THE RELATIONSHIP BETWEEN CONCEPTUAL AND PROCEDURAL KNOWLEDGE IN CALCULUS

by

Janine Esther Hechter

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PROMOTOR: PROF. G. STOLS

CO-SUPERVISOR: DR. C. COMBRINCK

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Declaration

I, Janine Esther Hechter, student number u04367138, hereby declare that this thesis, "*The relationship between conceptual and procedural knowledge in calculus,*" submitted in accordance with the requirements for the Philosophiae Doctor degree to the University of Pretoria, is my own original work and has not previously been submitted to any other institution of higher learning. All sources cited or quoted in this research paper are indicated and acknowledged through a comprehensive list of references.

Ethics statement

The author, whose name appears on the title page of this thesis, has obtained, for the research described in this work, the applicable research ethics approval. The author declares that he/she has observed the ethical standards required in terms of the University of Pretoria's *Code of ethics for researchers and the Policy guidelines for responsible research.*

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Janine Esther Hechter

6 April 2020

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CC Ms Bronwynne Swarts
Prof Gerrit Stols
Dr Celeste Combrinck

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- Informed consent/assent,
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- Data storage requirements.

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Abstract

Literature describes different stances with respect to conceptual and procedural mathematical knowledge. The concept-driven versus skills-orientated perspectives have led to "*math wars"* between researchers, while some mathematics education specialists advocate that the five strands of mathematical proficiency should be seen as interconnected. Conceptual knowledge is the knowledge of concepts or principles, and procedural knowledge the knowledge of procedures. Both types of knowledge are critical components of mathematical proficiency. This study used a mixed methods design to analyse the relationship between conceptual and procedural knowledge. The qualitative content analysis investigated relations between procedural and conceptual knowledge *within* the solutions of 33 calculus items. The analysis included the number of procedural and conceptual steps needed to answer the item, item label and item classification into one of four knowledge classes based on the type and quality of knowledge. The items were included in a data collection instrument used for quantitative analysis. Rasch analysis was performed to measure item difficulty and person proficiency, and describe the underlying cognitive construct *between* items. The Rasch person–item map confirmed that items were not clustered together per class and that item difficulty was not linked to the number of procedural and/or conceptual steps needed to do the mathematics. Confirmatory factor analysis showed over-correlation between classes and that defined classes cannot be separated, confirming integration of procedural and conceptual cognitive processes. The relationship between procedural and conceptual knowledge *within* and *between* items is complex. Findings indicated that item solutions drew on both procedural and conceptual components that cannot be separated. Solutions could follow more than one approach and analyses could differ, since what is conceptual for one student could be procedural for another. Therefore, teaching strategies should navigate between concepts and procedures, methods and representations.

Key Terms: Procedural knowledge, conceptual knowledge, calculus, mixed methods, content analysis, Rasch analysis, confirmatory factor analysis.

Language editor

M.B. BRADLEY

P.O. Box 37326
Faerie Glen
Pretoria 0043

072 369 5149

DECLARATION ON EDITING

Student: J.E. Hechter

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M.B. BRADLEY (MA) - Language editor

List of abbreviations

AVGE MEAS	average measure
C+P-	conceptual+ procedural- knowledge category
C+P+	conceptual+ procedural+ knowledge category
CFA	confirmatory factor analysis
C-P-	conceptual- procedural- knowledge category
C-P+	conceptual- procedural+ knowledge category
CPI	conceptual performance index
DIF	differential item functioning
DoE	Department of Education
FTOC	fundamental theorem of calculus
MCQ	multiple-choice questions
MNSQ	mean squares residuals
PCA	principal component analysis
PPI	procedural performance index
PTMA	point biserial correlations
RMT	Rasch measurement theory
SD	standard deviation
SEM	structural equation modelling
UNWTD	number of items per category
ZSTD	mean standardised fit statistics

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1. CHAPTER 1: INTRODUCTION

1.1 BACKGROUND AND LEARNING THEORIES

Learning theories describe the way in which knowledge is organised and conceptual development occurs in individuals. The theories could be applied to understand student learning and inform teaching. The trends in teaching and learning moved from behaviourism and passed through cognitivism, towards constructivism (Abdulwahed, Jaworski & Crawford, 2012).

The learning theory of behaviourism, which developed during the nineteenth century, assumes the transmission of knowledge through direct instruction and focusses on atomisation and a mastery-oriented curriculum (Woodward, 2004). Teaching focusses on practice, repetition, memorisation and testing of discrete, basic skills, therefore the focus is on content validity rather than construct validity (Gipps, 1994).

Constructivism advocates that pedagogies should focus on concepts and contextualisation instead of isolated facts and argues that learning is evidenced through increasingly organised conceptual knowledge schemas. The role of the teacher is not to transmit knowledge, but to facilitate the construction of new knowledge through the promotion of discourse and cognitive conflict among peers. Jean Piaget, a Swiss developmental psychologist, advanced the cognitive development theory in 1936 and Len Vygotsky, a Russian psychologist, developed the sociocultural theory of cognitive development in the early twentieth century (The Psychology Notes Headquarters, 2020a; The Psychology Notes Headquarters, 2020b).

Piaget advocated the construction of knowledge when students connect new knowledge to prior knowledge and map it into their existing knowledge schema (Case, 1998; Gipps, 1994) through the processes of assimilation and accommodation. Assimilation is the process of taking new information into existing schemas. Accommodation involves modifying existing schemas, or ideas, as a result of new information or new experiences. The process may include the development of new schemes. The construction of knowledge occurs through the process of discovery and metacognition (Schoenfeld, 1992) when students reflect on learning and meaning through asking questions and searching for connections and conflicts. Metacognition occurs through self-reflection or engagement with peers or a significant other. The capability to assimilate and accommodate new information and construct knowledge develops with the biological age of the student. Piaget described four developmental stages: the sensorimotor, preoperational, concrete operational and formal operational stages.

Vygotsky explained and emphasised learning through social interaction, discourse and language patterns. He referred to children's acquisition of language and the use of language for social (interpersonal) and intra-personal (self-regulatory) purposes (Case, 1998). The construction of language and knowledge is embedded in culture and social interaction, including formal environments such as schooling. According to Vygotsky's theory, the novice is developed or scaffolded by peers who are more capable. He introduced the term zone of proximal development, which refers to the gap between a student's actual developmental level and his/her potential developmental level (Gipps, 1994). The more capable peer scaffolds the student to perform at a higher level. Vygotsky's model requires conscious reflection upon existing conceptual knowledge (Baker & Czarnocha, 2002; Engelbrecht, Harding & Potgieter, 2005).

According to Woodward (2004), students grapple with ambiguous or seemingly contradictory concepts and occasional misunderstandings are unavoidable. Misconceptions develop when a student attempts to make an early generalisation (Swan, 2001). Misconceptions do not refer to sporadic errors, but to an extension of a schema of concepts wrongly applied to an extended domain (Nesher, 1987). Although constructivism explains the difference between errors and misconceptions, it cannot explain how the social environment affects development.

Case (1998) describes three epistemological frameworks for conceptual development, namely the empiricist view (developed in England), the rationalist position (evolved in continental Europe) and the socio-historical view (developed in post-revolutionary Russia). The three frameworks for the development of knowledge could be seen as complementary, since they developed approximately simultaneously, but in different geographical areas and against the background of different cultures and worldviews. The empiricist view connects to behaviourism, since it focusses on empirical experience and the response to environmental stimuli. Abdulwahed *et al.* (2012) distinguish between radical and social constructivism. Radical constructivism connects to the rationalist framework (Case, 1998) and to Piaget's theory that describes learning as a logical construct where existing cognitive schema are adapted or new schema emerge from an internal process of reflection. Social constructivism ties to the socio-historical framework (Case, 1998) and the learning theory advocated by Vygotsky. According to this view, the primary origin of learning is social interaction and culture.

1.2 THE HISTORY OF MATHEMATICS EDUCATION

The learning theories of behaviourism and constructivism influenced the development of mathematics education. Behaviourism advocates the transmission of knowledge through explicit teaching and direct instruction; on the other hand, constructivism supports the view that knowledge is constructed when new knowledge is integrated into an existing knowledge schema. The history of the reforms in mathematics education over the past century could be described in terms of two paradigms: the procedural-formalist paradigm and the cognitive-cultural paradigm (Ellis & Berry III, 2005; Sorensen, 2013). The procedural-formalist paradigm with its behaviourist foundation assumes transmission of knowledge and procedures, while the cognitive-cultural paradigm builds on a socio-constructivist foundation that promotes the active role of the student and improvement of conceptual understanding through reflection and shared experiences.

Reforms transform and attempt to redefine the epistemological position within a field (Ellis & Berry III, 2005). Reforms ask questions about core beliefs of mathematics education within a paradigm and advance to restructure thinking about the nature, teaching and learning of and access to mathematics. A revision is a renewal effort to address certain components within the boundaries of an accepted paradigm. These revisions could be seen as a *quick fix* approach or surface-level modifications, but the revisions generally fail to address critical issues within a paradigm.

1.2.1 Procedural-formalist paradigm

The superficial revisions within the procedural-formalist paradigm (Ellis & Berry III, 2005; Sorensen, 2013) focussed on rules and procedures. Mathematical learning was connected to the learning theory of behaviourism and mathematics was seen as apart and isolated from human experience. At the beginning of the twentieth century, the Thorndike stimulus-response bond theory viewed mathematics as a subject that had to be explicitly taught in a sequenced manner, and stated that learning best occurred through drill and practice. This theory ignored mathematical thinking skills applied when doing mathematical problem-solving. In the 1920s the progressive education association movement noted the importance of society and agreed that learning was connected to students' experiences and interests. This social efficiency movement (Sorensen, 2013) claimed that certain students were better able to do mathematics and that mathematics was meant for a small elite group – predominantly white, middle-class males. This revision resulted in only 25% of high school students choosing algebra as a subject in the 1940s and 1950s. The new mathematics movement in the mid-twentieth century advocated the teaching of modern, advanced mathematics to all students. After the failure of the new mathematics revision, the back-to-basics approach in the 1970s again focussed on drill and practice of rote skills and procedural mathematical knowledge. The paradigm failed to develop critical learning and teaching of mathematical concepts to all students.

1.2.2 Cognitive-cultural paradigm

The paradigm shift towards constructivism and the cognitive-cultural paradigm started in the mid- to late 1980s. Besides the mathematical content and curricula, this paradigm focussed on *how* students learn mathematics (Ellis & Berry III, 2005; Sorensen, 2013). According to Woodward (2004), Skemp and Hiebert and their colleagues underlined the development of conceptual understanding in a well-developed schema, Skemp and Schoenfeld emphasised metacognition, and Schoenfeld focussed on problem-solving. The goal was to enhance the construction of knowledge, not merely focus on knowledge transmission and practising of procedural skills. The active role of the student in knowledge construction was highlighted and students were encouraged to share experiences, engage in critical thinking and make meaningful connections. The teaching and learning of mathematics was embedded in culture, human experience and social interaction.

1.2.3 Tension between the paradigms

Sowder (2007) claims that the two paradigms have opposing views about what mathematics is, since they represent different value systems - different beliefs and understandings about what it is to know, understand, learn and teach mathematics.

The concept-driven versus skills-orientated perspectives have led to the so-called "*math wars*" between mathematics education researchers in the United States (Brown, Seidelmann & Zimmermann, 2002; Hiebert, Morris & Glass, 2003; Sowder, 2007; Star, 2005). According to Star (2005:404), "whether developing skills with symbols leads to conceptual understanding, or whether the presence of basic understanding should precede symbolic representation and skill practice, is one of the basic disagreements between the opposing sides of the so-called math wars."

The mathematics education researchers that advocate concept-driven reform accentuate understanding of mathematics and the development of competent citizens who can use mathematics to solve problems and who can "*make sense*" mathematically of their world. The emphasis is on reasoning, critical thinking and the development of problem-solving skills. On the other hand, skills-orientated mathematics education researchers promote the development of skills. They argue that the new mathematics is fuzzy and lacks rigour, and that skills are best learned through drill and practice of procedures. A study conducted by Yates (2009) in Australia showed that teachers were resistant to change and still held strong beliefs deeply rooted in the ideas presented by the back-to-basics movement. The study showed that teachers believe that students must have a strong foundation in basic skills before they can manage learning that is more complex. According to this study, basic skills can only be learnt through direct instruction, rote learning and repetition.

After South Africa became a constitutional democracy in 1994, education specialists had concerns that many learners did not obtain the necessary problem-solving and reasoning skills during their learning processes. In 1998 the Department of Education (DoE) introduced Curriculum 2005, a new outcomes-based education system, to improve these cognitive skills (Engelbrecht, Harding & Phiri, 2010). However, the focus of high school mathematics teaching in South Africa remains on the development of procedural knowledge and skills-orientated teaching (Engelbrecht, Bergsten & Kagesten, 2009; Engelbrecht *et al.*, 2005). This is reflected in the structure of high school mathematics textbooks approved by the DoE (Leshota, 2019). Definitions, procedures and examples are found at the beginning of textbook chapters and chapters are ended with application problems (Brown *et al.*, 2002; Engelbrecht *et al.*, 2005). The result is that first-year students have more training in algebraic manipulation than conceptual understanding. Lecturers for first-year students often claim that these students have little understanding of pre-calculus, and that stronger students are mostly procedurally inclined (Engelbrecht *et al.*, 2009).

1.2.4 Intertwined strands of mathematical proficiency

Kilpatrick, Swafford and Findell (2001) advocate that the strands of mathematical proficiency, namely conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition, should not be seen as independent of each other, but as interwoven and interconnected. The National Research Council in the United States recommended the above-mentioned five strands as mathematical learning goals for twenty-first-century students (Hiebert *et al.*, 2003).

Kilpatrick *et al.* (2001:116) describe conceptual understanding as the "comprehension of mathematical concepts, operation and relations" and procedural fluency refers to the "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately." Strategic competence refers to the skill to formulate, represent and solve mathematical problems, in other words problem formulation and problem-solving. Problem-solving involves creative, flexible critical thinking (Schoenfeld, 1992). Different representations, including numerical, symbolical, verbal and graphical versions (Ball, Bass & Hill, 2004; Brown *et al.*, 2002; Kilpatrick *et al.*, 2001) could be used to describe, understand and solve problems. Adaptive reasoning is the ability to think logically about the relationship among concepts and circumstances (Kilpatrick *et al.*, 2001), therefore the skill to see relations between concepts within a schema. Deductive reasoning includes explanation of a concept or procedure, and reflection, evaluation and justification of a procedure chosen to solve a problem; on the other hand, intuitive, inductive reasoning includes making conjectures based on patterns, analogy and metaphors. Productive disposition is the tendency to see mathematics as sensible, useful, and worthwhile, together with a belief in hard work and one's own efficacy as a learner and doer of mathematics (Kilpatrick *et al.*, 2001).

1.3 RATIONALE

The discussion on "*math wars*", which focusses on the concept-driven and skills-orientated argument, together with the view on the five intertwined strands of mathematical proficiency, emphasises the need to investigate, analyse and describe procedural and conceptual knowledge needed to solve mathematical problems.

The concept-driven versus skills-orientated argument is important for the researcher's own work as a first-year engineering mathematics lecturer on an extended programme. The focus of the extended mathematics module is on filling the possible background knowledge gaps from high school, and on developing conceptual understanding and cognitive skills needed to pass the mainstream engineering modules (Grayson, 2010). Given the procedural focus of high school mathematics teaching and learning, the question is what the focus of the module should be to produce optimal results with respect to deep, flexible learning of mathematics and creative, critical problem-solving skills. The answer to the question led the researcher to the problem statement in the next section.

1.4 RESEARCH QUESTIONS

This study investigates first-year engineering mathematics students' learning path of calculus by reflecting on procedural and conceptual knowledge and analysing relations between the types of knowledge as part of a PhD research project. The main research question is:

What is the nature of conceptual and procedural knowledge in calculus?

Sub-questions:

1. How can calculus items be analysed and classified with respect to conceptual and procedural knowledge?
2. How are conceptual and procedural constructs related in calculus?

1.5 CONCEPTUAL AND PROCEDURAL KNOWLEDGE

Literature describes different stances with respect to definitions of and the relations between conceptual and procedural mathematical knowledge. Mathematics education researchers and psychologists interested in mathematics teaching and learning both use the terminology conceptual and procedural knowledge, but the two disciplines have different perspectives on the terms, which are deeply rooted in each discipline's research practices (Star & Stylianides, 2013). Star, a mathematics education researcher, and Rittle-Johnson, a developmental psychologist, explore relations between psychology and mathematics education and advocate that researchers in the two disciplines should work together and that research in one discipline should inform research in the other (Star & Rittle-Johnson, 2016). Mathematics education researchers tend to refer to these constructs as *qualities* of knowledge, whereas psychologists tend to refer to conceptual and procedural knowledge as *types* of knowledge.Star (2005) states that conceptual and procedural knowledge refers not only to *what* is known (the type), but also to *how* the knowledge is known, therefore whether the quality of knowledge is deep or superficial.

In order to answer the research questions, the researcher had to investigate the learning and teaching of mathematics in relation to conceptual and procedural knowledge. This included an investigation into the measurement of conceptual and procedural knowledge.

1.6 RESEARCH DESIGN AND APPROACH

The research study followed the postpositivist paradigm and used an explanatory, exploratory, sequential mixed methods design (Christensen, Johnson & Turner, 2015; Creswell & Creswell, 2017) to answer the research questions. The design placed equal emphasis on the qualitative and quantitative sections and answered the research questions in three layers. The first layer followed a qualitative approach; the second and third layers followed a quantitative approach. The first layer of analysis investigated relations between procedural and conceptual knowledge *within* the solutions of calculus items; the second layer of analysis described and measured the underlying construct between the solutions of a selection of calculus items and the third layer of analysis investigated relations *among* four knowledge classes defined for the research study.

1.6.1 Qualitative research approach

Qualitative analysis included the development of calculus items on the conceptual topics of pre-calculus (functions), differentiation and integration. Content analysis informed item labelling and the classification of the calculus items into four knowledge classes based on the type and quality of knowledge (Star, 2005) used to do the problem-solving. The classified items were taken up in a data collection instrument that informed the quantitative data analysis. The trustworthiness and credibility of the content analysis were determined through two triangulation sessions (researcher and mathematics colleague; researcher and supervisor).

1.6.2 Quantitative research approach

A data collection instrument was developed using the calculus items that were developed, analysed, labelled and classified in the qualitative section of the study. The test was administered to first-year extended degree engineering students. The voluntary data collection was done in the augmented mathematics module that runs parallel to the mainstream mathematics module. Data obtained from the administered test were complete, and the quality was excellent.

Rasch measurement theory (RMT), a recognised technique used in psychometric analysis, was used to model the relationship between item difficulty and person proficiency, and to describe and measure the underlying latent cognitive calculus construct embedded in the solutions of the calculus items. Furthermore, the analysis produced item statistics to evaluate the data collection instrument that was developed for the research study. The second part of the quantitative analysis used structural equation modelling (SEM) to apply confirmatory factor analysis (CFA). The analysis investigated possible relations *among* the four knowledge classes.

The quantitative analysis required the establishment of validity and reliability on three levels, namely validity of the content analysis, instrument validity and construct validity and reliability. Content analysis validity was established through the above-mentioned two triangulation rounds. Apart from the triangulation rounds mentioned, instrument validation required a third session between the researcher, supervisor and statistics specialist, where adaptations to the instrument were suggested. Evidence of reliable and valid construct inferences were derived from Rasch statistics.

1.7 VALUE OF THE RESEARCH

The research project generated new academic knowledge that may empower calculus students and lecturers to develop good learning and teaching practice. This information could be disseminated by publishing academic articles and by making presentations at conferences.

1.7.1 Academic value

The scientific importance of this research project includes qualitative and quantitative academic research output.

1.7.1.1 Qualitative research output

The qualitative section of this project includes the development of a content analysis framework to measure the integrated relations between procedural and conceptual knowledge *within* calculus tasks. The research study will show how calculus tasks can be analysed pertaining to procedural and conceptual components.

1.7.1.2 Quantitative research section

The quantitative section of this project includes the development of an instrument to measure the integrated relations between procedural and conceptual knowledge *between* calculus tasks. Rasch analysis was used to describe and measure the construct validity of item solutions and to evaluate the data collection instrument. CFA was implemented to investigate possible relations between items in four knowledge classes.

1.7.2 Developing good practice

The newfound data could be used to solve the research problem and make recommendations on good practice to learn and teach calculus. The target stakeholders included engineering students learning calculus and lecturers teaching calculus. The broader mathematics education community may also benefit from the findings.

1.8 ETHICAL CLEARANCE

Ethical clearance was obtained from the Faculty of Education for this research study. The study was conducted according to ethical guidelines set out by the university. A letter of approval from the dean of the Faculty of Engineering, Built Environment and Information Technology at the University of Pretoria was attached as part of the ethical clearance application submitted to the research ethics committee of the Faculty of Education. The informed consent form (Appendix 1) requested students to agree voluntarily to participate in the research project and to give permission for data collection, analysis and publication that might follow from the study. The ethical clearance process is summarised in Table 1.1:

Table 1.1: Ethical clearance process

Ethical clearance process	
Researcher's responsibility	
Ethical clearance from Faculty of Education Informed consent form for students	Appendix 1

1.9 RESEARCH STRUCTURE

The chapters were outlined to assure a well-structured research report in which the content flows in a logical order, and research aims and questions are addressed and answered satisfactorily. The remaining thesis chapters are introduced below, with a description of each chapter.

Chapter 2: Literature review

This chapter focusses on definitions for conceptual and procedural knowledge and possible relations between the types of knowledge, the preferred approach to learn and teach mathematics, and methods to measure conceptual and procedural knowledge.

Chapter 3: Research methodology

In this chapter, the focus is on important aspects related to research methodology used to answer the research questions. The choice of a postpositivist research paradigm and mixed methods approach is argued and explained. The assumptions and the process followed to do the qualitative content analysis are described. In the quantitative approach section, the development of a data collection instrument, data collection, RMT and CFA are explained.

Chapter 4: Qualitative results

The first layer of analysis investigated relations between procedural and conceptual knowledge *within* the solutions of calculus items. This chapter shares the results of the qualitative content analysis with the reader.

Chapter 5: Quantitative results

In a second layer of analysis a test was administered and data was collected and analysed to describe relations between the labelled, classified calculus items. The results of an initial Rasch analysis showed items with significantly disordered categories. Marks and mark categories were collapsed for these items. The data were re-analysed by applying the Rasch model to the revised instrument. In the third layer of analysis relations *between* the defined knowledge classes were investigated using CFA. The results of the Rasch analysis and CFA are shared in this chapter.

Chapter 6: Discussion, findings and recommendations

This chapter discusses the research findings and provides answers to the research questions. Recommendations on further research are shared with the reader.
The next chapter discusses the literature review that was needed to answer the research questions.

2. CHAPTER 2: LITERATURE REVIEW AND CONCEPTUAL FRAMEWORK

2.1 INTRODUCTION

Chapter 1 presents background on the history of mathematics education and the rationale for the research study, and provides an overview of key concepts, the methodology followed and the research structure. In general, this chapter presents the different theories underpinning the study, as well as a review of the literature of previous studies conducted on the topic of conceptual and procedural knowledge. More specifically, it presents the definitions of and the relations between the two kinds of mathematical knowledge.

In order to investigate the nature of conceptual and procedural knowledge, the researcher had to evaluate literature on the development and teaching of conceptual understanding and procedural fluency to enhance mathematical proficiency. Furthermore, the measurement of conceptual and procedural knowledge had to be defined in order to measure and describe conceptual and procedural relations *within* and *between* calculus item solutions.

The chapter is concluded with a description of the conceptual framework used for the research study. The conceptual framework integrates literature with the researcher's empirical experience; views on the relations between conceptual and procedural knowledge when teaching and learning mathematics, in particular calculus, are shared.

2.2 PROCEDURAL AND CONCEPTUAL KNOWLEDGE

Literature describes the different stances with respect to definitions of and relations between conceptual and procedural knowledge when teaching and learning mathematics. This section critically analyses and discusses the views of various mathematics education researchers in order to understand the relationship between conceptual and procedural knowledge better.

2.2.1 Procedural knowledge

Skemp (1976) distinguishes between relational and instrumental understanding. Instrumental understanding refers to both knowledge of something and of procedures for doing something. The procedures are fixed plans with ”no awareness of the overall relationship between successive stages and the final goal” (Skemp, 1976:14), therefore without reasoning. Students are dependent on outside guidance from a teacher. Hiebert and Wearne (1986:7) define procedural knowledge as follows:

> One kind of procedural knowledge is a familiarity with the individual *symbols* of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of *rules or procedures* for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols.

Hiebert and Carpenter (1992) describe procedural knowledge as a sequence of actions. Procedural knowledge, as described by Hiebert and Lefevre (1986), is only meaningful if it is linked to a conceptual base (Engelbrecht *et al.*, 2005). Star (2005) states that Hiebert and Lefevre’s definition of procedural knowledge refers to sequential relationships where one step in a procedure is linked to the next step, in other words an algorithm where steps are performed in a specific, predetermined order. He argues that this definition of procedural knowledge implies the common use of procedural knowledge. However, the definition does not imply rich connections, but rather a superficial connection. Star explains in a personal communication with Baroody (23 December 2005) that “procedural knowledge (or skill) is valuable in and of itself, not solely because of its connections with and integration to conceptual knowledge.” Star (2005) claims that procedural fluency is not limited to superficial procedural knowledge without relations, since procedures can be well-connected. There are therefore different kinds or levels of procedural knowledge.

Star (2005:408) describes deep procedural knowledge as knowledge of procedures associated with “comprehension, flexibility, and critical judgement and that is distinct from (but possibly related to) knowledge of concepts”. The author describes heuristic procedures as deep procedural knowledge, since these “rule of thumb” procedures are general and abstract, and may be useful in problem-solving. Heuristic strategies require the student to make informed choices; therefore, these require the use of common sense knowledge that is integral to a specific situation or context. Heuristic strategies require thought and do not fit satisfactorily into either the procedural or conceptual knowledge category, but lie at the intersection of the two knowledge types (Hiebert & Wearne, 1986).

Baroody, Feil and Johnson (2007) agree with Hiebert and Wearne (1986) that procedures may be connected to or embedded in other procedures and agree with Star (2005) that procedures are not limited to knowledge without relations. According to Baroody *et al.* (2007:117), superficial procedural knowledge or a weak procedural scheme is “unconnected, disembodied, meaningless, context-bound, or mechanical procedures”. Deep procedural knowledge or a strong procedural scheme is described as “well-connected, contextualized, integrated, meaningful, general, or strategic procedural knowledge.” Star and Baroody disagree on the definition of deep procedural knowledge. Baroody claims that deep procedural knowledge is embedded in conceptual understanding. In contrast, Star (2007:133) states that deep procedural knowledge is knowledge in itself and not necessarily the result of connections to conceptual knowledge: “Procedures can be known deeply, flexibly, and with critical judgment - positive outcomes that are exclusively about students' knowledge of procedures and not necessarily a result of connections to conceptual knowledge.”

Procedural fluency has been one of the main objectives of traditional mathematics education in South Africa (Engelbrecht *et al.*, 2005) and internationally (Engelbrecht *et al.*, 2009). Flexibility in working with procedures is seen as an indicator of deep understanding of procedural knowledge. According to Star (2005:409):

> A more flexible solver - one with a deep knowledge of procedures - can navigate his or her way through this procedural domain, using techniques other than ones that are overpracticed, to produce solutions that best match problem conditions or solving goals.

The flexible student uses deep procedural knowledge to choose a preferred problem-solving method that best suits the mathematical problem. Some students stay algebraically grounded, while others prefer graphical or hybrid methods. The increasing availability and efficiency of technology, e.g. calculators and computers, emphasises the possibility of “number crunching” and procedural fluency that can be achieved in the absence of procedural flexibility (Engelbrecht *et al.*, 2009).

Kilpatrick *et al.* (2001:121) describe procedural fluency as the “knowledge of procedures, knowledge of when and how to use them appropriately, skill in performing them flexibly, accurately and efficiently”. Procedural fluency includes the ability to estimate the result of a procedure, solve problems in more than one way, as well as the success in choosing the most appropriate problem-solving procedure. Novices have fragmented knowledge and do not see how different pieces of knowledge relate to one another. Expert mathematicians know and use more procedures, and have the skill to adapt procedures more easily than beginners. Algorithms are powerful tools that provide students with the awareness that mathematics is well structured, organised and filled with patterns. Accuracy and efficiency of procedural fluency will be improved with practice.

2.2.2 Conceptual knowledge

Literature distinguishes between the concept definition and the concept image of a concept or principle. The notion of the concept definition refers to the formal, traditional definition of a concept. The concept image is seen as a personal interpretation of the mathematical definition of a concept (Brown *et al.*, 2002; Kabael, 2014; Scheja & Pettersson, 2010). The concept image is made up of mental pictures, examples and non-examples, processes and properties. The development of a concept image is a long-term, dynamic process. This interpretation forms part of the total cognitive structure pertaining to a particular concept. Brown *et al.* (2002) state:

> A strong concept image is a rich, integrated, mental representation that allows the student to flexibly move between multiple formulations and representations of an idea. A student who has connected mathematical ideas in this way can create and use a model to analyse a situation, uncover patterns and synthesize them to form an integrated picture. They can also use symbols meaningfully to describe generalizations.

A student with a weak concept image lacks flexibility and adaptive expertise with respect to a particular concept. A weak concept image for a limit of a function may include the misconception that a limit can never be reached (Scheja & Pettersson, 2010).

As explained earlier, Skemp (1976) distinguishes between relational and instrumental understanding. Relational mathematics is a “build-up of a conceptual structure or schema” (Skemp, 1976:14); therefore knowledge of *what* to do, *how* to do it and *why*. The goal of relational understanding is to build a schema of knowledge, but the schema is never complete. In contrast to instrumental understanding, the knowledge quality of relational understanding is labelled as deep (Baroody *et al.*, 2007). According to Scheja and Pettersson (2010), Skemp indicated that procedural knowledge may be an obstacle in the development of conceptual knowledge, and that teaching should focus on conceptual rather than instrumental understanding.

Conceptual understanding (Kilpatrick *et al.*, 2001) refers to a connected and functional handle on mathematical ideas. Students with well-developed conceptual understanding have organised facts and methods in a coherent whole; they learn new ideas by connecting them to existing ideas. Meaningful development of conceptual understanding is indicated when a student has the ability to use different representations to represent a mathematical situation. Conceptual understanding provides the basis for generation of new knowledge, therefore expanding an existing schema, and solving new and unfamiliar problems. Conceptual understanding helps students to estimate and avoid errors of magnitude. Furthermore, conceptual understanding helps students to see similarities between seemingly unrelated situations. This could result in students having less to learn, since understanding is encapsulated in clusters of facts and principles. The development of conceptual understanding may lead to higher levels of student confidence. Hiebert and Wearne (1986:3) define conceptual knowledge:

> Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the *linking relationships* are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network.

Hiebert and Wearne (1986) state that a conceptual knowledge network grows as new knowledge is related to prior knowledge. Crooks and Alibali (2014) classify the type of conceptual knowledge described by Hiebert and Lefevre (1986) as connection knowledge, since it focusses on the understanding of relationships and connections within a domain. However, Star (2005) explains that conceptual knowledge does not necessarily imply richness of relationships or indicate connected knowledge. Star argues that Hiebert and Lefevre's definition of conceptual knowledge refers to deep connected conceptual knowledge that is rich in connections. This description refers to the common use of conceptual knowledge. Star (2005:407) distinguishes between weakly connected conceptual knowledge where the connections are "limited and superficial"; and deep connected conceptual knowledge where connections may be described as "extensive and deep". Baroody *et al.* (2007:117) agree with Star's (2005) view when they contrast "sparsely connected conceptual knowledge" with "well-connected conceptual knowledge". Strong schemas that are embedded in deep conceptual knowledge involve "generalizations broad in scope, high standards of logical consistency, principle-driven comprehension, … and predictions that are derived logically" (Baroody *et al.*, 2007:117). In contrast, weak schemas that imply superficial conceptual knowledge include "generalizations local in scope, low standards of logical consistency, precedent-driven comprehension and … predictions that are looked up" (not derived logically) (Baroody *et al.*, 2007:117). The authors describe Hiebert and Lefevre's definition of conceptual knowledge with two categories, namely primary-level concepts that are less abstract, since they are tied to a specific context, and reflective-level concepts that are tied to multiple contexts.

Crooks and Alibali (2014) conducted a literature review on the definitions of conceptual and procedural knowledge and the relations between the two kinds of knowledge in literature. Crooks and Alibali (2014:371) categorised procedural knowledge as a subset of conceptual knowledge when they proposed two types of conceptual knowledge:

1. General principle knowledge involves understanding of mathematical ideas without relation to specific problems or procedures.
2. Knowledge of principles underlying procedures involves connecting concepts to specific procedures.

The second type of conceptual knowledge relates to the choice of procedures and the purpose of steps within a procedure, therefore the *how* and the *why* in problem-solving.

2.2.3 Relations between procedural and conceptual knowledge

Hiebert and Carpenter (1992) state that both kinds of knowledge are important and are required for mathematical expertise. Hiebert and Carpenter (1992:78) claim that "if a learner connects the procedure with some of the conceptual knowledge on which it is based, then the procedure becomes part of a larger network, closely related to conceptual knowledge." Conceptual knowledge contributes to mathematical proficiency through its relationships with procedural knowledge; on the other hand, all mathematical procedures are potentially connected in a knowledge network.

Engelbrecht *et al.* (2009) and Engelbrecht, Bergsten and Kågesten (2012) describe relations between procedural and conceptual knowledge as complex. The authors describe mathematical operativity with Bergsten's model of dynamic relationships of mathematical work in Figure 2.1.

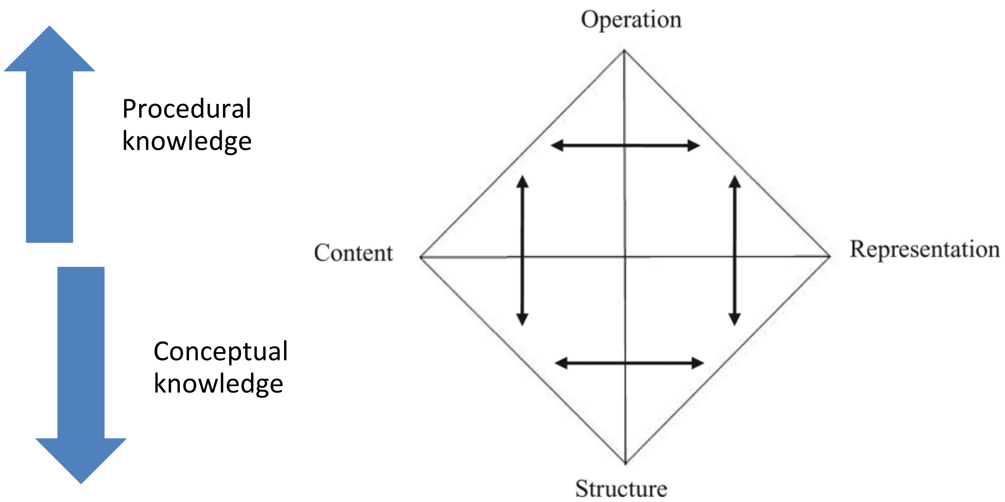


Figure 2.1: Bergsten's model of dynamic relationships of mathematical work (1990)

The assumption is made that a student performs mathematical operations on and within mathematical structures (vertical axis). The model places conceptual knowledge in the lower and procedural knowledge in the upper part of the diagram. Conceptual and procedural knowledge are strongly connected in mathematical work, since both of kinds of knowledge draw on representation and mathematical content (horizontal axis), and the relations between these two parts. A student's mathematical operativity may focus on either the procedural or conceptual part of the model.

As mentioned in chapter 1, Kilpatrick *et al.* (2001) describe the five strands of mathematical proficiency, namely conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition, as intertwined and interconnected. Hiebert *et al.* (2003) highlight the notion that mathematical proficiency will improve when all five strands of mathematical proficiency (Kilpatrick *et al.*, 2001) are developed, and not only one strand is focussed on. The use of procedures can strengthen and develop conceptual understanding. In contrast, compartmentalisation of procedures develops when skills are learned in isolation. This practice could also result in separation, where students cannot apply what they learnt in school to solve real-life problems. Students make mental representations (mind maps/conceptual schema) on how they connect and organise different pieces of knowledge (concepts and procedures). The schema will provide the basis for understanding, critical and logical reasoning, problem-solving and the tendency to see mathematics as sensible and useful. Mathematical understanding and proficiency will mature over time.

Star (2005) claims that both conceptual and procedural knowledge are critical components of mathematical proficiency. Both kinds of knowledge refer to *what* is known (the type) and to *how* the knowledge is known (the quality). The quality of conceptual and procedural knowledge may be described as either deep or superficial. According to Baroody *et al.* (2007:116), Star explains the common use of procedural and conceptual knowledge as follows (Star, personal communication with Baroody, 23 December 2005):

> The adjectives *conceptual* and *procedural* are typically used to refer to the [number] of connections (knowledge quality ...), with *conceptual* meaning richly connected. In common usage, *procedural* knowledge does not mean knowledge of procedures, but instead it refers to knowledge that is not richly connected.

The above quotation points out that, according to Star, knowledge quality might be linked to the number of connections. It also confirms that, as mentioned previously, the common use of conceptual knowledge refers to rich, deep connections; conversely, the common use of procedural knowledge indicates an absence of rich connections. Table 2.1 illustrates Star (2005) take on procedural and conceptual knowledge categories.

Table 2.1: Framework of procedural and conceptual knowledge (Star, 2005)

Framework of procedural and conceptual knowledge categories		
	Knowledge quality (how)	
Knowledge type (what)	Superficial	Deep
Procedural	**Common** use of procedural knowledge	*Rich, well-connected procedural knowledge*
Conceptual	*Weakly connected conceptual knowledge*	**Common** use of conceptual knowledge

According to Baroody *et al.* (2007), Star observes conceptual and procedural knowledge as separate parts, and not necessarily on a continuum. Star suggests that deep flexible procedural knowledge can develop with or without conceptual knowledge. The researcher presents an interpretation of Star's perspective on mathematical knowledge (according to Baroody) in Table 2.2.

Table 2.2: Star's view on mathematical knowledge relations (Baroody *et al.*, 2007)

Relations between conceptual and procedural knowledge

Conceptual knowledge		Procedural knowledge: Not connected	Sparsely connected	Somewhat connected	Well-connected	Richly connected
	Richly connected					**INTERCONNECTED$_{cp}$**
	Well-connected					**P_3**
	Somewhat connected					**P_2**
	Sparsely connected		**interconnected$_{cp}$**			**P_1**
	Not connected	**isolated**	**PP**			**P_0**

Key:	
INTERCONNECTED$_{cp}$	Deep (interconnected or integrated) conceptual and procedural knowledge
interconnected$_{cp}$	Conceptual and procedural knowledge sparsely interconnected
P_0-P_3	Deep procedural knowledge not connected, sparsely, somewhat or well-connected to conceptual knowledge
PP	Procedural knowledge sparsely connected with other procedural knowledge
isolated	Isolated procedural or conceptual knowledge

According to Baroody *et al.* (2007), Star claims that deep procedural knowledge could develop separately from conceptual knowledge when he defines P_0 as richly connected procedural knowledge that is not connected to conceptual knowledge. The researcher disagrees with this view, since deep procedural knowledge cannot exist in the absence of conceptual knowledge.

Baroody's framework (Baroody *et al.*, 2007) describes the mutually dependent relationship between procedural and conceptual knowledge along a continuum. Baroody's view on the development of procedural and conceptual knowledge over a continuum connects to the different levels of mathematical operativity as described by Bergsten's model of dynamic relationships of mathematical work (Engelbrecht *et al.*, 2012).

Baroody *et al.* (2007) proposed an additional framework with respect to mathematical knowledge, illustrated in Table 2.3. The degree of connectedness or mutual interdependence between procedural and conceptual knowledge is presented on the *x*-axis. Other aspects of knowledge quality are presented on the *y*-axis. The other aspects of knowledge quality take into account the degree of connectedness of the knowledge type to real-world applications, degree of knowledge structure or organisation, the generality or abstractness of the knowledge, and the accuracy of the knowledge.

Table 2.3: Procedural and conceptual knowledge vs knowledge quality (Baroody *et al.*, 2007).

Procedural and conceptual knowledge vs knowledge quality

Other aspects of knowledge quality	• Relatively complete (rich connections within knowledge type and with real world applications) • well-structured • abstract, general • accurate knowledge No knowledge					
		Not connected	Sparsely connected	Somewhat connected	Well-connected	Richly connected

Connectedness of procedural and conceptual knowledge

Baroody states that superficial procedural and conceptual knowledge may exist independently; however, deep procedural knowledge cannot exist without deep conceptual knowledge or vice versa (Baroody *et al.*, 2007). At one end of this continuum, understanding is sparsely connected and paired with superficial procedural knowledge and lack of conceptual knowledge. At the other end one finds richly connected procedural and conceptual knowledge (Pettersson & Scheja, 2008). In contrast with Star, Baroody *et al.* (2007) argue that procedural flexibility implies the integration of procedural knowledge with conceptual knowledge.

Table 2.4 shows a comparison between Star's framework (2005) and Baroody's framework (2007) on the relations between procedural and conceptual knowledge.

Table 2.4: Comparison of mathematical knowledge: Star vs Baroody

Comparison of mathematical knowledge: Star vs Baroody	
Star (Star, 2005)	**Baroody (Baroody *et al.*, 2007)**
Two types of knowledge – procedural, conceptual	Three types of knowledge – procedural, conceptual and a combination of procedural and conceptual
Different types of knowledge described as separate parts	Different types of knowledge described as a continuum
Describes relationship between connectedness of procedural AND conceptual knowledge *x*-axis: connectedness of procedural knowledge *y*-axis: connectedness of conceptual knowledge	**Describes relationship between connectedness of procedural and conceptual knowledge AND other aspects of knowledge quality** *x*-axis: **connectedness of procedural and conceptual knowledge** *y*-axis: other aspects of knowledge quality: - connections with knowledge type to real-world situations and applications - degree of knowledge structure and organisation - abstractness of knowledge (specific/local or abstract/generalisation) - accuracy of knowledge
Deep procedural knowledge is possible without connections to conceptual knowledge, therefore it can exist without the presence of deep conceptual knowledge	Deep procedural knowledge is embedded in conceptual understanding, therefore it cannot exist without deep conceptual knowledge

The differences and progression between Star's and Baroody's viewpoints are evident.

2.2.4 Routine and adaptive expertise

Hatano (1988) distinguishes between routine and adaptive expertise. Routine expertise refers to knowledge that was attained by rote learning and that is suitable for use in familiar tasks. Adaptive expertise is meaningful, well-connected knowledge that can be applied flexibly to familiar and unfamiliar tasks. Baroody *et al.* (2007) claim that routine expertise involves superficial procedural and conceptual knowledge; conversely, adaptive expertise implies deep procedural and deep conceptual knowledge. As discussed in the previous sections on procedural and conceptual knowledge, Baroody *et al.* (2007) differentiate between strong and weak schemas pertaining to procedural and conceptual knowledge. Baroody *et al.* (2007) claim that deep conceptual and deep procedural knowledge connect to strong schemas and adaptive expertise, therefore meaningful knowledge that can be applied flexibly and appropriately to new and familiar tasks. Hatano, in Baroody and Dowker (2009), comments on the product and on the process that leads to adaptive expertise: "Flexibility and adaptability seem to be possible only when there is some corresponding conceptual knowledge to give meaning to each step of the skill and provide criteria for selection among alternative possibilities for each step within the procedures." Baroody claims that conceptual knowledge is the basis of procedural flexibility, since it informs *how* and *why* a procedure can be applied in problem-solving (Baroody, 2009; Baroody & Dowker, 2009). According to Baroody (2009:13), conceptual knowledge usually underlies procedural innovations and can play either a direct or indirect role in the invention of procedures. Adaptive expertise requires productive thinking and is developed when:

- different procedures are evaluated and the most effective problem-solving strategy is then selected (Baroody *et al.*, 2007; Crooks & Alibali, 2014; Kilpatrick *et al.*, 2001; Star, 2005)
- a known procedure is modified to solve a unfamiliar problem
- a new procedure is invented.

Taraban, DeFinis, Brown, Anderson and Sharma (2007) distinguish between shallow and deep knowledge with respect to knowledge needed for engineering education. Shallow knowledge relates to routine expertise, since it entails lists of concepts, simple facts of concepts, simple definitions of key terms and major steps in a procedure. Deep knowledge encompasses coherent explanations that enable students to generate inferences, solve problems, make decisions, integrate and decompose ideas, apply knowledge in practice and make predictions. Deep knowledge relates to adaptive expertise and involves extended cognitive representations that will be retained longer than shallow knowledge. The authors claim that novices show backward reasoning, since they start at an unknown variable and link possible concepts that come to mind to it. Experts start with the given information, categorise and classify the problem, anticipate calculations and identify different methods to solve the problem, therefore experts show forward reasoning. The authors claim that engineering students often master shallow knowledge, e.g. manipulate symbols and equations after extensive practice, but they lack deep mastery of content, therefore deep knowledge.

It is evident from the previous discussion that there are connections between Hatano's definition of routine and adaptive expertise, Baroody's weak and strong schemas and Taraban *et al.* (2007) description of shallow and deep knowledge. All of these categories include both procedural and conceptual knowledge.

2.2.5 Summary

The views of some education researchers with respect to relations between conceptual and procedural knowledge could be summarised on a continuum of connectedness:

1. Conceptual and procedural knowledge are not necessarily connected:

Vygotsky: Conceptual understanding may develop in the absence of procedural fluency.

Skemp: Procedural knowledge might be an obstacle in the development of conceptual understanding.

Star: Procedural knowledge can develop in the absence of conceptual knowledge.

2. Conceptual and procedural knowledge are connected:

Piaget: Conceptual knowledge grows as new connections are formed though assimilation and integration (including procedural knowledge).

Hiebert and Lefevre: Procedural knowledge is meaningful if connected to a conceptual base.

Hiebert and Carpenter: Procedures are connected to a conceptual base.

Hiebert, Morris and Glass: Mathematical proficiency requires focus on all five strands of mathematical proficiency.

Kilpatrick: The five strands of mathematical procifiency are interconnected and intertwined.

Hatano: Flexibility is only possible when conceptual knowledge gives meaning to procedures.

Baroody: Conceptual knowledge is the basis for mathematical proficiency

Crooks and Alibabi: Procedural knowledge is seen as a subset of conceptual knowledge.

3. Integrated conceptual and procedural knowledge:

Hatano: The knowledge involves routine and adaptive expertise.

Baroody: Weak and strong schemas are involved.

Taraban: Shallow and deep engineering knowledge could be obtained.

The researcher agrees with the following views on relations between procedural and conceptual knowledge:

1. The five strands for mathematical proficiency are interconnected and intertwined (Hiebert *et al.*, 2003; Kilpatrick *et al.*, 2001).
2. The classification into superficial and deep procedural and conceptual knowledge, therefore classification according to *type* and *quality* of knowledge, is valid (Star, 2007).
3. The development of procedural and conceptual knowledge is mutually dependent and along a continuum (Baroody *et al.*, 2007).
4. Conceptual knowledge underpins the development of mathematical proficiency and the modification and invention of procedures (Baroody, 2009; Baroody & Dowker, 2009; Baroody *et al.*, 2007; Hatano, 1988; Hiebert & Carpenter, 1992; Hiebert & Wearne, 1986).
5. Deep procedural knowledge is not necessarily the result of connections to conceptual knowledge (Star, 2005; Star, 2007), e.g. the procedural application of integration techniques.

The essence of mathematics education research is closely related to questions about how students learn and how one should teach to promote students' understanding. Engelbrecht *et al.* (2009) describe relations between conceptual and procedural knowledge as “complex, with respect to both the meanings of the notions themselves as well as the relationships between them in the context of doing, teaching and learning mathematics.” The next section examines the development and teaching of conceptual understanding and procedural skills.

2.3 DEVELOPMENT OF MATHEMATICAL KNOWLEDGE

Mathematical students who aim to master mathematical proficiency need to develop adaptive expertise and flexibility with respect to conceptual and procedural knowledge. The previous section highlighted the importance of both conceptual knowledge and procedural skills; however, how do students develop these? This section focusses on the development of conceptual understanding and procedural skills.

2.3.1 Different representations

Representations of a contextual or mathematical problem include physical representations, graphs, numbers and symbols, written and verbal representations (Ball *et al.*, 2004; Brown *et al.*, 2002; Kilpatrick *et al.*, 2001). The use of different representations forms part of mathematical problem-solving practice that will enhance mathematical understanding and will develop the concept image of a concept. Davis (2005) illustrates how a real-world problem could be solved using different representations, namely by manipulation of equations, tables and graphs. The move between different representations provides the opportunity to develop different procedures. A specific representation allows for exploration of concepts linked to that particular representation. Furthermore, the representation also connects the concepts to particular, related procedures. The practice of different presentations will therefore enhance the development of both procedural and conceptual knowledge. The connection between concepts and procedures will often result in procedures that are correctly retained for longer periods. Davis (2005) claims that mathematically proficient students can be described as flexible and resourceful problem solvers, since they use common sense and different representations and procedures to solve problems.

2.3.2 Threshold concepts

Threshold concepts are key concepts related to a specific topic or knowledge field. Scheja and Pettersson (2010) introduced the idea of a threshold concept, which can be described as a *conceptual gateway* or *portal* for other concepts (Pettersson & Scheja, 2008; Scheja & Pettersson, 2010). Threshold concepts are described as integrative, since these concepts bind different smaller topics together. In the context of calculus, a threshold concept, for example, is the area enclosed by the graph of a function and the x-axis. It is also integrative in the sense that the area enclosed by the graph of a function and the x-axis can be found by evaluating the definite integral or the Riemann sum (which is a limit of a sum of areas), over a specific interval. Threshold concepts are transformative, since understanding these concepts will enhance integration with prior knowledge and improve learning (Scheja & Pettersson, 2010). The integral concept is transformative, since concepts for procedural evaluation of the integral, e.g. integration techniques, connect to the understanding of the fundamental theorem of calculus.

2.3.3 The order of development: conceptual vs procedural knowledge

Students learn through development of the concept image of a mathematical concept using different representations. Teachers should identify key threshold concepts and facilitate knowledge construction by connecting new knowledge to prior knowledge. As discussed previously, the nature of mathematical knowledge may be procedural or conceptual. Literature reveals different views on how procedural fluency and conceptual understanding should develop. Baroody (2009) shared four views with respect to the developmental order of conceptual and procedural knowledge, namely:

1. Concepts-first development view: Children initially develop conceptual knowledge, for example through parents' descriptions and explanations, and then acquire procedural knowledge from it through repeated problem-solving.
2. Procedures-first development view: Children first learn procedures for solving problems, and then gradually develop conceptual knowledge from abstraction of repeated procedures.
3. Iterative development view: Conceptual understanding can lead to the development of procedural fluency, which then leads to conceptual advance,

and vice versa. Relations between the two types of knowledge could be described as bidirectional.

4. Simultaneous development view: Conceptual and procedural knowledge develop concurrently.

When Rittle-Johnson and Schneider (2016) and Schneider, Rittle-Johnson and Star (2011) discussed views with respect to the developmental order of mathematical knowledge, they agreed with Baroody on the first three views, but ignored the simultaneous development view. The authors added the inactivation development view, where conceptual and procedural knowledge develop independently. In her later work Rittle-Johnson (2017) only mentioned Baroody's first three views and stated that the iterative perspective is supported by evidence and is the most well-accepted view among researchers.

Initial knowledge is partial and it is difficult to determine whether it could be described as the result of conceptual understanding or procedural experience. Prior knowledge and experience will determine which type of knowledge emerges first. It could be assumed that one type of knowledge is not well developed before the other emerges. Gradual improvements in each type of knowledge are accommodated over time, in no specific order. Rittle-Johnson, Schneider and Star (2015:11) state:

> Evidence indicates that there are causal, bidirectional links between the two types of knowledge; improving procedural knowledge can lead to improved conceptual knowledge and vice versa, especially if potential links between the two are made salient (e.g. through conceptually sequencing problems).

According to this view, causal relations could be described as bidirectional. One type of knowledge is a good and reliable predictor of the other. Teaching using one type of mathematical knowledge will lead to development of the other, and vice versa. The strength of relations may vary, also among individuals. The study conducted by Schneider *et al.* (2011) indicated symmetrical relations between conceptual and procedural knowledge.

In the next section, the researcher investigates whether teaching to enhance bidirectional relations should be focussed on conceptual understanding or procedural fluency, or both.

2.4 TEACHING OF MATHEMATICAL KNOWLEDGE

Mathematics education researchers agree that the teaching and learning of conceptual understanding and procedural fluency are both important, but there are different views on the order in which procedures and concepts should be taught. The relations between conceptual and procedural knowledge could be described as bidirectional and teaching practices should facilitate the iterative development of procedural and conceptual knowledge. Conceptual knowledge could help with the selection and adaptation of procedures; on the other hand, explaining procedures may help students to make deeper conceptual connections. The teaching of mathematics can focus on the development of either procedural fluency or conceptual understanding, or both.

Engelbrecht *et al.* (2012:5) distinguish between teaching for procedural knowledge and teaching for conceptual understanding:

> Teaching for procedural knowledge would mean presenting, for the students, ready-made definitions, notations and procedures without first providing the deeper meanings to the concepts involved. Teaching for conceptual understanding, on the other hand, would start with problems that require an initial reasoning from the students, to make connections to their prior knowledge.

This view is also shared by Brown *et al.* (2002).

The difference in teaching approach is reflected in the structure of textbooks chosen as study material for a particular course. When a skills-orientated teaching approach is followed, a textbook will be chosen where definitions, procedures and examples are found at the beginning of textbook chapters and chapters are ended with application problems (Brown *et al.*, 2002; Engelbrecht *et al.*, 2005). Conversely, concept-driven teachers will start with applications and draw the mathematics needed to solve the problem from the real-life context.

Traditional calculus teaching focussses on definitions, theorems and proofs and algebraic proficiency (Engelbrecht *et al.*, 2012; Engelbrecht *et al.*, 2005). In contrast, Abdulwahed *et al.* (2012:55) claim that "engineering students … prefer an experiential learning style; hence, student-centred experiential learning approaches with real-world problems such as project-based learning are more compatible to their learning style than classical abstract methods of teaching mathematics."

There appears to be a gap between the focus on traditional calculus teaching approaches and the preferred learning style of engineering students. In the following sections, the researcher discusses different teaching approaches that could lead to the development of mathematical proficiency.

2.4.1 Direct instruction approach

Direct instruction and the teaching of definitions, symbols and procedural skills were emphasised by the learning theory of behaviourism and the procedural-formalist paradigm. This approach connects to instrumental understanding and rules without reasoning. Wu (1999:7) states: “deep understanding of mathematics ultimately lies within the skills.” Traditional teaching methods refer to skills that are taught by teachers and practised in isolation by students until mastery is achieved. Mathematics are broken down into a smaller group of isolated skills, facts and procedures (Sorensen, 2013). These approaches assume that conceptual understanding will develop from computational proficiency and procedural fluency. Students must first develop skills before they can understand concepts, and skills are best learnt through memorising and practising procedures, without calculators, until they are automatic (Sowder, 2007). Yates (2009) states that the importance of automaticity in basic computational skills becomes apparent when it is absent. Procedures and understanding are seen as intertwined, but procedural fluency acts as a vehicle to develop conceptual knowledge. As mentioned previously, a study among Australian teachers indicated a preference for strong basics skills before students can manage more complex learning (Yates, 2009).

Wu (1999:5) states that “the standard algorithms embody conceptual understanding”; therefore the formulation and learning of algorithms will enhance the development of conceptual knowledge. Wu claims that when a skill or procedure is ignored in favour of a conceptual approach, the result could be superficial conceptual understanding. On the other hand, it has been argued that the approach of “path smoothing”, where teachers tend to simplify the mathematics by requiring students merely to fill the gaps with an arithmetical answer or low-level recall of facts, does not promote sustainable learning (Yates, 2009).

A study by Pettersson and Scheja (2008) identified the limit and integral as threshold concepts for the understanding of calculus. Studies by Pettersson and Scheja (2008) and Scheja and Pettersson (2010) involved written reflections and interviews regarding students' understanding of the mathematical concepts limit and integral. According to Pettersson and Scheja (2008:779), "students were communicating their understanding of limit and integral within an algorithmic context in which the very operations of these concepts were seen as defining features and a basis for understanding of those concepts." Pettersson and Scheja (2008) and Scheja and Pettersson (2010) found that students preferred algebraic methods (integration techniques) above geometric interpretations (Riemann sum) to evaluate definite integrals. The authors claim that the students who formed part of this study chose an algebraic frame for the learning of calculus. The research study indicated that student understanding of calculus was portrayed within an algorithmic context with a distinct focus on procedures and techniques (Pettersson & Scheja, 2008; Scheja & Pettersson, 2010). Procedural fluency was fore-grounded above conceptual connections. Student explanations formed part of a "wider algorithmic contextualization" (Pettersson & Scheja, 2008; Scheja & Pettersson, 2010), which may potentially facilitate conceptual understanding of calculus. The interviewer asked probing questions to challenge procedural understanding and open up discussion for conceptual reflection. In this process the nature of the contextualisation shifted from algebraic to conceptual; therefore the procedures and operations were transformed to become structures that acted as a frame for the development of conceptual understanding (Scheja & Pettersson, 2010). Pettersson and Scheja (2008:781) argue:

> Students who develop their understanding of calculus using an algorithmic context of interpretation do so, not because of a misconception, but, because it is functional for them and enables them to deal pragmatically and often successfully with learning tasks that they are confronted with in teaching and exams.

Both studies of Pettersson and Scheja (2008 and 2010) serve as an example where procedural fluency is fore-grounded and mentioned as the preferred focus for the facilitation of conceptual understanding of calculus.

2.4.2 Reform-minded approach

Reformed-minded teaching approaches accentuate the development of conceptual understanding as promoted by the cognitive-cultural paradigm. The aim of the approach is to develop citizens who can make sense of the world around them mathematically and use mathematics to solve problems (Sowder, 2007). According to this approach, teaching focusses on development of problem-solving skills and flexible reasoning, uncovering of relationships and patterns, making sense of mathematics and applications (Brown *et al.*, 2002; Engelbrecht *et al.*, 2005). Although computational skills are important, these are not the main focus.

This approach prefers constructivist pedagogical approaches to promote students' conceptual understanding (Abdulwahed *et al.*, 2012). Students expand their prior knowledge and apply it to new contexts. The teacher's role is to facilitate discussions and meaningful connections. The reflection, critical thinking and problem-solving process will lead to the construction of knowledge and generalisations. According to Sorensen (2013:16), any constructivist classroom will include "complex and relevant learning situations, social dialogue, multiple perspectives and multiple modes of learning, student ownership in learning and students who are self-aware of the knowledge they have constructed." Brown *et al.* (2002) share the view that student learning occurs when teaching firstly focusses on the development of conceptual knowledge and thereafter the acquisition of procedural knowledge, but the converse is not true.

Baker and Czarnocha (2002) conducted a study with students who had difficulty with computations. The authors indicated that conceptual development occurs through conscious reflection upon existing conceptual knowledge. Metacognition (Schoenfeld, 1992) and development of conceptual understanding may occur in the absence of procedural fluency. The study agrees with Vygotsky's stance that conceptual development is possible in the absence of reflection on repeated exercises. On the other hand, this study takes a stand against Piaget's view that procedural fluency is required for the construction of conceptual knowledge (Baker & Czarnocha, 2002; Engelbrecht *et al.*, 2005). Baker and Czarnocha (2002) claim that students have the ability to think and reflect on a method when doing written mathematics before they are competent in applying the method. Brown *et al.* (2002) argue that if procedures are taught before concepts are explained and understood, students might struggle to develop conceptual knowledge.

Chappell and Killpatrick (2003) performed a study where calculus students were taught following either a procedural or a conceptual teaching approach. The students subjected to the conceptual approach scored significantly higher than the students who followed the procedural approach on both procedural and conceptual assessment. The study provided evidence that a concept-based environment can develop conceptual understanding without sacrificing proficiency skills (Chappell & Killpatrick, 2003; Engelbrecht *et al.*, 2012). Knowledge gained in a concept-based environment can be extended better to unknown situations (Engelbrecht *et al.*, 2012).

Engelbrecht *et al.* (2005) investigated students' performance in procedural and conceptual questions in a first-year calculus module. The module applied a reform-minded approach; therefore, the focus was on understanding and on cultivating conceptual thinking. New concepts were introduced using different representations - verbal, numeric, algebraic and visual - which could enhance conceptual understanding (Ball *et al.*, 2004; Brown *et al.*, 2002; Kilpatrick *et al.*, 2001). Findings showed that students performed better in conceptual that in procedural problems. In addition, students were more confident with conceptual problems than procedural problems, and had fewer misconceptions about concepts than about procedures.

2.4.3 The iterative, integrated teaching approach

The view of basic procedural skills versus conceptual understanding has led to a misconception that the demand for precision and fluency in the execution of basic mathematical skills runs counter to the development of conceptual understanding. Mathematics skills, understanding, and proficiency are completely intertwined (Kilpatrick *et al.*, 2001; Wu, 1999; Yates, 2009). It has been noted before that the neglect of basic skills acquisition in favour of a conceptual approach may result in superficial conceptual understanding, but too much emphasis on basics skills may leave students without effective mathematical understanding (Yates, 2009).

Baroody (2009:23) claims that the investigative teaching approach involving "purposeful, meaningful, and inquiry-based instruction" is the best approach to achieve mathematical proficiency. The aim of this teaching approach is to promote procedural fluency and conceptual understanding in an integrated manner.

Mathematical inquiry aims to develop problem-solving, reasoning, representation and communications skills. The role of the teacher is to plan and facilitate teachable moments by means of appropriate tasks, probing questions, reflection and class discussions. The student is an active participant who benefits from the inquiry-based learning environment. The selection of appropriate tasks will create metacognition and cognitive conflict (Baroody, 2009; Swan, 2001) and provide the student with a platform for the construction of knowledge and conceptual understanding.

Brown *et al.* (2002:3) state that teachers should plan their lessons and describe integrated teaching strategies:

> Conceptual understanding does not "just happen." Lessons have to be carefully designed so that students have experiences that will help them make connections. Tasks must be chosen with a particular mathematical idea as a focus and they must connect closely to students' prior knowledge. A fruitful task allows the development of multiple strategies and the use of more than one representation. This way, students are asked to explain their thinking, compare their methods, and justify their results. The focus should not only centre on the correct answer, but on the valid reasoning and vital connections elicited.

The studies by Rittle-Johnson and Alibali (1999), Rittle-Johnson, Siegler and Alibali (2001), Rittle-Johnson and Koedinger (2009), Schneider *et al.* (2011), Rittle-Johnson and Schneider (2016), Rittle-Johnson *et al.* (2015), Rittle-Johnson, Fyfe and Loehr (2016) and Rittle-Johnson (2017) agree that causal, bidirectional connections exist between procedural and conceptual knowledge. The following quotation from Rittle-Johnson and Schneider (2016:11) summarises the iterative nature of relations between procedural and conceptual knowledge when referring to the teaching and learning of mathematics:

> There is extensive evidence indicating that the development of conceptual and procedural knowledge of mathematics is often iterative, with one type of knowledge supporting gains in the other knowledge, which in turn supports gains in the other type of knowledge. Conceptual knowledge may help with the construction, selection, and appropriate execution of problem solving procedures. At the same time, practice implementing procedures may help students develop and deepen understanding of concepts, especially if the practice is designed to make underlying concepts more apparent. Both kinds of knowledge are intertwined and can strengthen each other over time.

The development of conceptual and procedural mathematical knowledge is iterative and bidirectional. Therefore, teaching instruction should not only focus on one of the types of knowledge, since procedures and concepts are intertwined and will strengthen each other over time. Conceptual knowledge may help with the development, selection and correct implementation of problem-solving procedures; on the other hand, procedural fluency may help students observe, develop and deepen conceptual understanding.

Rittle-Johnson collaborated with various experts and wrote a number of articles on the teaching and learning of mathematics, focussing on relations between procedural and conceptual knowledge. Rittle-Johnson and Alibali (1999) highlight the causal, bidirectional relations between conceptual and procedural knowledge in children learning mathematical equivalence. Conceptual instruction leads to increased conceptual understanding and the development of an appropriate procedure; conversely, procedural instruction leads to increased conceptual understanding but to limited transfer of the instructed procedure. The authors recommend that learners should be prompted to explain why procedures are correct, and that this should enhance conceptual understanding. Rittle-Johnson *et al.* (2001) pointed out that conceptual understanding would be enhanced by the development and choice of appropriate procedures when using suitable representations. Rittle-Johnson and Koedinger (2009) claim that the iterative teaching approach appears to show greater mathematical knowledge gains, particularly with respect to procedures, than the concepts-first teaching approach. Schneider *et al.* (2011) conducted the first successful attempt to model the longitudinal relations between conceptual knowledge and procedural knowledge by means of latent variable analyses. The findings suggested that conceptual understanding and procedural fluency develop in an iterative manner, and that both types of knowledge support the development of procedural flexibility. The study found clear evidence of symmetrical bidirectional predictive relations between conceptual and procedural knowledge. The methodology used in this study will be discussed in the next section. A study by Rittle-Johnson *et al.* (2016) investigated whether conceptual and procedural instruction should both occur in a single lesson. It was found that two interventions of conceptual instruction lead to better retention of procedural and conceptual knowledge than a combination of procedural and conceptual instruction.

Rittle-Johnson *et al.* (2015) state that there is no optimal sequencing of instruction and that there are multiple routes to mathematical proficiency. It was confirmed that the order of procedural and conceptual instruction did not make a difference to the results of the study by Rittle-Johnson *et al.* (2016).

Rittle-Johnson (2017) claimed that mathematical proficiency requires conceptual and procedural knowledge as well as procedural flexibility. The author suggested that comparing, explaining and exploring are learning techniques that will support the development of both conceptual and procedural knowledge. Students should compare correct or incorrect procedures and should have opportunities to engage with and explore problems before instruction.

Engelbrecht *et al.* (2009), Engelbrecht *et al.* (2012), Bergsten, Engelbrecht and Kagesten (2015) and Bergsten, Engelbrecht and Kågesten (2017) collaborated on a project between two institutions, one in South Africa and one in Sweden. The project investigated whether the emphasis in undergraduate mathematics courses for engineering students should remain the traditional, procedural approach, or whether the course would benefit from a more conceptual orientation. Engelbrecht *et al.* (2009:1) describe the relationship between procedural and conceptual knowledge as a "complex interdependence of these constructs when doing mathematical work." The authors claim that engineering faculties traditionally emphasised the teaching of procedural and computational skills, but simultaneously expected creative, analytic conceptual skills (Engelbrecht *et al.*, 2009). Engelbrecht *et al.* (2012:7) ask: "Should we carry on with equipping students with fluent manipulation skills or should the emphasis move towards a more conceptual presentation of the mathematical concepts and ideas to develop deeper understanding of these concepts?" Engelbrecht *et al.* (2012) reported that students performed better on procedural tasks than conceptual tasks and that undergraduate students viewed mathematics as procedural, since procedural questions were more common than conceptual questions in their modules. A paper by Bergsten *et al.* (2015) reported that two qualified engineers who were interviewed saw the role of mathematics in engineering as more conceptual than procedural. However, one of the engineers stated that the procedural foundation underpins important conceptual understanding of mathematics. According to a study by Bergsten *et al.* (2017), junior and senior engineering students viewed both procedural and conceptual aspects as essential to the mathematics curricula. Bergsten *et al.* (2015:12) reported: "Teaching in the basic mathematics courses could benefit from keeping a balance between the conceptual and procedural approaches, emphasising the value and distinct feature of both for the development of mathematical knowledge, as well as the dynamics between them.” The studies by Engelbrecht, Bergsten and Kågesten emphasised that the integration of both conceptual and procedural aspects of mathematical activity are important for the understanding of mathematics when studying engineering.

2.4.4 Summary

The above discussion indicates that researchers have different views on teaching strategies for developing mathematical knowledge. The direct instruction approach is promoted by Wu, Yates and Pettersson and Scheja; on the other hand, the authors Baker and Czarnocha, Chappell and Killpatrick, and Engelbrecht, Harding and Potgieter prefer the reform-minded approach. The researcher agrees with the iterative, integrated teaching approach promoted by Baroody, Rittle-Johnson and co-authors, and Engelbrecht, Bergsten and Kågesten.

2.5 MEASUREMENT OF MATHEMATICAL KNOWLEDGE

The above sections emphasised the importance of the development of conceptual understanding and procedural fluency, and using an iterative, integrated teaching and learning approach to enhance mathematical proficiency. In order to measure conceptual and procedural relations, measurement of conceptual and procedural knowledge has to be defined. Researchers define different measures for conceptual and procedural knowledge. The choice of measure is critical for the interpretation of evidence on relations between conceptual and procedural knowledge. The first part of this section discusses how the measurement of procedural and conceptual knowledge can be defined. The second part of the section focusses on four studies that have attempted to measure relations between procedural and conceptual knowledge.

2.5.1 Defining measurement of procedural and conceptual knowledge

Baker and Czarnocha (2002) measured procedural knowledge using students’ course average and conceptual knowledge using scores on students’ writing exercises as a result of metacognition. The study found that meta-cognitive reflection occurred through both writing about mathematics and doing repeated exercises.

According to Hiebert and Wearne (1986), the presence of links and connections is fundamental to the nature of conceptual knowledge. Star and Stylianides (2013) stated that conceptual knowledge could be assessed using concept maps and interviews. The authors advocated that multiple-choice questions (MCQ) are not suitable for assessing the quality of students' conceptual knowledge. Furthermore, the study indicated that teachers had different views on whether a particular task measured conceptual knowledge, procedural knowledge, neither or both. According to teachers, measurement of conceptual knowledge connected to whether a student understood a definition or merely memorised it. A correct answer might indicate conceptual knowledge in one student, but procedural knowledge in another.

Rittle-Johnson and Schneider (2016) evaluated different studies on the relations between conceptual and procedural knowledge and discussed ways to measure procedural and conceptual knowledge. The authors described procedural knowledge tasks mostly as solving problems that are familiar to students. A reasonably unfamiliar task might require identification of a known procedure and a small adjustment to a known procedure. The time required to complete a procedural task is important in the measurement of the task, since an automated response is anticipated. Tasks do not involve mindful reflection and are often independent of conceptual knowledge. The outcome measure is accuracy of answers or procedures and time required to complete the task. Flexibility of procedural knowledge could be determined by measuring the student's knowledge of multiple procedures and the person's ability to choose the best procedure to solve a problem.

Rittle-Johnson and Schneider (2016) described conceptual knowledge tasks as relatively unfamiliar to students; students need to access their conceptual knowledge framework to derive an answer. Conceptual tasks often require knowledge of many related concepts; therefore, the task has a multi-dimensional construct. These tasks should be completed as pen-and paper assessments or answered during interviews. Interviews provide information on the quality of students' understanding of a topic or concept. Conceptual measures are stronger if multiple tasks are used to assess the same concept, since this method reduces the impact of task-specific properties. Table 2.5 summarises the distinction between implicit or explicit measures of concepts (Rittle-Johnson & Schneider, 2016:5-6).

Table 2.5: Measurement of conceptual knowledge (Rittle-Johnson & Schneider, 2016)

Measurement of conceptual knowledge	
Implicit measures (evaluate)	**Explicit measures (explain)**
Evaluate unfamiliar procedures	*Explain* judgements
Evaluate examples of concepts	Generate or select definitions of concepts
Evaluate quality of answers given by others	*Explain* why procedures work
Translate quantities between representational systems	Draw concept maps
Compare quantities	
Invent principle-based shortcut procedures (rules of thumb)	
Encode key features	
Sort examples into categories	

Implicit measures are often evaluation tasks that measure the correctness of answers and the quality of unfamiliar procedures, examples of concepts, translation between representations or comparisons of quantities. Explicit measures explain why procedures work, give judgement on unfamiliar procedures, and explain the generation of concept definitions and concept maps. Concept maps are cognitive structures where connections between concepts are made explicit. Concept maps are linked to the personal concept image of a particular student; therefore, concept maps may differ among students. Interviewers may use probing questions as a tool to prompt thinking and conceptual reflection pertaining to a topic (Pettersson & Scheja, 2008). To summarise: *implicit* measures relate to *evaluation* of procedures, answers, examples of concepts, etc.; *explicit* measures relate to *explanation* of procedures, answers, definitions and concepts.

The application and evaluation of (familiar) procedures traditionally fall in the domain of procedural knowledge. As pointed out before, Crooks and Alibali (2014) classify procedural knowledge under conceptual knowledge; furthermore, the authors do not distinguish between familiar and unfamiliar procedures. Table 2.6 indicates the types of measures of conceptual knowledge described by Crooks and Alibali (2014:364) and connects the types of measures to the classification of implicit and explicit measurement of conceptual knowledge described by Rittle-Johnson and Schneider (2016).

Table 2.6: Measurement of conceptual knowledge (Crooks & Alibali, 2014)

Measurement of conceptual knowledge	
Types of measures	**Explanation (explicit) and evaluation (implicit)**
Application of procedures (focus on answer)	Carrying out procedure (implicit) Justification of procedure (explicit)
Evaluation of (different) procedures	Explanation (explicit) Rating of procedure (implicit)
Evaluation of examples	Explanation (explicit) Rating of example (implicit)
Explanation of concepts	Explanation of a concept (explicit)
Miscellaneous	Varied

Rittle-Johnson *et al.* (2015) build their study on the measurement of conceptual knowledge shared by Rittle-Johnson and Schneider (2016) and Crooks and Alibali (2014). The authors state that conceptual knowledge can be measured, ranging from evaluating the correctness of an example or relatively unfamiliar procedure to providing definitions and explanations of concepts. The authors state that a feature of conceptual tasks should be that the task is relatively unfamiliar to participants, since students have to access conceptual knowledge rather than use a known procedure to derive the answer. Procedural tasks involve familiar procedures or procedures that require small adaptations to known procedures. Measures of procedural knowledge mostly involve solving problems using relatively familiar procedures and measuring determines the accuracy of answers. Table 2.7 provides an overview of the measurement of procedural and conceptual knowledge in literature.

Table 2.7: Overview on the measurement of mathematical knowledge

Overview on the measurement of mathematical knowledge		
Author	**Measure of procedural knowledge**	**Measure of conceptual knowledge**
Baker and Czarnocha (2002)	Course average (assessment)	Type: scores on writing exercises as a result of metacognition (e.g. concept maps)
Star and Stylianides (2013)		Type: concept maps and interviews Multiple-choice questions not suitable
Crooks and Alibali (2014)	Procedural knowledge is classified under conceptual knowledge	1. **Apply** familiar and unfamiliar procedures and justify **answers:** doing (implicit), justifying (explicit) 2. Evaluate (different) familiar and unfamiliar **procedures**:-explain (explicit), rate (implicit) 3. Explain a **concept** (explicit) 4. Evaluate **examples**: explanation (explicit), rating (implicit)
Rittle - Johnson and Schneider (2016)	Accuracy of answers Accuracy of familiar procedures Time required to complete the task	Type: pen-and paper assessment or interviews 1. Evaluate correctness of **answers** (implicit) 2. Evaluate (implicit) and explain/judge (explicit) unfamiliar **procedures** 3. Generate/select definitions of **concepts** and **concept maps** (explicit) 4. Evaluate **examples** of concepts (implicit) Other: Translate between representations (implicit) Compare quantities (implicit)
Rittle - Johnson, Schneider, Star (2015)	Accuracy of answers Accuracy of familiar procedures	1. Evaluate the correctness of an **answer**/example 2. Evaluate correctness of relatively unfamiliar **procedures** 3. Provide **definitions** and explain **concepts**

The level of procedural fluency and flexibility will be reflected by measurement of the accuracy of answers and familiar procedures, and time spent on the task. On the other hand, conceptual understanding, strategic competence and adaptive reasoning (Kilpatrick *et al.*, 2001) will be assessed when conceptual knowledge is measured through evaluation (implicit measures) and explanations (explicit measures) of concepts and procedures.

2.5.2 Measurement of procedural and conceptual relations

Literature provides evidence of attempts to measure relations between procedural and conceptual knowledge. In this study the researcher highlights four studies. The first study was by psychologists, the second study was a collaboration between psychologists Rittle-Johnson and Schneider, and Star, researcher in mathematics education. The third and fourth studies were conducted by science and mathematics education experts.

2.5.2.1 Studies conducted from a psychological perspective

An article by Schneider and Stern (2010) was based on Schneider's doctoral thesis. This was the first time in literature where validities of hypothetical measures of conceptual or procedural knowledge were tested (Schneider & Stern, 2010). The study followed a multimethod approach and attempted to measure the development of relations between conceptual and procedural knowledge of 289 fifth- and sixth-graders by four common hypothetical measures of each knowledge type. The first part of the study was an experimental pre-test and post-test design with two treatment groups and a control group. The conceptual intervention group received treatment to express mainly conceptual knowledge; the procedural intervention group was guided to express mainly procedural knowledge and the control group engaged in non-mathematical activity to minimise the probability of transfer to the tests. Part one measured whether the treatments affected the conceptual and procedural groups of measures in consistent ways. The authors measured conceptual knowledge using a group of four measures discussed in the previous section, more specifically:

1. Evaluation of problem-solving strategies
2. Concept linked to representation on a pie chart
3. Comparison of ordinal relations between numbers
4. Verbal (written) explanations of general principles.

The group of four procedural knowledge measures were:

1. Accuracy of procedure followed
2. Speed used to solve the problem
3. Asymmetry – this indicates whether an increase in procedural knowledge decreases the solution time for task completion
4. Dual-task costs – students with better procedural knowledge need less cognitive input to solve a task, therefore they are better able to solve a second task simultaneously.

Asymmetry and dual-task costs were unique to this study and were not included in the discussion on the measurement of procedural knowledge in the previous section.

The second part of the study had three measurement points as part of a longitudinal design. The study assessed whether conceptual and procedural knowledge could be modelled as latent factors underlying the conceptual and procedural groups of measures. Validities were explored using CFAs. The study showed significant problems with measurement validity, since for each kind of knowledge, and at each measuring point, the latent factor explained less than 50% of the variance (Schneider *et al.*, 2011; Schneider & Stern, 2010). The authors suggested that latent variable analysis could be a useful measurement tool; however, theoretical and practical progress can only be established once validity issues have been dealt with.

A study by Schneider *et al.* (2011) evaluated the linear equation knowledge of 200 middle-school learners before and after lessons on the topic. The study used latent variable analyses to investigate predictive relations between conceptual and procedural knowledge. The authors successfully modelled longitudinal relations between the constructs conceptual and procedural knowledge, and procedural flexibility with SEM at two different measurement points. Procedural flexibility was modelled only for the second measurement point, since students in general display little or no flexibility prior to problem-solving experience within a specific domain. The study also investigated whether prior knowledge moderated relations between conceptual and procedural knowledge. Data were collected from two samples (N = 228 and 304) differing in prior knowledge.

The instrument consisted of nine procedural knowledge items (three familiar, six unfamiliar), 13 conceptual knowledge items and 20 items that measured procedural flexibility. The sample items mentioned by Schneider *et al.* (2011) indicated a distinct difference in the nature of procedural and conceptual items. Procedural items asked students to solve for the unknown and the focus was on the accuracy of procedures and answers. Conceptual knowledge items asked students to evaluate and compare methods (implicit measures) and explain or justify their reasoning (explicit measures) (Rittle-Johnson & Schneider, 2016). The procedural flexibility items asked students to generate multiple methods, recognise multiple methods, evaluate unconventional methods and explain their reasoning (Schneider *et al.*, 2011).

The study showed that the data for the latent constructs conceptual knowledge, procedural knowledge and procedural flexibility were an excellent fit to the model. The relatively high factor loadings indicated that constructs were measured with acceptable reliability. The study suggested that in the overall context of a longitudinal study, these three constructs could be modelled as interrelated, but separate latent factors. High correlations between latent factors assessing knowledge structures indicated that these knowledge structures often appear together. This does not necessarily imply a high level of cognitive integration of the two knowledge structures. Analyses of the data yielded the following findings:

1. There was clear evidence of symmetrical, bidirectional predictive relations between conceptual and procedural knowledge.
2. The relations were stable over two large samples differing in prior knowledge; therefore, students' prior knowledge did not influence the predictive relations between conceptual and procedural knowledge.
3. Conceptual and procedural knowledge independently predicted students' procedural flexibility. This highlights the importance of both types of knowledge for increasing flexibility.

The multimethod approach used by Schneider and Stern (2010) to measure the development of relations showed low levels of validity (Rittle-Johnson & Schneider, 2016) and could not be regarded as a possible method to answer the research questions. The methodology used in the study conducted by Schneider *et al.* (2011) could not be considered a possible methodology framework, since the model measured longitudinal relations between the constructs over two different measuring points. However, SEM, in particular CFA, could be implemented as part of the quantitative analysis to investigate expected relations between different knowledge classes in this study.

2.5.2.2 Studies conducted from a mathematical education perspective

Engelbrecht *et al.* (2005) conducted a study to compare the conceptual and procedural skills of first-year calculus students taught using a reform-minded teaching approach. The aim of the study was to determine whether there was any relationship between student's conceptual understanding and procedural fluency, and to investigate possible relations between students' confidence levels when dealing with procedural and conceptual tasks and performance of tasks. The study used the following working definitions for procedural and conceptual knowledge (Engelbrecht *et al.*, 2005:704) to construct 10 procedural and 10 conceptual test items for the study:

> **Procedural knowledge** is the ability to physically solve a problem through the manipulation of mathematical skills, such as procedures, rules, formulae, algorithms and symbols used in mathematics.
>
> **Conceptual knowledge** is the ability to show understanding of mathematical concepts by being able to interpret and apply them correctly to a variety of situations, as well as the ability to translate these concepts between verbal statements and their equivalent mathematical expressions. It is a connected network in which linking relationships are as prominent as the separate bits of information.

The paper presents examples of a conceptual and a procedural item. The example of the conceptual item in the study connects to translations between algebraic mathematical expressions and graphs (visual), therefore translations between different representational systems (Rittle-Johnson & Schneider, 2016). On the other hand, the example of the procedural task involves differentiation rules and linking relationships when the equation of the tangent line is determined. The procedural task connects to the accuracy of procedures and answers (Rittle-Johnson & Schneider, 2016). The authors claim that most items require both procedural and conceptual knowledge, but items could be classified as predominantly procedural or conceptual. The authors pointed out that the nature of a conceptual problem could become procedural if a student became familiar with the type of problem (Engelbrecht *et al.*, 2009; Engelbrecht *et al.*, 2005). Construct validity was established when the test items were independently evaluated by a panel of experts in the mathematics department where the course was presented. The views of the panel were combined to obtain a triangulated opinion of the percentage of procedural and conceptual knowledge needed to complete an item $(k_p \text{ and } k_c)$ successfully. The panel also indicated the level of item difficulty on a procedural and conceptual level $(d_p \text{ and } d_c)$. Engelbrecht *et al.* (2005) used the procedural performance index (PPI) and conceptual performance index (CPI) to calculate each students' performance.

$$\text{PPI} = \frac{\sum \alpha^* k_p^* d_p^*}{\sum k_p^* d_p^*}$$

$$\text{CPI} = \frac{\sum \alpha^* k_c^* d_c^*}{\sum k_c^* d_c^*}$$

$k_p^* =$ is the percentage of procedural knowledge needed to complete an item * successfully according to the definition of the approach used to solve a mathematical task, and the panel

$k_c^* =$ is the percentage of conceptual knowledge needed to complete an item * successfully according to the definition of the approach used to solve a mathematical task, and the panel

$d_p^* =$ is the level of difficulty on a procedural level of an item * according to the panel item

$d_c^* =$ is the level of difficulty on a conceptual level of an item * according to the panel item

$\alpha^* = 1$ if item * is answered correctly

$\alpha^* = 0$ if item * is answered incorrectly

The sum was taken over the number of relevant items. Both PPI and CPI were expressed as percentages. The study could be criticised, since these calculations allowed for subjective interpretation and possible bias, as the paper does not provide a scientific method to determine the percentage of procedural and conceptual knowledge needed to complete an item successfully, or to determine the procedural or conceptual difficulty per item. The study indicated that students do not perform better on procedural items than conceptual items, that students are more confident to handle conceptual rather than procedural problems and that students do not have more misconceptions about concepts than procedures.

The fourth study or project relates to ongoing collaboration between mathematical education experts at a university in South Africa and one in Sweden. The project investigates whether the teaching emphasis in undergraduate mathematics courses for engineering students should remain the traditional, procedural approach, or whether students would benefit from a more conceptual orientation (Bergsten *et al.*, 2015; Bergsten *et al.*, 2017; Engelbrecht *et al.*, 2012). Engelbrecht, Bergsten and Kågesten classified the approach used to solve a mathematical task as mainly conceptual or mainly procedural. The authors initially developed working definitions to describe the approaches used to solve mathematical tasks (Engelbrecht *et al.*, 2009), but re-worked the definitions to construct test items that were used in the project. The following definitions describe the conceptual and procedural approaches used to solve mathematical tasks (Bergsten *et al.*, 2015; Bergsten *et al.*, 2017; Engelbrecht *et al.*, 2012):

> **Conceptual approach**: This includes translations between verbal, visual (graphical), numerical, and formal/algebraic mathematical expressions (representations); linking relationships; and interpretations and applications of concepts (for example by way of diagrams) to mathematical situations.
>
> **Procedural approach**: This includes symbolic and numerical calculations, employing (given) rules, algorithms, formulae, and symbols.

The definitions for teaching approaches could be linked to the measurement of procedural and conceptual knowledge discussed earlier. The outcome measure for the procedural approach focusses on accuracy of answers and familiar procedures. The outcome measure for the conceptual approach is conceptual understanding that relates to translations between representations, linking relationships, interpretations and applications. The conceptual measures link to evaluation (implicit measures) and explanation (explicit measures) of conceptual tasks.

Engelbrecht *et al.* (2012) developed a draft test to investigate the mathematical problem-solving approaches used by undergraduate engineering students in South Africa and Sweden. The authors used the above definitions to classify the approach used to solve a mathematical task as either conceptual or procedural. Mathematics colleagues at both universities thoroughly and independently scrutinised the questions to ensure construct validity and obtain an unbiased view of the levels of procedural or conceptual knowledge needed to complete each item successfully. Cohesive opinions were obtained; however, it was noted that thinking among mathematical colleagues pertaining to problem-solving approaches was not unique.

A pilot study was conducted to determine whether a selection of students in South Africa and Sweden shared the same view on the nature of the tasks as the panel. In 2008 the students wrote a draft test consisting of 16 pairs of conceptual and procedural multiple-choice items. Each pair of questions included a conceptual and a procedural question in no specific order; in total 32 questions (Engelbrecht, 2008). The questions were answered in open-ended format and students' solutions were studied. The qualitative analysis of students' solutions indicated that some solutions on conceptual tasks (as classified by the panel) were "*proceduralised*" by the students (Engelbrecht *et al.*, 2009:938). This emphasised the notion that being "*conceptual"* is not necessarily a property of the task, but rather of the solution to the task. The outcome of the pilot test indicated a good fit between the expected solution strategy and actual student solutions for many of the items. In a second part of the instrument development process, the multiple-choice questions were given to a large group of South African students in order to do an item analysis with respect to the discrimination index and level of difficulty of questions.

The final test instrument was developed in 2009 and only included items in which the solution approaches of students and experts were agreed upon as either conceptual or procedural (Engelbrecht, 2009). The test instrument included the following calculus topics: differentiation of a single variable function, applications (interpretation) of the derivative, differential equations and integration (Engelbrecht *et al.*, 2012). The test included eight items (four pairs). The following conditions applied with respect to the items:

1. Each pair included a procedural and a conceptual approach item.
2. The solution approaches of students and of experts were agreed on as either conceptual or procedural.
3. Items were used with a good discrimination index.
4. Items with the same level of difficulty were paired.
5. No exceptionally easy or difficult questions were used.
6. The order of the conceptual and procedural items was different.

The procedural and conceptual performance indices used to measure student performance in procedural and conceptual items were based on the classical test theory. This could be seen as a limitation of their study, since indices will not necessarily give consistent results across different samples.

The procedural and conceptual approaches developed by Engelbrecht *et al.* (2012), Bergsten *et al.* (2015) and Bergsten *et al.* (2017) provided sensible definitions to measure knowledge in mathematical tasks. The approaches were used as a baseline to analise relations between conceptual and procedural knowledge in calculus item solutions. Some of the classified calculus items were incorporated into the data collection instrument used in this research study.

The conceptual framework used to inform the research study is discussed in the next section.

2.6 CONCEPTUAL FRAMEWORK

2.6.1 Introduction

The conceptual framework for this research study is embedded in the individual learning theory of constructivism. Mathematical knowledge is developed when students connect new knowledge to prior knowledge, and map it into their existing knowledge schema. Learning will be enhanced through the use of language for interpersonal and intra-personal communication in a social schooling environment. The study agrees with the view that the five strands for mathematical proficiency are interconnected and that teaching and learning should focus on all five stands (Hiebert *et al.*, 2003; Kilpatrick *et al.*, 2001). The development of procedural knowledge becomes meaningful when it is connected to a conceptual base (Hiebert & Carpenter, 1992; Hiebert & Wearne, 1986), since conceptual knowledge provides the basis for mathematical proficiency (Baroody *et al.*, 2007:127). Furthermore, the development of both conceptual understanding and procedural fluency underpins the development of flexibility and adaptive expertise (Baroody, 2009:24).

The researcher agrees with Star (2007) classification of mathematical knowledge according to *type* and *quality* of knowledge and that deep procedural knowledge can develop independently from conceptual knowledge (Star, 2005; Star, 2007); in other words, procedures can be understood and known deeply, flexibly, and with critical judgement, in the absence of conceptual understanding. The researcher used Star's description of superficial and deep procedural knowledge and superficial and deep conceptual knowledge to classify calculus items into four knowledge classes, namely conceptual- procedural- (C-P-), conceptual+ procedural- (C+P-), conceptual- procedural+ (C-P+) and conceptual+ procedural+ (C+P+). The classification of types of knowledge is seen over a continuum, not as separate parts, as indicated in Baroody's presentation of Star's framework (Baroody *et al.*, 2007). Figure 2.2 provides a visual presentation of the quantitative conceptual framework the researcher used for the study.

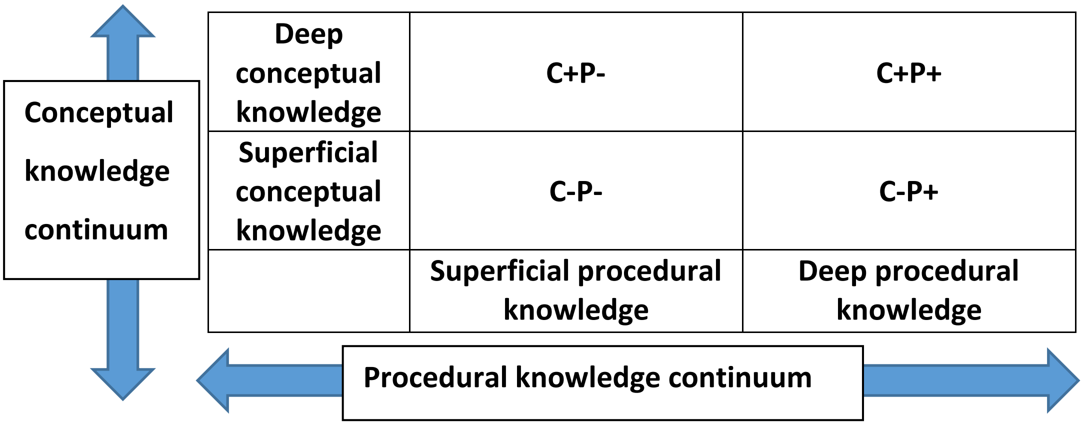


Figure 2.2: Conceptual framework

The researcher expanded the definitions of procedural and conceptual problem-solving approaches described by Engelbrecht *et al.* (2012), Bergsten *et al.* (2015) and Bergsten *et al.* (2017) for calculus items. As discussed earlier, the procedural approach focussses on accuracy of answers and familiar procedures; on the other hand, the conceptual approach connects to translations, relationships, interpretations and applications that connect to evaluation and explanation of conceptual tasks. In the study, the expanded definitions were used to analyse calculus items in order to measure conceptual understanding and procedural fluency *within* items, following a qualitative approach. The items were categorised using the notion proposed by Star (2005) that knowledge quality might be connected to the number of connections made to do the mathematical calculations. The classified items were incorporated into a data collection instrument and Rasch analysis was performed to evaluate the instrument and describe how conceptual and procedural constructs were related *between* calculus items.

The researcher's view is that an iterative, integrated teaching and learning approach should be followed to develop mathematical understanding. Teaching should be bidirectional, focussing alternatively on the development of conceptual understanding and procedural fluency, in no specific order. Students need to develop conceptual understanding of specific threshold concepts, and develop procedural fluency in distinctive mathematical procedures to maximise the development of their mathematical understanding.

2.6.2 The case of calculus

Calculus textbooks, e.g. that of Stewart (2016), and the researcher's empirical experience as a proficient mathematics teacher and lecturer provided guidance to identify threshold concepts and algebraic procedural skills that have to be mastered as a prerequisite for understanding calculus. The threshold concepts are hierarchical, since students need to understand a preceding threshold concept before a new concept can be introduced and understood. The concepts can then be connected and new knowledge can be constructed through assimilation, accommodation and reflective abstraction. These concepts, in synchronous order, include:

1. Pre-calculus – the behaviour of functions
2. Limits
3. Continuity
4. Differentiation
5. Curve sketching
6. Calculation of the net area enclosed by a graph of a function, *x*-axis and a restricted domain. This can be calculated by estimation of area calculated by geometric formulae, calculation of the Riemann sum (which includes the concepts of sigma notation and limits), and the definite integral.
7. The fundamental theorem of calculus (FTOC) (Stewart, 2016:394, 396)

Part 1: If f is continuous on [*a*, *b*], then the function defined by $g(x) = \int_a^x f(t)\,dt,\ a \leq x \leq b$ is also continuous on [*a, b*] and differentiable on (*a*, *b*), and $g'(x) = f(x)$ on (*a*, *b*).

Part 2: If f is continuous on [*a*, *b*] then $\int_a^b f(x)\,dx = F(b) - F(a)$, where *F* is any anti-derivative of *f* i.e. $F' = f$.

8. The application of real-life contextual problems involving calculus knowledge is essential for engineering students. Application problems include relations between the following concepts: displacement and total distance travelled, velocity and speed (rate of change), and acceleration, volume and rate of flow.

Furthermore, the algebraic, procedural skills students need to master include:

1. Algebraic manipulation of expressions

- Factorisation
- Adding, subtracting, multiplication and division of fractions
- Decomposition of fractions through long division, remainder theorem and partial fractions

2. Limit laws and special limits
3. Differentiation rules
4. Sigma notation
5. Integration techniques.

Students will develop the concept image and concept definition when they cultivate conceptual understanding and procedural fluency pertaining to the above threshold concepts and procedural skills. The development of the concept image will be enhanced by the use of different representations that may include the use of technology.

The iterative, integrated approach is a sound teaching practice to develop calculus understanding. Teaching and learning should focus alternatively on the development of conceptual understanding and procedural fluency, in no specific order. The point of view is illustrated with two practical teaching examples. Example 1 moves from an algebraic frame, therefore procedural fluency, to the development of conceptual understanding; conversely, example 2 connects concepts with procedures.

Example 1

The concept of rate of change can be explained by using graphs to show relations between a function $f(x)=\frac{x^2}{2}-2$ and its gradient function $f'(x)=x$ (Figure 2.3). The derivative $f'(x)$ is found using differentiation rules, therefore algebraic manipulation of symbols. The derivative at a point *x* represents the gradient function or rate of change of the function at the given point *x,* e.g. the gradient of the function *f* is 0 at *x* = 0, since the gradient function $f'(0)=0$. The function *f* takes on a local minimum at *x* = 0, therefore where $f'(0)=0$. The researcher prefers to teach the use of differentiation rules first, therefore procedural knowledge, and then connect the derivative to the concept of rate of change using graphical representations.

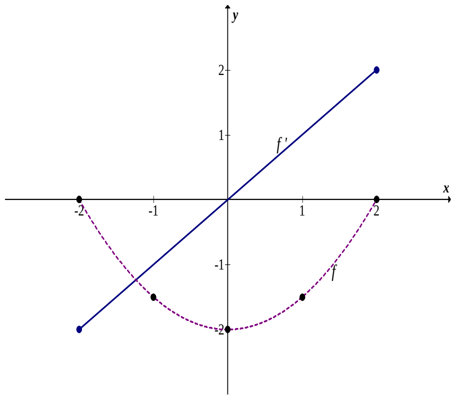


Figure 2.3: Graphical illustration of rate of change for *f*.

Example 2

The area enclosed by the graph of a function and the *x*-axis can be estimated using geometric formulas or finding the Riemann sum. These methods will provide a conceptual, graphical understanding of finding the enclosed area. Alternatively, if the function is known, the area can be found by evaluation of the definite integral on a specific interval. The procedural evaluation of the definite integral involves integration techniques. If the initial teaching emphasis is on conceptual and graphical methods, and thereafter on the evaluation of the definite integral and integration techniques, it may be deduced that conceptual understanding was introduced before the development of procedural fluency. If the calculation of the enclosed area was first introduced with the evaluation of the definite integral, and thereafter the Riemann sum, the focus would be an algebraic frame to establish procedural fluency, which, in turn, may lead to conceptual advance (Pettersson & Scheja, 2008; Scheja & Pettersson, 2010). In the case of this example, the researcher's choice of teaching focus is first on the conceptual understanding, followed by procedural evaluation of the definite integral.

It is important to focus teaching and learning on both conceptual understanding and procedural fluency, since algebraic techniques do not necessarily provide a solution to a problem. The following problem cannot be solved using integration techniques and students have to calculate the area with the Riemann sum (Hechter, Billman & Du Preez, 2018):

Use the fundamental theorem of calculus (part 2) to calculate the exact area enclosed by the curve of $k(x) = 1 - \ln x \cos 2x + x$, the x-axis, $x = 2$ and $x = 8$ by evaluating the definite integral: $\int_2^8 (1 - \ln x \cos 2x + x)dx$, if possible.

The manipulation or fixing of the constant of the integrand when applying integration techniques connects integration to differentiation using the FTOC (part 1). The linking of relationships in the process of integration could be interpreted as following a conceptual problem-solving approach; however, the finding of the constant may also be interpreted as deep procedural knowledge, since the constant is changed in order to facilitate integration using reversed thinking. The example below illustrates the researcher's argument:

$$\begin{aligned}
&\int x\sqrt{1+2x^2}\,dx \\
&= \tfrac{1}{4}\int 4x(1+2x^2)^{\frac{1}{2}}\,dx \\
&= \tfrac{1}{4}.\frac{(1+2x^2)^{\frac{3}{2}}}{\frac{3}{2}} + c \\
&= \tfrac{1}{4}.\tfrac{2}{3}.(1+2x^2)^{\frac{3}{2}} + c \\
&= \tfrac{1}{6}(1+2x^2)^{\frac{3}{2}} + c \quad .
\end{aligned}$$

Procedural fluency will help students to observe, develop and deepen conceptual understanding of concepts. Conversely, conceptual understanding may help with the development, selection, and correct implementation of problem-solving procedures.

2.7 SUMMARY

Mathematics education researchers broadly agree on the definition of procedural and conceptual knowledge, perhaps with the exception of Crooks and Alibali (2014), who include procedural knowledge as a subset of conceptual knowledge. However, from the above discussion it is clear that literature describes different perspectives with respect to the relations between conceptual and procedural knowledge. According to Engelbrecht *et al.* (2005) and Baker and Czarnocha (2002), Piaget's learning theory assumes existing conceptual knowledge according to which actions are performed. Conceptual knowledge grows as new connections are formed though assimilation and integration into a more comprehensive cognitive schema (Baroody *et al.*, 2007). Star suggests that procedural knowledge may develop in the absence of the development of conceptual understanding (Star, 2005); on the other hand, Vygotsky states that conceptual understanding may develop independently of procedural fluency (Baker & Czarnocha, 2002; Engelbrecht *et al.*, 2005). Kilpatrick *et al.* (2001) state that the five strands of mathematical proficiency are intertwined, therefore the relationship between conceptual and procedural knowledge is connected and interdependent. Hatano and Baroody state that conceptual understanding is the basis for adaptive expertise and procedural fluency (Baroody, 2009; Baroody & Dowker, 2009; Baroody *et al.*, 2007; Hatano, 1988). Furthermore, Hatano, Baroody and Taraban classify conceptual and procedural knowledge together when routine and adaptive expertise, weak and strong schemas and shallow and deep knowledge are discussed.

The use of different representations will enhance conceptual understanding and procedural fluency and develop the concept image concerning a concept or principle. The iterative perspective is the view supported by most mathematics education researchers on the developmental order of conceptual and procedural knowledge. Teaching strategies advocated by researchers focus on direct instruction using a procedural frame, a reform-minded, conceptual approach or an iterative, integrated teaching approach.

Literature informs different measures for mathematical knowledge. The measurement of procedural knowledge could focus on course marks, accuracy of answers, accuracy of familiar procedures and time required to complete the task. Conceptual knowledge could be measured through pen-and-paper assessment where calculations are shown, concept maps, interviews, etc. The suggested measurement for conceptual knowledge includes implicit and explicit measures of conceptual tasks. Implicit measures include evaluation of correctness of answers and procedures, and translation between representations. Explicit measures include explanations of why procedures work and the generation of concept maps.

The research methodology used to answer the research questions will be discussed in the next chapter.

3. CHAPTER 3: RESEARCH METHODOLOGY

3.1 INTRODUCTION

In chapter 2 the literature study that informed the study was discussed. This chapter focusses on important aspects with respect to research methodology used to answer the research questions. This study used a mixed methods approach to investigate, analyse and describe relations between conceptual and procedural knowledge in solutions of calculus items in three layers. The first layer followed a qualitative approach; the second and third layers followed a quantitative approach.

The first layer of analysis investigated relations between procedural and conceptual knowledge *within* the solutions of calculus items. Calculus items were developed and qualitative content item analysis was performed using the approaches described to answer mathematical items defined by Engelbrecht, Bergsten and Kågesten in various papers discussed in chapter 2. The analysed items were classified into four knowledge classes analogous to the conceptual and procedural knowledge framework suggested by Star (2005).

In a second layer of analysis a data collection instrument was developed using the classified calculus items. The test was administered to first-year engineering students. Rasch measurement theory was used to describe and measure the underlying construct for all items. The analysis compared item difficulty and person proficiency and produced item statistics, indicating potentially problematic items and response categories.

In the third layer of analysis relations *among* items in the four knowledge classes were investigated using CFA.

3.2 RESEARCH APPROACH

A mixed methods approach was chosen for the research study. The discussion and motivation for the approach will be discussed referring to the research paradigm, design and the choice of research methods for the study.

3.2.1 Research paradigm

Literature indicates that different paradigms hold different philosophical views concerning ontological, epistemological, and value and belief assumptions. The choice of research paradigm chosen for a mixed methods approach is complex, since the methodogy involves both a qualitative and a quantitative section. According to Maxwell and Mittapalli (Maxwell & Mittapalli, 2010:5):

> The main argument for combining qualitative and quantitative paradigmatic positions, as well as methods, in mixed methods research has traditionally been their complementarity — that they have different strengths and limitations and that using them together allows the researcher to draw conclusions that would not be possible using either method alone.

Hall (2012) investigates three possible paradigms for mixed methods research. He argues that the a-paradigmatic stance is not a sustainable option, since no research is paradigm-free. The use of multiple paradigms is not an option either, since each paradigm choice assumes a specific philosophy. The author argues that the only viable option is to choose a single paradigm. He argues against pragmatism and a transformative paradigm, but suggests the realist perspective as a viable option for mixed methods research. Scientific realism or postpositivism holds the ontology (world view/research perception) of an objective reality that exists, but acknowledges that there may be more than one approach to understanding reality in terms of the epistemological view (knowledge view) of constructivism (Maxwell & Mittapalli, 2010).

The research perspective chosen for this study is the postpositivist paradigm. The underlying assumptions of the postpositivist paradigm is that objective truth exists in the world, but that absolute truth cannot be found (Creswell & Creswell, 2017). There is a real, objective and measurable reality, although one has limited access to reality and measurement is always accompanied by error. The combination of qualitative and quantitative approaches provided a more complete understanding and investigation of the research problem than either of the approaches on its own, since it examined evidence from different angles (Christensen *et al.*, 2015; Clark & Ivankova, 2016; Creswell & Creswell, 2017). The postpositivist philosophy underpins both the qualitative and quantitative analysis for this study. The qualitative analysis was content analysis that was triangulated by mathematics education experts, but other specialists in the field may have classified items differently or held diverse opinions on the constructs. The quantitative analysis aimed to investigate whether reliable and valid inferences could be drawn by the instrument overall, the items and the sub-scales.

3.2.2 Research design

Star and Rittle-Johnson (2016) state that in the recent past mathematics education researchers focussed on *why* certain teaching practices work and preferred qualitative approaches; conversely, psychologists emphasised quantitative approaches in order to answer the question of *what* works. The authors advocate the blurring of boundaries through combining qualitative with quantitative methods. The main strength of mixed methods research is the complementary, efficient integration of the strengths and the reduction of bias and weaknesses of qualitative and quantitative research design in a single research study (Christensen *et al.*, 2015). Major methods for qualitative research design include narrative research, phenomenology, ethnography, case studies and grounded theory. Content analysis is described as a qualitative method used in educational research (Cohen, Manion & Morrison, 2011; Maree, 2016). Quantitative research design is broadly described by experimental and non-experimental designs (Christensen *et al.*, 2015; Creswell & Creswell, 2017; Maree, 2016).

The researcher used an exploratory sequential mixed methods design (Christensen *et al.*, 2015; Creswell & Creswell, 2017) in order to provide deeper and more complete understanding and comprehensive analysis of the research problem. Qualitative data were explored and analysed using content analysis (Cohen *et al.*, 2011; Maree, 2016), and the findings were used to develop an instrument that was implemented in the second quantitative data phase. The quantitative study was a cross-sectional study, since data were collected as a *snapshot* in a single, brief period of time (Saunders & Tosey, 2013). The paradigm emphasis provided equal status to the qualitative and quantitative approaches, since equal emphasis was placed on the methodologies (Christensen *et al.*, 2015).

Christensen *et al.* (2015) describe weakness minimisation validity when conducting mixed methods studies as the degree to which the researcher combines approaches to minimise weakness of the analysis and results. The authors mentioned the checking of the operation and meaning of measurement instruments as one of the strengths of a mixed methods approach. After the quantitative analysis had been completed, disordered categories were identified for certain items and the qualitative analysis was used to inform the collapse of marks and categories of these items. Rasch analysis was performed again after the collapse of the categories.

The term “multiple validities” refers to the evaluation of data quality and the results of qualitative and quantitative strands (Christensen *et al.*, 2015; Clark & Ivankova, 2016; Creswell & Creswell, 2017). In qualitative approaches validity refers to trustworthiness, credibility and transferability of findings (Clark & Ivankova, 2016; Maree, 2016). Trustworthiness refers to whether the findings are persuasive and credible for others. Credibility and transferability are described as a subset of trustworthiness. Credibility refers to the congruency and accuracy of findings as interpreted by research partners, e.g. using triangulation. Christensen *et al.* (2015:69) define triangulation as:

> Use of multiple data sources, research methods, investigators, and/or theories/perspectives to cross-check and corroborate research data and conclusions.

Transferability refers to connections between research findings and the researcher’s own experience (Maree, 2016).

In quantitative approaches the validity and reliability of data and analysis procedures have to be investigated. Validity refers to the accuracy of inferences, interpretations, or actions based on test scores and other measures. Reliability refers to the accuracy of measurement procedures to produce consistent or stable scores (Christensen *et al.*, 2015; Clark & Ivankova, 2016; Creswell & Creswell, 2017). Triangulation processes were used to reduce bias in both qualitative and quantitative analysis and will be explained in the following sections.

3.2.3 Research methods

Research methods refer to the research questions, data collection, data analysis, interpretation and validation (Creswell & Creswell, 2017). According to Christensen *et al.* (2015), the major methods of data collection include tests, questionnaires, interviews, focus groups, observations and existing data. Creswell and Creswell (2017) describe surveys and experiments as quantitative data collection methods, and observations, interviews, documents and audio-visual material as qualitative methods.

In the qualitative part of the research study calculus items were developed and content analysis was conducted on the approaches used to answer the items. In the quantitative section the data collection instrument was given to students as a test with three sections. The results of the test were analysed using Rasch analysis (Bond & Fox, 2015) and CFA (Cohen *et al.*, 2011; Maree, 2016).

The qualitative and quantitative approaches will be explained concerning methodology and analysis in the next two sections.

3.3 QUALITATIVE ANALYSIS

3.3.1 Introduction

Qualitative analysis describes and measures relations between procedural and conceptual knowledge *within* the solutions for calculus items. A large selection of calculus items was developed and a complete memorandum, including, if possible, various ways of solving the problems, were developed for each item. The memorandum was analysed according to newly developed definitions on conceptual and procedural approaches to answer calculus items. Thereafter each task was labelled and classified into one of four knowledge classes (Star, 2005) based on the type and quality of knowledge used to do the problem-solving. Mathematics education specialists were involved with the development, analysis and classification of calculus items through a process of triangulation.

3.3.2 Content analysis

Content analysis was conducted to evaluate how calculus items can be analysed and classified with respect to conceptual and procedural knowledge. Content analysis is described as a systematic, rigorous analysis, derived from theoretical constructs, which researchers can use to make inferences. According to Cohen *et al.* (2011:564):

> Content analysis takes text and analyses, reduces and interrogates them into summary form through the use of both pre-existing categories and emergent themes in order to generate or test a theory. It uses systematic, replicable, observable and rule-governered forms of analysis in a theory-dependent system for the application of those categories.

The content analysis performed for this study will be described by the steps suggested by Cohen *et al.* (2011).

1. The population is mathematical items, but the sample and context are calculus items.
2. The units of analysis are the solutions of the calculus items.
3. The coding of categories refers to procedural or conceptual problem-solving approaches used to answer calculus items.
4. The construction and description of categories for analysis refer to the breakdown of procedural and conceptual problem-solving approaches in order to create expanded, refined procedural and conceptual analysis categories.
5. The data analysis was conducted when the frequency of each code (category) in the calculus answer was counted and the item was labelled.
6. Summarising included searching for patterns and relations *within* answers to calculus items and the inductive development of four knowledge classes.

The researcher will refer to the above steps when expanding on the steps in the following sections. The studies by Bergsten *et al.* (2015); Bergsten *et al.* (2017); Engelbrecht *et al.* (2009); Engelbrecht *et al.* (2012) provided guidelines for the development and analysis of the calculus items.

3.3.2.1 Development of items

The items developed for the test used in the empirical studies done by Engelbrecht, Bergsten and Kågesten were on the mathematics topics of differentiation, application of the derivative, differential equations and integration. The calculus items for this study covered three topics: functions (pre-calculus), differentiation and application of derivatives (curve sketching and contextual problems), and integration and application of integration. Questions on differential equations fell outside the scope of the curriculum for first-year engineering students and were therefore not included in the study. The research instrument items included 17 multiple-choice and 16 open-ended questions. Three of the six applicable items from the final test developed by Engelbrecht (2009) were included as part of the 33 calculus items that were analysed and classified as part of this study. Nine of the items included in this study were part of the 24 applicable items in the authors' draft instrument (Engelbrecht, 2008). The selected items were familiar to students through their first-year curriculum and coursework, however, procedures and concepts were not specifically targeted in the university module. The rest of the items were developed as part of course assessment of the first-year mathematics module (Hechter, 2017).

3.3.2.2 Coding, construction and description of problem-solving categories

The researcher used the conceptual and procedural approaches to solve mathematical tasks defined by Engelbrecht *et al.* (2012), Bergsten *et al.* (2015) and Bergsten *et al.* (2017) to code and describe problem-solving categories. The definitions were further classified into sub-categories for conceptual and procedural approaches. The sub-categories for the conceptual approaches were:

- C_1 = translations between verbal, visual (graphical), numerical, and formal/algebraic mathematical expressions (representations)
- C_2 = linking relationships
- C_3 = interpretations of concepts
- C_4 = applications of concepts to mathematical situations (contextual problems).

The sub-categories for procedural approaches were:

- P_1 = symbolic and numerical calculations
- P_2 = (given) rules
- P_3 = algorithms
- P_4 = formulae
- P_5 = symbols.

The above definitions for conceptual and procedural approaches were expanded and refined for calculus tasks. The breakdown of problem-solving approaches was coded and described for functions, differentiation and integration in Table 3.1.

Table 3.1: Conceptual and procedural problem-solving categories

Conceptual and procedural problem-solving categories	
C_1	**translations** between verbal, visual (graphical), numerical, and formal/algebraic mathematical expressions (representations)
[1]C_{2F} [1]C_{2D} [1]C_{2I}	**linking** relationships wrt *functions*: functions $\Leftrightarrow$ inverse functions, equation of a function **linking** relationships wrt *differentiation*: $f \Leftrightarrow f' \Leftrightarrow f'', D_{f'} \subseteq D_f$, $f'(x) = 0 \Rightarrow$ f local extrema, $f'(x) > 0 \Rightarrow f$ increasing, $f'(x) < 0 \Rightarrow f$ decreasing, $f''(x) = 0 \Rightarrow$ possible point of inflection, $f''(x) > 0 \Rightarrow f$ concave up, $f''(x) < 0 \Rightarrow f$ concave down, link position function (displacement) $\Rightarrow$ velocity (speed) $\Rightarrow$ acceleration **linking** relationships wrt *integration*: FTOC (part 1): $g'(x) = f(x)$, fix constant with integration techniques, link acceleration $\Rightarrow$ velocity (speed) $\Rightarrow$ position function (displacement) and total distance
[1]C_{3F} [1]C_{3D} [1]C_{3I}	**interpretation of concepts** wrt *functions*: definitions, functions and relations, inverses, domain and range, restrictions, inequalities, concept of intersection and union **interpretation of concepts** wrt *differentiation*: gradient, continuity, differentiability, point of inflection, concavity **interpretation of concepts** wrt *integration*: FTOC, definite integral = enclosed net area
C_4	**applications** of concepts to mathematical situations (contextual problems)
P_1	symbolic and numerical **calculations**, substitution
[1]P_{2F} [1]P_{2D} [1]P_{2I}	**rules** wrt *functions*, expressions (e.g. division by 0), equations (e.g. $a.b = 0 \Rightarrow a = 0 \text{ or } b = 0$), inequalities (e.g. division by -1), exponential laws (e.g. $a^0 = 1$), log laws, graph of the parabola, factorisation, $\sqrt{a^2} = \|a\|$ *differentiation* **rules** *integration* **techniques**, FTOC (part 2)
P_3	**algorithms** (set of rules)
P_4	**formulae**
P_5	**symbols** (including notation)

[1] F = functions, D = differentiation, I = integration

Examples of interpretation of codes and categories:

C_{3D} refers to a conceptual category that involves the **interpretation** of concepts related to the topic of **differentiation.**

P_{2I} refers to a procedural category that involves **integration techniques**.

3.3.2.3 Content analysis, categories, and labels

The solutions of the 33 calculus items were analysed and labelled according to the number of conceptual and procedural approaches used to solve each task. The following guidelines were followed to analyse, label and classify each item:

1. All possible item solutions were presented.
2. Each solution was analysed and the number of procedural and conceptual categories used to solve the item was counted.
3. If a category was repeated in the solution to a specific item, it was only counted once.
4. If there were more than one method to solve a problem, the analysis was repeated.
5. The number of conceptual and procedural categories used to solve a task was used to label each item.
6. Each item was classified into one of four knowledge classes, namely: C-P-, C+P-, C-P+ and C+P+.

The content analysis, number of categories and item labels informed the relationship between procedural and conceptual knowledge *within* the solutions of calculus items.

3.3.2.4 Knowledge classes and relations between classes

The item analysis of the 33 items informed the inductive classification of the calculus items into four knowledge classes according to the conceptual framework for conceptual and procedural knowledge based on the type and quality of knowledge (Star, 2005). If the analysis and classification of an item resulted in two different knowledge classes, both results were shared in the analysis. The knowledge classes are reflected in Table 3.2.

Table 3.2: Knowledge classes for the research study (adapted from Star, 2005)

Knowledge classes for the research study		
Knowledge classes	**Number of conceptual categories**	**Number of procedural categories**
Conceptual-Procedural-	2 or fewer	2 or fewer
Conceptual+Procedural-	3 or more	2 or fewer
Conceptual-Procedural+	2 or fewer	3 or more
Conceptual+Procedural+	3 or more	3 or more

The relations *between* items in the four knowledge categories were investigated using CFA as part of the quantitative analysis.

3.3.2.5 Relations between knowledge classes

In a discussion on the most efficient strategy to follow to solve a problem, Star (2005, p. 409) states that the “easiest to do is the one with the fewest steps.”. The notion is in line with the researcher’s expectation that the easiest item will be the one with the least number of steps. Furthermore, the researcher expected procedural steps to be easier than conceptual steps. Therefore, the expectation was that a C-P- item would be easier than a C-P+, C+P- or C+P+ item.

3.3.3 Trustworthiness, credibility, transferability and limitations

The validity of the content analysis refers to the trustworthiness, credibility and transferability of findings. Triangulation processes compared the views of different research investigators to ensure validity of the choice of items, analysis of item solutions, item labels and knowledge classes. The analyses of calculus items were triangulated between the researcher, a mathematics colleague who co-teaches the module and the supervisor, a mathematics education expert. The process of triangulation was done in two sessions. The first session of triangulation of the item analysis was between the researcher and a mathematics colleague that co-teach the module. The process involved agreement on item selection and the guidelines followed to analyse, label and classify the items:

- comparison of item solutions and different methods to do the mathematics, therefore agreement on item solutions
- agreement on the analysis of items, therefore agreement on the number and choice of conceptual and procedural categories per item
- agreement on item label and classification per knowledge class.

The second triangulation session between the researcher and supervisor involved:

- agreement on the development and refinement of conceptual and procedural categories
- agreement on the definition of the four knowledge classes based on the work of Star (2005).

The process of intensive iterative reflection led to agreement on the congruency and accuracy of findings for both rounds of triangulation, therefore crystallisation (Maree, 2016; McMillan & Schumacher, 2014) was reached. Trustworthiness and credibility of findings could be established. The findings were in line with what the researcher expected from her own experience as mathematics lecturer, therefore transferability between the research findings and her own experience was established.

Content analysis, in contrast with a grounded theory approach, is useful for testing or confirming a pre-existing theory rather than building a new one (Cohen *et al.*, 2011). The reliability of content analysis, therefore the accuracy to produce consistent results, may be a limitation to the study, since category descriptions and/or interpretations are subjective and dependent on the views and experiences of the coders.

The validity and reliability of the analysis would have been greater if a panel of experts had been involved in the selection, analysis, labelling and classification of items, as was the case with the studies conducted by Bergsten *et al.* (2017); Engelbrecht *et al.* (2009); Engelbrecht *et al.* (2012). This can be seen as a limitation to the study.

3.3.4 Summary

The 33 calculus items were developed and content analysis was performed to label and classify each item into one of four knowledge classes. A data collection instrument that included the items of the qualitative study was developed and implemented in the quantitative section of the research study. The development of the instrument and a description of items that were included are discussed in the quantitative analysis section.

3.4 QUANTITATIVE ANALYSIS

3.4.1 Introduction

The first part of the quantitative data analysis models describes and measures the latent cognitive construct of conceptual and procedural knowledge embedded in the solutions of calculus items. A data collection instrument was developed and the data were collected and analysed using RMT. The findings gave information on the instrument and modelled the relationship between item difficulty and person proficiency. The qualitative analysis informed and explained the interpretation of the Rasch analysis through a process of triangulation. During this process mathematical knowledge informed decisions on whether and how marks and mark categories of calculus items used for the analysis could be collapsed. The initial Rasch analysis was adjusted and a second tailored Rash analysis was performed.

The second part of the quantitative analysis used SEM to do CFA. The analysis investigated possible relations *between* items in the four knowledge classes, namely C-P-, C+P-, C-P+ and C+P+.

3.4.2 Previous studies

It was indicated in the previous chapter that the study implemented by Schneider and Stern (2010) showed significant problems with measurement validity. The study was therefore not suitable to form the methodological basis for this research study. Schneider *et al.* (2011) successfully modelled the longitudinal relations between conceptual knowledge and procedural knowledge by means of SEM and analysed the data using latent variable analyses. The longitudinal study was conducted at two measurement points and included two groups of participants. The participants were divided into two groups: participants with no prior knowledge and participants with some prior knowledge. The purpose of this separation was to determine whether prior knowledge had an influence on procedural flexibility of calculus knowledge or not. The methodology chosen for the study was not directly applied in the current research; however, SEM was used when the CFA was done.

3.4.3 Participants in the study

The students invited to participate in the research study were 205 first-year engineering students who enrolled for an extended degree in engineering in 2017. The participants were all engineering students and the students for the degree programme were selected according to specific minimum entry-level requirements, therefore students' school results had to meet a certain minimum criterion. According to ethical clearance guidelines, each student who voluntarily signed an informed consent form was included in the study. All students were invited to participate in the research study; however only 192 students formed part of the study. Furthermore, 21 students did not complete all three sections of the test. The frequency of the participants who completed sections A, B and C of the test is given in Table 3.3.

Table 3.3: Frequency of participants who completed sections A, B and C

Frequency of participants who completed sections A, B and C of the test		
Test section completed	**Frequency**	**Percentage**
A, B, C	171	89%
A only	5	2.6%
B only	1	0.005%
C only	0	0%
A, B	10	5.2%
A, C	4	0.02%
B, C	1	0.005%
TOTAL	192	100%

The description of the 192 participants who took part in the research study is shown in Table 3.4. The number of male students was considerably higher that the number of female students. The reason is that traditionally the enrolment of male students is higher than that of female students for engineering courses.

Table 3.4: Participants in the study

Participants in the study			
Description	**Category**	**Frequency**	**Percentage**
Gender	Male	146	76%
	Female	46	24%
Race	African	101	52.6%
	Coloured	4	2.1%
	Indian	11	5.7%
	White	69	35.9%
	Other	7	3.6%
Study field	Chemical	18	9.4%
	Civil	54	28.1%
	Computer and electrical	38	19.8%
	Industrial	14	7.3%
	Electronic, metallurgical, mining	16	8.3%
	Mechanical	52	27.1%
	Total	192	

The race category *Other* included Chinese, Asian and Portuguese students. Computer and electrical students, as well as electronic, metallurgical and mining students, were grouped together for analysis purposes. These groups of students were small in number and were grouped together on the engineering faculty timetable.

3.4.4 Development of the instrument

The items developed and classified for the qualitative analysis were included in a data collection instrument. The development of the data collection instrument involved three triangulation sessions. Triangulation between the researcher and a mathematics colleague that co-teach the module, led to agreement on the following guidelines:

- Items and mark allocation had to reflect module assessment.
- Knowledge classes had to be represented approximately equally.
- At least one contextual item had to be included for each topic.

A triangulation session between researcher, supervisor and co-supervisor produced the following recommendations:

- More items had to developed.
- The quality of the items were most important for a well-funtioning instrument.
- The number of items did not need to be equal per knowledge class or per topic.
- The ratio of multiple-choice items to open-ended items did not need not be equal.
- Students would be required to show calculations and reasoning for all items (including multiple-choice items).

During a third round of triangulation between the researcher and the mathematics colleague, the recommendations of the supervisors were incorporated and the data collection instrument (Appendices 2, 3 and 4) was finalised.

The data collection instrument included 33 items that were classified into four knowledge classes: C-P-, C+P, C-P+ and C+P+. The number of questions per knowledge class were chosen to be approximately equal for the development of the instrument. The content analysis revealed that item 30 could be dealt with using at least two problem-solving approaches, therefore the item was for the interim classified as C–P– and C–P+. The decision was made that the qualitative analysis of student responses would inform the category for the item.

The instrument included 17 multiple-choice and 16 open-ended questions. Each multiple-choice item had five possible answers and 14 multiple-choice items were assessed with partial credits. The open-ended items were assessed according to mark allocation used in module assessment. Table 3.5 presents an overview of the instrument.

Table 3.5: Overview of data collection instrument

Overview of data collection instrument			
	Information	**Number of items**	**Marks**
Knowledge categories	Conceptual- Procedural-	9+(1[2])=10	17 (18)
	Conceptual+ Procedural-	8	16
	Conceptual- Procedural+	7+(1[2])=8	16 (15)
	Conceptual+ Procedural+	8	25
Types of questions	Multiple-choice questions	17	30
	Open-ended questions	16	44
Topics covered	Functions	9	21
	Differentiation	13	32
	Integration	11	21
Total		33	74

The number of items and the mark allocation for items were not equal for the different topics of functions, differentiation and integration. This notion was acceptable, since topics develop hierarchically and flexible knowledge on functions has to precede the topics of differentiation and integration. Furthermore, flexible knowledge on functions and differentiation have to precede integration.

3.4.5 Data collection strategies

The data were collected from the engineering students towards the end of the academic year after concepts in the test were covered. Data were collected by the researcher and the mathematics colleage that co-teach the module during a brief

[2] Item 30 belongs to two knowledge classes, depending on the solution strategy followed

period of time in three consecutive data collection sessions. Students received the informed consent form and section A, the multiple-choice questions, in the first session. The open-ended questions (sections B and C) were distributed in sessions two and three. The informed consent form and the three sections of the test are added in the addendum as Appendices 2, 3, 4 and 5. The data collection process can be described as follows:

1. Students were invited to participate in the PhD research project aimed at investigating first-year extended degree engineering students' learning path of calculus by analysing the procedural and conceptual knowledge components of student solutions.
2. It was pointed out that participation in the research project was voluntary and students could withdraw at any point without giving reasons.
3. Confidentiality and anonymity were guaranteed; participants would not be identifiable in research findings.
4. It was explained that the focus of the research study was on how the questions were answered, therefore on the problem-solving method chosen and the calculations used to derive answers.
5. Students were asked to hand in the informed consent form together with section A of the test if they voluntarily agreed to be a participant in the PhD research project.
6. Students were asked to copy answers (not solutions) to the questions on the last, separate page that was provided with each section of the test. This page was to be torn off and submitted as part of the continuous course assessment.
7. If students preferred not to be part of the research study, they still had to do the tasks and submit the last page as course assessment.
8. Students were requested to answer 17 multiple-choice items and 16 open-ended items as part of a test with three sections: A, B and C.
9. Participants were required to show all calculations for both types of questions. Student responses were selectively analysed and used to further inform the classification of the calculus items into knowledge classes.
10. Students were requested to supply additional biographical information when they submitted test C. The aim was to use the additional information in the description of the research participants.

11. Student responses to multiple-choice item answers were captured in an Excel document. The open-ended items were marked according to a marking memorandum and the marks obtained for the questions were captured in the same Excel document.
12. Besides the marking of the open-ended items, the researcher also noted some general or repeated errors. The error analysis was used to inform the statistical analysis.
13. Data cleaning (Combrinck, 2018) was done when 20 of the 192 (10.4%) students' responses to multiple-choice and open-ended items were re-checked and compared to the captured data in the Excel document. No errors were found when quality assurance of the data capturing process was done.
14. The data on the Excel spreadsheet was imported into the Statistical Package for the Social Sciences (SPSS) version 25 (IBM Corporation, 2017). The software program was used to generate the descriptive statistics that are included in chapter 5.
15. The data in the SPSS file were imported into Winsteps 3.93.1 (Linacre, 2016) to do the Rasch analysis.
16. The Rasch analysis process included the investigation of infit and outfit statistics. Infit statistics refer to the overall pattern of persons and items in relation to the model. Outfit statistics reveal possible outliers regarding items and persons (Bond & Fox, 2015; Combrinck, 2018; Linacre, 2002; Linacre, 2018b). The Rasch analysis reported on:
    1. Global fit statistics and summary statistics, in particular reliability and separation indices, etc.
    2. Item and person fit statistics. These statistics indicated whether an item or person contributed to the measurement model.

    The Rasch analysis process was an iterative process, since initial infit and outfit statistics pointed out that marks and categories of some items had to be collapsed to ensure better fit of the data to the model.

3.4.6 Rasch measurement theory

According to Lange, Verhulst, Roberts and Dorsey (2015), RMT provides a bridge between the shortcomings of qualitative and quantitative research approaches in psychometric analysis. RMT assumes the existence of a "latent trait" (Andrich, 1988;

Bansilal, 2015; Bond & Fox, 2015; Dunne, Craig & Long, 2012; Lange *et al.*, 2015); there is a hidden construct, trail, attribute or dimension that is inferable from respondents' answers or observations. The construct of interest for the study is calculus. The construct was evaluated through the operationalisation of relations between conceptual and procedural knowledge in calculus items. Construct validity implies that answers or observations are reflections of a single construct, therefore undimensionality is a requirement of the Rasch model (Bond & Fox, 2015; Linacre, 1996; Wright & Stone, 1979). In addition, RMT evaluates instrument validity, and indicates whether the underlying calculus construct or measured data fit the Rasch model (Bond & Fox, 2015; Dunne *et al.*, 2012; Stols, Long & Dunne, 2015).

According to Dunne *et al.* (2012:7), RMT requires the measurement of a coherent set of appropriate items. It is not necessary for the items to be equal in difficulty. The authors affirm what the Rasch model requires:

> Each item is conceptually relevant to the purpose of the test: it consistently gives partial information about the ability which we seek to measure (justifying a possible inclusion of the item), it enriches the information provided by all the other items collectively (contradicting possible redundancy and exclusion of the item), and it is substantially free of characteristics which might obscure the information obtainable from the instrument (contributing to the precision, rather than to uncertainty, of the instrument, and being free of bias).

The Rasch model measures person ability and item difficulty along a continuum. Person ability, denoted by β_v, and item difficulty, denoted by δ_i, are represented on the same logit scale or linear dimension. The assumption is made that all differences between these numbers, such as $(\beta_v - \delta_i), (\beta_v - \beta_w)$ and $(\delta_i - \delta_j)$, are meaningful. The differences are used to calculate item outcome probabilities and align persons and items on the same scale (Boone & Noltemeyer, 2017; Linacre, 1996; Wright & Stone, 1979). The probability of any person answering any item correctly is calculated using the difference between the log-odds of the locations of ability of the specified person and the log-odds of the difficulty of the particular item $(\beta_v - \delta_i)$ (Bansilal, 2015; Bond & Fox, 2015; Dunne *et al.*, 2012).

Dunne *et al.* (2012) note the following assumptions considering items used for Rasch analysis:

1. The items and item memorandum have to be constructed by experts (Mallinckrodt, Miles & Recabarren, 2016). According to Dunne *et al.* (2012:6), the development of a data collection instrument to be used in an educational context requires "subject expertise, teaching experience and pedagogical insights into the learning journeys particular to the subject."
2. The expectation is that higher item scores will be associated with higher person abilities β_v, and lower item scores will be associated with lower person abilities.
3. Mark categories (1, 2,…*m*) are distinct and uniquely ordered.
4. Item difficulties have a mean of 0, and then the relative difficulties of the items are located accordingly.
5. Person proficiencies are estimated in relation to the corresponding person performance on each of the items and the difficulty of the items.

The simple logistic model was developed by Rasch for the analysis of dichotomously scored test items. Bansilal (2015:2) shares the Rasch simple logistic model for dichotomous items in the following equation:

$$P\{X_{vi} = 1\} = \frac{e^{\beta_v - \delta_i}}{1 + e^{\beta_v - \delta_i}}$$

This function expresses the probability of a person *v* with ability β_v responding successfully on a dichotomous item *i*, with two ordered categories labelled as 0 and 1. *P* is the probability of a correct answer; X_{vi}, the item score variable allocated to a response of person *v*, on dichotomous item *i* , and δ_i is the difficulty of item *i*. If a person *v* is placed at the same location on the scale as an item labelled *i*, then $\beta_v = \delta_i$ or $\beta_v - \delta_i = 0$ and then $P\{X_{vi} = 1\} = 50\%$. Therefore it could be noted that a person will have a 50% chance of achieving a correct response to an item with a difficulty level equal to the person's ability level. If the difficulty of an item is above a person's ability, then the person's chance is less than 50% to have the item correct; if the difficulty of the item is below a person's ability, then the person's chance is more than 50% to have the item correct.

The partial credit model is useful in educational settings, since partial credits could responsibly be awarded for partial success or progression between complete failure and complete success on an item. The model creates the possibility of having a different number of responses for different items on the same test, therefore test items with a mixture of scales (Bond & Fox, 2015). Every item has its own scale - some items are dichotomous and other items have more than two scores. Bansilal (2015:2) explains that the Rasch partial credit model was developed by Andrich (1978); it is given in the next equation:

$$P\{X_{vi}=1\}=\frac{e^{(X\beta_v-\delta_i)-\sum_{k=1}^{X}\tau_{ki}}}{\sum_{X=0}^{mi} e^{(X\beta_v-\delta_i)-\sum_{k=1}^{X}\tau_{ki}}}$$

Thresholds, τ_k with $X \in \{1, 2, ..., m\}$, refer to the transition between two adjacent categories that are naturally ordered. This implies that the threshold of a higher achievement category is more difficult to achieve than the threshold of a lower achievement category. In the above formula the threshold parameters τ_{ki} are not assumed identical across all items, but thresholds τ_{ki} are assumed to be different for each item *i* with categories $X \in \{1, 2, ..., m\}$.

Rasch analysis requires checking whether the data fit the model and whether valid and reliable inferences can be derived from the instrument, or whether measurement should be improved (Andrich, 1988; Bansilal, 2015; Bond & Fox, 2015; Dunne *et al.*, 2012; Linacre, 2017; Stols *et al.*, 2015). The efficacy of the instrument is determined by the theoretical work that has informed the development of the instrument. The qualitative analysis informs the data analysis and the interpretations and extrapolations to be made from the analysis. RMT requires interactive engagement between theoretical and empirical components of the research study. This process will enhance the validity and reliability of the instrument and results.

3.4.7 Rasch measures

Bond and Fox (2015) suggest the examination of Rasch measures to determine to what extent the data fit the model, the general functioning of the instrument and the individual fit of items and persons to the Rasch model and identification of anomalies. The Rasch measures are discussed in the subsections below.

3.4.7.1 Differential item functioning

Differential item functioning (DIF) may indicate potential item bias, since it designates the loss of item invariance across subsamples of respondents, e.g. according to gender, race or study field. To interpret DIF, the researcher first investigated the significance of the *p*-value per item, for each subsample. A significant difference between the model and the data is implied for items with a *p* value of less than 0.05. Thereafter, the researcher investigated the DIF size of the selected items. Only if an item is both significantly different in the subsamples and the DIF size/contrast is greater than 0.64, is DIF present (Boone, Staver & Yale, 2014).

3.4.7.2 Global fit statistics

Global fit statistics were investigated to check whether the data fit the model. The study requires a non-significant chi-squared probability result of $p > 0.05$ to deduce that the data fit the model; $p < 0.05$ means the data do not fit the model (Andrich & Marais, 2019; Bond & Fox, 2015; Stenner, Fisher Jr, Stone & Burdick, 2013).

3.4.7.3 Summary statistics

The person separation index is an estimate of the spread or separation of persons on the measured calculus construct. The person reliability index is an indication of the replicability of person ordering if the persons were given a similar set of items to measure the construct (Bond & Fox, 2015). A high person separation index of > 2.00 and reliability index of > .80 indicate reliable measurement of a wide range of person abilities (Bond & Fox, 2015; Linacre, 2017). If the person separation index is low it indicates poor separation of proficiencies, since there is a limited spread of person locations. The person separation index may be a result of a homogeneous group or response dependency between questions.

Item separation and reliability refer to the ability of the instrument to define a distinct hierarchy of items along the measured construct (Bond & Fox, 2015). The item separation index is an estimate of the spread or separation of items on the measured calculus construct. If the item separation index is low it indicates that items are relatively similar in difficulty. The item reliability index indicates the duplication of item positions if the same items were given to a comparative sample. Linacre (2017) indicates that data can separate persons into three ability categories if the item separation index is > 3 and the item reliability is > 0.9.

3.4.7.4 Dimensionality, item polarity and principal component analysis

The dimensionality measure evaluates whether all aspects of the test are pulling in the same direction. It is expected that the test instrument and items reflect a single underlying dimension or construct, therefore that unidimensionality is established. Dimensionality is assessed in terms of degrees, but reality is complex and constructs are multi-faceted (Linacre, 2008). The question is not whether the instrument measures a unidimensional construct, but rather if the unidimensionality is sufficient. Unidimensionality is supported when the overall fit of person and item mean squares (MNSQ) is close to 1.00 and mean standardised fit statistics (ZSTD) are close to 0.00 (Peoples, O'Dwyer, Wang, Brown & Rosca, 2014).

RMT is based on the principles of measurement (Combrinck, 2018). A basic principle of measurement involves directionality and polarity of items. It is assumed that the underlying construct or latent trait is measured along a linear continuum (Bond & Fox, 2015; Boone *et al.*, 2014). Item polarity examines whether all items are aligned in the same direction and positively correlate to the underlying construct. The point biserial correlations (PTMA) indicate whether the items are positively or negatively correlated to the underlying construct (Linacre, 2017).

Principal component analysis (PCA) identifies possible substructures or secondary dimensions in the data by performing a principal components decomposition of the observation residuals. If there are large substructures, then it may be suggested that the data be divided into two measurement instruments or subsets. According to Linacre (2017), an Eigenvalue for the first contrast of less than 3 and unexplained variance of less than 5% are the minimum criteria to be considered a dimension.

Combrinck (2018) suggests that Eigenvalues above 2 should be investigated further, since two or more items could indicate another dimension or construct. However, the identified items may correlate owing to having the same number of marks or categories, the same topic or same technique, or the correlation could even be a random event. Instruments with more items are more likely to form clusters.

3.4.7.5 Infit and outfit statistics

Infit statistics are sensitive to the overall pattern of item responses and indicate inlier-sensitive items or information-weighted fit. Infit statistics refer to whether persons and items align with the expectation that persons with higher ability will answer more difficult items correctly, and persons with lower ability will not answer difficult items correctly (Combrinck, 2018). Infit statistics indicate where items fit the Guttman pattern too well or where there are unexpected responses to items that lie near the person's ability level (Linacre, 2002). An example is incorrectly answering a question that is slightly below the person's ability, therefore an item that should be easier for the person to answer correctly.

Outfit item statistics distinguish between items that fit the model and anomalies, therefore give an indication of outliers (Boone & Rogan, 2005; Linacre, 2002). Outfit statistics suggest extremely high or low score items (Combrinck, 2018). These items are too easy or too difficult and should be reconsidered, since they do not inform the analysis or provide significant information about the underlying construct. These items do not contribute to discrimination between items and might distort measures. Overfit items are too good to be true items and are probably over-discriminating items as a result of item dependence, e.g. where a correct answer to a prior item increases the probability of a correct answer to a current item. Underfit items are described as noisy and unexpected and may be a result of guessing, careless mistakes, special knowledge or a poor item. Item category probability curves could be investigated to determine the fit of items. If item categories are disordered, an explanation for the contrast should be investigated against the theoretical component that supports the instrument.

Person fit statistics inform anomalies or misfit persons.The outliers could include a person who correctly answers an item that should have been too difficult for him/her (guessing), or an item that should have been easy, but not answered correctly by persons of higher ability (careless mistakes, problems with item phrasing). Person anomalies do not contribute to the understanding of the relative difficulty of items and could be eliminated from the data set by treating them as missing responses.

According to Linacre (2002), the ideal range for infit MNSQ and outfit MNSQ values is between 0.50 and 1.50. Values outside the range could be considered suspicious. It is common practice to keep all items and persons, unless there is a very good reason to remove them. Values of MNSQ < 0.50 are useful, but duplicative and may produce misleadingly high reliability and separation coefficients. Items and persons with values between 1.50 to 2.00 are useful, but noisy, and unproductive for construction of measurement. Items with values > 2.00 have unexpected responses that could be overpowering and distort the measurement system. MNSQ scores should be considered first, thereafter the standardised ZSTD infit and outfit scores. The ZSTD range from -2.00 to 2.00 is considered to be reasonably predictable and acceptable. ZSTD scores above 2.00 are unpredictable and ones of less than -2.00 are too predictable.

3.4.7.6 Distractors

This section refers to the functioning of the item distractors chosen for multiple-choice questions. The researcher did not do a full analysis on the choice of distractors for the data collection test, since the development and functioning of the instrument was not the main focus of the study.

In case of problematic measures, items, persons and/or categories can be identified and resolved in concurrence with the qualitative analysis. The revised, tailored Rasch analysis could produce a better fit of the data to the model and valid and reliable results.

3.4.8 Rasch analysis and collapse of marks and mark categories

Linacre (2017), Linacre (2018c) and Iramaneerat, Smith Jr and Smith (2008) suggest the following item requirements for RMT:

1. More than 10 observations per category (UNWTD measure). If there are fewer than 10 observations per category the findings may be unstable, i.e. non-replicable.
2. A smooth distribution of category frequencies as observed with the item category probability curves. The frequency distribution should not be jagged. Jaggedness can indicate categories that are very narrow, or perhaps indicate that category transitions have been defined as categories and should be collapsed. This will depend on the distribution of the sample and the theoretical underpinning of the qualitative analysis.
3. Clearly advancing average measures and average measures that are not disordered. As indicated in the previous section, the expectation is that higher item scores will be associated with higher person abilities, and the converse must also hold.
4. Average measures near their expected values (AVGE MEAS).
5. Observations that fit their categories. The aim is that the outfit mean squares (OUTFIT MNSQ) should be close to 1.00. Values much above 1.00 are more problematic than values below 1.00.

The collapse of categories is explained per item when the content analysis is examined in chapter 4; however, two examples are given here to illustrate the process of collapse of marks and mark categories.

Example 1:

The category probability curves for item 7, a multiple-choice item, is presented before and after collapse in Figure 3.1. It is clear that category 1 could be described as a category transition and should be collapsed. Category 1 was collapsed to become part of category 0, and category 2 became category 1. The rationale followed was that credit could not be awarded for partial success on this item.

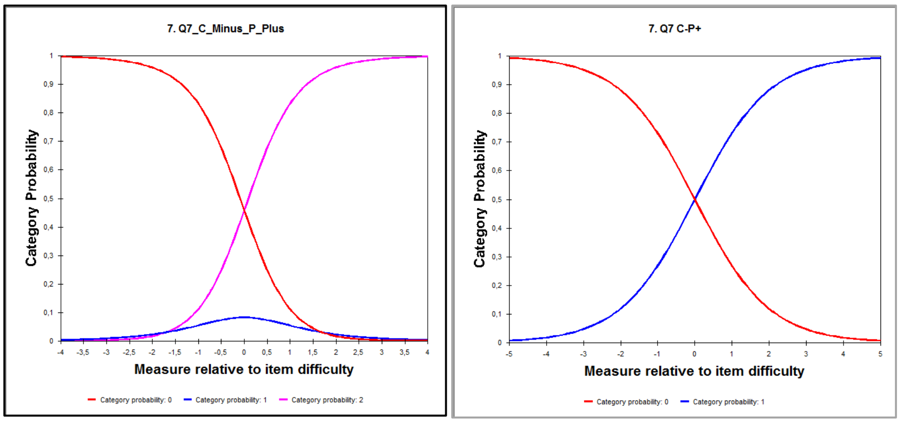


Figure 3.1: Item 7 probability curves (before and after collapse)

Example 2:

Figure 3.2 is the category probability curves for item 22, an open-ended item, before and after collapse. The item category probability curves for categories 1 and 2 are disordered and indicate that categories 1 or 2 should be collapsed into one category.

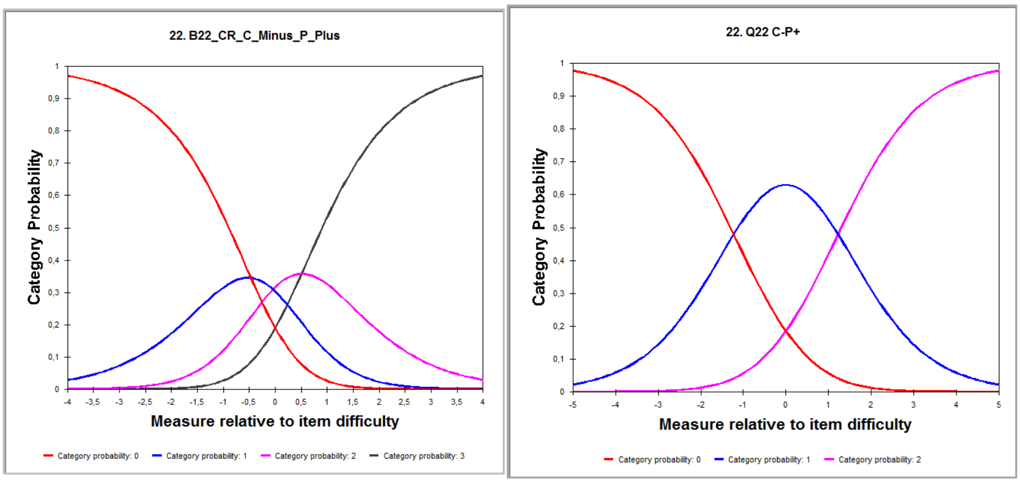


Figure 3.2: Item 22 probability curves (before and after collapse)

The rationale followed for the collapse was that marks for category 1 and category 2 were initially awarded for evidence of the same mathematical skill, namely applying differentiation rules. The result of the collapse in categories can be seen in Figure 3.3.

The suggested collapse of categories for item 22 is summarised below:

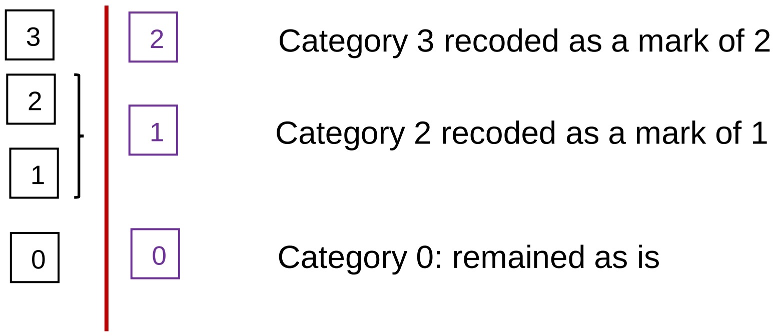


Figure 3.3: Collapse of categories for item 22

The collapse of marks and mark categories resulted in adaptation of the data collection instrument. After the category collapse, the Rasch analysis was repeated. The collapsed items mostly revealed ordered categories and advancing average measures. Smooth distribution of category frequencies was observed with the category probability curves. The adapted instrument and results of the second, tailored Rasch analysis are discussed in chapter 5.

3.4.9 Confirmatory factor analysis

Factor analysis groups together variables that load onto the same hypothetical latent trait (Cohen *et al.*, 2011) and takes on variance that is common to the identified latent factors, but excludes random noise in the data (Schneider *et al.*, 2011). Latent variable analysis models latent factors separately if these are significantly different from each other; therefore, the process detects structures and commonalities in the relationships between variables, and indicates which items are predicted by an underlying trait (regression weight). Factor analysis includes exploratory factor analysis and CFA. Exploratory factor analysis seeks to detect underlying patterns, since it measures whether and how variables or items are connected to different latent factors or traits (Blunch, 2013). CFA investigates whether and how specified variables or items are connected to distinct and specified latent factors, therefore it examines a confirmatory hypothesis. According to Blunch (2013), CFA assumes that items (X_1, X_2, etc in Figure 3.4) are considered functions of the latent factors or traits (F_1, F_2 and F_3 in Figure 3.4). In the CFA model the arrows point from the latent factor to the items, and the assumption is that the grouped items are predicted by the latent factor.

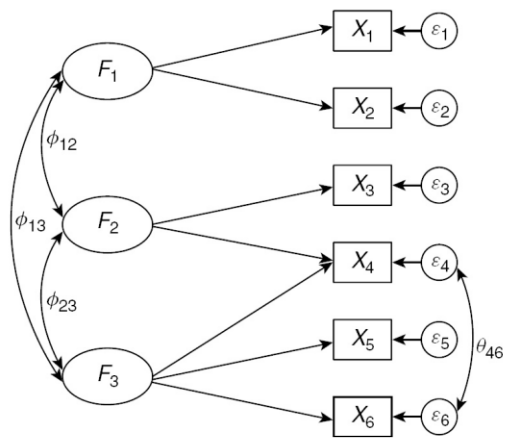


Figure 3.4: Confirmatory factor analysis (Blunch, 2013:4)

Bond and Fox (2015) suggest that Rasch analysis should first be performed to guide the construction and quality control of measurement scales for each variable or item. After the Rasch analysis, the rescaled items (due to collapsed categories) were imported into SEM software as raw, unweighted data to evaluate relationships between the measures for each item. SEM was implemented to simulate relationships between the classified calculus items (variables) and the four knowledge classes C-P-, C+P-, C-P+ and C+P+ (latent factors).

Maree (2016) suggests that the sample size for CFA should be between five and 10 times the number of item categories in the instrument. The revised instrument (Table 5.1) includes 45 marks and 81 item categories (options), therefore the number of participants of 192 is below the minimum suggested specification of 405 respondents for the study. The number of items per factor is suggested to be between eight and 10. The number of items per knowledge class was eight and nine respectively.

CFA was applied using the IBM Amos (Version 25.0) computer program (Arbuckle, 2017a) to test the existence of the hypothesised subscales. The four knowledge classes that were theorised in the development of the test were specified as four distinct latent factors in the model. The CFA evaluated according to guidelines by (Arbuckle, 2017b) and (Schumacker, 2010):

- The fit between the CFA model and the data requires a non-significant chi-squared probability result of $p > 0.05$ to conclude that the data fit the model.

The noncentrality parameter and the root mean square error of approximation were investigated to check the model fit. A noncentrality parameter of 0 indicates a perfect fit and an root mean square error of approximation less than 0.05 indicates a good fit (Schumacker, 2010). The expectation was that the model would fit the data.

- Expected regression weights between items connected to each latent factor were investigated. The expectation was that factor loadings (regression weights) would be above 0.700.
- Expected correlations between the four identified latent factors were considered. Strong latent factor correlations between 0.8 and less than 1 were expected.

3.4.10 Validity, reliability and limitations to quantitative analysis

Rasch analysis has an advantage over classical test theory, since it involves item response theory. Internal consistencies across samples are more stable and the analysis yields significantly fewer measurement errors than classical test theory analysis.

Quantitative analysis required the establishment of the validity of the quantitative content analysis, as well as construct and instrument validity and reliability. The validity of the content analysis of calculus items was discussed in the first part of the study. The two triangulation sessions between the researcher, a mathematics colleague and the supervisor included agreement on chosen items, analysis and item labels, description of knowledge categories and the definitions of the four knowledge classes.

Construct and instrument validity and reliability were assessed by applying the Rasch model when global fit statistics, summary statistics, item and person fit statistics, category options, category probability curves, etc. were investigated. The initial Rasch analysis revealed disordered categories for a number of items. The investigation of the empirical results and the mathematical construct led to the collapsing of marks and mark categories for some of the items during a third triangulation session between the researcher, supervisor and psychometrician.

The 192 students who participated in the study could be seen as a limitation, since it is a relatively small number of respondents. According to Stols *et al.* (2015), the rule of thumb is that the number of participants for the Rasch analysis should be equal to 10 times the number of marks allocated in the data collection instrument. In the case of this study the suggested number of participants would be 450. As mentioned in the previous section, the number of respondents is also a limitation on the CFA. A further limitation to the study is that engineering students are a homogeneous group with similar aptitude. Students who enrolled for this academic programme had to have an admission point score of more than 25.

3.5 ETHICAL ISSUES

The dean of the Faculty of Engineering, Built Environment and Information Technology at the University of Pretoria gave permission for the research to be conducted in his faculty. The dean’s approval of the PhD research project was attached as part of the ethical clearance application submitted to the research ethics committee of the Faculty of Education. Ethical clearance was obtained from the Faculty of Education on 18 September 2017.

As stated in the description on data collection strategies, the objective of the quantitative section of the research study was explained to all students before data collection took place. Students were informed that participation was voluntary and could be terminated at any time. The informed consent form (Appendix 1) was distributed, explained and signed by students before data collection commenced. A parent/guardian informed consent form was not applicable, since all participants were independent and not minors. This was confirmed by including a question on the age of the participant in the informed consent form.

3.6 SUMMARY

The research study followed a postpositivist paradigm and used an exploratory sequential mixed methods design in order to describe the nature of conceptual and procedural knowledge in solutions of calculus items. The combination of qualitative and sequential quantitative analysis provided deeper and more complete understanding and comprehensive analysis of the research problem. The emphasis on qualitative and quantitative analysis was equal.

The qualitative analysis answered the first research question regarding how calculus items can be analysed and classified with respect to conceptual and procedural knowledge. The content analysis investigated relations between procedural and conceptual knowledge *within* solutions of items. Calculus items were developed and the solutions to the 33 items were analysed and labelled according to the number of conceptual and procedural approaches used to solve each task. The content analysis informed the classification of the items into one of four knowledge classes.

The findings of the qualitative analysis were used to develop an instrument that was implemented in the sequential cross-sectional quantitative study. Rasch analysis was implemented to answer the second research question on how conceptual and procedural constructs are related in calculus. Rasch analysis measured the calculus construct for all the items. The analysis focussed on relations between item difficulty and person proficiency. The first round of Rasch analysis revealed disordered categories for a number of items. The empirical results were investigated together with the qualitative content analysis. Categories were collapsed for some items and the analysis was repeated. The second, tailored analysis provided a better fit of the data to the model and the underlying calculus construct. The CFA in the third layer of analysis investigated relations *between* classified items in the four knowledge classes.

Chapter 4 attempts to answer the first research question and focusses on the findings for the qualitative content analysis; thereafter chapter 5 focusses on the second research question and the results of the quantitative Rasch analysis and CFA.

4. CHAPTER 4: QUALITATIVE RESULTS

4.1 INTRODUCTION

In chapter 3, the research methodology that informed the study was discussed. Chapter 4 attempts to answer the first research question and shares the results for the qualitative content analysis:

> How can calculus items be analysed and classified with respect to conceptual and procedural knowledge?

Part one of the qualitative analysis provides a tool to analyse the procedural and conceptual knowledge *within* a selection of calculus item solutions. Each item solution was analysed and labelled according to the type and number of approach categories used to solve it (Table 3.1).

In the second part of the analysis, each labelled item was classified according to one of four knowledge classes: C-P-, C-P+, C+P- and C+P+ (Table 3.2). The classes, based on the notion of type and quality of knowledge described by Star (2005), were defined to describe and classify each item according to the emphasis in mathematical approach required to solve it.

4.2 ITEM ANALYSIS: ANALYSIS, LABEL AND KNOWLEDGE CLASS

4.2.1 Introduction

The literature study emphasised the notion that being conceptual or procedural is not necessarily a property of the task, but rather of the ***solution to the task*** (Engelbrecht *et al.*, 2009). The method used by students to complete the tasks indicates which type of knowledge or approach they used (Crooks & Alibali, 2014).

The researcher's expectation is that the easiest item will be the one with the fewest number of steps (Star, 2005); therefore, the expectation is that a C-P- item will be easier than a C-P+, C+P- or C+P+ item. The motivation for this expectation is that procedural and conceptual items with a "+" label indicates three or more non-repeated steps to solve the item; conversely procedural and conceptual items with a "–" label indicates two or fewer non-repeated steps to do the mathematics (Table 3.2).

The approach used by a student to solve a mathematical task is not absolute (Engelbrecht *et al.*, 2009), since a task is conceptual or procedural following a chosen solution approach given a student's prior knowledge and experience. The same task might be a relatively unfamiliar problem for a person with little or no prior knowledge, since he/she has to access conceptual knowledge rather than use a known procedure to derive the answer. Alternatively, the task might be a familiar problem that measures procedural knowledge for a person with more prior knowledge (Rittle-Johnson *et al.*, 2015; Schneider *et al.*, 2011). The nature of a conceptual problem could become procedural if a student is introduced to the problem repeatedly and he/she becomes familiar with the type of problem. The ambivalent nature of solution approaches could become evident in the application or contextual problems, since the application might be familiar to some students and unfamiliar to others.

Educators' perspectives on conceptual and procedural knowledge are not unique and each may hold different views on the nature of mathematical tasks depending on his/her understanding and experience. According to Star and Stylianides (2013:178):

> If mathematics educators were asked whether this item assessed conceptual knowledge, they likely would answer: "It depends." …. an assessment question may indicate the presence of conceptual knowledge in one student but fail to do so in another student, even if both students answered the question correctly.

An educator might have the opinion that a particular task assesses conceptual knowledge, procedural knowledge, or both.

The qualitative analysis for this study is a product of the triangulated view of three mathematical experts, since the researcher triangulated the analysis with two mathematics education specialists. It can be concluded that the labelling and classification of items analysed for the quantitative section of this study are not unique.

Star (2005); Star (2007) describes procedural knowledge on a spectrum from rote learning to deep procedural knowledge. Deep procedural knowledge involves flexibility and critical judgement related to procedures, not necessarily embedded in conceptual understanding. Star (2007:133) claims that deep procedural knowledge is evident when “a student can provide a cogent explanation of how steps are interrelated to achieve a goal.” This type of knowledge relates to procedural fluency (Kilpatrick *et al.*, 2001).

The analysed calculus items were included in a data collection instrument administrated for the quantitative section of the study. The test included 17 multiple-choice questions with well-developed distractors and 16 open-ended questions, therefore a total of 33 items. The distractors were developed based on the researcher’s experience as a mathematics education specialist. Rittle-Johnson and Schneider (2016) suggested that items that measure conceptual and/or procedural knowledge in tasks should be completed as pen-and paper assessments. The multiple-choice items were not standard multiple-choice items, since students were required to show all calculations on the answer sheet provided.

The test results were analysed with Rasch analysis to describe and measure the underlying calculus construct. The initial Rasch analysis indicated that some item category probability curves appeared jagged and categories were disordered. These items were examined in concurrence with the qualitative content analysis (Mallinckrodt *et al.*, 2016). The analysis resulted in the collapsing of marks and mark categories for 27 of the 33 items, since it made sense in terms of the mathematics. The revised, tailored Rasch analysis produced a better fit of the data to the model and generated valid, reliable results. The qualitative content analysis will refer to both the initial and collapsed marks.

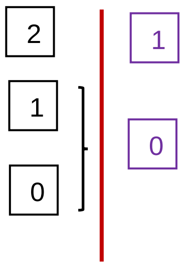
The analysed multiple-choice and open-ended items are discussed in the next two sections. The analysis of each item describes:

- Complete solution(s) to answer/solve the item
- For some items, more than one solution method or approach
- Mark allocation for solutions before the collapse of marks
- Visual representation of collapse of marks for use in the quantitative analysis
- Comments concerning student responses and error analysis
- The results, including:
    - Item label(s). More than one solution method or approach resulted in more than one item label.

- Analysis of the solution(s) concerning conceptual and procedural approaches followed to do the item. Repeated categories, shaded in grey, were counted once when the analysis was done.
- Classification according to knowledge class. More than item label could result in more than one knowledge class.

4.2.2 Multiple-choice items (1 – 17)

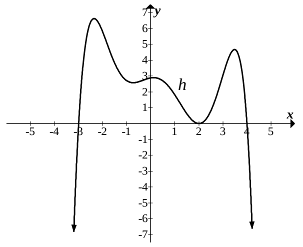
Questions 3, 13, 14 and 15 had a mark allocation of one mark. The rest of the multiple-choice items had an initial mark allocation of two marks, and the award of partial marks was possible for items where evidence of some degree of understanding was visible. The collapse of item marks from two to one was implemented for items 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16 and 17. The collapse of marks could be motivated in terms of the disordered categories and the mathematics, since students were unable to answer the items successfully. Visual representation of the collapse of marks for multiple-choice items:



Comments concerning distractors used for the items are included for some items. If fewer than 5% of students (10) chose an option, the choice was considered a weak distractor that could enlarge the guessing parameter in an analysis. If more than 30% of students (58) chose an option, it was regarded as a strong distractor (Linacre, 2017). The analysis noted the weak distractors, but the distractors were not replaced or altered for the revised version of the test. It could be suggested that the identified weak distractors could be changed if the test is repeated for another research project. Option E, *None of these*, was the same option for all multiple-choice items.

4.2.2.1 Item 1

Consider $h(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ shown in the figure. Which statement is true for h? Explain.



Answer and initial mark allocation: 2 marks

Approach 1:

n is even C1 C2F

$a_n < 0$ C1 C2F

Approach 2:

Similar thinking to shape of the parabola C1 P2F

Mark allocation after collapse of marks: 1 mark

A. n is odd, $a_n > 0$

B. n is odd, $a_n < 0$

C. n is even, $a_n < 0$

D. n is even, $a_n > 0$

E. None of these

Comments concerning student responses and error analysis:

- 26.1% of students answered this item successfully.
- Option A was a strong distractor, since 31.6% of the students chose A. These students chose n to be odd and $a_n > 0$.

Results:

Approach 1:

Label: C=2P=0

C_1: translations (graph)

C_{2F}: link equation of a function

Category: C-P-

Approach 2:

Label: C=1P=1

C_1: translations (graph)

P_{2F}: rule graph of the parabola (a < 0)

Category: C-P-

4.2.2.2 Item 2

Solve the inequality: $4x - 1 < 3x < 5x - 4$. Write your final answer in interval notation.

Answer and initial mark allocation: 2 marks

$$4x - 1 < 3x < 5x - 4$$

$$4x - 1 < 3x \text{ and } 3x < 5x - 4 \qquad C_{3F}$$

$$x < 1 \text{ and } -2x < -4 \qquad P_1 \quad P_1$$

$$x < 1 \text{ and } x > 2 \qquad P_{2F}$$

$$x \in \varnothing \qquad C_1 \quad P_5$$

Mark allocation after collapse of marks: 1 mark

A. $x \in (-\infty, 1]$

B. $x \in (-\infty, 1] \cup (-\infty, 2)$

C. $x \in \emptyset$

D. $x \in (-\infty, 1] \cap (2, \infty)$

E. None of these

Comments concerning student responses and error analysis:

- 29.5% of students answered this item successfully.
- Some students were not familiar with the symbol $\varnothing$ (denotes empty set) - P_5 notation.
- Option A was a weak distractor, since no students chose this option.
- Option B was a weak distractor, since only 1.6% of students chose this option.
- Option D was a weak distractor, since the intervals should have been defined on open intervals to test the ‘and’ and ‘or’ concept in mathematics.

Results:

Label: C=2P=3

C_{3F}: interpretation intersection

P_1: symbolic and numerical calculations

P_{2F}: inequality rules (linear)

C_1: translations (graph)

P_5: notation and symbols

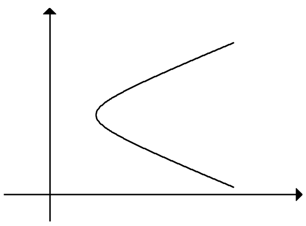
Category: C-P+

4.2.2.3 Item 3

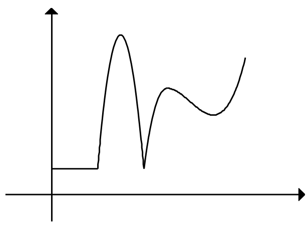
Which of the following graphs are graphs of a function $f(x)$? Explain.

Answer and mark allocation: 1 mark:

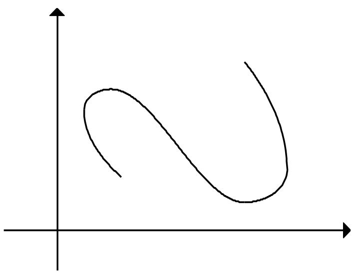
A



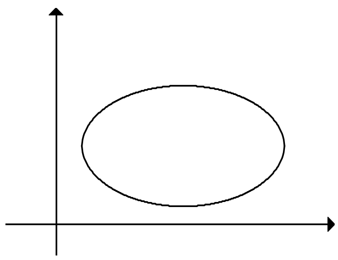
B.



C.



D.



E. None of these

Approach 1:

Vertical line test P_{2F}

Approach 2:

Function – each x has only one y C_{3F}

Comments concerning student responses and error analysis:

- This item was included in the draft data collection instrument (32 questions) developed by Engelbrecht (2008).

- Although two different approaches were suggested to answer the item, the knowledge class remained the same.
- 79.9% of students answered this item successfully.
- Option C was a weak distractor, since only 1% of students chose this option.
- Option D was a weak distractor, since only 2.1% of students chose this option.

Results:

Approach 1:

Label: C=0P=1

P_{2F}: function rule

Category: C-P-

Approach 2:

Label: C=1P=0

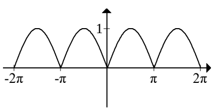
C_{3F}: interpretation functions

Category: C-P-

4.2.2.4 Item 4

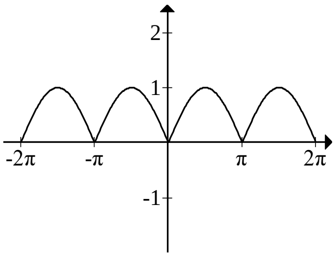
Let $f(x) = \sqrt{x}$ and $g(x) = 1 - \cos^2 x$. The graph of $(f \circ g)(x)$ is represented by the following graph:

Answer and initial mark allocation: 2 marks

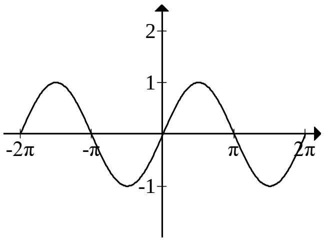
		Alternative analysis:
$(f \circ g)(x) = \sqrt{1-\cos^2 x}$	C_{3F}	C_{3F}
$= \sqrt{\sin^2 x}$	P_4	P_4
$= \lvert\sin x\rvert$	P_{2F}	C_{2F}
$= \begin{cases} \sin x & if \ \sin x \geq 0 \\ -\sin x & if \ \sin x < 0 \end{cases}$	C_{3F}	
Graph 	C_1	C_1

Mark allocation after collapse of marks: 1 mark

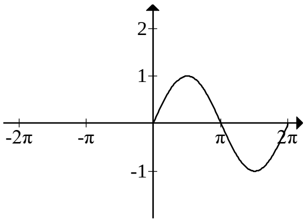
A.



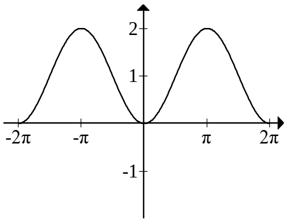
B.



C.



D.



E. None of these

Comments concerning student responses and error analysis:

- 40% of students answered this item successfully.
- 23.4% of the students chose B, therefore did not show evidence of conceptual understanding related to P_{2F}: $\sqrt{a^2} = |a|$ and C_{3F}: absolute value function, alternatively C_{2F}: square root function non-negative.

Results:

Analysis 1

Label: C=3P=2

C_{3F}: interpretation composite function

P_4: trig formula

P_{2F}: $\sqrt{a^2} = |a|$

C_{3F}: interpretation absolute value function

C_1: translations (graph)

Category: C+P-

Alternative analysis for the same method:

Analysis 2

Label: C=3P=1

C_{3F}: interpretation composite function

P_{4}: trig formula

C_{2F}: link square root function non-negative

C_{1}: translations (graph)

Category: C+P-

The analysis and label are different for the same method, but the knowledge class remains the same.

4.2.2.5 Item 5

The domain of $f(x) = \frac{1}{e^{x+1}-1}$ is:

Answer and initial mark allocation: 2 marks

$$
\begin{array}{ll}
e^{x+1} - 1 \neq 0 & \mathbf{P_{2F}} \\
\Rightarrow e^{x+1} \neq 1 \quad \checkmark & \mathbf{P_{1}} \\
\Rightarrow e^{x+1} \neq 1^{0} & \mathbf{P_{2F}} \\
\Rightarrow x+1 \neq 0 & \\
 & \mathbf{P_{1}} \\
\Rightarrow x \neq -1 & \\
D\left(\frac{1}{e^{x+1}-1}\right) = \mathbb{R} \setminus \{-1\} \quad \checkmark & \mathbf{C_{3F}}
\end{array}
$$

Mark allocation after collapse of marks: 1 mark

A. $\mathbb{R}$

B. $\mathbb{R}\setminus\{1\}$

C. $\mathbb{R}\setminus\{0\}$

✓ D. $\mathbb{R}\setminus\{-1\}$

E. None of these

Comments concerning student responses and error analysis:

- 85.8% of students answered this item successfully.
- Option A was a weak distractor, since only 2.1% of students chose this option.
- Option B was a weak distractor, since only 2.1% of students chose this option.
- Option C was a weak distractor, since only 3.1% of students chose this option.

Results:

Label: C=1P=3

P_{2F}: division by 0

P_1: symbolic and numerical calculations

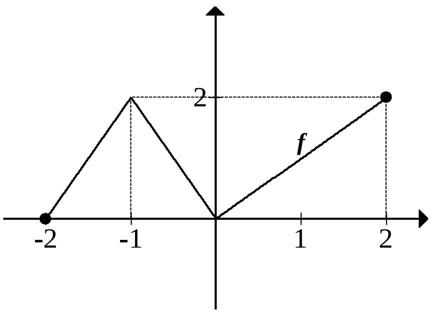
P_{2F}: exponential laws (e.g. $a^0 = 1$)

C_{3F}: interpretation domain

Category: C-P+

4.2.2.6 Item 6

The graph of a function f is given. Draw the graph of the derivative function f'.



Answer and initial mark allocation: 2 marks

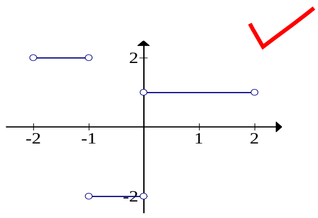
$(-2,-1)$ gradient $= \frac{2}{1} = 2$ — P_4

$(-1,0)$ gradient $= -\frac{2}{1} = -2$ — P_4

$(0,2)$ gradient $= \frac{2}{2} = 1$ — P_4

f is not differentiable at $x = -2, -1, 0$ and 2 — C_{3D}

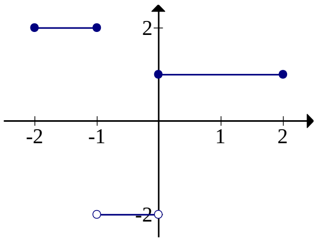
Graph



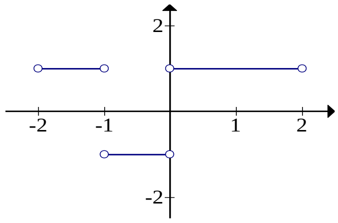
C_1

Mark allocation after collapse of marks: 1 mark

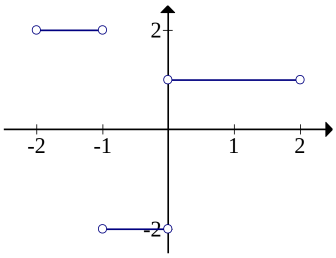
A.



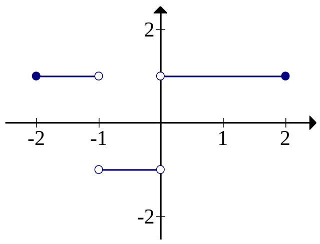
B.



C.



D.



E. None of these

Comments concerning student responses and error analysis:

- This item was included in the draft data collection instrument (32 questions) developed by Engelbrecht (2008).
- 12.6% of students answered this item successfully.
- Option D was a strong distractor, since 45.3% of the students chose D. These students calculated the incorrect gradients and did not realise that *f* is not differentiable at $x = -2$ or $x = 2$ ($f'(-2)$, $f'(2)$ not defined).
- Option A was a strong distractor, since 35.4% of the students chose A. These students calculated the correct gradients, but did not realise that *f* is not differentiable at $x = -2$ or $x = 2$ ($f'(-2)$, $f'(2)$ not defined).
- Option B was a weak distractor, since only 1% of students chose this option.

Results:

Label: C=2P=1

P_4: formula for gradient

C_{3D}: interpretation differentiability

C_1: translations (graph)

Category: C-P-

4.2.2.7 Item 7

Given $f(x) = x^3$ what is the equation of the tangent line at $x = 1$?

Answer and initial mark allocation: 2 marks

$f'(x) = 3x^2$	P_{2D}
$\Rightarrow m = 3$	C_{3D}
$\Rightarrow y = 3x + c$ ✓	C_{2F}
$f(1) = 1$	
$\Rightarrow 1 = 3(1) + c$	P_1
$\Rightarrow c = -2$	P_1
$\Rightarrow y = 3x - 2$ ✓	C_{2F}

Mark allocation after collapse of marks: 1 mark

A. $y = x + 2$

✓ B. $y = 3x - 2$

C. $y = x$

D. $y = 3x + 2$

E. None of the above

Comments concerning student responses and error analysis:

- 66.1% of students answered this item successfully.
- Option A was a weak distractor, since only 3.7% of students chose this option.

Results:

Label: C=2P=3

P_{2D}: differentiation rules

C_{3D}: interpretation gradient

C_{2F}: link equation of a function

P_1: substitution

P_1: symbolic and numerical calculations

Category: C-P+

4.2.2.8 Item 8

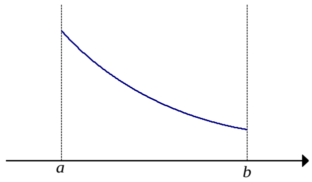
Which of the following functions f in the sketch has the properties that $f'(x) < 0$ and $f''(x) > 0$ for all $x \in [a, b]$? Explain.

Answer and initial mark allocation: 2 marks

$f'(x) < 0 \Rightarrow f$ descending for all $x \in [a,b]$ C_{2D}

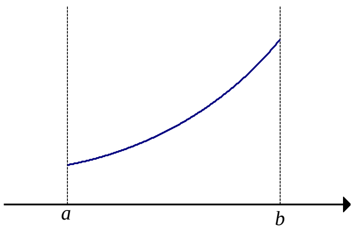
$f''(x) > 0 \Rightarrow f$ concave up for all $x \in [a,b]$ C_{2D}

Graph C_1

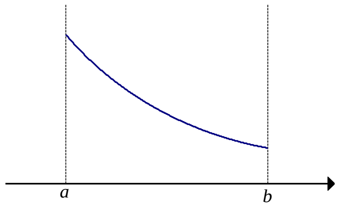


Mark allocation after collapse of marks: 1 mark

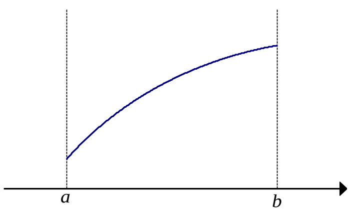
A.



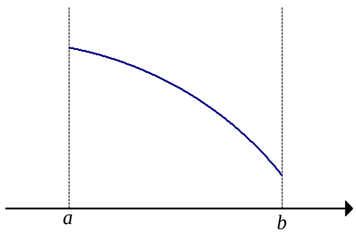
B.



C.



D.



E. None of these

Comments concerning student responses and error analysis:

- This item was included in the draft (32 questions) and final (8 questions) data collection instrument developed by Engelbrecht (2008) and Engelbrecht (2009).
- 66.3% of students answered this item successfully.

Results:

Label: C=3P=0

C_{2D}: link $f'(x) < 0 \Rightarrow f$ decreasing

C_{2D}: link $f''(x) > 0 \Rightarrow f$ concave up

C_1: translations (graph)

Category: C+P-

4.2.2.9 Item 9

Which of the following graphs $f(x)$ satisfies both given conditions:

(i) $f'(x) > 0$ on $(-\infty, 0)$ and $f'(x) \leq 0$ elsewhere

(ii) $f''(x) > 0$ on $(-\infty, -1)$ and $f''(x) \leq 0$ elsewhere

Answer and initial mark allocation: 2 marks

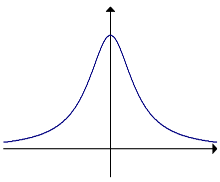
$f(x)$ increasing on $(-\infty,0)$, decreasing on $(0,\infty)$ ✓ C_{2D}

$f(x)$ concave up on $(-\infty,-1)$, concave down on $(-1,\infty)$ ✓ C_{2D}

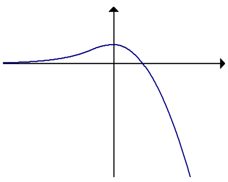
Graph C_1

Mark allocation after collapse of marks: 1 mark

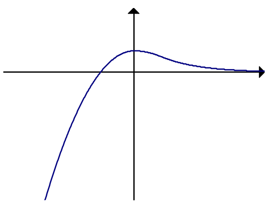
A.



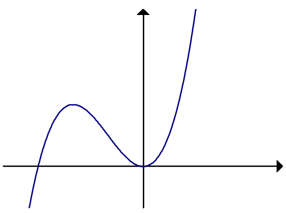
✓ B.



C.



D.



E. None of these

Comments concerning student responses and error analysis:

- This item was discussed in the article by Engelbrecht *et al.* (2005).
- 53.2% of students answered this item successfully.

Results:

Label: C=3P=0

C_{2D}: link link $f\,'(x) > 0 \Rightarrow f$ increasing, $f\,'(x) < 0 \Rightarrow f$ decreasing

C_{2D}: link $f\,''(x) > 0 \Rightarrow f$ concave up, $f\,''(x) < 0 \Rightarrow$ concave down

C_1: translations (graph)

Category: C+P-

4.2.2.10 Item 10

Let $f(x) = |2x|$. The graph of f' is given in:

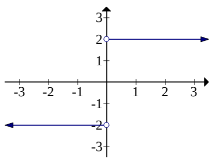
Answer and initial mark allocation: 2 marks

$$f(x) = |2x|$$

$$= \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases} \qquad C_{3F}$$

$$f\,'(x) = \begin{cases} 2, & x > 0 \\ -2, & x < 0 \end{cases} \qquad P_{2D}$$

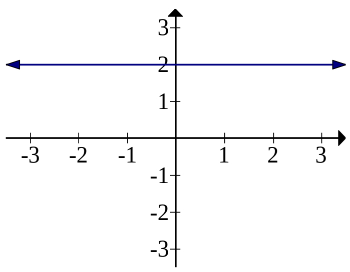
f not differentiable at $x = 0, x \neq 0$ $\qquad C_{3D}$



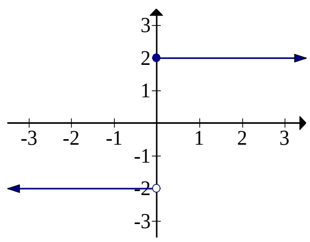
C_1

Mark allocation after collapse of marks: 1 mark

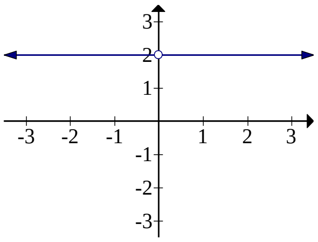
A



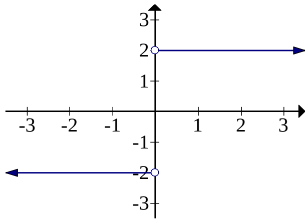
B.



C.



D.



E. None of these

Comments concerning student responses and error analysis:

- 30.5% of students answered this item successfully.
- Option B was a strong distractor, since 45.3% of the students chose B. These students could find *f'* (using differentiation rules), but did not realise that *f* is not differentiable at $x = 0$ ($f'(0)$ not defined).

Results:

Label: C=3P=1

C_{3F}: interpretation absolute value function

P_{2D}: differentiation rules

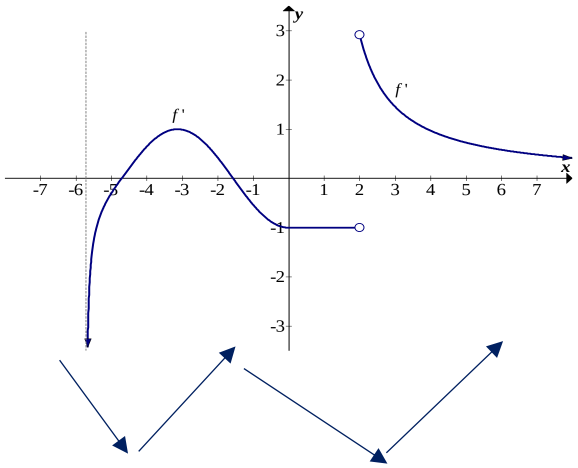
C_{3D}: interpretation differentiability

C_1: translations (graph)

Category: C+P-

4.2.2.11 Item 11

Given: the graph of f', which of the following is a possible graph for f? Explain.



Answer and initial mark allocation: 2 marks

$f'(x) > 0$ for $x \in (-4.7, -1.5)$ or $(2, \infty)$ C1 C2D

$\Rightarrow f$ increasing on $(-4.7, -1.5)$ or $(2, \infty)$

$f'(x) < 0$ for $x \in (-6, -4.7)$ or $(-1.5, 2)$ C1 C2D

$\Rightarrow f$ decreasing on $(-6, -4.7)$ or $(-1.5, 2)$

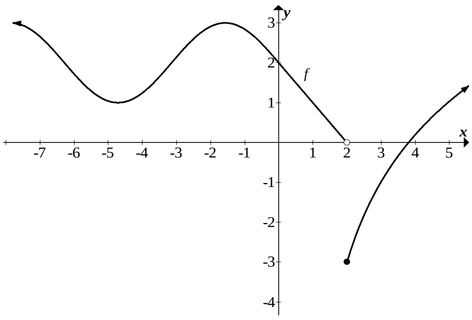
$f'(2)$ not defined C1 C3D

$\Rightarrow f$ not differentiable at $x = 2$

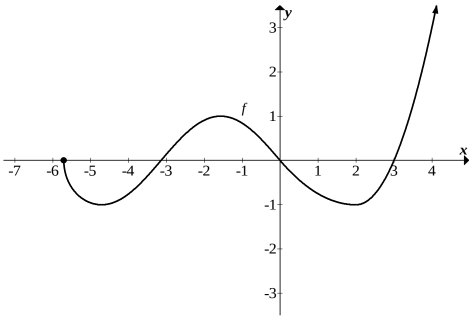
$D_{f'} \subseteq D_f = (-6, \infty) \setminus \{2\}$ C1 C2D

Mark allocation after collapse of marks: 1 mark

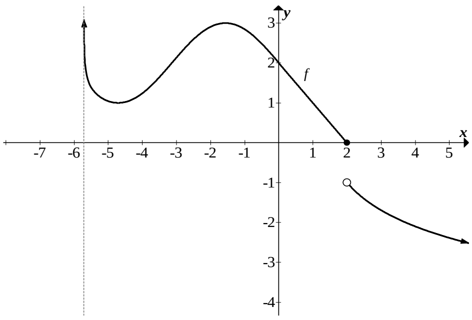
A.



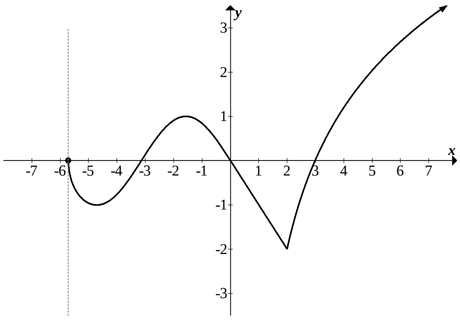
B.



C.



D.



E. None of these

Comments concerning student responses and error analysis:

- 60.3% of students answered this item successfully.
- Option C was chosen by 17.8% of the students. These students missed that the function *f* is increasing from $(2, \infty)$.

- Option A was chosen by 11% of the students. These students missed that $D_{f'} \subseteq D_f$.
- Option B was a weak distractor, since only 3.7% of students chose this option. The function f in option B is continuous and differentiable on the restricted domain.

Results:

Label: C=6P=0

C_1: translation (graph)

C_{2D}: link $f'(x) > 0 \Rightarrow f$ increasing, $f'(x) < 0 \Rightarrow$ decreasing

C_1: translations (*f*'(2) not defined)

C_{3D}: interpretation differentiability

C_1: translations (domain *f*')

C_{2D}: link $D_{f'} \subseteq D_f$

Category: C+P-

4.2.2.12 Item 12

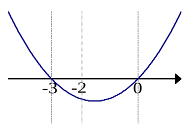
Let $f(x) = x^3 + 6x^2 + 9x + 2$. Which of the following statements is true with respect to the graph of f? Show your reasoning.

Answer and initial mark allocation: 2 marks

$f'(x) = 3x^2 + 12x + 9 = 0$ **P_{2D}** **C_{2D}**

$\Rightarrow 3(x+3)(x+1) = 0$ **P_{2F}**

$\Rightarrow x = -3$ or $x = -1$ **P_1**

Graph  **C_1**

$f'(x) < 0$ for $x \in (-3, -1)$ **C_{3F}**

$\Rightarrow f$ is decreasing on $(-3, -1)$ ✓ **C_{2D}**

$f''(x) = 6x + 12 = 0$ P2D C2D

$\Rightarrow x = -2$ P1

P2F

$f'' > 0$ for $x \in (-2, \infty)$

$\Rightarrow f$ concave up on $(-2, \infty)$ C2D

Mark allocation after collapse of marks: 1 mark

A. f is decreasing on $(-3,-1)$, concave down on $(-2,\infty)$

B. f is increasing on $(-3,-1)$, concave up on $(-2,\infty)$

C. f is decreasing on $(-3,-1)$, concave up on $(-2,\infty)$

D. f is increasing on $(-3,-1)$, concave down on $(-2,\infty)$

E. None of these

Comments concerning student responses and error analysis:

- 46.3% of students answered this item successfully.
- Option B was chosen by 23.4% of the students. These students incorrectly indicated whether the function is increasing/decreasing, but correctly indicated the concavity of the function.
- Option A was chosen by 16.7% of the students. These students were correct that the function is decreasing, but were wrong regarding the concavity of the function.

Results:

Label: C=6P=4

P_{2D}: differentiation rules

C_{2D}: link local extrema

P_{2F}: factorisation

P_1: symbolic and numerical calculations

C_1: translations (graph)

C_{3F}: interpretation quadratic inequality

C_{2D}: link *f* decreasing

C_{2D}: link possible point of inflection

P_{2F}: inequality rules (linear)

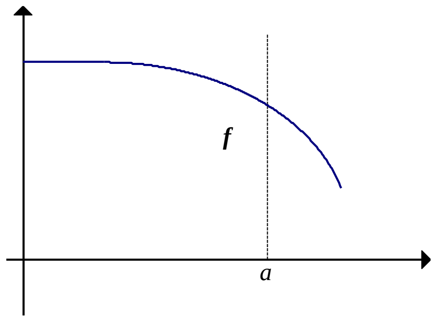
C_{2D}: link *f* concave up

Category: C+P+

4.2.2.13 Item 13

The graph of the function f is given. Consider: $g(x) = \int_0^x f(x)dx \ (0 \leq x \leq a)$.

What can we say about the function g? Explain.



Answer and mark allocation: 1 mark

$f(x) > 0$ **C_1**

$g'(x) = f(x)$ FTOC **C_{2I}**

$f(x) > 0 \Rightarrow g'(x) > 0 \Rightarrow g$ increasing ✓ **C_{2D}**

A. It is decreasing

✓ B. It is increasing

C. It is constant

D. We cannot say whether it increases or decreases

E. None of these

Comments concerning student responses and error analysis:

- This item was included in the draft (32 questions) and final (8 questions) data collection instrument developed by Engelbrecht (2008) and Engelbrecht (2009).
- 48.4% of students answered this item successfully.
- Option C was a weak distractor, since only 4.7% of students chose this option.

Results:

Label: C=3P=0

C_1: translation (graph)

C_{2I}: link $g'(x) = f(x)$

C_{2D}: link $g'(x) > 0 \Rightarrow g$ increasing

Category: C+P-

4.2.2.14 Item 14

The function $v(t)$ is a velocity function on the time interval $[a, b]$. Select the best interpretation of the integral $\int_a^b v(t)dt$.

Answer and mark allocation: 1 mark

A. The displacement between $t = a$ and $t = b$

B. The total distance travelled from $t = a$ to $t = b$

C. The acceleration between $t = a$ and $t = b$

D. The average speed attained between $t = a$ and $t = b$

E. None of these

Answer: displacement C_{2I}

Comments concerning student responses and error analysis:

- 76.8% of students answered this item successfully.
- Option C was a weak distractor, since only 4.7% of students chose this option.
- Option D was a weak distractor, since only 1.6% of students chose this option.

Results:

Label: C=1P=0

C_{2I}: link velocity (speed) $\Rightarrow$ position function (displacement)

Category: C-P-

4.2.2.15 Item 15

Given the function f in the sketch. Estimate the value of $\int_{-2}^{6} f(x)dx$?

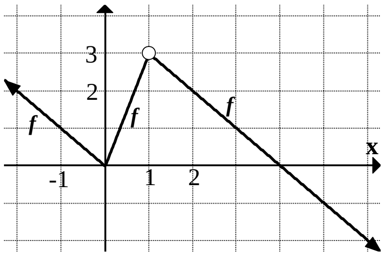
Answer and mark allocation: 1 mark

$$\int_{-2}^{6} f(x)dx$$
$$= \tfrac{1}{2}(2)(2) + \tfrac{1}{2}(1)(3) + \tfrac{1}{2}(3+2)(1)$$
$$= 2 + \tfrac{3}{2} + \tfrac{5}{2}$$
$$= 6$$

C1 C3I

P4

P1



A. 4

B. 6

C. 8

D. 10

E. None of these

Comments concerning student responses and error analysis:

- This item was included in the draft data collection instrument (32 questions) developed by Engelbrecht (2008).
- 51.1% of students answered this item successfully.
- Option D was a weak distractor, since only 4.7% of students chose this option.
- Option E (none of these) was chosen by 26% of the students. This could have happened because students struggle with fraction calculations.
- The method of evaluating the definite integral was not presented as a possible solution, since the question states *estimate*, not calculate.

Results:

Label: C=2P=2

C_1: translation (graph)

C_{3I}: interpretation definite integral = enclosed net area

P_4: formulae of geometric figures

P_1: symbolic and numerical calculations

Category: C-P-

4.2.2.16 Item 16

Evaluate the integral: $\int_1^e \frac{1}{x}dx$.

Answer and initial mark allocation: 2 marks

$$\int_1^e \frac{1}{x}dx$$

$= \ln|x|\Big|_1^e$ P21

$= \ln e - \ln 1$ P21

$= 1 - 0 = 1$ P1

Mark allocation after collapse of marks: 1 mark

A. e

B. -1

C. 0

D. 1

E. None of these

Comments concerning student responses and error analysis:

- 87.8% of students could evaluate the integral, but not the definite integral.
- 78.9% of students answered this item successfully (evaluate definite integral).
- Option A was a weak distractor, since only 2.1% of students chose this option.
- Option B was a weak distractor, since only 1.6% of students chose this option.
- Option C was a weak distractor, since only 1.6% of students chose this option.

Results:

Label: C=0P=3

P_{2I}: integration techniques

P_{2I}: FTOC (part 2)

P_1: symbolic and numerical calculations

Category: C-P+

4.2.2.17 Item 17

If $f''(\theta) = e^{2\theta}$, $f'(0) = 2$ and $f(0) = \frac{1}{4}$, find $f(\theta)$.

Answer and initial mark allocation: 2 marks

$\int e^{2\theta}d\theta = \frac{e^{2\theta}}{2} + c$ P21

$f'(0) = \frac{e^{2(0)}}{2} + c = 2$ C2D P1

$\Rightarrow \frac{1}{2} + c = 2$ **P1**

$\Rightarrow c = \frac{3}{2}$

$\Rightarrow f'(\theta) = \frac{e^{2\vartheta}}{2} + \frac{3}{2}$ **C2F**

$\int \frac{e^{2\vartheta}}{2} + \frac{3}{2} d\theta = \frac{e^{2\vartheta}}{4} + \frac{3}{2}\theta + c$ **P2I**

$f(0) = \frac{e^{2(0)}}{4} + \frac{3}{2}(0) + c = \frac{1}{4}$ **C2D** **P1**

$\Rightarrow \frac{1}{4} + c = \frac{1}{4}$ **P1**

$\Rightarrow c = 0$

$\Rightarrow f(\theta) = \frac{e^{2\vartheta}}{4} + \frac{3}{2}\theta$ **C2F**

Mark allocation after collapse of categories: 1 mark

A. $e^{2\theta}$

B. $\frac{e^{2\theta}}{2} + \frac{3}{2}\theta - \frac{1}{4}$

C. $\frac{e^{2\theta}}{4} + \frac{3}{2}\theta$

D. $\frac{e^{2\theta}}{4} + \frac{3}{4}\theta$

E. None of these

Comments concerning student responses and error analysis:

- This item was included in the draft (32 questions) and final (8 questions) data collection instrument developed by Engelbrecht (2008) and Engelbrecht (2009).
- 49.2% of students answered this item successfully.

Results:

Label: C=2P=3

P_{2I}: integration techniques

C_{2D}: link *f,f',f''*

P_1: substitution

P_1: symbolic and numerical calculations

C_{2F}: link equation of a function

Category: C-P+

4.2.3 Open-ended questions (18 – 33)

Collapse of item marks and categories was implemented for all items, excluding items 30 and 33. The new marks and mark categories used for the revised Rasch analysis are shared per item. The researcher expected items 21 and 25 to follow a single solution approach. However, three student solutions indicated a graphical approach to solve the inequality in item 21 and five students used the turning point formula to do item 25. The alternative methods were not analysed and labelled, but were included in the item analysis.

4.2.3.1 Item 18

Let $f(x) = e^{2x} - 1$ and $g(x) = \ln(x+1)$. Find $(f \circ g)(x)$ and simplify. Indicate all restrictions on x (if any).

Answer and initial mark allocation: 3 marks

$(f \circ g)(x)$

$= e^{2(\ln(x+1))} - 1$ $\quad C_{3F}$

$= e^{\ln(x+1)^2} - 1$ $\quad P_{2F}$

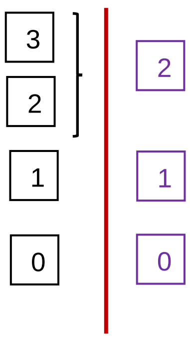
$= x^2 + 2x + 1 - 1$ $\quad C_{2F}$

$= x^2 + 2x$ $\quad P_1$

$\boldsymbol{x} + 1 > 0$ $\quad C_{3F}$

$\boldsymbol{x} > -1$

Mark allocation after collapse of marks: 2 marks



Comments concerning student responses and error analysis:

- 46.2% of students answered this item successfully - the focus was on the composite function, restriction ignored.
- 8.8% of students could not expand as composite functions.

Results:

Label: C=3P=2

C_{3F}: interpretation composite functions

P_{2F}: log laws

C_{2F}: link inverse functions

P_1: symbolic and numerical calculations

C_{3F}: interpretation domain

Category: C+P-

4.2.3.2 Item 19

Use the definition of the absolute value to rewrite the following function in expanded form and simplify: $k(x) = |3x + 1| - 2$

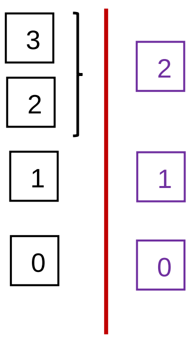
Answer and initial mark allocation: 3 marks

$$k(x) = |3x+1| - 2$$

$$= \begin{cases} 3x+1-2 & \text{if } 3x+1 \geq 0 \\ -(3x+1)-2 & \text{if } 3x+1 < 0 \end{cases} \qquad C_{3F}$$

$$= \begin{cases} 3x-1 & \text{if } x \geq \frac{-1}{3} \\ -3x-3 & \text{if } x < \frac{-1}{3} \end{cases} \qquad P_1 \quad P_{2F}$$

Mark allocation after collapse of marks: 2 marks



Comments concerning student responses and error analysis:

- 68.1% of students answered this item successfully.
- 7.1% of students could not use the definition of the absolute value to expand the function.
- 92.9% of students could use the definition of the absolute value to expand the function, but could not simplify correctly.

Results:

Label: C=1P=2

C_{3F}: interpretation absolute value function

P_1: symbolic and numerical calculations

P_{2F}: inequality rules (linear)

Category: C-P-

4.2.3.3 Item 20

If a ball is thrown upwards from the top of a building 25 metres high at an initial velocity of $20\ m/s$, then the height h above the ground t seconds later will be: $h(t) = -5t^2 + 20t + 25$. During what time interval will the ball be more than 40 m above the ground?

Answer and initial mark allocation: 3 marks

$-5t^2 + 20t + 25 > 40$ C_4

$-5t^2 + 20t - 15 > 0$ P_1

$-t^2 + 4t - 3 > 0$ ✓ P_1

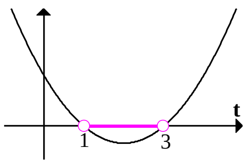
P_{2F}

$t^2 - 4t + 3 < 0$

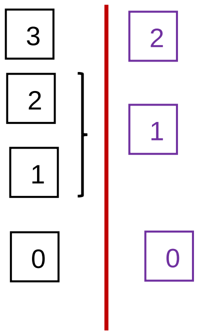
$(t-3)(t-1) < 0$ ✓ P_{2F}

Graph of parabola C_1

$t \in (1,3)$ ✓ C_{3F}



Mark allocation after collapse of marks: 2 marks



Comments concerning student responses and error analysis:

- 30.1% of students answered this item successfully.
- 82.5% of students could transfer the context to an equation or inequality.

Results:

Label: C=3P=3

C_4: contextual applications (height ball)

P_1: symbolic and numerical calculations

P_{2F}: rules and inequalities

P_{2F}: factorisation

C_1: translations (graph)

C_{3F}: interpretation of quadratic inequality

Category: C+P+

4.2.3.4 Item 21

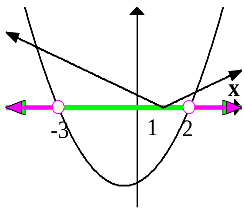
Solve $\frac{|x-1|}{x^2+x-6} \geq 0$. Write your final answer in interval notation.

Answer and initial mark allocation: 3 marks

Approach 1: Algebraic method

$$\frac{|x-1|}{x^2+x-6} \geq 0$$ P_{2F}

$(x-2)(x+3) > 0 \quad$ and $\quad |x-1| \geq 0$ P_{2F}

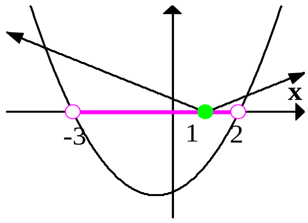


Graph of parabola C_1

$x \in (-\infty,-3) \cup (2,\infty) \quad$ and $\quad x \in \mathbb{R}$ C_{3F} C_{3F}

$\Rightarrow x \in (-\infty,-3) \cup (2,\infty)$ C_{3F}

OR



$$\frac{|x-1|}{x^2+x-6} \geq 0$$

$(x-2)(x+3) < 0$ and $|x-1| \leq 0$ P_{2F}

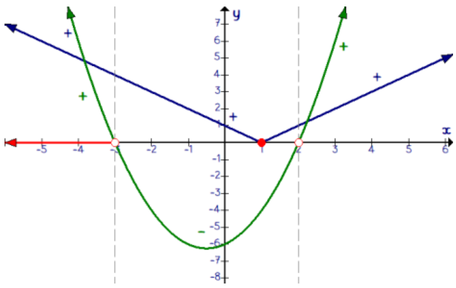
Graph of parabola $x-1=0 \Rightarrow x=1$ C_1 C_{3F} C_{3F} P_1

$x \in (-3, 2)$

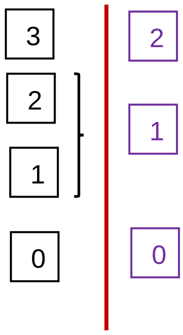
$\Rightarrow x = 1$ C_{3F}

Final answer: $x \in (-\infty, -3) \cup \{1\} \cup (2, \infty)$ C_{3F} P_5

Approach 2: Graphical method



Mark allocation after collapse of marks: 2 marks



Comments concerning student responses and error analysis:

- 1.7% of students answered this item successfully.
- The three students who answered item 21 successfully used the graphical approach to answer the question. Although the algebraic approach was taught as preferable method in class, these students showed the flexibility to use graphs instead of an algebraic approach. The presence of flexibility is an indicator of adaptive expertise (Hatano, 1988). This information can be used to inform future teaching practice.

- Fifty-nine students (30.7%) changed the absolute value function to a straight line function. This is seen as a mathematical and conceptual error, since changing the function changes the nature of the question.

Results:

Label: C=5P=4

P_{2F}: inequality rules ($\frac{+}{+}$ or $\frac{-}{-}$)

P_{2F}: factorisation

C_1: translations (graph)

C_{3F}: interpretation quadratic inequality

C_{3F}: interpretation absolute value inequality

C_{3F}: interpretation inequality (and)

P_1: symbolic and numerical calculations

C_{3F}: interpretation inequality (or)

P_5: notation

Category: C+P+

4.2.3.5 Item 22

If $f(x) = \sin(\frac{\pi x}{2})$, then find the second derivative $f''(1)$.

Answer and initial mark allocation: 3 marks

$f'(x) = \cos(\frac{\pi x}{2}).\frac{\pi}{2}$ **P_{2D}**

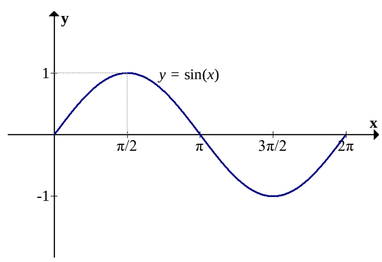
$f''(x) = -\sin(\frac{\pi x}{2}).\frac{\pi}{2}.\frac{\pi}{2}$ **P_{2D}**

$= -\frac{\pi^2}{4}\sin(\frac{\pi x}{2})$ **P_1**

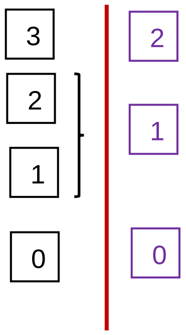
$f''(1) = -\frac{\pi^2}{4}\sin(\frac{\pi}{2})$ **P_1**

$= -\frac{\pi^2}{4}(1)$ **C_1**

$= -\frac{\pi^2}{4}$



Mark allocation after collapse of marks: 2 marks



Comments concerning student responses and error analysis:

- This item was included in the draft data collection instrument (32 questions) developed by Engelbrecht (2008).
- 41.5% of students answered this item successfully.
- 50.3% of students could find the second derivative, but could not calculate $f''(1)$.

Results:

Label: C=1P=3

P_{2D}: differentiation rules

P_1: symbolic and numerical calculations

P_1: substitution

C_1: translations (graph)

Category: C-P+

4.2.3.6 Item 23

Find the derivative of $m(x) = \frac{\operatorname{cosec} x}{e^x - 1}$

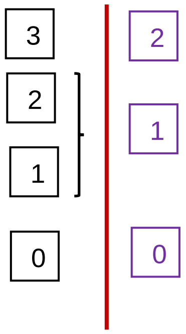
Answer and initial mark allocation: 3 marks

$$m'(x) = \frac{-\operatorname{cosec} x.\cot x.(e^x - 1) - \operatorname{cosec} x.e^x}{(e^x - 1)^2}$$

P_{2D}

P_{2D}

Mark allocation after collapse of marks: 2 marks



Comments concerning student responses and error analysis:

- 32.4% of students answered this item successfully.
- 24% of students could not calculate any part of the derivative.

Results:

Label: C=0P=2

P_{2D}: differentiation rules

P_{2D}: differentiation rules (quotient rule)

Category: C-P-

4.2.3.7 Item 24

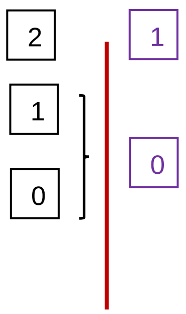
Find the derivative of $h(x) = \sqrt{x^3 - 7}.\tan\left(\frac{x}{2}\right)$.

Answer and initial mark allocation: 2 marks

$$h(x) = (x^3 - 7)^{\frac{1}{2}}.\tan(\tfrac{x}{2})$$

$$h'(x) = \tfrac{1}{2}(x^3 - 7)^{-\frac{1}{2}}.3x^2.\tan(\tfrac{x}{2}) + (x^3 - 7)^{\frac{1}{2}}.\sec^2(\tfrac{x}{2}).\tfrac{1}{2}$$ **P_{2D} P_{2D}**

Mark allocation after collapse of marks: 1 mark



Comments concerning student responses and error analysis:

- 43.3% of students answered this item successfully
- 29.2% of students could not calculate any part of the derivative.

Results:

Label: C=0P=2

P_{2D}: differentiation rules

P_{2D}: differentiation rules (product rule)

Category: C-P-

4.2.3.8 Item 25

If a stone is thrown vertically upwards, the position function of the stone is given by $s(t) = 30t - 5t^2 + 20$, where s is in metres and t is in seconds.

Calculate:

1. the time t when the stone will reach its maximum height
2. the maximum height of the stone (before it falls to the ground).

Answer and initial mark allocation: 3 marks

Approach 1:

$s'(t) = 30 - 10t$ (velocity function) ✓ C_4 P_{2D}

$s'(t) = 0$ (velocity function = 0) C_4 C_{2D}

$\Rightarrow 30 - 10t = 0$

$\Rightarrow -10t = -30$ ✓ P_1

$\Rightarrow t = 3s$

$s(3) = 30t - 5t^2 + 20$ (position function at t = 3) C_4 C_{2D}

$\Rightarrow s(3) = 90 - 45 + 20$ ✓ P_1

$\Rightarrow s(3) = 65m$

Approach 2: turning point formula to calculate (time; maximum height):

$s(t) = 30t - 5t^2 + 20 = 0$

$\Rightarrow t^2 - 6t - 4 = 0$

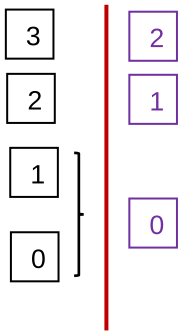
$\Rightarrow (t - 6t + 9) - 4 - 9 = 0$

$\Rightarrow (t-3)^2 - 13 = 0$

TP: (3,-13)

$t = 3s \Rightarrow s(3) = 65m$

Mark allocation after collapse of marks: 2 marks



Comments concerning student responses and error analysis:

- 74.2% of students answered this item successfully.
- Five students used the second approach (turning point formula).
- This item is an example of a contextual problem that could be seen as unfamiliar and conceptual to some students, but familiar and procedural to students who had seen a similar task before.
- The categories 0 and 1 were collapsed to 0, since many students (procedurally) know that $s'(t) = 0$ represents where the local extreme value(s) will be found.

Results:

Label: C=5P=3

C_4: contextual applications (velocity stone)

P_{2D}: differentiation rules

C_4: contextual applications (time $_{\text{max height}} \Rightarrow$ time $_{\text{velocity 0}}$)

C_{2D}: link $f'(x) = 0 \Rightarrow f$ local extrema

P_1: symbolic and numerical calculations

C_4: contextual applications (max height)

C_{2D}: link displacement and velocity

P_1: substitution

Category: C+P+

4.2.3.9 Item 26

Determine the point of inflection (if any) of $f(x) = e^x(x-1)$. Motivate your answer.

Answer and initial mark allocation: 4 marks

$$f'(x) = e^x(x-1) + e^x$$ **P2D**

$$= e^x.x$$ **P1**

$$f''(x) = e^x.x + e^x$$ **P2D**

$$= e^x(x+1)$$ **P2F**

$f''(x) = 0$

$\Rightarrow e^x(x+1) = 0$ **C2D**

$e^x > 0, x \in \mathbb{R}$ **C1**

$\Rightarrow x + 1 = 0$

$\Rightarrow x = -1$ (possible poi) **P2F**

Concavity change?

$f''(x) > 0 \Rightarrow f$ concave up **C2D**

$e^x(x+1) > 0$

$e^x > 0, x \in \mathbb{R}$ **C1**

$x + 1 > 0 \Rightarrow x > -1$ **P2F** **P2F**

f concave down $\Rightarrow f''(x) < 0$ **C2D**

$e^x(x+1) < 0$ **C1**

$e^x > 0, x \in \mathbb{R}$ **P2F** **P2F**

$x + 1 < 0 \Rightarrow x < -1$

$\Rightarrow (-1, -\frac{2}{e})$ poi since: **C3D**

$f''(-1) = 0$ and concavity change at $x = -1$

Mark allocation after collapse of marks: 2 marks

Initial marks	Marks after collapse
4, 3	2
2, 1	1
0	0

Comments concerning student responses and error analysis:

- 42.2% of students could find the first and second derivative, but could not find the point of inflection.
- 35.8% of students could find the point of inflection, but only 2.3% students could motivate their answers.

Results:

Label: C=4P=6

P_{2D}: differentiation rules

P_1: symbolic and numerical calculations

P_{2F}: factorisation

C_{2D}: link $f''(x)=0 \Rightarrow$ possible point of inflection

C_1: translations ($\boldsymbol{e}^x$ graph)

P_{2F}: rules and equations $(a.b=0 \Rightarrow a=0 \text{ or } b=0)$

C_{2D}: link $f''(x)>0 \Rightarrow f$ concave up, $f''(x)<0 \Rightarrow f$ concave down

P_{2F}: rules and inequalities $(a.b>0)$; $(a.b<0)$

P_{2F}: inequality rules (linear)

C_{3D}: interpretation point of inflection

Category: C+P+

4.2.3.10 Item 27

A mass, hanging in equilibrium at the end of a spring, is pulled and released. The position (in metres) of the mass is described by $x(t) = 0.4\cos(3t)$. Determine the acceleration (in m/s^2) of the mass, after π seconds.

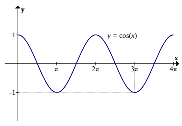
Answer and initial mark allocation: 3 marks

$x'(t) = -1.2\sin(3t)$ ✓ **C_4 C_{2D} P_{2D}**

$x''(t) = -3.6\cos(3t)$ ✓ **C_{2D} P_{2D}**

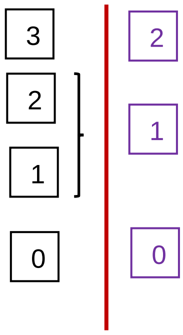
$x''(\pi) = -3.6\cos(3\pi)$ **P1**

$= -3.6(-1)$ **C1**



$= 3.6\,m/s^2$ **P1**

Mark allocation after collapse of marks: 2 marks



Comments concerning student responses and error analysis:

- 45.9% of students answered this item successfully.
- 44.8% of students could find the second derivative, but could not calculate $x''(\pi)$.

Results:

Label: C=4P=3

C_4: contextual application (spring)

C_{2D}: link displacement $\Rightarrow$ velocity

P_{2D}: differentiation rules

C_{2D}: link velocity $\Rightarrow$ acceleration

P_1: substitution

C_1: translations (graph)

P_1: symbolic and numerical calculations

Category: C+P+

4.2.3.11 Item 28

Evaluate the integral: $\int x\sqrt{1+2x^2}\,dx$

Answer and initial mark allocation: 2 marks

$$\int x\sqrt{1+2x^2}\,dx$$

$$=\tfrac{1}{4}\int 4x(1+2x^2)^{\frac{1}{2}}\,dx \qquad\qquad C_{2I}$$

$$=\tfrac{1}{4}.\frac{(1+2x^2)^{\frac{3}{2}}}{\frac{3}{2}}+c \qquad\qquad P_{2I}$$

$$=\tfrac{1}{4}.\tfrac{2}{3}.(1+2x^2)^{\frac{3}{2}}+c \qquad\qquad P_{1}$$

$$=\tfrac{1}{6}(1+2x^2)^{\frac{3}{2}}+c$$

Mark allocation after collapse of marks: 1 mark

2	1
1	0
0	

Comments concerning student responses and error analysis:

- This item was included in the draft data collection instrument (32 questions) developed by Engelbrecht (2008).
- 56.9% of students answered this item successfully (evaluate the integral).

Results:

Analysis 1

Label: C=1P=2

C_{2I}: link - fix constant to do integration

P_{2I}: integration techniques

P_1: symbolic and numerical calculations

Category: C-P-

The fixing of the constant to do integration could be classified as deep procedural knowledge (Star, 2005), since the student needed a clear and coherent explanation of how the constant should be altered in order to apply the integration techniques. If the category of deep procedural knowledge is preferred, the knowledge class will change to C=0P=2 and the category will remain C-P-.

Analysis 2

Label: C=0P=2

P_{2I}: integration techniques (deep procedural knowledge)

P_1: symbolic and numerical calculations

Category: C-P-

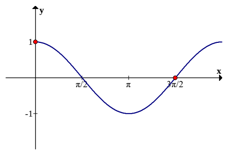
4.2.3.12 Item 29

Evaluate the integral: $\int_0^{\frac{3\pi}{2}} \sin\theta \, d\theta$

Answer and initial mark allocation: 2 marks

Approach 1:

$$\int_0^{\frac{3\pi}{2}} \sin\theta \, d\theta$$
$$= -\cos x \Big|_0^{\frac{3\pi}{2}}$$
$$= -[\cos\tfrac{3\pi}{2} - \cos 0]$$
$$= -[0-1]$$
$$= 1$$



P_{2I}

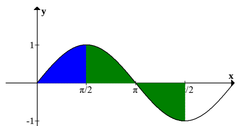
P_{2I}

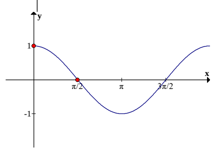
C_1

P_1

Approach 2:

$$\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta$$
$$= -\cos x \Big|_0^{\frac{\pi}{2}}$$
$$= -[\cos\tfrac{\pi}{2} - \cos 0]$$
$$= -[0-1]$$
$$= 1$$





C_{3I}

P_{2I}

P_{2I}

C_1

P_1

Mark allocation after collapse of marks: 1 mark

2	1
1	
0	0

Comments concerning student responses and error analysis:

- 97.7% of students could evaluate the integral.
- 65.9% of students answered this item successfully (evaluate definite integral).

Results:

Approach 1

Label: C=1P=3

P_{2I}: integration techniques

P_{2I}: FTOC (part 2)

C_1: translations (cos graph)

P_1: symbolic and numerical calculations

Category: C-P+

Approach 2

Label: C=2P=3

C_{3I}: interpretation definite integral = enclosed net area

P_{2I}: integration techniques

P_{2I}: FTOC (part 2)

C_1: translations (cos graph)

Category: C-P+

4.2.3.13 Item 30

Find $\frac{d}{dx}\int_0^x t^2 dt$

During the triangulation process it was identified that the following task could be done using at least two problem-solving approaches. According to the analysis, this task falls into either of two categories, namely C-P- or C-P+.

Answer and mark allocation: 1 mark (no collapse):

Approach 1:

$$\frac{d}{dx}\int_0^x t^2 dt$$
$$= x^2 \quad \checkmark \qquad C_{2I}$$

Approach 2:

$\int_0^x t^2\,dt$

$= \frac{t^3}{3}\Big|_0^x$ **P2I**

$= \frac{x^3}{3} - 0$ **P2I**

$= \frac{x^3}{3}$

$\Rightarrow \frac{d}{dx}\frac{x^3}{3}$ **P2D**

$= x^2$ ✓

Comments concerning student responses and error analysis:

- This item was included in the draft data collection instrument (32 questions) developed by Engelbrecht (2008).
- 56.9% of students answered this item successfully.

Results:

Approach 1:

Label: C=1P=0

C_{2I}: link FTOC (part 1): $g'(x) = f(x)$

Category: C-P-

Analysis of student responses indicated 51 students chose to do this item applying the procedural second approach and 47 students chose the conceptual first approach (FTOC part 1). As a result, item 30 was placed in the C-P+ knowledge class for the quantitative Rasch analysis.

Approach 2:

Label: C=0P=3

P_{2I}: integration techniques

P_{2I}: FTOC (part 2)

P_{2D}: differentiation rules

Category: C-P+

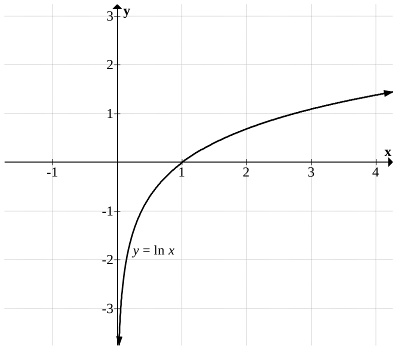
4.2.3.14 Item 31

Evaluate the integral $\int_{\frac{1}{2}}^{1} |\ln x|\,dx$ and indicate the calculated area on the graph.

Answer and initial mark allocation: 4 marks

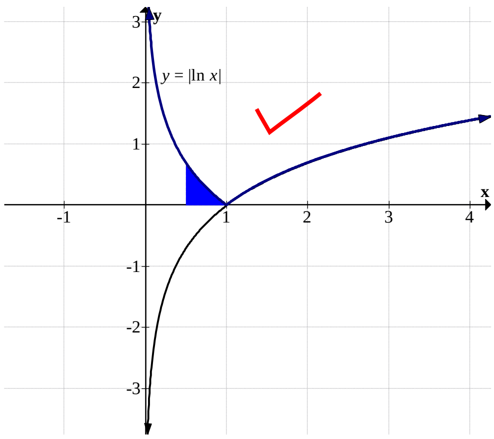
$$\int \ln x \, dx$$
$$= \int 1.\ln x \, dx$$
$$= x.\ln x - \int x.\tfrac{1}{x} \, dx$$
$$= x.\ln x - x + c$$

$$|\ln x|$$
$$= \begin{cases} \ln x, & \ln x \geq 0 \\ -\ln x, & \ln x < 0 \end{cases}$$

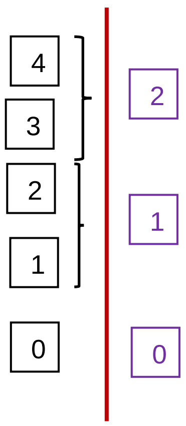


$$\int_{\frac{1}{2}}^{1} |\ln x| \, dx$$
$$= \int_{\frac{1}{2}}^{1} -\ln x \, dx \quad \text{C3F}$$
$$= -\int_{\frac{1}{2}}^{1} \ln x \, dx$$
$$= -(x.\ln x - x)\Big|_{\frac{1}{2}}^{1} \quad \text{P2I}$$
$$= -[(1.\ln 1 - 1) - (\tfrac{1}{2}.\ln \tfrac{1}{2} - \tfrac{1}{2})] \quad \text{P2I}$$
$$= -[0 - 1 - \tfrac{1}{2}\ln \tfrac{1}{2} + \tfrac{1}{2})] \quad \text{C1}$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\ln \tfrac{1}{2} \quad \text{P1}$$

Graph C1



Mark allocation after collapse of marks: 2 marks



Comments concerning student responses and error analysis:

- 11.7% of students answered this item successfully.
- 51.5% of students could evaluate the integral.
- Forty-five (26.3%) students ignored the absolute value in the function $|\ln x|$.
- Thirty-five (20.4%) students indicated the integral of $|\ln x|$ is $\frac{1}{x}$ (the derivative).

- Critique and suggestion to improve the question:

  Students could indicate correct area for definite integral $\int_{\frac{1}{2}}^{1} |\ln x| dx$ conceptually, without using procedures. Suggestion: leave out the graph of the function in the question.

Results:

Label: C=3P=3

C_{3F}: interpretation absolute value function

P_{2I}: integration techniques (by parts)

P_{2I}: FTOC (part 2)

C_1: translations (ln graph)

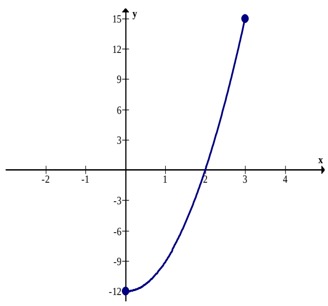
P_1: symbolic and numerical calculations

C_1: translations (graph)

Category: C+P+

4.2.3.15 Item 32

The velocity function of a particle is given by $v(t) = 3t^2 - 12\ (0 \leq t \leq 3)$, where v is in m/s and t is in seconds. The graph of $v(t)$ is given in the figure. Calculate the total distance travelled by the particle at $t = 2$.



Answer and initial mark allocation: 3 marks

Context — C_4

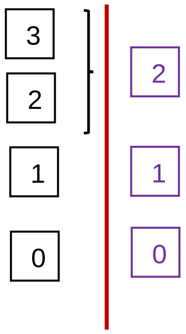
$\text{Total distance} = \int_0^2 -(3t^2 - 12)dt$ — C_{2I} C_{3F}

$= -(t^3 - 12t)\Big|_0^2$ — P_{2I}

$= -((8-24)-0)$ — P_{2I}

$= 16\,m$ — P_1

Mark allocation after collapse of marks: 2 marks



Comments concerning student responses and error analysis:

- 67% of students answered this item successfully.
- Critique and suggestions to improve the question:
    - Students gave a positive answer without motivation. The reason could be that students knew that distance is always positive.
    - Suggestion to change question to:
      Calculate the total distance travelled by the particle at t = 3, therefore:
      $\text{Total distance} = \int_0^2 -(3t^2 - 12)dt + \int_2^3 (3t^2 - 12)dt$. This new question discriminates better between total distance and displacement.
    - If the question is kept as is, the graph should not be given.

Results:

Label: C=3P=3

C_4: contextual applications (velocity particle)

C_{2I}: link velocity and total distance

C_{3F}: interpretation absolute value function

P_{2I}: integration techniques

P_{2I}: FTOC (part 2)

P_1: symbolic and numerical calculations

Category: C+P+

4.2.3.16 Item 33

Let $g(x) = \int_0^{\sin x} \sqrt{1 + t^3}\, dt$.

1. Determine the domain of $f(t) = \sqrt{1+t^3}$.
2. Determine an expression for $g'(x)$

Answer and mark allocation: 2 marks (no collapse):

1. $1+t^3 \geq 0$ **C_{3F}**

$\Rightarrow t^3 \geq -1$ **P_1**

$\Rightarrow t \geq -1$

$\Rightarrow D\left(\sqrt{1+t^3}\right) = [-1, \infty)$ ✓ **C_{3F}**

2. $g'(x) = \sqrt{1+(\sin x)^3}.\cos x$ ✓ **C_{2I}** **P_{2D}**

Comments concerning student responses and error analysis:

- This item was included in the draft data collection instrument (32 questions) developed by Engelbrecht (2008).
- 10.9% of students answered both 1 and 2 successfully.
- 48.6% of students answered either 1 or 2 successfully.

Results:

Label: C=3P=2

C_{3F}: interpretation restriction on function

P_1: symbolic and numerical calculations

C_{3F}: interpretation domain

C_{2I}: link $g'(x) = f(x)$

P_{2D}: differentiation rules (chain)

Category: C+P-

4.3 ANALYSIS PER KNOWLEDGE CLASS

Table 4.1-Table 4.4 present the items per knowledge class. If the content analysis indicated two solution approaches per item, both approaches were included in the table. The main conceptual problem-solving categories are: C_1: translations, C_2: linking relationships, C_3: interpretation of concepts and C_4: application of concepts (contextual problems); furthermore, the main procedural problem-solving categories are P_1: calculations and substitution, P_2: rules and techniques, P_3: algorithms (set of rules), P_4: formulae and P_5: symbols and notation. The category P_3: algorithms or set of rules, was never applied as category for the calculus items.

4.3.1 C-P- knowledge class item analysis

In this section, C-P- items are analysed further (Table 4.1). As indicated in the table, items 1 and 3 have more than one solution approach and label; however, the knowledge class for the two items remained C-P-.

Table 4.1: C-P- item analysis

C-P- item analysis											
Item	**Label**	**Breakdown of analysis**	**C_1**	**C_2**	**C_3**	**C_4**	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**
Multiple-choice questions											
1 Functions	C2P0	C_1: translations (graph) C_{2F}: link equation of a function	•	•							
	C1P1	C_1: translations (graph) P_{2F}: graph of the parabola (a < 0)	•					•			
3 Functions	C0P1	P_{2F}: function rule						•			
	C1P0	C_{3F}: interpretation functions			•						
6 Derivatives	C2P1	P_4: formula for gradient C_{3D}: interpretation differentiability C_1: translations (graph)	•		•					•	
14 Integration	C1P0	C_{2I}: link velocity $\Rightarrow$ displacement		•							
15 Integration	C2P2	C_1: translation (graph) C_{3I}: interpretation definite integral = area P_4: formulae of geometric figures P_1: symbolic and numerical calculations	•		•		•			•	
Open-ended questions											
19 Functions	C1P2	C_{3F}: interpretation absolute value funct P_1: symbolic and numerical calculations P_{2F}: inequality rules (linear)			•		•	•			
23 Derivatives	C0P2	P_{2D}: differentiation rules P_{2D}: differentiation rules (quotient rule)						• •			
24 Derivatives	C0P2	P_{2D}: differentiation rules P_{2D}: differentiation rules (product rule)						• •			
28 Integration	C1P2	C_{2I}: link - fix constant to do integration P_{2I}: integration techniques P_1: symbolic and numerical calculations		•			•	•			

According to Table 4.1, 24 steps were applied to do the items in the C-P- class. The highest number of steps per category was in the procedural category P_2: rules and techniques ($\frac{8}{24}$ steps, 33%).

4.3.2 C-P+ knowledge class item analysis

Table 4.2 describes the analyses for C-P+ items. The open-ended items 29 and 30 had more than one solution approach, analysis and label. Both labels for item 29 placed the item in class C-P+; however, the two solution approaches for item 30 indicated two labels and two different knowledge classes. As indicated in the item analysis, students preferred the procedural approach above the conceptual approach.

Table 4.2: C-P+ item analysis

C-P+ item analysis											
Item	**Label**	**Breakdown of analysis**	**C_1**	**C_2**	**C_3**	**C_4**	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**
Multiple-choice questions											
2 Functions	C2P3	C_{3F}: interpretation intersection P_1: symbolic and numerical calculations P_{2F}: inequality rules (linear) C_1: translations (graph) P_5: notation and symbols	•		•		•	•			•
5 Functions	C1P3	P_{2F}: division by 0 P_1: symbolic and numerical calculations P_{2F}: exponential laws (e.g. $a^0 = 1$) C_{3F}: interpretation domain			•		•	• •			
7 Derivatives	C2P3	P_{2D}: differentiation rules C_{3D}: interpretation gradient C_{2F}: link equation of a function P_1: substitution P_1: symbolic and numerical calculations		•	•		• •	•			
16 Integration	C0P3	P_{2I}: integration techniques P_{2I}: FTOC (part 2) P_1: symbolic and numerical calculations					•	• •			
17 Integration	C2P3	P_{2I}: integration techniques C_{2D}: link f,f',f'' P_1: substitution P_1: symbolic and numerical calculations C_{2F}: link equation of a function		• •			• •	•			
Open-ended questions											
22 Derivatives	C1P3	P_{2D}: differentiation rules P_1: symbolic and numerical calculations P_1: substitution C_1: translations (graph)	•				• •	•			

C-P+ item analysis

Item	Label	Breakdown of analysis	C_1	C_2	C_3	C_4	P_1	P_2	P_3	P_4	P_5
29 Integration	C1P3	P_{2I}: integration techniques P_{2I}: FTOC (part 2) C_1: translations (cos graph) P_1: symbolic and numerical calculations	•				•	• •			
	C2P3	C_{3I}: interpretation definite integral =area P_{2I}: integration techniques P_{2I}: FTOC (part 2) C_1: translations (cos graph) P_1: symbolic and numerical calculations	•		•		•	• •			
30 Integration	C1P0[3]	$C_{2I:}$ link FTOC (part 1): $g'(x) = f(x)$		•							
	C0P3	$P_{2I:}$ integration techniques $P_{2I:}$ FTOC (part 2) $P_{2D:}$ differentiation rules						• • •			

The C-P+ ítems contributed to 39 steps. The procedural category P_2: rules and techniques, indicated the highest number of steps per category ($^{15}\!/_{39}$, 38.5%), followed by the P_1: calculations and substitution category ($^{11}\!/_{39}$ steps, 28.2%). The P_1 and P_2 procedural problem-solving categories connect strongly to the definition of superficial procedural knowledge that Star (2005) describes as the common use of procedural knowledge.

4.3.3 C+P- knowledge class item analysis

In this section the analysis of the C+P- items is shared in Table 4.3. Item 4, a multiple-choice item, indicated one method, but two alternative analyses and labels. The two item labels for this item both belong to knowledge class C+P-.

Table 4.3: C+P- item analysis

C+P- item analysis

Item	Label	Breakdown of analysis	C_1	C_2	C_3	C_4	P_1	P_2	P_3	P_4	P_5
Multiple-choice questions											
4 Functions	C3P2	C_{3F}: interpretation composite function $P_{4:}$ trig formula	•		• •			•		•	

[3] C-P- knowledge class

C+P- item analysis											
Item	**Label**	**Breakdown of analysis**	**C_1**	**C_2**	**C_3**	**C_4**	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**
		P_{2F}: $\sqrt{a^2} = \lvert a \rvert$ C_{3F}: interpretation absolute value C_1: translations (graph)									
	C3P1	C_{3F}: interpretation composite function P_{4}: trig formula C_{2F}: link square root function non-negative C_1: translations (graph)	•	•	•					•	
8 Derivatives	C3P0	C_{2D}: link $f'(x) < 0 \Rightarrow f$ decreasing C_{2D}: link $f''(x) > 0 \Rightarrow f$ concave up C_1: translations (graph)	•	• •							
9 Derivatives	C3P0	C_{2D}: link $f'(x) > 0 \Rightarrow f$ increasing, $f'(x) < 0 \Rightarrow$ decreasing C_{2D}: link $f''(x) > 0 \Rightarrow f$ concave up, $f''(x) < 0 \Rightarrow$ concave down C_1: translations (graph)	•	• •							
10 Derivatives	C3P1	C_{3F}: interpretation absolute value function P_{2D}: differentiation rules C_{3D}: interpretation differentiability C_1: translations (graph)	•		• •			•			
11 Derivatives	C6P0	C_1: translation (graph) C_{2D}: link $f'(x) > 0 \Rightarrow f$ increasing, $f'(x) < 0 \Rightarrow$ decreasing C_1: translations ($f'(2)$ not defined) C_{3D}: interpretation differentiability C_1: translations (domain f') C_{2D}: link $D_{f'} \subseteq D_f$	• • •	• •	•						
13 Integration	C3P0	C_1: translation (graph) C_{2I}: link $g'(x) = f(x)$ C_{2D}: link $g'(x) > 0 \Rightarrow g$ increasing	•	• •							
Open-ended questions											
18 functions	C3P2	C_{3F}: interpretation composite functions P_{2F}: log laws C_{2F}: link inverse functions C_{3F}: interpretation domain P_1: symbolic and numerical calculations		•	• •		•	•			
33 Integration	C3P2	C_{3F}: interpretation restriction on function P_1: symbolic and numerical calculations C_{3F}: interpretation domain		•	• •		•	•			

C+P- item analysis											
Item	**Label**	**Breakdown of analysis**	**C_1**	**C_2**	**C_3**	**C_4**	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**
		C_{2I}: link $g'(x) = f(x)$ P_{2D}: differentiation rules (chain)									

The total number of steps for C+P- ítems was 38; the highest number of steps was found in conceptual categories, as expected. The highest frequency was for linking relationships ($^{11}\!/_{38}$ steps, 28.9%), followed by interpretation of concepts ($^{10}\!/_{38}$ steps, 26.3%) and translations ($^{9}\!/_{38}$ steps, 23.7%).

4.3.4 C+P+ knowledge class item analysis

Table 4.4 describes the analyses of C+P+ items. The alternative methods used by students for items 21 and 25 were not analysed, and therefore not included in Table 4.4. In total 62 steps were performed to solve the items in the C+P+ class.

Table 4.4: C+P+ item analysis

C+P+ item analysis											
Item	**Label**	**Breakdown of analysis**	**C_1**	**C_2**	**C_3**	**C_4**	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**
Multiple-choice questions											
12 Derivatives	C6P4	P_{2D}: differentiation rules C_{2D}: link local extrema P_{2F}: factorisation P_1: symbolic and numerical calculations C_1: translations (graph) C_{3F}: interpretation quadratic inequality C_{2D}: link *f* decreasing C_{2D}: link possible point of inflection P_{2F}: inequality rules (linear) C_{2D}: link *f* concave up	•	• • • •	•		•	• • •			
Open-ended questions											
20 Functions	C3P3	C_4: contextual application (height ball) P_1: symbolic and numerical calculations P_{2F}: rules and inequalities P_{2F}: factorisation C_1: translations (graph) C_{3F}: interpretation quadratic inequality	•		•	•	•	• •			

C+P+ item analysis

Item	Label	Breakdown of analysis	C_1	C_2	C_3	C_4	P_1	P_2	P_3	P_4	P_5
Open-ended questions											
21 Functions	C5P4	P_{2F}: inequality rules P_{2F}: factorisation C_1: translations (graph) C_{3F}: interpretation absolute value ineq C_{3F}: interpretation quadratic inequality C_{3F}: interpretation inequality (and) P_1: symbolic and numerical calculations C_{3F}: interpretation inequality (or) P_5: notation	•		• • • •		•	• •			•
25 Derivatives	C5P3	C_4: contextual application (velocity stone) P_{2DF}: differentiation rules C_4: contextual application (time $_{\text{max height}}$ $\Rightarrow$ time $_{\text{velocity 0}}$) C_{2D}: link $f'(x)=0 \Rightarrow f$ local extrema P_1: symbolic and numerical calculations C_4: contextual applications (max height) C_{2D}: link displacement and velocity P_1: substitution		• •		• • •	• •	•			
26 Derivatives	C4P6	P_{2D}: differentiation rules P_1: symbolic and numerical calculations P_{2F}: factorisation C_{2D}: link $f''(x)=0 \Rightarrow$ possible poi C_1: translations ($\boldsymbol{e}^x$ graph) P_{2F}: rules and equations C_{2D}: link $f''(x)>0 \Rightarrow f$ concave up, $f''(x)<0 \Rightarrow f$ concave down P_{2F}: rules and inequalities P_{2F}: inequality rules (linear) C_{3D}: interpretation point of inflection	•	• •	•		•	• • • • •			
27 Derivatives	C4P3	C_4: contextual application (spring) C_{2D}: link displacement $\Rightarrow$ velocity P_{2D}: differentiation rules C_{2D}: link velocity $\Rightarrow$ acceleration P_1: substitution C_1: translations (graph) P_1: symbolic and numerical calculations	•	• •		•	• •	•			

C+P+ item analysis											
Item	**Label**	**Breakdown of analysis**	**C_1**	**C_2**	**C_3**	**C_4**	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**
Open-ended questions											
31 Integration	C3P3	C_{3F}: interpretation absolute value P_{2I}: integration techniques P_{2I}: FTOC (part 2) C_1: translations (ln graph) P_1: symbolic and numerical calculations C_1: translations (graph)	● ●		●		●	● ●			
32 Integration	C3P3	C_4: contextual application (particle) C_{2I} link velocity and total distance C_{3F}: interpretation absolute value P_{2I}: integration techniques P_{2I}: FTOC (part 2) P_1: symbolic and numerical calculations		●	●	●	●	● ●			

The highest number of steps per category for the C+P+ class was found alternatively in the conceptual and procedural categories: P_2: rules and techniques category ($^{18}/_{62}$ steps, 29%), followed by C_2: linking relationships ($^{11}/_{62}$ steps, 17.7%), P_1: calculations and substitution ($^{10}/_{62}$ steps, 16.1%) and C_3: interpretation of concepts ($^{9}/_{62}$ steps, 14.5%). Although the P_2 category had the highest number of steps, the total number of conceptual steps was higher than the number of procedural steps ($^{33}/_{62}$ steps, 53.2%) vs ($^{29}/_{62}$ steps, 46.8%). The contextual applications contributed to $^{6}/_{62}$ steps (0.97%) for C_4: applications of concepts to mathematical situations.

4.4 DISCUSSION

The further analysis and discussion in this section relates to all items across all four knowledge classes. The analysis includes reference to items that could be solved using multiple approaches, item clusters that assess the same concept, and items with a multi-dimensional conceptual construct.

4.4.1 Multiple solutions per item

Table 4.5 illustrates that items 1, 3, 4, 29 and 30 have more than one solution method or analysis, therefore more than one item label. The different labels for items 1, 3, 4, and 29 belong to the same class. Item 4 has one solution, but two different analyses of the same method. Item 30 has two labels that belong to different classes. For the purposes of the Rasch analysis the class for item 30 was moved from C-P- and C-P+, to C-P+, since student solutions indicated that participants preferred to solve this item using a procedural rather than a conceptual approach.

Table 4.5: Items with multiple solutions

Items with multiple solutions (labels)		
Item	**Label**	**Knowledge class**
1	C2P0 or C1P1	C-P-
3	C0P1 or C1P0	C-P-
4	C3P2 or C3P1	C+P-
29	C1P3 or C2P3	C-P+
30	C1P0 or C0P3	C-P- or C-P+

4.4.2 Item clusters that assess the same concept

Some items could be clustered together, since they measure the same concept. Rittle-Johnson and Schneider (2016) argue that conceptual measures are stronger if multiple tasks are used to assess the same concept, since doing this reduces the impact of task-specific properties. Table 4.6 indicates item clusters that assess the same notion or concept.

Table 4.6: Item clusters that assess the same concept

Item clusters that assess the same concept			
Items	**Class**	**Concept assessed**	**Further analysis**
Inequalities			
2	C-P+	Solve the inequality (linear)	29.5% (item 2) and 1.7% (item 21) of students could solve the inequality. Students have not mastered inequalities.
21	C+P+	Solve the inequality (quadratic)	
Differentiation			
6	C-P-	Graph of function *f* is given; the graph of the derivative function *f'* is required	12.5% (item 6), 30.5% (item 10) of students could select the correct graph for *f'*. Many students could find the

10	C-P+	Function f is given; the graph of the derivative function f' is required	derivative, but did not know that a function is not differentiable at a vertex.
Differentiation			
22	C-P+	f is given and f' and $f''(1)$ required	43.3% (item 24) of students could apply product rule, 32.4% (item 23) quotient rule, 50.3% (item 22) could find second derivative of sin x. It appears as if students have not mastered differentiation.
23	C-P-	f is given and f' required (quotient rule)	
24	C-P-	f is given and f'' required (product rule)	
Curve sketching			
8	C+P-	Information on f' and f'' are given; the graph of the function f is required	66.3% (item 8), 53.2% (item 9) and 60.3% (item 11) of students could select the correct graph for f, therefore students appear to have mastered finding f if information on f' and f'' is given.
9	C+P-	Information on f' and f'' is given; the graph of the function f is required	
11	C+P-	The graph on f'' is given; the graph of the function f is required	
Curve sketching			
12	C+P+	Function f is given, description of the graph of f is required (gradient and concavity)	46.3% (item 12), 35.8% (item 26) of students could describe f if f was given. Students did not master describing f.
26	C+P+	Function f is given, description of the graph of f is required (point of inflection)	
Integration techniques			
16	C-P+	Integrand given, definite integral required	87.8% (item 16), 56.9% (item 28, 97.7% (item 29) and 51.5% (item 31) of students could evaluate the integral, therefore students appear to have mastered integration techniques.
28	C-P-	Integrand given, general antiderivative required	
29	C-P+	Integrand given, definite integral required	
31	C+P+	Integrand given, definite integral required	
Contextual applications			
20	C+P+	Contextual application	Items were all contextual problems, but percentage of students who answered successfully varied: Item 20 (30.1%), item 25 (74.2%), item 27 (45.9%) and item 32 (67%).
25	C+P+	Contextual application	
27	C+P+	Contextual application	
32	C+P+	Contextual application	

Items involving contextual application yielded varied results; however, findings for the other clustered items correlated and conjectures were made that could inform teaching practices. The analysis of the clustered items indicated that students:

- have not yet mastered linear or quadratic inequalities
- have not yet fully mastered the topic of differentiation
- have mastered curve sketching to some extent
- have mastered integration techniques.

4.4.3 Items with a multi-dimensional conceptual construct

Conceptual tasks often require knowledge of many related concepts, therefore the tasks have a multi-dimensional construct (Rittle-Johnson & Schneider, 2016). Table 4.7 discusses items that require knowledge of six or more related conceptual calculus concepts or categories. Both items relate to the multidimensional construct of curve sketching. Item 11 involves moving between graphical and algebraic representations (translations), link and interpretation between differentiation and gradient function, and link between domain of *f* and *f'*. Item 11 had no procedural categories. Item 12 included linking the concepts local extrema, gradient function, point of inflection and concavity, as well as the graph and interpretation of a quadratic inequality, and four procedural steps.

Table 4.7: Examples of items that include various related concepts

Items that include related concepts		
Item & label	**Related conceptual concepts**	**Analysis**
11 C6P0	C_1	Translate between graph *f* and graph *f'* - gradient concept
	C_1	Translate between graph and differentiability concept
	C_1	Translate between graph and domain concept
	C_{2D}, C_{3D}	Link and interpret derivative and gradient
	C_{2D}	Link domain *f* and *f'*
12 C6P4	C_{2D}	Link local extrema (plus P_{2D} differentiation rules)
	C_{2D}	Link f decreasing
	C_{2D}	Link possible point of inflection
	C_{2D}	Link concave up (plus P_{2F} linear inequality rules)
	C_1	Translate graph quadratic inequality (and P_{2F} factorisation, P_1 calculations)
	C_{3F}	Interpretation quadratic inequality

The various related concepts placed items 11 and 12 into a category with more conceptual integration. The researcher expected that these items would be experienced as difficult items.

The validity of the qualitative content analysis was discussed in the previous chapter. The triangulation session between the researcher and a mathematics colleague, and the triangulation session between the researcher and supervisor were discussed in detail. Iterative reflection led to agreement and crystallisation (Maree, 2016; McMillan & Schumacher, 2014) was reached.

4.5 SUMMARY

The qualitative content analysis answered the research question on how calculus items could be analysed and classified with respect to conceptual and procedural knowledge. The analysis confirmed the notion that the nature of calculus item solutions draws on both procedural and conceptual components. The nature of some items emphasises procedures and rules, while other items focus on translations, and/or linking and interpretations of concepts. The nature of the approach used to solve a calculus item is not absolute, since it may differ among students, educators or mathematics education experts (Star & Stylianides, 2013). The analysis indicated that there could be more than one method to solve an item. The researcher anticipated different methods for items 1, 3, 4, 29 and 30, but additional methods, suggested by student work, were included in the study (items 21 and 25). What complicates the categorisation further is that what is conceptual for one student could be procedural for another, depending on whether the person had seen the item or approach before. The researcher expected that the easiest and most efficient strategy to solve a problem would be the appoach with the lowest number of steps, therefore the expectation was that a C-P- item would be easier than a C-P+, C+P- or C+P+ item.

The analysis per knowledge class indicated that items in the C-P- knowledge class have the highest number of steps for rules and techniques (Table 4.1). C-P+ items require mostly rules and techniques, followed by calculations and substitution (Table 4.2). As noted previously, these procedural problem-solving categories connect strongly to the common use of procedural knowledge or superficial procedural knowledge described by Star (2005). C+P- items highlighted linking relationships, interpretation of concepts and translations (Table 4.3). It is interesting to note that C+P+ items indicated a high number of steps alternatively in conceptual and procedural categories: rules and techniques, followed by linking relationships, calculations and interpretation of concepts; however, in overview the number of conceptual steps was higher than the number of procedural steps (Table 4.4).

The analysis indicated that five items had more than one solution approach or analysis, and therefore more than one item label (Table 2.5). According to the analysis, seven item clusters assessed the same notion or concept (Table 4.6). The findings of the clustered items informed further analysis and indicated that students had not yet mastered inequalities, were not fully proficient in differentiation and had partial understanding of curve sketching, but were skilled in integration techniques. The outcomes were diagnostic and could inform teaching strategies. Furthermore, the analysis reflected on two curve-sketching tasks that each required knowledge of more than six related concepts (Table 4.7). These items were described as multi-dimensional construct items and it was expected that they would be considered difficult items.

The choice of procedural or conceptual approach connects to the prior knowledge of the students and the choice of teaching strategy, learning and assessment. A procedural approach will focus on accuracy of procedures and answers, rules and techniques and tasks that involve familiar procedures. A conceptual approach will involve evaluation and explanation of unfamiliar tasks that could lead to metacognition, reflection and discussion. Teaching should include both types of problem-solving approaches.

The items developed and classified for the qualitative analysis were included in a data collection instrument. The initial Rasch analysis indicated that some item categories were disordered and items were re-examined against the construct of the mathematics. The collapsed marks and mark categories were used in the revised Rasch analysis that describes the cognitive construct of conceptual and procedural knowledge *between* calculus items. The revised, tailored Rasch analysis produced a better fit of the data to the model and valid and reliable results. Table

4.8 gives a summary of the item analysis of marks and mark categories, before and after collapse.

Table 4.8: Summary: Item analysis of items (before and after collapse)

Item analysis of items - before and after collapse						
Item	**Label**	**Class**	**Marks**		**Number of categories**	
			Before collapse	**After collapse**	**Before collapse**	**After collapse**
Multiple-choice questions						
Functions						
1	C2P0 or C1P1	C-P-	2	1	3	2
2	C2P3	C-P+	2	1	3	2
3	C0P1 or C1P0	C-P-	1	1	2	2
4	C3P2 or C3P1	C+P-	2	1	3	2
5	C1P3	C-P+	2	1	3	2
Derivatives						
6	C2P1	C-P-	2	1	3	2
7	C2P3	C-P+	2	1	3	2
8	C3P0	C+P-	2	1	3	2
9	C3P0	C+P-	2	1	3	2
10	C3P1	C+P-	2	1	3	2
11	C6P0	C+P-	2	1	3	2
12	C6P4	C+P+	2	1	3	2
Integration						
13	C3P0	C+P-	1	1	2	2
14	C1P0	C-P-	1	1	2	2
15	C2P2	C-P-	1	1	2	2
16	C0P3	C-P+	2	1	3	2
17	C2P3	C-P+	2	1	3	2
Open-ended questions						
Functions						
18	C3P2	C+P-	3	2	4	3
19	C1P2	C-P-	3	2	4	3
20	C3P3	C+P+	3	2	4	3

Item analysis of items - before and after collapse						
Item	**Label**	**Class**	**Marks**		**Number of categories**	
			Before collapse	**After collapse**	**Before collapse**	**After collapse**
21	C5P4	C+P+	3	2	4	3
Derivatives						
22	C1P3	C-P+	3	2	4	3
23	C0P2	C-P-	3	2	4	3
24	C0P2	C-P-	2	1	3	2
25	C5P3	C+P+	3	2	4	3
26	C4P6	C+P+	4	2	5	3
27	C4P3	C+P+	3	2	4	3
Integration						
28	C1P2	C-P-	2	1	3	2
29	C1P3 or C2P3	C-P+	2	1	3	2
30	C1P0 or C0P3	C-P-	1	1	2	2
31	C3P3	C+P+	4	2	5	3
32	C3P3	C+P+	3	2	4	3
33	C3P2	C+P-	2	2	3	3
Total			74	45		

Chapter 5 will focus on the results obtained from the quantitative analysis. Chapter 4 and chapter 5 will be used to inform chapter 6, the final discussion chapter on findings and recommendations.

5. CHAPTER 5: QUANTITATIVE RESULTS

5.1 INTRODUCTION

In chapter 3 the research methodology that informed the study was discussed. Chapter 5 focusses on the results obtained from the quantitative analysis and presents results for the second research question that assesses the cognitive constructs of mathematical knowledge in calculus:

How are conceptual and procedural constructs related in calculus?

In the first part of the quantitative section Rasch analysis will investigate the overall pattern of the cognitive calculus construct when focussing on relations between person proficiency and item difficulty. The analysis will also examine whether Rasch statistics support the hypothesis that valid and reliable inferences can be derived from construct measures, i.e. whether the items in the instrument represent true interval scale measurement of one underlying construct. The results focus on descriptive statistics, general functioning of the instrument and thereafter the fit of individual items and persons to the model (Combrinck, 2018). Investigation of the item and person fit statistics informs whether each item and each person contribute to the measurement model as a whole. Infit statistics report on the overall pattern of whether persons and items align with the expected Rasch model, and outfit statistics identify possible outliers.

The results of the initial Rasch analysis indicated items with disordered categories. These items were investigated through a process of triangulation between researcher, mathematics education specialist and psychometrician. The investigation resulted in the collapse of marks and mark categories for identified items and a revised data collection instrument. The outcomes shared in this chapter are the results obtained from the revised Rasch analysis.

In the second part of the quantitative analysis the results obtained from the Rasch analysis, in particular the descriptive statistics and the person-item map, are compared to the CFA results. The study investigates whether the CFA results support the calculus items as classified a priori.

5.2 CATEGORY COLLAPSE

5.2.1 Introduction

The initial application of the Rasch Partial Credit Model with Winsteps 3.93.1 (Linacre, 2016) indicated that some categories were disordered and did not indicate clearly advancing average measures or sufficient observations per category (UNWTD measure, AVGE MEAS, Appendix 5). Furthermore, some item category probability curves appeared jagged (Item category probability curves, Appendix 6). The items were identified according to the guidelines suggested by Linacre (2017) and Iramaneerat *et al.* (2008), discussed in chapter 3.

The statistics specialist suggested that the disordered categories be investigated by comparing the outcome of the data analysis with the mathematical construct. The investigation was in conjunction with the qualitative content analysis (Mallinckrodt *et al.*, 2016), therefore the theoretical component that supported the instrument (Stols *et al.*, 2015). The triangulation between researcher, supervisor and statistics specialist informed the collapse of marks and mark categories for 27 of the 33 items. The collapsing of marks and mark categories was only implemented if it made sense in terms of the calculus construct. The multiple-choice and open-ended items are discussed under separate headings.

5.2.2 Multiple-choice items (1 – 17)

Items 3, 13, 14 and 15 were dichotomous items with two response categories and possible scores of 0 and 1. The other items had scores of 0, 1 and 2, since credits were given for a partially correct answer. In mathematics, the marker gives partial credits to students when partial success is obtained, or when the marker continues to mark with a calculation error (not mathematical or conceptual error). The marker therefore gives partial credit to the student for showing some understanding, even though the correct final answer is not reached. It could be argued that this mathematical practice may not yield a true indication of ability where statistical results show significantly disordered categories, i.e. that the partial mark does not indicate more ability. Analysis of the multiple-choice item responses revealed the following statistical data:

1. Disordered categories were found where clear advancing average measures were not present between categories (AVGE MEAS, Appendix 5). The progression in average measure was too small to be of statistical significance for items 2, 4, 5, 6 ,7, 9, 10, 11 and 17.
2. Disordered categories were also seen on the item category probability curves (Appendix 6). The disordered category item curves appeared narrow and jagged, since category transitions were initially defined as categories. This was the case for items 4, 5, 7, 9, 11, 12, 16 and 17. Collapse of categories was done through inputs from mathematical subject specialists, on the basis of the theory.
3. The percentage observations in some categories were less than 10% for items 5, 7, 8 and 16 (UNWTD %, Appendix 5). The percentage observations were too small to be statistically significant.
4. The outfit mean squares were not close to 1 for items 4, 7 and 11 (OUTFIT MNSQ, Appendix 5), but did not exceed the recommended limit of 1.50 (Linacre, 2017).

It could be motivated that item one remained in three categories, but since all the categories of the other multiple-choice items were collapsed from three to two categories, it made sense to collapse the categories for this item as well. As a result, all 17 multiple-choice items were changed to dichotomous items with only two response categories. Scores of 0 and 1 were collapsed to 0; the score of 2 was changed to 1. Each item was considered on a case-to-case basis and the collapsing was done based on conceptual and theoretical alignment. The details for the collapse can be found in section 4.2.2 in chapter 4.

5.2.3 Open-ended items (18 – 33)

The statistical analysis of the open-ended items gave indications of potential problems with mark allocation and mark categories. Analysis of the open-ended item responses revealed the following statistical data:

1. Disordered categories were found for items 19, 20, 21, 23, 24, 25, 26, 28 and 31 where clear advancing item average measures (AVGE MEAS, Appendix 5) were not visible between categories or where the item category probability graphs were jagged (Appendix 6).

2. The percentage observations in some categories were less than 10% for items 18, 19, 21, 22, 25, 26, 27, 29, 31 and 32 (UNWTD %, Appendix 5).
3. The outfit mean squares were not close to 1 one for items 20, 21, 23 and 25 (OUTFIT MNSQ, Appendix 5).
4. The mathematical practice of continued marking when assessing multi-step procedures resulted in more than one way to obtain two marks for items 31 and 32.

The results of the initial Rasch analysis, together with the theoretical calculus construct, informed the collapse of marks and mark categories for 14 of the 16 items. It was not necessary to collapse the two categories for item 30 or the three categories for item 33, as they were not disordered.

5.3 REVISED DATA COLLECTION INSTRUMENT

The tailored Rasch analysis implied that the analysis had to be repeated for the recoded data. The original raw data for all items except items 3, 13, 14, 15, 30 and 33 were recoded according to the summary of the collapsed marks and mark categories for items shared in chapter 4 (Table 4.8). The recoded Excel data were checked and imported into SPSS version 25 and descriptive statistics were produced. Thereafter, the SPSS data file was imported into Winsteps 3.93.1 and the Rasch analysis was repeated.

Table 5.1 presents an overview of the initial and revised data collection instrument that was used for the second tailored Rasch analysis. According to chapter 4, item 30 was moved from the C-P- category into the C-P+ category, since analysis of student responses indicated that more students prefered a procedural approach above a conceptual approach.

Table 5.1 shows that the number of items did not change from the original to the revised instrument. The number in brackets is the percentage of item marks per category, and the percentage of item marks per question type, respectively. The percentage mark allocation per category changed marginally: the percentage marks for C-P- and C+P- items increased, and those for C-P+ and C+P+ items decreased. It was not anticipated that the difficulty per category would change, since the change in percentage mark allocation was not significant.

Table 5.1: Overview of the initial and revised data collection instrument

Data collection instrument				
	Item type	**No. of items**	**Marks (before collapse)**	**Marks (after collapse)**
Category	C-P-	9	17 (22.9%)	11 (24.4%)
	C+P-	8	16 (21.6%)	10 (22.2%)
	C-P+	8	16 (21.6%)	9 (20.1%)
	C+P+	8	25 (33.8%)	15 (33.3%)
Item types	MCQ	17	30 (40.5%)	17 (37.8%)
	Open ended	16	44 (59.5%)	28 (62.2%)
Topics	Functions	9	21	13
	Differentiation	13	32	18
	Integration	11	21	14
Total		33	74	45

Table 5.1 reveals that the percentage marks per question type decreased for multiple-choice questions and increased for open-ended questions. It is not significant to report on the percentage for item marks per topic, since the topics build on one another.

The revised data collection instrument yielded a better fit of the data to the model than the original instrument. The results going forward were based on the collapsed data. The next section reveals the results of the Rasch analysis and the CFA.

5.4 RASCH ANALYSIS

5.4.1 Introduction

The first part of the results will focus on descriptive statistics and the second part will report on the general functioning of the instrument. The bigger picture relates to investigating the global fit statistics, summary statistics, most notably the reliability and separation indices, dimensionality and the item map. The third part of the results will focus on the individual fit of items and persons to the Rasch model, the identification of anomalies, and the evaluation of empirical item-category measures and category functioning. Finally, CFA between the four knowledge categories' results will be interpreted in relation to Rasch analysis results, in particular to the results of the item-person map.

5.4.2 Descriptive statistics

The descriptive statistics describe the results for sample items and persons.

5.4.2.1 Item statistics

Table 5.2 shows the descriptive statistics for the test, per item category and per type of question.

Table 5.2: Descriptive statistics for the test

Descriptive statistics for the test, per item category and type of question					
	Mean %	**Minimum**	**Maximum**	**SD**	**Valid N**
Test	54.98%	22.22%	86.67%	13.54%	192
Category					
C-P-	54.98%	18.18%	90.91%	17.53%	192
C-P+	62.41%	11.11%	100.00%	19.28%	192
C+P-	50.13%	0.00%	90.00%	18.36%	192
C+P+	53.21%	0.00%	100.00%	18.62%	192
Type of question					
Multiple-choice items	52.88%	11.76%	100.00%	14.93%	190
Open-ended items	56.71%	11.11%	92.86%	16.15%	187

The descriptive statistics in Table 5.2 show that the mean test score is 54.98%. The standard deviation (SD) provides information on the spread of the data, since 68.2% of the data lie within one SD from the mean of the test (Field, 2018). The SD is 13.54%, therefore the data are located relatively close to the mean. This could be expected, since the first-year engineering students who wrote the test form a homogeneous group.

The researcher expected C-P- items to be the easiest, followed by C-P+, C+P- and then C+P+ items; therefore, the expectation was that the easiest items would lie mostly in category C-P- and the most difficult items mostly in category C+P+. Furthermore, it was anticipated that C-P- items would have the highest average and C+P+ items the lowest average, for the reasons stated above.

The highest mean percentage was for C-P+ items at 62.41%. These items included no, one or two conceptual steps, but three or more non-repeated procedural steps. On average, students performed slightly better in C-P- items (54.98%) compared to C+P+ items (53.21%) and C+P- items (50.13%), therefore the test indicated a lower mean for items with three or more conceptual steps. The SD of the four categories was within the expected range. The analysis indicated that students performed better in the open-ended items (56.71%) than in the multiple-choice items (52.88%).

Table 5.3 contains the descriptive statistics per item, ordered from difficult to easy. The item statistics revealed that item difficulty was not necessarily linked to the number of procedural and/or conceptual steps needed to do the mathematical calculation. The frequencies per item options are given in Appendix 12.

Items 21, 6 and 1 (in order) were the most difficult items; students scored mean percentages below 30% (Table 5.3). It was expected that the three most difficult items would lie in knowledge class C+P+, since the solving of these items required several conceptual and procedural steps. The most difficult item was an open-ended item labelled C+P+; however, items 6 and 1 were multiple-choice items in the C-P- class. It was not expected that items in the C-P- class would be experienced as difficult, since fewer steps were required to do the mathematics.

According to the descriptive statistics, the easiest items were items 5, 25 and 19 (in order) and the mean percentages were above 80%. It was expected that the easiest items would be in knowledge class C-P-, but the easiest item, item 5, was a multiple-choice question labelled C-P+. Item 25 and item 19 were open-ended items labelled respectively C+P+ and C-P-.

Table 5.3: Descriptive statistics per item - ordered from difficult to easy

Descriptive statistics per item			
Item	**Category**	**Valid n**	**Mean %**
21	C+P+	178	10.67%
6	C-P-	190	12.63%
1	C-P-	188	26.06%
2	C-P+	190	29.47%
10	C+P-	190	30.53%
33	C+P-	175	35.14%
31	C+P+	171	37.43%
4	C+P-	190	40.00%
24	C-P-	178	43.26%
12	C+P+	190	46.32%
13	C+P-	190	48.42%
17	C-P+	187	49.20%
15	C-P-	190	51.05%
9	C+P-	190	53.16%
23	C-P-	179	54.19%
20	C+P+	183	56.28%
28	C-P-	174	56.90%
30	C-P+	174	56.90%
26	C+P+	173	56.94%
11	C+P-	189	60.32%
29	C-P+	176	65.91%
7	C-P+	189	66.14%
8	C+P-	190	66.32%
22	C-P+	183	66.67%
27	C+P+	183	68.31%
18	C+P-	182	68.68%
14	C-P-	190	76.84%
32	C+P+	176	78.69%
16	C-P+	190	78.95%
3	C-P-	189	79.89%
19	C-P-	182	80.49%
25	C+P+	182	81.32%
5	C-P+	190	85.79%

Furthermore, item 11 (C6P0) and item 12 (C6P4) included various related concepts and were expected to be difficult items, however, the mean percentage for both items were close to the test mean percentage.

5.4.2.2 Person statistics

Person statistics include analysis according to subsamples of respondents, e.g. males and females, different races and engineering study fields (Appendix 13). The researcher analysed the data for differential item functioning (DIF). DIF may indicate potential item bias, since it investigates the loss of item invariance across gender, race and study field subsamples (Appendix 14). DIF is only significant if people from different groups with the same underlying ability have a different probability of answering an item correctly. This occurrence might suggest item bias for a particular subsample.

The person statistics and DIF indicated that item 6 showed potential item bias and was significantly more difficult for African students (average measure = 3.44) than for white students (average measure = 1.67). The reason for the potential bias for this item is not evident to the researcher. No significant item bias was found for the subsamples of gender or engineering study field. Items were considered invariant across subsamples as per Rasch model requirements, since all items, except item 6, showed no significant DIF for groupings of participants.

5.4.3 General functioning of the instrument

5.4.3.1 Global fit statistics

In traditional statistical modelling, models are structured to fit the data. When the model does not fit the data, the model is modified or an alternative model is devised. RMT requires that the data fit the model, not vice versa, as found in traditional statistical modelling. Rasch uses the principles of measurement and the model is predetermined, following a Guttmann format where item difficulty and person ability align (Bond & Fox, 2015; Linacre, 2016; Wright & Stone, 1979). Measurement is defined as equally ordered, linear, additive and meaningful estimates of the latent trait (Engelhard Jr, 2013). Model fit statics are conducted to estimate the degree to which the data fit the Rasch model. The global fit statistics give an indication of the fit of the data to the model; however, the general model fit is less important than item fit.

In the study, the log-likelihood chi-squared was 7960.39, with approximately 7955 degrees of freedom, $p = .4809$. The probability is much greater than the required $p > 0.05$, therefore a significant difference between the model and the data was not established and it can be deduced that the data fit the Rasch model (Appendix 7).

5.4.3.2 Summary statistics

Both items and persons are on a linear continuum of difficulty and ability and are aligned after application of the Rasch model (Wilson, 2005). The separation indices indicate whether the items discriminate between different levels of person proficiency. Ideally, the difficulty of items should cover a wide range and the persons should present a wide range of abilities so that the construct is covered comprehensively.Table 5.4 indicates the summary statistics for persons and items.

Table 5.4 Summary statistics for persons and items

Summary statistics for persons and items							
Summary of 192 measured persons				**Summary of 33 measured items**			
Separation 1.60		Reliability 0.72		Separation 6.21		Reliability 0.97	
Infit		Outfit		Infit		Outfit	
MNSQ 1.00	ZSTD 0.00	MNSQ 1.01	ZSTD 0.00	MNSQ 1.00	ZSTD 0.00	MNSQ 1.00	ZSTD 0.00

The summary statistics show that the person separation index is low, 1.60 (< 2.00), and the person reliability is also lower than desirable, 0.72 (< .80). These results can be expected, since the participants in this research study have approximately the same aptitude and range of abilities within the group, therefore the range of abilities measured is restricted.

The item separation index for this study is large at 6.21 (> 2.00), therefore items can discriminate between three or more person ability groups. The high item reliability of 0.97 (>.80) indicates that the sample size was large enough for the number of items, and the items measured a wide range of difficulties. High item reliability does not indicate that the test was of high quality.

An overall fit of person and item MNSQ close to 1.00 and ZSTD close to 0.00 support the notion of unidimensionality (Peoples *et al.*, 2014). Table 5.4 shows that the summary person infit MNSQ and outfit MNSQ were 1.00 and 1.01 respectively, and the standardised summary person infit and outfit were both 0. The summary item infit MNSQ and outfit MNSQ were both 1.00, and the standardised summary item infit and outfit were both 0.00. Mean squares and standardised fit statistics support the notion of unidimensionality.

In general, the global and summary statistics confirm that the data demonstrate a very good fit to the Rasch model.

5.4.3.3 Dimensionality, item polarity and principal component analysis

The Rasch model assumes a single latent trait and that the instrument is assessing a unidimensional construct (Andrich, 1988; Bond & Fox, 2015). Dimensionality is a complicated aspect of measurement that investigates whether all items share the same dimension. Besides the person and item mean squares and standardised fit statistics discussed in the previous section, evidence of unidimensionality could also be supported by investigating item polarity and principal component analysis (PCA). Table 5.5 shows all items ordered by item measure and point biserial correlations (PTMA). Item measure indicates item difficulty: the higher the score, the higer the difficulty of the item. Item measure will again be referred to in the discussion on the person-item map (5.4.3.4). PTMA gives evidence of whether items share the same dimension. The table reveals that all items had a positive PTMA score (PTMA, IFILE Appendix 8). It could therefore be established that all the items were positively correlated to the underlying construct of calculus.

Table 5.5: Items ordered by item measure and PTMA

Items ordered by item measure and PTMA			
Item	**Category**	**Measure**	**PTMA**
21	**C+P+**	**2.52**	**0.36**
6	**C-P-**	**2.32**	**0.23**
1	**C-P-**	**1.36**	**0.26**
2	C-P+	1.18	0.24
10	C+P-	1.12	0.16
33	C+P-	1.02	0.46
31	C+P+	0.94	0.45
4	C+P-	0.66	0.04
24	C-P-	0.53	0.33
12	C+P+	0.38	0.33
13	C+P-	0.28	0.35
17	C-P+	0.25	0.40
15	C-P-	0.16	0.21
9	C+P-	0.07	0.31
23	C-P-	0.06	0.46
26	C+P+	-0.04	0.45
30	C-P+	-0.06	0.38
28	C-P-	-0.07	0.44
20	C+P+	-0.1	0.30
11	C+P-	-0.26	0.24
29	C-P+	-0.49	0.33
7	C-P+	-0.53	0.19
8	C+P-	-0.54	0.25
22	C-P+	-0.73	0.46
27	C+P+	-0.73	0.51
18	C+P-	-0.76	0.43
25	C+P+	-0.97	0.26
32	C+P+	-0.97	0.45
14	C-P-	-1.11	0.35
19	C-P-	-1.17	0.41
16	**C-P+**	**-1.24**	**0.29**
3	**C-P-**	**-1.31**	**0.20**
5	**C-P+**	**-1.76**	**0.24**

Table 5.6 indicates the nearly identical observed raw unexplained variance (67.6%) and expected raw unexplained variance (67.8%), and the acceptable Eigenvalue score of 33% for the total unexplained raw variance. Furthermore, Table 5.6 shows that the first contrast of the PCA for this instrument had an Eigen value of 2.31

(< 3.00) and unexplained variance of 4.7% (< 5%), safely within the criteria limits suggested by Linacre (2017). The first contrast Eigenvalue was above 2.0, therefore the researcher decided to investigate the first contrast further according to the more rigorous guidelines suggested by Combrinck (2018).

Table 5.6: Standardised residual variance in Eigenvalue units

Standardised residual variance in Eigenvalue units			
	Eigenvalue	**Observed**	**Expected**
Total raw variance in observations	48.82	100.0%	100.0%
Raw variance explained by measures	15.82	32.4%	32.2%
Raw variance explained by persons	4.04	8.3%	8.2%
Raw variance explained by items	11.78	24.1%	24.0%
Raw unexplained variance (total)	**33.0%**	**67.6%**	**67.8%**
Unexplained variance in 1st contrast	**2.31**	**4.7%**	**7.0%**
Unexplained variance in 2nd contrast	1.94	4.0%	5.9%
Unexplained variance in 3rd contrast	1.79	3.7%	5.4%
Unexplained variance in 4th contrast	1.75	3.6%	5.3%
Unexplained variance in 5th contrast	1.54	3.1%	4.7%

Table 5.7 expands further on the first contrast from the PCA. The researcher considered the minor threat to dimensionality for items with loading greater than 0.4 and less than - 0.4 separately. The first contrast identified five items in Table 5.7 that possibly formed separate components or constructs that needed to be investigated. The table showed that items 6 and 13 had negative standardised residual loadings less than - 0.4 and items 23, 26 and 28 had positive standardised residual loadings above 0.4. The researcher examined the items to see if they displayed a common trend that made them different from all other items, or whether there was any clear reason why these items should be clustered together. Items 23, 26 and 28 did not have the same mark allocation or item label, and no common theme was found in the questions. The three items did not represent a different, separate construct. Items 6 and 13 had the same mark allocation and the embedded theme of the questions was the linking of the relationship between a function and the gradient function of the function. However, the direction of the theme of the relationship was different in the two items, therefore the items could not be clustered together to form an independent construct. In was concluded that the occurrence of

the standardised residual loadings above 0.4 and below - 0.4 may have been due to similar rating scales, a similar topic or technique, or could have been random.

Table 5.7: Contrast 1 from PCA

Contrast 1 from PCA				
Item	**Residual loadings**	**Marks**	**Item label**	**Theme of question**
23	0.58	2	C-P-	Find the first derivative
26	0.48	2	C+P+	Determine the point of inflection (incl. 2nd derivative)
28	0.44	1	C-P-	Evaluate the integral
6	-0.46	1	C-P-	Given the graph of *f*, draw the graph of *f''* ($f \Rightarrow f'$)
13	-0.42	1	C+P-	Given graph of *f* and $g(x) = \int_0^x f(x)dx \ (0 \leq x \leq a)$, describe g $(f' \Rightarrow f)$

Although unidimensionality is difficult to establish, mean squares and ZSTD, positive point biserial values and relatively low Eigenvalue for the first contrast of the PCA offered evidence that the instrument was sufficiently unidimensional.

5.4.3.4 Person-item map

The person-item map or Wright map (Bond & Fox, 2015; Dunne *et al.*, 2012; Stols *et al.*, 2015; Wright & Stone, 1979) provides a summary of person proficiency estimated in relation to item difficulty, and indicates difficulty and construct validity of the data collection instrument as a whole. The person-item map locates items, response categories and the persons in one picture. The mean of the items are located at 0 and the SD is 1 (Boone, 2016). The item difficulties and person proficiencies are estimated and located on the same linear scale. The logit scale tends to range between -5 and 5.

Figure 5.1 is the person-item map for this study. The item mean is slightly lower than the person mean, which indicates that the test was somewhat too easy for the participants. The average person has a 50% chance to answer items that lie above the items of average difficulty correctly. The spread of items along the item difficulty variable (the right of the y-axis) has some gaps, but most of the items are targeted and aligned with the persons. The instrument is well targeted and tests the calculus construct well, although there are gaps that indicate less well-defined or less tested

```
MEASURE     PERSON - MAP - ITEM

                <more>|<rare>
   3                  +
                      |
                      |
                      |
                      |  Q21  C+P+
                      |  Q6  C-P-
                 .    |
                      |
   2             .    +T
                 .    |
                      |
                 .  T|
                 .    |
                 .    |  Q1  C-P-
                      |
              .##    |  Q10  C+P-    Q2  C-P+
   1          ###    +S Q31  C+P+    Q33  C+P-
               ## S|
             .###    |
            .####    |  Q4  C+P-
             ####    |  Q24  C-P-
           .#####    |  Q12  C+P+
             .### M|  Q13  C+P-    Q17  C-P+
        #########    |  Q15  C-P-    Q9  C+P-
   0         .###    +M Q23  C-P-    Q26  C+P+
             .###    |  Q20  C+P+    Q28  C-P-    Q30  C-P+
              .##    |  Q11  C+P-
               ##    |
             .### S|  Q29  C-P+    Q7  C-P+    Q8  C+P-
                #    |
              .##    |  Q18  C+P-    Q22  C-P+    Q27  C+P+
               .#    |
  -1             .    +S Q25  C+P+    Q32  C+P+
                 .    |  Q14  C-P-    Q19  C-P-
                 .  T|  Q16  C-P+    Q3  C-P-
                 .    |
               .#    |
                      |
                      |  Q5  C-P+
                      |
  -2                  +T
                <less>|<freq>
```

Key:

```
EACH "#" IS 3: EACH "." IS 1 TO 2
C-P-: C-P- items
C-P+: C-P+ items
C+P-: C+P- items
C+P+: C+P+ items
```

Figure 5.1: Person-item map: person proficiency vs item difficulty

regions of the construct. The spread of persons along the left of the y-axis suggests a bell curve closely grouped together, since the population is homogeneous. There is an outlier (. #) to the left of the spread of the normal curve, indicating four or five persons with low proficiency.

As noted previously when the descriptive statistics per item (Table 5.3) were discussed, the item measure hierarchy according to the category labels was not as anticipated. This was also confirmed in Table 5.5, where the difficulty or measure per item was shared. The person-item map confirms that items in the four knowledge categories are not grouped together according to knowledge class label, but are scattered over the map. The spread of C-P+ items appears closer to the bottom of the map and seems to entail the easier items. On the other hand, the spread of C+P+ items is closer to the top of the map and these appear to be the more difficult items. The C+P- items appear to cluster closer to the mean item than the C-P- items.

The most difficult items, in order of difficulty, were items 21, 6 and 1. This is confirmed by the descriptive statistics (Table 5.3) and the Rasch analysis (Table 5.5, Figure 5.1). Items 6 and 1 are classified as C-P- items and it was not expected that items in this class would be among the most difficult items. It was indicated in chapter 4 that item 6 was answered correctly by only 12.6% of the students. Option D was chosen by 45.3% of the students. These students calculated the incorrect gradients of the lines and did not realise that f is not differentiable at $x = -2$ and at $x = 2$. $f'(-2)$ and $f'(2)$ are not defined, since the graph of f makes a sharp point or vertex at $x = -2$ and at $x = 2$. Option A was chosen by 35.4% of the students. These students calculated the correct gradients of the lines, but did not realise that f is not differentiable at $x = -2$ and at $x = 2$, therefore that $f'(-2)$ and $f'(2)$ are not defined. Item 1 required students to show the link between the equation of a polynomial and the graph of the function. The question was successfully answered by only 26.1% students, since students could not make the theoretical connection between the polynomial and the graph.

The descriptive statistics (Table 5.3) indicated that the highest mean was for items 5, 25 and 19 (in order); however the Rasch analysis indicated that items 5, 3 and 16 were the easiest items (Table 5.5, Figure 5.1). Item 5, the easiest item, is classified in the C-P+ class and 85.8% of the students successfully answered the multiple-choice question. Students were asked to find the domain of a function, but could substitute possible item options to obtain the correct answer. It could be suggested that the item be changed to an open-ended question.

5.4.4 Specific item and person fit

Infit statistics relate to the overall pattern of item responses and suggest whether persons and items conform to the expected model of higher ability, which implies answering more difficult items correctly. Lower ability makes correct answers less likely. Outfit statistics identify item and person outliers.

5.4.4.1 Item fit and item polarity

Item fit to the Rasch model is considered to be an important indicator of distortion that may be caused by items (Linacre, 2017). Item misfits are identified by checking that items cooperate to measure. Table 5.8 (IFILE, Appendix 8) indicated that there were no items with infit MNSQ and outfit MNSQ out of the ideal range between 0.50 and 1.50. Item 4 was the only item that showed potential misfit. The infit MNSQ and outfit MNSQ for item 4 are 1.18 and 1.26 respectively; however, the standardised ZSTD infit is 3.42 and the standardised ZSTD outfit is 3.68. A standardised infit and outfit score from -2.00 to 2.00 is considered acceptable. Scores outside this range may indicate an item that fails to discriminate and therefore does not contribute to measurement, but rather causes noise in the data. In general, the items fit the model and there are no misbehaving items. Item polarity was established, since PTMA indicated that all items were positively correlated to the underlying calculus construct (Table 5.8).

5.4.4.2 Person fit

The ideal range for person fit, similar to item fit, is between 0.50 and 1.50. According to the PFILE (Appendix 9) only one person has an MNSQ value above 2.00. Person 78 has a MNSQ outfit value of 2.25 and can be seen as an outlier. Removing outliers is tempting, as it improves model functioning; however, outliers also give useful information. It was decided to keep person 78 as part of the analysis, since there was only one outlier and it is common practice to keep all persons unless there is a very good reason to remove them.

Table 5.8: Item infit, outfit and and PTMA

Item measure						
Item	**Label**	**Infit MNSQ**	**Infit ZSTD**	**Outfit MNSQ**	**Outfit ZSTD**	**PTMA**
1	C-P-	1.01	0.17	1.05	0.50	0.26
2	C-P+	1.04	0.57	1.05	0.50	0.24
3	C-P-	1.03	0.34	1.09	0.65	0.20
4	**C+P-**	**1.18**	**3.42**	**1.26**	**3.68**	**0.04**
5	C-P+	1.00	0.04	0.96	-0.14	0.24
6	C-P-	0.99	-0.02	1.05	0.29	0.23
7	C-P+	1.08	1.29	1.09	1.09	0.19
8	C+P-	1.04	0.63	1.03	0.35	0.25
9	C+P-	1.01	0.15	0.99	-0.13	0.31
10	C+P-	1.09	1.22	1.16	1.62	0.16
11	C+P-	1.04	0.78	1.12	1.77	0.24
12	C+P+	0.99	-0.17	0.98	-0.42	0.33
13	C+P-	0.97	-0.63	0.97	-0.48	0.35
14	C-P-	0.97	-0.33	0.87	-1.03	0.35
15	C-P-	1.08	1.88	1.08	1.46	0.21
16	C-P+	0.98	-0.13	0.98	-0.07	0.29
17	C-P+	0.94	-1.33	0.92	-1.44	0.40
18	C+P-	0.96	-0.45	0.99	-0.05	0.43
19	C-P-	0.97	-0.22	0.94	-0.33	0.41
20	C+P+	1.10	1.16	1.10	1.14	0.30
21	C+P+	0.96	-0.23	0.91	-0.43	0.36
22	C-P+	0.92	-0.92	0.94	-0.63	0.46
23	C-P-	0.96	-0.51	0.97	-0.31	0.46
24	C-P-	0.99	-0.18	0.99	-0.08	0.33
25	C+P+	1.20	1.54	1.27	1.32	0.26
26	C+P+	0.96	-0.45	0.95	-0.55	0.45
27	C+P+	0.87	-1.46	0.90	-1.06	0.51
28	C-P-	0.91	-1.77	0.89	-1.78	0.44
29	C-P+	0.98	-0.28	0.95	-0.52	0.33
30	C-P+	0.96	-0.84	0.92	-1.20	0.38
31	C+P+	0.94	-0.59	0.94	-0.62	0.45
32	C+P+	0.94	-0.47	0.87	-0.83	0.45
33	C+P-	0.94	-0.64	0.92	-0.82	0.46

5.4.4.3 Item categories

The analysis of item categories includes the investigation of whether all categories for all items are aligned in the same direction, whether categories show clearly advancing average person measures, and whether categories are ordered. Clearly advancing average person measures imply that correct answers and higher category values correspond to higher average person measures, and vice versa (Linacre, 2018b).

According to Linacre (2017) at least 10 observations per category are required, average measures should increase with each follow-up category in turn, and infit and outfit MNSQ should be close to 1.00 for each category. The correct option should have the highest PTMA correlation and average measure; the least correct option should have a negative PTMA correlation and the lowest average measure.

Item 21 was earlier identified as the most difficult item of the test (Figure 5.1, Table 5.3 and Table 5.5). The item was identified as problematic when the researcher used the category options file (Appendix 10) to investigate the item categories. An abstract of Appendix 10 is given in Table 5.9.

Table 5.9: Item 21 category options (from Category options, Appendix 10)

Item 21 Category information						
Category	**UNWTD**	**UNWTD%**	**AVGE MEAS**	**INFIT MNSQ**	**OUTFIT MNSQ**	**PTMA**
0	143	80.34	0.12	0.92	0.94	-0.41
1	32	17.98	0.90	0.74	0.65	0.41
2	**3**	**1.69**	**0.47**	**1.92**	**1.68**	**0.04**

It can be seen in Table 5.9 that item 21 results in only three people being placed in category two (UNWTD < 10). Furthermore, the three persons have a lower average person measure or ability (0.47) than students in category one (0.9), therefore the average person measures for category two do not advance from category one to category two. The infit and outfit MNSQ for category two are 1.92 and 1.68 respectively, therefore the scores are > 1.50. It can be noted that the correct option does not have the highest PTMA correlation. In addition, it is clear from Figure 5.2 that the average person measure categories for item 21 are disordered.

```
-4   -3    -2    -1     0     1     2     3     4     5
|-----+-----+-----+-----+-----+-----+-----+-----+-----|  NUM   ITEM
|                   0 2 1                         |  21  Q21 C+P+
|                   0  1                          |   6  Q6 C-P-
|                   0 1                           |   1  Q1 C-P-
|                   0 1                           |   2  Q2 C-P+
|                   0 1                           |  10  Q10 C+P-
|                  0  1 2                         |  33  Q33 C+P-
|                  0 1   2                        |  31  Q31 C+P+
|                   01                            |   4  Q4 C+P-
|                  0  1                           |  12  Q12 C+P+
|                  0  1                           |  13  Q13 C+P-
|                  0  1                           |  17  Q17 C-P+
|                   01                            |  15  Q15 C-P-
|                  0  1                           |   9  Q9 C+P-
|                0   1 2                          |  23  Q23 C-P-
|                 0  1 2                          |  26  Q26 C+P+
|                  0  1                           |  30  Q30 C-P+
|                  0  1                           |  28  Q28 C-P-
|                 0 1 2                           |  20  Q20 C+P+
|                  0 1                            |  11  Q11 C+P-
|                  0  1                           |  29  Q29 C-P+
|                  0 1                            |   7  Q7 C-P+
|                  0 1                            |   8  Q8 C+P-
|                0 1   2                          |  22  Q22 C-P+
|               0  1   2                          |  27  Q27 C+P+
|                0    1                           |  24  Q24 C-P-
|                 01   2                          |  18  Q18 C+P-
|               0  1  2                           |  32  Q32 C+P+
|                 0 12                            |  25  Q25 C+P+
|                 0  1                            |  14  Q14 C-P-
|                01   2                           |  19  Q19 C-P-
|                 0  1                            |  16  Q16 C-P+
|                  0 1                            |   3  Q3 C-P-
|                 0  1                            |   5  Q5 C-P+
|-----+-----+-----+-----+-----+-----+-----+-----+-----|  NUM   ITEM
-4   -3    -2    -1     0     1     2     3     4     5
```

Figure 5.2: Average person measure categories per item

Figure 5.3 shows the category probability curves (Appendix 11) for item 21. The categories are increasing as expected, although the second and third categories remain disordered.

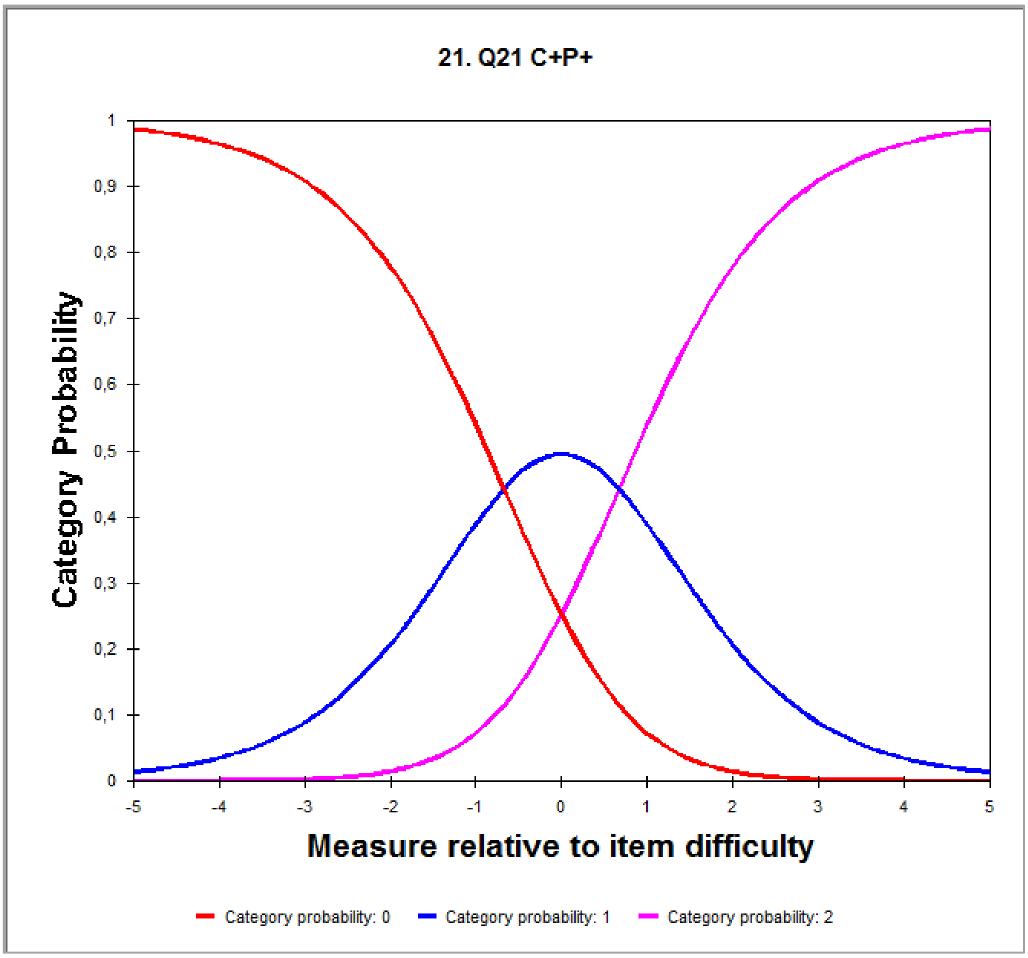


Figure 5.3: Item 21 category probability curves

The researcher investigated the mathematical content analysis and student solutions and decided to retain the three categories to inform the study. Student solutions indicated that the three students in category two who correctly answered the item were the only students who used graphical methods to solve the inequality. Furthermore, these students with lower ability performed better in this item than students with higher ability. If categories two and three were collapsed to a single category, the average person measure would make sense and categories would not appear disordered. However, keeping the three categories informed the analysis and made sense against the background of the mathematics and the study. The interpretation of the analysis could be seen as diagnostic, considering the teaching of inequalities, since the results indicate that the teaching of inequalities should preferably focus on graphical methods rather than algebraic methods.

5.4.5 Confirmatory factor analysis

The CFA resulted in a chi-square goodness of fit *p*-value of 0.000, which indicated that there was a significant difference between the model and the data, therefore the CFA model and the data did not show a good fit. A non-significant chi-squared probability result of $p > 0.05$ would have provided evidence that the data fit the model. Furthermore, the log-likelihood chi-squared was high at 780.942 with approximately 489 degrees of freedom.

Two SEM model fit statistics indicate that the CFA model and the data did not show a good fit. The analysis indicated that the noncentrality parameter was 291.942, but values close to 0.95 or 0.90 reflect a good fit and a value of 0 indicates a perfect fit (Schumacker, 2010). Furthermore, the root mean square error of approximation was 0.056, but according to Schumacker (2010) a value less than 0.05 indicates a good fit.

The CFA produced low factor loadings, therefore the expectation of the assumption that the grouped items are predicted by the latent factor is not met. According to guidelines by Arbuckle (2017b) and Schumacker (2010), the expected factor loadings had to be above 0.70. Figure 5.4 indicates that the regression weights between the latent factors and their corresponding items were between -.06 and 0.51, well below 0.70. Furthermore, the CFA indicates overly large correlations between the latent factors, i.e. knowledge classes. It was anticipated that the expected correlations between the four identified latent factors would be above 0.8 and less than 1. Figure 5.4 reveals that four of the six correlations were more than 1. This implies over-correlation between the latent factors, suggesting that the factors cannot be separated into the four knowledge classes defined in the development of the test.

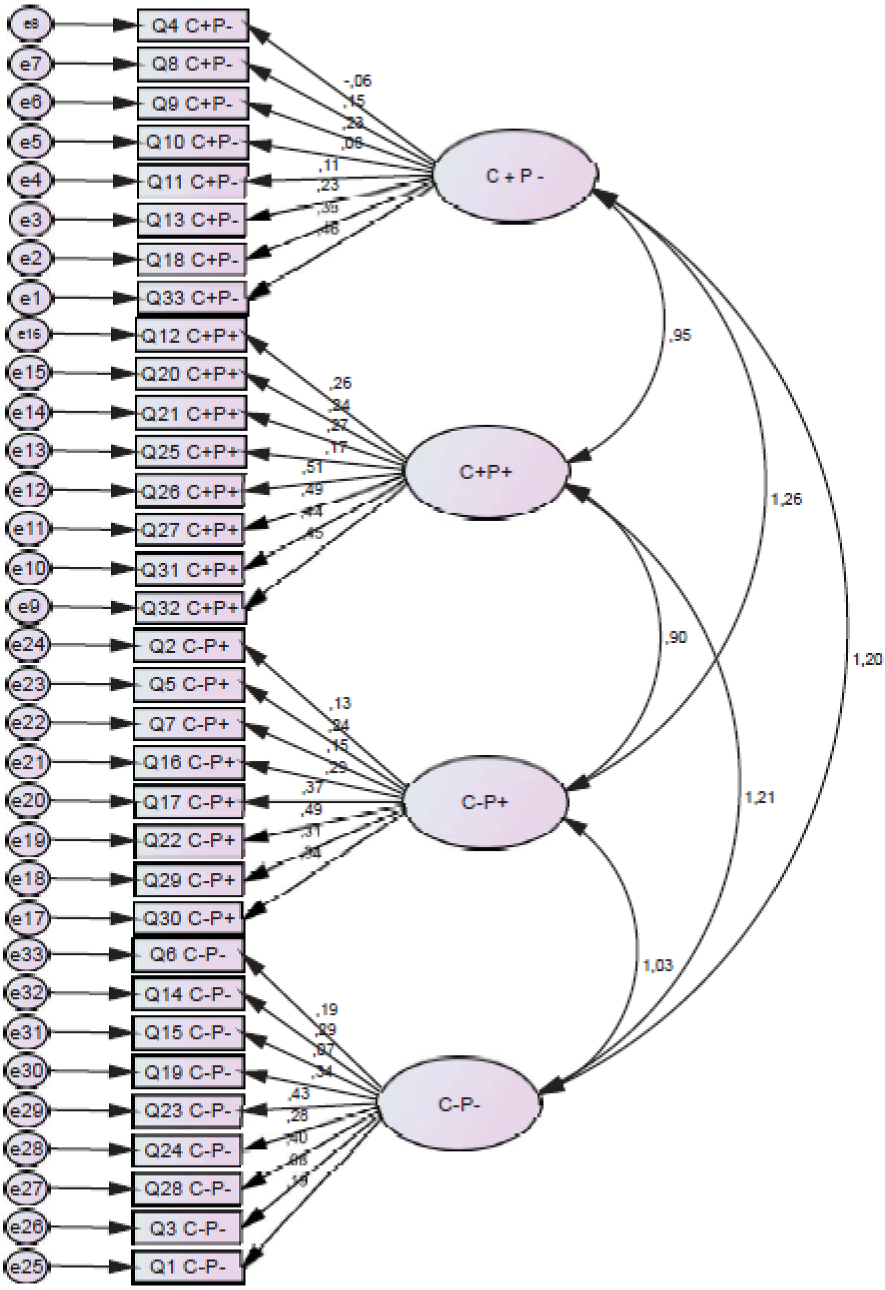


Figure 5.4: Outcome of the CFA for this study

The result of over-correlation between latent factors aligns with Rasch analysis that gave evidence of a single latent trait and a unidimensional calculus construct when mean squares and standardised fit statistics, item polarity and PCA were investigated. Furthermore, the integrated location of knowledge class labels in the person-item map (Figure 5.1) illustrates that the position of items appears to be independent of the defined knowledge classes.

The study by Schneider *et al.* (2011) used SEM to model procedural and conceptual knowledge, and procedural flexibility as interrelated, but separate, latent factors. The study by Schneider *et al.* (2011) is differerent from this research study, since the items used for this study clearly separated the construct of procedural and conceptual knowledge. This could be seen in the middle school sample items that formed part of the appendix to the paper. In contrast to the study by Schneider *et al.* (2011), the items used in the content analysis for this study indicated integration of conceptual and procedural knowledge. Furthermore, the CFA correlations above 1 confirm multicollinearity due to latent traits not being separate constructs.

5.5 VALIDITY AND RELIABILITY

Stols *et al.* (2015) stated that Rasch analysis informs the researcher of the extent to which the data fit the model, identifies and explains misfits and gives information on targeting and the reliability of the instrument. The targeting is good and the reliability of the instrument is high. The development of the instrument was theoretically well grounded and validated through various processes of triangulation. The instrument had a slightly lower than desirable person separation and person reliability due to the homogeneous sample. The high item reliability (0.97) indicates that the sample size was large enough for the number of items, and the items measured a wide range of difficulties. The non-significance of the χ^2 value indicated that the data had a good fit to the Rasch model.

Item 4 was identified as a potential misfitting item, but although the item was noisy and failed to contribute to measurement, it did not distort measurement. In general, the items fit the model and there were no misfits. Item polarity was established and all items correlated positively to the underlying calculus construct.

The analysis identified one possible person misfit, but it was decided to keep the anomaly, since there was no good reason to remove the outlier.

Item 21 did not show clear advancing average measures for the categories that resulted in disordered categories. However, the mathematical construct informed the analysis and provided suitable reasons to keep the original categories.

5.6 SUMMARY

The initial Rasch analysis indicated that some of the categories appeared to be disordered. The disordered categories were investigated and the analysis resulted in the collapse of marks and mark categories for identified items. The reasons for the collapse of categories were discussed in detail. A revised data collection instrument was used to perform the second, tailored Rasch analysis. The analysis revealed a better fit of the data to the model, since items indicated improved ordered categories, mostly advancing average person measures and improved smooth category probability curves.

The global fit and summary statistics indicated that the data had a good fit to the Rasch model. The chi-squared test resulted in a $p > 0.05$. The large item separation index showed that items could discriminate between three or more person ability groups and the high item reliability indicated that the sample size was large enough for the number of items. The relatively low person separation index was expected, since the population of engineering students is relatively homogeneous. The positive point biserial values indicated that items were positively correlated to the underlying calculus construct and item polarity was established. Unidimensionality was confirmed with the overall fit of person and item mean squares and standardised fit statistics, item polarity and a relatively low Eigenvalue for the first contrast of the PCA.

The person-item map revealed that the item mean was slightly lower than the person mean, therefore, the test was somewhat too easy for the participants. The spread of items disclosed that most of the items were well targeted and tested the calculus construct sufficiently. The closely grouped spread of persons confirmed the homogeneous population of participants.

The infit and outfit statistics for individual items yielded no misfit items or persons. Item 21 was identified as the only item with no clearly advancing average person measures and disordered categories; however, the mathematical analysis explained and justified the results.

The pupose of this chapter was to measure and describe conceptual and procedural construct relations between calculus items. The researcher expected that items in the four knowledge classes would be clustered together and the easiest items would be C-P- items, followed by C-P+, C+P- and C+P+ items, the most difficult items. The findings did not support this view, since item measure difficulty according to knowledge class was not as anticipated. This was confirmed by:

- The mean per knowledge class was the highest for C-P+ items, followed by C-P- , C+P+ and the lowest for C+P- items. On average, students performed better in items with fewer conceptual and more procedural steps needed to do the mathematics (Table 5.2).
- Descriptive item statistics (Table 5.3) and item difficulty or measure (Table 5.5) revealed that item difficulty is not necessarily linked to the number of procedural and/or conceptual steps needed to do the mathematics, resulting in the knowledge classes defined for the study.
- The person-item map (Figure 5.1) confirmed that items in the four knowledge classes were not grouped together per knowledge class, but were scattered over the map. This endorsed the finding that item difficulty was not linked to the number of procedural and/or conceptual steps needed to do the mathematics.
- The Rasch analysis gave evidence of a single latent trait and a unidimensional calculus construct through mean squares and standardised fit statistics (Table 5.4), item polarity (Table 5.5) and PCA (Table 5.6).
- The CFA over-correlation between latent factors did not support separate, underlying latent traits of the defined knowledge classes as specified a priori, but rather a single unidimensional construct and connectedness of conceptual and procedural problem-solving categories (Figure 5.4).

In conclusion, the cognitive processes of procedural fluency and conceptual understanding appear to be intertwined and integrated, therefore procedural and conceptual knowledge constructs cannot be separated. The results of the quantitative Rasch analysis confirmed targeting and reliability of the instrument and endorsed that the data fit the model. The validity and reliability of the analysis and results were supported by the statistics reported in the current chapter.

6. CHAPTER 6: DISCUSSION OF FINDINGS AND RECOMMENDATIONS

This study reflect on the relationship between procedural and conceptual knowledge by analysing calculus items and their solutions. The research study was embedded in the postpositivist paradigm. Equal status was provided to the qualitative and quantitative approaches. The mixed research method design provided deeper and more comprehensive analysis and understanding of the research problem.

In the first part of the study 33 calculus items were selected from previous studies (Bergsten *et al.*, 2015; Bergsten *et al.*, 2017; Engelbrecht *et al.*, 2012) or developed by the researcher (Hechter, 2017). The solutions of these items represent the construct and sub-constructs of conceptual and procedural calculus knowledge (see chapter 4.2). The solution of each item was analysed and labelled according to the type and number of categories required to solve the problem (see chapter 3.3). Each labelled item was accordingly classified in one of four knowledge classes (see chapter 3.3): C-P-, C-P+, C+P- and C+P+, based on the notion of type and quality of knowledge (Star, 2005). The rationale behind this was to describe and classify each item according to the mathematical approach required to solve it.

The analysed items were included in a data collection instrument that was implemented in a single cross-sectional quantitative study. The test was administered to a cohort of first-year mathematics students. The collected data were investigated and analysed with Rasch measurement theory (RMT) to describe the overall pattern of the calculus construct, focussing on relations between person proficiency and item difficulty (see chapter 5.4). The results of the Rasch analysis and confirmatory factor analysis (CFA) were compared to investigate whether the classification of calculus items into four knowledge classes was supported (see chapter 5.4). The findings confirmed the validity and reliability of the construct and the instrument.

In this chapter the researcher focussed on four issues:

- How do the findings fit into and contribute to the current research literature?
- How do the findings contradict the current research literature?
- What are the unique contributions to the current research literature?
- What are the gaps in the current research literature?

6.1 INTRODUCTION

The literature describes different positions with respect to conceptual and procedural mathematical knowledge. The procedural-formalist and the cognitive-cultural paradigms (Ellis & Berry III, 2005; Sorensen, 2013) have opposing views about what mathematics is, since they represent different beliefs and understandings about what it is to know, understand, learn and teach mathematics. The concept-driven versus skills-orientated perspectives have led to the so-called "*math wars"* between researchers globally (Brown *et al.*, 2002; Hiebert *et al.*, 2003; Sowder, 2007; Star, 2005), including South Africa (Bergsten *et al.*, 2015; Bergsten *et al.*, 2017; Engelbrecht *et al.*, 2009; Engelbrecht *et al.*, 2012; Engelbrecht *et al.*, 2005). The concept-driven reform emphasises understanding mathematics to apply and solve contextual problems; alternatively, skills-orientated mathematics education specialists promote fluency in mathematical procedures, best learned through direct instruction, rote learning, drill and practice.

Kilpatrick *et al.* (2001) advocate that the strands of mathematical proficiency, namely conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition, should not be seen as independent of one another, but as interwoven and interconnected. Hiebert *et al.* (2003) claim that mathematical proficiency will improve when all five strands of mathematical proficiency are developed, and not only one strand is focussed on. Procedures and skills should not be learned in isolation, but should be connected to different methods and representations to develop the concept image and definition considering a certain concept. Students should organise mathematical knowledge into a conceptual schema that provides the basis for understanding, reasoning and problem-solving.

The concept-driven and skills-focussed argument, together with the view on the five intertwined strands of mathematical proficiency, emphasised the need to investigate and analyse procedural and conceptual knowledge needed to do calculus as part of a mathematics module in an extended curriculum engineering degree programme.

6.2 FINDINGS

The study posed and answered the main research question: ***What is the nature of conceptual and procedural knowledge in calculus?*** The researcher answered the main question by answering the sub-questions through highlighting the findings that emerged from the research study.

6.2.1 Sub-question 1: How can calculus items be analysed and classified with respect to conceptual and procedural knowledge?

Qualitative content analysis provided a way to analyse 33 calculus items solutions in terms of their procedural and conceptual nature. Evidence of the findings will be illustrated with examples and reflections on literature.

Finding 1: Item solutions drew on both procedural and conceptual knowledge

The analysis found that most of the item solutions require both procedural and conceptual steps. The finding is illustrated using items 18 and 20:

Item 18 appears to be a procedural item, but when the solution is analysed, it is clear that in order to solve the problem alternate conceptual and procedural steps are required.

Let $f(x) = e^{2x} - 1$ and $g(x) = \ln(x+1)$. Find $(f \circ g)(x)$ and simplify. Indicate all restrictions on x (if any).

$(f \circ g)(x)$	
$= e^{2(\ln(x+1))} - 1$	C3F
$= e^{\ln(x+1)^2} - 1$	P2F
$= x^2 + 2x + 1 - 1$	C2F
$= x^2 + 2x$	P1
$\boldsymbol{x} + 1 > 0$	C3F
$\boldsymbol{x} > -1$	

Results:

Label: C=3P=2

C_{3F}: interpretation composite functions

P_{2F}: log laws

C_{2F}: link inverse functions

P_1: symbolic and numerical calculations

C_{3F}: interpretation domain

The item category, based on the solution, is therefore C+P-.

The alternative conceptual and procedural steps are clearly visible in the analysis of the solution to the contextual **item 20:** If a ball is thrown upwards from the top of a building 25 metres high with an initial velocity of $20\ m/s$, then the height h above the ground t seconds later will be: $h(t) = -5t^2 + 20t + 25$. During what time interval will the ball be more than 40 m above the ground?

$-5t^2 + 20t + 25 > 40$ **C_4**

$-5t^2 + 20t - 15 > 0$ **P_1**

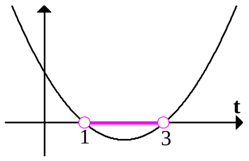
$-t^2 + 4t - 3 > 0$ **P_1**

$t^2 - 4t + 3 < 0$ **P_{2F}**

$(t-3)(t-1) < 0$ **P_{2F}**

Graph of parabola **C_1**

$t \in (1,3)$ **C_{3F}**



Results:

Label: C=3P=3

C_4: contextual applications (height ball)

P_1: symbolic and numerical calculations

P_{2F}: rules and inequalities

P_{2F}: factorisation

C_1: translations (graph)

C_{3F}: interpretation of quadratic inequality

The item category, based on the solution, is therefore C+P+.

According to the analysis, items 8, 9, 11, 13 and 14 involved only a conceptual approach and items 16, 23 and 24 required procedural steps; however, the uniqeness of the solving approach and analysis of the approach could be questioned, since it appears not to be absolute (Star & Stylianides, 2013). The research study provides evidence and examples for this statement that are explained further under the heading of findings 2 and 3.

Finding 2: Item solutions could follow more than one approach

The researcher anticipated different methods for items 1, 3, 29 and 30, but students provided additional, unexpected solution methods for items 21 and 25. This confirms the work done by Star and Stylianides (2013), which indicates that the nature of the approach used to solve mathematics is not absolute, since it may differ among students, educators or mathematics education experts. The different solution approaches could therefore possibly produce different item labels and knowledge classes for the same item. The finding will be supported by the evidence provided through items 21, 29 and 30.

Item 29 could be solved using different methods and producing two different labels, C=1P=3 and C=2P=3. The labels remain in the same knowledge class, C-P+.

Evaluate the integral: $\int_0^{\frac{3\pi}{2}} \sin\theta\, d\theta$.

Approach 1:

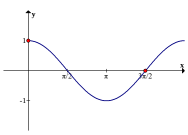
$\int_0^{\frac{3\pi}{2}} \sin\theta\, d\theta$

$= -\cos x\Big|_0^{\frac{3\pi}{2}}$ P_{2I}

$= -[\cos\frac{3\pi}{2} - \cos 0]$ P_{2I}

$= -[0-1]$ C_1

$= 1$ P_1



Approach 2:

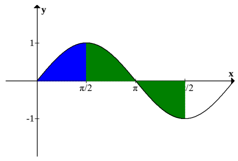
$$\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta$$

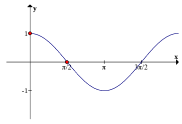
$$= -\cos x \Big|_0^{\frac{\pi}{2}}$$

$$= -[\cos\tfrac{\pi}{2} - \cos 0]$$

$$= -[0-1]$$

$$= 1$$





C_{3I}

P_{2I}

P_{2I}

C_1

P_1

Results:

Approach 1:

Label: C=1P=3

P_{2I}:integration techniques

P_{2I}: FTOC (part 2)

C_1: translations (cos graph)

P_1: symbolic and numerical calculations

The item category, based on the solution, is C-P+.

Approach 2:

Label: C=2P=3

C_{3I}: interpretation definite integral = enclosed net area

P_{2I}:integration techniques

P_{2I}: FTOC (part 2)

C_1: translations (cos graph)

P_1: symbolic and numerical calculations

The item category, based on the solution, is C-P+.

The analysis above indicates that the method is not unique, since at least two approaches are possible. With the first approach the definite integral is on the interval $0 \le \theta \le \frac{3\pi}{2}$. With the second approach the integral is evaluated for $0 \le \theta \le \frac{\pi}{2}$. This approach is valid, since the definite integral on the net area for $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ is 0.

The intention of **item 30** was to assess the application of the FTOC; however, it was discussed in chapter 4 that more students preferred to solve this item with integration and differentiation techniques (approach 2), rather than directly applying the FTOC.

Find $\frac{d}{dx}\int_0^x t^2 dt$

Approach 1:

$\frac{d}{dx}\int_0^x t^2 dt$ **C2I**

$= x^2$

Approach 2:

$\int_0^x t^2\, dt$

$= \frac{t^3}{3}\Big|_0^x$ **P2I**

$= \frac{x^3}{3} - 0$ **P2I**

$= \frac{x^3}{3}$

$\Rightarrow \frac{d}{dx}\frac{x^3}{3}$ **P2D**

$= x^2$

Results:

Approach 1:

Label: C=1P=0

C2I: link FTOC (part 1): $g'(x) = f(x)$

The item category, based on the solution, is C-P-.

Approach 2:

Label: C=0P=3

P2I: integration techniques

P2I: FTOC (part 2)

P2D: differentiation rules

The item category, based on the solution, is C-P+.

According to the analysis of item 30, the task belongs to C-P- or C-P+, but the researcher decided to place the item in the C-P+ knowledge class for the implementation of the Rasch analysis.

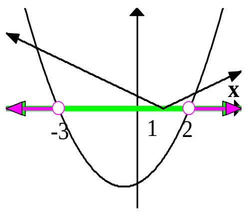
The lecturer taught the algebraic approach as preferable method to solve **item 21**. The algebraic approach had nine steps (C=5P=4) and the item was placed in the C+P+ knowledge class. However, an alternative graphical, more conceptual method was suggested when students' solutions were investigated.

Solve $\frac{|x-1|}{x^2+x-6} \geq 0$. Write your final answer in interval notation.

Approach 1: Algebraic method

$$\frac{|x-1|}{x^2+x-6} \geq 0$$ P2F

$(x-2)(x+3) > 0$ and $|x-1| \geq 0$ P2F



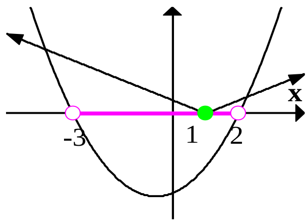
Graph of parabola C1

$x \in (-\infty,-3) \cup (2,\infty)$ and $x \in \mathbb{R}$ C3F C3F

$\Rightarrow x \in (-\infty,-3) \cup (2,\infty)$ C3F

OR

$$\frac{|x-1|}{x^2+x-6} \geq 0$$



$(x-2)(x+3) < 0$ and $|x-1| \leq 0$ P2F

Graph of parabola $x-1=0 \Rightarrow x=1$ C1 C3F C3F P1

$x \in (-3,2)$

$\Rightarrow x=1$ C3F

Final answer: $x \in (-\infty,-3) \cup \{1\} \cup (2,\infty)$ C3F P5

Results

Label: C=5P=4

P_{2F}: inequality rules ($\frac{+}{+}$ or $\frac{-}{-}$)

P_{2F}: factorisation

C_1: translations (graph)

C_{3F}: interpretation quadratic inequality

C_{3F}: interpretation absolute value inequality

C_{3F}: interpretation inequality (and)

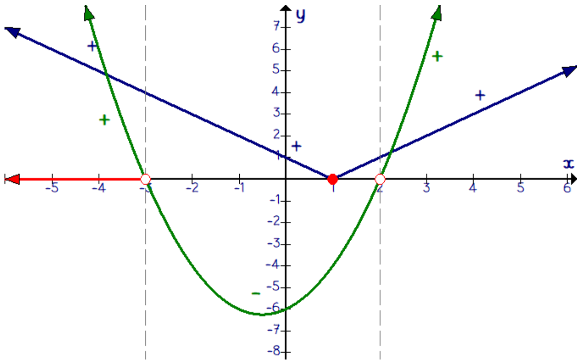
P_1: symbolic and numerical calculations

C_{3F}: interpretation inequality (or)

P_5: notation

The item category, based on the solution, is C+P+.

Approach 2 (graphical):



The analysis of the graphical approach that emerged from the students' solutions is as follows:

Label: C=4P=1

C_1: translations (draw graph of absolute value)

C_1: translations (draw graph of parabola)

C_{3F}: interpretation *x*-intercepts parabola

C_{3F}: interpretation inequality (graphical)

P_5: notation

The item category, based on the solution, is C+P-.

The analysis of approach 2, the graphical approach, indicates a shorter method with fewer steps and fewer procedural obstacles. This example will be discussed further under the recommendations for teaching calculus (6.6.2).

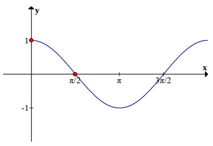
Finding 3: Analyses of the same item solution could differ depending on the prior knowledge or experience of a person

The current study found that being conceptual or procedural is not necessarily a property of the task, but rather of the solution of the task, as was also reported by

Engelbrecht *et al.* (2009). It could be argued that an item is familiar and therefore procedural for some students, but unfamiliar and conceptual to other students. The item analysis of the same solution approach could differ, since the interpretation of the analysis connects to the specific view held by a student, educator or mathematics education expert (Star & Stylianides, 2013). Again, this finding could possibly produce different item labels and knowledge classes for the same item solution. This finding is illustrated using item 29 and item 4.

Since **item 29** was discussed with the previous finding, the researcher only focusses on the section of the solution approach where the interpretation of the analysis could differ. The analysis presented above uses C_1: translations (cos graph) for both approaches 1 and 2:

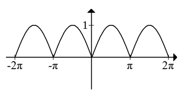
$$\begin{aligned}&\cos\tfrac{\pi}{2}-\cos 0\\&=0-1\\&=-1\end{aligned}$$



However, another person could categorise the trigonometric substitution of known values as procedural, since the values of $\cos\frac{\pi}{2}=0$ and $\cos 0=-1$ are procedural knowledge to him/her (Engelbrecht *et al.*, 2009).

Item 4 serves as another example where the interpretation of the analysis is not unique. The item requests the student to choose the graph for the composite function where $f(x)=\sqrt{x}$ and $g(x)=1-\cos^2 x$.

$$\begin{aligned}(f\circ g)(x)&=\sqrt{1-\cos^2 x} && \mathbf{C_{3F}}\\&=\sqrt{\sin^2 x} && \mathbf{P_4}\\&=|\sin x| && \mathbf{P_{2F}}\\&=\begin{cases}\sin x & if\ \sin x\geq 0\\ -\sin x & if\ \sin x<0\end{cases} && \mathbf{C_{3F}}\end{aligned}\quad \mathbf{C_{2F}}$$

The analysis for the graphical interpretation for |sin*x*| as  could be considered the procedural step P_{2F}: $\sqrt{a^2}=|a|$ and the conceptual step

C_{3F}: interpretation absolute value function. Alternatively, it could be argued that the answer of a square root function (C_{2F}) is always non-negative.

6.2.2 Sub-question 2: How are conceptual and procedural constructs related in calculus?

The researcher investigated the relationship connecting conceptual and procedural constructs *between* calculus items with the quantitative analysis presented in chapter 5. The 33 calculus items were taken up in a test that was administered to 205 first-year mathematics students. The initial Rasch analysis showed some disordered categories and the data collection instrument was adapted by collapsing marks and mark categories for some items. The statistics for the second Rasch analysis indicated that the data had a good fit to the Rasch model. The items were positively correlated to the underlying construct, unidimensionality was confirmed, the person-item map indicated that the items aligned to persons and the analysis indicated no misfitting items or persons. The analysis gave evidence of the following two findings:

Finding 4: Procedural and conceptual components are interconnected and cannot be separated

Conceptual and procedural knowledge are defined in literature as two distict bodies of knowledge. The research study was embedded in the assumption presented by literature and two distinct mathematical constructs were assumed. The study provides evidence that, in practice, the two knowledge components are interconnected and cannot be separated. This finding could be linked to the work done by Kilpatrick *et al.* (2001) on the interconnected strands of mathematical proficiency. Evidence of this finding is present on three levels.

The Rasch analysis statistics showed evidence that a single latent trait was present. The single unidimensional construct is interpreted in the context of this study as evidence of the integration of the cognitive processes. Procedural fluency and conceptual understanding cannot be separated.

The person-item map (Figure 6.1) provides evidence that items are scattered over the map and that the position of items is independent of the defined knowledge classes, in other words, the number of procedural and conceptual steps used to solve the problem.

MEASURE   PERSON - MAP - ITEM

```
                 <more>|<rare>
   3                   +
                       |
                       |
                       |
                       |  Q21  C+P+
                       |  Q6  C-P-
                    .  |
                       |
   2                .  +T
                    .  |
                       |
                    . T|
                    .  |
                    .  |  Q1  C-P-
                       |
                  .##  |  Q10  C+P-   Q2  C-P+
   1              ###  +S Q31  C+P+   Q33  C+P-
                   ## S|
                 .###  |
                .####  |  Q4  C+P-
                 ####  |  Q24  C-P-
               .#####  |  Q12  C+P+
                 .### M|  Q13  C+P-   Q17  C-P+
            #########  |  Q15  C-P-   Q9  C+P-
   0             .###  +M Q23  C-P-   Q26  C+P+
                 .###  |  Q20  C+P+   Q28  C-P-   Q30  C-P+
                  .##  |  Q11  C+P-
                   ##  |
                 .### S|  Q29  C-P+   Q7  C-P+    Q8  C+P-
                    #  |
                  .##  |  Q18  C+P-   Q22  C-P+   Q27  C+P+
                   .#  |
  -1                .  +S Q25  C+P+   Q32  C+P+
                    .  |  Q14  C-P-   Q19  C-P-
                    . T|  Q16  C-P+   Q3  C-P-
                    .  |
                   .#  |
                       |
                       |  Q5  C-P+
                       |
  -2                   +T
                 <less>|<freq>
```

Key:

```
EACH "#" IS 3: EACH "." IS 1 TO 2
C-P-: C-P- items
C-P+: C-P+ items
C+P-: C+P- items
C+P+: C+P+ items
```

Figure 6.1: Person-item map: person proficiency vs item difficulty (repeat 5.1)

The finding is further supported by the CFA, since the analysis showed that sub-constructs were poorly specified and that there was over-correlation of latent factors (knowledge classes) – confirmed in Figure 6.2.

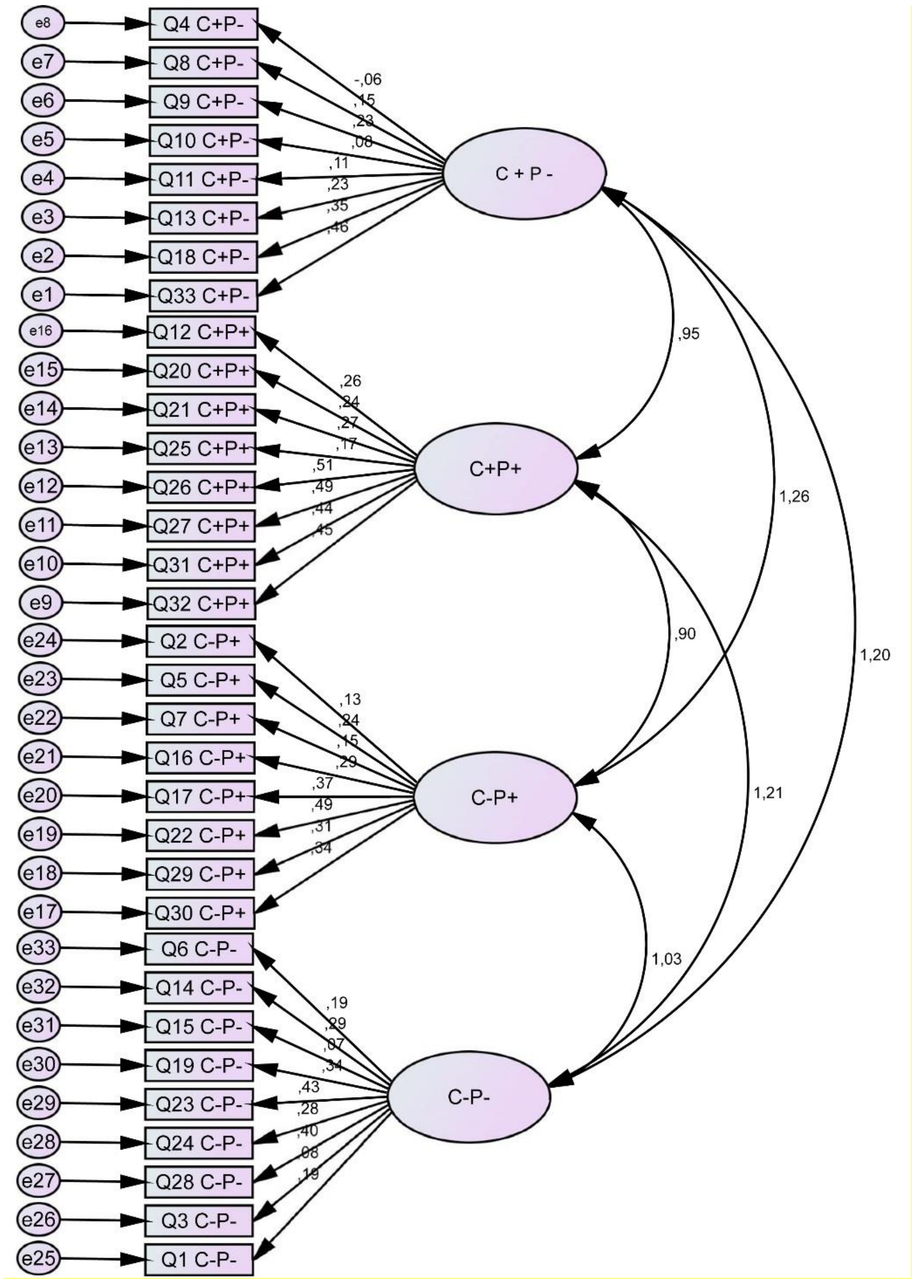


Figure 6.2: Outcome of the CFA for this study (repeat 5.4)

The Rasch analysis confirmation of a unidimensional calculus construct, the integrated location of items in the person-item map and the CFA provide evidence that, in practice, procedural and conceptual components cannot be separated.

Finding 5: Item difficulty does not only depend on the number of procedural or conceptual steps

Based on the work of (Star, 2005), the researcher expected that items in the four knowledge classes would be clustered together in terms of difficulty, since it was expected that the easiest items would be those with the fewest steps. The expectation was that the easiest items would be C-P- items, followed by C-P+, C+P- and C+P+ items; however, the descriptive statistics and person-item map, indicated that item difficulty could not be linked to the number of procedural and/or conceptual steps needed to do the mathematics or the knowledge classes defined for the study.

Item 6, in the C-P- category, had an item mean of 12.63%, indicating it was very difficult. On the other side of the difficulty spectrum, item 25, in the C+P+ category, had a mean of 81.32%, which made it one of the easiest items. Furthermore, the mean per knowledge class was highest for C-P+ items (62.41%), followed by C-P- (54.98%) and C+P+ (53.21%), and lowest for C+P- (50.13%) items. On average, students performed better in items with fewer conceptual steps needed to do the mathematics, that is, the C- items. It appears as if there is no correlation between item difficulty and the number of steps performed. These findings were not expected and could be seen as contradictory to the current research literature.

In the next section the academic and practical contributions to the mathematics education community will be discussed.

6.3 CONTRIBUTION OF THE RESEARCH STUDY

The study made unique contributions on academic and practice levels.

6.3.1 Academic contribution related to methodology

The researcher developed a content analysis framework (chapter 3.3) to measure the integrated relations between procedural and conceptual knowledge components *within* calculus items. The quantitative section of this project includes the development of a pilot instrument to describe procedural and conceptual constructs that are related in calculus. The researcher developed a test of 33 calculus items that showed procedural and conceptual steps. The instrument did not undergo piloting phases, but could be identified as a draft instrument that could be refined to represent the calculus construct. Rasch analysis was implemented to describe the knowledge constructs.

6.3.2 Academic contribution related to mathematical knowledge

The mixed-methods study, which firstly focussed on the relations *within* individual items and secondly on the relations *between* items, developed and positioned the researcher to a point where the complex nature of procedural and conceptual knowledge in calculus items could be analysed and described to answer the main research question and benefit mathematics education scholars in general.

6.3.3 Developing good practice

The rationale of the research study in the introductory chapter included the development of academic knowledge to enhance calculus learning and contribute to teaching strategies. The choice of a procedural or conceptual approach connects to the choice of teaching strategy, learning and assessment. A procedural approach focusses on accuracy of procedures and answers, rules and techniques and tasks that involve familiar procedures. A conceptual approach involves evaluation and explanation of unfamiliar tasks that could lead to metacognition, reflection and discussion. Teaching should include both types of problem-solving approaches. The significant findings of the study position the researcher to make recommendations on the teaching and learning of calculus. This will be discussed further in the section on recommendations.

The next two sections reflect on the methodology used in the research study. The first section will reflect on the content analysis and the subsequent section on the Rasch analysis.

6.4 REFLECTIONS ON METHODOLOGY: CONTENT ANALYSIS

6.4.1 Definition of problem-solving categories

The definition and re-definition of conceptual and procedural categories were the researcher's interpretation of the definitions to describe problem-solving approaches to solve mathematical tasks (Bergsten *et al.*, 2015; Bergsten *et al.*, 2017; Engelbrecht *et al.*, 2012). The breakdown into categories could be contested, since the category P_3: Algorithms (set of rules) was not used in the analysis of the 33 calculus items. This could be seen as a limitation on the defined categories; however, the limitation could be overcome if P_3 should be included in P_2 or P_4, when the procedural knowledge components are revised.

6.4.2 Uniqueness of analysis: methods and interpretation of analysis

Notwithstanding the triangulation between the researcher and two mathematics education specialists, the interpretation of problem-solving categories and item analysis could be seen as not objective. The nature of the approach used to solve a calculus item is not absolute, since it may differ among students, educators or mathematics education experts (Star & Stylianides, 2013) and this may be seen as a limitation to the study.

- The analysis indicated that there could be more than one method to solve an item, e.g. for items 1, 3, 21, 25, 29 and 30.
- The analysis of the same method could result in two different interpretations, e.g. items 4 and 29.

The nature of an unseen item could be regarded as conceptual for one student, but procedural to a student who is familiar with the task.

6.5 REFLECTIONS ON METHODOLOGY: RASCH ANALYSIS

6.5.1 Test design/data collection instrument

In the current study the initial data collection instrument was revised by collapsing mark allocation and mark categories, as described in chapters 3 and 5. The test can be improved according to findings from the current study before repeating it, involving a follow-up group of first-year engineering students. The revision could include the following:

- Improvement and replacement of weak multiple-choice question distractors, therefore distractors that were chosen by less than 5% of students.
- Including student methods and investigation of more alternative methods to solve items.
- Revision on the formulation of open-ended items 31 and 32, as suggested in chapter 4.

6.5.2 External validity

Internal validity and reliability were assessed by the statistics presented by the Rasch analysis. The external validity of the instrument was not examined and remains an area for investigation.

6.5.3 Homogeneous participants

The study investigated the calculus construct, in other words the calculus item solutions. The nature of RMT assumes that the implementation of the Rasch model is independent of the sample of participants; however, the homogeneous nature of first-year engineering students who participated in the study could be seen as a limitation to the study. The relatively small number of respondents have similar aptitude, since the students satisfied the same admission requirements. This concern is less important; however, the findings cannot be generalised to the broad population of first-year mathematics students.

6.6 RECOMMENDATIONS

Arising from findings in this study, the researcher suggests that the test that was used to collect data be revised according to the suggestions presented in 6.5.1. The researcher suggests that a follow-up to this study could include deeper investigation into and analyses of student solutions.

6.6.1 Learning/development of calculus

Students have different learning preferences and learning styles; some prefer to stay algebraically and procedurally grounded, while others prefer to work with graphs or images, therefore more conceptually. Conceptual understanding can lead to the development of procedural fluency, which then leads to conceptual advance, and vice versa. Students could also use graphical computer software, e.g. GeoGebra, to combine the algebraic and graphical view on one screen, and toggle between the two representations (Escuder & Furner, 2012). This practice could improve the linking of relationships and interpretation between differentiation and integration (when functions are known), e.g. for items 10, 12, 16, 22, 26, 29 and 31.

6.6.2 The teaching of calculus

The researcher agrees with the literature that the investigative, iterative, integrated teaching approach (Baroody, 2009; Rittle-Johnson, 2017) is the preferred teaching strategy to follow when teaching (and assessing) mathematics. Mathematical inquiry, metacognition and class discussion will enhance reasoning and problem-solving skills. The teacher is a facilitator who plans teachable moments with carefully designed, appropriate tasks and probing questions that should direct his/her teaching to the prior knowledge of the students. The student is an active participant who develops procedural fluency and conceptual understanding through discussion and reasoning, comparison, and justification of methods and results.

The use of different methods and representations (Ball *et al.*, 2004; Brown *et al.*, 2002; Kilpatrick *et al.*, 2001) will enhance mathematical understanding; combining methods and representations will develop the concept image for a particular mathematical concept. The movement between different representations will provide the opportunity to discuss various methods and relate the procedures to connected concepts (Davis, 2005). Bidirectional relations where concepts and procedures are both emphasised are recommended.

The researcher describes the conceptual, algebraic and integrated teaching approaches with examples from the study in the three sections below.

6.6.2.1 Items for which a conceptual teaching approach is preferred

Item 21, discussed in 6.2.1, serves as an excellent example to illustrate that it is preferable to teach the solving of quadratic inequalities with a graphical or conceptual approach rather than an algebraic approach. Item 21 requested the following:

Solve $\frac{|x-1|}{x^2+x-6} \geq 0$. Write your final answer in interval notation.

Although the algebraic approach was suggested as preferable method in class, the conceptual, graphical method suggested by students indicates a shorter, less complicated method (C=4P=1).

According to the content analysis, descriptive analysis and person-item map, item 21 was the most difficult item of the test, since only 1.7% of students answered this item correctly. The three students who answered the item correctly used graphs instead of algebra to solve the inequality. Furthermore, the three students who scored 100% had an average person measure of 0.47; on the other hand, the students who scored 50% had an average person measure of 0.9. The result showed disordered categories, since students with lower person proficiency were not expected to perform better than students with higher person proficiency. The researcher retained the disordered categories, since an explanation for the result could be found in the mathematical approach used to solve the item.

The evidence found in the example could be regarded as diagnostic, since the analysis of student responses indicated that the teaching of quadratic inequalities should preferably focus on graphical methods, but it could be supported by the algebraic approach. This emphasises the finding that conceptual and procedural components cannot be separated in the teaching and learning of calculus.

Furthermore, chapter 2.6.2 provides an example of a problem where conceptual teaching is inevitable:

> Use the fundamental theorem of calculus (part 2) to calculate the exact area enclosed by the curve of $k(x) = 1 - \ln x \cos 2x + x$, the x-axis, $x = 2$ and $x = 8$ by evaluating the definite integral: $\int_2^8 (1 - \ln x \cos 2x + x)dx$, if possible.

The definite integral cannot be solved with integration techniques, and the Riemann sum has to be calculated (Hechter *et al.*, 2018) to evaluate the definite integral.

6.6.2.2 Items for which an algebraic teaching approach is preferred

Based on the researcher's analysis and the analysis of students' solutions, the algebraic teaching approach (product and quotient rule for differentiation) was preferred for items 23 and 24:

Item 23: Find the derivative of $m(x) = \frac{\operatorname{cosec} x}{e^x - 1}$

Item 24: Find the derivative of $h(x) = \sqrt{x^3 - 7}.\tan\left(\frac{x}{2}\right)$.

The procedural approach is preferred, since the graphs of the functions *m(x)* and *h(x)* are not familiar to the students.

6.6.2.3 Items for which an integrated teaching approach is preferred

Item 30, discussed in 6.2.1, is an example of an item that requires a cross between an algebraic and conceptual teaching approach. It was noted earlier that item 30 could be approached with integration and differentiation techniques, or applying the FTOC. However, navigation between procedural and conceptual approaches would enhance conceptual understanding of the FTOC and the relationship between differentiation and integration, as explained in Figure 6.3 (Hechter *et al.*, 2018):

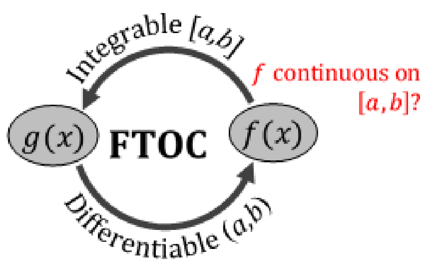


Figure 6.3: The relationship between differentiation and integration

Students preferred to evaluate the definite integral in items 29 (discussed in 6.2.1) and item 31 with integration techniques above geometric interpretations using the Riemann sum (Pettersson & Scheja, 2008; Scheja & Pettersson, 2010).

Item 29: Evaluate the integral: $\int_0^{\frac{3\pi}{2}} \sin\theta \, d\theta$

Item 31: Evaluate the integral $\int_{\frac{1}{2}}^{1} |\ln x| dx$ and indicate the calculated area on the graph.

Student understanding of definite integrals was connected to an algorithmic context with a distinct focus on procedures and techniques; procedural fluency was foregrounded above conceptual connections; however, the solving of both tasks requires some conceptual steps.

To summarise: Mathematical proficiency requires both types of mathematical knowledge and procedural flexibility (Rittle-Johnson, 2017). It is important to focus teaching and learning on both conceptual understanding and procedural fluency, since procedural fluency will help students to deepen conceptual understanding, and conversely, conceptual understanding may help with the development of procedural fluency. Lecturers should navigate their teaching between concepts and procedures, different solution methods and different representations, e.g. algebra and graphs. Conceptual knowledge could help with the selection and adaptation of procedures; on the other hand, explaining procedures may help students to make deeper conceptual connections. Teaching strategies should emphasise that there are more than one method to do the mathematics (Star & Stylianides, 2013) and that the order of sequencing teaching instruction (procedural vs conceptual), is not important (Rittle-Johnson *et al.*, 2015).

6.6.3 Identification of calculus topics of concern

The content analysis, descriptive statistics and item-person map could be used as a diagnostic analysis to identify calculus topics that need more teaching support.

The content analysis indicated that seven item clusters assessed the same notion or concept (Table 4.6). Further analysis of the descriptive statistics of the clustered items indicated that students had not yet mastered linear or quadratic inequalities, were not fully proficient in differentiation, had partial understanding of curve sketching and were relatively skilled in integration techniques. Items involving contextual application gave varied results.

The person-item map, together with the zone of proximal development described by Vygotsky, is a suitable educational theory that could be applied to identify areas of additional teaching support (Dunne *et al.*, 2012). The identification of more difficult items or topics above the zero-difficulty level could be used to structure learning opportunities and inform teaching strategies.

6.6.4 Mathematics teacher training

Teachers should be trained to develop conceptual understanding of mathematics through focussing on both procedural and conceptual knowledge. The topics algebra, trigonometry and geometry were historically taught as separate knowledge domains, but teachers need to be made aware that mathematics should be seen as a coherent, connected body of knowledge. Teachers should be trained to include the use of different representations to explain a particular concept. This will give access to both procedural and conceptual knowledge in a spontaneous, natural way, leading to the development of procedural fluency and conceptual understanding. The researcher would also like to recommend exposure to graphical software, e.g. GeoGebra, to be included in teacher training.

6.6.5 The topic of calculus

The items included in the instrument were on the topic of calculus – including pre-calculus, differentiation and integration. It could be recommended that the instrument should be altered to focus on different mathematics topics.

6.6.6 The level of mathematics

The level of items included in the instrument were for first-year mathematics students. However, the instrument could be revised to include items for primary, secondary or tertiary levels.

6.7 CONCLUSION

As a first-year mathematics lecturer on an extended engineering programme, it is important for the researcher to know that what she teaches her students contributes significantly to the development of mathematical proficiency. Merely assuming that what she does in class will fill the possible background knowledge gaps from high school and develop conceptual understanding and cognitive skills is not enough. The drive to generate deep, flexible mathematical learning and develop critical problem-solving skills led to the investigation of literature on the so-called "*math wars"* and the formulation of the research question on the nature of conceptual and procedural knowledge in calculus. The study was rooted in the investigation of the concept-driven and skills-orientated argument, as well as Kilpatrick *et al.* (2001) view on intertwined strands of mathematical proficiency.

The findings indicate an integrated, complex relationship between procedural and conceptual knowledge *within* and *between* calculus items when solving calculus tasks. The findings fit in and contribute to the view of interconnected strands of mathematical proficiency shared by Kilpatrick *et al.* (2001). The integrated relationship is visible on two levels:

- The content analysis reveals that the 33 calculus items follow conceptual and procedural steps to achieve problem-solving; however, the uniqueness of a problem-solving approach and the interpretation of an analysis are not absolute (Star & Stylianides, 2013).

• The Rasch analysis and person-item map confirmed the integrated relationship between procedural and conceptual knowledge and indicated that the items had a range of difficulties and did not cluster together according to item label or class. To summarise the findings:

- ***Item solutions drew on both procedural and conceptual knowledge.***
- ***Item solutions could follow more than one approach.***
- ***Analyses of the same item solution could differ depending on the prior knowledge or experience of a person.***
- ***Procedural and conceptual components are interconnected and cannot be separated.***
- ***Item difficulty does not only depend on the number of procedural or conceptual steps.***

The findings are significant for the teaching and learning of calculus. Procedural skills should be connected to different methods and representations, and mathematical knowledge should be organised and connected in a conceptual schema in order to develop mathematical proficiency over time. Teaching should include both types of problem-solving approaches and multiple methods; however, the order of teaching is not important.

Future research could include deeper investigation into student methods and solutions, the revision of the data collection instrument (altered questions, other mathematics topics, different levels of education), as well as investigation into teacher training and the use of graphical software.

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8. APPENDICES

8.1 APPENDIX 1: INFORMED CONSENT FORM

Ms Janine Hechter
ENGAGE
Natural Sciences 2
University of Pretoria
Lynnwood Road
Pretoria
0002
janine.hechter@up.ac.za
Cell: 082 414 8530

19 October 2017

Letter of Consent to the Additional Mathematics student

Dear Sir/Madam

You are invited to participate in a research study aimed at investigating Additional Mathematics students' learning path of calculus by analysing the procedural and conceptual knowledge components of calculus tasks and solutions, and the relationship between the two types of knowledge. This research will be reported upon in my PhD (Mathematics Education) thesis at the University of Pretoria.

It is proposed that you form part of this study's data collection phase by completing questionnaires during class time. The focus of the research study is on *how* the questions are answered, hence the problem-solving method chosen and the calculations used to derive the answers. The informed consent form will be submitted together with completed Questionnaire A, B and C if you agree to be a participant in the PhD research project. The answers (not solutions) to the questions will be submitted as part of JPO 126 continuous assessment, on a separate page.

Your participation in this research project is absolutely voluntary and confidential. You may withdraw from the study at any point. You do not have to provide reasons for your decision. Confidentiality and anonymity will be guaranteed at all times; participants will therefore not be identifiable in the findings of my research. Only my supervisor and I will have access to the collected data. The collected data will only be used for academic purposes. I intend to find patterns relating to the group and not to any individuals. After the successful completion of my PhD (Mathematics Education) I will share my findings in my thesis, academic articles, and at conferences.

If you are willing to participate in this study, please sign this letter as a declaration of your consent, i.e. that you participate willingly and that you understand that you may withdraw at any time.

Yours sincerely

______________________________ 2017/10/19

Researcher: Me. Janine Hechter Date

______________________________ 2017/10/19

Supervisor: Prof. Gerrit Stols Date

I the undersigned, hereby consent to (please select the correct option for both statements by indicating it with an X in the appropriate block):

1) completing Questionnaire A

YES	NO

2) completing Questionnaire B

YES	NO

3) completing Questionnaire C

YES	NO

Participant's name and surname: ____

Participant's student number: __________________________

Study field: ____________________________

Participant's age: ____________ Participant's gender: | M | F |

Participant's signature: _________________________ Date: ______________

E-mail address: _______________________________

Contact number: ______________________________

8.2 APPENDIX 2: QUESTIONNAIRE A

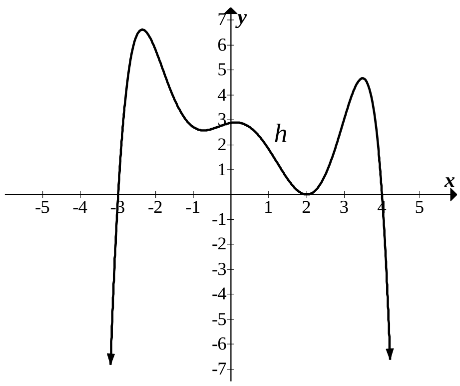
Questionnaire A: Calculus										**Date:**								
Student number:										**Discussion group:**	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**

Questions 1- 17 will be answered for research and analysis purposes.

Instructions:

- Answer questions 1-17 below. **Show all your reasoning and calculations. Circle the correct answer.**
- Re-write your final answer to each question (A, B, etc) in the space provided on page 9 (Class test 7).
- Tear off page 9 (Class test 7) and hand it in separately as JPO 126 continuous assessment.
- Hand in the informed consent form together with Questionnaire A if you voluntarily agree to be a participant in the PhD research project.

1. Consider $h(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ shown in the figure. (2)
The following is true for h? Explain.



A. n is odd, $a_n > 0$
B. n is odd, $a_n < 0$
C. n is even, $a_n < 0$
D. n is even, $a_n > 0$
E. None of these

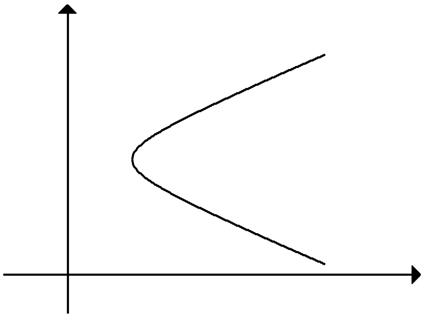
---

2. Solve the inequality: $4x - 1 < 3x < 5x - 4$. Write your final answer in interval notation. (2)

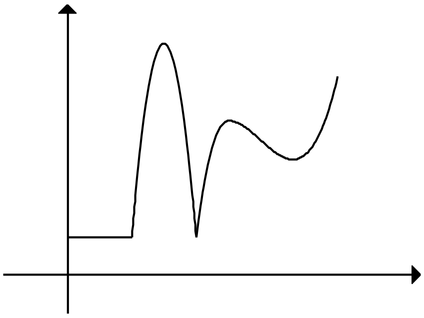
A. $x \in (-\infty, 1]$
B. $x \in (-\infty, 1] \cup (-\infty, 2)$
C. $x \in \emptyset$
D. $x \in (-\infty, 1] \cap (2, \infty)$
E. None of these

3. Which of the following graphs are graphs of a function $f(x)$? Explain. (1)

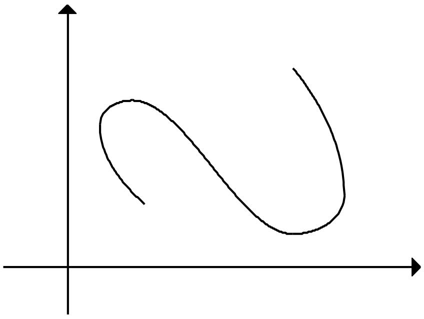
A.



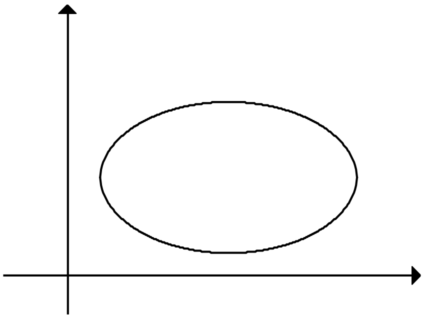
B.



C.



D.

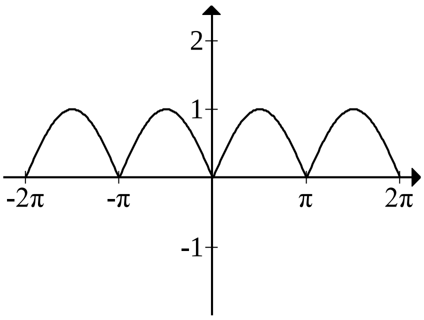


E. None of these

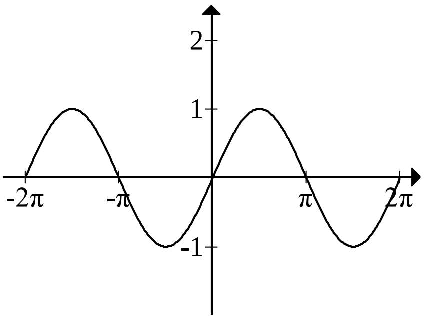
---

4. Let $f(x) = \sqrt{x}$ and $g(x) = 1 - \cos^2 x$. The graph of $(f \circ g)(x)$ is represented by: (2)

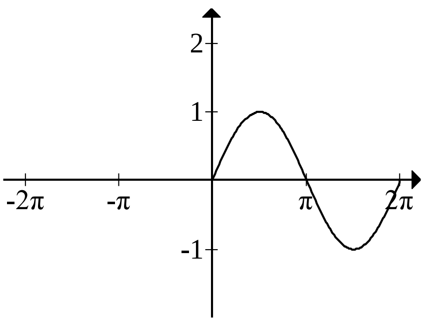
A.



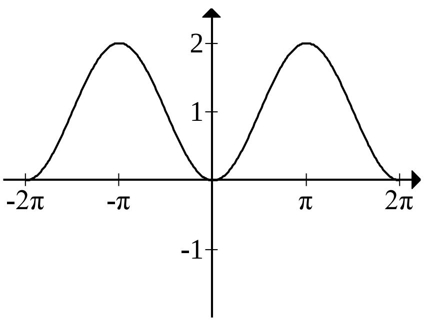
B.



C.



D.



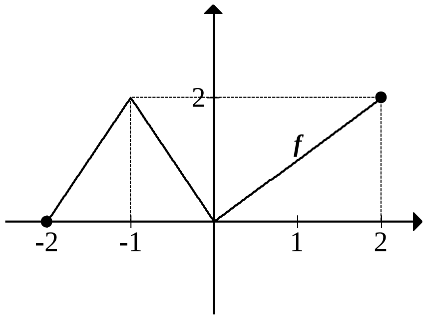
E. None of these

5. The domain of $f(x) = \frac{1}{e^{x+1}-1}$ is: (2)

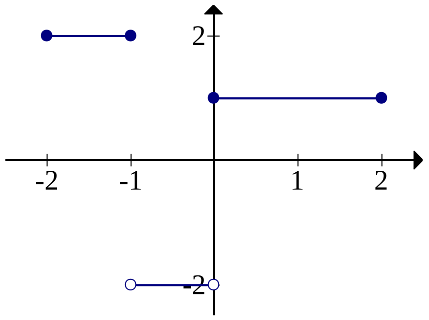
A. $\mathbb{R}$
B. $\mathbb{R}\backslash\{1\}$
C. $\mathbb{R}\backslash\{0\}$
D. $\mathbb{R}\backslash\{-1\}$
E. None of these

---

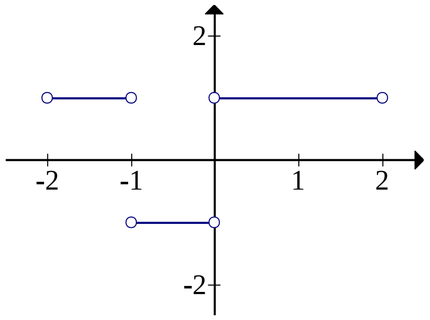
6. The graph of a function f is given. Draw the graph of the derivative function f'. (2)



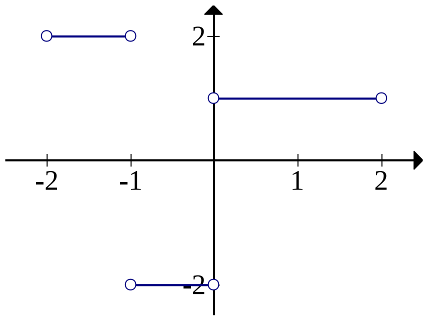
A.



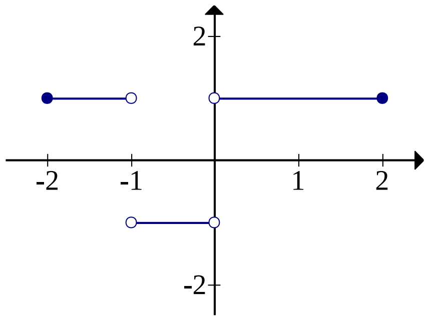
B.



C.



D.



E. None of these

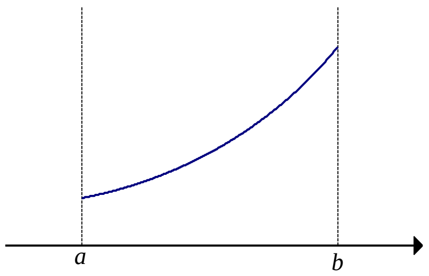
7. Given $f(x) = x^3$, what is the equation of the tangent line at $x = 1$? (2)

A. $y = x + 2$
B. $y = 3x - 2$
C. $y = x$
D. $y = 3x + 2$
E. None of the above

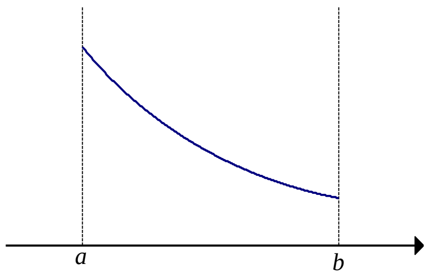
---

8. Which of the following functions f in the sketch has the properties that $f'(x) < 0$ and $f''(x) > 0$ for all $x \in [a, b]$. Explain. (2)

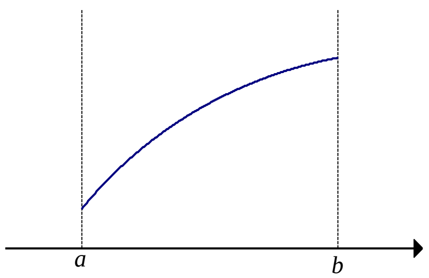
A.



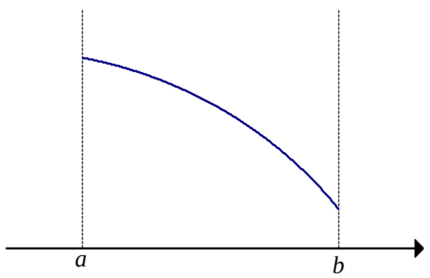
B.



C.



D.



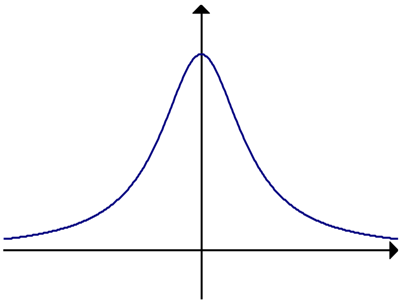
E. None of these

9. Which of the following graphs $f(x)$ satisfies both given conditions: (2)

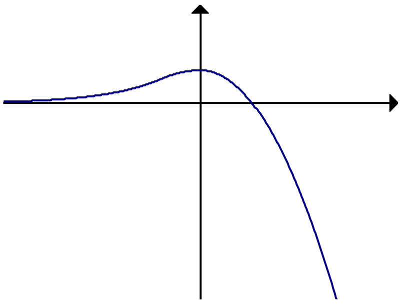
(i) $f'(x) > 0$ on $(-\infty, 0)$ and $f'(x) \leq 0$ elsewhere

(ii) $f''(x) > 0$ on $(-\infty, -1)$ and $f''(x) \leq 0$ elsewhere

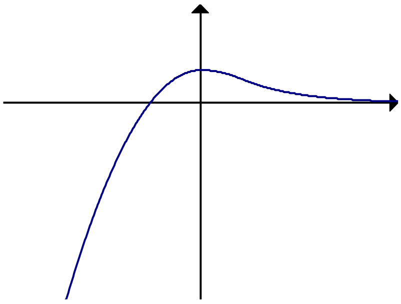
A.



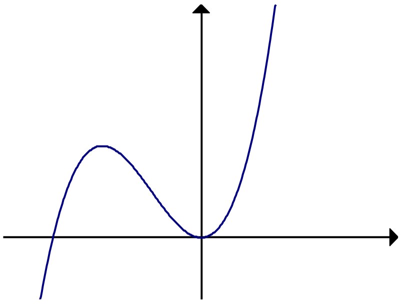
B.



C.



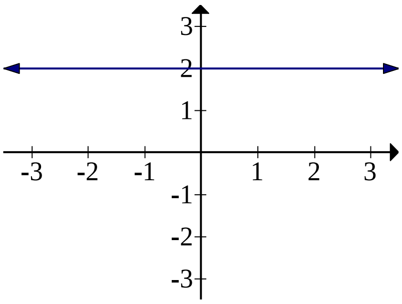
D.



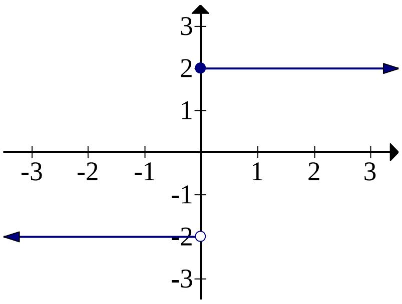
E. None of these

10. Let $f(x) = |2x|$. The graph of f' is given in: (2)

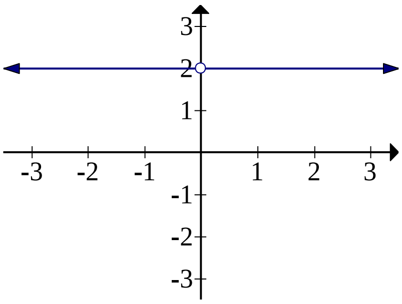
A.



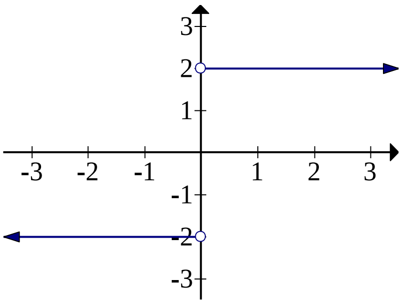
B.



C.

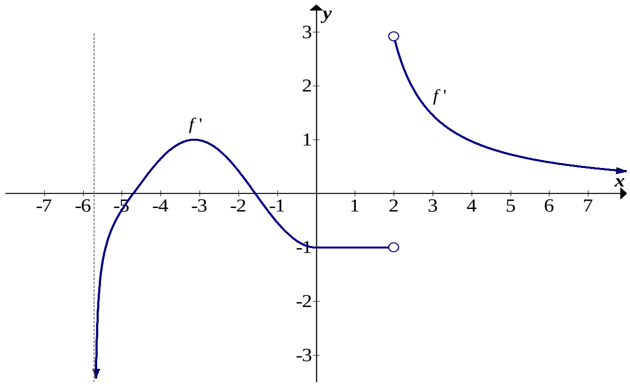


D.



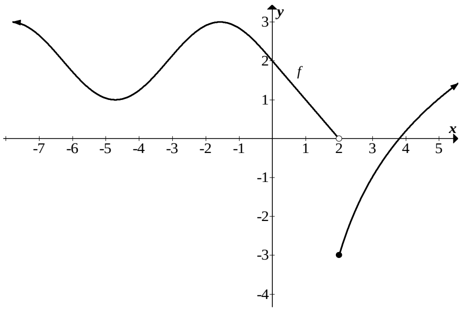
E. None of these

11. Given: the graph of f' : (2)

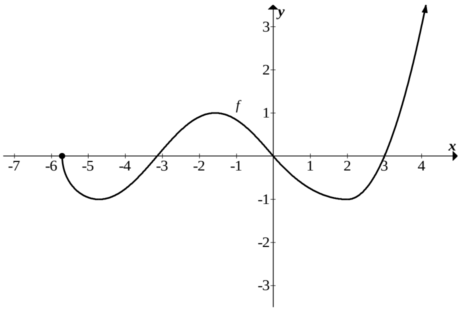


Which of the following is a possible graph for f? Explain.

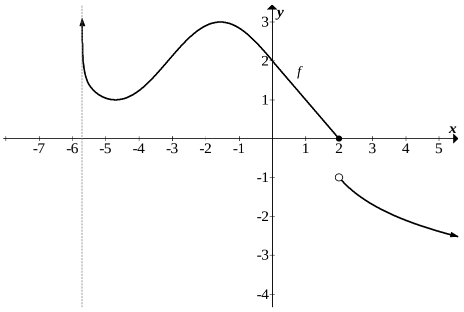
A.



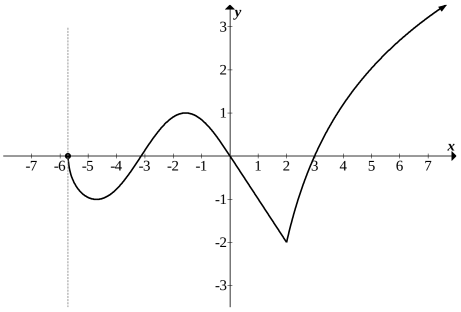
B.



C.



D.



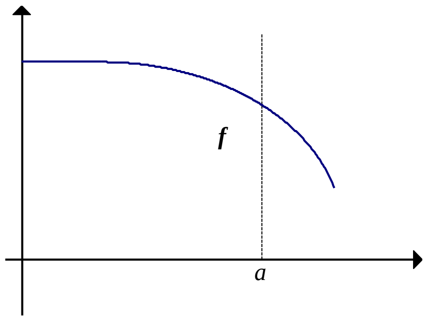
E. None of these

12. Let $f(x) = x^3 + 6x^2 + 9x + 2$. Which of the following statements is true with respect to the graph of f? Show your reasoning. (2)

A. f is decreasing on $(-3, -1)$, concave down on $(-2, \infty)$
B. f is increasing on $(-3, -1)$, concave up on $(-2, \infty)$
C. f is decreasing on $(-3, -1)$, concave up on $(-2, \infty)$
D. f is increasing on $(-3, -1)$, concave down on $(-2, \infty)$
E. None of these

---

13. The graph of the function f is given. Consider the function $g(x) = \int_0^x f(x)dx$ $(0 \leq x \leq a)$. What can we say about the function g? Explain. (1)



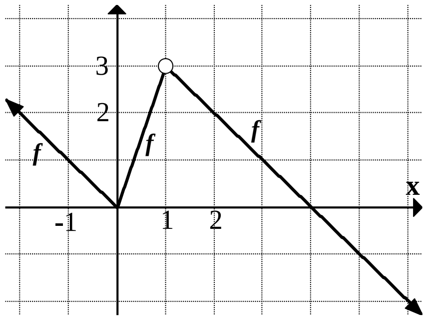
A. It is decreasing
B. It is increasing
C. It is constant
D. We cannot say whether it increases or decreases
E. None of these

---

14. The function $v(t)$ is a velocity function on the time interval $[a, b]$. Select the best interpretation of the integral $\int_a^b v(t)dt$. (1)

A. The displacement between $t = a$ and $t = b$
B. The total distance travelled from $t = a$ to $t = b$
C. The acceleration between $t = a$ and $t = b$
D. The average speed attained between $t = a$ and $t = b$
E. None of these

15. Given the function f in the sketch, estimate the value of $\int_{-2}^{6} f(x)dx$. (2)



A. 4
B. 6
C. 8
D. 10
E. None of these

---

16. Evaluate the integral: $\int_1^e \frac{1}{x} dx$ (1)

A. e
B. -1
C. 0
D. 1
E. None of these

---

17. If $f''(\theta) = e^{2\theta}$, $f'(0) = 2$ and $f(0) = \frac{1}{4}$, find $f(\theta)$. (2)

A. $e^{2\theta}$
B. $\frac{e^{2\theta}}{2} + \frac{3}{2}\theta - \frac{1}{4}$
C. $\frac{e^{2\theta}}{4} + \frac{3}{2}\theta$
D. $\frac{e^{2\theta}}{4} + \frac{3}{4}\theta$
E. None of these

8.3 APPENDIX 3: QUESTIONNAIRE B

Questionnaire B: Calculus										**Date:**								
Student number:										**Discussion group:**	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**

Questions 1- 10 will be answered for research and analysis purposes.

Instructions:

- Answer questions 1-10 below. **Show all your reasoning and calculations.**
- Re-write your final answer to each question in the space provided on page 5 (Class test 8).
- Tear off page 5 (Class test 8) and hand it in separately as JPO 126 continuous assessment.
- Hand in Questionnaire B if you voluntarily agree to be a participant in the PhD research project.

1. Let $f(x) = e^{2x} - 1$ and $g(x) = \ln(x+1)$ (3)

   Find $(f \circ g)(x)$ and simplify. Indicate all restrictions on x (if any).

2. Use the definition of the absolute value to rewrite the following function in expanded form and simplify: (3)

   $k(x) = |3x + 1| - 2$

$$= \begin{cases} \text{..............................} & \text{if } \text{........................} \\ \text{..............................} & \text{if } \text{........................} \end{cases}$$

$$= \begin{cases} \text{........................} & \text{if } \text{..........} \\ \text{........................} & \text{if } \text{..........} \end{cases}$$

3. If a ball is thrown upwards from the top of a building 25 metres high with an initial velocity of 20 m/s, then the height h above the ground t seconds later will be: $h(t) = -5t^2 + 20t + 25$. During what time interval will the ball be more than 40 m above the ground? (3)

---

4. Solve $\frac{|x-1|}{x^2+x-6} \geq 0$. Write your final answer in interval notation. (3)

5. If $f(x) = \sin\left(\frac{\pi x}{2}\right)$, then find the second derivative $f''(1)$ (3)

6. Find the derivative of $m(x) = \frac{\operatorname{cosec} x}{e^x - 1}$ (3)

7. Find the derivative of $h(x) = \sqrt{x^3 - 7}.\tan\left(\frac{x}{2}\right)$. (2)

8. If a stone is thrown vertically upwards, the position function of the stone is given by

$s(t) = 30t - 5t^2 + 20$, where s is in metres and t is in seconds.

Calculate:

1. the time t when the stone will reach its maximum height
2. the maximum height of the stone (before it falls to the ground) (3)

9. Determine the point of inflection (if any) of $f(x) = e^x(x - 1)$. Motivate your answer. (4)

---

10. A mass, hanging in equilibrium at the end of a spring, is pulled and released. The position (in metres) of the mass is described by $x(t) = 0.4\cos(3t)$. Determine the acceleration (in $\mathrm{m/s^2}$) of the mass, after π seconds. (3)

8.4 APPENDIX 4: QUESTIONNAIRE C

Questionnaire C: Calculus										**Date:**								
Student number:										**Discussion group:**	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**

Biographical Information

1. Ethnicity:

African	☐	Coloured	☐	Indian	☐	White	☐	Other	☐

If other, please specify: ______

2. Home language: (If more than one, choose language spoken most)

Afrikaans	☐	English	☐	Sepedi	☐	IsiZulu	☐
Sesotho	☐	SiSwati	☐	IsiXhosa	☐	IsiNdebele	☐
Setswana	☐	Tshivenda	☐	Xitsonga	☐	Other	☐

If other, please specify: ______

3. In what year did you write your grade 12 examination? ______

4. In which country did you write your grade 12 examination? ______

5. If in SA, in which city and province did you write your grade 12 examination? ______

6. At which school did you write your grade 12 examination? ______

7. Which grade 12 examinations did you write?

NSC	☐	IEB	☐	IE	☐	Cambridge	☐	Other	☐

If other, please specify: ______

8. What was your grade 12 Mathematics mark?

50 – 59	☐	60 – 69	☐	70 – 79	☐	80 - 89	☐	90 - 100	☐

9. What was your grade 12 Physical Science mark?

50 – 59	☐	60 – 69	☐	70 – 79	☐	80 - 89	☐	90 - 100	☐

10. Are you a first-generation university student?

Yes	☐	No	☐

Questions 1- 6 will be answered for research and analysis purposes.

Instructions:

- Answer questions 1 - 6 below. **Show all your reasoning and calculations**.
- Re-write your final answer to each question in the space provided on page 5 (Class test 9).
- Tear off page 5 (Class test 9) and hand it in separately as JPO 126 continuous assessment.
- Hand in Questionnaire B if you voluntarily agree to be a participant in the PhD research project.

1. Evaluate the integral: $\int x\sqrt{1+2x^2}dx$ (2)

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2. Evaluate the integral: $\int_0^{\frac{3\pi}{2}} \sin\theta \, d\theta$ (2)

3. Find $\frac{d}{dx}\int_0^x t^2\,dt$ (1)

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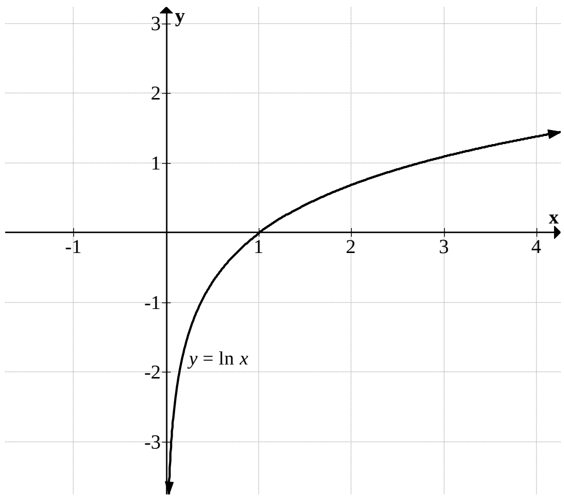
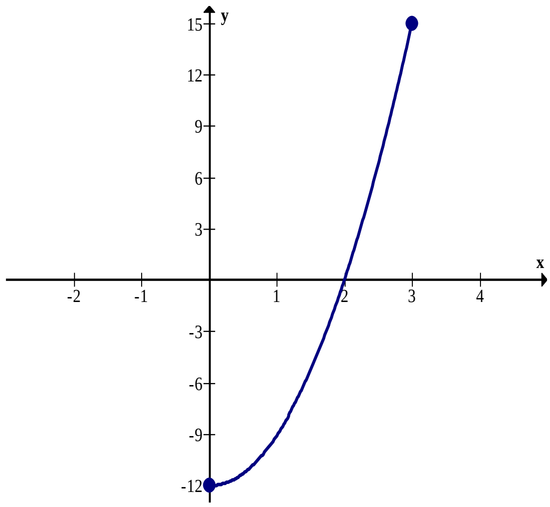
4. Evaluate the integral $\int_{\frac{1}{2}}^{1} |\ln x|\,dx$ and indicate the calculated area on the graph.

(4)



5. The velocity function of a particle is given by $v(t) = 3t^2 - 12\ (0 \leq t \leq 3)$, where v is in m/s and t is in seconds. The graph of $v(t)$ is given in the figure. Calculate the total distance travelled by the particle at $t = 2$. (3)



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6. Let $g(x) = \int_0^{\sin x} \sqrt{1 + t^3}\, dt$.

1. Determine the domain of $f(t) = \sqrt{1 + t^3}$.
2. Determine an expression for $g'(x)$ (2)

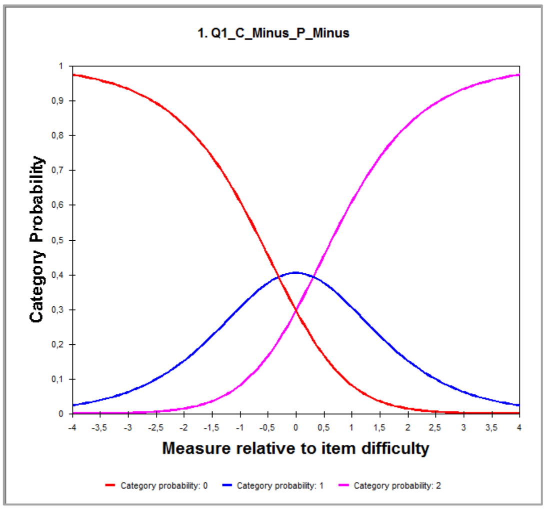
8.5 APPENDIX 5 CATEGORY OPTIONS (BEFORE COLLAPSE)

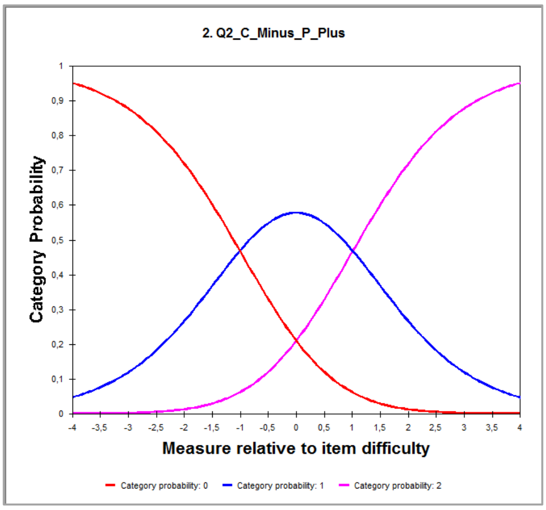
ITEM	CODE	VALUE	SCORE	UNWTD	UNWTD %	WTD	WTD %	AVGE MEAS	P.SD MEAS	S.E. MEAS	INFIT MNSQ	OUTFIT MNSQ	PTMA	LABEL
1	0	0	0	68	36,17	68	36,1702	0,13	0,42	0,05	0,99	0,99	-0,2657	Q1 C-P-
1	1	1	1	71	37,77	71	37,766	0,31	0,45	0,05	1,04	1,02	0,0097	Q1 C-P-
1	2	2	2	49	26,06	49	26,0638	0,53	0,52	0,07	1,03	1,03	0,2802	Q1 C-P-
2	0	0	0	31	16,32	31	16,3158	0,23	0,42	0,08	1,13	1,13	-0,0684	Q2 C-P+
2	1	1	1	103	54,21	103	54,2105	0,25	0,48	0,05	1,03	1,00	-0,1399	Q2 C-P+
2	2	2	2	56	29,47	56	29,4737	0,47	0,49	0,07	1,03	1,03	0,2083	Q2 C-P+
3	0	0	0	38	20,11	38	20,1058	0,16	0,51	0,08	1,02	1,04	-0,1574	Q3 C-P-
3	1	1	1	151	79,89	151	79,8942	0,35	0,47	0,04	0,99	1,00	0,1574	Q3 C-P-
4	0	0	0	69	36,32	69	36,3158	0,25	0,38	0,05	1,16	1,16	-0,0877	Q4 C+P-
4	1	1	1	45	23,68	45	23,6842	0,36	0,51	0,08	1,30	1,41	0,0564	Q4 C+P-
4	2	2	2	76	40,00	76	40	0,33	0,54	0,06	1,33	1,38	0,0372	Q4 C+P-
5	0	0	0	18	9,47	18	9,4737	0,01	0,44	0,11	1,15	1,04	-0,2006	Q5 C-P+
5	1	1	1	9	4,74	9	4,7368	0,01	0,45	0,16	0,71	0,83	-0,1364	Q5 C-P+
5	2	2	2	163	85,79	163	85,7895	0,36	0,47	0,04	0,99	1,00	0,2512	Q5 C-P+
6	0	0	0	96	50,53	96	50,5263	0,27	0,46	0,05	1,10	1,11	-0,0807	Q6 C-P-
6	1	1	1	70	36,84	70	36,8421	0,26	0,46	0,06	1,29	1,30	-0,0808	Q6 C-P-
6	2	2	2	24	12,63	24	12,6316	0,61	0,54	0,11	0,99	1,04	0,2388	Q6 C-P-
7	0	0	0	51	26,98	51	26,9841	0,07	0,41	0,06	1,05	0,99	-0,3002	Q7 C-P+
7	1	1	1	13	6,88	13	6,8783	0,50	0,48	0,14	1,92	2,15	0,1088	Q7 C-P+
7	2	2	2	125	66,14	125	66,1376	0,39	0,48	0,04	1,10	1,11	0,2234	Q7 C-P+
8	0	0	0	18	9,47	18	9,4737	0,03	0,49	0,12	1,04	1,10	-0,1887	Q8 C+P-
8	1	1	1	46	24,21	46	24,2105	0,23	0,42	0,06	1,10	1,03	-0,0875	Q8 C+P-
8	2	2	2	126	66,32	126	66,3158	0,38	0,48	0,04	1,04	1,03	0,1962	Q8 C+P-
9	0	0	0	47	24,74	47	24,7368	0,19	0,45	0,07	1,19	1,22	-0,1455	Q9 C+P-
9	1	1	1	42	22,11	42	22,1053	0,14	0,50	0,08	0,98	0,93	-0,1867	Q9 C+P-
9	2	2	2	101	53,16	101	53,1579	0,44	0,45	0,05	1,00	1,00	0,2811	Q9 C+P-
10	0	0	0	45	23,68	45	23,6842	0,05	0,46	0,07	0,96	0,98	-0,2998	Q10 C+P-
10	1	1	1	87	45,79	87	45,7895	0,38	0,43	0,05	1,04	1,01	0,1257	Q10 C+P-
10	2	2	2	58	30,53	58	30,5263	0,41	0,51	0,07	1,11	1,11	0,1408	Q10 C+P-
11	0	0	0	54	28,57	54	28,5714	0,17	0,52	0,07	1,17	1,53	-0,1851	Q11 C+P-
11	1	1	1	21	11,11	21	11,1111	0,20	0,45	0,10	0,98	0,87	-0,0819	Q11 C+P-
11	2	2	2	114	60,32	114	60,3175	0,40	0,45	0,04	1,06	1,09	0,2235	Q11 C+P-
12	0	0	0	70	36,84	70	36,8421	0,12	0,46	0,06	1,07	1,06	-0,2989	Q12 C+P+
12	1	1	1	32	16,84	32	16,8421	0,25	0,44	0,08	0,86	1,10	-0,0582	Q12 C+P+

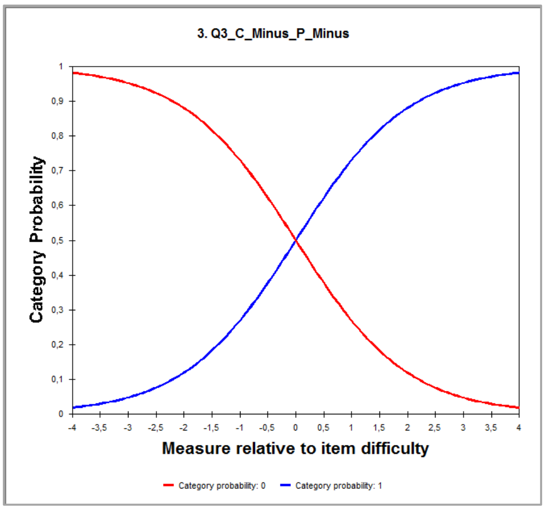
12	2	2	2	88	46,32	88	46,3158	0,48	0,45	0,05	1,00	0,98	0,3329	Q12 C+P+
13	0	0	0	98	51,58	98	51,5789	0,17	0,43	0,04	0,96	0,96	-0,299	Q13 C+P-
13	1	1	1	92	48,42	92	48,4211	0,46	0,49	0,05	0,97	0,97	0,299	Q13 C+P-
14	0	0	0	44	23,16	44	23,1579	0,03	0,37	0,06	0,90	0,86	-0,3198	Q14 C-P-
14	1	1	1	146	76,84	146	76,8421	0,39	0,48	0,04	0,98	0,97	0,3198	Q14 C-P-
15	0	0	0	93	48,95	93	48,9474	0,24	0,45	0,05	1,03	1,03	-0,1497	Q15 C-P-
15	1	1	1	97	51,05	97	51,0526	0,38	0,51	0,05	1,05	1,05	0,1497	Q15 C-P-
16	0	0	0	23	12,11	23	12,1053	-0,01	0,45	0,10	1,02	1,09	-0,2436	Q16 C+P+
16	1	1	1	17	8,95	17	8,9474	0,13	0,46	0,11	0,93	0,89	-0,1166	Q16 C+P+
16	2	2	2	150	78,95	150	78,9474	0,38	0,47	0,04	1,00	1,00	0,2766	Q16 C+P+
17	0	0	0	73	39,04	73	39,0374	0,12	0,43	0,05	1,04	1,02	-0,3074	Q17 C+P+
17	1	1	1	22	11,76	22	11,7647	0,13	0,43	0,09	1,04	0,84	-0,1361	Q17 C+P+
17	2	2	2	92	49,20	92	49,1979	0,50	0,46	0,05	0,98	0,96	0,3877	Q17 C+P+
18	0	0	0	16	8,79	16	8,7912	0,04	0,48	0,12	1,02	1,02	-0,1767	Q18 C+P-
18	1	1	1	82	45,05	82	45,0549	0,14	0,46	0,05	0,90	0,89	-0,3317	Q18 C+P-
18	2	2	2	51	28,02	51	28,022	0,46	0,39	0,05	0,77	0,77	0,1833	Q18 C+P-
18	3	3	3	33	18,13	33	18,1319	0,68	0,44	0,08	0,92	0,90	0,3446	Q18 C+P-
19	0	0	0	13	7,14	13	7,1429	-0,07	0,48	0,14	1,05	1,12	-0,2216	Q19 C-P-
19	1	1	1	45	24,73	45	24,7253	0,07	0,42	0,06	0,94	0,86	-0,2954	Q19 C-P-
19	2	2	2	22	12,09	22	12,0879	0,30	0,41	0,09	1,07	0,86	-0,0128	Q19 C-P-
19	3	3	3	102	56,04	102	56,044	0,49	0,46	0,05	1,00	1,00	0,3802	Q19 C-P-
20	0	0	0	32	17,49	32	17,4863	0,08	0,50	0,09	1,17	1,15	-0,2215	Q20 C+P+
20	1	1	1	51	27,87	51	27,8689	0,27	0,39	0,06	1,14	1,08	-0,0592	Q20 C+P+
20	2	2	2	45	24,59	45	24,5902	0,27	0,44	0,07	1,26	1,31	-0,0521	Q20 C+P+
20	3	3	3	55	30,05	55	30,0546	0,54	0,52	0,07	1,13	1,14	0,2903	Q20 C+P+
21	0	0	0	143	80,34	143	80,3371	0,23	0,46	0,04	0,97	0,96	-0,3822	Q21 C+P+
21	1	1	1	22	12,36	22	12,3596	0,65	0,39	0,09	0,69	0,58	0,2527	Q21 C+P+
21	2	2	2	10	5,62	10	5,618	0,92	0,45	0,15	0,72	0,63	0,2934	Q21 C+P+
21	3	3	3	3	1,69	3	1,6854	0,36	0,36	0,25	2,85	2,36	0,0092	Q21 C+P+
22	0	0	0	15	8,20	15	8,1967	-0,14	0,45	0,12	0,96	0,94	-0,2802	Q22 C-P+
22	1	1	1	33	18,03	33	18,0328	-0,14	0,37	0,07	0,58	0,52	-0,4329	Q22 C-P+
22	2	2	2	59	32,24	59	32,2404	0,39	0,42	0,06	0,81	1,11	0,0949	Q22 C-P+
22	3	3	3	76	41,53	76	41,5301	0,55	0,39	0,04	0,88	0,90	0,4037	Q22 C-P+
23	0	0	0	43	24,02	43	24,0223	-0,09	0,49	0,08	0,91	0,92	-0,4835	Q23 C-P-
23	1	1	1	28	15,64	28	15,6425	0,30	0,39	0,08	1,01	1,44	-0,0225	Q23 C-P-
23	2	2	2	50	27,93	50	27,933	0,47	0,42	0,06	0,78	0,81	0,1768	Q23 C-P-
23	3	3	3	58	32,40	58	32,4022	0,53	0,38	0,05	0,97	0,98	0,2893	Q23 C-P-
24	0	0	0	52	29,21	52	29,2135	-0,04	0,47	0,07	0,86	0,88	-0,4686	Q24 C-P-
24	1	1	1	49	27,53	49	27,5281	0,41	0,41	0,06	1,06	1,12	0,1098	Q24 C-P-

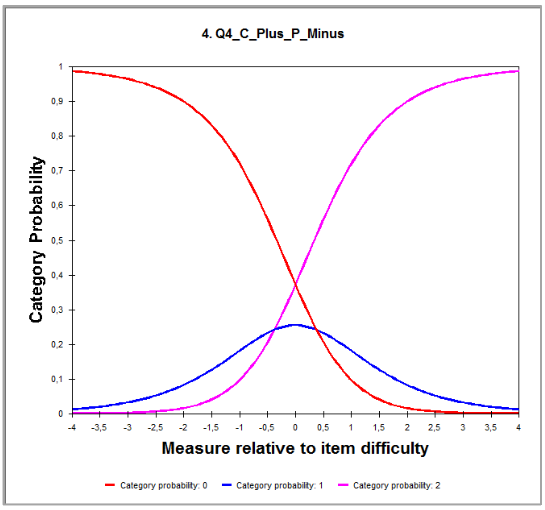
24	2	2	2	77	43,26	77	43,2584	0,51	0,43	0,05	0,95	0,95	0,3311	Q24 C-P-
25	0	0	0	16	8,79	16	8,7912	0,02	0,38	0,10	1,46	1,40	-0,1894	Q25 C+P+
25	1	1	1	5	2,75	5	2,7473	0,06	0,38	0,19	1,16	0,93	-0,0893	Q25 C+P+
25	2	2	2	26	14,29	26	14,2857	0,23	0,51	0,10	0,89	1,39	-0,0712	Q25 C+P+
25	3	3	3	135	74,18	135	74,1758	0,38	0,48	0,04	1,29	1,19	0,2128	Q25 C+P+
26	0	0	0	38	21,97	38	21,9653	-0,05	0,41	0,07	0,89	0,86	-0,4274	Q26 C+P+
26	1	1	1	23	13,29	23	13,2948	0,21	0,44	0,09	1,11	1,11	-0,1051	Q26 C+P+
26	2	2	2	50	28,90	50	28,9017	0,39	0,37	0,05	0,80	0,74	0,0632	Q26 C+P+
26	3	3	3	58	33,53	58	33,526	0,57	0,42	0,06	0,94	0,93	0,3427	Q26 C+P+
26	4	4	4	4	2,31	4	2,3121	0,80	0,61	0,35	1,00	1,00	0,1477	Q26 C+P+
27	0	0	0	17	9,29	17	9,2896	-0,12	0,48	0,12	0,96	1,04	-0,2848	Q27 C+P+
27	1	1	1	29	15,85	29	15,847	0,01	0,45	0,08	0,82	0,83	-0,2741	Q27 C+P+
27	2	2	2	53	28,96	53	28,9617	0,22	0,43	0,06	0,81	0,91	-0,1303	Q27 C+P+
27	3	3	3	84	45,90	84	45,9016	0,58	0,39	0,04	0,82	0,85	0,4853	Q27 C+P+
28	0	0	0	42	24,14	42	24,1379	-0,04	0,39	0,06	0,85	0,80	-0,4303	Q28 C-P-
28	1	1	1	33	18,97	33	18,9655	0,21	0,38	0,07	0,81	0,68	-0,1168	Q28 C-P-
28	2	2	2	99	56,90	99	56,8966	0,53	0,45	0,05	0,91	0,91	0,4643	Q28 C-P-
29	0	0	0	4	2,27	4	2,2727	-0,39	0,18	0,10	0,74	0,70	-0,2264	Q29 C-P+
29	1	1	1	56	31,82	56	31,8182	0,15	0,43	0,06	0,95	0,92	-0,2527	Q29 C-P+
29	2	2	2	116	65,91	116	65,9091	0,44	0,47	0,04	0,97	0,97	0,3194	Q29 C-P+
30	0	0	0	75	43,10	75	43,1034	0,14	0,43	0,05	0,93	0,91	-0,3506	Q30 C-P+
30	1	1	1	99	56,90	99	56,8966	0,48	0,48	0,05	0,95	0,94	0,3506	Q30 C-P+
31	0	0	0	63	36,84	63	36,8421	0,09	0,44	0,06	1,00	0,97	-0,4004	Q31 C+P+
31	1	1	1	45	26,32	45	26,3158	0,35	0,37	0,06	0,74	0,79	0,0039	Q31 C+P+
31	2	2	2	43	25,15	43	25,1462	0,45	0,31	0,05	0,96	0,88	0,1265	Q31 C+P+
31	3	3	3	12	7,02	12	7,0175	0,77	0,54	0,16	0,95	0,91	0,2439	Q31 C+P+
31	4	4	4	8	4,68	8	4,6784	1,10	0,47	0,18	0,76	0,76	0,3516	Q31 C+P+
32	0	0	0	17	9,66	17	9,6591	-0,14	0,38	0,10	0,86	0,85	-0,3147	Q32 C+P+
32	1	1	1	41	23,30	41	23,2955	0,11	0,43	0,07	0,86	0,84	-0,2549	Q32 C+P+
32	2	2	2	93	52,84	93	52,8409	0,42	0,43	0,04	0,86	0,86	0,1846	Q32 C+P+
32	3	3	3	25	14,20	25	14,2045	0,70	0,45	0,09	0,92	0,93	0,311	Q32 C+P+
33	0	0	0	71	40,57	71	40,5714	0,10	0,43	0,05	0,92	0,92	-0,3859	Q33 C+P-
33	1	1	1	85	48,57	85	48,5714	0,43	0,45	0,05	0,90	0,90	0,1952	Q33 C+P-
33	2	2	2	19	10,86	19	10,8571	0,74	0,43	0,10	0,89	0,88	0,2954	Q33 C+P-

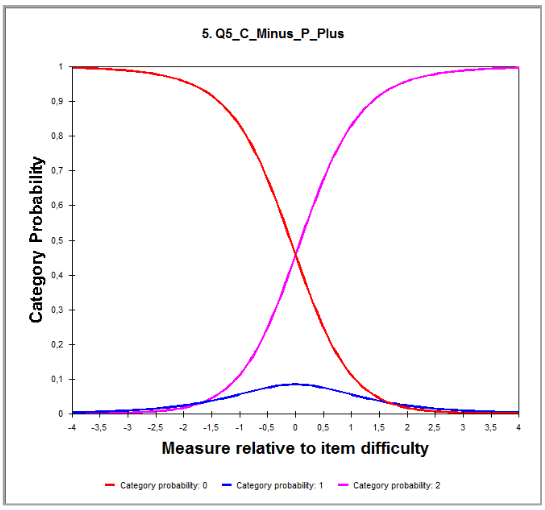
8.6 APPENDIX 6: ITEM CATEGORY PROBABILITY CURVES (BEFORE COLLAPSE)

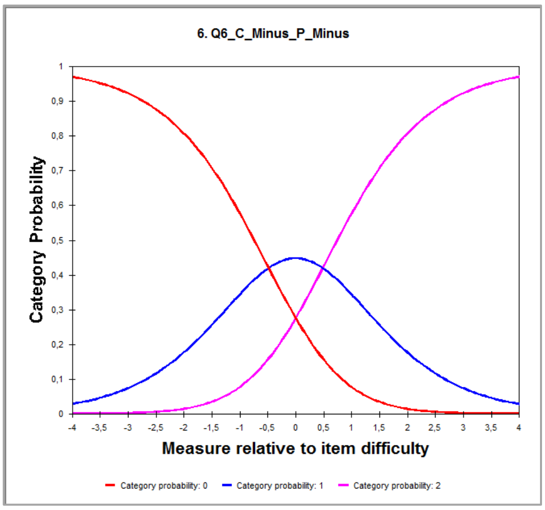


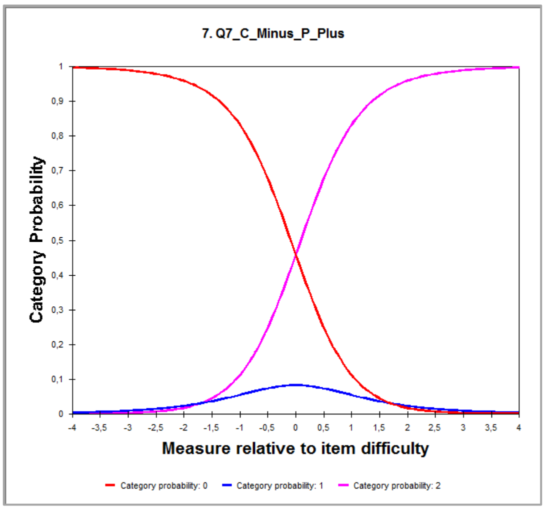


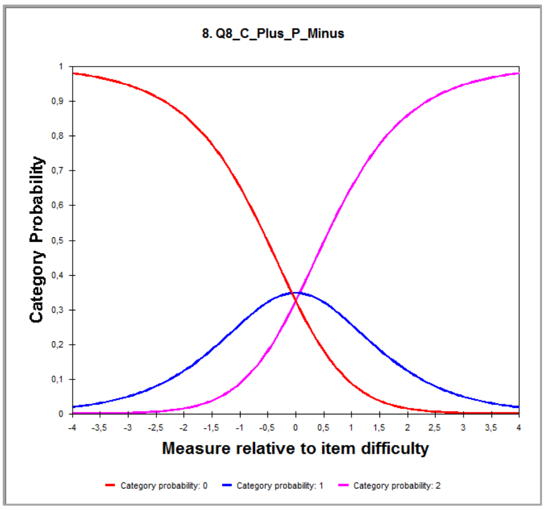


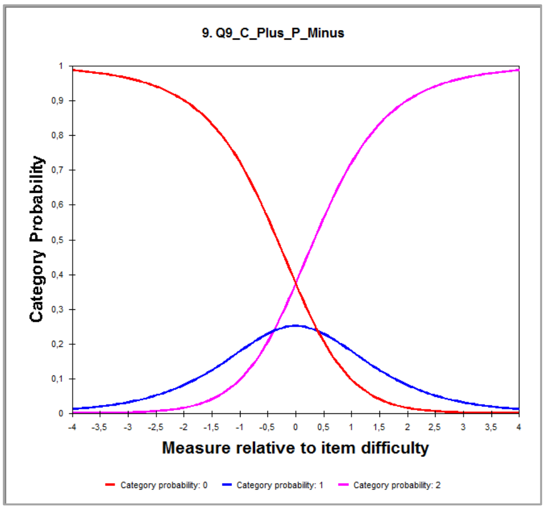


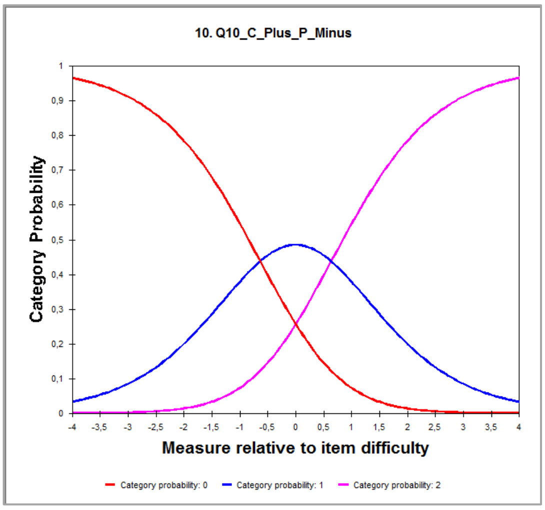


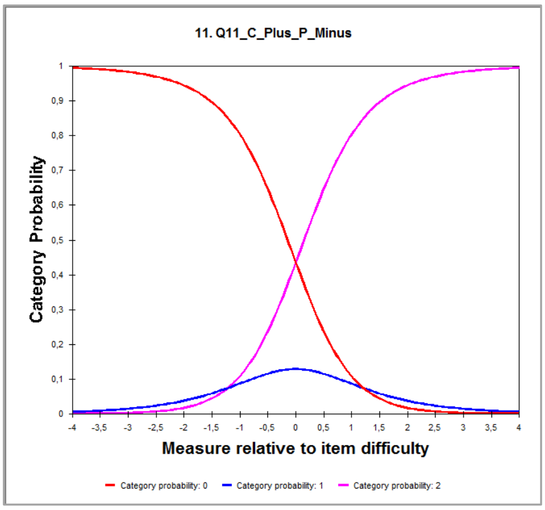


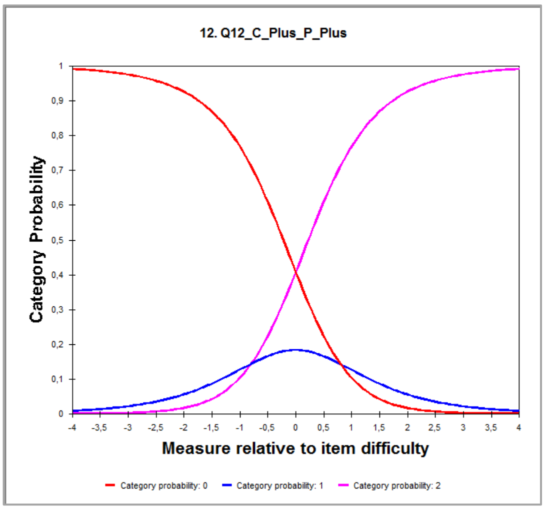

7. Q7_C_Minus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

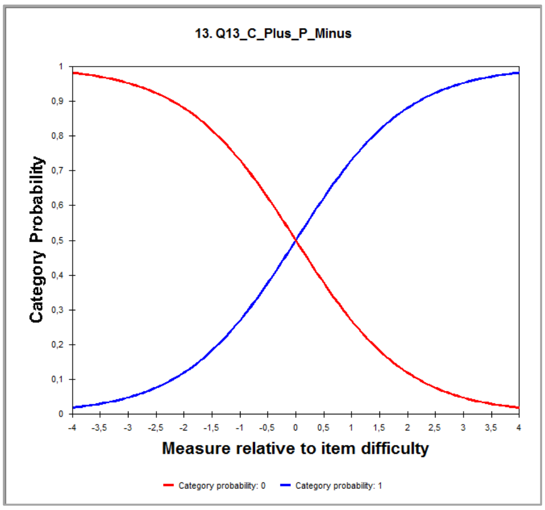

8. Q8_C_Plus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

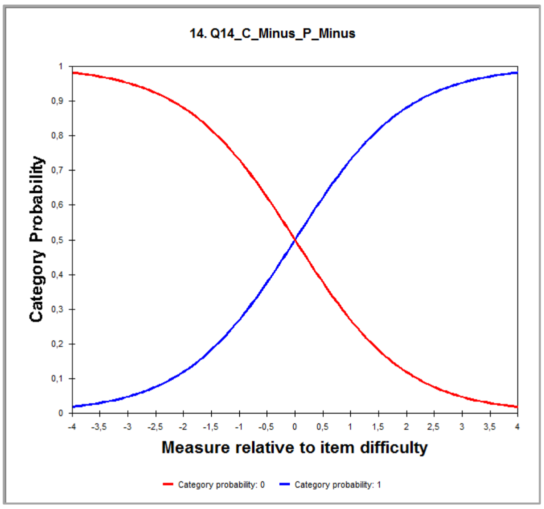

9. Q9_C_Plus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

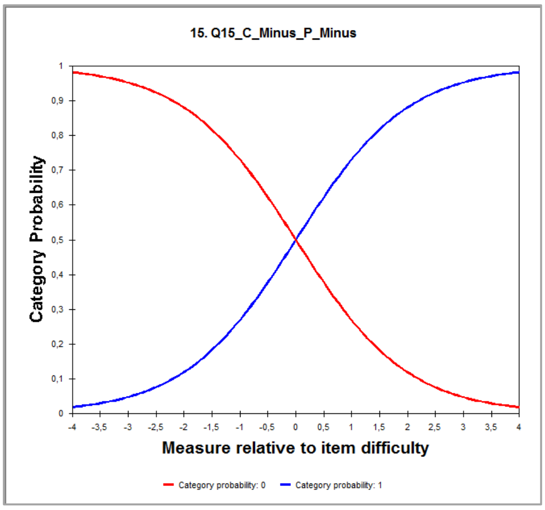

10. Q10_C_Plus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

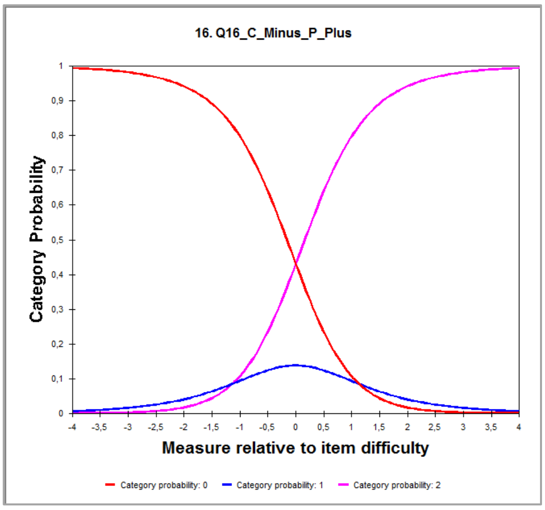

11. Q11_C_Plus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

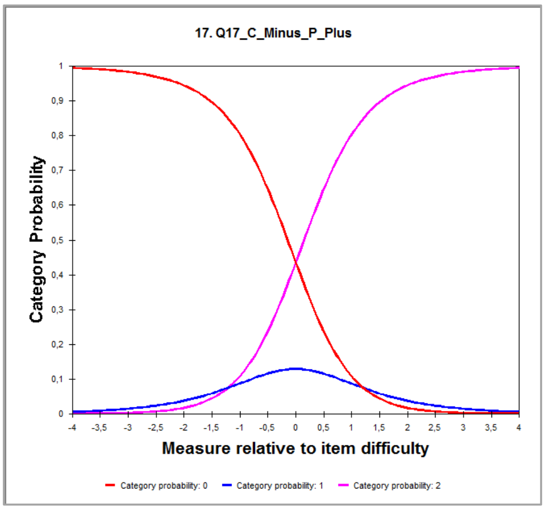

12. Q12_C_Plus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

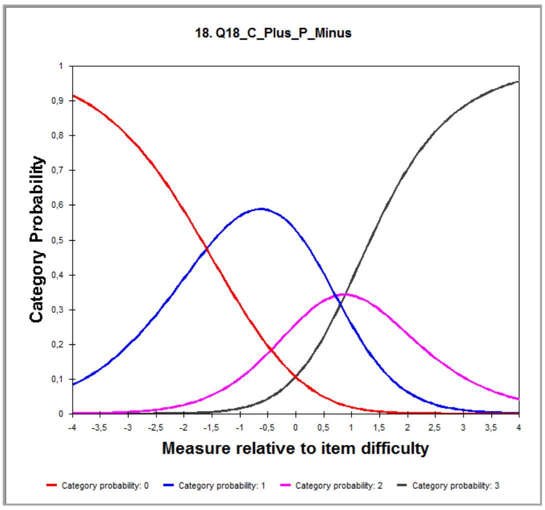

13. Q13_C_Plus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1

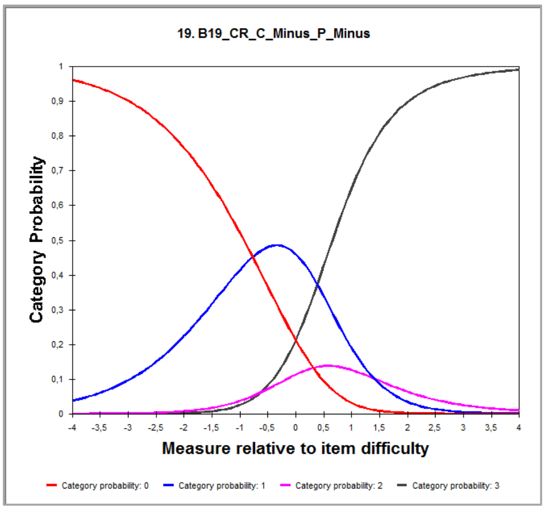

14. Q14_C_Minus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1

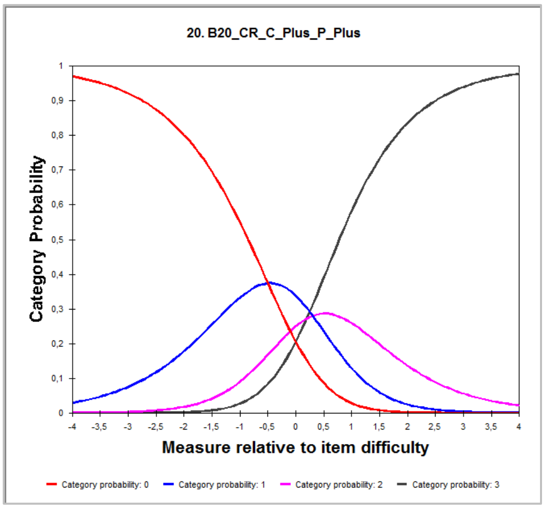

15. Q15_C_Minus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1

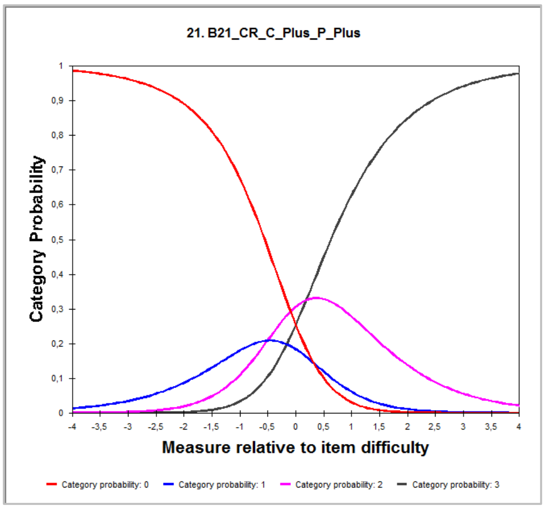

16. Q16_C_Minus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

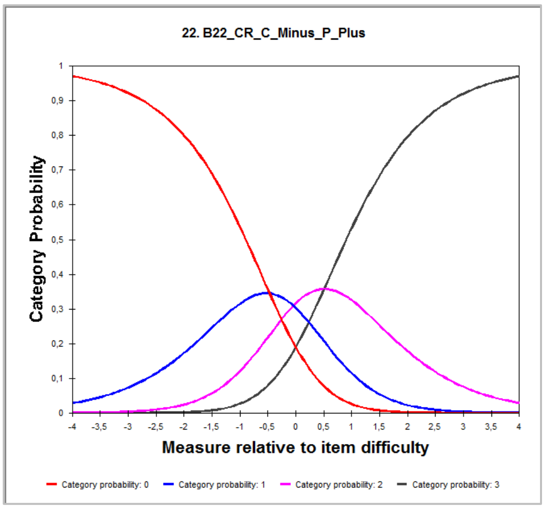

17. Q17_C_Minus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

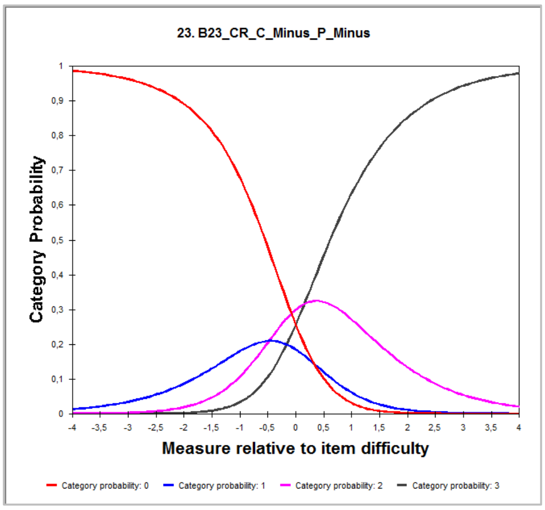

18. Q18_C_Plus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2
Category probability: 3

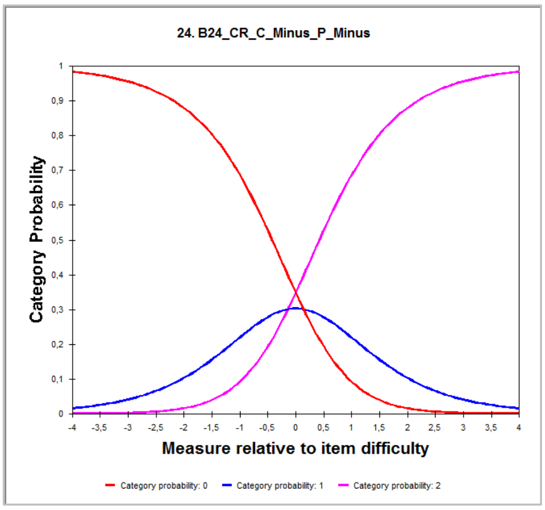

19. B19_CR_C_Minus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2
Category probability: 3

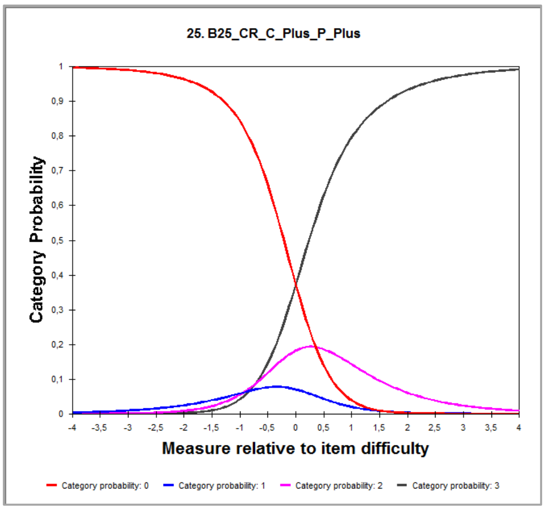

20. B20_CR_C_Plus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2
Category probability: 3

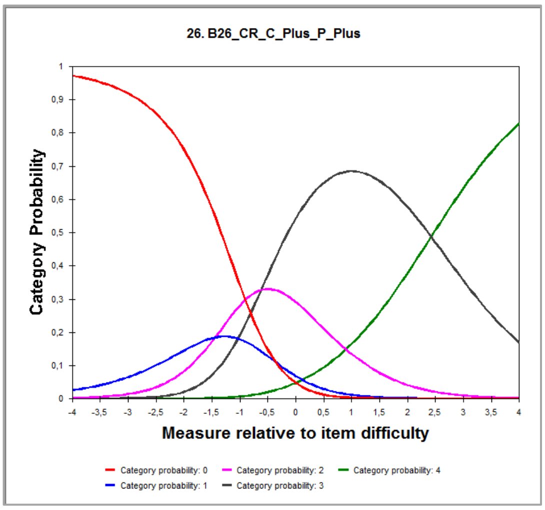

21. B21_CR_C_Plus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2
Category probability: 3

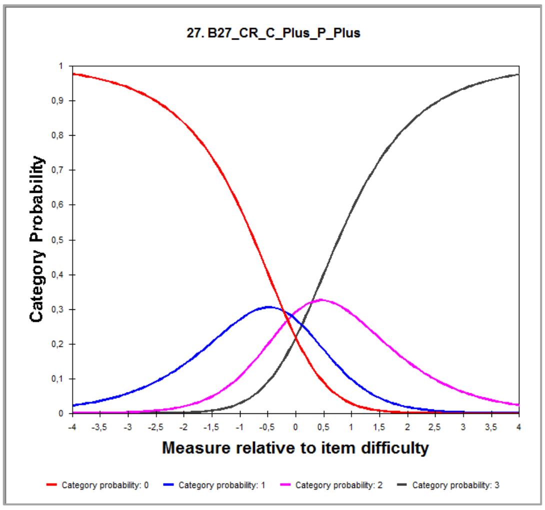

22. B22_CR_C_Minus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2
Category probability: 3

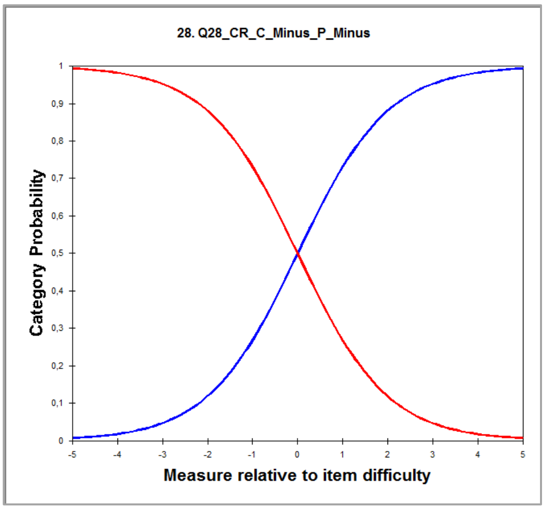

23. B23_CR_C_Minus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2
Category probability: 3

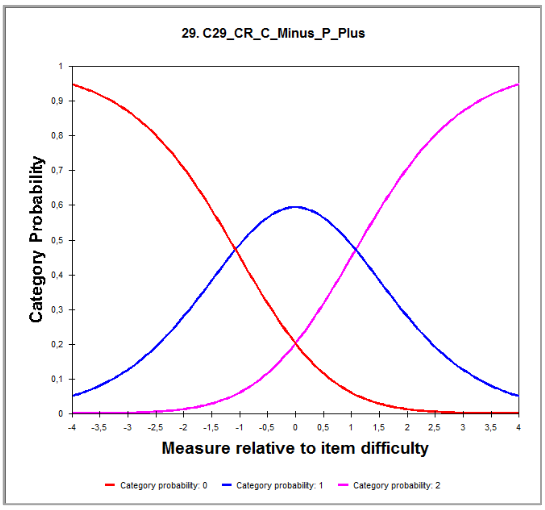

24. B24_CR_C_Minus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

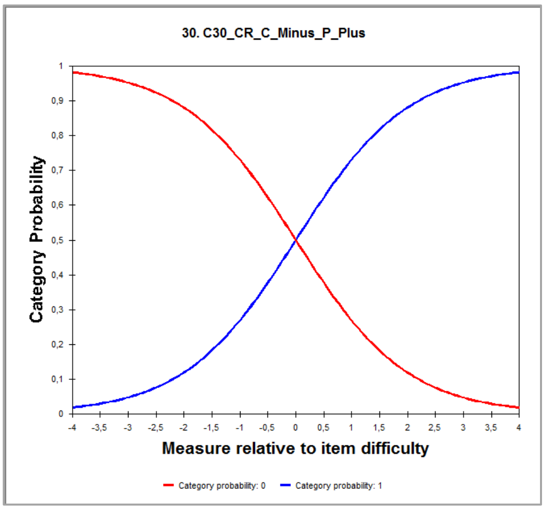

25. B25_CR_C_Plus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2
Category probability: 3

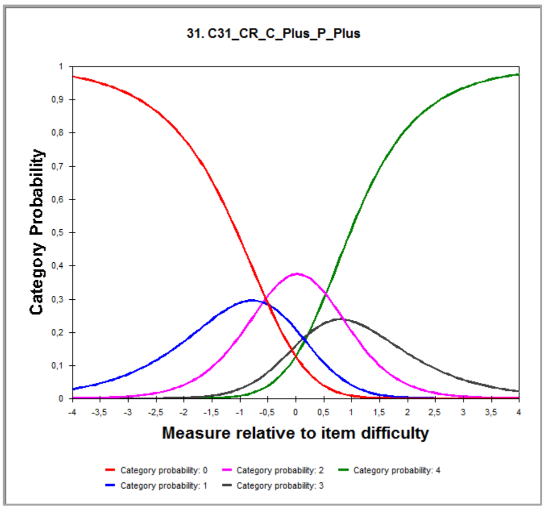

26. B26_CR_C_Plus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2
Category probability: 3
Category probability: 4

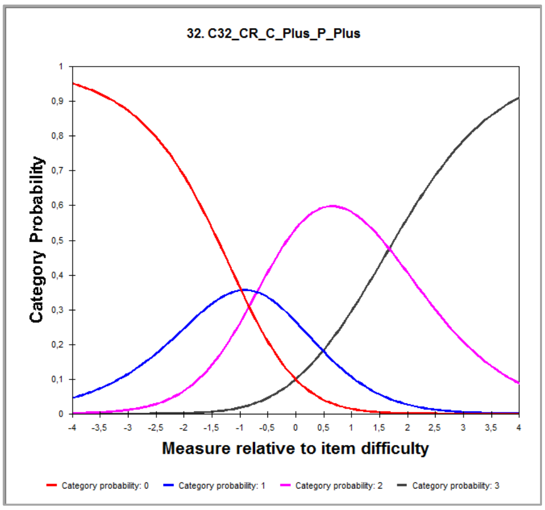

27. B27_CR_C_Plus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2
Category probability: 3

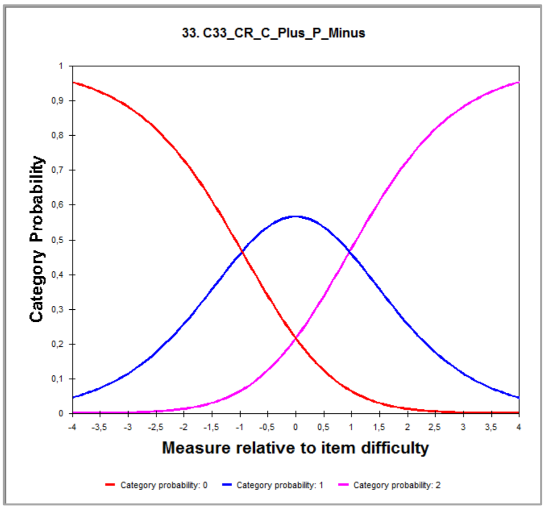

28. Q28_CR_C_Minus_P_Minus
Category Probability
Measure relative to item difficulty


29. C29_CR_C_Minus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2


30. C30_CR_C_Minus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1


31. C31_CR_C_Plus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 2
Category probability: 4
Category probability: 1
Category probability: 3


32. C32_CR_C_Plus_P_Plus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2
Category probability: 3


33. C33_CR_C_Plus_P_Minus
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

8.7 APPENDIX 7: GLOBAL FIT STATISTICS

Global fit statistics Collapse 08 November 2018

TABLE 44.1 Winsteps_Collapsed_data_08.11.2018.xl ZOU841WS.TXT Nov 8 2018 14:47

INPUT: 192 PERSON 33 ITEM REPORTED: 192 PERSON 33 ITEM 78 CATS WINSTEPS 3.93.1

Global Statistics:

Active PERSON: 192

Active ITEM: 33

Active datapoints: 6071 = 95.8% of Active+Missing datapoints

Non-extreme datapoints: 6071

Missing datapoints: 265 = 4.2% of Active+Missing datapoints

Standardised residuals N(0,1): mean: .00 P.SD: 1.00

Log-likelihood chi-squared: 7960.3905 with approximately 7955 d.f., probability = .4809

Global Root-Mean-Square Residual: .5008 with expected value: .5015

Capped Binomial Deviance: .2499 for 3924.0 dichotomies with expected value: .2468

8.8 APPENDIX 8: IFILE

ENTRY	MEASURE	COUNT	SCORE	MODLSE	IN.MSQ	IN.ZSTD	OUT.MSQ	OUT.ZSTD	PTMA	OBSMATCH	EXPMATCH	PTMA-E	RMSR	WMLE	NAME
1	1,36	188	49	0,17	1,0123	0,171	1,0544	0,5011	0,258	75	74,939	0,285	0,4226	1,35	Q1 C-P-
2	1,18	190	56	0,17	1,04	0,571	1,0467	0,501	0,2363	71,579	71,909	0,2926	0,4439	1,17	Q2 C-P+
3	-1,31	189	151	0,19	1,0328	0,341	1,0916	0,6511	0,2018	80,423	80,048	0,2638	0,3918	-1,3	Q3 C-P-
4	0,66	190	76	0,16	1,1827	3,4212	1,2634	3,6813	0,0445	56,842	65,006	0,3105	0,5062	0,66	Q4 C+P-
5	-1,76	190	163	0,21	0,9983	0,041	0,9605	-0,139	0,2419	85,789	85,784	0,2339	0,3381	-1,74	Q5 C-P+
6	2,32	190	24	0,22	0,988	-0,019	1,0484	0,291	0,2289	87,368	87,36	0,2239	0,3208	2,3	Q6 C-P-
7	-0,53	189	125	0,16	1,0817	1,2911	1,0917	1,0911	0,1947	66,138	68,96	0,3035	0,4684	-0,53	Q7 C-P+
8	-0,54	190	126	0,16	1,0386	0,631	1,0277	0,351	0,2544	68,421	69,085	0,303	0,4584	-0,54	Q8 C+P-
9	0,07	190	101	0,15	1,0061	0,151	0,9917	-0,129	0,3122	65,263	63,121	0,3159	0,4747	0,07	Q9 C+P-
10	1,12	190	58	0,17	1,0859	1,2211	1,1581	1,6212	0,1594	69,474	71,074	0,295	0,4579	1,12	Q10 C+P-
11	-0,26	189	114	0,16	1,0406	0,781	1,1226	1,7711	0,2358	65,079	65,689	0,3117	0,4738	-0,25	Q11 C+P-
12	0,38	190	88	0,15	0,9916	-0,169	0,9753	-0,419	0,3317	62,632	63,069	0,3153	0,4711	0,37	Q12 C+P+
13	0,28	190	92	0,15	0,9718	-0,629	0,9727	-0,479	0,3523	62,632	62,85	0,316	0,4673	0,28	Q13 C+P-
14	-1,11	190	146	0,18	0,9663	-0,329	0,8672	-1,0291	0,3484	76,316	77,304	0,2759	0,3976	-1,1	Q14 C-P-
15	0,16	190	97	0,15	1,0848	1,8811	1,0818	1,4611	0,2055	56,842	62,81	0,3162	0,4938	0,16	Q15 C-P-
16	-1,24	190	150	0,19	0,9833	-0,129	0,9831	-0,069	0,2899	78,947	79,198	0,268	0,3884	-1,24	Q16 C-P+
17	0,25	187	92	0,15	0,942	-1,3291	0,9219	-1,4391	0,3982	65,241	62,732	0,3142	0,4605	0,25	Q17 C-P+
18	-0,76	182	250	0,13	0,957	-0,449	0,992	-0,049	0,426	63,187	56,311	0,3859	0,5768	-0,76	Q18 C+P-
19	-1,17	182	293	0,13	0,9696	-0,219	0,9439	-0,3291	0,4138	68,132	66,261	0,3762	0,558	-1,16	Q19 C-P-
20	-0,1	183	206	0,12	1,1021	1,1611	1,1012	1,1411	0,3042	57,377	54,753	0,4019	0,6515	-0,1	Q20 C+P+
21	2,52	178	38	0,18	0,9576	-0,229	0,9077	-0,4291	0,3639	81,461	80,269	0,2907	0,4178	2,51	Q21 C+P+
22	-0,73	183	244	0,13	0,9166	-0,9191	0,9418	-0,6291	0,4592	60,656	57,115	0,3769	0,5506	-0,73	Q22 C-P+
23	0,06	179	194	0,11	0,9565	-0,509	0,9713	-0,309	0,4595	48,045	48,06	0,4242	0,6605	0,06	Q23 C-P-
24	0,53	178	77	0,16	0,9903	-0,179	0,9938	-0,079	0,3283	61,798	63,786	0,3163	0,4675	0,53	Q24 C-P-
25	-0,97	182	296	0,12	1,2002	1,5412	1,2658	1,3213	0,2623	65,385	67,658	0,4039	0,672	-0,96	Q25 C+P+
26	-0,04	173	197	0,11	0,9608	-0,449	0,9503	-0,549	0,4533	46,821	47,825	0,417	0,6651	-0,04	Q26 C+P+
27	-0,73	183	250	0,12	0,8709	-1,4591	0,9014	-1,0591	0,5112	59,016	55,231	0,3881	0,555	-0,73	Q27 C+P+
28	-0,07	174	99	0,16	0,9131	-1,7691	0,8899	-1,7791	0,4363	68,391	64,267	0,3153	0,4488	-0,06	Q28 C-P-
29	-0,49	176	116	0,17	0,9809	-0,279	0,954	-0,519	0,3345	69,318	68,674	0,3028	0,4468	-0,48	Q29 C-P+
30	-0,06	174	99	0,16	0,9576	-0,839	0,9243	-1,1991	0,3817	63,218	64,223	0,3146	0,4597	-0,06	Q30 C-P+
31	0,94	171	128	0,13	0,9436	-0,5891	0,9411	-0,6191	0,4483	59,649	56,353	0,3842	0,5825	0,94	Q31 C+P+
32	-0,97	176	277	0,13	0,9432	-0,4691	0,8717	-0,8291	0,4539	65,341	63,606	0,3912	0,5855	-0,96	Q32 C+P+
33	1,02	175	123	0,13	0,9398	-0,6391	0,9229	-0,8191	0,456	60	55,64	0,3906	0,5821	1,02	Q33 C+P-

8.9 APPENDIX 9: PFILE

ENTRY	MEASURE	IN.MSQ	IN.ZSTD	OUT.MSQ	OUT.ZSTD	OBSMATCH	EXPMATCH	RMSR	WMLE
1	0,65	0,8445	-0,6492	0,9018	-0,3191	66,667	66,718	0,4552	0,63
2	1,18	1,2401	0,9612	1,0046	0,121	72,727	72,632	0,5088	1,15
3	0,2	0,6999	-1,5493	0,6246	-1,4194	88,235	68,136	0,3751	0,19
4	1,49	0,9554	-0,069	1,0636	0,2911	81,818	76,125	0,4212	1,45
5	0,9	0,6063	-1,8494	0,5645	-1,7294	78,788	69,204	0,3723	0,88
6	0,36	0,8466	-0,6492	0,8681	-0,5191	53,125	64,42	0,471	0,35
7	-0,92	0,9812	-0,029	0,9288	-0,1591	57,576	65,811	0,5116	-0,91
8	-1,32	1,2528	0,8713	1,6746	1,1617	81,25	76,33	0,4448	-1,25
9	0,54	0,985	0,011	0,8762	-0,4391	67,742	66,92	0,4861	0,52
10	0,13	1,2234	1,0312	1,1359	0,6711	56,25	63,91	0,577	0,12
11	0,52	0,7564	-1,1292	0,8122	-0,7692	75,758	65,408	0,4371	0,51
12	-0,48	0,6493	-1,9394	0,667	-1,5493	72,727	62,468	0,4267	-0,48
13	-0,14	0,7581	-1,2092	0,7367	-1,1493	78,261	63,468	0,4191	-0,14
14	0,41	1,2041	0,9512	1,293	1,2613	57,576	64,509	0,5584	0,39
15	0,16	1,0998	0,4811	1,2756	1,0913	53,846	65,594	0,5263	0,15
16	0,18	0,8745	-0,5491	0,888	-0,4691	66,667	63,714	0,4854	0,17
17	0,41	1,274	1,2313	1,259	1,1313	66,667	64,509	0,5744	0,39
18	0,14	1,0908	0,4811	1,1292	0,6511	56,25	63,98	0,5403	0,13
19	0,11	0,9233	-0,2891	1,1717	0,8212	65,625	63,71	0,502	0,1
20	0,52	1,2186	0,9912	1,3481	1,4013	60,606	65,408	0,5548	0,51
21	0,29	1,1883	0,9012	1,158	0,7612	54,545	64,012	0,5608	0,28
22	1,84	1,003	0,111	0,9852	0,131	75,758	80,116	0,3999	1,79
23	-0,04	0,6788	-1,6793	0,6936	-1,5593	72,727	63,414	0,4334	-0,05
24	0,07	1,3371	1,5213	1,3017	1,3813	66,667	63,534	0,6047	0,06
25	0,77	0,8924	-0,3991	0,9196	-0,2191	81,818	68,121	0,4603	0,75
26	0,9	1,0502	0,2911	0,844	-0,4792	66,667	69,204	0,49	0,88
27	-0,15	0,8906	-0,4791	1,0098	0,121	60,606	62,557	0,4985	-0,16
28	0,07	0,9101	-0,3691	0,8714	-0,5691	63,636	63,534	0,4989	0,06
29	-0,24	1,216	1,0212	1,2023	0,9112	58,065	62,13	0,5786	-0,25
30	0,41	1,162	0,7812	1,1438	0,6811	63,636	64,509	0,5486	0,39
31	0,41	1,2142	0,9912	1,3749	1,5614	57,576	64,509	0,5608	0,39
32	2,04	0,9652	0,001	0,788	-0,2492	78,788	81,922	0,3741	1,98
33	1,18	1,3516	1,3314	1,0202	0,171	57,576	72,632	0,5311	1,15
34	0,07	1,1439	0,7211	1,1228	0,6311	63,636	63,534	0,5593	0,06
35	-0,81	0,8847	-0,5391	0,9135	-0,2391	78,788	65,098	0,4906	-0,8
36	0,07	1,0081	0,111	1,0664	0,3811	69,697	63,534	0,5251	0,06

37	-0,48	1,0979	0,5411	1,0673	0,3611	54,545	62,468	0,5549	-0,48
38	-0,92	1,2681	1,3013	1,7251	2,1717	51,515	65,811	0,5816	-0,91
39	0,41	1,059	0,3411	1,2193	0,9812	57,576	64,509	0,5237	0,39
40	0,65	1,2573	1,1313	1,3416	1,3213	60,606	66,718	0,5554	0,63
41	0,77	0,9138	-0,3091	0,8386	-0,5492	69,697	68,121	0,4658	0,75
42	0,52	1,0327	0,221	0,9724	-0,039	63,636	65,408	0,5107	0,51
43	-0,04	1,1025	0,5511	1,092	0,4911	60,606	63,414	0,5523	-0,05
44	-0,37	0,769	-1,1792	0,797	-0,8992	72,727	61,512	0,4648	-0,37
45	0,18	0,8218	-0,8192	0,9612	-0,109	60,606	63,714	0,4706	0,17
46	-0,13	1,152	0,7012	1,0626	0,3311	66,667	65,874	0,5574	-0,14
47	-0,48	0,7039	-1,5893	0,7454	-1,1193	72,727	62,468	0,4443	-0,48
48	1,84	1,0789	0,3511	1,2903	0,7213	75,758	80,116	0,4147	1,79
49	-0,48	0,8979	-0,4591	0,9447	-0,1591	60,606	62,468	0,5018	-0,48
50	0,52	0,9218	-0,2891	0,8282	-0,6892	63,636	65,408	0,4825	0,51
51	-0,37	0,9942	0,041	0,8684	-0,5391	66,667	61,512	0,5285	-0,37
52	0,23	1,1063	0,5411	1,1961	0,8512	56,522	65,966	0,5019	0,23
53	-0,26	1,3224	1,5013	1,8092	3,0918	63,636	60,544	0,609	-0,26
54	2,26	0,7807	-0,5492	0,8094	-0,1392	87,879	83,566	0,318	2,19
55	0,18	1,4073	1,7714	1,1673	0,8112	51,515	63,714	0,6158	0,17
56	0,77	0,9875	0,021	0,9254	-0,1991	63,636	68,121	0,4842	0,75
57	-0,92	1,218	1,0812	1,2898	1,0213	63,636	65,811	0,57	-0,91
58	0,65	1,076	0,4011	1,1595	0,6912	75,758	66,718	0,5138	0,63
59	-0,09	1,2861	1,3313	1,2264	0,8212	47,059	68,965	0,5088	-0,09
60	0,29	0,8426	-0,6192	0,801	-0,7692	62,963	64,69	0,465	0,27
61	0,9	1,2963	1,2213	1,6528	2,0217	66,667	69,204	0,5444	0,88
62	0,07	0,7129	-1,4493	0,7107	-1,4593	75,758	63,534	0,4415	0,06
63	0,65	1,4549	1,8415	1,3438	1,3213	66,667	66,718	0,5975	0,63
64	0,77	0,9768	-0,019	0,9331	-0,1691	63,636	68,121	0,4816	0,75
65	-0,04	1,041	0,261	1,1331	0,6811	63,636	63,414	0,5367	-0,05
66	0,61	1,1298	0,6211	1,0586	0,3111	68,75	66,332	0,5302	0,59
67	0,41	1,4842	2,0115	1,2708	1,1813	60,606	64,509	0,62	0,39
68	0,41	1,0934	0,4911	1,0645	0,3511	63,636	64,509	0,5322	0,39
69	-0,59	0,7495	-1,3193	0,8554	-0,5391	66,667	63,327	0,4572	-0,58
70	-0,04	1,016	0,151	0,9744	-0,049	60,606	63,414	0,5302	-0,05
71	0,77	1,0041	0,091	0,9911	0,061	63,636	68,121	0,4882	0,75
72	1,18	1,0227	0,171	1,248	0,8012	66,667	72,632	0,462	1,15
73	-0,82	0,8244	-0,7492	0,7402	-0,7793	66,667	66,269	0,4661	-0,81
74	1,65	0,9936	0,081	0,8812	-0,1391	72,727	78,176	0,4146	1,62
75	1,18	0,9276	-0,2091	1,0295	0,201	72,727	72,632	0,44	1,15
76	0,41	0,8526	-0,6391	0,8648	-0,5491	63,636	64,509	0,4699	0,39
77	0,65	0,5901	-2,0594	0,5934	-1,8294	78,788	66,718	0,3805	0,63

78	-0,96	1,5401	2,3615	2,2546	3,4023	54,839	66,036	0,6401	-0,95
79	-0,41	1,0895	0,4511	0,9911	0,061	62,963	64,237	0,5443	-0,41
80	0,31	0,9477	-0,1391	0,9279	-0,2191	53,846	64,065	0,4949	0,3
81	-0,7	1,2351	1,1712	1,0969	0,4511	57,576	64,269	0,5839	-0,69
82	0,52	0,9917	0,041	1,0487	0,281	63,636	65,408	0,5005	0,51
83	-0,03	0,5927	-2,2394	0,6107	-2,1894	71,875	62,723	0,4079	-0,04
84	-0,48	0,825	-0,8592	0,9054	-0,3391	66,667	62,468	0,481	-0,48
85	-0,7	1,2866	1,1913	1,3346	0,9213	64,706	72,082	0,491	-0,66
86	0,18	0,8299	-0,7792	0,9423	-0,1991	66,667	63,714	0,4729	0,17
87	-0,48	0,9292	-0,2991	0,9576	-0,109	60,606	62,468	0,5105	-0,48
88	1,04	0,7583	-0,9892	0,8429	-0,4392	84,848	70,84	0,4076	1,01
89	-0,7	0,8822	-0,5491	0,9942	0,061	63,636	64,269	0,4935	-0,69
90	0,65	0,8416	-0,6692	0,8188	-0,6892	66,667	66,718	0,4544	0,63
91	-0,48	1,0595	0,3611	0,9013	-0,3591	66,667	62,468	0,5451	-0,48
92	-0,3	1,2443	1,1712	1,0729	0,4111	61,29	60,735	0,5952	-0,3
93	0,74	0,8477	-0,5592	0,7818	-0,7092	59,259	67,57	0,4438	0,72
94	0,29	1,0528	0,3111	1,0352	0,231	66,667	64,012	0,5278	0,28
95	-0,15	0,9498	-0,1791	0,8487	-0,6792	66,667	62,557	0,5148	-0,16
96	1,18	1,2275	0,9212	1,0138	0,151	69,697	72,632	0,5062	1,15
97	0,36	0,9103	-0,3391	0,9742	-0,039	59,375	64,418	0,4884	0,35
98	0,18	0,9514	-0,159	0,953	-0,149	60,606	63,714	0,5063	0,17
99	1,65	1,1389	0,5511	0,7143	-0,5893	78,788	78,176	0,4438	1,62
100	0,52	1,2384	1,0712	1,2194	0,9412	60,606	65,408	0,5593	0,51
101	0,65	0,8795	-0,4791	0,7923	-0,8092	72,727	66,718	0,4646	0,63
102	0,52	0,9265	-0,2691	0,8495	-0,5892	69,697	65,408	0,4838	0,51
103	-0,15	0,6668	-1,7693	0,7313	-1,3193	66,667	62,557	0,4313	-0,16
104	-0,59	1,0049	0,091	0,8628	-0,5091	66,667	63,327	0,5294	-0,58
105	-0,26	0,8472	-0,7192	0,7856	-0,9892	66,667	60,544	0,4875	-0,26
106	0,9	1,4843	1,8515	1,6138	1,9216	57,576	69,204	0,5825	0,88
107	0,19	0,8701	-0,5491	1,0583	0,3311	65,625	64,273	0,4767	0,18
108	-0,81	0,9029	-0,4391	1,058	0,3011	66,667	65,098	0,4957	-0,8
109	1,05	0,8956	-0,3091	0,7234	-0,7293	73,913	73,729	0,4168	1,02
110	0,41	0,8084	-0,8692	0,8403	-0,6692	63,636	64,509	0,4576	0,39
111	-0,48	1,0427	0,281	1,1352	0,6311	60,606	62,468	0,5408	-0,48
112	-1,1	0,821	-0,8392	0,8024	-0,5392	71,875	67,656	0,4551	-1,08
113	-0,37	0,8203	-0,8792	0,7768	-1,0092	72,727	61,512	0,4801	-0,37
114	-0,04	0,7777	-1,0892	0,745	-1,2593	60,606	63,414	0,4639	-0,05
115	1,33	0,6541	-1,3993	0,5793	-1,2694	84,848	74,767	0,3595	1,3
116	-0,26	0,8957	-0,4591	0,9697	-0,069	72,727	60,544	0,5012	-0,26
117	-0,59	1,0726	0,4211	1,1286	0,5911	69,697	63,327	0,5469	-0,58
118	-0,81	0,9707	-0,079	0,9431	-0,1191	66,667	65,098	0,5139	-0,8

119	0,65	1,015	0,141	1,0615	0,3211	66,667	66,718	0,4991	0,63
120	-1,54	0,8333	-0,6092	0,6236	-0,9294	78,788	72,105	0,4256	-1,51
121	0,29	0,9108	-0,3591	0,8526	-0,6391	69,697	64,012	0,4909	0,28
122	0,18	0,9145	-0,3391	0,9505	-0,159	60,606	63,714	0,4964	0,17
123	-0,48	1,2948	1,4213	1,1709	0,7712	60,606	62,468	0,6026	-0,48
124	-0,81	0,6685	-1,8293	0,6581	-1,3493	72,727	65,098	0,4265	-0,8
125	-1,29	1,0746	0,3711	0,8248	-0,2692	64	70,243	0,4944	-1,26
126	-0,19	1,6832	1,9917	1,5131	1,5515	37,5	55,54	0,7825	-0,2
127	-1,13	0,8665	-0,4891	1,0618	0,2911	69,565	72,712	0,4131	-1,09
128	0,41	0,9357	-0,2291	0,8709	-0,5191	63,636	64,509	0,4923	0,39
129	-0,37	1,0729	0,4111	1,4255	1,6514	46,667	62,714	0,5399	-0,37
130	0,18	1,3167	1,4313	1,2545	1,1713	63,636	63,714	0,5956	0,17
131	0,77	0,8441	-0,6292	0,8328	-0,5792	63,636	68,121	0,4476	0,75
132	1,18	0,6484	-1,4994	0,5912	-1,3394	78,788	72,632	0,3679	1,15
133	-0,15	1,0678	0,3911	1,1077	0,5611	36,364	62,557	0,5458	-0,16
134	-0,04	0,6633	-1,7693	0,7288	-1,3493	72,727	63,414	0,4284	-0,05
135	0,07	0,9987	0,061	0,8991	-0,4191	63,636	63,534	0,5226	0,06
136	-0,44	1,1009	0,5511	1,2544	1,0813	59,375	63,513	0,5447	-0,43
137	0,41	1,4828	2,0015	1,2943	1,2613	60,606	64,509	0,6197	0,39
138	0,77	1,2058	0,9112	1,4224	1,4914	78,788	68,121	0,535	0,75
139	0,52	0,9685	-0,069	0,9453	-0,1591	57,576	65,408	0,4946	0,51
140	0,8	0,6186	-1,6994	0,5671	-1,7094	72,414	67,801	0,3855	0,77
141	1,04	1,1154	0,5311	1,3038	1,0113	66,667	70,84	0,4943	1,01
142	1,04	1,064	0,3411	0,8319	-0,4792	75,758	70,84	0,4828	1,01
143	0,65	1,5377	2,1215	1,572	2,0316	63,636	66,718	0,6143	0,63
144	-1,54	1,0249	0,181	1,0224	0,191	72,727	72,105	0,472	-1,51
145	-0,01	0,8905	-0,4291	0,9583	-0,109	70	64,506	0,482	-0,01
146	1,04	1,0007	0,081	0,9088	-0,2091	72,727	70,84	0,4682	1,01
147	0,18	0,8116	-0,8792	0,7743	-1,0792	72,727	63,714	0,4677	0,17
148	0,52	0,6285	-1,8694	0,6328	-1,7094	69,697	65,408	0,3984	0,51
149	0,07	1,0254	0,191	0,9437	-0,1991	72,727	63,534	0,5295	0,06
150	0,18	0,8302	-0,7792	0,8533	-0,6491	60,606	63,714	0,473	0,17
151	-1,45	0,8802	-0,1691	0,6295	-0,4194	80	59,702	0,5518	-1,39
152	-0,48	0,9403	-0,2391	0,969	-0,059	60,606	62,468	0,5135	-0,48
153	0,77	1,7093	2,6117	1,8671	2,7019	54,545	68,121	0,637	0,75
154	0,66	0,92	-0,2291	0,8013	-0,6392	67,857	67,464	0,4721	0,64
155	1,11	1,1682	0,7112	1,0018	0,111	78,125	71,777	0,4939	1,09
156	0,29	0,9402	-0,2091	0,9164	-0,3191	57,576	64,012	0,4988	0,28
157	0,29	0,8079	-0,8792	0,7226	-1,3293	75,758	64,012	0,4624	0,28
158	-0,04	1,1839	0,9012	1,2182	1,0412	60,606	63,414	0,5723	-0,05
159	-0,09	0,9063	-0,3891	0,9239	-0,1691	70,588	68,965	0,4271	-0,09

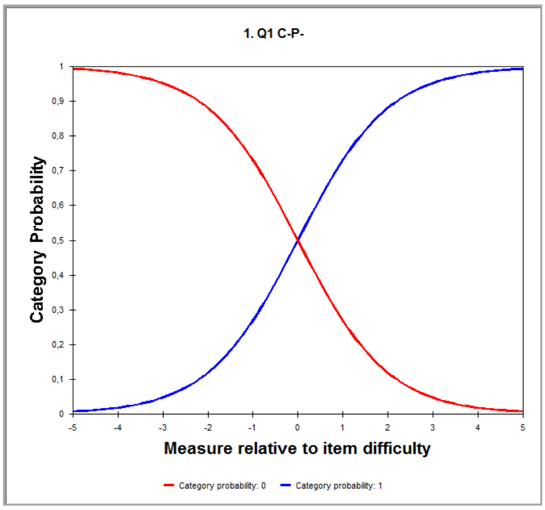
160	0,9	0,8255	-0,6992	1,0278	0,191	78,788	69,204	0,4344	0,88
161	-1,32	1,1467	0,6811	1,0245	0,191	70,968	69,74	0,5202	-1,29
162	1,04	1,1608	0,7012	1,2017	0,7212	72,727	70,84	0,5043	1,01
163	-0,73	1,3984	1,8414	1,2474	0,9512	56,25	65,27	0,6134	-0,73
164	0,41	0,901	-0,3991	0,8629	-0,5591	57,576	64,509	0,4831	0,39
165	-0,26	1,0982	0,5311	1,0002	0,071	60,606	60,544	0,555	-0,26
166	0,07	0,7722	-1,1092	0,7802	-1,0592	69,697	63,534	0,4596	0,06
167	0,9	1,1324	0,6111	1,0819	0,3811	63,636	69,204	0,5088	0,88
168	-0,15	1,1124	0,5911	1,1703	0,8312	54,545	62,557	0,5571	-0,16
169	-0,15	0,8831	-0,5191	0,7898	-0,9992	66,667	62,557	0,4964	-0,16
170	0,18	0,8326	-0,7592	0,7696	-1,0992	66,667	63,714	0,4737	0,17
171	0,07	0,8205	-0,8392	0,8878	-0,4791	63,636	63,534	0,4737	0,06
172	1,04	0,6448	-1,5794	0,6243	-1,3194	84,848	70,84	0,3758	1,01
173	0,41	0,9439	-0,1891	0,8431	-0,6592	69,697	64,509	0,4944	0,39
174	1,04	1,4286	1,6214	1,9384	2,5219	54,545	70,84	0,5594	1,01
175	-1,03	0,8462	-0,7192	0,7912	-0,6292	63,636	67,033	0,469	-1,02
176	0,14	0,6497	-1,6194	0,6492	-1,5694	77,778	64,96	0,4128	0,13
177	0,18	1,1451	0,7211	1,169	0,8212	75,758	63,714	0,5555	0,17
178	0,07	0,7072	-1,4893	0,6934	-1,5593	81,818	63,534	0,4398	0,06
179	-0,23	1,4162	1,8414	1,6557	2,5617	43,75	61,648	0,632	-0,23
180	-1,46	1,2694	1,0813	0,9182	-0,0691	64,516	71,308	0,5329	-1,43
181	1,04	1,178	0,7712	1,1163	0,4711	60,606	70,84	0,508	1,01
182	2,04	0,9162	-0,1491	0,8547	-0,1091	78,788	81,922	0,3645	1,98
183	0,14	0,8038	-0,8092	0,8461	-0,5792	77,778	64,96	0,4591	0,13
184	-0,04	0,5669	-2,4094	0,715	-1,4393	84,848	63,414	0,3961	-0,05
185	0,29	0,8422	-0,6992	0,8166	-0,8192	75,758	64,012	0,4721	0,28
186	-0,92	0,752	-1,2892	1,3781	1,2714	75,758	65,811	0,4479	-0,91
187	0,52	0,9985	0,071	1,0872	0,4411	63,636	65,408	0,5022	0,51
188	0,65	1,0881	0,4511	0,9838	0,021	60,606	66,718	0,5167	0,63
189	1,18	0,9646	-0,059	0,8449	-0,3792	81,818	72,632	0,4487	1,15
190	0,77	0,792	-0,8892	0,9653	-0,039	75,758	68,121	0,4336	0,75
191	0,65	1,0394	0,251	1,0479	0,271	60,606	66,718	0,505	0,63
192	0,52	0,9301	-0,2491	0,8986	-0,3591	63,636	65,408	0,4847	0,51

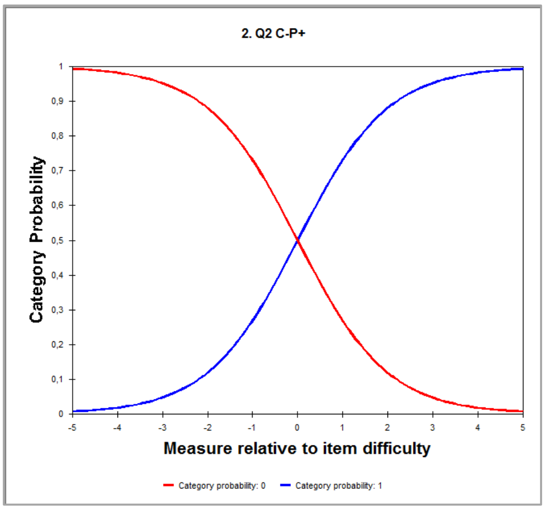
8.10 APPENDIX 10: CATEGORY OPTIONS (AFTER COLLAPSE)

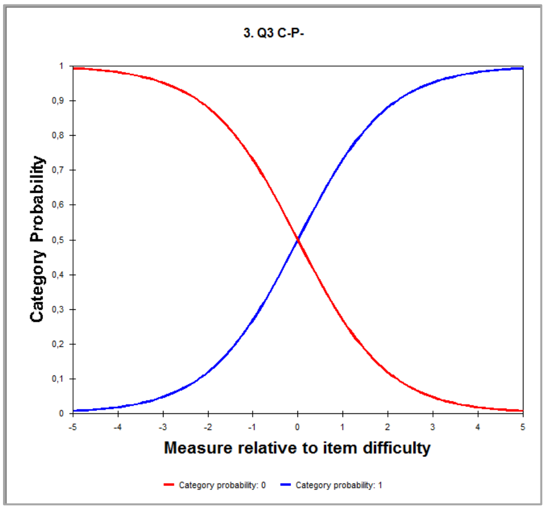
ITEM	CODE	VALUE	SCORE	UNWTD	UNWTD %	WTD	WTD %	AVGE MEAS	P.SD MEAS	S.E. MEAS	INFIT MNSQ	OUTFIT MNSQ	PTMA	LABEL
1	0	0	0	139	73,94	139	73,94	0,13	0,68	0,06	1,00	1,00	-0,26	Q1 C-P-
1	1	1	1	49	26,06	49	26,06	0,56	0,75	0,11	1,03	1,08	0,26	Q1 C-P-
2	0	0	0	134	70,53	134	70,53	0,14	0,69	0,06	1,02	1,02	-0,24	Q2 C-P+
2	1	1	1	56	29,47	56	29,47	0,52	0,71	0,10	1,07	1,05	0,24	Q2 C-P+
3	0	0	0	38	20,11	38	20,11	-0,04	0,74	0,12	1,09	1,11	-0,20	Q3 C-P-
3	1	1	1	151	79,89	151	79,89	0,32	0,70	0,06	1,02	1,02	0,20	Q3 C-P-
4	0	0	0	114	60,00	114	60,00	0,23	0,66	0,06	1,12	1,14	-0,04	Q4 C+P-
4	1	1	1	76	40,00	76	40,00	0,29	0,80	0,09	1,28	1,36	0,04	Q4 C+P-
5	0	0	0	27	14,21	27	14,21	-0,18	0,67	0,13	1,00	0,96	-0,24	Q5 C-P+
5	1	1	1	163	85,79	163	85,79	0,32	0,70	0,06	1,00	1,00	0,24	Q5 C-P+
6	0	0	0	166	87,37	166	87,37	0,19	0,69	0,05	0,99	0,99	-0,23	Q6 C-P-
6	1	1	1	24	12,63	24	12,63	0,68	0,78	0,16	0,98	1,07	0,23	Q6 C-P-
7	0	0	0	64	33,86	64	33,86	0,04	0,67	0,08	1,09	1,09	-0,21	Q7 C-P+
7	1	1	1	125	66,14	125	66,14	0,36	0,72	0,06	1,08	1,08	0,21	Q7 C-P+
8	0	0	0	64	33,68	64	33,68	0,00	0,68	0,09	1,06	1,03	-0,25	Q8 C+P-
8	1	1	1	126	66,32	126	66,32	0,38	0,70	0,06	1,04	1,04	0,25	Q8 C+P-
9	0	0	0	89	46,84	89	46,84	0,01	0,72	0,08	1,03	1,02	-0,31	Q9 C+P-
9	1	1	1	101	53,16	101	53,16	0,46	0,66	0,07	0,99	0,98	0,31	Q9 C+P-
10	0	0	0	132	69,47	132	69,47	0,18	0,70	0,06	1,07	1,09	-0,15	Q10 C+P-
10	1	1	1	58	30,53	58	30,53	0,42	0,74	0,10	1,15	1,21	0,15	Q10 C+P-
11	0	0	0	75	39,68	75	39,68	0,04	0,78	0,09	1,08	1,18	-0,24	Q11 C+P-
11	1	1	1	114	60,32	114	60,32	0,39	0,64	0,06	1,01	1,03	0,24	Q11 C+P-
12	0	0	0	102	53,68	102	53,68	0,03	0,70	0,07	1,01	1,00	-0,33	Q12 C+P+
12	1	1	1	88	46,32	88	46,32	0,51	0,65	0,07	0,98	0,96	0,33	Q12 C+P+
13	0	0	0	98	51,58	98	51,58	0,01	0,62	0,06	0,95	0,94	-0,35	Q13 C+P-
13	1	1	1	92	48,42	92	48,42	0,51	0,72	0,08	1,01	1,01	0,35	Q13 C+P-
14	0	0	0	44	23,16	44	23,16	-0,20	0,55	0,08	0,92	0,83	-0,35	Q14 C-P-
14	1	1	1	146	76,84	146	76,84	0,39	0,71	0,06	1,01	0,99	0,35	Q14 C-P-
15	0	0	0	93	48,95	93	48,95	0,10	0,68	0,07	1,08	1,07	-0,20	Q15 C-P-
15	1	1	1	97	51,05	97	51,05	0,39	0,73	0,07	1,11	1,11	0,20	Q15 C-P-
16	0	0	0	40	21,05	40	21,05	-0,13	0,71	0,11	0,99	1,02	-0,28	Q16 C-P+
16	1	1	1	150	78,95	150	78,95	0,35	0,69	0,06	1,00	1,00	0,28	Q16 C-P+
17	0	0	0	95	50,80	95	50,80	-0,03	0,64	0,07	0,94	0,91	-0,40	Q17 C-P+
17	1	1	1	92	49,20	92	49,20	0,55	0,67	0,07	0,95	0,93	0,40	Q17 C-P+
18	0	0	0	16	8,79	16	8,79	-0,14	0,74	0,19	1,19	1,20	-0,17	Q18 C+P-
18	1	1	1	82	45,05	82	45,05	-0,01	0,68	0,08	0,83	0,89	-0,34	Q18 C+P-
18	2	2	2	84	46,15	84	46,15	0,61	0,59	0,07	0,90	0,91	0,44	Q18 C+P-

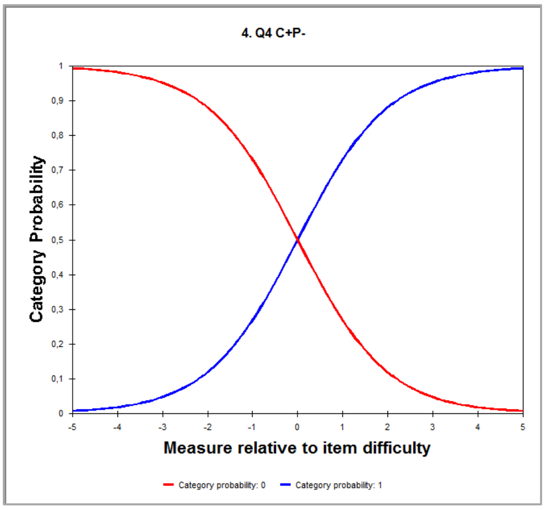
19	0	0	0	13	7,14	13	7,14	-0,40	0,73	0,21	1,07	1,15	-0,25	Q19 C-P-
19	1	1	1	45	24,73	45	24,73	-0,10	0,61	0,09	0,87	0,76	-0,29	Q19 C-P-
19	2	2	2	124	68,13	124	68,13	0,47	0,67	0,06	0,97	0,96	0,41	Q19 C-P-
20	0	0	0	32	17,49	32	17,49	-0,11	0,75	0,13	1,13	1,13	-0,23	Q20 C+P+
20	1	1	1	96	52,46	96	52,46	0,21	0,62	0,06	0,91	0,91	-0,07	Q20 C+P+
20	2	2	2	55	30,05	55	30,05	0,56	0,76	0,10	1,12	1,12	0,27	Q20 C+P+
21	0	0	0	143	80,34	143	80,34	0,12	0,67	0,06	0,92	0,94	-0,41	Q21 C+P+
21	1	1	1	32	17,98	32	17,98	0,90	0,61	0,11	0,74	0,65	0,41	Q21 C+P+
21	2	2	2	3	1,69	3	1,69	0,47	0,45	0,32	1,92	1,68	0,04	Q21 C+P+
22	0	0	0	15	8,20	15	8,20	-0,41	0,75	0,20	0,97	0,97	-0,28	Q22 C-P+
22	1	1	1	92	50,27	92	50,27	0,08	0,69	0,07	0,93	1,03	-0,26	Q22 C-P+
22	2	2	2	76	41,53	76	41,53	0,62	0,56	0,07	0,90	0,91	0,42	Q22 C-P+
23	0	0	0	43	24,02	43	24,02	-0,32	0,73	0,11	0,93	0,92	-0,47	Q23 C-P-
23	1	1	1	78	43,58	78	43,58	0,38	0,58	0,07	0,93	1,02	0,12	Q23 C-P-
23	2	2	2	58	32,40	58	32,40	0,59	0,59	0,08	1,01	1,02	0,30	Q23 C-P-
24	0	0	0	101	56,74	101	56,74	0,02	0,71	0,07	1,01	1,02	-0,33	Q24 C-P-
24	1	1	1	77	43,26	77	43,26	0,50	0,63	0,07	0,97	0,97	0,33	Q24 C-P-
25	0	0	0	21	11,54	21	11,54	-0,22	0,59	0,13	1,33	1,20	-0,24	Q25 C+P+
25	1	1	1	26	14,29	26	14,29	0,11	0,69	0,14	1,10	1,37	-0,09	Q25 C+P+
25	2	2	2	135	74,18	135	74,18	0,37	0,71	0,06	1,27	1,23	0,24	Q25 C+P+
26	0	0	0	38	21,97	38	21,97	-0,21	0,61	0,10	0,94	0,93	-0,38	Q26 C+P+
26	1	1	1	73	42,20	73	42,20	0,26	0,61	0,07	1,00	0,97	-0,05	Q26 C+P+
26	2	2	2	62	35,84	62	35,84	0,65	0,64	0,08	0,97	0,97	0,38	Q26 C+P+
27	0	0	0	17	9,29	17	9,29	-0,43	0,69	0,17	0,93	0,96	-0,31	Q27 C+P+
27	1	1	1	82	44,81	82	44,81	0,01	0,67	0,07	0,84	0,90	-0,32	Q27 C+P+
27	2	2	2	84	45,90	84	45,90	0,65	0,54	0,06	0,83	0,85	0,49	Q27 C+P+
28	0	0	0	75	43,10	75	43,10	-0,08	0,64	0,07	0,91	0,87	-0,43	Q28 C-P-
28	1	1	1	99	56,90	99	56,90	0,55	0,66	0,07	0,92	0,92	0,43	Q28 C-P-
29	0	0	0	60	34,09	60	34,09	-0,06	0,65	0,09	0,97	0,93	-0,34	Q29 C-P+
29	1	1	1	116	65,91	116	65,91	0,46	0,69	0,06	0,98	0,98	0,34	Q29 C-P+
30	0	0	0	75	43,10	75	43,10	-0,03	0,66	0,08	0,96	0,92	-0,38	Q30 C-P+
30	1	1	1	99	56,90	99	56,90	0,52	0,67	0,07	0,97	0,94	0,38	Q30 C-P+
31	0	0	0	63	36,84	63	36,84	-0,03	0,70	0,09	1,02	1,01	-0,36	Q31 C+P+
31	1	1	1	88	51,46	88	51,46	0,38	0,53	0,06	0,86	0,84	0,11	Q31 C+P+
31	2	2	2	20	11,70	20	11,70	1,02	0,77	0,18	0,91	0,90	0,37	Q31 C+P+
32	0	0	0	17	9,66	17	9,66	-0,44	0,56	0,14	0,89	0,89	-0,33	Q32 C+P+
32	1	1	1	41	23,30	41	23,30	-0,05	0,59	0,09	0,92	0,74	-0,25	Q32 C+P+
32	2	2	2	118	67,05	118	67,05	0,50	0,66	0,06	0,96	0,96	0,43	Q32 C+P+
33	0	0	0	71	40,57	71	40,57	-0,08	0,65	0,08	0,95	0,95	-0,41	Q33 C+P-
33	1	1	1	85	48,57	85	48,57	0,44	0,66	0,07	0,98	0,95	0,22	Q33 C+P-
33	2	2	2	19	10,86	19	10,86	0,90	0,51	0,12	0,92	0,90	0,30	Q33 C+P-

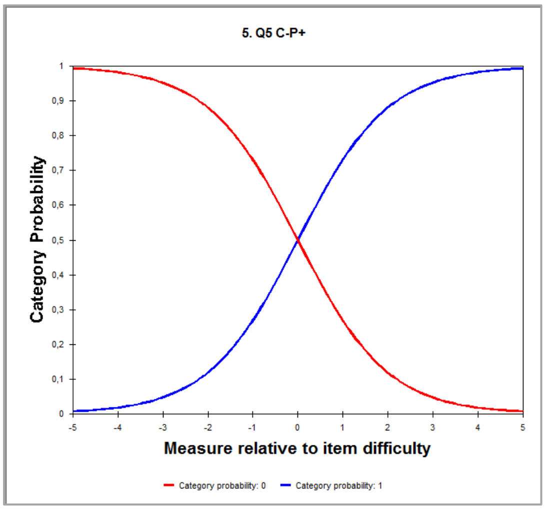
8.11 APPENDIX 11: ITEM CATEGORY PROBABILITY CURVES (AFTER COLLAPSE)

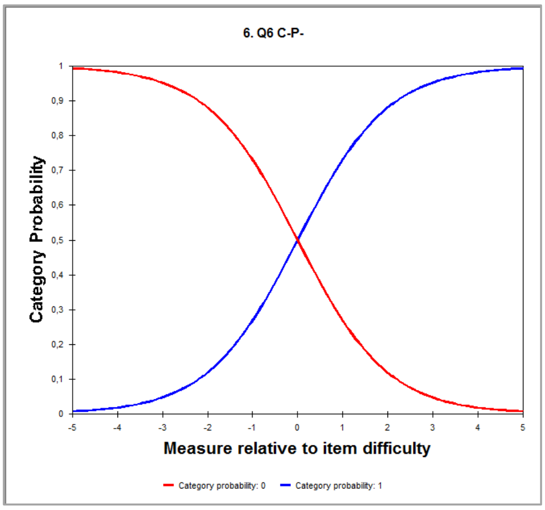


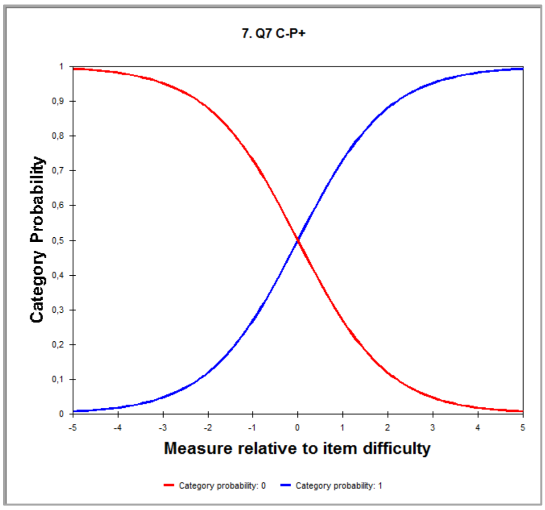


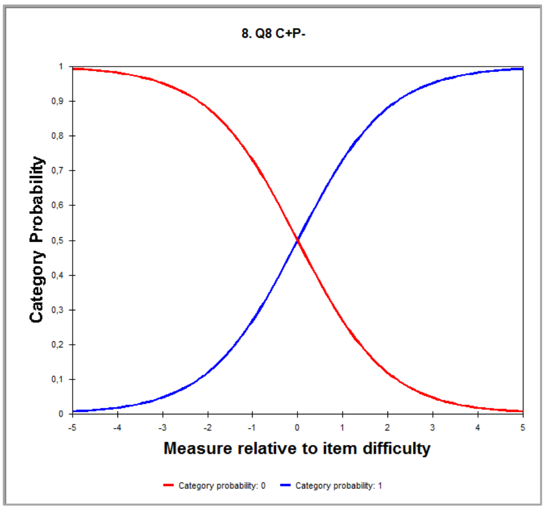


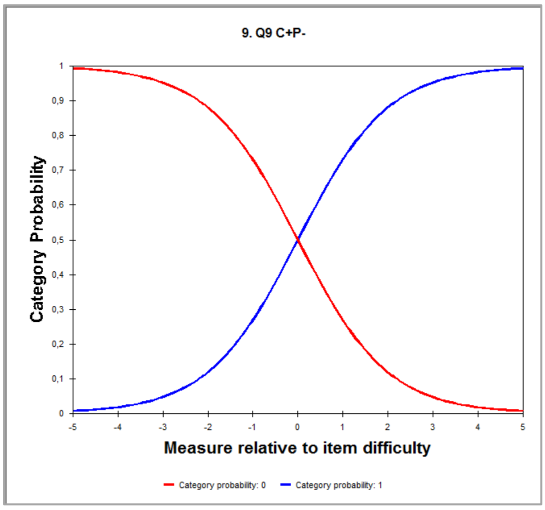


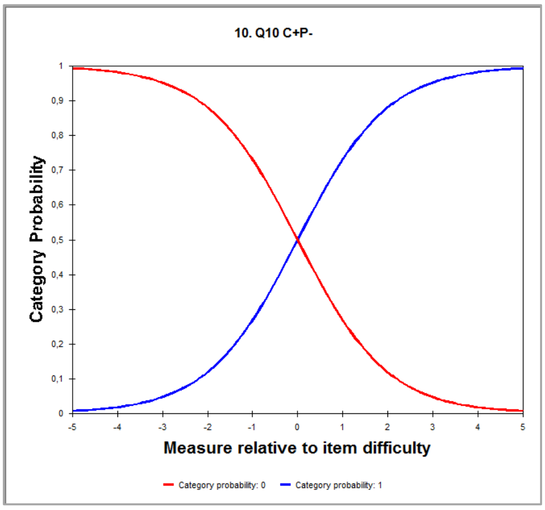


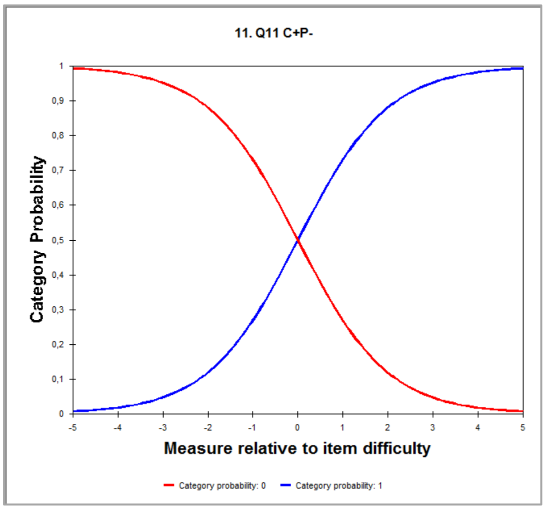


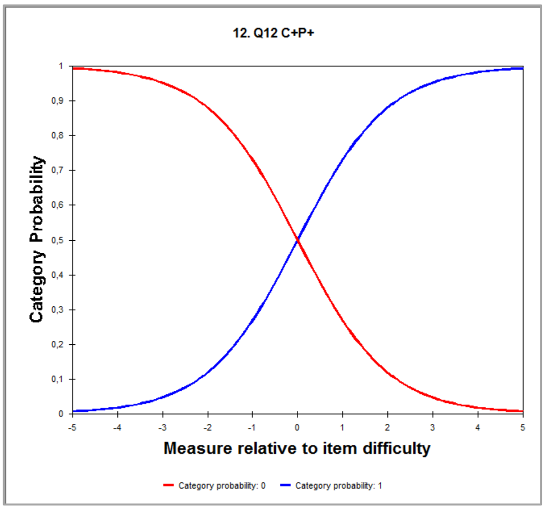

7. Q7 C-P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1

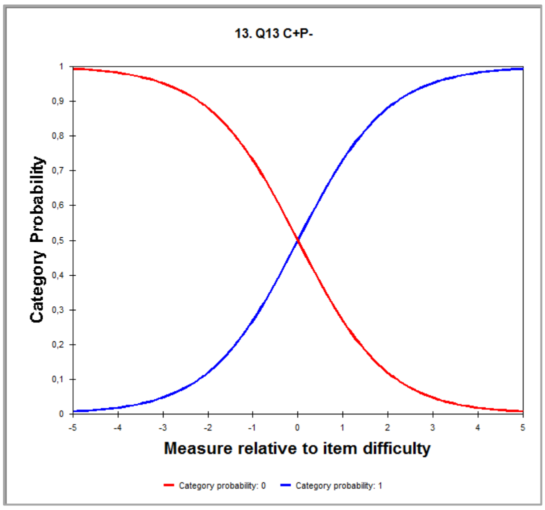

8. Q8 C+P-
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1

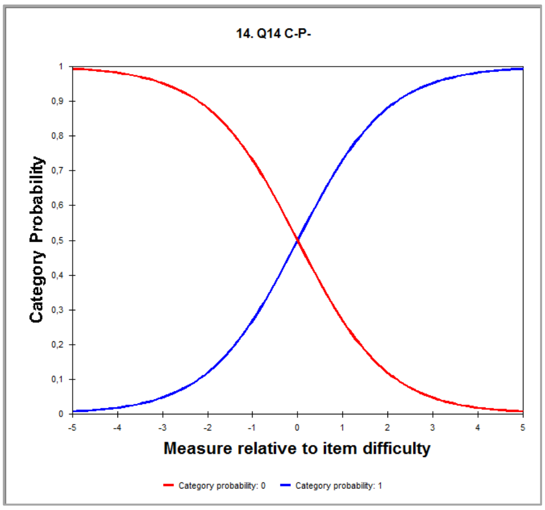

9. Q9 C+P-
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1

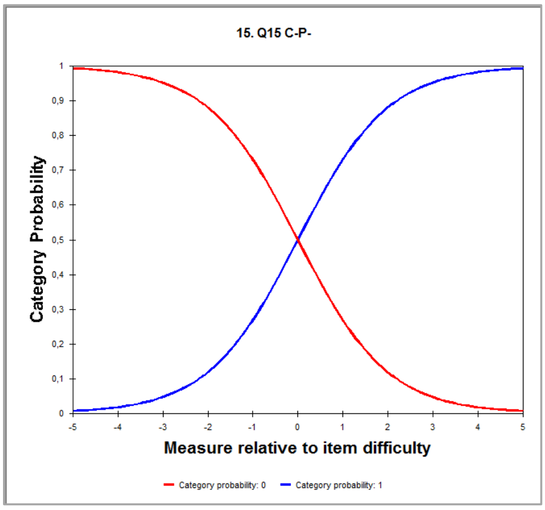

10. Q10 C+P-
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1

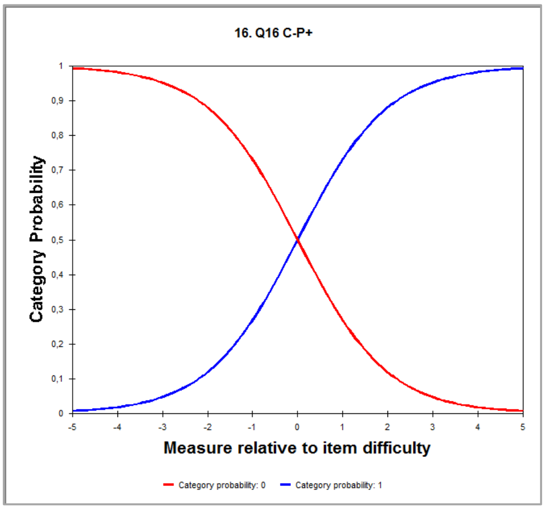

11. Q11 C+P-
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1

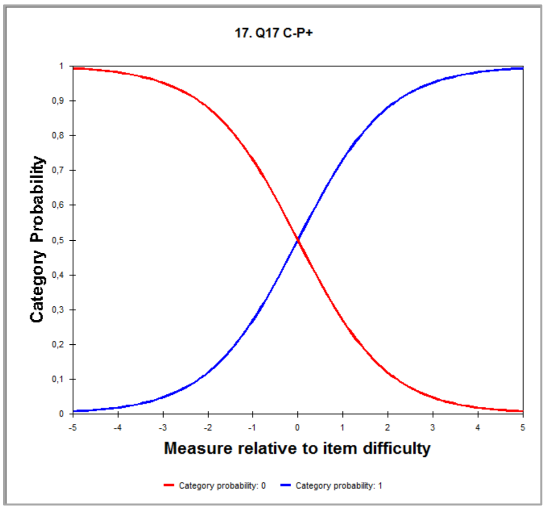

12. Q12 C+P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1

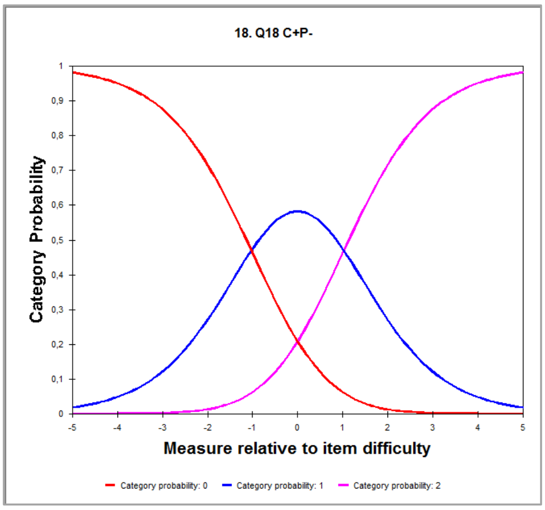

13. Q13 C+P-
Category Probability
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
-5 -4 -3 -2 -1 0 1 2 3 4 5
Measure relative to item difficulty
Category probability: 0
Category probability: 1

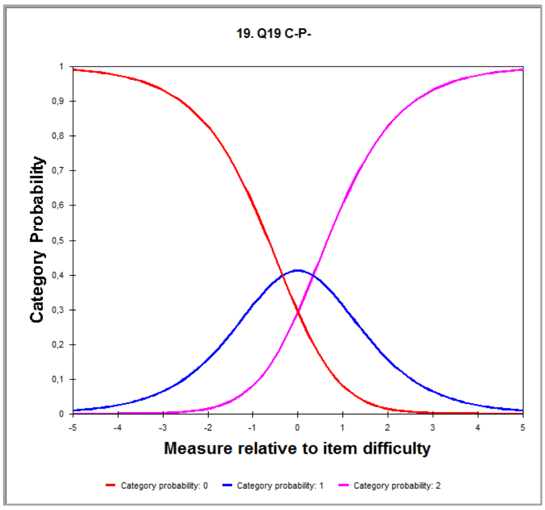

14. Q14 C-P-
Category Probability
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
-5 -4 -3 -2 -1 0 1 2 3 4 5
Measure relative to item difficulty
Category probability: 0
Category probability: 1

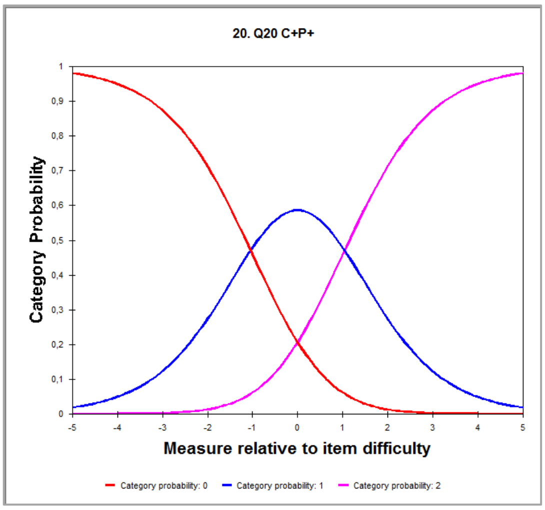

15. Q15 C-P-
Category Probability
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
-5 -4 -3 -2 -1 0 1 2 3 4 5
Measure relative to item difficulty
Category probability: 0
Category probability: 1

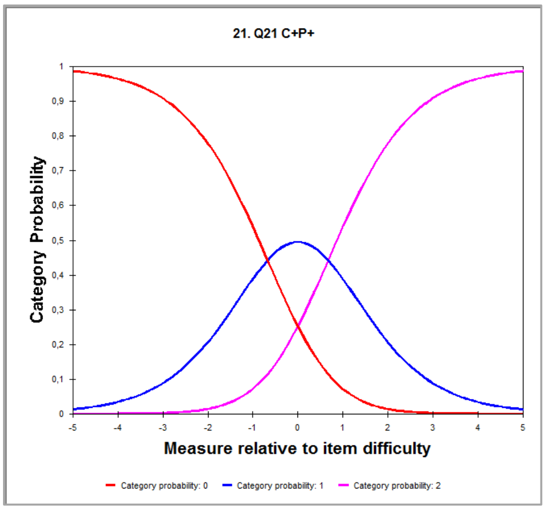

16. Q16 C-P+
Category Probability
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
-5 -4 -3 -2 -1 0 1 2 3 4 5
Measure relative to item difficulty
Category probability: 0
Category probability: 1

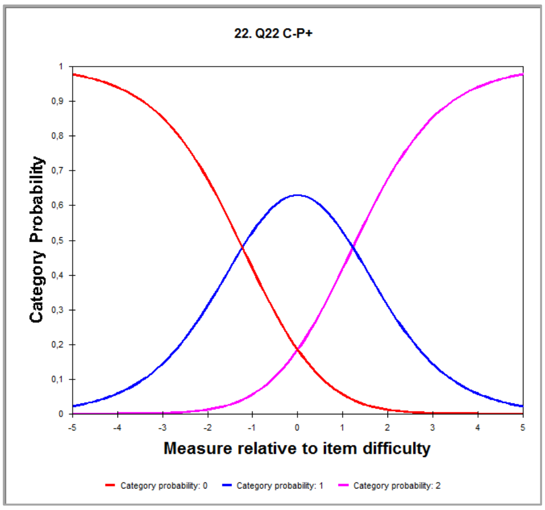

17. Q17 C-P+
Category Probability
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
-5 -4 -3 -2 -1 0 1 2 3 4 5
Measure relative to item difficulty
Category probability: 0
Category probability: 1

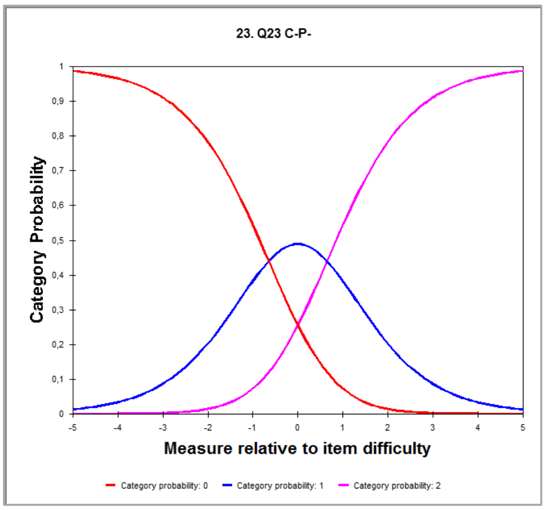

18. Q18 C+P-
Category Probability
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
-5 -4 -3 -2 -1 0 1 2 3 4 5
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

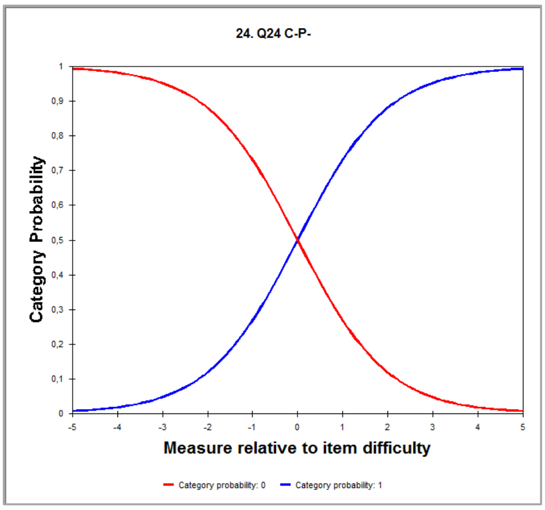

19. Q19 C-P-
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

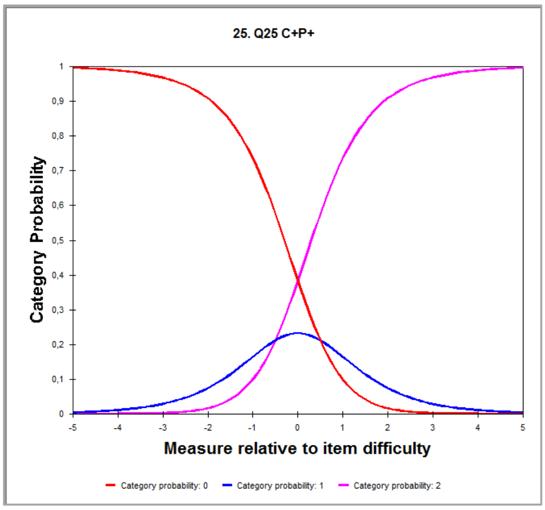

20. Q20 C+P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

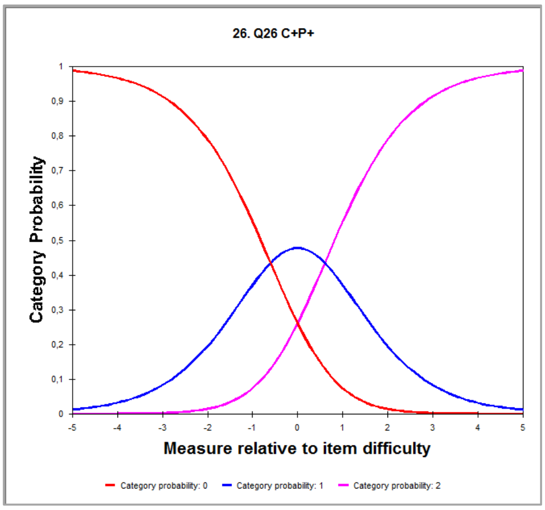

21. Q21 C+P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

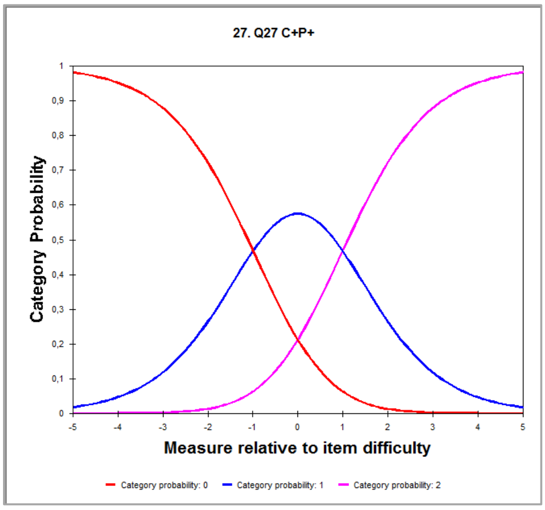

22. Q22 C-P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

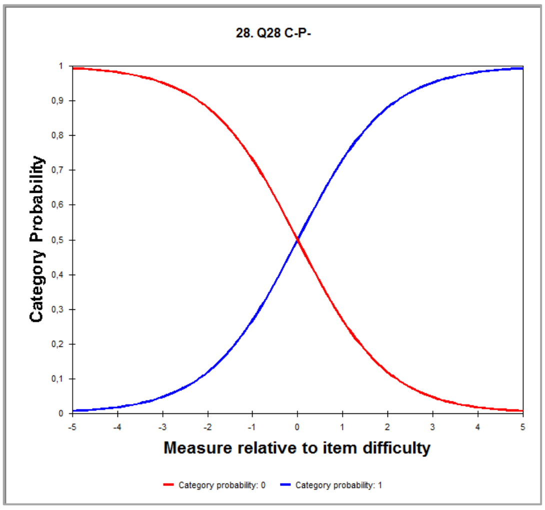

23. Q23 C-P-
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

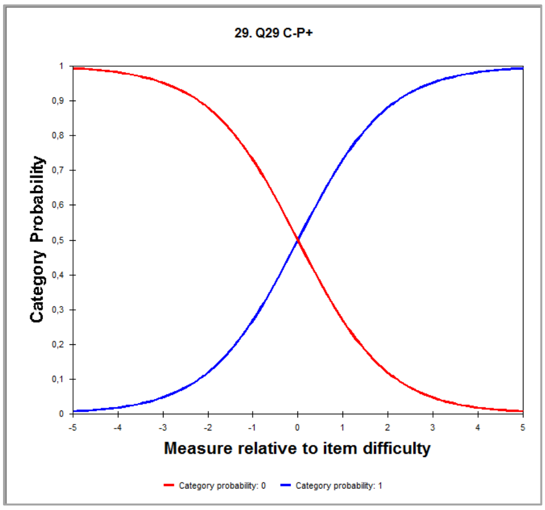

24. Q24 C-P-
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1

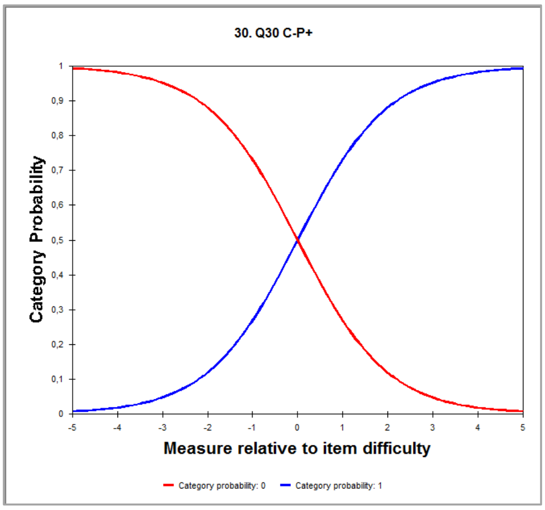

25. Q25 C+P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

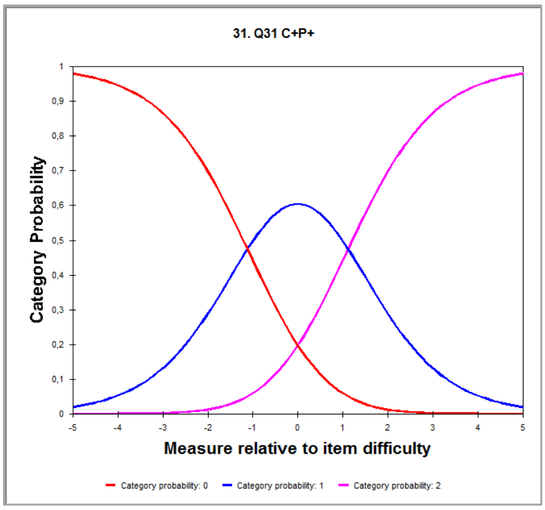

26. Q26 C+P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

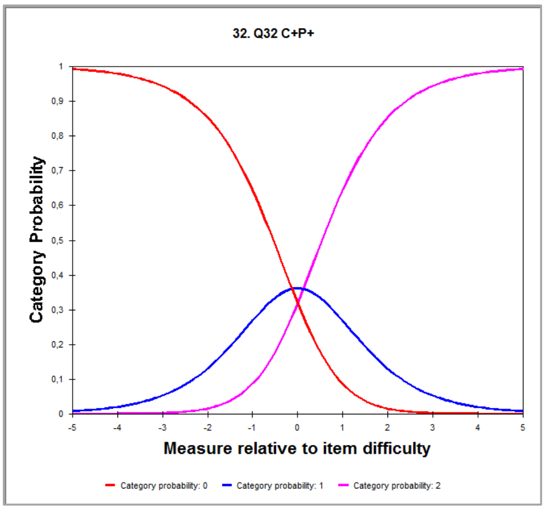

27. Q27 C+P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

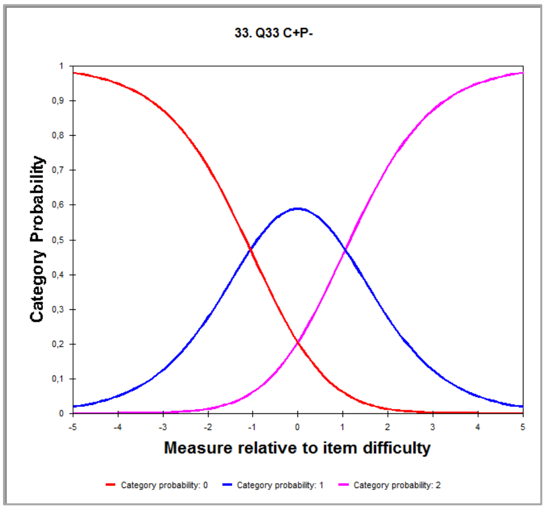

28. Q28 C-P-
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1


29. Q29 C-P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1


30. Q30 C-P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1


31. Q31 C+P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2


32. Q32 C+P+
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2


33. Q33 C+P-
Category Probability
Measure relative to item difficulty
Category probability: 0
Category probability: 1
Category probability: 2

8.12 APPENDIX 12: FREQUENCY PER ITEM OPTIONS

Multiple-choice questions				
Item	**Category**	**Mark**	**Freq**	**n**
1	C-P-	0	139	188
		1	49	
2	C-P+	0	134	190
		1	56	
3	C-P-	0	38	189
		1	151	
4	C+P-	0	114	190
		1	76	
5	C-P+	0	27	190
		1	163	
6	C-P-	0	166	190
		1	24	
7	C-P+	0	64	189
		1	125	
8	C+P-	0	64	190
		1	126	
9	C+P-	0	89	190
		1	101	
10	C+P-	0	132	190
		1	58	
11	C+P-	0	75	189
		1	114	
12	C+P+	0	102	190
		1	88	
13	C+P-	0	98	190
		1	92	
14	C-P-	0	44	190
		1	146	
15	C-P-	0	93	190
		1	97	
16	C-P+	0	40	190
		1	150	
17	C-P+	0	95	187
		1	92	
Open-ended questions				
18	C+P-	0	16	182
		1	82	
		2	84	
19	C-P-	0	13	182
		1	45	
		2	124	

Open-ended questions				
Item	**Category**	**Mark**	**Freq**	**n**
20	C+P+	0	32	183
		1	96	
		2	55	
21	C+P+	0	143	178
		1	32	
		2	3	
22	C-P+	0	15	183
		1	92	
		2	76	
23	C-P-	0	43	179
		1	78	
		2	58	
24	C-P-	0	101	178
		1	77	
25	C+P+	0	21	182
		1	26	
		2	135	
26	C+P+	0	38	173
		1	73	
		2	62	
27	C+P+	0	17	183
		1	82	
		2	84	
28	C-P-	0	75	174
		1	99	
29	C-P+	0	60	176
		1	116	
30	C-P+	0	75	174
		1	99	
31	C+P+	0	63	171
		1	88	
		2	20	
32	C+P+	0	17	176
		1	41	
		2	118	
33	C+P-	0	71	175
		1	85	
		2	19	

8.13 APPENDIX 13: PERSON STATISTICS SUBSAMPLES (GENDER, RACE, STUDY FIELD)

Table 1: Descriptive gender statistics per item category and type of question

	Mean %		Minimum		Maximum		SD		Valid N	
Gender	M	F	M	F	M	F	M	F	M	F
Test	56.17%	51.22%	22.22%	22.22%	86.67%	68.89%	13.97%	11.42%	146 (76.04%)	46 (23.96%)
Category										
C-P-	55.64%	52.87%	18.18%	18.18%	90.91%	81.82%	18.40%	14.39%	146	46
C-P+	63.63%	58.56%	11.11%	11.11%	100.00%	88.89%	19.48%	18.27%	146	46
C+P-	50.51%	48.91%	0.00%	20.00%	90.00%	90.00%	19.02%	16.22%	146	46
C+P+	55.19%	46.96%	0.00%	13.33%	100.00%	73.33%	19.24%	15.00%	146	46
Question type										
MCQ	54.21%	48.72%	11.76%	17.65%	100.00%	76.47%	15.45%	12.44%	144	46
Open	58.06%	52.60%	11.11%	17.86%	92.86%	78.57%	16.42%	14.71%	141	46

Table 1 reflects on the data analysis in relation to the gender statistics with respect to the test, item categories and the type of question. Of the students who wrote the test, 23.96% were female and 76.04% male. The statistics in Table 1 indicate that males performed better than females in the test in general, in every knowledge category and for both types of questions. Furthermore, the SD for females was lower than the SD for males for each heading. This indicated that the data for females were located closer to the mean for the test, for each knowledge category and for both types of questions.

Table 2: Descriptive race statistics per item category and type of question

	Mean%			Minimum			Maximum			SD			Valid N		
Race	African	White	Other	African	White	Other	African	White	Other	African	White	Other	African	White	Other
Test	52.96%	58.92%	51.92%	22.22%	22.22%	25.00%	75.56%	86.67%	80.00%	12.01%	14.50%	12.85	101 (52.60%)	69 (35.94%)	22 (11.46%)
Category															
C-P-	52.92%	57.31%	57.12%	18.18%	18.18%	18.18%	90.91%	90.91%	90.91%	15.89%	19.40%	16.83	101	69	22
C-P+	59.53%	67.68%	59.09%	11.11%	11.11%	22.22%	100.00%	100.00%	88.89%	17.81%	20.33%	18.51	101	69	22
C+P-	49.03%	52.92%	46.40%	0.00%	16.67%	0.00%	90.00%	90.00%	80.00%	16.91%	19.40%	20.52	101	69	22
C+P+	51.10%	58.59%	46.06%	0.00%	0.00%	0.00%	86.67%	100.00%	80.00%	15.79%	19.37%	23.63	101	69	21
Question type															
MCQ	49.47%	57.54%	53.78%	11.76%	17.65%	29.41%	76.47%	100.00%	88.24%	13.35%	16.23%	13.37	100	69	21
Open	55.48%	60.55%	50.55%	17.86%	21.43%	11.11%	82.14%	92.86%	78.57%	14.51%	16.11%	17.57	100	66	21

The descriptive race statistics in Table 2, race statistics, informed the study for three categories, namely African, White and Other category. The Other category included data for Coloured, Indian, Chinese, Asian and Portuguese students. The white students achieved a higher average than students in the other two categories for the test in general, in every knowledge category and for both types of questions. The spread of data for the white students is further from the mean than for the other two race categories, since the SD is higher for the white students for the test in general, for C-P- and C-P+ knowledge categories and for the MCQ.

Table 3: Descriptive engineering study field statistics per item category and type of question

	Mean%				SD				Valid N			
Study field	Civil	Computer, Electrical	Mechanical	Chemical, Industrial, Electronic, Metallurgy, Mining	Civil	Computer, Electrical	Mechanical	Chemical, Industrial, Electronic, Metallurgy, Mining	Civil	Computer, Electrical	Mechanical	Chemical, Industrial, Electronic, Metallurgy, Mining
Test	55.49%	57.68%	54.74%	52.54%	14.30%	12.74%	14.38%	12.14%	54 (28.13%)	38 (19.79%)	52 (27.08%)	48 (25.00%)
Category												
C-P-	56.04%	56.72%	53.55%	53.94%	16.80%	17.78%	19.78%	15.66%	54	38	52	48
C-P+	60.88%	66.35%	62.82%	60.58%	20.56%	21.66%	18.59%	16.34%	54	38	52	48
C+P-	52.02%	53.22%	48.97%	46.79%	16.63%	18.47%	19.85%	18.24%	54	38	52	48
C+P+	53.19%	56.27%	54.18%	49.79%	19.17%	18.45%	17.94%	19.10%	54	38	52	48
Question type												
MCQ	55.01%	54.64%	51.53%	50.49%	15.63%	14.79%	15.79%	12.95%	54	38	50	48
Open	55.45%	60.45%	58.08%	53.79%	17.29%	14.96%	16.00%	15.42%	53	36	51	47

The statistics per engineering study fields are shared in Table 3. The table shows that the performance of the engineering students in most study fields was similar for the test in general, in every knowledge category and for both types of questions. Similar performance was expected, since students belonged to a homogeneous population.

8.14 APPENDIX 14: DIFF PER SUBSAMPLE (GENDER, RACE, STUDY FIELD)

Table 1: Gender

PERSON	Obs-Exp	DIF	DIF	PERSON	Obs-Exp	DIF	DIF	DIF	JOIN	Rasch-Welch			Size	Active	ITEM	
CLASS	Average	MEASURE	S.E.	CLASS	Average	MEASURE	S.E.	CONTRAST	S.E.	t	d.f.	Prob.	CUMLOR	Slices	Number	Name
Female	0	1.36	0.37	Male	0.00	1.36	0.20	0.00	0.42	0.00	73.00	1.00	0.15	17	1	Q1 C-P-
Female	0.1	0.66	0.32	Male	-0.03	1.35	0.20	-0.68	0.37	-1.83	81.00	0.07	-0.53	17	2	Q2 C-P+
Female	-0.05	-1.01	0.34	Male	0.02	-1.44	0.23	0.42	0.41	1.04	89.00	0.30	0.37	17	3	Q3 C-P-
Female	0.02	0.56	0.32	Male	-0.01	0.69	0.18	-0.13	0.36	-0.36	76.00	0.72	0.60	17	4	Q4 C+P-
Female	-0.05	-1.38	0.37	Male	0.02	-1.92	0.27	0.54	0.45	1.18	95.00	0.24	0.11	17	5	Q5 C-P+
Female	-0.05	3.21	0.73	Male	0.02	2.17	0.24	1.03	0.77	1.35	55.00	0.18	0.76	17	6	Q6 C-P-
Female	-0.05	-0.29	0.31	Male	0.02	-0.62	0.19	0.33	0.36	0.91	82.00	0.36	0.15	17	7	Q7 C-P+
Female	0.01	-0.58	0.32	Male	0.00	-0.54	0.19	-0.05	0.37	-0.12	78.00	0.90	-0.13	17	8	Q8 C+P-
Female	0.08	-0.29	0.31	Male	-0.03	0.19	0.18	-0.48	0.36	-1.36	76.00	0.18	0.09	17	9	Q9 C+P-
Female	0	1.1	0.34	Male	0.00	1.12	0.19	-0.02	0.39	-0.05	73.00	0.96	-0.11	17	10	Q10 C+P-
Female	0.05	-0.49	0.31	Male	-0.02	-0.18	0.18	-0.31	0.36	-0.85	77.00	0.40	-0.27	17	11	Q11 C+P-
Female	-0.09	0.77	0.32	Male	0.03	0.25	0.18	0.51	0.37	1.39	73.00	0.17	0.35	17	12	Q12 C+P+
Female	0.05	0.08	0.31	Male	-0.01	0.35	0.18	-0.26	0.35	-0.75	77.00	0.46	-0.53	17	13	Q13 C+P-
Female	0.07	-1.53	0.38	Male	-0.02	-0.98	0.20	-0.55	0.43	-1.26	72.00	0.21	-0.34	17	14	Q14 C-P-
Female	-0.05	0.37	0.31	Male	0.01	0.10	0.18	0.27	0.36	0.76	76.00	0.45	0.66	17	15	Q15 C-P-
Female	0.04	-1.53	0.38	Male	-0.01	-1.15	0.21	-0.37	0.44	-0.86	74.00	0.39	-0.40	17	16	Q16 C-P+
Female	-0.09	0.66	0.32	Male	0.03	0.11	0.18	0.55	0.37	1.50	75.00	0.14	0.08	17	17	Q17 C-P+
Female	-0.07	-0.58	0.24	Male	0.02	-0.83	0.15	0.25	0.28	0.90	80.00	0.37	0.68	16	18	Q18 C+P-
Female	0.08	-1.4	0.26	Male	-0.03	-1.08	0.15	-0.32	0.30	-1.04	75.00	0.30	-0.45	16	19	Q19 C-P-
Female	0.1	-0.35	0.24	Male	-0.03	-0.02	0.14	-0.34	0.27	-1.23	75.00	0.22	-0.44	16	20	Q20 C+P+
Female	-0.07	3.16	0.51	Male	0.02	2.41	0.19	0.75	0.55	1.37	56.00	0.17	1.10	16	21	Q21 C+P+
Female	0.06	-0.89	0.26	Male	-0.02	-0.68	0.15	-0.21	0.30	-0.73	76.00	0.47	-0.20	16	22	Q22 C-P+
Female	0.14	-0.23	0.22	Male	-0.04	0.16	0.13	-0.39	0.25	-1.53	74.00	0.13	-0.36	16	23	Q23 C-P-
Female	0.03	0.38	0.32	Male	-0.01	0.58	0.18	-0.20	0.37	-0.55	74.00	0.58	-0.09	16	24	Q24 C-P-
Female	-0.17	-0.63	0.2	Male	0.05	-1.13	0.15	0.50	0.26	1.95	94.00	0.05	0.45	16	25	Q25 C+P+
Female	-0.04	0.05	0.22	Male	0.01	-0.07	0.13	0.12	0.26	0.47	73.00	0.64	0.30	16	26	Q26 C+P+
Female	-0.06	-0.57	0.24	Male	0.02	-0.79	0.15	0.22	0.28	0.80	79.00	0.43	0.36	16	27	Q27 C+P+
Female	0.01	-0.1	0.31	Male	0.00	-0.07	0.19	-0.04	0.36	-0.10	81.00	0.92	-0.51	16	28	Q28 C-P-
Female	-0.02	-0.39	0.31	Male	0.01	-0.53	0.20	0.14	0.37	0.38	84.00	0.71	-0.41	16	29	Q29 C-P+
Female	0.01	-0.1	0.31	Male	0.00	-0.06	0.19	-0.04	0.36	-0.11	81.00	0.91	-0.26	16	30	Q30 C-P+
Female	-0.04	1.05	0.25	Male	0.01	0.90	0.15	0.15	0.29	0.50	77.00	0.62	-0.15	16	31	Q31 C+P+
Female	-0.1	-0.74	0.21	Male	0.04	-1.07	0.16	0.33	0.26	1.25	95.00	0.21	0.84	16	32	Q32 C+P+
Female	0.15	0.6	0.24	Male	-0.06	1.18	0.15	-0.58	0.28	-2.06	81.00	0.04	-1.00	16	33	Q33 C+P-

Table 2: Race

CLASS	COUNT	AVERAGE	EXPECT	MEASURE	SCORE	MEASURE	DIF SIZE	S.E.	t	Prob.	Number	Name
African	98	0.26	0.24	1.36	0.02	1.25	-0.11	0.24	-0.47	0.64210	1	Q1 C-P-
Other	21	0.29	0.24	1.36	0.05	1.08	-0.28	0.51	-0.54	0.59490	1	Q1 C-P-
White	69	0.26	0.30	1.36	-0.04	1.59	0.23	0.29	0.81	0.42260	1	Q1 C-P-
African	100	0.23	0.27	1.18	-0.04	1.40	0.23	0.24	0.93	0.35370	2	Q2 C-P+
Other	21	0.29	0.27	1.18	0.02	1.08	-0.10	0.51	-0.19	0.85360	2	Q2 C-P+
White	69	0.39	0.34	1.18	0.05	0.92	-0.26	0.26	-0.98	0.33180	2	Q2 C-P+
African	99	0.74	0.79	-1.31	-0.05	-1.02	0.29	0.24	1.24	0.21760	3	Q3 C-P-
Other	21	0.86	0.77	-1.31	0.09	-1.93	-0.62	0.64	-0.97	0.34460	3	Q3 C-P-
White	69	0.87	0.82	-1.31	0.05	-1.69	-0.38	0.37	-1.03	0.30690	3	Q3 C-P-
African	100	0.46	0.37	0.66	0.09	0.28	-0.38	0.21	-1.83	0.07040	4	Q4 C+P-
Other	21	0.38	0.37	0.66	0.01	0.59	-0.07	0.48	-0.14	0.89070	4	Q4 C+P-
White	69	0.32	0.45	0.66	-0.13	1.28	0.61	0.27	2.25	0.02760	4	Q4 C+P-
African	100	0.84	0.85	-1.76	-0.01	-1.68	0.08	0.28	0.29	0.77360	5	Q5 C-P+
Other	21	0.86	0.84	-1.76	0.02	-1.93	-0.18	0.64	-0.28	0.78510	5	Q5 C-P+
White	69	0.88	0.88	-1.76	0.01	-1.84	-0.08	0.39	-0.21	0.83190	5	Q5 C-P+
African	100	0.04	0.11	2.32	-0.07	3.44	1.12	0.51	2.17	0.03220	6	Q6 C-P-
Other	21	0.14	0.11	2.32	0.03	2.04	-0.28	0.65	-0.43	0.67400	6	Q6 C-P-
White	69	0.25	0.15	2.32	0.09	1.67	-0.65	0.29	-2.20	0.03110	6	Q6 C-P-
African	99	0.65	0.64	-0.53	0.00	-0.55	-0.02	0.22	-0.10	0.92270	7	Q7 C-P+
Other	21	0.76	0.63	-0.53	0.14	-1.26	-0.72	0.54	-1.35	0.19340	7	Q7 C-P+
White	69	0.65	0.70	-0.53	-0.05	-0.29	0.25	0.27	0.92	0.36110	7	Q7 C-P+
African	100	0.69	0.64	-0.54	0.05	-0.76	-0.22	0.22	-0.99	0.32610	8	Q8 C+P-
Other	21	0.62	0.63	-0.54	-0.01	-0.50	0.04	0.48	0.07	0.94190	8	Q8 C+P-
White	69	0.64	0.70	-0.54	-0.06	-0.22	0.32	0.27	1.22	0.22730	8	Q8 C+P-
African	100	0.52	0.51	0.07	0.01	0.02	-0.05	0.21	-0.25	0.80200	9	Q9 C+P-
Other	21	0.43	0.49	0.07	-0.07	0.37	0.30	0.47	0.64	0.53140	9	Q9 C+P-
White	69	0.58	0.58	0.07	0.00	0.07	0.00	0.26	0.00	1.00000	9	Q9 C+P-
African	100	0.24	0.28	1.12	-0.04	1.35	0.22	0.24	0.93	0.35540	10	Q10 C+P-
Other	21	0.24	0.28	1.12	-0.04	1.36	0.23	0.54	0.43	0.66850	10	Q10 C+P-
White	69	0.42	0.35	1.12	0.07	0.79	-0.34	0.26	-1.30	0.19880	10	Q10 C+P-
African	100	0.56	0.58	-0.26	-0.02	-0.16	0.10	0.21	0.47	0.63620	11	Q11 C+P-
Other	21	0.71	0.57	-0.26	0.15	-0.98	-0.73	0.51	-1.44	0.16750	11	Q11 C+P-
White	68	0.63	0.65	-0.26	-0.01	-0.19	0.06	0.27	0.23	0.81690	11	Q11 C+P-
African	100	0.41	0.44	0.38	-0.03	0.50	0.12	0.21	0.59	0.55790	12	Q12 C+P+
Other	21	0.38	0.43	0.38	-0.05	0.59	0.22	0.48	0.46	0.65170	12	Q12 C+P+

White	69	0.57	0.51	0.38	0.05	0.13	-0.25	0.26	-0.97	0.33800	12	Q12 C+P+
African	100	0.41	0.46	0.28	-0.05	0.50	0.22	0.21	1.03	0.30480	13	Q13 C+P-
Other	21	0.52	0.45	0.28	0.08	-0.06	-0.35	0.47	-0.74	0.46610	13	Q13 C+P-
White	69	0.58	0.53	0.28	0.05	0.06	-0.22	0.26	-0.86	0.39440	13	Q13 C+P-
African	100	0.75	0.76	-1.11	-0.01	-1.08	0.03	0.24	0.13	0.89910	14	Q14 C-P-
Other	21	0.86	0.74	-1.11	0.12	-1.93	-0.82	0.64	-1.28	0.21660	14	Q14 C-P-
White	69	0.77	0.80	-1.11	-0.03	-0.92	0.19	0.30	0.62	0.53630	14	Q14 C-P-
African	100	0.42	0.49	0.16	-0.07	0.46	0.29	0.21	1.38	0.17110	15	Q15 C-P-
Other	21	0.62	0.47	0.16	0.15	-0.50	-0.67	0.48	-1.41	0.17600	15	Q15 C-P-
White	69	0.61	0.56	0.16	0.05	-0.08	-0.24	0.26	-0.92	0.36110	15	Q15 C-P-
African	100	0.79	0.78	-1.24	0.01	-1.32	-0.08	0.25	-0.31	0.75820	16	Q16 C-P+
Other	21	0.76	0.76	-1.24	0.00	-1.24	0.00	0.54	0.00	1.00000	16	Q16 C-P+
White	69	0.8	0.82	-1.24	-0.02	-1.11	0.13	0.31	0.41	0.68000	16	Q16 C-P+
African	98	0.44	0.46	0.25	-0.03	0.36	0.11	0.21	0.52	0.60420	17	Q17 C-P+
Other	21	0.43	0.46	0.25	-0.03	0.37	0.12	0.47	0.26	0.79830	17	Q17 C-P+
White	68	0.59	0.54	0.25	0.04	0.05	-0.20	0.26	-0.78	0.43940	17	Q17 C-P+
African	96	1.31	1.33	-0.76	-0.02	-0.71	0.05	0.17	0.31	0.75720	18	Q18 C+P-
Other	20	1.15	1.33	-0.76	-0.18	-0.29	0.48	0.36	1.33	0.20100	18	Q18 C+P-
White	66	1.53	1.45	-0.76	0.08	-1.03	-0.27	0.23	-1.18	0.24400	18	Q18 C+P-
African	96	1.56	1.58	-1.17	-0.01	-1.13	0.04	0.17	0.24	0.81280	19	Q19 C-P-
Other	20	1.7	1.57	-1.17	0.13	-1.60	-0.44	0.43	-1.01	0.32780	19	Q19 C-P-
White	66	1.65	1.67	-1.17	-0.02	-1.10	0.07	0.23	0.31	0.75900	19	Q19 C-P-
African	97	1.1	1.08	-0.10	0.03	-0.17	-0.07	0.16	-0.43	0.66460	20	Q20 C+P+
Other	20	1.05	1.08	-0.10	-0.03	-0.03	0.07	0.36	0.21	0.83820	20	Q20 C+P+
White	66	1.18	1.21	-0.10	-0.03	-0.02	0.08	0.20	0.42	0.67420	20	Q20 C+P+
African	93	0.12	0.18	2.52	-0.06	2.99	0.46	0.31	1.48	0.14120	21	Q21 C+P+
Other	20	0.3	0.20	2.52	0.10	2.01	-0.51	0.46	-1.12	0.27760	21	Q21 C+P+
White	65	0.32	0.27	2.52	0.06	2.28	-0.24	0.25	-0.96	0.34020	21	Q21 C+P+
African	97	1.32	1.29	-0.73	0.03	-0.82	-0.08	0.17	-0.47	0.63800	22	Q22 C-P+
Other	20	1.2	1.29	-0.73	-0.09	-0.46	0.27	0.38	0.71	0.48530	22	Q22 C-P+
White	66	1.39	1.41	-0.73	-0.01	-0.69	0.04	0.22	0.18	0.85850	22	Q22 C-P+
African	96	1.19	1.02	0.06	0.17	-0.30	-0.36	0.15	-2.37	0.01970	23	Q23 C-P-
Other	19	0.89	1.05	0.06	-0.16	0.40	0.34	0.34	1.00	0.33350	23	Q23 C-P-
White	64	0.98	1.19	0.06	-0.20	0.51	0.46	0.19	2.45	0.01710	23	Q23 C-P-
African	95	0.41	0.40	0.53	0.01	0.48	-0.05	0.22	-0.23	0.81500	24	Q24 C-P-
Other	19	0.47	0.41	0.53	0.06	0.24	-0.29	0.49	-0.60	0.55750	24	Q24 C-P-
White	64	0.45	0.49	0.53	-0.04	0.69	0.16	0.27	0.61	0.54550	24	Q24 C-P-
African	96	1.51	1.59	-0.97	-0.08	-0.79	0.18	0.15	1.21	0.22810	25	Q25 C+P+

Other	20	1.65	1.57	-0.97	0.08	-1.17	-0.20	0.37	-0.53	0.60010	25	Q25 C+P+
White	66	1.79	1.70	-0.97	0.09	-1.32	-0.35	0.26	-1.34	0.18410	25	Q25 C+P+
African	93	1.16	1.07	-0.04	0.09	-0.24	-0.19	0.15	-1.28	0.20390	26	Q26 C+P+
Other	19	0.84	1.10	-0.04	-0.26	0.51	0.55	0.34	1.62	0.12370	26	Q26 C+P+
White	61	1.2	1.26	-0.04	-0.06	0.09	0.14	0.19	0.71	0.48070	26	Q26 C+P+
African	97	1.37	1.32	-0.73	0.05	-0.87	-0.13	0.17	-0.78	0.43600	27	Q27 C+P+
Other	20	1.15	1.32	-0.73	-0.17	-0.28	0.45	0.36	1.26	0.22240	27	Q27 C+P+
White	66	1.42	1.44	-0.73	-0.02	-0.68	0.06	0.21	0.26	0.79420	27	Q27 C+P+
African	96	0.57	0.54	-0.07	0.03	-0.20	-0.14	0.22	-0.65	0.52020	28	Q28 C-P-
Other	19	0.53	0.54	-0.07	-0.01	-0.02	0.05	0.49	0.10	0.92160	28	Q28 C-P-
White	59	0.58	0.62	-0.07	-0.05	0.16	0.23	0.28	0.81	0.42280	28	Q28 C-P-
African	97	0.62	0.64	-0.49	-0.02	-0.41	0.08	0.22	0.37	0.71200	29	Q29 C-P+
Other	19	0.53	0.63	-0.49	-0.10	-0.02	0.47	0.49	0.97	0.34780	29	Q29 C-P+
White	60	0.77	0.71	-0.49	0.06	-0.84	-0.35	0.32	-1.08	0.28290	29	Q29 C-P+
African	95	0.52	0.54	-0.06	-0.03	0.05	0.11	0.21	0.53	0.59580	30	Q30 C-P+
Other	19	0.47	0.54	-0.06	-0.06	0.22	0.28	0.49	0.58	0.56960	30	Q30 C-P+
White	60	0.68	0.62	-0.06	0.06	-0.37	-0.30	0.29	-1.03	0.30620	30	Q30 C-P+
African	94	0.62	0.70	0.94	-0.08	1.17	0.23	0.18	1.30	0.19650	31	Q31 C+P+
Other	18	0.67	0.70	0.94	-0.03	1.02	0.09	0.40	0.21	0.83270	31	Q31 C+P+
White	59	0.98	0.85	0.94	0.14	0.57	-0.36	0.21	-1.71	0.09280	31	Q31 C+P+
African	97	1.56	1.54	-0.97	0.02	-1.01	-0.04	0.16	-0.26	0.79170	32	Q32 C+P+
Other	19	1.42	1.52	-0.97	-0.09	-0.75	0.22	0.34	0.65	0.52650	32	Q32 C+P+
White	60	1.65	1.65	-0.97	0.00	-0.97	0.00	0.24	0.00	1.00000	32	Q32 C+P+
African	97	0.76	0.65	1.02	0.11	0.73	-0.30	0.17	-1.80	0.07430	33	Q33 C+P-
Other	19	0.74	0.66	1.02	0.08	0.80	-0.22	0.38	-0.58	0.56730	33	Q33 C+P-
White	59	0.59	0.80	1.02	-0.21	1.61	0.59	0.23	2.59	0.01200	33	Q33 C+P-

Table 3: Study field

CLASS	COUNT	AVERAGE	EXPECT	MEASURE	SCORE	MEASURE	DIF SIZE	S.E.	t	Prob.	Number	Name
Chemical	18	0.17	0.22	1.36	-0.05	1.72	0.36	0.66	0.55	0.58980	1	Q1 C-P-
Civil	53	0.21	0.26	1.36	-0.06	1.71	0.35	0.35	0.99	0.32800	1	Q1 C-P-
Computer Electrical	37	0.24	0.28	1.36	-0.04	1.59	0.23	0.40	0.58	0.56780	1	Q1 C-P-
Electronic Metallurgical Mining	16	0.25	0.25	1.36	0.00	1.36	0.00	0.61	0.00	1.00000	1	Q1 C-P-
Industrial	14	0.14	0.24	1.36	-0.09	1.99	0.63	0.77	0.81	0.43150	1	Q1 C-P-
Mechanical	50	0.40	0.27	1.36	0.13	0.68	-0.67	0.30	-2.21	0.03150	1	Q1 C-P-
Chemical	18	0.33	0.25	1.18	0.09	0.72	-0.46	0.53	-0.87	0.39490	2	Q2 C-P+
Civil	54	0.26	0.30	1.18	-0.04	1.40	0.22	0.33	0.68	0.49780	2	Q2 C-P+

Computer Electrical	38	0.34	0.32	1.18	0.02	1.07	-0.10	0.36	-0.29	0.77280	2	Q2 C-P+
Electronic Metallurgical Mining	16	0.44	0.28	1.18	0.16	0.41	-0.77	0.53	-1.46	0.16680	2	Q2 C-P+
Industrial	14	0.21	0.27	1.18	-0.05	1.48	0.30	0.66	0.46	0.65420	2	Q2 C-P+
Mechanical	50	0.26	0.30	1.18	-0.04	1.39	0.21	0.34	0.63	0.53020	2	Q2 C-P+
Chemical	18	0.61	0.76	-1.31	-0.14	-0.57	0.74	0.51	1.45	0.16530	3	Q3 C-P-
Civil	53	0.89	0.80	-1.31	0.09	-2.06	-0.75	0.45	-1.67	0.10020	3	Q3 C-P-
Computer Electrical	38	0.82	0.82	-1.31	-0.01	-1.27	0.04	0.43	0.09	0.92510	3	Q3 C-P-
Electronic Metallurgical Mining	16	0.94	0.80	-1.31	0.14	-2.70	-1.39	1.04	-1.33	0.20390	3	Q3 C-P-
Industrial	14	0.79	0.80	-1.31	-0.01	-1.23	0.08	0.66	0.12	0.90530	3	Q3 C-P-
Mechanical	50	0.72	0.80	-1.31	-0.08	-0.82	0.49	0.33	1.49	0.14400	3	Q3 C-P-
Chemical	18	0.50	0.34	0.66	0.16	-0.06	-0.72	0.50	-1.45	0.16690	4	Q4 C+P-
Civil	54	0.39	0.40	0.66	-0.02	0.73	0.07	0.30	0.25	0.80590	4	Q4 C+P-
Computer Electrical	38	0.26	0.43	0.66	-0.17	1.48	0.82	0.38	2.14	0.03930	4	Q4 C+P-
Electronic Metallurgical Mining	16	0.50	0.38	0.66	0.12	0.13	-0.53	0.52	-1.01	0.32900	4	Q4 C+P-
Industrial	14	0.29	0.37	0.66	-0.09	1.08	0.42	0.60	0.70	0.49700	4	Q4 C+P-
Mechanical	50	0.48	0.40	0.66	0.08	0.32	-0.34	0.30	-1.14	0.26010	4	Q4 C+P-
Chemical	18	0.89	0.82	-1.76	0.07	-2.33	-0.57	0.77	-0.74	0.46770	5	Q5 C-P+
Civil	54	0.94	0.86	-1.76	0.09	-2.86	-1.10	0.60	-1.83	0.07350	5	Q5 C-P+
Computer Electrical	38	0.84	0.87	-1.76	-0.03	-1.47	0.29	0.46	0.62	0.53760	5	Q5 C-P+
Electronic Metallurgical Mining	16	0.94	0.86	-1.76	0.08	-2.70	-0.94	1.04	-0.90	0.38080	5	Q5 C-P+
Industrial	14	0.71	0.86	-1.76	-0.15	-0.83	0.92	0.60	1.53	0.15200	5	Q5 C-P+
Mechanical	50	0.78	0.86	-1.76	-0.08	-1.17	0.58	0.36	1.64	0.10780	5	Q5 C-P+
Chemical	18	0.11	0.10	2.32	0.01	2.22	-0.10	0.77	-0.14	0.89350	6	Q6 C-P-
Civil	54	0.15	0.13	2.32	0.02	2.16	-0.16	0.40	-0.41	0.68460	6	Q6 C-P-
Computer Electrical	38	0.13	0.14	2.32	-0.01	2.39	0.07	0.49	0.14	0.88740	6	Q6 C-P-
Electronic Metallurgical Mining	16	0.00	0.12	2.32	-0.12	4.42	2.10	1.87	1.13	0.27930	6	Q6 C-P-
Industrial	14	0.07	0.11	2.32	-0.04	2.77	0.45	1.04	0.43	0.67320	6	Q6 C-P-
Mechanical	50	0.16	0.13	2.32	0.03	2.05	-0.27	0.40	-0.68	0.49930	6	Q6 C-P-
Chemical	18	0.67	0.60	-0.53	0.06	-0.84	-0.30	0.53	-0.58	0.57280	7	Q7 C-P+
Civil	54	0.72	0.66	-0.53	0.06	-0.85	-0.31	0.32	-0.98	0.33330	7	Q7 C-P+
Computer Electrical	37	0.57	0.69	-0.53	-0.12	0.05	0.58	0.35	1.67	0.10340	7	Q7 C-P+
Electronic Metallurgical Mining	16	0.75	0.65	-0.53	0.10	-1.04	-0.50	0.59	-0.85	0.40800	7	Q7 C-P+
Industrial	14	0.64	0.65	-0.53	-0.01	-0.49	0.04	0.57	0.08	0.93950	7	Q7 C-P+
Mechanical	50	0.64	0.67	-0.53	-0.03	-0.41	0.12	0.31	0.40	0.69410	7	Q7 C-P+
Chemical	18	0.56	0.60	-0.54	-0.05	-0.31	0.23	0.50	0.45	0.65730	8	Q8 C+P-
Civil	54	0.74	0.66	-0.54	0.08	-0.95	-0.41	0.33	-1.26	0.21370	8	Q8 C+P-
Computer Electrical	38	0.68	0.69	-0.54	-0.01	-0.49	0.05	0.37	0.13	0.89710	8	Q8 C+P-
Electronic Metallurgical Mining	16	0.75	0.65	-0.54	0.10	-1.04	-0.50	0.59	-0.84	0.41350	8	Q8 C+P-
Industrial	14	0.36	0.65	-0.54	-0.30	0.74	1.28	0.57	2.25	0.04410	8	Q8 C+P-

Mechanical	50	0.66	0.67	-0.54	-0.01	-0.51	0.03	0.31	0.10	0.92120	8	Q8 C+P-
Chemical	18	0.33	0.47	0.07	-0.14	0.72	0.64	0.53	1.22	0.23990	9	Q9 C+P-
Civil	54	0.59	0.53	0.07	0.06	-0.20	-0.27	0.29	-0.91	0.36810	9	Q9 C+P-
Computer Electrical	38	0.50	0.56	0.07	-0.06	0.36	0.29	0.34	0.84	0.40560	9	Q9 C+P-
Electronic Metallurgical Mining	16	0.69	0.51	0.07	0.17	-0.71	-0.78	0.56	-1.41	0.18100	9	Q9 C+P-
Industrial	14	0.50	0.51	0.07	-0.01	0.13	0.06	0.55	0.10	0.92080	9	Q9 C+P-
Mechanical	50	0.52	0.54	0.07	-0.02	0.14	0.07	0.30	0.24	0.81360	9	Q9 C+P-
Chemical	18	0.28	0.26	1.12	0.02	1.01	-0.12	0.55	-0.21	0.83730	10	Q10 C+P-
Civil	54	0.35	0.31	1.12	0.04	0.91	-0.21	0.30	-0.70	0.49010	10	Q10 C+P-
Computer Electrical	38	0.34	0.33	1.12	0.01	1.07	-0.05	0.36	-0.14	0.89210	10	Q10 C+P-
Electronic Metallurgical Mining	16	0.25	0.29	1.12	-0.04	1.34	0.22	0.61	0.36	0.72150	10	Q10 C+P-
Industrial	14	0.29	0.28	1.12	0.01	1.08	-0.04	0.60	-0.06	0.95140	10	Q10 C+P-
Mechanical	50	0.26	0.31	1.12	-0.05	1.39	0.27	0.34	0.80	0.43020	10	Q10 C+P-
Chemical	18	0.44	0.54	-0.26	-0.10	0.19	0.44	0.50	0.88	0.38970	11	Q11 C+P-
Civil	54	0.63	0.61	-0.26	0.02	-0.37	-0.12	0.30	-0.39	0.70090	11	Q11 C+P-
Computer Electrical	37	0.65	0.63	-0.26	0.01	-0.33	-0.07	0.36	-0.19	0.84850	11	Q11 C+P-
Electronic Metallurgical Mining	16	0.50	0.59	-0.26	-0.09	0.13	0.39	0.52	0.75	0.46550	11	Q11 C+P-
Industrial	14	0.50	0.59	-0.26	-0.09	0.13	0.38	0.55	0.70	0.49800	11	Q11 C+P-
Mechanical	50	0.66	0.61	-0.26	0.05	-0.51	-0.25	0.31	-0.80	0.42610	11	Q11 C+P-
Chemical	18	0.61	0.40	0.38	0.21	-0.57	-0.94	0.51	-1.85	0.08300	12	Q12 C+P+
Civil	54	0.41	0.47	0.38	-0.06	0.65	0.27	0.29	0.93	0.35750	12	Q12 C+P+
Computer Electrical	38	0.58	0.50	0.38	0.08	0.01	-0.37	0.34	-1.07	0.29190	12	Q12 C+P+
Electronic Metallurgical Mining	16	0.44	0.44	0.38	-0.01	0.41	0.03	0.53	0.06	0.95110	12	Q12 C+P+
Industrial	14	0.50	0.44	0.38	0.06	0.13	-0.25	0.55	-0.45	0.65730	12	Q12 C+P+
Mechanical	50	0.38	0.47	0.38	-0.09	0.78	0.40	0.31	1.31	0.19740	12	Q12 C+P+
Chemical	18	0.33	0.42	0.28	-0.09	0.72	0.43	0.53	0.82	0.42360	13	Q13 C+P-
Civil	54	0.50	0.49	0.28	0.01	0.23	-0.06	0.29	-0.19	0.84890	13	Q13 C+P-
Computer Electrical	38	0.58	0.52	0.28	0.06	0.01	-0.27	0.34	-0.80	0.43050	13	Q13 C+P-
Electronic Metallurgical Mining	16	0.44	0.47	0.28	-0.03	0.41	0.13	0.53	0.24	0.81340	13	Q13 C+P-
Industrial	14	0.43	0.46	0.28	-0.03	0.43	0.15	0.55	0.26	0.79570	13	Q13 C+P-
Mechanical	50	0.48	0.49	0.28	-0.01	0.32	0.04	0.30	0.13	0.89690	13	Q13 C+P-
Chemical	18	0.83	0.72	-1.11	0.11	-1.83	-0.72	0.65	-1.10	0.28680	14	Q14 C-P-
Civil	54	0.80	0.77	-1.11	0.03	-1.30	-0.19	0.35	-0.52	0.60320	14	Q14 C-P-
Computer Electrical	38	0.84	0.79	-1.11	0.05	-1.47	-0.36	0.46	-0.78	0.44280	14	Q14 C-P-
Electronic Metallurgical Mining	16	0.69	0.76	-1.11	-0.08	-0.71	0.40	0.56	0.72	0.48240	14	Q14 C-P-
Industrial	14	0.79	0.77	-1.11	0.02	-1.23	-0.12	0.66	-0.18	0.86140	14	Q14 C-P-
Mechanical	50	0.68	0.77	-1.11	-0.09	-0.61	0.50	0.32	1.58	0.12140	14	Q14 C-P-
Chemical	18	0.56	0.45	0.16	0.11	-0.31	-0.48	0.50	-0.95	0.35590	15	Q15 C-P-
Civil	54	0.54	0.51	0.16	0.02	0.06	-0.11	0.29	-0.36	0.71700	15	Q15 C-P-

Computer Electrical	38	0.50	0.54	0.16	-0.04	0.36	0.19	0.34	0.57	0.57500	15	Q15 C-P-
Electronic Metallurgical Mining	16	0.50	0.49	0.16	0.01	0.13	-0.03	0.52	-0.06	0.95500	15	Q15 C-P-
Industrial	14	0.71	0.49	0.16	0.22	-0.83	-1.00	0.60	-1.65	0.12460	15	Q15 C-P-
Mechanical	50	0.42	0.51	0.16	-0.09	0.59	0.43	0.30	1.41	0.16460	15	Q15 C-P-
Chemical	18	0.78	0.74	-1.24	0.03	-1.45	-0.20	0.59	-0.35	0.73410	16	Q16 C-P+
Civil	54	0.80	0.79	-1.24	0.01	-1.30	-0.05	0.35	-0.15	0.88320	16	Q16 C-P+
Computer Electrical	38	0.84	0.81	-1.24	0.03	-1.47	-0.22	0.46	-0.49	0.62870	16	Q16 C-P+
Electronic Metallurgical Mining	16	0.75	0.79	-1.24	-0.04	-1.04	0.21	0.59	0.35	0.73280	16	Q16 C-P+
Industrial	14	0.71	0.79	-1.24	-0.07	-0.83	0.41	0.60	0.68	0.50770	16	Q16 C-P+
Mechanical	50	0.78	0.79	-1.24	-0.01	-1.17	0.07	0.36	0.20	0.83880	16	Q16 C-P+
Chemical	18	0.39	0.43	0.25	-0.04	0.45	0.20	0.51	0.38	0.70560	17	Q17 C-P+
Civil	53	0.47	0.49	0.25	-0.02	0.34	0.09	0.29	0.32	0.74980	17	Q17 C-P+
Computer Electrical	37	0.62	0.52	0.25	0.10	-0.20	-0.45	0.36	-1.26	0.21670	17	Q17 C-P+
Electronic Metallurgical Mining	16	0.50	0.47	0.25	0.03	0.13	-0.11	0.52	-0.22	0.83000	17	Q17 C-P+
Industrial	14	0.36	0.47	0.25	-0.11	0.74	0.49	0.57	0.87	0.40380	17	Q17 C-P+
Mechanical	49	0.49	0.50	0.25	-0.01	0.31	0.06	0.30	0.19	0.85000	17	Q17 C-P+
Chemical	18	1.06	1.27	-0.76	-0.21	-0.20	0.56	0.38	1.49	0.15570	18	Q18 C+P-
Civil	51	1.31	1.37	-0.76	-0.06	-0.59	0.17	0.23	0.74	0.46310	18	Q18 C+P-
Computer Electrical	35	1.51	1.44	-0.76	0.08	-1.01	-0.24	0.31	-0.80	0.42920	18	Q18 C+P-
Electronic Metallurgical Mining	16	1.38	1.35	-0.76	0.03	-0.83	-0.07	0.42	-0.17	0.86960	18	Q18 C+P-
Industrial	13	1.46	1.36	-0.76	0.10	-1.06	-0.30	0.48	-0.62	0.54640	18	Q18 C+P-
Mechanical	49	1.43	1.38	-0.76	0.05	-0.92	-0.15	0.25	-0.61	0.54280	18	Q18 C+P-
Chemical	18	1.67	1.51	-1.17	0.16	-1.64	-0.48	0.44	-1.09	0.29330	19	Q19 C-P-
Civil	51	1.61	1.61	-1.17	0.00	-1.17	0.00	0.25	0.00	1.00000	19	Q19 C-P-
Computer Electrical	35	1.71	1.67	-1.17	0.05	-1.34	-0.18	0.34	-0.52	0.60870	19	Q19 C-P-
Electronic Metallurgical Mining	16	1.50	1.60	-1.17	-0.10	-0.90	0.27	0.39	0.68	0.50490	19	Q19 C-P-
Industrial	13	1.92	1.61	-1.17	0.31	-2.86	-1.69	1.00	-1.69	0.11950	19	Q19 C-P-
Mechanical	49	1.47	1.61	-1.17	-0.14	-0.77	0.40	0.23	1.74	0.08760	19	Q19 C-P-
Chemical	18	0.89	1.02	-0.10	-0.13	0.23	0.33	0.38	0.86	0.40010	20	Q20 C+P+
Civil	51	1.04	1.13	-0.10	-0.09	0.13	0.24	0.23	1.05	0.30030	20	Q20 C+P+
Computer Electrical	35	1.09	1.20	-0.10	-0.11	0.18	0.29	0.27	1.06	0.29890	20	Q20 C+P+
Electronic Metallurgical Mining	16	1.31	1.10	-0.10	0.22	-0.67	-0.57	0.41	-1.37	0.19260	20	Q20 C+P+
Industrial	13	1.23	1.10	-0.10	0.13	-0.44	-0.33	0.45	-0.75	0.46950	20	Q20 C+P+
Mechanical	50	1.24	1.13	-0.10	0.11	-0.40	-0.30	0.23	-1.28	0.20540	20	Q20 C+P+
Chemical	18	0.17	0.17	2.52	0.00	2.52	0.00	0.61	0.00	1.00000	21	Q21 C+P+
Civil	49	0.31	0.22	2.52	0.09	2.12	-0.40	0.29	-1.39	0.17030	21	Q21 C+P+
Computer Electrical	34	0.12	0.24	2.52	-0.13	3.34	0.82	0.52	1.57	0.12680	21	Q21 C+P+
Electronic Metallurgical Mining	15	0.20	0.21	2.52	-0.01	2.57	0.04	0.64	0.07	0.94470	21	Q21 C+P+
Industrial	13	0.15	0.18	2.52	-0.03	2.69	0.17	0.74	0.23	0.82260	21	Q21 C+P+

Mechanical	49	0.22	0.22	2.52	0.01	2.48	-0.04	0.33	-0.14	0.89140	21	Q21 C+P+
Chemical	18	1.28	1.24	-0.73	0.04	-0.85	-0.12	0.41	-0.29	0.77700	22	Q22 C-P+
Civil	51	1.20	1.34	-0.73	-0.14	-0.32	0.41	0.24	1.74	0.08860	22	Q22 C-P+
Computer Electrical	35	1.43	1.39	-0.73	0.04	-0.84	-0.11	0.30	-0.37	0.71500	22	Q22 C-P+
Electronic Metallurgical Mining	16	1.31	1.31	-0.73	0.00	-0.73	0.00	0.43	0.00	1.00000	22	Q22 C-P+
Industrial	13	1.31	1.31	-0.73	-0.01	-0.71	0.02	0.47	0.04	0.96600	22	Q22 C-P+
Mechanical	50	1.44	1.33	-0.73	0.11	-1.07	-0.33	0.26	-1.30	0.19900	22	Q22 C-P+
Chemical	16	0.81	1.01	0.06	-0.20	0.49	0.43	0.37	1.15	0.27080	23	Q23 C-P-
Civil	51	0.90	1.08	0.06	-0.18	0.45	0.39	0.21	1.89	0.06480	23	Q23 C-P-
Computer Electrical	35	1.26	1.16	0.06	0.10	-0.16	-0.22	0.26	-0.86	0.39830	23	Q23 C-P-
Electronic Metallurgical Mining	16	1.31	1.04	0.06	0.28	-0.55	-0.61	0.38	-1.59	0.13350	23	Q23 C-P-
Industrial	13	1.00	1.04	0.06	-0.04	0.14	0.09	0.40	0.22	0.83270	23	Q23 C-P-
Mechanical	48	1.19	1.08	0.06	0.10	-0.17	-0.23	0.22	-1.06	0.29400	23	Q23 C-P-
Chemical	17	0.41	0.37	0.53	0.04	0.35	-0.18	0.52	-0.34	0.74060	24	Q24 C-P-
Civil	51	0.51	0.44	0.53	0.07	0.19	-0.34	0.30	-1.14	0.26180	24	Q24 C-P-
Computer Electrical	34	0.38	0.47	0.53	-0.09	0.92	0.39	0.37	1.07	0.29340	24	Q24 C-P-
Electronic Metallurgical Mining	16	0.38	0.41	0.53	-0.04	0.69	0.16	0.54	0.30	0.76650	24	Q24 C-P-
Industrial	13	0.38	0.41	0.53	-0.02	0.64	0.11	0.58	0.19	0.85530	24	Q24 C-P-
Mechanical	47	0.43	0.44	0.53	-0.01	0.59	0.06	0.31	0.18	0.85770	24	Q24 C-P-
Chemical	17	1.47	1.50	-0.97	-0.03	-0.90	0.07	0.35	0.21	0.83760	25	Q25 C+P+
Civil	51	1.73	1.62	-0.97	0.10	-1.28	-0.31	0.26	-1.20	0.23580	25	Q25 C+P+
Computer Electrical	35	1.71	1.70	-0.97	0.02	-1.03	-0.06	0.31	-0.20	0.84410	25	Q25 C+P+
Electronic Metallurgical Mining	16	1.38	1.62	-0.97	-0.25	-0.48	0.49	0.33	1.50	0.15510	25	Q25 C+P+
Industrial	13	1.62	1.64	-0.97	-0.02	-0.91	0.06	0.43	0.14	0.89330	25	Q25 C+P+
Mechanical	50	1.60	1.62	-0.97	-0.02	-0.91	0.06	0.23	0.26	0.79890	25	Q25 C+P+
Chemical	16	0.69	0.99	-0.04	-0.30	0.62	0.67	0.38	1.75	0.10280	26	Q26 C+P+
Civil	49	1.10	1.15	-0.04	-0.05	0.07	0.11	0.21	0.54	0.59120	26	Q26 C+P+
Computer Electrical	33	1.06	1.21	-0.04	-0.15	0.28	0.32	0.25	1.27	0.21250	26	Q26 C+P+
Electronic Metallurgical Mining	16	1.00	1.08	-0.04	-0.08	0.13	0.18	0.36	0.49	0.63390	26	Q26 C+P+
Industrial	11	1.64	1.13	-0.04	0.51	-1.29	-1.24	0.55	-2.25	0.05100	26	Q26 C+P+
Mechanical	48	1.31	1.14	-0.04	0.17	-0.43	-0.38	0.22	-1.73	0.09080	26	Q26 C+P+
Chemical	18	1.11	1.26	-0.73	-0.15	-0.34	0.39	0.38	1.04	0.31280	27	Q27 C+P+
Civil	51	1.37	1.37	-0.73	0.00	-0.73	0.00	0.24	0.00	1.00000	27	Q27 C+P+
Computer Electrical	35	1.46	1.43	-0.73	0.03	-0.81	-0.08	0.29	-0.27	0.79150	27	Q27 C+P+
Electronic Metallurgical Mining	16	1.25	1.34	-0.73	-0.09	-0.49	0.25	0.40	0.61	0.54900	27	Q27 C+P+
Industrial	13	1.54	1.35	-0.73	0.19	-1.28	-0.55	0.50	-1.10	0.29410	27	Q27 C+P+
Mechanical	50	1.38	1.37	-0.73	0.01	-0.77	-0.04	0.24	-0.16	0.87330	27	Q27 C+P+
Chemical	16	0.56	0.51	-0.07	0.06	-0.32	-0.25	0.53	-0.47	0.64240	28	Q28 C-P-
Civil	49	0.63	0.57	-0.07	0.06	-0.36	-0.30	0.32	-0.94	0.35320	28	Q28 C-P-

Computer Electrical	34	0.50	0.61	-0.07	-0.11	0.41	0.48	0.36	1.32	0.19470	28	Q28 C-P-
Electronic Metallurgical Mining	15	0.47	0.55	-0.07	-0.08	0.30	0.37	0.54	0.68	0.51090	28	Q28 C-P-
Industrial	13	0.46	0.55	-0.07	-0.09	0.31	0.37	0.57	0.65	0.52730	28	Q28 C-P-
Mechanical	47	0.62	0.57	-0.07	0.04	-0.27	-0.20	0.32	-0.64	0.52680	28	Q28 C-P-
Chemical	16	0.69	0.60	-0.49	0.09	-0.90	-0.42	0.56	-0.74	0.47060	29	Q29 C-P+
Civil	50	0.62	0.66	-0.49	-0.04	-0.30	0.19	0.31	0.60	0.55050	29	Q29 C-P+
Computer Electrical	34	0.76	0.69	-0.49	0.07	-0.89	-0.40	0.42	-0.94	0.35420	29	Q29 C-P+
Electronic Metallurgical Mining	15	0.53	0.64	-0.49	-0.11	0.01	0.50	0.54	0.92	0.37370	29	Q29 C-P+
Industrial	13	0.46	0.65	-0.49	-0.18	0.31	0.79	0.57	1.39	0.19110	29	Q29 C-P+
Mechanical	48	0.71	0.66	-0.49	0.04	-0.72	-0.23	0.33	-0.69	0.49640	29	Q29 C-P+
Chemical	16	0.50	0.51	-0.06	-0.01	-0.04	0.02	0.53	0.04	0.96510	30	Q30 C-P+
Civil	50	0.48	0.57	-0.06	-0.09	0.34	0.41	0.30	1.35	0.18400	30	Q30 C-P+
Computer Electrical	34	0.68	0.61	-0.06	0.07	-0.40	-0.34	0.38	-0.88	0.38450	30	Q30 C-P+
Electronic Metallurgical Mining	15	0.47	0.55	-0.06	-0.08	0.30	0.36	0.54	0.67	0.51450	30	Q30 C-P+
Industrial	13	0.62	0.55	-0.06	0.07	-0.35	-0.29	0.58	-0.49	0.63280	30	Q30 C-P+
Mechanical	46	0.63	0.58	-0.06	0.05	-0.32	-0.25	0.32	-0.79	0.43460	30	Q30 C-P+
Chemical	16	0.56	0.64	0.94	-0.08	1.18	0.24	0.44	0.55	0.58780	31	Q31 C+P+
Civil	49	0.73	0.76	0.94	-0.02	1.00	0.06	0.24	0.27	0.78870	31	Q31 C+P+
Computer Electrical	33	0.94	0.80	0.94	0.14	0.56	-0.38	0.28	-1.34	0.19140	31	Q31 C+P+
Electronic Metallurgical Mining	15	0.40	0.71	0.94	-0.31	1.94	1.00	0.50	1.99	0.06790	31	Q31 C+P+
Industrial	12	0.75	0.70	0.94	0.05	0.80	-0.14	0.47	-0.30	0.77390	31	Q31 C+P+
Mechanical	46	0.80	0.77	0.94	0.04	0.83	-0.11	0.24	-0.43	0.66640	31	Q31 C+P+
Chemical	16	1.75	1.47	-0.97	0.28	-1.80	-0.83	0.50	-1.65	0.12210	32	Q32 C+P+
Civil	50	1.58	1.56	-0.97	0.02	-1.01	-0.04	0.24	-0.18	0.85480	32	Q32 C+P+
Computer Electrical	34	1.68	1.63	-0.97	0.04	-1.11	-0.14	0.32	-0.44	0.66390	32	Q32 C+P+
Electronic Metallurgical Mining	15	1.60	1.56	-0.97	0.04	-1.07	-0.11	0.43	-0.25	0.80760	32	Q32 C+P+
Industrial	13	1.38	1.57	-0.97	-0.19	-0.55	0.42	0.39	1.06	0.31130	32	Q32 C+P+
Mechanical	48	1.48	1.58	-0.97	-0.10	-0.70	0.27	0.22	1.19	0.23890	32	Q32 C+P+
Chemical	16	0.81	0.60	1.02	0.21	0.45	-0.57	0.41	-1.42	0.17860	33	Q33 C+P-
Civil	50	0.68	0.71	1.02	-0.03	1.11	0.08	0.24	0.35	0.72580	33	Q33 C+P-
Computer Electrical	33	0.85	0.76	1.02	0.09	0.79	-0.23	0.28	-0.82	0.42020	33	Q33 C+P-
Electronic Metallurgical Mining	15	0.73	0.67	1.02	0.06	0.86	-0.16	0.43	-0.38	0.70950	33	Q33 C+P-
Industrial	13	0.77	0.66	1.02	0.11	0.73	-0.30	0.45	-0.67	0.51780	33	Q33 C+P-
Mechanical	48	0.56	0.71	1.02	-0.15	1.45	0.43	0.25	1.70	0.09550	33	Q33 C+P-