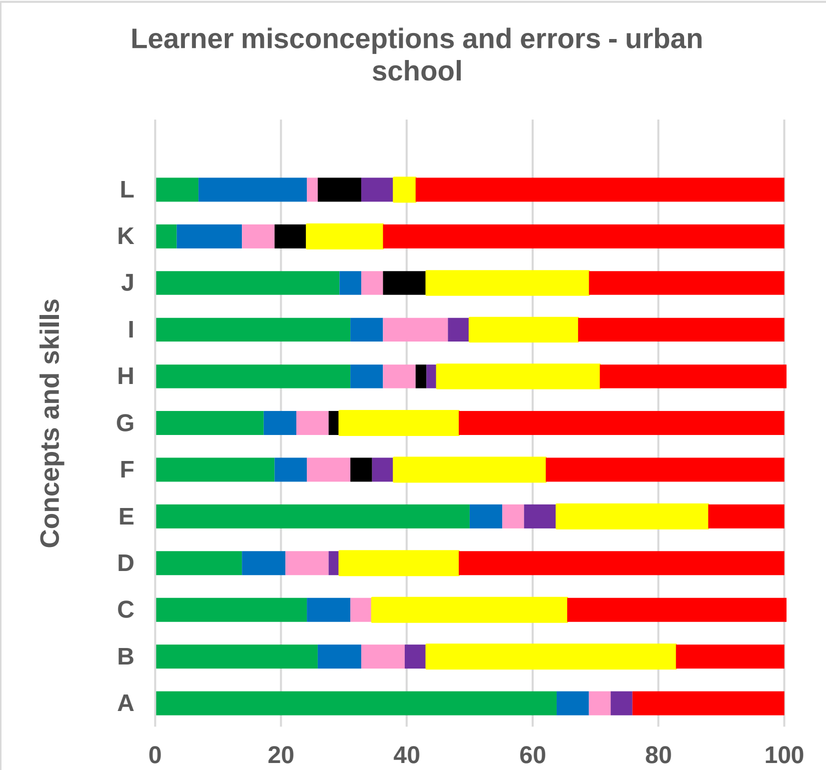
Learners displayed errors in almost all the questions. Some learners displayed more than one type of error within a problem. When a learner displayed more than one type of error, the more intricate error would determine the error category. As indicated in the previous section that learners' scripts were further analysed according to categories as presented in Tables 1 and 7, Figure 6 represents the findings of the data in terms of categories, including the number of learners who got correct responses and who did not write the question. For the purpose of this study and for easier interpretation of the graphs in Figures 6 and 7, the errors were indicated as conceptual (CO), procedural (P), factual encoding (F), encoding (E) and careless error (CA). The graph would also indicate the number of learners' correct responses (COR) and the number of learners who did not write (DNW) the question. Figure 6 shows the number of learners who committed different types of errors in the diagnostic test and the summative test.



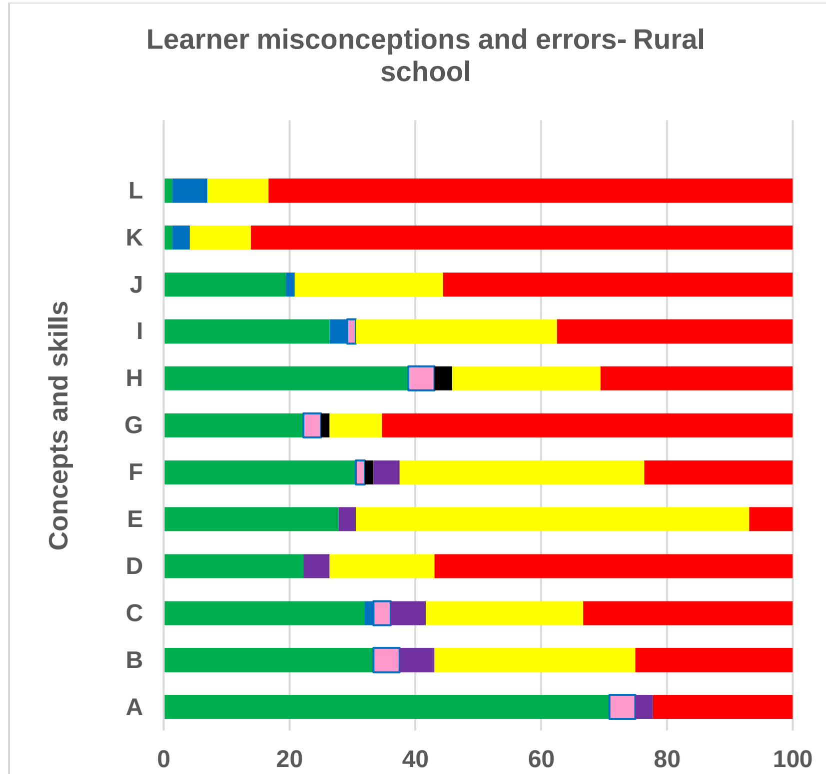
	A	B	C	D	E	F	G	H	I	J	K	L
COR- %	68	30	28	18	38	25	20	35	28	24	2	4
DNW- %	2	3	4	3	2	2	2	2	4	2	6	11
CA- %	4	5	3	3	2	4	4	5	5	2	2	1
E- %	0	0	0	0	0	2	2	2	0	3	2	3
F- %	3	5	3	3	4	4	0	1	2	0	0	2
P- %	0	35	27	18	45	32	13	24	25	25	11	7
CO- %	23	22	35	55	9	30	59	31	35	45	76	72

Figure 6: Graph showing learner misconceptions and errors

From the graph in Figure 6, it is evident that learners mostly committed conceptual and procedural errors. Although learners generally did not do well in problems and calculations involving fractions, as indicated by the number of procedural errors and conceptual errors, learners from the urban school committed more careless errors than learners from the rural school. Figure 7 shows misconceptions and errors committed by learners according to their location. The graph also revealed that learners from the urban school committed more conceptual errors than learners from the rural school in the diagnostic test. This was opposite to the findings from the summative test.



	A	B	C	D	E	F	G	H	I	J	K	L
COR- %	64	26	24	14	50	19	17	31	31	29	3	7
DNW- %	5	7	7	7	5	5	5	5	5	3	10	17
CA- %	3	7	3	7	3	7	5	5	10	3	5	2
E- %	0	0	0	0	0	3	2	2	0	7	5	7
F- %	3	3	0	2	5	3	0	2	3	0	0	5
P- %	0	40	31	19	24	24	19	26	17	26	12	3
CO- %	24	17	38	52	12	38	52	31	33	31	64	59



	A	B	C	D	E	F	G	H	I	J	K	L
COR- %	71	33	32	22	28	31	22	39	26	19	1	1
DNW- %	0	0	1	0	0	0	0	0	3	1	3	6
CA- %	4	4	3	0	0	1	3	4	1	0	0	0
E- %	0	0	0	0	0	1	1	3	0	0	0	0
F- %	3	6	6	4	3	4	0	0	0	0	0	0
P- %	0	32	25	17	63	39	8	24	32	24	10	10
CO- %	22	25	33	57	7	24	65	31	38	56	86	83

Figure 7: Graphs showing learner misconceptions and errors by location

Even though all the errors as tabulated in Table 7 were evident in the learner's scripts, this study focused on conceptual, procedural errors and symbolic errors. Encoding errors could be as a result of careless errors. As indicated earlier, misconceptions and errors that emerged from the tests and interviews would be presented in an integrated manner (i.e. presenting the finding emanating from the learner response from the test and corroborate it with the excerpt of the interview transcript where applicable). The learner interviews were conducted to gather explanations relating to the emerged errors, for corroboration, or to gain an in-depth understanding thereof. Unless otherwise indicated, in each category, the learners' errors from the diagnostic test are presented before errors from the summative test. The transcripts in brackets indicate the translation of the conversation from isiXhosa (one of the indigenous languages spoken in South Africa) to English.

4.3.1 Conceptual errors

This is the category of learners who hold misconceptions about mathematical concepts, display misunderstanding of the principles of solving a problem, or do not know what to do and use procedures used in a different concept.

(i) Treating fractions as whole numbers

a) Errors from the diagnostic test

When subtracting common fractions in $\frac{5}{7} - \frac{2}{3}$, some learners from both schools subtracted the numerators and denominators as they were, without making the denominators the same. Figure 8 shows different extracts from different learners' vignettes.

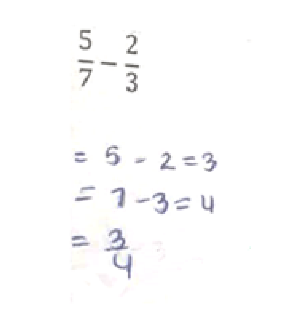
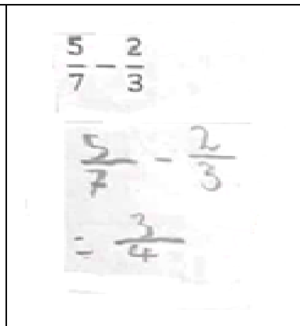
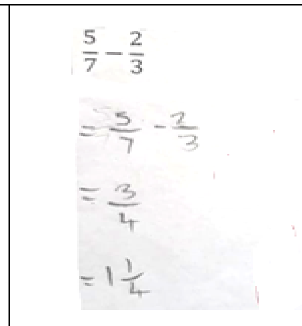
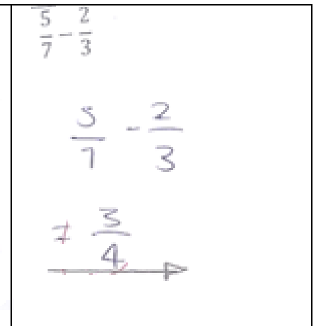
			
Learner U52	Learner U39	Learner U35	Learner R20

Figure 8: Misconceptions on subtraction of common fractions by Learners U52, U39, U35 and R20

From Figure 8, it is evident that Learner U52 subtracted 2 from 5 (the numerators) and 3 from 7 (the denominators) to obtain 3 and 4, respectively. Learner U39, Learner U35 and Learner R20 seemed to have adopted a similar approach as Learner U52, however, Learner U35's work revealed another misconception as this learner erroneously implied that $\frac{3}{4} = 1\frac{1}{4}$. Learner U52 also used the equal signs incorrectly.

When solving $\frac{4}{5} \times 2\frac{3}{8}$ in $3\frac{4}{7} \div \frac{5}{7}$, Learner U45 multiplied and divided the numerators with the whole numbers, respectively. Figure 9 shows how Learner U45 solved problems involving mixed numbers.

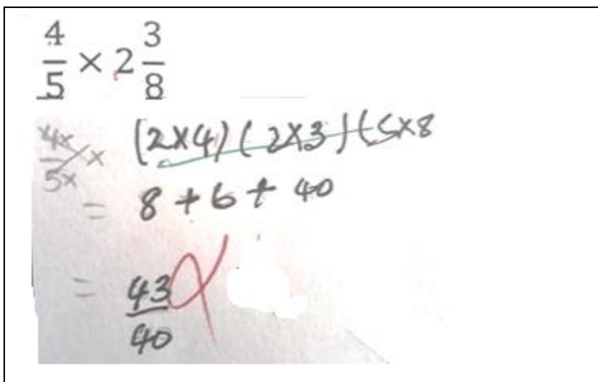
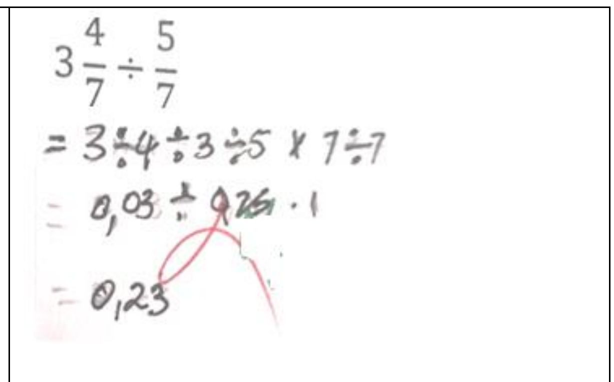
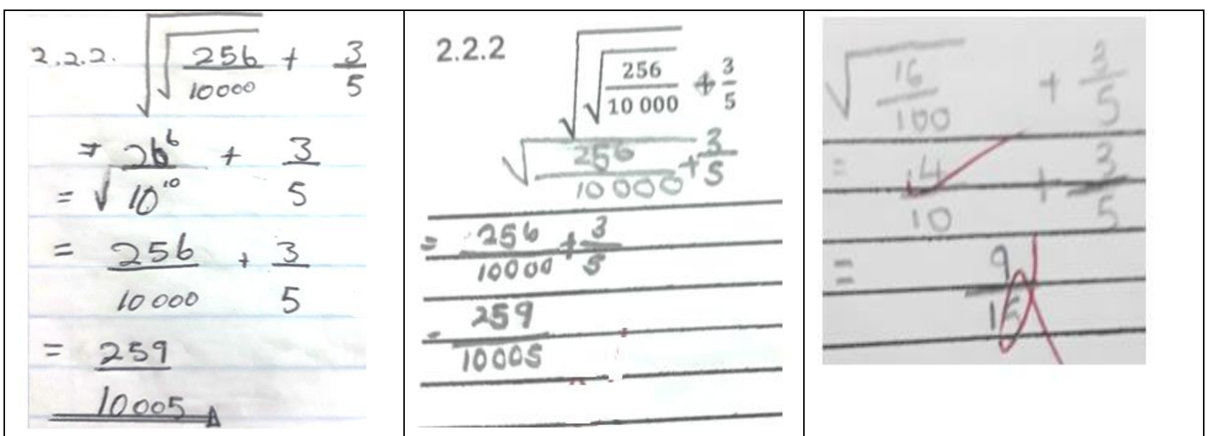
	
Learner U45	Learner U45

Figure 9: Misconceptions on converting mixed numbers, Learner U45

In $\frac{4}{5} \times 2\frac{3}{8}$, Learner U45 multiplied both the numerators 4 and 3 by 2 to get 8 and 6, respectively. The learner also multiplied 5 and 8 to get 40. In addition, the learner erroneously implied that the brackets mean addition as the learners separated the products 8, 6 and 40 from each bracket by an addition sign. It is not traceable how the learner got 43 of the solution, $\frac{43}{40}$, from. Similarly, this learner followed the same approach with other calculations involving mixed numbers using division. In $3\frac{4}{7} \div \frac{5}{7}$, the learner divided the whole number of the mixed number by the numerators and lastly divided the denominators. It is however not traceable how the learner got 0,03 and 0,25, and subsequently 0.23.

b) Errors form summative test

It is not traceable what influenced Learner R08 to write $\sqrt{\sqrt{\frac{256}{10\,000}}} + \frac{3}{5}$ as $\frac{26^6}{10^{10}} + \frac{3}{5}$, however, in the step that followed the learner ignored the square root signs and wrote $\frac{256}{1000} + \frac{3}{5}$. Learner U02 literally removed the first square root sign to get $\sqrt{\frac{256}{10\,000}} + \frac{3}{5}$, and the second square root sign, hence got$\frac{256}{10\,000} + \frac{3}{5}$. Both learners added the numerators and the denominators without finding the common denominators to get $\frac{259}{10\,005}$. Learner U45 determined $\sqrt{\sqrt{\frac{256}{10\,000}}}$ as $\frac{4}{10}$ then solved $\frac{4}{10} + \frac{3}{5}$ to be $\frac{9}{15}$. This learner further showed a careless error by implying that $4 + 3$ is 9.



Learner R08	Learner U02	Learner U45

Figure 10: Misconceptions on non-routine problems, Learners U08, U02 and U45

When Learner U45 was asked to explain her solution, the learner acknowledged her mistake of writing $\frac{9}{15}$ though whilst trying to correct her mistake, the correction was about her adding 4 and 3 to get 9 and not the misconception of adding the denominators and the numerators without making the denominators the same. Below is the conversation between the learner and me:

Learner: So I first calculated $\sqrt{256}$ and then I got 16…then $\sqrt{10000}$ and I got 100 so I said $\frac{16}{100} + \frac{3}{5}$ and then …I said it is 4.

Researcher: The square root?

Learner: Yes, oh square root of $\sqrt{16}$ is 4 and then the $\sqrt{100}$ is 10 and I still added these ones when I added them, I got …7 hayibo (oops)! It was supposed to be $\frac{7}{15}$.

(ii) Misunderstanding of the principles of the concept

a) Errors form diagnostic test

- *Ignoring root signs and lack of understanding of the meaning of exponents*

Generally, most learners showed a lack of understanding of the concept of squares, cubes, square roots and cube roots of fractions. Figure 11 presents common errors committed by learners when responding to the question that required them (learners) to calculate $\sqrt{\frac{4}{25} + \left(\frac{2}{3}\right)^2}$.

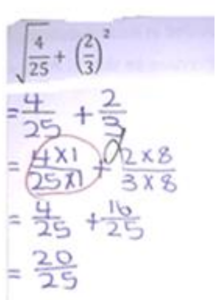
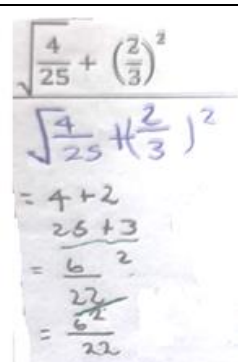
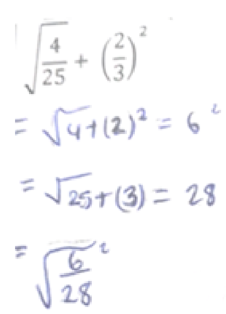
		
Learner R29	Learner U40	Learner U52

Figure 11: Misconceptions on addition squares and square roots of fractions, Learners R29, U40 and U52

In simplifying $\sqrt{\frac{4}{25}}+\left(\frac{2}{3}\right)^2$, Learner R29 ignored the exponents to get $\frac{4}{25}$ and $\frac{2}{3}$. In addition, Learner R29 multiplied $\frac{2}{3}$ by $\frac{8}{8}$ and erroneously ended up having $\frac{16}{25}$. The learner correctly calculated $\frac{4}{25}+\frac{16}{25}$ to get $\frac{20}{25}$. Learner U40 re-wrote the fractions as whole numbers and added the numerators 4 and 2 to get 6 and subtracted the denominators 25 and 3 to get 22. Similarly, Learner U40, followed the same approach as Learner R29 of literally ignoring the exponents $\frac{1}{2}$ and 2 in $\sqrt{\frac{4}{25}}$ and $\left(\frac{2}{3}\right)^2$, respectively. However, Learner U40 wrote the answer as $\frac{6}{22}^2$ where only the numerator was raised to exponent 2. In addition, Learner U40 erroneously added 25 and 3 to get 22. Learner U52 followed the same approach of adding numerators and denominators as Learner U40, though Learner U52 did not remove the square root sign in $\sqrt{\frac{4}{25}}$ and only the numerator for $\left(\frac{2}{3}\right)^2$, was raised to exponent 2. This was evident when the learner wrote $\sqrt{4}$ and $\sqrt{25}$ and only raised 2 to exponent 2 and not the whole fraction $\frac{2}{3}$. The learner then added $\sqrt{4}$ and $(2)^2$ to get 6, and $\sqrt{25}$ and 3 to get 28. Evidently this learner showed no understanding of the square roots and squares. Lastly, Learner U52 wrote the final answer as $\sqrt{\frac{6}{28}^2}$. Learner U40 and Learner U52 committed more or less the same error. Learner U40 and Learner U52 also treated the numbers as whole numbers like the learners in Figures 8 and 10. All the learners demonstrated no knowledge of using multiples and factors to simplify fractions after calculations.

When calculating $\sqrt[3]{\frac{8}{27}}-\left(\frac{1}{2}\right)^3$, Learner U40 and Learner U52 in Figure 12 followed the same procedures they used when solving fractions involving square roots and squares in Figure 12. Learner U40 re-wrote the fractions as whole numbers, ignoring the exponents $\frac{1}{3}$ and 3 in $\sqrt[3]{\frac{8}{27}}$and $\left(\frac{1}{2}\right)^3$, respectively. The learner then subtracted 1 from 8 to get 7 and 2 from 27 to get 25 and wrote the answer as $\frac{7}{25}^3$ where only the numerator was raised to 3. Learner U52 followed the same approach of subtracting 1 from 8 to get 7 and 2 from 27 to get 25, however the learner confused the cube root sign with the square root sign, hence the learner wrote $\sqrt{8}$ and $\sqrt{27}$. The learner only raised 1 to exponent 3 and not the whole fraction $\frac{1}{2}$, showing no understanding of cubes. The learner then subtracted $(1)^3$ from $\sqrt{8}$ to get 7^3 and 2 from $\sqrt{27}$ to get 25. Evidently this learner showed no understanding of the square root sign. Lastly, Learner U52 wrote the final answer as. $\sqrt{\frac{7}{25}^3}$.

Learner R08 multiplied 8 and 1, the numerators, with 3 to get 24 and 3, respectively, however it could not be traced how 14 was obtained. Learner R08 then subtracted 3 from 24 to get 21 and 2 from 14 to get 12. Learner U42 multiplied 8 and 27 by 3 for the cube root to get $\frac{24}{81}$ and 1 and 2 by exponent 3 to get $\frac{3}{6}$. The learner then subtracted 3 from 24 and 6 from 81 to get $\frac{21}{75}$. In addition, Learner R08 and Learner U42 also displayed no knowledge of using multiples and factors to simplify fractions after calculations. Figure 12 shows how learners solved fractions that involve cube root and cubes.

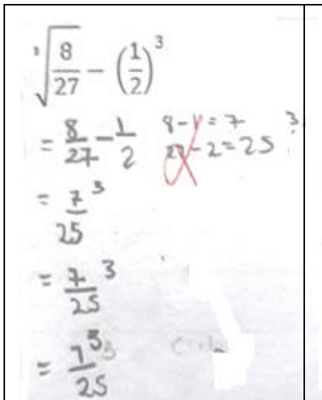
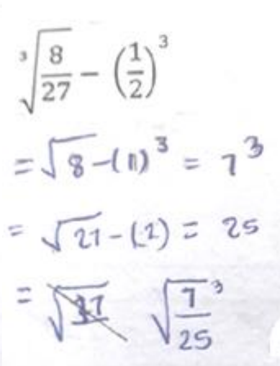
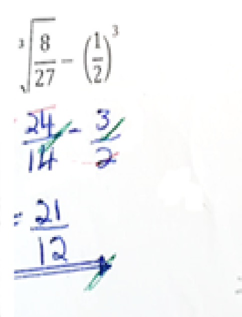
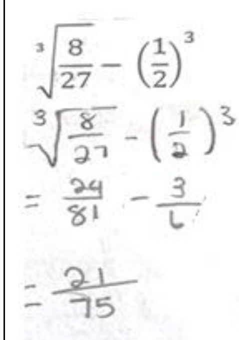
			
Learner U40	Learner U52	Learner R08	Learner U42

Figure 12: Misconceptions on subtracting cubes and cube roots of common fractions, Learners U40, U52, R08 and U42

Even though questions that were not attempted by learners would not be presented, researcher found it imperative to present Learner U22's response to this question as it contradicts with his statement.

Learner U22 did not attempt the problem and to substantiate the statement, the learner claimed during interviews that he was not familiar with the problem that involved two square root signs. Below is a conversation between Learner U22 and myself in response to why he did not attempt the question.

Learner: On this question I didn't understand what the square root of the square root was.

Researcher: Okay

Learner: Yes, ma'am so it wasn't easy for me to do it, because I wasn't familiar with how it was done.

Whilst the learner claimed that he did not understand the question on square root of a square root of a fraction because he was not familiar with the question, in the diagnostic test the learner did not correctly respond to questions with root signs.

Figure 13, followed by the conversation between myself and the learner, attest to the statement.

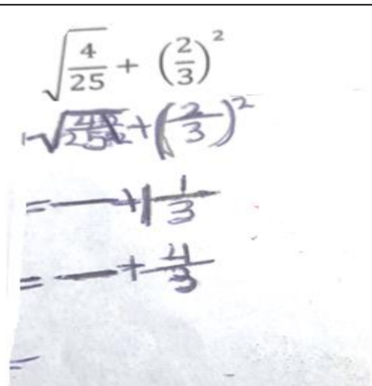
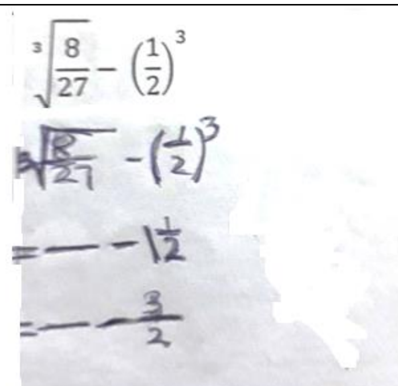
	
Learner U22	

Figure 13: Misconceptions on squares and square roots, cubes and cube roots, Learner U22

Learner: Here I first wrote the question. What I first did is try to do the ones that I do understand. So, here was $\left(\frac{2}{3}\right)^2$ then I said 2^2 was 4, then $2^3$was 6.

Researcher: Where did you write that, where is that?

Learner: Oh I didn't write 6 because I was leaving the denominator to remain 3, so then I wrote $\frac{4}{3}$, when I turned it into a mixed fraction it was $1\frac{1}{3}$ so I converted it into $\frac{4}{3}$ as it was before. And then by the square root here and the sign I couldn't use it because I wasn't using a calculator, so it was difficult for me to find the square root of the fraction.

Researcher: Okay, because you didn't have a calculator?

Even though Learner R08 and Learner U17 in $\sqrt{\frac{4}{25}} + \left(\frac{2}{3}\right)^2$, where learners were supposed to add a square and a square root of a fraction, displayed an understanding of square root unlike the learners in Figure 11 , the learners showed a lack of understanding of squares and addition of fractions where addition of fractions was treated as the addition of whole numbers, like learners in Figures 7 and 9. Learner R08 and Learner U17 correctly determined the square root of $\sqrt{\frac{4}{25}}$ as $\frac{2}{5}$, however Learner R08 multiplied the numerator of the square of the fraction by the exponent to get 4, and did not square the denominator 3.Learner U17 followed the same trend as Learner R08, however, Learners U17 multiplied both the numerator and the denominator of the square of a fraction by 2 to get $\frac{4}{6}$. Learner R08 and Learner U17 then added the numerators and the denominators to get $\frac{6}{8}$ and $\frac{6}{11}$, respectively.

Figure 14 shows how Learner R08 and Learner U17 solved fractions involving square roots and squares.

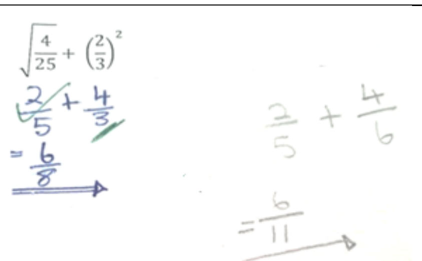
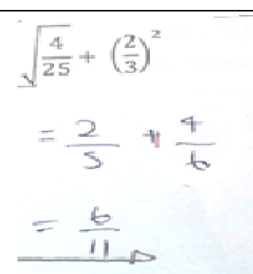
	
Learner R08	Learner U17

Figure 14: Misconceptions on squaring of fractions, Learner R08 and U17

This was evident during the interviews when Learner R08 explained that she thought what is inside is multiplied by what is outside but referred to numerators in both the problems that involved squares and cubes (see Figure 12 on cubes).

Researcher:	Ok, then let us come here (pointing at the $\sqrt{\frac{4}{25}}$) apha usicholile isquare root, kwenzeke ntoni apha (pointing at $\left(\frac{2}{3}\right)^2$) (here you correctly calculated the square root, what happened here?
Learner:	Bendicingu ba Miss u times le ngalena engaphandle. (I thought Miss you multiply by what is outside)
Researcher:	Ooh le ephezulu kuphela (Okay, the one on top only)
Learner:	Yes Miss
Researcher:	Ok, is the same thing oyenzile apha? (Ok is the same thing that you did here?), pointing at the problem that involved the cube root and cube)
Learner:	Yes Miss

- *Lack of understanding of mixed numbers*

In $\frac{4}{5} \times 2\frac{3}{8}$, learners did not know how to work with mixed numbers. Learner U52 and Learner U42 multiplied the numerators together and the denominators together. However, both these learners did not take into consideration that the multiplicand was a mixed number, and hence ignored 2. Learner U52 also erroneously multiplied 4 and 3 to get 16 instead of 12 and correctly multiplied 5 and 8 to get 40. Since the learner did not multiply by 2, the learner wrote $2\frac{16}{40}$ as the answer. The findings also revealed that Learner U52 made the same error with all problems that involved mixed numbers when converting them into improper fractions. Learner U42 followed the same procedure of multiplying the numerators and the denominators without first changing the mixed number to an improper fraction. However, Learner U42 multiplied 4 and 3 to get 12, and 5 and 8 to get 40. Similarly, since Learner U42 did not convert the mixed number, the learner wrote the answer as $2\frac{12}{40}$. Though Learner U42 went a step further than Learner U52 and simplified the fractions, Learner U42 showed lack of understanding of factors and multiples to simplify fractions before and after the calculations, as the answer was $2\frac{6}{20}$. Learner R23 multiplied the numerator and the denominator of the first fraction as well as the numerator of the mixed number by 2 to get $\frac{8}{10} \times \frac{6}{10}$. However, it is not understood how the learner came up with the denominator for $\frac{6}{10}$ and the solution of $\frac{1}{10}$, in which the learner showed a misconception in implying that $\frac{1}{10}$ was the same as $1\frac{9}{10}$.

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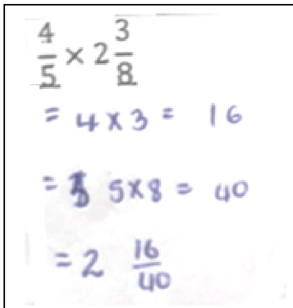
<td>
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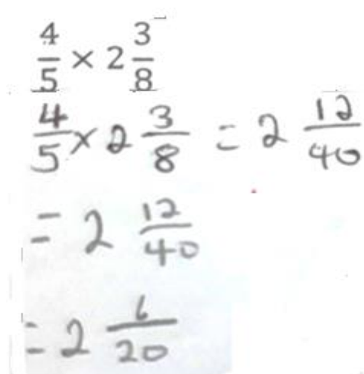
<td>
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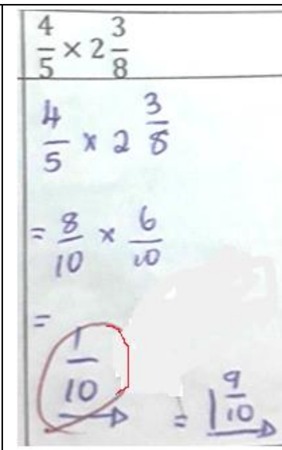
<td>
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<tr>
<td>Learner U52</td>
<td>Learner U42</td>
<td>Learner R23</td>
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Figure 15: Misconceptions on conversion of mixed numbers, Learner U52, U42 and R23

- *Representing whole numbers as percent and not knowing what to do*

When solving $\frac{5}{6}$ of 58, Learner R60 in Figure 16 wrote 58 as $\frac{58}{100}$. The findings revealed serious misconceptions when the learner wrote $\frac{5}{6}$ as $\frac{56}{100}$. Although $28 \div \frac{7}{4}$ does not fall under the category in discussion, this example shows that learners dealt with calculations involving whole numbers and fractions. Learner R60 also wrote 28 as $\frac{28}{100}$.The learner carried out the procedure of dividing $\frac{28}{100}$ by $\frac{7}{4}$ correctly to get $\frac{4}{25}$. When Learner R60 was asked to explain why she wrote 58 as hundredths, her reason was that she was not going to be able to multiply 58 alone without making it a hundredth. And she further said when multiplying a fraction by a whole number, one must make the whole number be out of 100. This learner made the same error with all calculations involving whole numbers as seen in Figure 16. Similarly, Learner R65 and Learner U46 followed the same trend of writing 58 as $\frac{58}{100}$. Learner R65 displayed an understanding of multiples and factors and wrote the answer as $\frac{5}{6} \times \frac{58}{100}$ as $\frac{29}{60}$. However, it is not traceable how the learners came up with the answers. Learner R65 followed the same procedure when calculating $28 \div \frac{7}{4}$ where the learner wrote the denominator for 28 as 100 to get $\frac{28}{100,}$. Thereafter, Learner R 65 correctly solved $\frac{28}{100} \div \frac{7}{4}$ to get $\frac{4}{25}$. Learner U18 multiplied 5 by 58 to get 290. The assumption was that the learner also multiplied 6 by 58, without computation but by inspection, and concluding that if $5 \times 58 = 290$, therefore $6 \times 58 = 291$, which is 1 more than 290 since 6 is one more than 5.

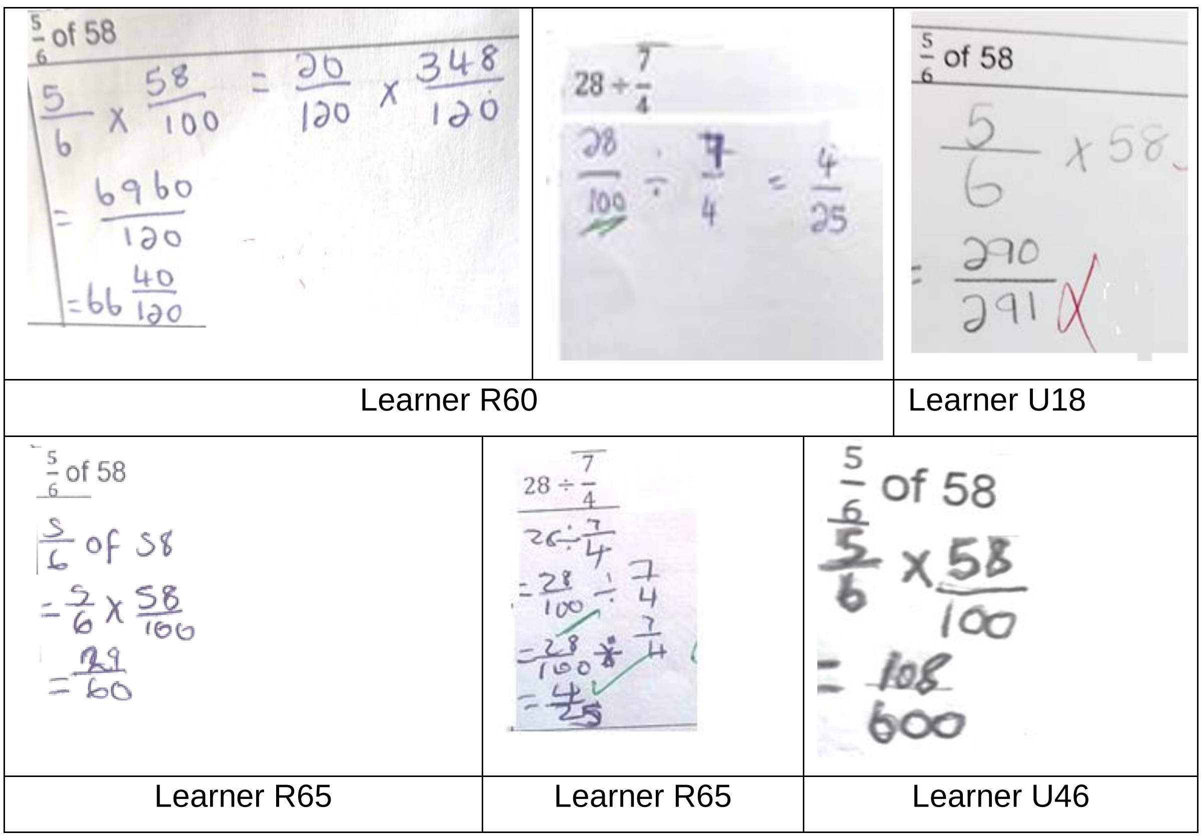


Figure 16: Misconceptions on calculations involving a whole number and a fraction, Learners R60, U18, R65 and U46

The conversation below between Learner R60 and myself attests to why the learner wrote whole numbers as out of 100.

Learner: Apha Miss ndithe 5 over 6, times 58 over hundred. (Here miss I said 5 over 6 × 58 over hundred.)

Researcher: Why over hundred?

Learner: Cause Miss u 58 lo bendingekhe ndikwazi uku mmaltiplaya yedwa ndingafakanga u 100. (Because Miss I would not be able to multiply 58 alone without inserting 100)

Researcher: Okay, does it mean xa umaltiplaya ifraction newhole number, i-whole number kufanale ibe out of hundred idenominator yakhona? (Ok, does it mean when you multiply a fraction by a whole number, the whole number must have a denominator which is 100)?

Learner: Yes miss

- *Inverting both fractions when dividing fractions*

The general finding with this concept was that learners did not know which fraction to invert. Some learners divided the whole number by a numerator and the denominator, whilst others inverted both fractions.

In Figure 17, Learner U22 inverted both fractions without changing the division sign. Learner U22 showed an understanding of the basic facts about writing whole numbers as fractions and converting mixed numbers into fractions. When he was asked the reason behind inverting fractions, he correctly answered that, in fraction form 28 is an improper fraction. He further said that he inverted $\frac{28}{1}$ to a proper fraction, $\frac{1}{28}$, so that it would be easier for him to calculate with. He felt that a fraction would be easier to work with if it is a proper fraction.

He did the same with $\frac{16}{15}$ to be $\frac{15}{16}$. One could insensibly think that the learner used an algorithm of dividing fractions incorrectly, not knowing that there is more behind the solution. When explaining why he changed $\frac{4}{5}$ to $\frac{5}{4}$ yet $\frac{4}{5}$ is already a proper fraction, interestingly he said he could not change only one fraction, but he had to change both, meaning you either change all or you do not change any. In $3\frac{4}{7} \div \frac{5}{7}$ he correctly converted the mixed number to an improper fraction $\frac{25}{7} \div \frac{5}{7}$ and applied his algorithm of changing the first fraction to be an *"easier one*", a proper fraction and did the same to the second fraction to get $\frac{7}{25} \div \frac{7}{5}$. When he was further asked if $\frac{25}{7} \div \frac{5}{7}$ would be difficult for him, he said it was not going to be difficult, but he was trying to use what he was taught about changing a fraction, and he forgot how it is called. Figure 17 shows Learner 33's calculation using his own "*algorithm*".

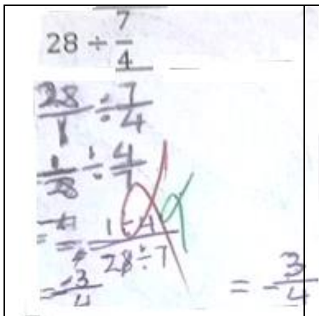
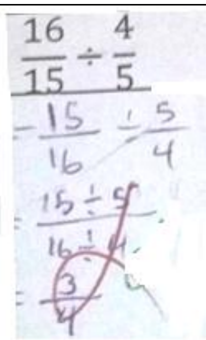
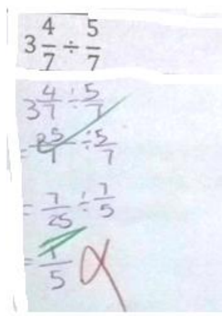
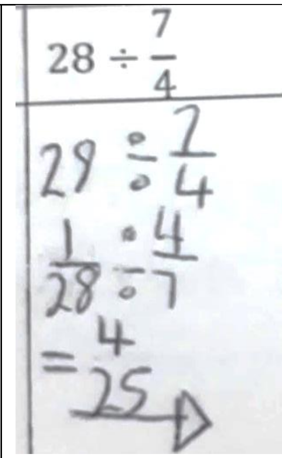
			
Learner U22			Learner R21

Figure 17: Misconceptions on inverting both fractions, LearnersU22 and R21

The following is the conversation between Learner U22 and myself:

Learner: Question 7. I first figured out that I need to change the number into an improper fraction because I'm dealing with fractions here. So, I changed 28 into a fraction which is $\frac{28}{1}$divided by $\frac{7}{4}$ then I turned it into a proper fraction which is $\frac{1}{28}$ divided by $\frac{4}{7}$

Researcher: Why?

Learner: Because I wanted to convert the denominators to be the same in an easy way

Learner: Here on question eight I first wrote $\frac{15}{16}$ because I tried to make denominators the same easier teacher. So, it was $\frac{15}{16}$ divided by $\frac{5}{4}$ the I said 15 divided by 5 then16 divided by 4.

Researcher: But you said in question 7 the reason why you changed the fraction, you wanted it to be easier.

Learner: Yes teacher

Researcher: So, it's only easier for you when it is…

Learner: When it's a proper fraction

Researcher: But this one is an improper fraction

Learner: Yes ma'am. If I change this side, I also have to change this side

Researcher: Okay

Learner: I couldn't change only one side. Then I came here with 15 divided by 5 which was 3, then 16 divided by 4 which was 4.

Learner: This side question 9, as I was taught how to change the mixed fraction to improper. I said 3 multiplied by 7 which gave me 21, 21 plus 4 is 25, 25 was my improper fraction divided by $\frac{5}{7}$. So, I changed again to make it easier for me to divide. So, I said 7 divided by 7 which gave 1, then 25 divided by 5 which gave me 5. That's how I got it.

Researcher: Let's look at this one, was it difficult for you to divide $\frac{25}{7} \div \frac{5}{7}$?

Learner: No ma'am it wasn't.

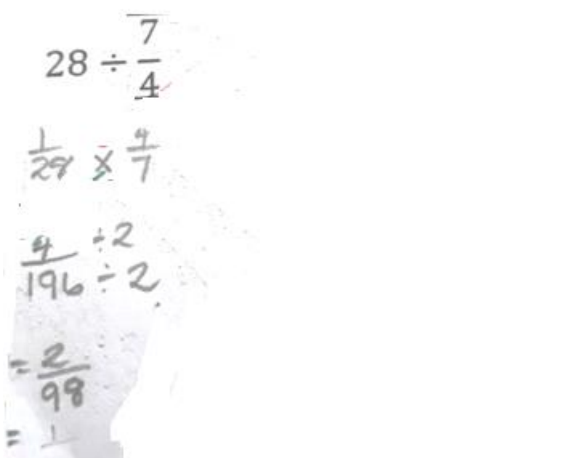
Researcher: But you said you changed it for it to be easier.

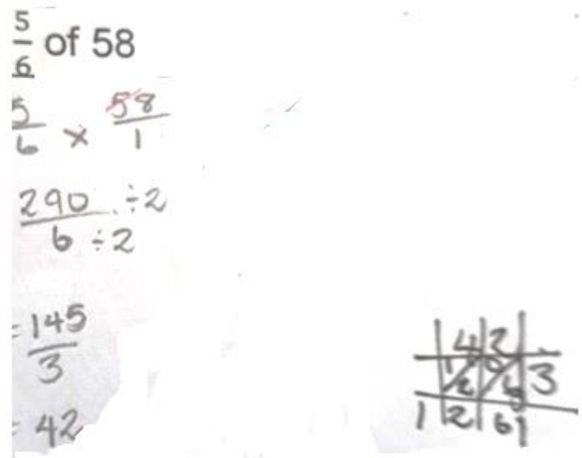
Learner: Yes ma'am… I don't remember what it was called to change the fraction, but I tried using that as I was taught. That's how I used it here

Researcher: So, they should be changed, that's how you were taught?

Learner: Yes ma'am, they must, the denominator and the numerator.

The findings also revealed that even though learners may commit the same error, their explanations may be different. In Figure 18, Learner U10 also inverted both fractions in $28 \div \frac{7}{4}$, but her explanation was different from that of Learner U22 in Figure 17. When she was asked why she changed $\frac{28}{1}$ to $\frac{1}{28}$, she said if a whole number is on the right of the fraction, its denominator must be 1, and if a whole number is on the left of the fraction, its numerator must be 1. Figure 18 is an example of how Learner U10 solved problems involving a whole number and a fraction with both division and multiplication.





Learner U10

Figure 18: A different view of 1 as a numerator and a denominator, Learner U10

Though the second problem, $\frac{5}{6}$ of 58, did not fall within the category under discussion, it is imperative to present the learner's understanding about calculations involving whole numbers which differed from Learner U22 in Figure 17. Learner U10 showed an understanding that $\frac{5}{6}$ of 58 meant $\frac{5}{6} \times 58$ and correctly carried out the procedure to get $\frac{290}{6}$, thus also showing the understanding that fractions are not inverted when they are multiplied. In addition, Learner U10 showed an understanding that when dividing fractions, the division sign changes to a multiplication sign and the second fraction is inverted.

The following conversation attests to the statement that if the whole number is on the right of the fraction, its denominator is 1 and if it is on the left of the fraction its numerator is 1.

Researcher: Okay, let's let go to this one. I'm interested in this one. This…, it was $28 \div \frac{7}{4}$

Learner: Aha

Researcher: How did you do it? Why is it that it's now $\frac{1}{28}$?, why did you change this division sign to a multiplication sign? …And why did you change this $\frac{7}{4}$ to be $\frac{4}{7}$?

Learner: So, in eeh. in order to divide fractions, you need to change the sign

Researcher: Okay.

Learner: And when you would've changed the sign, you change the order of your fraction

Researcher: Which fraction, all fractions?

Learner: No, the first…the second fraction, $\frac{7}{4}$ yah, so since on the first one which is…the multiplication sign …you have to multiply eeh $\frac{5}{6}$ of 58

Researcher: Okay

Learner: So the denominator had to be 1, so if it's on the other side of the fraction, you have to give it the numerator 1

Researcher: Okay…

Learner: So, then I changed the sign and then I multiplied. So, when you multiply you have to change the form of the fraction, so I get $\frac{4}{7}$

Researcher:	So, you are saying if the whole number is on the left, you have to give it a numerator that is 1?
Learner:	Yes
Researcher:	But if the whole number is on the right, you have to give it a denominator that is 1?
Learner:	Yes

b) Errors from summative test

In $\frac{x+1}{4} - \frac{x-1}{2} = 3$, where learners were required to solve the equation with denominators that are integers, Learner R21 added the denominators and equated their sum to x and wrote the answer as $x = 6$, whilst Learner R13 substituted for x with the denominators of each fraction 4 and 2 in the first and second fraction, respectively, to get $\frac{4+1}{4} - \frac{2-1}{2} = 3$. The learner then added the numbers that constitute the numerator for each fraction to get $\frac{5}{4} - \frac{1}{2}$ and left out "$= 3$". Evidently, the learner made the denominators to be same to get $\frac{5}{4} - \frac{2}{4}$, by multiplying $\frac{1}{2}$ by $\frac{2}{2}$ and carried out the procedure of subtracting common fractions to get $\frac{3}{4}$. Learner U14 substituted for x by 2 to get $\frac{2+1}{4} - \frac{2-1}{2}$ and wrote the answer as $x = 2$. Figure 19 shows misconceptions revealed by the three learners.

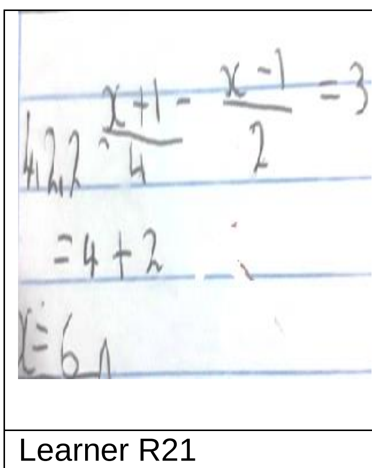
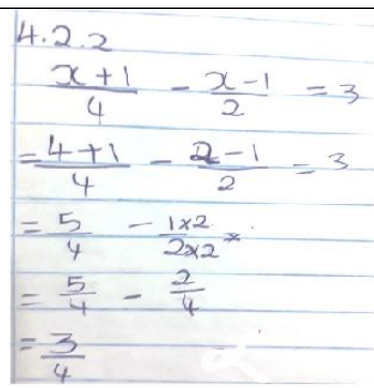
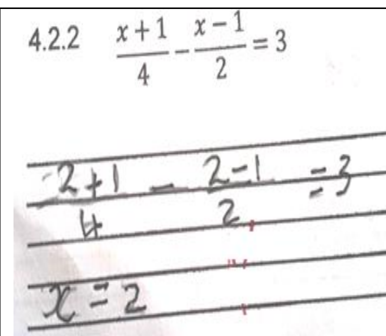
		
Learner R21	Learner R13	Learner U14

Figure 19: Misconceptions on solving equations, Learners R21, R13 and U14

4.3.2 Procedural errors

Learners in this category apply incorrect steps when solving problems or sometimes miss steps to complete the procedure. In addition, learners use incorrect use rules or algorithms or apply correct procedures in an incorrect situation.

(i) Misunderstanding and misapplication of procedures

- *Partial application of correct procedures*

a) Errors from diagnostic test

In Figure 20 learners who committed the same error when multiplying the common fraction by a common fraction. For instance, Learner R60, Learner R08 and Learner U46 correctly multiplied 4 by 3 to get 12 but did not multiply 5 by 5 to get 25. Instead, they applied the rule of addition or subtraction of common fractions and wrote 5 as the denominator of the product, since the denominators for both the numerators, 4 and 3 are 5. In addition, Learner R60 showed the understanding of the number of wholes and fraction parts in an improper fraction, and hence wrote $\frac{12}{5}$ as $2\frac{2}{5}$.

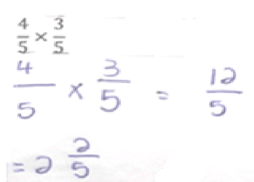
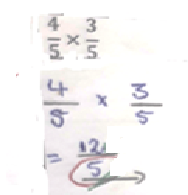
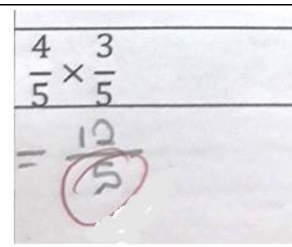
		
Learner R60	Learner R08	Learner U46

Figure 20: Partial application of correct procedures, Learners R60, R08 and U46

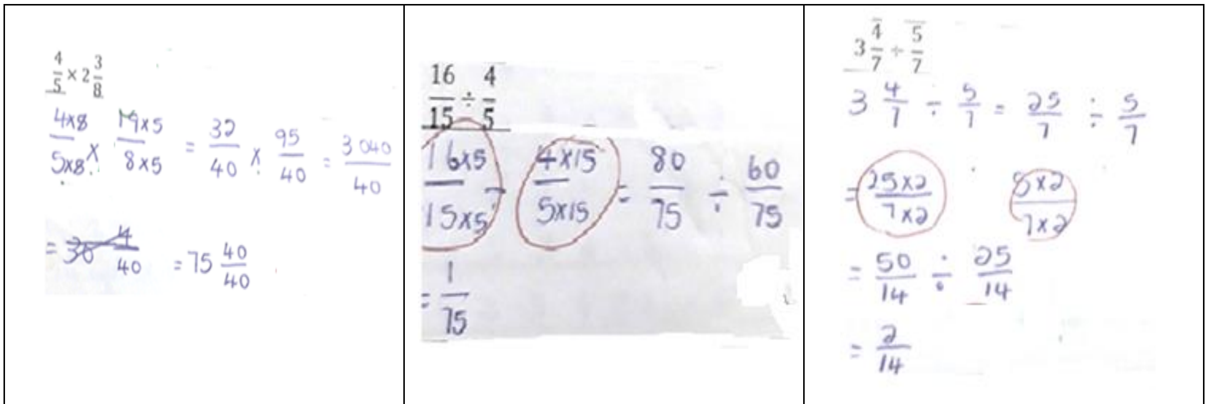
The conversation between myself and Learner R08 in Figure 20 attests to the statement mentioned earlier on.

Learner: Ndithe 4 times 3 kwaphuma u 12. So, ke idenominator kwath'uMiss siyishiya injalo, iba the same if iyinto enye siyikhipha injalo asiyi multiply. (I said 4× 3 and got 12. So, Miss (referring to the teacher in this case) said we leave the denominator as it is if it is the same thing, we do not multiply it).

Researcher: Not kwi addition? (Not in addition?)

Learner: Hayi I am not sure ke Miss (No, I am then not sure Miss).

The findings revealed that the misconception of not multiplying or dividing the denominators is deeply entrenched in Learner R60, as she displayed the same misconception with other examples involving division and multiplication of fractions. Figure 21 shows that Learner R60 set the denominators the same with all her calculations, even those with other operation signs.



Learner R60

Figure 21: Partial application of correct procedures, Learner R60

Learner R60 demonstrated the knowledge of converting mixed numbers into improper fraction by writing $2\frac{3}{8}$ as $\frac{19}{8}$, however made the denominators to be the same. Learner R60 also demonstrated the knowledge of the LCM for 5 and 8 to be 40. In addition, the learner demonstrated the knowledge of expressing each fraction, $\frac{4}{5}$ and $\frac{19}{8}$ as equivalent fractions whose denominator is 40 by correctly carrying out the procedure $\frac{4}{5} \times \frac{8}{8}$ and $\frac{19}{8} \times \frac{5}{5}$ to get $\frac{32}{40}$ and $\frac{95}{40}$, respectively. Although the learner correctly multiplied 32 by 95 to get 3 040, she did not multiply 40 by 40 to get 1 600. Instead, the learner incorrectly applied the rule of adding or subtracting common fractions by writing 40 as the common denominator to get $\frac{3\,040}{40}$ instead of $\frac{3\,040}{1\,600}$. To simplify $\frac{3\,040}{40}$, the learner showed a partial understanding of factors and multiples by writing 75 $\frac{40}{40}$, without considering that $\frac{40}{40}$ is 1, and hence the final answer of 76.

In $\frac{16}{15} \div \frac{4}{5}$, the learner made the denominators to be the same by multiplying $\frac{16}{15}$ by $\frac{5}{5}$ and $\frac{4}{5}$ by $\frac{15}{15}$ to get $\frac{80}{75}$ and $\frac{60}{75}$, respectively. It is evident that the learner made the denominators the same because he was influenced by the rule of adding and subtracting common fractions. In addition, the learner was also influenced by the rule of determining the LCM of two numbers a and b , where a is multiplied by b when neither a is a factor of b nor b a multiple of a or vice versa. The learner therefore did not consider that 5 is a factor of 15 or 15 a multiple of 5. The learner then erroneously divided 80 by 60 to get 1, however did not divide 75 by 75, instead writing the answer as $\frac{1}{75}$. When calculating $3\frac{4}{7} \div \frac{5}{7}$, the learner correctly converted $3\frac{4}{7}$ to get $\frac{25}{7}$. It is not traceable why the learner multiplied both $\frac{25}{7}$ and $\frac{5}{7}$ by $\frac{2}{2}$ to get $\frac{50}{14} \div \frac{25}{14}$ since 7 was already a common denominator, however the learner erroneously multiplied $\frac{5}{7}$ by $\frac{2}{2}$ to get $\frac{25}{14}$ instead of $\frac{10}{.14}$. The learner then divided 50 by 25 to get 2, however did not divide 14 by 14 to get 1. Also, she was influenced by the rule of adding and subtracting common fractions, hence the $\frac{2}{14}$. During interviews, the learner said she made the denominators to be the same but could not explain the choice of multiplying the denominators by 2. Learner R60 used the same procedure of making the denominators the same even with multiplication (see Figure 21) of common fractions, without multiplying or dividing the denominators, but writing the common denominator in the solution. Interestingly, when asked the reason behind writing the common denominator as it was in the solution, she said it would give her something bigger than the numerator.

The following were Learner R60's responses when asked to explain her solutions to problems involving both multiplication and division with fractions.

Learner: Apha Miss ndithe 16 over 15, ndathi 4 over 5. Ndathi 16 times yakhiph’u 80, ndathi 15 times 5 yakhiph’u 75. Ndathi 4 times 15 yakhiph’u60, ndathi 5 times 15 yakhiph’u 75. Ndathi 80 divided by 60 yakhiph’u 1 over 75. (Here Miss, I said 16 over 15, I said 4 over 5. I said 16 ×5 is 80, I said 15× 5 is 75. I said 4×15 is 60, I said 5×15 is 75. I said 80 b÷ 60 is 1 over 75.

Researcher: OK, why wangatsho ukuba 75 divided by 75? (ok, why did you not say 75 divide 75?

Learner: Kuba i-answer yakhona iphume inkulu kakhulu Miss. (Because its answer was too big Miss).

b) Errors from summative test

When solving $\sqrt{\sqrt{\frac{256}{10\,000}}} + \frac{3}{5}$, Learner R13 determined $\sqrt{\sqrt{\frac{256}{10\,000}}}$ as $\sqrt{\frac{4}{10}}$, then solved $\sqrt{\frac{4}{10}} + \frac{3}{5}$ as 0,2 + 0,6 which gave 0,8. It is evident that the learner had no knowledge of using multiples and factors to write fractions in simplest form before or after calculations. In addition to that, the learner could not write tenths as decimals, and hence wrote $\frac{4}{10}$ as 0,2.

In the diagnostic test, Learner R13 managed to solve the problem with one square root sign correctly which proves that she was not familiar with the problem.

Learner U10 correctly wrote $\sqrt{\sqrt{\frac{256}{10\,000}}}$ as 0,4, however this learner revealed a misconception when she wrote 0,4 as $\frac{1}{4}$. From there the learner made $\frac{1}{4} + \frac{3}{5}$ to have the same denominators by multiplying $\frac{1}{4}$ by $\frac{5}{5}$ and $\frac{3}{5}$ by $\frac{4}{4}$ to get $\frac{5}{20} + \frac{20}{10}$, then correctly added the fractions to get $\frac{17}{20}$. When asked to explain where she got $\frac{1}{4}$ from 0,4, she could not answer. Learner U08 only worked out $\sqrt{\frac{256}{10000}}$ as $\frac{16}{100}$ and did not determine $\sqrt{\frac{16}{100}}$. It is evident that the learner was also not familiar with the problem as she correctly determined the square root of a fractions in the diagnostic test. Learner U08 then made the denominators to be the same by multiplying $\frac{3}{5}$ by $\frac{20}{20}$ and correctly carried out the procedure when solving $\frac{16}{100} + \frac{60}{100}$ to get $\frac{76}{100}$, however, partially simplified $\frac{76}{100}$ to $\frac{38}{50}$.

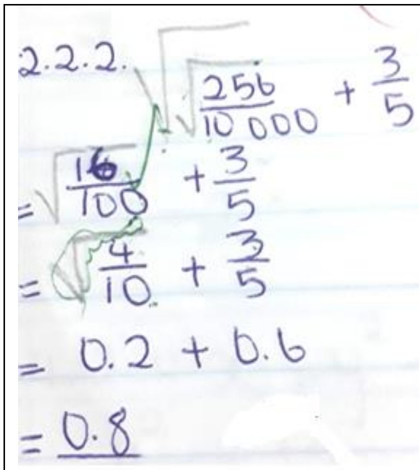
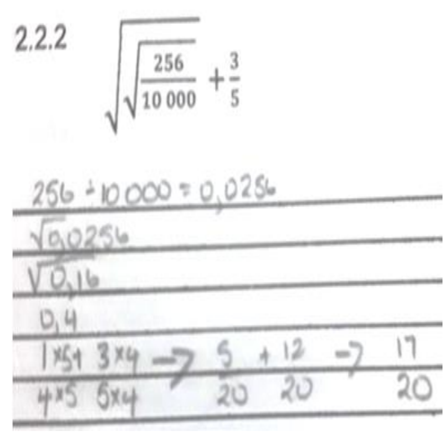
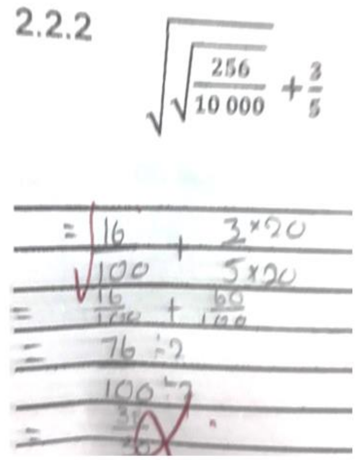
		
Learner R13	Learner U10	Learner U08

Figure 22 Inability to solve non-routine problems, Learners R13, U10 and U08

Here is the conversation between Learner U10 and myself:

Learner: Okay! It's 200… and…so I saw it as the square root of 256 out of 10000

Researcher: Yeah

Learner: And when you get the answer, you find the square root of that answer

Researcher: Excellent

Learner: That's how I took it, so it's $\frac{256}{10\ 000}$, and then it gave me 0. 0256 and then I found the square root of 0. 0256 and it was 0. 16 and then the square root of 0. 16 was 0.4

Researcher: Okay. Excellent and then what happened? Just continue, 0.4… I get this one

Learner: Mhm mm

Researcher: Now we are here (pointing at $\frac{1}{4}$)

Learner: Mhm mm, Oh God! (mumbling) I wonder

Researcher: (Laughing) You wonder?

Learner: I wonder how I did it, I just wonder

- *Inability to apply a correct procedure*

a) Errors from diagnostic test

When solving $\frac{4}{5} \times 2\frac{3}{8}$, some learners showed an inability to convert mixed numbers into an improper fraction and multiplied both the numerator and the denominator by the whole number. See Figure 23:

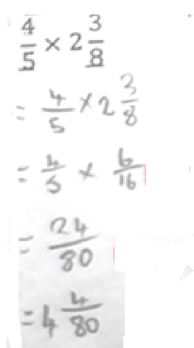
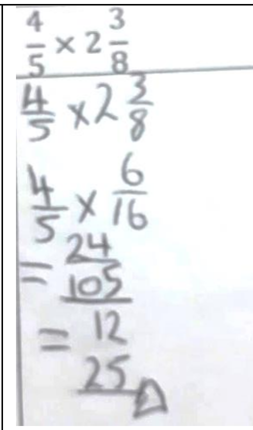
	
Learner U35	Learner R21

Figure 23: Inability to apply correct procedures, Learners U35 and R21

For instance, the misconception illustrated in Figure 23 shows that to calculate and simplify $2\frac{3}{8}$, Learners U35 and R21 multiplied 3 and 8 by 2 to get $\frac{6}{16}$. From there, Learners U35 correctly multiplied $\frac{4}{5} \times \frac{6}{16}$ to get $\frac{24}{80}$. However, Learner U35 revealed another misconception by erroneously implying that $\frac{24}{80} = 4\frac{4}{80}$. Learner R21 incorrectly multiplied $\frac{4}{5} \times \frac{6}{16}$ to get $\frac{24}{105}$. Similarly, Learner R 21 revealed another misconception by erroneously implying that $\frac{24}{105} = \frac{12}{25}$.

b) Errors from summative test

Learner R13 made the denominators the same by multiplying the denominator for $\frac{x-1}{2}$ by 2 to get $\frac{x-1}{4}$ without multiplying the numerator to make the fractions equivalent.

Evidently the learner wrote 4 as the denominator for the right-hand side to get $\frac{3}{4}$ without multiplying 3 by $\frac{4}{4}$. Learner R13 understood that the denominators must be the same, however the procedure was carried out incorrectly by implying that $\frac{x+1}{4} - \frac{x-1}{2}$ is the same as $\frac{x+1}{4} - \frac{x-1}{4}$ where the learner only multiplied the denominator of $\frac{x-1}{2}$ by 2. From there, the learner ignored the left-hand side and only wrote the final answer to the problem as $\frac{3}{4}$. Learner U10 also followed the approach of making the denominators the same, however differently from Learner R13. Learner U10 multiplied $\frac{x+1}{4}$ by $\frac{2}{2}$ and $\frac{x-1}{2}$ by $\frac{4}{4}$. It is perceptible that in making the denominators the same, the learner was influenced by the procedure followed when the denominators are not factors of each other. The learner revealed a misconception when multiplying $\frac{x+1}{4}$ by $\frac{2}{2}$ to get $\frac{x+2}{8}$. The same misconception was evident when the learner wrote the answer as $\frac{x-4}{8}$ from $\frac{x-1}{2} \times \frac{4}{4}$. The learner displayed a lack of knowledge of the distributive property, and hence could not display the understanding of multiplying a binomial by a monomial in both fractions on the left-hand side. In addition, the learner ignored 3 on the right-hand and equated the fractions to get $\frac{x+2}{8} = \frac{x-4}{8}$. From there, the learner added x and 2 for $\frac{x+2}{8}$ and got $\frac{2x}{8}$. The same approach was done with and x and -4 , hence $\frac{-4x}{8}$. In the next step, the learner again subtracted the fractions, ignoring the equal sign, and hence got $\frac{2x}{8} - \frac{-4x}{8}$ which resulted to $\frac{-2x}{8}$ as the final answer. The learner revealed another misconception as regards the subtraction of integers.

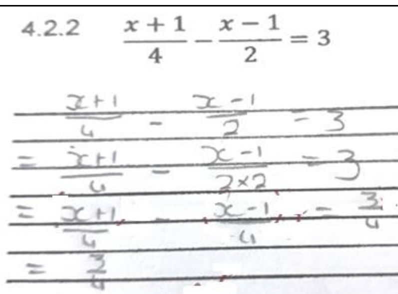
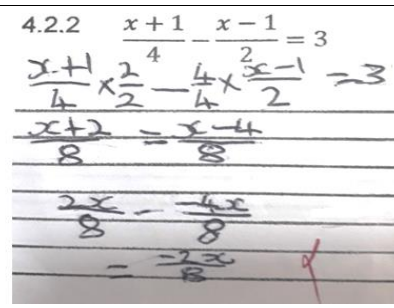
	
Learner R13	Learner U10

Figure 24: Misconceptions on algebraic equation, Learners R13 and U10

In explaining her solution, Learner U10 indicated that she got stuck and hence did not proceed. Here is the conversation between Learner U10 and myself:

Learner:	Then… I multiplied them by 2
Researcher:	Mhm
Learner:	Then I multiplied this side by 4
Researcher:	Mhm
Learner:	And then it gave me 3, ewe it gave me 3
Researcher:	Aha it was equal to 3, it did not give you 3
Learner:	Oh ewe! Then it's equal to 3, ewe ya (yes, yeah), Then I didn't get what's the next step. I was… confused.

In $\frac{2x^3}{3} = 5\frac{1}{3}$ where learners were supposed to solve exponential equations, Learner 06 seemed to have added the coefficient, exponent and the denominator to get 8 and wrote the answer as $\frac{8}{3}$. Also, with $5\frac{1}{3}$, the learner added the whole number, numerator and the denominator to get 9 and wrote the answer as $\frac{9}{3}$. The learner then changed the equal sign to a multiplication sign. It is evident that the learner was influenced by the rule of addition or subtraction of fractions. Surprisingly, the learner correctly multiplied the fractions with equal denominators in the diagnostic test. Evidently, Learner U12, followed the same approach as Learner U06 in changing the equal sign to a multiplication sign, however Learner U12 correctly converted $5\frac{1}{3}$ to $\frac{16}{3}$. In addition, Learner U06 and Learner R12 incorrectly carried out the procedure to get $\frac{72}{3}$ and $\frac{128}{3}$, respectively. It is indisputable that both learners were influenced by the rule of multiplying common fractions. Learner U16 correctly converted $5\frac{1}{3}$ to $\frac{16}{3}$, but could not solve the problem. Evidently, the learner lacks knowledge of the identity element of 1, and hence could not make x^3 the subject of the formula.

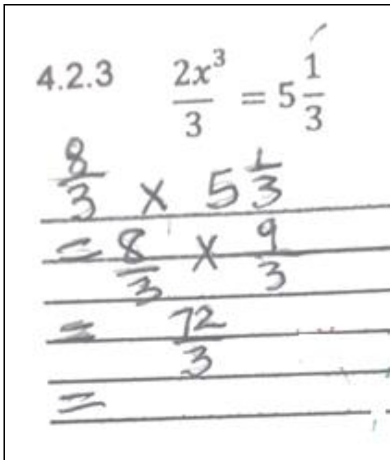
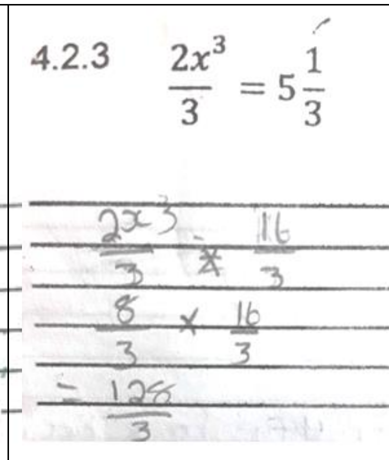
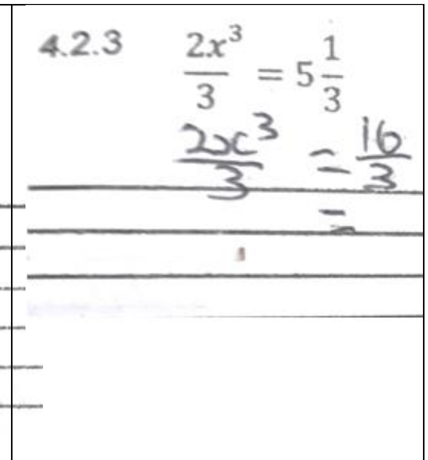
		
Learner U06	Learner U12	Learner U16

Figure 25:Learner misconceptions on solving exponential equations, Learners U06, U12 and U16

When Learner U16 was asked why he did not completely solve the problem, he claimed that he ran short of time. Evidently the learner did not know what to do, as he wrote something else that pertained to something that came after the one in question.

Researcher: You couldn't complete this one, what happened?

Learner: I ran short of time ma'am.

Another finding revealed that some learners missed other steps whilst others could not reach the solution when solving $\frac{2x^3}{3} = 5\frac{1}{3}$. Learner U10, without any calculation steps, correctly wrote $\frac{2x^3}{3} = 5\frac{1}{3}$ as $x^3 = 2^3 = 8$. However, the learner could not solve for x, and instead wrote 16 as 2×8 and $\frac{16}{3}$ as a mixed number $5\frac{1}{3}$. When asked how she came up with $x^3 = 2^3 = 8$, she said sometimes she would write without actually knowing what she was writing and her guts would tell her that she wrote the correct thing. Similarly, Learner R14 did not solve the equation as the learner did not equate the bases.

Below is the conversation between Learner U10 and me.

Learner:	Mhmm mm… Okay! Mhmm. Okay! 25 by 8 is 16. Mhmm
Researcher:	Mhmm, you are wondering?
Learner:	I'm still wondering. Sometimes I write, I'm just shocked that I actually got them right.
Researcher:	(Laughing) Okay, otherwise you're not sure how you got it?
Learner:	Well, I kind of believed I got it correct.

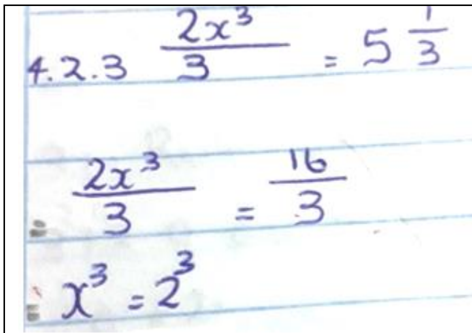
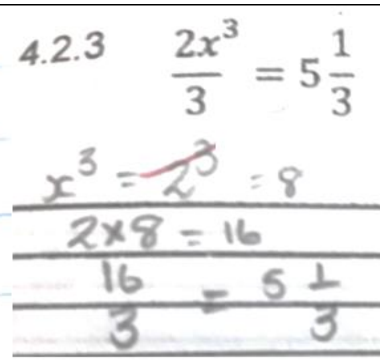
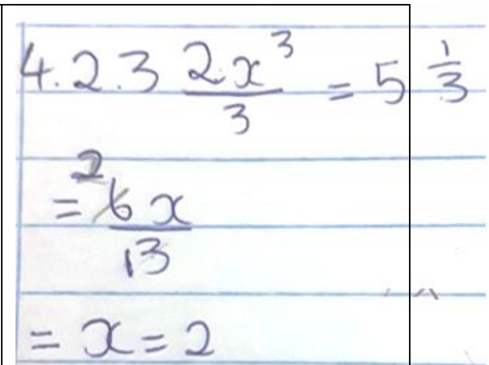
		
Learner R14	Learner U 10	Learner R12

Figure 26: Misconception on algebraic equations involving exponents, Learners R14 and U10

In Figure 26, Learner R12 multiplied the coefficient of x^3 which is 2, by the exponent 3 to get $\frac{6x}{3}$ and ignored $= 5\frac{1}{3}$. The learner displayed knowledge of simplification of fractions by representing $\frac{6x}{3}$ as $\frac{2x}{1}$, however, the learner accidentally solved for x as $x = 2$ albeit erroneously. When Learner R12 was asked what confused her not to change the mixed numbers into an improper fraction as she correctly did in the diagnostic test, she said she was confused by the variable x.

Below is the conversation between Learner R12 and me.

Researcher:	let's come to 4.2.3 with mixed numbers. Uyakwazi ukuzitshintsha uyabona ukuba uyitshintshile le? (You know how to change them; can you see you correct changed this one? [pointing at the problem in the diagnostic test])
Learner:	Yes mam
Researcher:	Waphinde wayitshintsha na le. (you also changed this one? [pointing at another problem in the diagnostic test])
Learner:	Yes miss
Researcher:	Kuye kwathini kule? (what happened to this one [pointing at the problem])
Learner:	(thinking… does not respond)
Researcher:	Mm? Uye wabhidwa yintoni? (what confused you?)
Learner:	Ngu x lona (It is this x)

4.3.3 Symbolic or factual errors

This category includes learners who lack basic mathematical facts, misidentify the operational signs or digits, and lack knowledge of formulas.

In Figure 27, Learner R08 changed the subtraction sign to a multiplication sign when solving $\frac{5}{7}-\frac{2}{3}$. It is evident that the incorrect practice of applying certain correct mathematical rules in inappropriate contexts is entrenched. Although Learner R08 displayed a misconception by changing the subtraction sign to a multiplication sign in $\frac{5}{7}-\frac{2}{3}$, the learner showed an understanding of the procedure of multiplication of common fractions. However, with $\frac{16}{15}\div\frac{4}{5}$, the learner erroneously subtracted the fractions to get $\frac{14}{15}$. It is evident that the learner was influenced by the rule of subtraction of common fractions. Learner R13 changed the multiplication sign to an addition sign when calculating $\frac{4}{5} \times \frac{3}{5}$, and wrote the answer as $\frac{7}{5}$. Even though Learner R18 did not change the sign in the same way as Learner R13 did, the solution of $\frac{7}{10}$ indicates that the learner also changed the sign. However, the learner revealed a misconception where the denominators were also added, hence $\frac{7}{10}$. It is evident that Learner R 13 and Learner R18 confused the multiplication sign with the addition sign.

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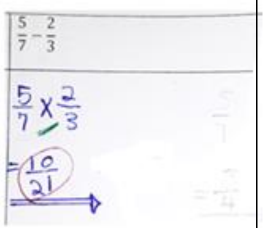
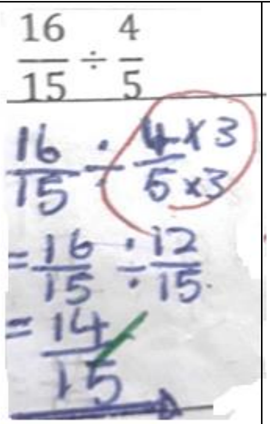
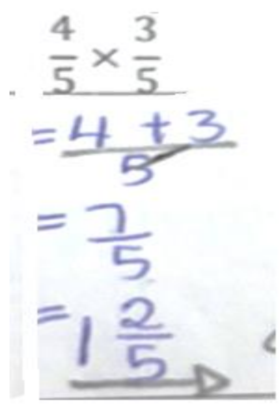
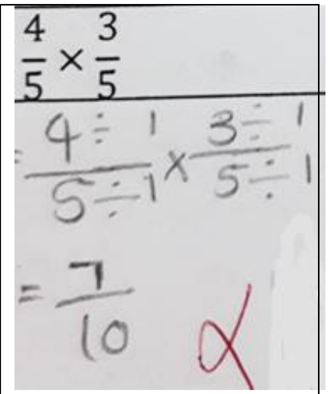
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Learner R08</td>
<td>Learner R08</td>
<td>Learner R13</td>
<td>Learner U18</td>
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Figure 27: Misconceptions involving changing the signs, Learners R08, R13 and U18

During the interviews, Learner R08 attested to using a multiplication sign instead of a subtraction sign and her reason was that she thought the sign changes to a multiplication sign. Below is the conversation, to attest to what Learner R08 wrote.

Learner: Apha Miss bendizixelela into yokuba kuyatshitshwa i-sign lena kube multiply.so ke ndiyenze ngo multiply, so ndayigeja. I was not sure ukuthi usubraction. (here Miss I told myself that the sign changes to be a multiply, so I did it with multiply, and I missed it. I was not sure that subtraction……)

Researcher: Akatshintshi. (does not change).

Learner: Yes

With the problem involving division of fractions, R08 multiplied $\frac{4}{5}$ by $\frac{5}{5}$ in order to make the denominators the same, and then erroneously subtracted 12 from 16 to get 14 and kept the denominator. When the learner was asked why she carried out the calculation as she did, she responded as follows:

Researcher: Apha uye wafuna ukuthini? Why umultiplye ngo 3 nale wayi multiply ngo 3? (here [pointing at the problem that involves the division sign], what did you multiply by 3 [pointing at the numerator], and this one by 3 [pointing at the denominator]?)

Learner: Ifana nalapha Miss bendifuna uba bendifuna uba idenominators zilingane. (it is the same as this one Miss [pointing at the denominator for the first fraction], I wanted to make the denominators to be the same).

Researcher: Ok, then what happened?

Learner: Ndiye ndasubtracter Miss. (I subtracted Miss)

Researcher: Why wangabi sazi subtracter apha idenominators? (Why did you not subtract the denominators)

Learner: Kuba ziyefana Miss. (because they are the same Miss)

All these different types of errors displayed by the learners will be discussed in the following chapter.

5 CHAPTER 5: DISCUSSION, CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

As indicated in the previous chapters, the purpose of the research was to explore the implications of the teachers' understanding of learner errors for mathematics learning. The study further sought to explore teachers' understanding of learner errors to inform learners' conceptual understanding and instructional decision making.

This chapter aims to respond to the research questions through discussion of the findings. In addition, it reflects on the study by drawing conclusions, demonstrating the utility of the theoretical framework (including whether it was fit for purpose as envisaged) and making recommendations. Lastly, it presents the limitations of the study.

The first part of this chapter presents the trends and the discussion of the findings of the teachers from the observations and interviews. The second part presents the trends and the discussion of the findings about how learners construct knowledge as collected from the tests and their explanation of the errors during the interviews.

5.2 Discussion of findings of data collected from teachers

As a way to preface the discussion of the findings related to teachers, Muthukrishnan et al. (2019) opine that error analysis aims at improving performance, by providing feedback and remediation thereof. Linking the previous knowledge of common fractions to the new topics that were taught during classroom observations seemed to be unsuccessful, as learners continued to have serious misconceptions with regards to fractions. The extent to which learners committed errors and the seriousness of the errors was indicative of the lack of effective remediation of errors and misconceptions. This contradicted the affirmation by teachers during the interviews that errors identified from the tests were corrected through revision. Setiawan and Koimah (2019) said that for effective learning to take place, learners must be given a chance to communicate their ideas, followed by feedback which not only provides a correct solution but also enables teachers to gain insight into the learners' mathematical thinking processes through the analysis of learners' responses.

The superficial analysis of learners' errors by concentrating on careless mistakes rather than errors that were resulting from misconceptions, led learners to cling to their misconceptions for a long period of time. Misconceptions were not diagnosed. Guerrero, (2014) and Hoosain and Chance (2004) also made the point that for effective learning to take place, the focus must be on the steps to come to the solution rather than the answer alone.

Focusing on the procedural process to solving problems where learners followed a routine by listening and copying correctly worked examples by the teachers, and were then then asked to complete the exercise, could be viewed as the traditional method of teaching. This practice seemed not to yield the desired competencies such as conceptual understanding. This is against the backdrop that learners continued to commit errors that could have been addressed by teachers. The approach that I viewed as traditional was not consistent with the prescripts or ideals of constructivist teaching (Bhattacharjee, 2015) and CGI practices (Carpenter, Fennema, Franke, Levi & Empson, 2015; Guerrero, 2014; Hoosain & Chance, 2004). These findings build on those of Trigwell (2012) who indicated that the traditional approach is teacher-focused and a teacher is a transmitter of knowledge. Specifically, (Saadati, Cerda, Giaconi, Reyes and Felmer (2019); Zakaria and Syamaun (2017), and Alsup (2004) reported that most teachers are still using the traditional approach where memorisation or recitation are common practices.

In addition, the tendency by teachers to follow the lesson step-by-step, could run a risk of disregarding the conceptual and procedural problems learners experience during teaching and learning. Teaching seemed not to be learner-centred, but rather teacher-centred, as learners seemed to be passively receiving information from the teachers, instead of actively involving themselves in knowledge creation as advocated by both CGI and constructivism.

Based on prior research indicating that the discovery of mathematical ideas by learners result in conceptual understanding and subsequently procedural fluency

(Grouws & Cebulla, 2000); and that traditional methods of teaching involve learners listening, copying and memorising what is said by the teacher, which sometimes does not make sense (Van de Walle et al., 2015), I postulate that the use of investigative approach, for instance, could have assisted learners in formulating rules for how to perform calculations with fractions and enhance conceptual understanding. Extending the research conducted by Carpenter, Fennema, Franke, Levi and Empson (2015) verifying that a CGI teacher should consider learners' previous knowledge as a powerful tool in enabling them to solve problems on their own, I opine that activities on previous knowledge should be purposefully selected so that they could lead to learners learning new concepts on their own. My suggestion supports the study conducted by Makonye (2016), which states that "learners should not be regarded as empty vessels that have to be filled with knowledge by the teachers" (p. 291). Similarly, Makhubele et al. (2015) concluded that constructivism does not dismiss the role of a teacher in class, but instead allows the teacher to become dominant in probing learners' critical thinking to uncover what they (learners) already know. Essentially, a constructivist teacher facilitates learning and advocates for interactive learning (Bada & Olusegun, 2015; Bhattacharjee, 2015). Therefore, my assertion about "…learning new concepts on their own." should not be misconstrued to mean that teachers neglect learners, forcing them to work on their own. Teachers are required to facilitate learning and allow learners to be critical and creative thinkers.

The fact that teachers rarely asked probing questions to encourage learners to justify their answers was indicative of teachers focusing on the answer rather than the process. Seemingly, not asking learners to justify their solutions implied that learners' solutions were not viewed as associated with, or as important as, the process involved in getting to the solution. This had implications on learning as the actual problem was not diagnosed, leading to teachers applying a blanket approach towards planning instruction guided by assumptions. Previous research findings have proved that asking learners questions to justify their answers, will lead to learners internalising information and correcting their mistakes (Pritchard, 2017). In addition, Milgram (2007) made a point that mathematics teachers treat reasoning and problem solving in isolation, and as different things, during instruction, whereas Kilpatrick (2011) identified reasoning, as one of the key components of teaching mathematics for understanding.

the tendency of teachers to call upon other learner to respond to the question when one learner gave an incorrect answer, is indicative of a disregard of learners' errors and could have far-reaching implications in terms of discouraging learners' active participation in class. Further, it could therefore be argued that not addressing learners' errors immediately could also have serious implications for the learning of new but related concepts. Learners could hold misconceptions over a long period of time. For instance, low learner performance on algebraic equations involving fractions, could be attributed to, among other factors, the fact that learners held misconceptions at the inception of computations with common fractions. Muthukrishnan et al. (2019) concurred and concluded that one of the potential threats to the acquisition of mathematics skills that can pose a challenge to the learning of mathematics, is teachers' lack of understanding of, and not attending to, learners' errors.

Even though the study was not about investigating teachers' pedagogical content knowledge (PCK), teachers appeared to hold some misconceptions too. I assert that marking learners incorrectly when having used a different strategy that yielded the correct answer could create gross confusion among learners as it could mean that they (learners) needed to adopt the strategy prescribed by the teacher since theirs (learners') is incorrect. Essentially confining learners to one strategy contradicts the prescripts of constructivism. Figure 28 taken from Section 4.2.2 of Chapter 4 show the teachers' uncertainty about the strategy used by the learner.

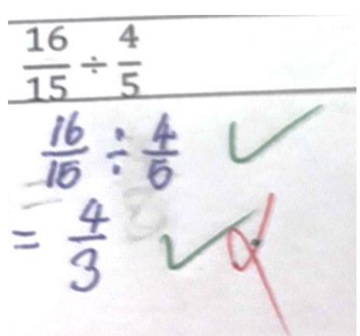

Figure 28: Incorrectly marking of learners

In addition, teachers marking learners' solution as incorrect due to not understanding the procedure used by learners, could be attributed to lack of PCK. Simultaneously, the inability to explain what I opine teachers view as the "golden strategy" when dividing fractions, "invert and multiply" could pose serious challenges to learners' conceptual understanding when learners invert both fractions. However, the silver lining in the scenario presented above is the teachers' acknowledgement, during interviews, that they do not know how to explain to learners why it is only the second fraction that is inverted when dividing fractions. Notwithstanding the assumed deficiency in PCK, the acknowledgement is the first step to reflective practice. These findings support the assertion by Isiksa and Cakiroglu (2011) that a reflective teacher fearlessly acknowledges that she or he holds misconceptions and the conclusion by Ratminingsih et al. (2017) and Jacobs (2016) that reflective teaching is a process where a teacher examines his or her own teaching attitudes and values in teaching with an intention to improve on them. By implication and according to my opinion, teaching of the concept of dividing fractions for understanding was rare.

Teachers claimed that they did not know how to explain the concept to learners. Thus, I assume that teachers opted to teach the concept through rehearsed algorithm, which might have led to learners formulating their own schemas of inverting the first fraction, or both fractions, as it was revealed in the diagnostic test. In addition, this could imply that there was no meaningful connection of the concepts to understand the procedure. As a result, it could not be expected of learners to correctly carry out the procedure of dividing fractions; and with the learners who carried out the procedure correctly it could not be guaranteed that they (learners) understood the concept. Muthukrishnan et al. (2019) made a point that conceptual understanding involves meaningful connection of concepts, and procedural knowledge does not imply conceptual knowledge, but conceptual knowledge can lead to correct procedures.

Based on the assertion that construction of knowledge involves assimilation and accommodation, (Bhattacharjee, 2015; Bada and Olusegun, 2015; Adom, Yeboah, and Ankrah, 2016 and Abrie et al., 2016) my opinion is that division of fractions could have been introduced through teaching for problem solving using the previous knowledge that learners might have acquired from the previous grades. This proposition is aligned with the study conducted by Hoosain and Chance (2004) which concluded that the duty of the teachers is to uncover and strengthen the mathematical knowledge that the learners possess. However, I propose that activities to teach the concept of division of fractions could have been designed in such a way that the previous knowledge would serve as a prerequisite for acquiring new information.

For example, assuming that in Grade 7, learners were taught to divide decimal fractions by a whole number as required by CAPS, a conjecture on division of a decimal fractions by a whole number was made. Thus, the algorithm of writing a decimal as an equivalent fraction where the denominator is a power of 10 was developed. Subsequently, the numerators are divided, and so are the denominators, after writing the divisor, which is a whole number, as a fraction.

Therefore, building on what was supposed to have been learned in Grade 7, the instructional decision that could have been made, I propose, would be to first divide a fraction by a whole number, where the divisor is a factor of the numerator of the dividend, such as $\frac{ab}{c} \div a$ or $\frac{ab}{c} \div b$. Then this concept could have been extended to division of a fraction by a fraction, where the numerator and the denominator of the divisor are factors of the numerator and the denominator of the dividend, respectively, i.e. $\frac{ab}{cd} \div \frac{a}{c}$ or $\frac{ab}{cd} \div \frac{b}{d}$. Such an example is illustrated in Figure 2 in Section 4.2.2, where 4 and 3 are factors of 16 and 15, respectively. When the concept of division with the above was understood, learners could have been taught division of fractions not limited to fractions, where the numerator and the denominator of the divisor are factors of the numerator and the denominator of the dividend, respectively. Then, through guided instruction, learners could have been required to set the denominators to be the same. Table 9 demonstrates that division of fractions could be done using different strategies.

Table 9: Different procedures of dividing common fractions

$\frac{9}{10} \div \frac{3}{50} = \frac{9}{10} \times \frac{5}{5} \div \frac{3}{50} = \frac{45}{50} \div \frac{3}{50} = \frac{15}{1} = 15$	$\frac{9}{10} \div \frac{3}{50} = \frac{9}{10} \times \frac{50}{3} = \frac{15}{1} = 15$
Making the denominators the same	Invert and multiply

Based on the explanation on the previous page, the strategy used by learners of making the denominators to be the same was correct. The only error the learners committed was not dividing the denominators, and the teacher could have utilised that error to teach for conceptual understanding. Figure 29 shows the method of calculation that was used by learners as extracted from Figure 1 of Section 4.2.2.

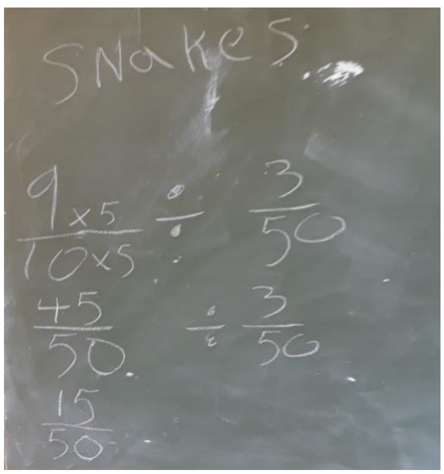

Figure 29: Learner method of calculation as extracted from Figure 1, Section 4.2.2

From the above explanation, I argue that the teachers seemed not notice that the procedure was correct due to the incorrect answer, which confirms that the teachers' focus was on the answer and not the process, contrary to CGI. Furthermore, I contend that there was a small gap between what learners know and what was not known, the ZPD, which could have been narrowed down though guiding questions. This proposition is also in line with the study conducted by Slavin (2019) which concluded that the teacher needs to provide guidance on mathematics tasks and activities that the learners cannot complete without assistance and supports prior research findings by Guerrero (2014) and Hoosain and Chance (2004), about what CGI advocates, namely an approach that focuses on the process and not the answer.

5.3 Learner misconceptions and errors

Since learners had the same assessment tasks, the results were analogous in both schools. The learners' concept of fractions was identified as the general misconception from the diagnostic test. The findings of the mistakes that emerged from the tests are discussed below. Learners' misconceptions and errors were discussed under the categories as discussed in Section 4.3. The discussions will focus on conceptual errors and procedural errors. The reason for focusing on these errors was that they were the most prevalent among learners. In addition, conceptual errors lead to a lack of understanding of procedures. This means that learners who have conceptual understanding are likely to carry out procedures fluently.

The discussion of learners' errors from both diagnostic and summative tests will be integrated.

5.3.1 Conceptual errors

There is a general tendency for learners to subtract the numerators and denominators, without setting the denominators equal. I argue that this indicates an entrenched conceptual error among learners. Brown et al. (2016) and Riccomini (2016) identified similar examples of procedural errors. The finding is also consistent with the study of Resnick, Jordan, Hansen, Rajan, Rodrigues, Siegler and Fuchs (2016) that when subtracting fractions, learners subtract the numerators and the denominators. However, I argue that errors such as the one mentioned above are conceptual errors because they reveal a lack of understanding of the concept of adding common fractions. Thus, treating fractions as whole numbers has more to do with a lack of conceptual understanding of a fraction as one number having one value and not two numbers with two values, which results in conceptual error. However, my argument is in line with Riccomini (2016) who also made the same point, that learners who commit errors as a result of a lack of understanding of concept have conceptual errors. Learners who worked with fractions or mixed numbers, as if they were working with whole numbers might have been influenced by the rules used when dealing with whole numbers and as such displayed *whole-number bias*. This finding is parallel to Kallai and Tzelgov (2009), and Schumacher and Malone (2017) who argue that when the learners view the numerator and the denominators as whole numbers, they approach fractions with *whole-number bias (WNB).*

Representing whole numbers in fraction form, as out of a hundred, implied that learners did not have a sense of what a hundredth is. by virtue of being a hundredth, a fraction cannot be equal to a whole number. I argue that that factual errors where learners lack basic facts that whole numbers are not equal to fractions contributed to the above statement. Brown et al. (2016) and Riccomini (2016) also made a point that factual errors result from a lack of basic mathematical facts. Learners could be made to understand that a fraction could be viewed as part of a whole as defined by Cortina and Visnovska (2015) or a part of a collection, and that by virtue of being a part of a whole the denominators should be the same or made the same if they are not (the concept of equivalence). Based on the above argument, I conclude that conceptual errors, if not addressed, can hinder the learning of all the concepts related to the ones which learners hold misconceptions about.

5.3.2 Procedural Errors

Other than conceptual errors, learners frequently commit procedural errors, which are likely to inhibit effective learning of mathematics if teachers do not understand and address them early on. As evidenced in the findings, procedural errors include multiplying only the numerators and not the denominators when multiplying fractions with similar denominators. Learners tend to apply an incorrect procedure that is used with calculations that involved addition and subtraction of fractions, where the denominator of the sum or difference of the fractions is the same as that of the fractions to be added or subtracted. In support of the findings, Brown, Skow and the IRIS Center (2016), Ricommini (2016) and Dlamini (2017) identified learners who use correct procedures in an incorrect situation as having procedural errors. Bilalić, McLeod and Gobet (2008) concluded that learners apply correct mathematical procedure in an inappropriate context because of their fixation on the previously introduced procedure and referred to the phenomena as the Einstellung effect. In other words, the Einstellung effect is likely to creep in if, for instance, a fraction is conceived as two whole numbers and not as one number.

The *Einstellung effect* implies that learners did not fully understand when to apply the procedure, and that interfered with the learning of the new procedure. My opinion is that the *Einstellung effect*, which is basically defined as cognitive fixation on the previously introduced procedure, and applying it to inappropriate contexts, is quite widespread in mathematics learning. One of the causes thereof could emanate from teaching, in the sense that if teachers do not assist learners in understanding the conditions under which certain mathematical rules or algorithms are applicable, learners are likely to generalise the applications thereof, even when it is not permitted. As I argued before, an investigative teaching approach could create an opportunity for learners to test the applicability of certain mathematical rules, thereby mitigating the *Einstellung effect* or cognitive fixation.

In addition, learners tend to make the denominators to be the same even when multiplying common fractions. Although the procedure is correct, I advance that it is not advisable for learners to follow that procedure to avoid working with big numbers which at the end need to be simplified by applying the knowledge of multiples and factors to simplify fractions, which learners generally do not do. After correctly making the denominators the same, learners would multiply the numerator and not the denominators, thus leaving the product with the denominator that is the same as one of the factors. Based on the above argument, I could infer that understanding of the rule of adding and subtraction of fractions has negatively influenced the learning of multiplication of fractions by learners. Chrysikou, Motyka, Nigro, Yang and Thompson-Schill (2016) also made a point that previous knowledge about solving new problems with similar features does not always lead to learning as it affects other procedures for computing with fractions. Munoz-Rubke, Olson, Will and James (2018) referred to this negative effect as functional fixedness. In addition, it was evident that learners seemed not to understand why the procedure of making the denominators to be the same works best with addition, subtraction, and division of fractions, and not with multiplication. This finding supports that learners seem not to understand that certain procedures are carried out only with certain problems and not all (Elbrink, 2008). I postulate that teaching of algorithms to learners and not allowing them (learners) to make conjectures to enhance meaningful knowledge construction, results in learners not knowing which procedure applies in which situation. This postulation is supported by Chan and Yeung (2001) who posit that an error can be classified as distorted information if learners show correct perceptible information.

To mitigate against the error mentioned above, I suggest that, in addition to many other strategies, teachers should consider teaching multiplication of fractions practically through paper folding and modelling, to enhance the understanding of the concept. The teacher could provide instruction of what learners should do, followed by deductions. This suggestion supports Pritchard's (2017) learning styles that learners learn through visual (seeing), auditory (listening) and kinaesthetic (doing). The instructional decision that could be employed to drive the suggestion would be to first teach multiplication of fractions through sharing using concrete objects, then draw diagrams and translate them to symbols. This could assist learners in understanding that sharing x number of sweets amongst b number of learners is the same as $\frac{x}{b}$ which could be represented as $\frac{1}{b} \times x$, where x and b in context are greater than zero. This could mean that two learners can receive $\frac{2x}{b}$ number of sweets which can be represented as $\frac{2}{b} \times x$, etc. This representation could be translated into a fraction of a whole. After having understood the concept of multiplying a fraction by a whole as discussed above, it could be extended to multiplication of a fraction by a fraction, using paper folding, modelling and symbols, simultaneously. This could be done by first multiplying two unitary fractions ($\frac{1}{a} \times \frac{1}{b}$), followed by multiplying a unitary by a non-unitary fraction ($\frac{1}{a} \times \frac{b}{c}$) and lastly multiplying two non-unitary fractions ($\frac{a}{b} \times \frac{c}{d}$), where non-unitary fractions are proper fractions and a, b, c and d are greater than zero. This proposition is consistent with the study conducted by Purwadi, Sudiarta and Suparta (2019) about the use of concrete materials, pictures and abstract symbols (CPA approach) as methods that can lead to conceptual understanding.

Other learners reported that they thought the subtraction sign was changed to a multiplication sign. This could be a result of learners being taught algorithms of computations with common fractions without understanding the reason for these algorithms, and this practice caused learners to come to their own conclusions that the algorithms apply in any problem type. Quinell (2010) also made the point that learners must be given many activities to explore in order for them to formulate conjectures, and not be given readily made rules or algorithms. However, in this study I wish to extend on Quinell ’s (2010) point above by advocating for purposeful activities designed such that they could drive the instruction towards clearing up learners' misconceptions.

Similarly, another common misconception regarding which I would like to share my insights, is that learners tend to incorrectly write a proper fraction as a mixed number. This reveals an inability to differentiate between the proper fraction and an improper fraction. These findings support the study by Milgram (2007) that one of the characteristics of mathematics is clear and precise definition of mathematical terms. Learners might have not understood when to write fractions as mixed numbers due to not making a meaningful connection between an improper fraction and a mixed number. Another reason could have been that converting mixed numbers to improper fractions was done in a very abstract way where rules were given without understanding where they (rules) come from– again this demonstrates that the implications on learning can be dire if teachers do not understand learners' errors. Since learners did not develop these rules on their own, they found it difficult to apply them correctly. Van de Walle et al. (2015) and Rushton (2018) also pointed out that affording learners with an opportunity to develop knowledge on their own is beneficial to the application of that knowledge.

Not understand the meaning of an equal sign supports the conclusion by Essien and Setati (2006) that learners view the equal sign as a means of writing the answer instead of a relational symbol. The implication for this error is that learners would not make sense of algebraic equations.

Based on prior research that learners' errors emanate from knowledge deficiencies or no understanding at all (Muthukrishnan, Kee & Sidhu, 2019), I conclude that learners who displayed the errors listed above could not solve algebraic expressions, algebraic equations and functions and relationships, as was the case in the summative test. Essentially, I contend that errors like the one involving the equal sign, point to lack of proficiency in mathematical language. If, for instance, a learner changes an equal sign to a multiplication sign in an equation, which in the lower grades was correctly referred to as number sentence, one can only conclude that the message communicated through the sentence expressed in numbers is not understood. Knuth, Stephens, McNeil & Alibali (2006) said that errors can hinder learning and lead to a lack of understanding regarding algebraic equations.

5.3.3 Symbolic errors

Learners committed symbolic errors by incorrectly assuming that mathematical problems that use identical symbols are similar. This was evident when learners ended up applying the rule of subtracting common fractions instead of the rule for division, and the rule of adding common fractions, instead of the rule of multiplying fractions. Learners must have been influenced by the fact that when they were taught multiplication and division of whole numbers the concept of multiplication and division were explained as repeated addition and repeated subtraction, respectively. I argue that the previous knowledge of the definition of these concepts, negatively influenced the learning of multiplication and division of common fractions. Importantly, the present research extends on earlier findings by Chrysikou, Motyka, Nigro, Yang and Thompson-Schill (2016) as alluded earlier on, that using previous knowledge to solve new problems with similar features sometimes has a negative effect on learning. This is what is referred to by Munoz-Rubke, Olson, Will and James (2018) as *functional fixedness*.

5.4 Implications of teacher’s understanding learner errors on learning

In all the misconceptions and errors that learners committed, several variations were noted, which showed a lack of conceptual understanding and resulted in a lack of procedural fluency. Research by Makhubele, Nkhoma and Luneta (2015); and Godden et al. (2013) concluded that a lack of conceptual understanding or misconceptions, mathematical generalisation, use of incorrect procedures and misapplication of a rule are some of the underlying causes of errors. This lack of conceptual knowledge was also noticed in the summative test with problems involving fractions. This problem could be attributed to a lack of remediation of misconceptions that learners held about fractions. When teachers do not understand, and effectively try to mitigate, learners’ errors, learners get glued to their misconceptions for a long period of time, and this subsequently leads to conceptual errors. This can have a negative effect on the learning of new concepts that is related to the concepts that learners hold misconceptions about and in relation to which they commit errors.

The tendency of teachers to focus on the answer and not the processes involved when solving a problem, as a result of failure to diagnose the causes of the misconceptions and errors, result in instructional decisions that do not target the errors. The use of the traditional approach by teachers where learners copy the solutions from the teachers’ examples, step by step, without understanding the procedure, may result in a misapplication of procedures, that leads to procedural errors. In addition, the over generalisation of rules which could be as a result of teaching algorithms, lead to symbolic errors as learners tend to associate related symbols with each other. Based on the above argument, a lack of understanding of the implications of teachers’ understanding of learners’ errors results in instruction that does not address errors. The lack of conceptual understanding and procedural knowledge are key elements of conceptual errors, procedural errors and symbolic errors.

5.5 Responding to the Research Questions

The primary aim of this section is to provide a brief reflection of data and findings associated with the four secondary research questions, and subsequently respond to the questions. I will also consolidate the responses to the secondary research questions and address the primary research question.

The study was directed by the primary research question: What are the implications of the teachers' understanding of learner errors on learning of school mathematics in the Senior Phase (specifically Grade 9)?

Four secondary research questions were used to explore the primary research question:

- How do teachers analyse learners' errors?
- What instructional decisions emanate from teachers' understanding of learners' errors?
- How do teachers' understanding of learners' errors inform learners' conceptual understanding in mathematics? and
- How do teachers' understanding of learners' errors inform learners' procedural knowledge in mathematics?

To answer the research questions, semi-structured interviews, document analysis and lesson observations were conducted. The research questions were aligned with the three data collection instruments and the theoretical lenses framing the study (see Table 6). The data analysis from tests was used to explore the implications of the teachers' understanding of learners' errors for mathematics learning and to get a first-hand understanding of learners' procedural and conceptual understanding. In addition, the data was used to explore the implications of teachers' understanding of learners' errors for teaching and learning, as identified through his or her analysis of the learner responses from the test. Data analysis from the teacher interviews was used to understand the teachers' awareness of the error, as well as procedural and conceptual explanations in relation to the error (Sapire et al., 2016). Data analysis from learner interviews were used gain more insights into, and corroborate, the errors that emerged. Data analysis from the observations was used to verify the extent to which the teacher addressed previously identified errors by the teacher, the promptness and the extent to which the emerging (unexpected) errors were addressed by the teacher, the classroom context, and the extent to which error analysis permeated learning. In other words, the implications of teachers' understanding of learners' errors as identified through his or her analysis of the learner responses from the test was explored in-depth using the aforementioned categories.

5.5.1 Secondary Research Question One

How do teachers analyse learners' errors?

Data related to this question was collected through documentary sources (learner responses from the tests), interviews and observations. Based on the discussion, teachers' analysis of learners' errors seemed to be superficial as the focus was on answers rather than processes, and careless or calculation errors rather than conceptual errors. In addition, as part of the analysis, teachers did not ask learners probing questions during teaching to get a deeper understanding of erroneous verbal responses given by learners. This is the view strongly advocated by CGI, where the focus is more on the thinking processes involved in obtaining the answer, than the answer itself (Guerrero, 2014; Hoosain & Chance, 2004). In other words, the analysis of learners' errors by teachers was conducted in such a way that it did not inform instruction and would not enhance conceptual understanding.

5.5.2 Secondary Research Question Two

What instructional decisions emanate from teachers' understanding of learners' errors?

The learner responses in the summative test showed that learners committed the same errors as in the diagnostic test. One reason that could be a contributing factor to this finding is a lack of feedback and appropriate instructional decision-making. Understanding learners' errors and understanding the implication of learners' errors on learning are two distinct enterprises. The latter is more intricate, albeit dependent on understanding learners' errors. Notwithstanding the fact that teachers could acknowledge their shortcomings regarding some aspects of error analysis without fear, which I view as the beginning of reflective practice, this could pose challenges for learning as instruction would not be planned in a manner as to demystify learners' misconceptions. To extend the research by Herholdt and Sapire (2014) that learners' errors may be effectively utilised for teaching and learning, I advance that this is possible only if teachers understand learners' errors and the appropriate instructional decisions are made to demystify learners' errors. Therefore, I am persuaded that, based on the above argument, teachers' analysis of learner errors did not inform teachers' instructional decisions.

5.5.3 Secondary Research Question Three:

How do teachers' understanding of learners' errors inform learners' conceptual understanding in mathematics?

Conceptual understanding involves making meaningful connections between concepts (Samuelsson, 2010). Given this assertion, I posit that conceptual understanding can make an essential contribution to mathematics proficiency among learners. Similarly, teachers' understanding of errors that may inhibit conceptual understanding, is vital. However, as discussed, teachers often highlighted computational errors and careless errors but fell short of penetrating to errors that were, according to my judgement, intricate in nature. In other words, teachers did not have a deep understanding of errors that that are of a conceptual nature. Because the misconceptions were not diagnosed and learners were not given a chance to reflect on their errors, they (learners) repeated these errors. Based on the above argument and observation, there was a lack of sufficient evidence suggesting that teachers' understanding of learners' errors informed learners' conceptual understanding.

5.5.4 Secondary Research Question Four

How do teachers' understanding of learners' errors inform learners' procedural knowledge in mathematics?

Procedural knowledge is an important aspect of mathematics teaching and learning. Through procedural knowledge, learners are able to use procedures in a flexible, correct and proper manner and perform basic calculation processes efficiently (Samuelsson, 2010). However, applying procedural knowledge blindly without understanding when and how to do so, is equally detrimental to mathematics proficiency.

Both teachers taught learners step-by-step procedures to solve problems and allowed them (learners) to copy the steps on the board, which was followed by activities to complete instead of allowing or encouraging learners to spend time solving problems on their own (Guerrero, 2014), and coming up with different ways to get to the solution (Carpenteret al., 2015).

Because learners did not meaningfully connect concepts and understand how they (concepts) link to procedures, they (learners) ended up using them in an inappropriate situation. The emerging trend of *Einstellung effect*, *whole-number bias* and *functional fixedness* that were revealed by learners' errors as discussed, are a sign that teaching was focused more on blind adherence to procedures without understanding. Therefore, the step-by-step procedures that were used by teachers to solve problems did not yield procedural fluency, but functional fixedness. The above submission would convince me that since teachers were possibly unaware that they were contributing to the entrenchment of cognitive fixation when dealing with mathematics procedures, their superficial understanding of procedural errors is unlikely to enhance procedural knowledge.

5.5.5 Responding to the Primary Research Question

What are the implications of the teachers' understanding of learner errors on learning of school mathematics in the Senior Phase (specifically Grade 9)?

In this study I investigated how teachers analysed learners' errors, including their understanding of learner errors, to inform learners' conceptual understanding, procedural fluency and instructional decision-making in mathematics. All these areas of investigation provided insights to answering the primary research question. This was done by collecting data through observations, interviews and tests. The two lenses framing the study, namely Constructivism and CGI, provided theoretical guidance on how mathematics learning should unfold. The reviewed literature provided a good sense of the research conducted in the field of my investigation. The integration of all the aspects guided data collection, analysis and discussions, and subsequently the response to the main research question.

Studies have shown that committing errors is part of learning and learners' errors present an opportunity to improve instruction (Freeman et al., 2014; Muthukrishnan, Kee & Sidhu, 2019). However, what was most salient in this study was the superficial way that teachers understand learners' errors and the non-utilisation of learners' errors to enhance effective learning of mathematics. A compelling practice of dealing with learners' errors, other than calling for the correct solution by both teachers, could neither be identified nor established. It could be argued, in this study, that the implications of this practice for mathematics learning are dire because:

- Teachers will continue to focus on correct answers, rather than the errors that learners made,
- In cases where errors are considered, the focus is likely to be on computational or symbolic or careless errors rather than deep conceptual errors that are likely to hinder effecting learning,
- Teachers are likely to resort to giving the correct solution, rather than utilising the learners' errors to clear up the misconception,

- The explanation of concepts is likely to assume a mere step-by-step approach without scaffolding to enhance conceptual understanding,
- Learners were taught algorithms, rather than being guided to discover the concept, and
- Learners are likely to use correct procedures inappropriately due to a lack of conceptual understanding (*Einstellung* effect).

Based on these implications, my assertion is that teachers did not view learners' errors as a tool for learning mathematics.

5.6 Nexus between the theoretical framework and the study

This section intends to demonstrate how the theoretical framework assisted me in conducting the study.

The reason I chose Cognitively-Guided Instruction and Constructivism as theoretical lenses underpinning this study was that both articulate factors that could benefit instruction and construction of knowledge. Thus, this framework addresses factors that serve as hindrances to the teaching-learning process, and subsequently student achievement. These theories assisted me in developing research questions, identifying the relevant literature for gaining insights into the study focus, and lastly, analysing and interpreting the research findings.

Cognitively Guided Instruction and Constructivism were key in responding to the main research question: *What are the implications of the teachers' understanding of learner errors on learning of school mathematics in the Senior Phase (specifically Grade 9)?* CGI is mainly concerned with teachers' understanding of the learners' cognitive processes, which, I argue, could be achieved through constructivism, the understanding of how learners construct knowledge.

Since the main objective of this study was to explore the implications of teachers' understanding of learner errors for the teaching and learning of mathematics, the study was mainly concerned with instructional practices and the construction of knowledge. Thus, there were two categories of factors that could benefit the study:

For instruction, these factors are “teachers’ *procedural* explanation in relation to the error, teachers’ *conceptual* explanation in relation to the error, teachers’ *awareness* of the error” (Sapire et al., 2016, p. 4). For construction of knowledge these factors include an in-depth understanding of how learners learn concepts meaningfully (Hox, & Boeije, 2005). All the above are situated in the learning theories that underpin the study.

Literature on errors and error analysis reviewed to conduct the study is closely related to instruction and construction of knowledge which are traits of both CGI and Constructivist respectively. Thus, classifying learner misconceptions and errors according to the views of different research studies, as discussed in the literature review, provided a broader lens of how the misconceptions and errors held or committed by the participants in this study can be viewed.

Data was collected from classroom observations, learners’ scripts (diagnostic and summative) and individual interviews with learners and teachers. CGI and Constructivist theory guided the data collection techniques. These techniques gave a broader picture of how instruction was carried out to construct knowledge. The framework was used during data analysis to determine whether causes of errors and misconceptions were because of instruction or knowledge construction. The responses from both teachers and learners in the interviews point to instruction and knowledge construction, as directly or indirectly linked to low learning achievements. This framework fits the purpose of this study, as I explored how teachers’ understanding of learner errors directly or indirectly influence learning in mathematics.

5.7 Limitations of the study

Despite the valuable findings yielded by the study, the limitations of the study are also acknowledged.

The time spent on lesson observations and learner interviews was insufficient due to other programmes that were taking place in the school such as extra mural activities and administration of quarterly tasks. Had I spent enough time on lesson observations and interviewing learners, I would have gained more insights into the teaching practice and how teachers deal with errors and misconceptions.

The study was a qualitative study and multiple case study design data was collected from only two schools within one sub-district. The small sample did not allow for generalisability. However, the findings are fascinating enough to make a case for a broad study to be conducted.

5.8 Recommendations

The study showed that learners have been taught algorithms with calculations involving fractions without understanding the concepts, resulting in procedural errors, and that teachers could not convince learners why certain, but not all, procedures work with certain calculations. Intensive training of educators on error analysis and how to handle and mitigate learners' errors on fractions is recommended.

This study also identified that, due to time constraints that hindered the learning of new concepts, learners could not be provided with feedback and remediation to errors that they (learners) committed in the diagnostic test. It is therefore recommended that policy makers consider the realistic factors, such as comparing the amount of work to be covered in the grade and the amount of time available, that affect learning in schools.

Due to the small sample used in this study, researchers who wish to investigate the effect of error analysis on mathematics instruction may expand on the findings of this study with other research methods that allow for generalisability.

5.9 Reflection

My aspiration is that this study will assist teachers to not use error analysis only when informing instruction to the next cohort of learners, but to use error analysis for the same cohort of learners as informed by interviews, so that the instruction can appropriately address the errors. Therefore, error analysis can best be utilised during the assessment for learning where each learner can be afforded an opportunity to explain the reasons behind his or her solutions that lead to errors and be able to adjust his or her understanding. This study had a major influence on how I view the relevance of error analysis for instruction and the importance of teachers understanding learner errors in mathematics.

5.10 Conclusion

This chapter concludes the study, which explored teachers' understanding of learner errors in mathematics. Since this was a qualitative study through the multiple case study design and located within the interpretivist paradigm, it was conducted through studying the participants in their natural settings, namely two Grade 9 schools from Eastern Cape Province. The theoretical lenses that guided the study were Cognitively Guided Instruction which is mainly concerned with teaching practice, and Constructivism which mainly focuses on knowledge construction.

The key findings of the study can be summarised as follows: Firstly, the analysis of learners' errors conducted by teachers seemed to be superficial as the focus was on answers rather than processes (Guerrero, 2014; Hoosain & Chance, 2004). Secondly, teachers' understanding of learners' errors did not enhance learners' conceptual understanding, and procedural fluency, as the intricacies of errors and misconceptions were not diagnosed. Finally, the conceptual and procedural errors that learners held in mathematics did not inform instructional decisions. The implications of the emerging trends on learning seemed to suggest that teachers' instructional decision-making was likely to be incompatible with learner errors or misconceptions. This could be attributed to the lack of nexus between what teachers perceived as errors and misconceptions and the reason for the errors emanating from learners' verbal explanations.

Even though I cannot make definitive conclusions apropos of the implications of teachers' understanding of learner errors in mathematics based on a case study, I can draw several lessons from the teachers' experiences with error analysis as a strategy to optimise mathematics teaching and learning:

Firstly, there is a call for a significant change of approach to error analysis within the mathematics education community, especially regarding the teachers. There is sufficient evidence that teacher focus on this area is minimal, and that considerable improvement in learner conceptual understanding and then procedural fluency would be possible with changes in teachers' instructional decision making. Teachers need to utilise learner errors to inform instructions. They seem to be constantly focusing on the utilisation of algorithms rather than enhancing the development of the conjectures though learner errors.

Secondly, effective teaching and learning involves continually analysing learner errors, where the teachers choose appropriate learning activities based on the errors, which provide scaffolding for demystifying learner misconceptions and support learning.

Lastly, if error analysis is to be an integral part of the teaching practice and learning process, and its status to improve learning is to be maintained, a major investment in teachers needs to be made. Muthukrishnan et al. (2019) summarises the effects of not utilising learner errors and say that the failure to analyse learner errors poses a potential threat to the acquisition of skills needed to learn mathematics. For effective teaching and learning of mathematics, error analysis should not only revolve around identifying errors, especially after summative assessment is administered, but also be engrained in formative assessments and inform instruction.

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ANNEXURES

ANNEXURE A: INTERVIEW SCHEDULE FOR LEARNERS

SCHOOL: ________________________ LEARNER CODE: _____________

The purpose of this schedule is to elicit the learner's explanation of the conceptual and/or procedural errors of the specific concept revealed from the error analysis.

……… thank you for your time to participate in this interview. *The purpose of the interview is to get your explanation to your answers as they appear on your answer sheet.* *As you have noted in the consent form you have completed, please be reminded that this interview will be recorded and all your responses will be kept confidential.* *The duration of this interview will be approximately* ***15 minutes****.*

N.B. Since the nature of the errors cannot be preempted, the interview schedule for learners will depend on the errors as identified from the learner's script.

1. Can you please explain how you came with your solution in this (these) problem (s).
2. Do you have any question you would like to ask pertaining to the questions from the tests?

Thank you for your time.

ANNEXURE B: OBSERVATION SHEET

SCHOOL: ____________________________________

The focus of the observation is to describe the classroom context as it appears at the time of observation and to gather data on how the lesson addressed the misconceptions that transpired from the learners.

1. Is the classroom conducive for learning to take place? | Y | N |

Comment on issues observed (e.g. number of learners, seating arrangement, availability of chalkboard and other resources________________________________

____

____

____

2. Does the teacher address the errors that are associated with the concept taught? | Y | N |

Comment on how this was or was not achieved____________________________

____

____

____

3. Were there learners' errors that emerged during the lesson? | Y | N |

Comment on how they were addressed by the teacher: _______________________

____

____

____

4. Did the teacher's handling of and attention to learners' errors seem to have assisted learners' conception? | Y | N |

Comment of how this was or was not achieved in relation to impacting on learning____

____

5. Were learners actively involved in the lesson? | Y | N |

Comment on how this was achieved or not achieved: _________________________

____

____

____

6. Were the objectives of the lesson met? | Y | N |

Detailed justification: ____

____

____

____

____

ANNEXURE C1: GRADE 9 MATHEMATICS TEST

FEBRUARY 2019 TIME: 45 minutes

SCHOOL CODE	:	
DATE	:	
LEARNER CODE	:	

This question paper consists of 5 pages, including the cover page.

Information and Instructions to the learner

1. This test consists of **9** questions.

2. Read the questions carefully.

3. Answer **ALL** the questions.

4. Write neatly and legibly.

5. Number your answers exactly as questions are numbered.

6. Clearly show **ALL** the calculations in the space provided in each question., If you need more space for your calculations please use the reverse side of the question paper.

7. Do not use any calculator.

Simplify the following. Show all the calculation steps used in obtaining the answer	
1.	$\frac{5}{7} - \frac{2}{3}$
2.	$\sqrt{\frac{4}{25}} + \left(\frac{2}{3}\right)^2$
3.	$\sqrt[3]{\frac{8}{27}} - \left(\frac{1}{2}\right)^3$

4.	$\frac{5}{6}$ of 58
5.	$\frac{4}{5} \times \frac{3}{5}$
6.	$\frac{4}{5} \times 2\frac{3}{8}$
7.	$28 \div \frac{7}{4}$
8.	$\frac{16}{15} \div \frac{4}{5}$

9.	$3\frac{4}{7} \div \frac{5}{7}$

ANNEXURE C2: GRADE 9 MATHEMATICS TEST

Province of the
EASTERN CAPE
Education

MATHEMATICS TEST: MARCH 2019

GRADE 9

MARKS: 50   **DURATION: 1 Hour**

This question paper consists of 7 pages, including the cover page.

Instructions and information to the learner

8. Read the questions carefully.

9. Answer **ALL** the questions.

10. Write neatly and legibly.

11. Number your answers exactly as questions are numbered.

12. Clearly show **ALL** the calculations, diagrams, graphs, etcetera you have used in determining the answers.

13. You may use an approved scientific calculator (non-programmable and non-graphical).

14. This question paper consists of **5** questions.

15. Diagrams are **NOT** drawn to scale.

QUESTION 1

In this question, write only the letter for the **correct answer** next to the corresponding number, e.g. If the correct answer in 1.1 is D, you should only write 1.1 D.

1.1 Complete: $\frac{1}{3}$ is… (1)

A. neither a real nor rational number

B. both a real and rational number

C. only a rational number

D. only a real number

1.2 What is the HCF of 324 and 540? (1)

A $5 \times 3 \times 3 \times 3 \times 3 \times 2$

B $5 \times 3 \times 3 \times 3 \times 2 \times 2$

C $3 \times 3 \times 3 \times 3 \times 2$

D $3 \times 3 \times 3 \times 2 \times 2$

1.3 What is $\frac{20x^2y-10xy^2}{5xy}$ when simplified? (1)

A. $15x - 5y$

B. $4x - 2y$

C. $6xy$

D. $2xy$

1.4 Complete: $\sqrt{\frac{900}{0,1\times0,1}} = \ldots$ (1)

A 3 000

B 300

C 30

D 3

1.5 What is the solution to $(x-2)(x+3)=0$? (1)

A $x=-2$ and $x=-3$

B $x=2$ and $x=-3$

C $x=-2$ and $x=3$

D $x=2$ and $x=3$

[05]

QUESTION 2

2.1 Write 0,0000000672 in scientific notation. (2)

2.2 Calculate without using a calculator. Show all the calculation steps.

2.2.1 $3{,}8 \times 10^{4} \times 2{,}3 \times 10^{-8}$ (leave your answer in scientific notation) (2)

2.2.2 $\sqrt{\sqrt{\frac{256}{10\,000}} + \frac{3}{5}}$ (2)

2.3 Evaluate:

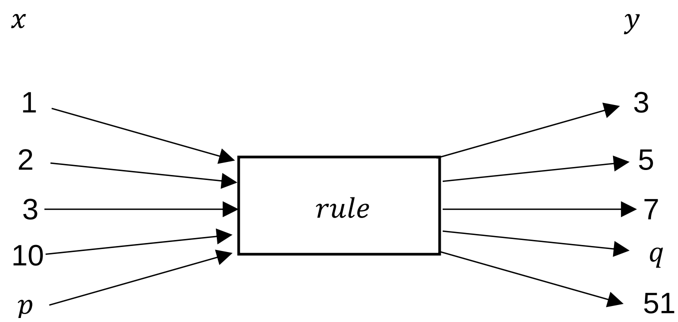
2.3.1 $2^{2018} - \frac{2^{2019}}{2}$ (2)

2.3.2 (12345655 × 12345653) – (12345651 × 12345657) (2)

[10]

QUESTION 3

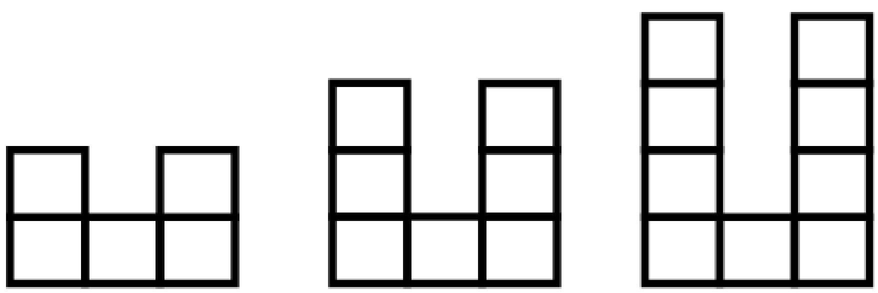
3.1 Study the flow diagram below and answer the questions that follow.



3.1.1 What is the general rule that describes the relationship between x and y? (2)

3.1.2 Complete the flow diagram by determining the value of q and p. (2)

3.2 The pattern below is formed by squares.



Stage 1 Stage 2 Stage 3

3.2.1 Write the general rule that describes the relationship between the stage number and the number of squares used in the form of (2)

$T_n = \dots$

3.2.2 How many squares will be in 10^{th} stage? (1)

[07]

QUESTION 4

4.1 Simplify:

4.1.1 $\sqrt{45q^6 - 9q^6}$ (2)

4.1.2 $(x-3)(2x-1) - 2(x-2)^2$ (3)

4.1.3 $\dfrac{x^2y^2 \times (x^{-2})^2}{x^{-2}}$ (3)

4.2 Solve for x.

4.2.1 $x + 8 = 12$ (1)

4.2.2 $\dfrac{x+1}{4} - \dfrac{x-1}{2} = 3$ (3)

4.2.3 $\dfrac{2x^3}{3} = 5\dfrac{1}{3}$ (2)

4.2.4 $x = y^2 - 4$, if $y = -2$ (2)

[16]

QUESTION 5

5.1 The table below represents the number of books of the same type and their mass in kg. Study the table and answer the questions that follow:

Number of books	3	6	9	12
Mass in kg	10	20	30	40

Is the number of books directly or indirectly proportional to the mass? Give a reason for your answer. (2)

5.2 How long will it take R2000 to yield to a simple interest of R480 if invested at 6% per annum? (3)

5.3 What amount will a loan of R5 200 yield to if invested at 7.5% per annum, compound interest for 3 years? Round off your answer to the nearest cents. (3)

5.4 Miss X can cover a certain distance in 6 hours at an average speed of 120km/h. At what average speed must she drive to complete her trip in 8 hours.? (4)

[12]

TOTAL = 50

ANNEXURE D: INTERVIEW SCHEDULE FOR TEACHERS

SCHOOL: ______________________________

Ms/Mr……… thank you for your time to participate in this interview.

The purpose of the interview is to get your understanding about learner's errors and error analysis.

As you have noted in the consent form you have completed, please be reminded that this interview will be recorded, and all your responses will be kept confidential.

The duration of this interview will be approximately ***30 minutes****.*

1. What do you understand by learners' errors in mathematics?
2. What is your understanding of error analysis in mathematics?
3. How would you conduct error analysis? (You may use an example to illustrate your answer).
4. How important is it for you to understand learners' errors in relation to:

(a) lesson planning?

(b) teaching the lesson?

(c) assessment (formal and informal)?

5. How important is your understanding of learners' errors for the learning of mathematics?
6. How would you use learners' common errors or common misconceptions when setting multiple choice questions?
7. When is it appropriate for you to conduct error analysis?
8. After identifying the nature of the error(s), what do you do with them?
9. Do you have any comment you would like to make pertaining to the importance of error analysis to the learning of mathematics?

Thank you for your time.

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

P.O. Box 1259
Matatiele
4730
01 October 2018

Superintendent-General
Eastern Cape Department of Education
BISHO
5605

Dear Sir

REQUEST FOR PERMISSION TO CONDUCT RESEARCH

I am currently enrolled for a Masters degree [Science, Mathematics & Technology Education (SMTE)] at the University of Pretoria. Part of the requirements for the awarding of this degree is the successful completion of a research project in the field of education.

The title of my approved research study is **"Exploring the implications of teachers' understanding of learner errors in mathematics".** This study is concerned with teaching and learning of mathematics through the analysis and utilisation of learners' errors.

I hereby request permission to conduct research in your province, which aims to address the following questions:

- How do teachers analyse learners' errors?
- What instructional decisions emanate from teachers' understanding of learners' errors?
- How does teachers' understanding of learners' errors inform learners' conceptual understanding in mathematics?
- How does teachers' understanding of learners' errors inform learners' procedural knowledge in mathematics?

I am planning to conduct my research in January to March 2019 with two schools (rural and urban schools) that are offering Grade 9 in Alfred Nzo West district.

It is my presumption that the research findings will make a creditable contribution towards improving learner conceptual understanding and procedural knowledge in mathematics.

Please note that should you grant me permission you will be requested to release a signed letter permitting this study to take place. Attached to this letter is the ECDoE Research Request Form, the research proposal and letters to all participants for their permission.

Signature:
Ms C.F. Mtumtum
Student Researcher
University of Pretoria
u18310398@tuks.co.za

Signature:
Dr R.D. Sekao
Supervisor
University of Pretoria
david.sekao@up.ac.za

ANNEXURE E2: LETTER TO THE DIRECTOR

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA
Faculty of Education

P.O. Box 1259

Matatiele

4730

01 October 2018

The District Director

Alfred Nzo West District

Eastern Cape Department of Education

Dear Sir

RE: REQUEST FOR PERMISSION TO CONDUCT RESEARCH

I am currently enrolled for a Masters degree [Science, Mathematics & Technology Education (SMTE)] at the University of Pretoria. Part of the requirements for the awarding of this degree is the successful completion of a significant research project in the field of education.

The title of my approved research study is "Exploring the implications of teachers' understanding of learner errors in mathematics". This study is concerned with teaching and learning of mathematics through the analysis and utilisation of learners' errors.

I hereby request permission to conduct research in your district, which aims to address the following questions:

- How do teachers analyse learners' errors?
- What instructional decisions emanate from teachers' understanding of learners' errors?
- How does teachers' understanding of learners' errors inform learners' conceptual understanding in mathematics?
- How does teachers' understanding of learners' errors inform learners' procedural knowledge in mathematics?

I am planning to conduct my research in January to March 2019 with two schools (rural and urban schools) that are offering Grade 9 in Alfred Nzo West district.

Attached are the letters to school principals, teachers and parents requesting permission to conduct research together with consent forms. It is my presumption that the research findings will make a creditable contribution towards improving learner conceptual understanding and procedural knowledge in mathematics.

Please note that should you grant me permission you will be requested to release a signed letter permitting this study to take place.

Signature:
Ms C.F. Mtumtum
Student Researcher
University of Pretoria
u18310398@tuks.co.za

Signature:
Dr R.D. Sekao
Supervisor
University of Pretoria
david.sekao@up.ac.za

ANNEXURE E3: LETTER TO THE PRINCIPAL

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

P.O. Box 1259

Matatiele

4730

01 October 2018

The Principal

Dear Sir/Madam

RE: PARTICIPATION IN A RESEARCH PROJECT

I am currently enrolled for a Masters degree [Science, Mathematics & Technology Education (SMTE)] at the University of Pretoria. Part of the requirements for the awarding of this degree is the successful completion of a significant research project in the field of education.

The title of my approved research study is "Exploring the implications of teachers' understanding of learner errors in mathematics". This study is concerned with the teaching and learning of mathematics through the analysis and utilisation of learners' errors.

Your school is hereby invited to participate in this research project, which aims to address the following research questions:

- How do teachers analyse learners' errors?
- What instructional decisions emanate from teachers' understanding of learners' errors?
- How does teachers' understanding of learners' errors inform learners' conceptual understanding in mathematics?
- How does teachers' understanding of learners' errors inform learners' procedural knowledge in mathematics?

Below is the scope and responsibility of your participation:

To gather information, this study will involve the observation of learners in the classroom during mathematics lessons in Grade 9 with minimal disruption of teaching and learning for the duration of Term 1 in 2019. I will be a passive participant who will do audio recordings and take field notes while the teacher and the learners are busy in class. The teacher(s) involved, will analyse some errors made by learners in order to understand the reason behind these errors.

This research project will also involve interviews with Grade 9 mathematics teachers and learners during teacher’s free periods, breaktime or after school hours. With the learners, the interviews will seek to get explanations to their solutions of mathematical problems in trying to gain an in-depth understanding of how they view certain mathematical concepts. With teachers the interviews will seek to understand their (teachers’) awareness of the errors that learners make. The interview, with your permission, will be audiotaped to ensure that accurate information is captured.

Please note that the decision for your school to participate is completely voluntary and that permission for your participation will also be protected by the Eastern Cape Department of Education. In addition, please note that each individual’s participation in the study will be completely voluntarily and will in no way either advantage or disadvantage them. Each participant will be free, at any stage during the process up to and including the stage at which they authenticate the transcript of their interview, to withdraw their consent to participate, in which case their participation will end immediately without any negative consequences. Any and all data collected from them up to that point in the study will then be discarded.

All the information obtained during the research study will be treated confidentially and will be utilised strictly for the purpose of the study. At no time will either your school or any of the individual participants be mentioned by name or be identified by any means in the research report.

At the end of the research study you will be provided with a copy of the research report containing both the findings of the study and recommendations. This research study presents a unique opportunity for your school to get involved in the process of research to improve the teaching and learning of mathematics.

If you grant permission for your school to participate in the study, kindly complete the consent form at the end of this letter.

Thanking you for your consideration in this research study.

Signature:
Ms CF Mtumtum
Student Researcher
University of Pretoria
u18310398@tuks.co.za

Signature:
Dr RD Sekao
Supervisor
University of Pretoria
david.sekao@up.ac.za

ANNEXURE E4: LETTER TO THE TEACHERS

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA
Faculty of Education

P.O. Box 1259
Matatiele
4730
01 October 2018

Dear Colleague

RE: PARTICIPATION IN A RESEARCH PROJECT

I am currently enrolled for a Masters degree [Science, Mathematics & Technology Education (SMTE)] at the University of Pretoria. Part of the requirements for the awarding of this degree is the successful completion of a significant research project in the field of education.

The title of my approved research study is "Exploring the implications of teachers' understanding of learner errors in mathematics". This study is concerned with the teaching and learning of mathematics through the analysis and utilisation of learners' errors.

You are hereby invited to participate in this research project, which aims to address the following research questions:

- How do teachers analyse learners' errors?
- What instructional decisions emanate from teachers' understanding of learners' errors?
- How does teachers' understanding of learners' errors inform learners' conceptual understanding in mathematics?
- How does teachers' understanding of learners' errors inform learners' procedural knowledge in mathematics?

Below is the scope and responsibility of your participation:

To gather information, this study will involve the observation of learners in the classroom during mathematics lessons in Grade 9 with minimal disruption of teaching and learning for the duration of Term 1 in 2019. I will be a passive participant who will do audio recordings and take field notes while you and your learners are busy in class. You will be requested to conduct error analysis for each test written by learners.

You and your learners will be interviewed during free periods or breaktime or after school hours. With the learners, the interviews will seek to get explanations to their solutions of mathematical problems in trying to gain an in-depth understanding of how they view certain mathematical concepts. In your case the interviews will seek to understand your awareness of the errors that learners make. The interview, with your permission, will be audiotaped to ensure that accurate information is captured.

Please note that the decision for you to participate is completely voluntary and that permission for your participation will also be protected by the Eastern Cape Department of Education. In addition, please note that your participation in the study will be completely voluntarily and will in no way either advantage or disadvantage you. As a participant you will be free, at any stage during the process up to and including the stage at which you authenticate the transcript of your interview, to withdraw your consent to participate, in which case your participation will end immediately without any negative consequences. Any and all data collected from you up to that point in the study will then be discarded.

All the information obtained during the research study will be treated confidentially and will be utilised strictly for the purpose of the study. At no time will either you, your school or any of the individual participants be mentioned by name or be identified by any means in the research report.

At the end of the research study you will be provided with a copy of the research report containing both the findings of the study and recommendations. This research study presents a unique opportunity for you and your school to get involved in the process of research to improve mathematics teaching and learning.

If you agree to participate in the study, kindly complete the consent form at the end of this letter.

Thanking you for your consideration to participate in this research study.

Signature:	Signature:
Miss CF Mtumtum	Dr RD Sekao
Student Researcher	Supervisor
University of Pretoria	University of Pretoria

ANNEXURE E5: LETTER TO THE PARENTS

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

P.O. Box 1259
Matatiele
4730
01 October 2018

Dear Parent

RE: REQUEST FOR YOUR CHILD TO PARTICIPATE IN THE RESEARCH PROJECT

I am currently enrolled for a Masters degree [Science, Mathematics & Technology Education (SMTE)] at the University of Pretoria. Part of the requirements for the awarding of this degree is the successful completion of a significant research project in the field of education

The title of my approved research study is "Exploring the implications of teachers' understanding of learner errors in mathematics". This study is concerned with teaching and learning of mathematics though the analysis and utilisation of learners' errors.

Your child is hereby invited to participate in this research project, which aims to understand:

- How do teachers analyse learners' errors?
- What instructional decisions emanate from teachers' understanding of learners' errors?
- How does teachers' understanding of learners' errors inform learners' conceptual understanding in mathematics?
- How does teachers' understanding of learners' errors inform learners' procedural knowledge in mathematics?

Below is the scope and responsibility of your child’s participation. To gather information, this study will involve the observation of children in the classroom during mathematics lessons in Grade 9 with minimal disruption of teaching and learning for the duration of Term 1 in 2019. I will be a passive participant who will do audio recordings and take field notes while the teacher and the learners are busy in class. Your child will be part of the learners in the class that I will be observing. I will not be teaching your child, but I will be present in class when his/her teacher teaches them. Together with your child’s teacher, we will analyse the errors made by all learners after assessment in order to understand these reasons behind these errors.

This research project will also involve interviews with Grade 9 mathematics learners during their free periods, breaktime or after school hours. The interviews will seek to get explanations to the learners’ solutions of mathematical problems so as to gain an in-depth understanding of how they view certain mathematical concepts. The interview, with your permission, will be audiotaped to ensure that accurate information is captured.

Please note that the decision for your child’s participation in the research is completely voluntary and will in no way either advantage or disadvantage your child. His/her permission for your participation will also be protected by the school. Your child will be free, at any stage during the process up to and including the stage at which he/she authenticates the transcript of his/her interview, to withdraw his/her consent to participate, in which case his/her participation will end immediately without any consequences. Any and all data collected from him/her up to that point in the study will then be discarded.

All the information obtained during the research study will be treated confidentially and will be utilised strictly for the purpose of the study. At no time will either your child, the school or any of the individual participants be mentioned by name or be identified by any means in the research report.

If you decide to allow your child’s participation, kindly complete the consent form at the end of this letter.

Thanking you for your consideration in this research study.

Signature:
Ms CF Mtumtum
Student Researcher
University of Pretoria
u18310398@tuks.co.za

Signature:
Dr RD Sekao
Supervisor
University of Pretoria
david.sekao@up.ac.za

ANNEXURE F: LETTER OF CONSENT - EASTERN CAPE DoE

Province of the
EASTERN CAPE
EDUCATION

STRATEGIC PLANNING POLICY RESEARCH AND SECRETARIAT SERVICES
Steve Vukile Tshwete Complex • Zone 6 • Zwelitsha • Eastern Cape
Private Bag X0032 • Bhisho • 5605 • REPUBLIC OF SOUTH AFRICA
Tel: +27 (0)40 608 4773/4035/4537 • Fax: +27 (0)40 608 4574 • Website: www.ecdoe.gov.za

Enquiries: B Pamla Email: babalwa.pamla@ecdoe.gov.za Date: 28 January 2019

Ms Cebisa Faith Mtumtum

Maluti District Office

Private Bag x 1835

Maluti

4740

Dear Ms. CF Mtumtum

PERMISSION TO UNDERTAKE A MASTERS' STUDY: EXPLORING THE IMPLICATIONS OF TEACHER'S UNDERSTANDING OF LEARNER'S ERRORS FOR MATHEMATICS LEARNING

1. Thank you for your application to conduct research.
2. Your application to conduct the above mentioned research from 2 selected schools of Maluti sub-district under the jurisdiction of Alfred Nzo West District of the Eastern Cape Department of Education (ECDoE) is hereby approved based on the following conditions:
    a. there will be no financial implications for the Department;
    b. institutions and respondents must not be identifiable in any way from the results of the investigation;
    c. you seek parents' consent for minors;
    d. it is not going to interrupt educators' time and task;
    e. you present a copy of the <u>written approval letter</u> of the Eastern Cape Department of Education (ECDoE) to the Cluster and District Directors before any research is undertaken at any institutions within that particular district;
    f. you will make all the arrangements concerning your research;
    g. the research may not be conducted during official contact time;

PGDP
EASTERN CAPE

Ilizwe elingaphambili!

building blocks for growth

Page 1 of 2

h. should you wish to extend the period of research after approval has been granted, an application to do this must be directed to Chief Director: Strategic Management Monitoring and Evaluation;

i. your research will be limited to those institutions for which approval has been granted, should changes be effected written permission must be obtained from the Chief Director: Strategic Management Monitoring and Evaluation;

j. you present the Department with a copy of your final paper/report/dissertation/thesis free of charge in hard copy and electronic format. This must be accompanied by a separate synopsis (maximum 2 – 3 typed pages) of the most important findings and recommendations if it does not already contain a synopsis.

k. you present the findings to the Research Committee and/or Senior Management of the Department when and/or where necessary.

l. you are requested to provide the above to the Chief Director: Strategic Management Monitoring and Evaluation upon completion of your research.

m. you comply with all the requirements as completed in the Terms and Conditions to conduct Research in the ECDoE document duly completed by you.

n. you comply with your ethical undertaking (commitment form).

o. You submit on a six monthly basis, from the date of permission of the research, concise reports to the Chief Director: Strategic Management Monitoring and Evaluation

3. The Department reserves a right to withdraw the permission should there not be compliance to the approval letter and contract signed in the Terms and Conditions to conduct Research in the ECDoE.

4. The Department will publish the completed Research on its website.

5. The Department wishes you well in your undertaking. You can contact the Director, Ms. NY Kanjana on the numbers indicated in the letterhead or email nelisa.kanjana@ecdoe.gov.za should you need any assistance.

NY KANJANA
DIRECTOR: STRATEGIC PLANNING POLICY AND RESEARCH

FOR SUPERINTENDENT-GENERAL: EDUCATION

ANNEXURE G1: LETTER OF CONSENT - SCHOOL PRINCIPAL

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

VOLUNTARY PARTICIPATION IN THE RESEARCH PROJECT

Topic: Exploring the implications of teachers' understanding of learner errors in mathematics

I,……………………………………….the principal of ………………………voluntarily and willingly agree that my school participates in the above-mentioned study introduced and explained to me by Ms CF Mtumtum, currently an enrolled Masters degree student at the University of Pretoria. I further declare that I understand, as was explained to me by the researcher, the aim, scope, purpose, possible consequences and benefits, and methods of collecting data. I understand that the researcher subscribes to the principles of:

- Voluntary participation in research, implying that the participants might withdraw from the research at any time without providing reasons.
- Informed consent, meaning that research participants must at all times be fully informed about the research process and purpose, and must give consent to their participation in the research.
- Safety in participation; that the human participants should not be placed at risk or harm of any kind.
- Privacy, meaning that the confidentiality and anonymity of human participants will be protected at all times.
- Trust, which implies that human participants will not be subjected to any acts of deception or betrayal in the research process or its published outcomes.

________________	________________	______________Date	School stamp
Principal	Signature		

ANNEXURE G2: LETTER OF CONSENT - TEACHER AS PARTICIPANT

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

VOLUNTARY PARTICIPATION IN THE RESEARCH PROJECT.

Topic: Exploring the implications of teachers' understanding of learner errors in mathematics

I,……………………………………………………..…the Grade 9 mathematics teacher at……………………………………voluntarily and willingly agree to participate in the above-mentioned study introduced and explained to me by Ms CF Mtumtum, currently an enrolled Masters degree student at the University of Pretoria.

I understand that I will participate in audio recorded interviews and that audio-recorded lessons will be done when I am teaching. I further declare that I understand, as was explained to me by the researcher, the aim, scope, purpose, possible consequences and benefits, and methods of collecting data. I understand that the researcher subscribes to the principles of:

- Voluntary participation in research, implying that the participants might withdraw from the research at any time without providing reasons.
- Informed consent, meaning that research participants must at all times be fully informed about the research process and purpose, and must give consent to their participation in the research.
- Safety in participation; that the human participants should not be placed at risk or harm of any kind.
- Privacy, meaning that the confidentiality and anonymity of human participants will be protected at all times.
- Trust, which implies that human participants will not be subjected to any acts of deception or betrayal in the research process or its published outcomes.

Teacher	Signature	Date	School Stamp

ANNEXURE H: PARENTAL CONSENT FOR THE CHILD'S PARTICIPATION

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

VOLUNTARY PARTICIPATION IN THE RESEARCH PROJECT.

Topic: Exploring the implications of teachers' understanding of learner errors in mathematics

I,…………………………………..……the parent of……………………………voluntarily and willingly agree that my child participates in in the above-mentioned study to be conducted in his/her school.

By signing below, I am consenting that my child participates in the lessons conducted by his/her teacher for research purposes and write any work given to him/her, which may be recorded by the researcher.

I understand that my child will be involved in audio recorded interviews and that photographs of his/her work will be taken (The photographs will only be of his/her work and not of him/her).

I further declare that I understand, as was explained to me by the researcher, the aim, scope, purpose, possible consequences and benefits, and methods of collecting data. I understand that the researcher subscribes to the principles of:

- **Voluntary participation in research** - implying that the participants might withdraw from the research at any time without providing reasons.
- **Informed consent** - meaning that research participants must at all times be fully informed about the research process and purpose and must give consent to their participation in the research.

- **Safety in participation** - that the human participants should not be placed at risk or harm of any kind.
- **Privacy** - meaning that the confidentiality and anonymity of human participants will be protected at all times.
- **Trust** - which implies that human participants will not be subjected to any acts of deception or betrayal in the research process or its published outcomes.

____________________	____________________	____________________
Child/learner	Signature	Date
____________________	____________________	____________________
Parent	Signature	Date