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Asymmetric generalizations of symmetric univariate probability distributions obtained through quantile splicing

by

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ABSTRACT

This thesis develops a skewing methodology for the formulation of two-piece families of distributions that can be defined through their cumulative distribution functions (CDFs), probability density functions (PDFs) or quantile functions. The advantage of this methodology is that the families of distributions constructed have skewness-invariant measures of kurtosis, allowing for the independent analysis of the skewness and kurtosis of a distribution.

The central contribution of this thesis is in the development of the quantile function of the two-piece family of distributions. This quantile function is constructed through the use of the quantile functions of half distributions developed from symmetric univariate distributions (henceforth referred to as the parent distribution). This quantile function is the used to derive a general formula for the r^{th} order L-moments of the two-piece family of distributions. The results of these L-moments will be in terms of the L-moments of both the parent distribution and the half distribution. The parameters of this new family of distributions can be estimated through the method of L-moments since closed form expressions exist for the L-moments and subsequently the estimators.

The results from the skewing methodology as well as from the formula for the r^{th} order L-moments will be applied to well-known symmetric univariate distributions. These include the arcsine, uniform, cosine, normal, logistic, hyperbolic secant and Student's $t(2)$ distributions, which do not have a shape parameter, as well as the quantile-based Tukey lamba distribution which has a kurtosis parameter.

KEYWORDS : Two-piece distribution, Half distribution, Quantile function, L-moment, Skewness-invariant kurtosis measure.

DECLARATION

I, Brenda Vuguza Mac'Oduol declare that the thesis, which I hereby submit for the degree PhD in Mathematical Statistics at the University of Pretoria, is my own work and has not been previously submitted by me for a degree at this or any other tertiary institution.

SIGNATURE: ________________________________

DATE: **03 FEBRUARY 2021** ________________________________

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LIST OF ABBREVIATIONS AND SYMBOLS

$\mathbb{I}^+$	The indicator function on $\mathbb{R}^+$.
$\mathbb{I}^-$	The indicator function on $\mathbb{R}^-$.
$\mathbb{N}$	The set of all natural numbers.
$\mathbb{R}$	The set of all real numbers.
$\mathbb{R}^+$	The set of all positive real numbers.
$\mathbb{R}^-$	The set of all negative real numbers.
$\mathbb{Z}^+$	The set of all positive integers.
$\mathrm{B}(\cdot,\cdot)$	Beta function.
G	Catalan's constant.
$\Gamma(\cdot)$	Gamma function.
${}_2F_1(\cdot,\cdot;\cdot,\cdot)$	Hypergeometric function.
$\psi^{(}m)(\cdot)$	Polygamma function.
$\zeta(\cdot)$	Zeta function.
erf^{-1}	Inverse error function.
$L_{X:r}$	r^{th} order L-moment for the parent distribution.
$L_{T:r}$	r^{th} order L-moment for the two-piece distribution.
$\tau_{X:r}$	r^{th} order L-moment ratio for the parent distribution.
$\tau_{T:r}$	r^{th} order L-moment ratio for the two-piece distribution.

INTRODUCTION

1.1 Aims and Objectives

The aim of this thesis is to develop a methodology that can be used to formulate of two-piece families of distributions that possess skewness-invariant measures of kurtosis. This methodology is demonstrated in Theorem 3.2.1, where the quantile function of a two-piece distribution is obtained by joining the quantile functions of two half distributions obtained from a univariate symmetric distribution at the median, with an asymmetry parameter being introduced to the quantile function left of the median. The piece-wise quantile function that is generated exhibits asymmetry below the median, with symmetry being attained when the asymmetry parameter has a unit value. This methodology will allow for the generalization of a distribution is either quantile-based or classically defined through the probability density functions (PDFs) and/or cumulative distribution functions (CDFs).

Furthermore, the results from the methodology are used to construct a general formulae for the r^{th} order L-moments that will be used to characterize a two-piece distribution. These L-moments will be functions of the L-moments of the parent distribution as well as the half distribution.

The methodology is used to develop two-piece families of distributions for the arcsine, uniform, cosine, normal, logistic, hyperbolic secant and Student's $t(2)$ distribution in Chapter 3, as well as for the quantile-based Tukey lambda distribution in Chapter 4.

1.2 The Evolution of Quantile Splicing

Quantile splicing is a term that was conceived to better describe the methodology proposed and developed in this thesis. Albeit splicing definitively means to join or intertwine, the integrated term in this case implies that two quantile functions are joined together.

The methodology was necessitated from the work of Balakrishnan et al. (2017), who proposed a skew logistic distribution as an alternative to the model proposed by van Staden and King (2015). Balakrishnan et al. (2017) used the CDF of the half logistic distribution as the building block to construct the CDF of the skew logistic distribution. This was achieved by joining the CDFs of the half logistic distribution to the right and left of the median, with the CDF to the left of the median possessing a single asymmetry parameter. In order to further characterize the skew logistic distribution, they made use of moments of order statistics from the standard half logistic distribution to obtain the single and product moments of order statistics. These results were then used to compute the means, variances and covariances of order statistics from the skew logistic distribution for any sample size, n.

The technique proposed by Balakrishnan et al. (2017) sought to introduce asymmetry to univariate distributions with the exemption of those from the quantile statistical universe, also referred to as quantile-based distributions. In order to alleviate this exclusion, the idea of using the quantile functions of half distributions arose. This would ensure that the method of introducing asymmetry would be used for distributions defined through their CDFs, PDFs or quantile functions. The result is a piecewise quantile function, joined at the point $0 < k < 1$, and consequent emergence of the two-piece family of distributions. The thesis will begin with the construction of the method of quantile splicing for the case $k = \frac{1}{2}$, whereas future research will consider the general form for $k \neq \frac{1}{2}$.

Furthermore, quantile splicing led to the derivation of the general formula for the r^{th} order L-moments. This allows for the characterization of the two-piece families of distributions in terms of L-moments as opposed to the moments of order statistics. The implication is on the computational ease with which results can be obtained especially with respect to parameter estimation, since the L-moments obtained are expressed in closed form and are simple in nature, as well as in the exploration of the properties of the distributions. Likewise, an estimation algorithm can be developed through the method of L-moments.

Another added advantage of using this method arises in the nature of its measures of kurtosis. Since asymmetry is introduced to essentially one side or piece of the distribution, the level of kurtosis of the two-piece distribution remains the same as that of the parent distribution. This means that the measures of kurtosis are skewness-invariant. This benefit enables the skewness and kurtosis of distributions to be analysed and reported seperately.

1.3 Outline of the Thesis

Chapter 2 takes a comprehensive look at the methods that have been used to introduce asymmetry in univariate distributions. These methods make use of the CDFs, PDFs or quantile functions as the kernels in the building of new distributional models. The main focus will be on the techniques used as well as the strengths and weaknesses of the results obtained.

Chapter 3 focuses on the development of the method of quantile splicing. This will entail constructing the two-piece quantile function of the resulting families of distributions from the quantile functions of the half distributions developed from their parent distributions. Emanating from this methodology will be a general formulae for the r^{th} order L-moments as mentioned in Section 1.1. This will comprise of the L-moments of both the parent and half distribution. In addition, this formulae can also be presented in terms of order statistics of the half distribution. Well-known quantile measures of distributional form are discussed, and their general forms for two-piece distributions are derived. These measures will also prove that the distributions constructed through the method of quantile splicing have skewness-invariant measures of kurtosis. An estimation algorithm is developed using the method of L-moments since we're able to obtain L-moments that are closed-form and simple in nature.

Chapter 4 makes use of the results in Chapter 3 to develop two-piece families of distributions for symmetric univariate distributions that do not have a shape parameter. Specifically, the arcsine, uniform, cosine, normal, logistic, hyperbolic secant and Student's $t(2)$ will be studied.

Chapter 5 uses the method of quantile splicing to construct the two-piece Tukey lambda distribution. This quantile-based parent distribution has a shape parameter that governs its level of kurtosis as compared to those in Chapter 3 that do not have any shape parameter.

Chapter 6 aims to summarise the techniques developed in the thesis as well as present possible areas for future research.

1.4 Contributions of the Thesis

The thesis aims to introduce a new methodology that can be used to construct two-piece families of distributions with skewness-invariant measures of kurtosis. Moreover, a general formula for the r^{th} order L-moments is derived. Other contributions from each chapter are hereby outlined.

Chapter 3

- An integral for calculating the r^{th} order L-moment of distributions that are split at a scaling point $0 < k < 1$ is derived in Section 3.3.3.

- The r^{th} order shifted scaled polynomials are derived in Section 3.3.3, with the first 4 polynomials being presented.

- In Section 3.3.4, the r^{th} order L-moments of half distributions are derived and also presented in terms of order statistics.

- Formula for the quantile measures of distributional form for the location, spread and shape for two-piece families of distributions are presented. As indicated, the quantile-based kurtosis measures are skewness-invariant.

- An estimation algorithm for two-piece families of distributions is developed in Section 3.5.

Chapter 4

- The quantile splicing methodology is used to derive the quantile functions, CDFs and PDFs of the two-piece distributions constructed from the Student's $t(2)$, hyperbolic secant, logistic, normal, cosine, uniform and arcsine distributions in Sections 4.2.2, 4.3.2, 4.4.2, 4.5.2, 4.6.2, 4.7.2 and 4.8.2, respectively.

- The quantile measures of distributional form for location, spread and shape for the Student's $t(2)$, hyperbolic secant, logistic, normal, cosine, uniform and arcsine distributions are derived in Sections 4.2.2, 4.3.2, 4.4.2, 4.5.2, 4.6.2, 4.7.2 and 4.8.2, respectively.

- The r^{th} order L-moments for the half distributions from the Student's $t(2)$, hyperbolic secant, logistic, normal, cosine, uniform and arcsine distributions are presented in Sections 4.2.2, 4.3.2, 4.4.2, 4.5.2, 4.6.2, 4.7.2 and 4.8.2, respectively. The results are derived in full in Section 4.9.

- The r^{th} order L-moments together with the L-skewness and L-kurtosis ratios for the two-piece families of distributions constructed from the Student's $t(2)$, hyperbolic secant, logistic, normal, cosine, uniform and arcsine distributions are presented in Sections 4.2.2, 4.3.2, 4.4.2, 4.5.2, 4.6.2, 4.7.2 and 4.8.2, respectively. The results are derived in full in Section 4.9.

Chapter 5

- The quantile function for the two-piece Tukey lambda distribution is derived in Section 5.3.
- The quantile measures of distributional form for location, spread and shape for the two-piece Tukey lambda distribution are derived in Section 5.4
- In Sections 5.5 and 5.6 the support and classes of the two-piece Tukey lambda distribution are presented.
- The r^{th} order L-moments for the half Tukey lambda distribution as well as the two-piece Tukey lambda distribution are derived in Section 5.7
- The values for the density and slope of the density curve for the two-piece Tukey lambda distribution are presented in Section 5.8.
- An estimation algorithm for two-piece Tukey lambda distribution is developed in Section 5.9.

Chapter 6

- The quantile function, CDF and PDF for the piecewise distribution constructed using the extended quantile splicing technique is derived.
- The general formula for the r^{th} order shifted scaled polynomials are derived in Section 6.3.2, with the first 4 polynomials being presented.
- In Section 6.3.2, the general formula for the r^{th} order L-moments of piecewise distributions are derived. These are presented in terms of order statistics as well as the L-moments of the k^{th} and $(1-k)^{th}$-piece distributioins.

METHODS OF SKEWING UNIVARIATE DISTRIBUTIONS

2.1 Introduction

In an effort to model asymmetric data, Pearson (1895) introduced procedures that made use of differential equations to generate univariate distributions. Different distributions were obtained from this Pearson system of differential equations that depended on the solutions obtained for each probability density function. Pearson then presented the various distributions obtained using this system.

Numerous asymmetric distributional families and skewing mechanisms have since been proposed and studied since the introduction of the Pearson system. These inculde the Burr types I - XII distributions (Burr (1942, 1968, 1973)), also studied in detail by Fry (1993) and Johnson et al. (1994). Johnson (1949) proposed a system of distributions by transforming (translating) the normal distribution, which yielded the lognormal family, a family of bounded distributions and a family of unbounded distributions. Some common univariate distributions that arise from these families are the normal, lognormal, gamma, beta and exponential distributions.

In the event a distribution is quantile-based, that is there is no closed-form expression for the cumulative distribution function (CDF) or probability density function (PDF) resulting in a distribution being defined through its quantile function, then quantile methods are applied. These methods were first documented on the early works on the lambda distribution (Hastings Jr et al. (1947), Tukey (1960)). Ramberg and Schmeiser (1972) and Ramberg and Schmeiser (1974) generalized those results to create the generalized lambda family of distributions (GLDs).

There are many methods that have since arisen with the aim of generating asymmetric distributions or skewing existing distributions. Depending on the existence of the PDF, CDF or quantile function of a distribution, these methods made use of them as the building blocks in the methodologies. The aim would be to increase the flexibility of distributions with regards to its distributional form, thereby improving the fit of the models to data sets.

Some of the existing methods of skewing univariate distributions will be discussed in this chapter, showing what has been done to introduce asymmetry, as well as the properties attained with each method.

2.2 Combining symmetric distributions

The skew normal distribution was introduced by Azzalini (1985) with the intention of introducing skewness to distributions through the combination of two symmetric distributions.

Definition 2.2.0.1. *A random variable X is said to follow a skew normal distribution, denoted as $X \sim SN(\delta)$, if its probability density function is given as*

$$f(x;\delta) = 2\phi(x)\Phi(\delta x), \qquad -\infty < x < \infty, \tag{2.1}$$

where $\phi(x)$ and $\Phi(x)$ are the standard normal probability density and cumulative distribution functions, respectively, whilst $\delta \in \mathbb{R}$ is the asymmetry parameter.

■

Let F and G be two independent standard normal random variables. X can be expressed in terms of F and G as

$$X = \frac{(\delta|U| + V)}{\sqrt{1+\delta^2}}.$$

2.2.1 Distributional properties of the skew normal distribution

The inclusion of a location parameter, μ, and scale parameter, σ, allows for X to be translated to $Y = \mu + \sigma X$. Some of the properties associated with the skew normal distribution are listed below.

Property 1: When $\delta = 0$, the PDF in Eq.(2.1) reduces to that of the standard normal distribution, $N(0,1)$.

Property 2: The random variable X^2 with parameter δ follows a χ^2-distribution with one degree of freedom.

Property 3: The moment generating function of X is $M_X(t) = 2\exp(t^2/2)\phi(\kappa t)$, where $t \in \mathbb{R}$ and $\kappa = \frac{\delta}{\sqrt{1+\delta^2}}$.

Property 4: The characteristic function of X is $\psi(t) = 2\exp(t^2/2)i\tau(\kappa t)$, where
$\tau(r) = \displaystyle\int_0^r \sqrt{\tfrac{2}{\pi}}\exp(y^2/2)dy$ and $\tau(-r) = -\tau(r)$ for $r \geq 0$.

This distribution can be used to fit data that exhibits unimodal behaviour with the presence of skewness.

2.2.2 Extensions on the combination of two symmetric distributions

The proposed method used to obtain the skew normal distribution was such that $f(x;\delta)$ took on the form $2h(\cdot)H(\cdot)$, where X is symmetric at 0 and $H(\cdot)$ is absolutely continuous, with h and H as the PDF and CDF of X, respectively. Since this proposed method catered to distributions with lighter tails than the normal distribution, a general form of Eq.(2.1) was introduced by Azzalini (1986), in order to accommodate heavy-tailed distributions as well.

The probability density functions of this broader class of distributions would be defined through the general form given as

$$f(x;\omega,\delta) = 2g(x;\omega)G(x;\delta), \quad \omega > 0, \tag{2.2}$$

where $g(\cdot)$ and $G(\cdot)$ are the PDF and CDF of any symmetric random variable, respectively, with $\omega > 0$ as an additional shape parameter. This means that the new generalization can be used to obtain different skewed distributions.

Extensive studies continued on Eq.(2.2) by Azzalini (2005) and even further advanced by Abtahi et al. (2012), whose results were used to generate the generalized skew t and the generalized skew Cauchy distributions.

Wang et al. (2004) also defined skew symmetric distributions as functions whose PDF takes on the form

$$f(x) = 2g(x)\xi(x), \tag{2.3}$$

where $\xi(x)$ is a non-constant asymmetry function, such that $\xi(-x) + \xi(x) = 1$ where $\xi : \mathbb{R} \to [0,1]$. This implies that a CDF can be chosen as the asymmetry function. The skew family of distributions is symmetric when $\xi = \frac{1}{2}$.

Arnold and Beaver (2000) and Arnold et al. (2002) have made further progress on Azzalini's work by suggesting the use of two non-normal symmetric densities to created a skewed distribution.

Definition 2.2.2.1. *Suppose U and V are two random variables with ρ_1, P_1 and ρ_2, P_2 as their probability density and cumulative distribution functions, respectively. Then if the conditional distribution of U given $\delta_0 + \delta_1 U > V$ is considered, the probability density function of U can be defined as*

$$f(u; \delta_0, \delta_1) = \frac{\rho_1(u) P_2(\delta_0 + \delta_1 u)}{P(\delta_0 + \delta_1 U > V)},$$

where $\delta_0 \in \mathbb{R}$ and $\delta_1 \in \mathbb{R}$ are shape parameters.

■

Kim (2005) introduced the two-piece skew normal as a further extension of the skew normal distribution.

Definition 2.2.2.2. *A random variable X_δ is said to have a two-piece skew normal distribution with parameter $\delta \in \mathbb{R}$, denoted by $X_\delta \sim TPSN(\delta)$, if its probability density function is given as*

$$f(x; \delta) = \frac{2\pi}{\pi + 2\tan^{-1}(\delta)} \phi(x) \Phi(\delta |x|), \qquad \delta \in \mathbb{R},\ x \in \mathbb{R}. \tag{2.4}$$

■

Eq.(2.4) reduces to the standard normal PDF when $\delta = 0$. The distribution exhibits unimodal and bimodal behaviour, and has been shown to be a mixture of two truncated skew normal distributions.

Through the introduction of two additional parameters, Jamalizadeh et al. (2011) proposed the three parameter generalized two-piece skew normal distribution, denoted as $X_{\delta_1,\delta_2,\rho} \sim GSTPSN(\delta_1, \delta_2, \rho)$ if the PDF is defined as

$$f(x; \delta_1, \delta_2, \rho) = \frac{1}{b(\delta_1, \delta_2, \rho)} \phi(x) \Phi_1(\delta_1 x, \delta_2 |x|; \rho), \qquad \delta_1, \delta_2 \in \mathbb{R},\ |\rho| < 1,\ x \in \mathbb{R}, \tag{2.5}$$

where $\Phi_1(\cdot, \cdot; \rho)$ denotes the PDF of a standard bivariate normal random variable with correlation, ρ, and

$$\frac{1}{b(\delta_1, \delta_2, \rho)} = \frac{1}{4\pi} \left\{ \cos^{-1}\left(\frac{-(\rho + \delta_1 \delta_2)}{\sqrt{1+\delta_1^2}\sqrt{1+\delta_2^2}} \right) + \cos^{-1}\left(\frac{-(\rho - \delta_1 \delta_2)}{\sqrt{1+\delta_1^2}\sqrt{1+\delta_2^2}} \right) + 2\tan^{-1}\delta_2 \right\}.$$

The PDF is unimodal when $\delta_2 \leq 2$, else it is bimodal when $\delta_2 > 2$.

2.3 Inverse Scaling Procedures

Fernández et al. (1995) introduced skewness to distributions by extending the work of Box and Tiao (1973). Skewness was introduced to the exponential power distribution in their attempt to tackle the problem of axial symmetry while modeling a new class of multivariate distributions. This resulted in their proposal of the so-called ν-spherical distributions, which was a solution to any impediment when it came to using them in actual modeling.

Let $\gamma \neq 1$ be defined as the spread parameter. When γ is split, two different density functions for positive and negative values of the random variable are obtained, despite sharing the same absolute values.

The PDF of a random variable from this proposed distribution would be defined as

$$f(x;\gamma,\kappa) = \begin{cases} k\exp(-\frac{1}{2}(x/\gamma)^{\kappa}), & \text{if } x \geq 0, \\ k\exp(-\frac{1}{2}(-\gamma x)^{\kappa}), & \text{if } x < 0, \end{cases} \tag{2.6}$$

where $\gamma > 0, \kappa > 0$ and $k = \frac{1}{2^{1/\kappa}\Gamma(1+\frac{1}{\kappa})(\gamma+\frac{1}{\gamma})}$ is an integrating constant that ensures $f(x;\gamma,\kappa)$ is a valid function. The exponential power class is obtained when $\gamma = 1$, else skewness is observed in the distribution.

Fernández and Steel (1998) introduced a general method of transforming symmetric distributions, by extending the results from Eq.(2.6). Consider a univariate random variable X that is symmetric around 0. Furthermore, let the probability density function, $f(x)$, be such that $f(x) = f(|x|)$, with $f(|x|)$ decreasing in $|x|$. Then a family of skewed distributions is generated if the PDF is given as

$$f_{\gamma}(x) = \frac{2}{1+\frac{1}{\gamma}}\left(f\left(\frac{x}{\gamma}\right)\mathbb{I}^{+}(x) + f(\gamma x)\mathbb{I}^{-}(x)\right), \quad \gamma > 0,$$

where $\mathbb{I}^{+}$ and $\mathbb{I}^{-}$ are indicator functions on $\mathbb{R}^{+}$ and $\mathbb{R}^{-}$, respectively.

Distributional properties

A few of the properties of this class of skewed distributions are:

Property 1: The mode of the scaled distribution is retained at 0.

Property 2: The distribution exhibits asymmetry when $\gamma \neq 1$.

Property 3: $f_{\gamma}(x) = f(x)$ when $\gamma = 1$, the original probability density function of the parent distributions.

Property 4: $f_{\gamma}(x) = f_{\frac{1}{\gamma}}(-x)$, such that a mirror image of the two scaled density functions is obtained at 0.

Fernández and Steel (1998) and Castillo et al. (2011) studied the class of skew normal distributions obtained when $f(x)$ is from a normal distribution.

The skew generalized secant hyperbolic (SGSH) distribution by Fischer and Vaughan (2002) was proposed as a skew generalization of the generalized secant hyperbolic (GSH) distribution of Vaughan (2002), by splitting the scale parameter. In order to introduce asymmetry into the GSH distribution, the half moments of the GSH are first computed. The procedure introduced by Gottschalk (1948) was used by Fischer and Vaughan (2002) to propose the half moments of the GSH. The full proof is documented in Fischer and Vaughan (2002)(p 8-11).

Definition 2.3.0.1. *Let $\delta > 1$ be defined, while $\mathbb{I}^{+}$ and $\mathbb{I}^{-}$ denote the indicator functions of x on $\mathbb{R}_{+}$ and $\mathbb{R}_{-}$ respectively. The PDF of a random variable X from the SGSH distribution is defined as*

$$\begin{aligned} f_{SGSH}(x;t,\delta) &= \frac{2}{\delta+\frac{1}{\delta}}\left\{ f_{SGSH}(x/\delta)\mathbb{I}^{-}(x) + f_{SGSH}(x\delta)\mathbb{I}^{+}(x)\right\} \\ &= \frac{2c_1}{\delta+\frac{1}{\delta}}\left\{ \frac{\exp(c_2x/\delta)\cdot\mathbb{I}^{-}(x)}{\exp(2c_2x/\delta)+2a\exp(c_2x/\delta)+1} + \frac{\exp(c_2\delta x)\cdot\mathbb{I}^{+}(x)}{\exp(2c_2\delta x)+2a\exp(c_2\delta x)+1}\right\} \end{aligned} \tag{2.7}$$

whilst the CDF and quantile functions are given in closed-form as

$$F_{SGSH}(x;t,\delta) = \frac{2\delta^2}{\delta^2+1}\left\{ F_{GSH}(x/\delta)\mathbb{I}^{-}(x) + \frac{\delta^2-1+2F_{GSH}(x\delta)}{2\delta^2}\mathbb{I}^{+}(x)\right\} \tag{2.8}$$

and

$$F^{-1}_{SGSH}(x;t,\delta) = \delta F^{-1}_{GSH}\left(x\cdot\frac{\delta^2+1}{2\delta^2}\right)\mathbb{I}^{A}(x) + \frac{1}{\delta}F^{-1}_{GSH}\left(x\cdot\frac{\delta^2+1}{2}-\frac{\delta^2-1}{2}\right)\mathbb{I}^{\bar{A}}(x)\Bigg\} \tag{2.9}$$

respectively, where

$$\mathbb{I}^{A}(x) = \begin{cases} 1, & if\ x < \frac{\delta^2}{\delta+1} \\ 0, & if\ x \geq \frac{\delta^2}{\delta+1} \end{cases} \quad \text{and} \quad \mathbb{I}^{\bar{A}}(x) = \begin{cases} 0, & if\ x < \frac{\delta^2}{\delta+1} \\ 1, & if\ x \geq \frac{\delta^2}{\delta+1} \end{cases} \tag{2.10}$$

■

Vicari and Kotz (2005) suggested that a more generalized form of $f_\gamma(x)$ can be obtained when using γ_1 and γ_2, instead of γ and $\frac{1}{\gamma}$. This will make the density functions more flexible in terms of their distributional shape through the inclusion of an additional shape parameter.

2.4 Beta-generated families of distributions

The beta-generated families of distributions were first introduced by the work of Eugene et al. (2002) when they presented the beta-normal distribution and its properties. This mechanism was intended to extend the flexibility of the distributional shape of both symmetric and skewed distributions. The skewness is attributed to a combination of shape parameters and not a single parameter as seen in the other methods already discussed.

Definition 2.4.0.1. *The CDF and PDF of a random variable from a beta-generated distribution takes the general forms*

$$G(x) = \int_0^{F(x)} r(t)dt$$

and

$$g(x) = \frac{1}{\mathrm{Beta}(\alpha, \beta)} f(x)F^{\alpha-1}(x)(1 - F(x))^{\beta-1},$$

respectively.

■

$G(x)$ and $g(x)$ are termed the beta-generated CDF and PDF, whereas $F(x)$ and $f(x)$ are the CDF and PDF of the parent distribution. The function $r(t)$ is the PDF of a Beta(α, β), otherwise known as the generator distribution.

There have been many beta-generated distributions proposed and studied since this method was introduced. These distributions arose from their ability to model skewed, heavy-tailed as well as bimodal data. These include but are not limited to

- The beta-normal (BN) distribution (Eugene et al. (2002), Famoye et al. (2004)) which has both unimodal and bimodal behaviour.

- The beta generalized logistic (BGL) distribution, introduced by Morais et al. (2013), compounds the beta distribution and the type IV generalized logistic distribution.

- The beta-Gumbel (BG) distribution (Nadarajah and Kotz (2004)) which was introduced with the hope that it would attract greater applicability in the engineering field where the Gumbel distribution is widely used. The hazard function of the BG distribution is an increasing function of the random variable.

- The beta-Weibull distribution (Famoye et al. (2005), Lee et al. (2007)) which has been applied to censored failure rate data. Famoye et al. (2005) showed that it is a unimodal distribution and presented some of its results on the non-central moments. Cordeiro and de Castro (2011) derive expansions for this distribution function and explicit closed form expressions for its moments.

- The beta-Pareto distribution (Akinsete et al. (2008)) whose hazard rate function presents unimodal and or decreasing behaviour.
- The beta-Cauchy distribution (Alshawarbeh et al. (2012) Alshawarbeh et al. (2013)) which has been found to model symmetric and skewed heavy-tailed distributions very well due to its flexibility.
- The beta-Student's t distribution by Jones and Faddy (2003) is proposed as a manageable skew t-distribution on the real line.

Distributional properties

This family of distributions displays various general properties, which are common to all of them.

Property 1: The values of the random variable X from a beta-generated family of distributions can be obtained from the inverse CDF, $F^{-1}(y)$, where $Y \sim \text{Beta}(\alpha, \beta)$.

Property 2: The PDF of $g(x)$ exhibits symmetry when $\alpha = \beta$, if $f(x)$ is symmetric. $g(x)$ is positively skewed when $\alpha > \beta$ and negatively skewed when $\alpha < \beta$.

Property 3: $g(x)$ is unimodal when $\alpha = \beta \geq 1$ and if $f(x)$ is also unimodal.

Property 4: The hazard function is defined as $h_g(x) = \frac{F^{\alpha-1}(x)(1-F(x))^{\beta-1}}{\mathbb{I}_{1-F(x)}(\beta,\alpha)}$ where $\mathbb{I}_x(\alpha, \beta)$ is the incomplete Beta function and $\mathbb{I}_x(\alpha, \beta) = B(\alpha, \beta) - \mathbb{I}_{1-x}(\beta, \alpha)$.

Property 5: The Shannon entropy of the density function $g(x)$ is obtained as $n_x = -E_X(\log(g(x))$, which for beta-generated distributions equates to $\log B(\alpha, \beta) + (\alpha - 1)\kappa(\alpha, \beta) + (\beta - 1)\kappa(\beta, \alpha) - E_Y(\log(F^{-1}(y)))$, where $\kappa(\alpha, \beta) = \psi(\alpha + \beta) - \psi(\alpha)$, $\psi(\cdot)$ is the digamma function and $Y \sim \text{Beta}(\alpha, \beta)$.

Generalizations of the beta-generated distributions

The initial results of Eugene et al. (2002) were then generalized to consider using any other generator other than the beta distribution. This generator would be defined on $[a, b]$ where $a, b \in \mathbb{R}$, and the domain then normalized to $(0, 1)$.

The Kumaraswamy distribution (Kw-distribution), proposed for doube-bounded random processes (Kumaraswamy (1980)), was used as the generator distribution in Eq.(2.11), giving rise to the Kw-generated (Kw-G) distribution.

Definition 2.4.0.2. *A random variable, X, from the Kw-G distribution is defined as having a probability density function, $g(x)$, that takes on the form*

$$g(x) = \alpha\beta f(x)F^{\alpha-1}(x)(1 - F^{\alpha}(x))^{\beta-1}, \quad \alpha, \beta > 0, \; x \in (0, 1), \tag{2.11}$$

where $f(x) = \alpha\beta x^{\alpha-1}(1 - x^{\alpha})^{\beta-1}$ and $F(x) = 1 - (1 - x^{\alpha})^{\beta}$ are the PDF and CDF of the Kw-distribution, respectively.

■

Cordeiro and de Castro (2011) and Nadarajah et al. (2012) studied distributions generated from this generalization. These included the Kw-normal, Kw-gamma, Kw-Weibull, Kw-Pareto and Kw-inverse gaussian distributions.

Another generalization known as the generalized beta-generated (GBG) distribution has also been studied by Alexander et al. (2012). This family of distributions uses the generalized beta Type I as the generator function.

Definition 2.4.0.3. *The PDF of a real-valued random variable, X, from the GBG distribution is defined as*

$$g(x) = \frac{\gamma}{B(\alpha, \beta)} f(x)F^{\alpha\gamma-1}(x)(1 - F^{\gamma}(x))^{\beta-1}, \quad \alpha, \beta, \gamma > 0, \; x \in (0, 1), \tag{2.12}$$

where $f(x) = \frac{\gamma}{B(\alpha,\beta)} x^{\alpha\gamma-1}(1-x^{\gamma})^{\beta-1}$ and $F(x)$ are the PDF and CDF of the beta Type I distribution, respectively.

■

The properties of the generated distributions were studied and documented by Alexander et al. (2012). When $\gamma = 1$, the GBG density function reduces to the beta generated density function, whereas the Kw-G density is obtained when $\alpha = 0$.

2.5 Asymmetric families of densities

Nassiri and Loris (2013) presented a general form of obtaining asymmetric density functions from distributions that were symmetric about zero. Two scale parameters, δ_1 $\delta_2 \in \mathbb{R}^+$, were introduced with the intention of retaining 0 as the mode to the symmetric density function as follows:

$$f_{\delta_1,\delta_2}(x) = \frac{2\delta_1\delta_2}{\delta_1+\delta_2}\begin{cases} f_X(\delta_1 x), & x \leq 0 \\ f_X(\delta_2 x), & x > 0. \end{cases} \tag{2.13}$$

With the introduction of a location parameter $\mu \in \mathbb{R}$, and scale parameter $\sigma > 0$, Eq.(2.13) can be redefined as

$$f_{\delta_1,\delta_2}(x;\mu,\sigma) = \frac{2\delta_1\delta_2}{\sigma(\delta_1+\delta_2)}\begin{cases} f_X\left(\delta_1\left(\frac{\mu-x}{\sigma}\right)\right), & x \leq \mu \\ f_X\left(\delta_2\left(\frac{x-\mu}{\sigma}\right)\right), & x > \mu. \end{cases} \tag{2.14}$$

Some of the properties from this family of distributions are:

Property 1: The PDF of $f_{\delta_1,\delta_2}(x;\mu,\sigma)$ exhibits symmetry when $\delta_1 = \delta_2$, is positively skewed when $\delta_1 > \delta_2$ and negatively skewed when $\delta_1 < \delta_2$.

Property 2: The special case $f_{\delta_1,\delta_2}(x;\mu,\sigma) = f(x;\mu,\sigma)$ when $\delta_1 = \delta_2$.

Property 3: The PDF in Eq.(2.6) is a special case when $\delta_1 = \frac{1}{\gamma}$, $\delta_2 = -\frac{1}{\delta_1}$ and $\kappa = 1$.

The CDF and quantile function are obtained as

$$F_{\delta_1,\delta_2}(x;\mu,\sigma) = \begin{cases} \frac{2\delta_2}{(\delta_1+\delta_2)}F_X\left(\delta_1\left(\frac{\mu-x}{\sigma}\right)\right), & x \leq \mu \\ \frac{\delta_2-\delta_1}{(\delta_1+\delta_2)} + \frac{2\delta_1}{(\delta_1+\delta_2)}F_X\left(\delta_2\left(\frac{x-\mu}{\sigma}\right)\right), & x > \mu \end{cases} \tag{2.15}$$

and

$$F^{-1}_{\delta_1,\delta_2}(p) = \begin{cases} \mu + \frac{\sigma}{\delta_1}F_X^{-1}\left(p\frac{(\delta_1+\delta_2)}{2\delta_2}\right), & p \leq \frac{\delta_2}{\delta_1+\delta_2} \\ \mu + \frac{\sigma}{\delta_2}F_X^{-1}\left(\frac{p(\delta_1+\delta_2)+(\delta_1-\delta_2)}{2\delta_1}\right), & p > \frac{\delta_2}{\delta_1+\delta_2}, \end{cases} \tag{2.16}$$

respectively.

Nassiri and Loris (2013) extensively used these results to derive the asymmetric Laplace and normal densities. Furthermore, they were able to prove that δ_1 and δ_2 control the allocation of mass to the density of the left and right tails, respectively. The r^{th} order finite moments exist for the asymmetric generalization if and only if the moments of the symmetric building block exist.

Arellano-Valle et al. (2005) introduced a general family of asymmetric distributions which includes the model by Fernández and Steel (1998) as a special case. The PDF and CDF are of the form

$$g(x|\beta) = \frac{2}{a(\beta)+b(\beta)}\left(f\left(\frac{x}{a(\beta)}\right)\mathbb{I}\{x \geq 0\} + f\left(\frac{x}{b(\beta)}\right)\mathbb{I}\{x < 0\}\right) \tag{2.17}$$

and

$$G(x|\beta) = \begin{cases} \frac{2b(\beta)}{a(\beta)+b(\beta)}F\left(\frac{x}{b(\beta)}\right), & \text{for } x < 0 \\ \frac{b(\beta)-a(\beta)}{a(\beta)+b(\beta)} + \frac{2a(\beta)}{a(\beta)+b(\beta)}F\left(\frac{x}{a(\beta)}\right), & \text{for } x \geq 0, \end{cases} \tag{2.18}$$

respectively, where β is an asymmetry parameter and $a(\beta)$ and $b(\beta)$ are known positive asymmetry functions. If $a(\beta) = b(\beta)$, then the whole class of symmetric densities is a special case of Eq.(2.17).

From the PDF and CDF defined above, the following properties hold for this family of asymmetric densities.

Property 1: The median of this family of distributions is:

$$G^{-1}\left(\frac{1}{2}\middle|\beta\right) = \begin{cases} b(\beta)F^{-1}\left(\frac{a(\beta)+b(\beta)}{4b\beta}\right) & a(\beta) < b(\beta) \\ a(\beta)F^{-1}\left(\frac{3a(\beta)-b(\beta)}{4a\beta}\right) & a(\beta) \geq b(\beta). \end{cases}$$

Property 2: When $x = 0$, then $G(0|\beta) = \frac{b(\beta)}{a(\beta)+b(\beta)}$.

Property 3: If X is a random variable from the skew family of distributions, then

$$\frac{P(X \geq 0|\beta)}{P(X < 0|\beta)} = \frac{a(\beta)}{b(\beta)}.$$

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2.6 Quantile modelling

2.6.1 Quantiles

Suppose X is a random variable defined by $F_X(x)$, such that $F_X(x)$ is right continuous. It follows that a quantile function, $Q_X(p)$, is defined such that

$$Q_X(p) = F_X^{-1}(p) = \inf\{x : F_X(x) \geq p\}, \tag{2.19}$$

where $0 < p < 1$ and $-\infty < x < \infty$. It can be observed that $F_X(Q_X(p)) = p$ is a composite function if X is a continuous random variable. Various quantile-based functions can be derived from the quantile function, and subsequently used to define a distribution.

- **Quantile density function**

The quantile-density function is also known as the sparsity function (Tukey (1965)). It is obtained as the first derivative of the quantile function with respect to p.
It is defined as

$$q_X(p) = \frac{d(Q_X(p))}{dp} = Q'_X(p), \quad 0 < p < 1. \tag{2.20}$$

- **Density quantile function**

As a result of the composite function $F_X(Q_X(p)) = p$ that arises from continuous random variables, the density quantile function is derived by taking derivatives on both sides of the function, such that

$$\begin{aligned} \frac{d(F_X(Q_X(p)))}{dp} &= \frac{dp}{dp} = 1 \\ \Rightarrow f_X(Q_X(p))q_X(p) &= 1 \\ \Rightarrow f_X(Q_X(p)) &= \frac{1}{q_X(p)} \end{aligned} \tag{2.21}$$

It can be denoted as $f_P(p)$ since it's derivative was obtained in terms of p and not X.

Quantile modelling rules

Gilchrist (2000) documented quantile modeling rules that can be used to construct quantile-based distributions. These rules include the addition, multiplication and reflection of quantile functions, while obtaining monotone non-decreasing quantile functions.

- **Addition rule**

Let X and Y be two random variables defined by their non-decreasing quantile functions $Q_X(p)$ and $Q_Y(p)$, respectively. Then $Q_Z(p) = Q_X(p) + Q_Y(p)$, is also non-decreasing and hence the quantile function of the random variable Z.

- **Reflection rule**

Consider $X \in (0, \infty)$ with quantile function $Q_X(p)$ and $Y \in (-\infty, 0)$ with quantile function $Q_Y(p) = -Q_X(1-p)$. Therefore, X and Y have distributions that are reflective of each other. Therefore, let $Q_Z(p)$ be the quantile function obtained from the sum of $Q_X(p)$ and $Q_Y(p)$ such that

$$\begin{aligned} Q_Z(p) &= Q_X(p) + Q_Y(p) \\ &= Q_X(p) - Q_X(1-p) \\ &= -Q_Y(1-p) - Q_X(1-p) \\ &= -Q_Z(1-p). \end{aligned} \tag{2.22}$$

This result shows that the sum of the quantile functions of two distributions which are reflective of each other gives rise to the quantile function of a symmetric distribution. Therefore, Z is symmetric.

2.6.2 Quantile-based distributions

Some well-known examples of quantile-based distributions include Tukey lambda distribution (Tukey (1960)), various types of lambda distributions (Ramberg and Schmeiser (1972); Freimer et al. (1988); van Staden (2014)) and the Davies distribution (Hankin and Lee (2006)).

van Staden and King (2015) made use of the quantile modeling techniques and developed the quantile-based skew logistic distribution (SLD_{QB}) by introducing a weighting parameter to the quantile function of the logistic distribution. They considered the quantile function of the logistic distribution, $Q_X(p) = \log\left[\frac{p}{1-p}\right]$, as the sum of the quantile functions of the standard reflected exponential and the standard exponential distributions, i.e.

$$Q_X(p) = \log\left[\frac{p}{1-p}\right] = \log(p) - \log(1-p), \quad 0 < p < 1,$$

afterwhich they then introduced an skewness parameter, $0 < \delta < 1$. The quantile function of the standard quantile-based skew logistic distribution was then defined as

$$Q_X(p) = (1-\delta)\log(p) - \delta\log(1-p), \quad 0 < p < 1. \tag{2.23}$$

With the inclusion of a location and scale parameter, $-\infty < \mu < \infty$ and $\sigma > 0$, respectively, Eq.(2.23) was generalized.

Definition 2.6.2.1. *A real-valued random variable X is said to follow the quantile-based skew logistic distribution, denoted $X \sim SLD_{QB}(\mu, \sigma, \delta)$, if its quantile function is given as*

$$Q_X(p) = \mu + \sigma\left((1-\delta)\log(p) - \delta\log(1-p)\right), \quad 0 < p < 1. \tag{2.24}$$

■

The distributional properties of the SLD_{QB} are:

Property 1: The distribution is symmetric for $\delta = \frac{1}{2}$, positively skewed for $\delta > \frac{1}{2}$ and negatively skewed for $\delta < \frac{1}{2}$.

Property 2: The SLD_{QB} possesses infinite support $(-\infty;\infty)$ when $(0 < \delta < 1)$, half-infinite support $(-\infty, \alpha]$ for $\delta = 0$, and half-infinite support $[\alpha, \infty)$ for $\delta = 1$.

Property 3: The reflected exponential, logistic and exponential distributions are special cases of the SLD_{QB} when $\delta = 0$, $\delta = \frac{1}{2}$ and $\delta = 1$, respectively.

Property 4: The range of values for the L-skewness, τ_3, is $[-\frac{1}{3}, \frac{1}{3}]$, while the L-kurtosis, τ_4, remains constant at $\frac{1}{6}$.

Balakrishnan and So (2015) presented a generalization of the SLD_{QB} model in Eq.(2.24) by van Staden and King (2015). This was done by introducing an additional shape parameter $\kappa > 0$, also termed the power parameter, to aid in increasing the flexibility of the distribution by providing a wider range of L-kurtosis values. This gave rise to the generalized quantile-based skew logistic distribution.

Definition 2.6.2.2. *A real-valued random variable X is said to have the generalized quantile-based skew logistic distribution, denoted $X \sim GSLD_{QB}(\mu, \sigma, \delta, \kappa)$, if its quantile function is given as*

$$Q_X(p) = \mu + \sigma\left((1-\delta)\log(p^{\kappa}) - \delta\log(1-p^{\kappa})\right), \quad 0 < p < 1, \tag{2.25}$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are the location and scale parameters, respectively, whereas $0 < \delta < 1$ and $\kappa > 0$ are the shape parameters.

■

The distributional properties of the SLD_{QB} are:

Property 1: The distribution is positively skewed as $\kappa \to 0$. As κ increases, the skewness of the $GSLD_{QB}$ decreases for $\delta < \frac{1}{2}$ and increases then decreases for $\frac{1}{2} < \delta < 1$.

Property 2: The $GSLD_{QB}$ is positively skewed for $\delta > \frac{1}{2}$ and negatively skewed for $\delta < \frac{1}{2}$.

Property 3: The $GSLD_{QB}$ will be reduced to the $GSLD_{QB}$ in Eq.(2.24) when $\kappa = 1$.

Property 4: The range of values for the L-skewness, τ_3, is $[-1, 1]$, which is the widest possible range.

Property 5: The range of values for the L-kurtosis, τ_4, is $[0.1504, 1]$.

2.6.3 Combining half distributions

Balakrishnan et al. (2017) introduced an alternative model to the skew logistic distribution proposed by van Staden and King (2015). They proposed taking the PDF of the half logistic distribution to the right of its location parameter, μ, and joining it to the PDF of the half logistic distribution to the left of μ. A single asymmetry parameter, $\alpha > 0$, was introduced to obtain the skew logistic distribution.

Definition 2.6.3.1. *A real-valued random variable from the skew logistic distribution, denoted as $X \sim SL(\mu, \sigma, \alpha)$, is defined by its PDF, CDF and quantile function as*

$$f_X(x) = \begin{cases} \frac{e^{-\frac{x-\mu}{\sigma}}}{\sigma\left(1+e^{\frac{-x-\mu}{\sigma}}\right)^2}, & x \geq \mu \\ \frac{e^{\frac{x-\mu}{\alpha\sigma}}}{\alpha\sigma\left(1+e^{\frac{x-\mu}{\alpha\sigma}}\right)^2}, & x \leq \mu, \end{cases}$$

$$F_X(x) = \begin{cases} \frac{1}{1+e^{\frac{-x-\mu}{\sigma}}}, & x \geq \mu \\ \frac{1}{1+e^{\frac{x-\mu}{\alpha\sigma}}}, & x \leq \mu, \end{cases}$$

$$Q_X(p) = \begin{cases} \mu - \sigma(\log(1-p) - \log(p)), & p \geq \frac{1}{2} \\ \mu - \alpha\sigma(\log(1-p) - \log(p)), & p < \frac{1}{2}, \end{cases}$$

respectively, while the PDF and CDF of the standard half logistic random variable, Y, is given by Balakrishnan (1985) as

$$f_Y(x) = \frac{2e^{-x}}{(1+e^{-x})^2}, \quad x > 0 \tag{2.26}$$

and

$$F_Y(x) = \frac{1-e^{-x}}{1+e^{-x}}, \quad x > 0,$$

respectively.

■

Balakrishnan et al. (2017) indicated existing relationships between X and Y, through their PDFs and CDFs, as follows:

Property 1: $F_X(x) = 1 - F_X(x/\alpha)$

Property 2: $f_X(-x) = \frac{1}{\alpha} f_X(x/\alpha)$

Property 3: $F_X(x) = \frac{1+F_Y(x)}{2}$

Property 4: $f_X(x) = \frac{f_Y(x)}{2}$

The single and product moments of the order statistics of the skew logistic distribution were then obtained for any sample size, and were used to compute the means, variances and covariances of all the order statistics.

QUANTILE SPLICING

3.1 Introduction

This chapter presents the quantile splicing methodology. This method involves the joining of the quantile functions of two half distributions at the median, resulting in two-piece families of distributions. Balakrishnan et al. (2017) introduced a skew logistic distribution by taking the half logistic distribution to the right of its location parameter and joining it to the half logistic distribution to the left of the location parameter which has an inclusion of a single asymmetry parameter, $\alpha > 0$.

The methodology made use of the CDF and PDF of the logistic distribution, as the kernel to obtain the skew logistic distribution. They then made use of order statistics from the half logistic distribution and their moments to obtain the single and product moments of the proposed skew logistic distribution. Furthermore, those results were used to obtain the means, variances and covariances of order statistics for any sample size.

This chapter proposes a technique that generalizes the results of Balakrishnan et al. (2017). The quantile functions of half distributions of symmetric parent distributions are used as kernels in place of the CDFs and PDFs of parent distributions to be skewed. This mechanism results in two-piece families of distributions that can be defined through closed-form expressions for the CDF and PDF if they exist, or through the quantile function in the case of quantile-based distributions. In essence, the methodology allows for either the CDF, PDF or quantile function of a half distribution from a symmetric distribution to be used as a kernel to obtain asymmetric distributions.

Moreover, the results from the proposition have led to the derivation of a general form for the r^{th} order L-moments of two-piece distributions. These results will make use of the r^{th} order L-moments from both the half distribution as well as the parent distribution. The results will enable the avoidance of tedious computations in obtaining single and product moments for the distributions.

The skewing mechanism used to obtain the two-piece family of distributions will be presented and discussed in Section 3.2. The process will begin with the derivation of the quantile function of the half distribution for any symmetric univariate distribution. This result is then used as the kernel for the mechanism that generates the two-piece families of distributions.

The general formulae for the r^{th} order L-moments of the two-piece distributions are derived in Section 3.3. This formulae is expressed in terms of the L-moments of the half distributions as well as the parent distributions. It will be used to obtain the first four L-moments of a two-piece distribution as well as the L-skewness and L-kurtosis moment ratios. In addition, the r^{th} order shifted scaled polynomials needed to derive the L-moments of the half distributions are also derived.

General results for the quantile measures of distributional form for the location, spread and shape of two-piece distributions are derived in Section 3.4. Conventional measures such as the median for location, the spread function for spread and the γ-functional for shape will be used to summarise the properties of these distributions. Skewness-invariant measures of kurtosis will also be derived.

In Section 3.5, the method for L-moments will be used to estimate the parameters for the two-piece distributions. This is because the results in Section 3.3 enable the derivation of L-moments that are simpler in form in comparison to the central moments.

The model validation procedures of these two-piece families of distributions when fitted to data sets will be studied in Section 3.6. In Section 3.7, the investigation of the tail behaviour of these families of distributions is presented. Finally, the conclusion of this chapter is given in Section 3.8.

3.2 Method of Quantile Splicing

Suppose X is a real-valued random variable from any univariate symmetric distribution, with a defined CDF, PDF or quantile function. X is standardized such that the first two L-moments, that is L_1 (L-location) and L_2 (L-scale) are equal to 0 and 1, respectively. If L_1 and L_2 depend on the location and scale parameters, then the parameters are subsequently redefined to account for the standardizing requirements imposed. This will ensure that any symmetric distribution can be generalized into a two-piece distribution, despite being centered at a location point other than 0. In the case of bounded distributions, the boundaries of the distribution are reparametrized such that the condition of $L_1 = 0$ and $L_2 = 1$ are also achieved.

Theorem 3.2.1. *Suppose Y is a folded random variable such that $Y = |X|$, where $0 < y < \infty$, and $Z = -Y$. The quantile functions of Y and Z are obtained as*

$$Q_Y(p) = Q_X\left(\frac{1+p}{2}\right), \quad 0 < p < 1$$

and

$$Q_Z(p) = Q_X\left(\frac{p}{2}\right), \quad 0 < p < 1$$

respectively.

Proof. The CDF of Y follows from the results below as

$$\begin{aligned} G_Y(y) &= P(Y \leq y) \\ &= P(|X| \leq y) \\ &= P(-y \leq X \leq y) \\ &= F_X(y) - F_X(-y) \\ &= F_X(y) - (1 - F_X(y)) \\ &= 2F_X(y) - 1. \end{aligned}$$

This implies that $F_X(y) = \frac{1+p}{2}$, where p is the depth in G, yielding the corresponding quantile function of Y as

$$Q_Y(p) = F_X^{-1}\left(\frac{1+p}{2}\right) = Q_X\left(\frac{1+p}{2}\right), \tag{3.1}$$

where $0 < p < 1$. Through the use of the reflection rules of quantile functions documented by Gilchrist (2000), the quantile function of Z is $Q_Z(p) = -Q_Y(1-p)$.

Therefore,

$$\begin{aligned}
Q_Z(p) &= -Q_Y(1-p) \\
&= -Q_X\left(\frac{1+(1-p)}{2}\right) \\
&= -Q_X\left(\frac{2-p}{2}\right) \\
&= Q_X\left(1-\frac{2-p}{2}\right) \\
&= Q_X\left(\frac{p}{2}\right),
\end{aligned} \tag{3.2}$$

where $0 < p < 1$. ■

Remark. *Eq.(3.2) and Eq.(3.1) are the quantile functions of two half distributions, from a standard symmetric parent distribution, whose domains are both* $0 < p < 1$.

Since Theorem 3.2.1 makes use of the quantile functions of half distributions as the kernels, the method can be used for any symmetric univariate distribution in the event that it has an unknown CDF. In order for quantile splicing to be established, the domains of the quantile functions have to be obtained for both the left side of the location parameter, $-\infty < \mu < \infty$, as well as the right side.

Consider Eq.(3.2) whose range of values is $0 < p < 1$. Let $s = \frac{p}{2}$, hence Eq.(3.2) yields $Q_X(s)$ where $0 < s \leq \frac{1}{2}$. In the same way, by replacing $\frac{1+p}{2}$ with s, the range of values for the quantile function in Eq.(3.1) is $\frac{1}{2} < s < 1$.

Definition 3.2.0.1. *A real-valued random variable T, denoted as* $T \sim TP(\mu, \sigma, \alpha)$*, follows a two-piece distribution if its quantile function is defined as*

$$Q_T(s) = \begin{cases} \mu + \alpha\sigma Q_{X:0}(s), & s \leq \frac{1}{2}, \\ \mu + \sigma Q_{X:0}(s), & s > \frac{1}{2}, \end{cases} \tag{3.3}$$

where $Q_{X:0}$ *is the standard quantile function of* X, $-\infty < \mu < \infty$, $\sigma > 0$ *and* $\alpha > 0$ *are the location, spread and asymmetry parameters, respectively.*

■

3.3 r^{th} Order L-moments

L-moments, as defined by Hosking (1990), are expectations of linear combinations of order statistics. They can be defined for any distribution that has an existing mean, even when all its other conventional moments do not exist. Moreover, they summarize a wider range of univariate distributions.

L-moments form the basis for general theory with regards to existing procedures, such as the use of order statistics and the Gini's mean difference statistic (Gini (1912)). The use of these results range from uniquely defining and summarizing the properties of a probability distribution in terms of location, spread and shape, to creating estimation procedures for parameters, as well as hypothesis tests.

Since they are linear functions of data, L-moments tend to be more robust to the effects of sampling variability in comparison to the conventional moments. Moreover, when estimated from a sample, they tend to exhibit more robustness to outliers. L-moments sometimes yield more efficient parameter estimates that are less subject to bias through estimation procedures such as the method of L-moments, as compared to the conventional maximum likelihood or method of moments estimates.

The general formulae for the r^{th} order L-moment of a two-piece distribution will consist of the L-moments of both the parent distribution as well as the half distribution. As a result of the domain intervals for the quantile functions in Eq.(3.3), the conventional L-moments developed by Hosking (1990) will have to be adapted to accomodate this new change. This will include deriving r^{th} order shifted scaled Legendre polynomials that will be used to obtain the L-moments of half distributions, which in turn will be used to develop the L-moments of the two-piece distribution.

3.3.1 Definition of L-moments

Definition 3.3.1.1. *Suppose that X is a real-valued random variable with CDF, $F_X(x)$, and quantile function, $Q_X(p)$, where $0 < p < 1$. Let $X_{1:n} \leq X_{2:n} \leq X_{3:n} \cdots \leq X_{n:n}$ be given as the order statistics of a random sample of size n. The L-moments can be defined in terms of the order statistics as*

$$L_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r-k:r}), \qquad r \in \mathbb{Z}^+. \tag{3.4}$$

■

Hosking (1990) defined the expectation of order statistics in terms of a distribution's quantile function as

$$E(X_{j:r}) = \frac{r!}{(j-1)!(r-j)!}\int_0^1 Q_X(p)p^{j-1}(1-p)^{r-j}dp, \qquad 0<p<1, \tag{3.5}$$

by using the definition of an expectation of an order statistic by David (1981).

The first four L-moments are obtained by combining the results obtained from substituting the respective value of $r \geq 0$ into Eqs.(3.4 and 3.5) to obtain

$$\begin{aligned} L_1 = E(X_{1:1}) &= \int_0^1 Q(p)dp \\ L_2 = \frac{1}{2}E(X_{2:2} - X_{1:2}) &= \int_0^1 Q(p)(2p-1)dp \\ L_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3}) &= \int_0^1 Q(p)(6p^2 - 6p + 1)dp \\ L_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}) &= \int_0^1 Q(p)(20p^3 - 30p^2 + 12p - 1)dp. \end{aligned} \tag{3.6}$$

L_1 is referred to as the L-location and it is equivalent to the mean. L_2 is referred to as the L-scale since it's a measure of spread. It is also the expectation of the Gini's mean difference statistic by Gini (1912). It is related to the "*totaltimetotest*" statistic of Gail and Gastwirth (1978), which they used to test for exponentiality in distributions against other alternatives such as the sample Lorenz statistic.

Gilchrist (2000) showed that L-moments of an order $r > 2$ have an increased susceptibility to large variability. For this reason, they are transformed into L-moment ratios so that they are independent of the measurement units of the random variable.

The L-moment ratios are defined as

$$\tau_r = \frac{L_r}{L_2}, \qquad r \geq 3. \tag{3.7}$$

When $r = 3$, $\tau_3 = \frac{L_3}{L_2}$ is obtained as a measure of skewness, referred to as the L-skewness ratio. Similarly, when $r = 4$, $\tau_4 = \frac{L_4}{L_2}$ is defined as the L-kurtosis ratio, which is a measure of kurtosis. Hosking (1989) and Jones (2004) showed these two L-moment ratios are bounded by the constraints $-1 < \tau_3 < 1$ and $\frac{1}{4}(5\tau_3^2 - 1) < \tau_4 < 1$, respectively.

By making use of (Sillitto (1951),Eq.(3.9)), τ_3 can be rewritten in terms of order statistics as

$$\tau_3 = \frac{E(X_{3:3} - 2X_{2:3} + X_{1:3})}{E(X_{3:3} - X_{1:3})}. \tag{3.8}$$

The L-skewness ratio has an advantage over the conventional skewness moment-ratio due to its extreme sensitivity in the extreme tail weight of a distribution. Due to its boundaries of $(-1, 1)$, it also yields smaller values of skewness as compared to the conventional moments which can take on values that tend to infinity. The closer τ_3 is to 1, the more positively skewed the distribution is. Similarly, the distribution will exhibit negative skewness when $-1 < \tau_3 < 0$, with extreme cases being noted when it is closest to -1. Symmetric distributions have a value of $\tau_3 = 0$.

3.3.2 r^{th} Order Shifted Legendre Polynomials

Legendre polynomials were first introduced by Legendre (1786) as a sequence of orthogonal polynomials that are obtained as solutions to the Legendre differential equations. Rodrigues' formula (Rodrigues (1816)) was used to redefine the Legendre polynomials such that

$$P_{r-1}(p) = \frac{1}{2^{r-1}(r-1)!} \frac{d^{r-1}}{dp^{r-1}} (p^2 - 1)^{r-1}, \quad r \in \mathbb{Z}^+. \tag{3.9}$$

The r^{th} order shifted Legendre polynomials are obtained by adding a scale parameter value of 2 and a location-shifting value of -1 in order to obtain their general form from the ordinary Legendre polynomials. As a result, the relationship between the shifted Legendre polynomials and the Legendre polynomials is

$$P^*_{r-1}(p) = P_{r-1}(2p - 1). \tag{3.10}$$

Lemma 3.3.1. *The r^{th} order shifted Legendre polynomials, for $r \in \mathbb{Z}^+$, are defined as*

$$P^*_{r-1}(p) = \frac{1}{(r-1)!} \frac{d^n}{dp^{r-1}} (p^2 - p)^{r-1}, \qquad 0 < p < 1. \tag{3.11}$$

The first 4 shifted Legendre polynomials are obtained as

$$\begin{aligned} P^*_0(p) &= 1, \\ P^*_1(p) &= 2p - 1, \\ P^*_2(p) &= 6p^2 - 6p + 1 \\ P^*_3(p) &= 20p^3 - 30p^2 + 12p - 1, \end{aligned} \tag{3.12}$$

respectively.

Proof. Through the use of Eq.(3.9) and $P^*_{r-1}(p) = P_{r-1}(2p-1)$, the r^{th} order shifted Legendre polynomials are derived as

$$\begin{aligned}
P^*_{r-1}(p) &= P_{r-1}(2p-1) \\
&= \frac{1}{2^{r-1}(r-1)!}\frac{d^{r-1}}{d(2p-1)^{r-1}}((2p-1)^2-1)^{r-1} \\
&= \frac{1}{2^{r-1}(r-1)!}\frac{d^{r-1}}{2^{r-1}dp^{r-1}}(4p^2-4p+1-1)^{r-1} \\
&= \frac{1}{2^{r-1}(r-1)!}\frac{d^{r-1}}{2^{r-1}dp^{r-1}}(4p^2-4p)^{r-1} \\
&= \frac{1}{2^{r-1}(r-1)!}\frac{d^{r-1}}{2^{r-1}dp^{r-1}}(4(p^2-p))^{r-1} \\
&= \frac{1}{2^{r-1}(r-1)!}\frac{d^{r-1}}{2^{r-1}dp^{r-1}}4^{r-1}(p^2-p)^{r-1} \\
&= \frac{1}{(r-1)!}\frac{d^{r-1}}{dp^{r-1}}(p^2-p)^{r-1}.
\end{aligned}$$

In order to generate the polynomials in Eq.(3.12), the positive integer values for r are sequentially substituted into Eq.(3.11). These shifted Legendre polynomials in Eq.(3.12) were used to derive the r^{th} order L-moments as shown by Hosking (1990). ■

Definition 3.3.2.1. *The r^{th} order L-moments can be defined in terms of a quantile function as*

$$L_r = \int_0^1 Q_X(p)P^*_{r-1}(p)dp, \qquad r \in \mathbb{Z}^+, \tag{3.13}$$

where

$$P^*_{r-1}(p) = \sum_{k=0}^{r-1}(-1)^{r-k-1}\binom{r-1}{k}\binom{r+k-1}{k}p^k \tag{3.14}$$

is the r^{th} order shifted Legendre polynomial. The first four L-moments are subsequently obtained as

$$\begin{aligned}
L_1 &= \int_0^1 Q_X(p)dp, \\
L_2 &= \int_0^1 Q_X(p)(2p-1)dp, \\
L_3 &= \int_0^1 Q_X(p)(6p^2-6p+1)dp,
\end{aligned}$$

and

$$L_4 = \int_0^1 Q_X(p)(20p^3-30p^2+12p-1)dp, \tag{3.15}$$

respectively. ■

Applications of the r^{th} order shifted Legendre polynomials

Some applications of the r^{th} order shifted Legendre polynomials are presented.

Application 1: Multipole function expansions:

Multipole functions are mathematical expansions of functions that are defined in terms of angles. Legendre polynomials can be used to expand functions that are of the form:

$$\frac{1}{\sqrt{1+\eta^2-2\eta p}} = \sum_{k=0}^{\infty} \eta^k p_k(p).$$

Application 2: Trigonometry:

The Chebychev polynomials, $T_r(\cos(\theta)) = \cos(r\theta)$ can also be expanded by the Legendre polynomials, $P_r(\cos(\theta))$, for $r \in Z_{\geq}0$.

3.3.3 r^{th} Order Shifted Scaled Polynomials

The integrals for the r^{th} order L-moments, as can be seen from Eq.(3.15), yield the respective results for a random variable when the boundaries are from 0 to 1. In the event that an integral with the same structure as that of Eq.(3.13) does not have the required boundaries to be termed an L-moment, the following results can be utilized to account for the difference in boundaries.

Lemma 3.3.2. *Let X be a real valued random variable with a quantile function defined as* $Q_X(p)$, *where* $0 < p < 1$ *and* $0 < k < 1$. *It follows that:*

$$\int_0^k Q_X(p)P^*_{r-1}(p)dp = k\int_0^1 Q_X\left(ku\right)P^*_{r-1}\left(2ku-1\right)du \tag{3.16}$$

Proof. Since the integrals of L-moments are bounde on $(0,1)$, consider the transformation $u = \frac{p}{k}$, such that $du = \frac{dp}{k}$. Therefore it follows that

$$\begin{aligned}\int_0^k Q_X(p)P^*_{r-1}(p)\ dp &= k\int_0^1 Q_X\left(ku\right)P^*_{r-1}\left(2ku-1\right)du\\ &= k\int_0^1 Q_X\left(ku\right)P^*_{r-1}\left(2ku-1\right)du.\end{aligned} \tag{3.17}$$

■

Remark. *The results in Lemma (3.3.2) imply that the transformation used also affects the* r^{th} *order shifted Legendre polynomials presented in Eq.(3.15). A scaling factor is introduced that decreases the domain over which the integral is to be obtained, ultimately affecting the shifted Legendre polynomials.*

Theorem 3.3.3. *The r^{th} order shifted Legendre polynomials, with a scaling factor $0 < k < 1$, is denoted as*

$$P_{r-1}^{*}(ku) = P_{r-1}(2ku-1) = \frac{1}{(r-1)!}\frac{d^{r-1}}{du^{r-1}}\left(ku^2-u\right)^{r-1} \tag{3.18}$$

where $r = 1, 2, 3, ...,$ with the first 4 shifted scaled polynomials obtained as

$$\begin{aligned}
&P_0^{*}(ku) = 1,\\
&P_1^{*}(ku) = 2(ku) - 1,\\
&P_2^{*}(ku) = 6(ku)^2 - 6(ku) + 1,\\
&\qquad \textit{and}\\
&P_3^{*}(ku) = 20(ku)^3 - 30(ku)^2 + 12(ku) - 1,
\end{aligned} \tag{3.19}$$

respectively.

Proof. Eq.(3.18) follows from Eq.(3.11) by replacing ku in place of p to obtain

$$\begin{aligned}
P_{r-1}^{*}(ku) &= \frac{1}{(r-1)!}\frac{d^{r-1}}{d(ku)^{r-1}}\left((ku)^2-(ku)\right)^{r-1}\\
&= \frac{1}{(r-1)!}\frac{d^{r-1}}{(k)^{r-1}du^{r-1}}\left(k\left(ku^2-u\right)\right)^{r-1}\\
&= \frac{1}{(r-1)!}\frac{d^{r-1}}{(k)^{r-1}du^{r-1}}k^{r-1}\left(ku^2-u\right)^{r-1}\\
&= \frac{1}{(r-1)!}\frac{d^{r-1}}{du^{r-1}}\left(ku^2-u\right)^{r-1}
\end{aligned} \tag{3.20}$$

The shifted scaled polynomials are obtained when values of $r = 1, 2, 3$ and 4 are substituted into Eq.(3.20) and the r^{th} derivative obtained accordingly. The first polynomial, when $r = 1$ is

$$\begin{aligned}
P_0^{*}(ku) &= \frac{1}{(1-1)!}\frac{d^{1-1}}{d(ku)^{1-1}}\left((ku)^2-(ku)\right)^{1-1}\\
&= \frac{1}{0!}\frac{d^0}{d(ku)^0}\left((ku)^2-(ku)\right)^{0}\\
&= 1.
\end{aligned}$$

The second polynomial , when $r = 2$, is

$$\begin{aligned}
P_1^{*}(ku) &= \frac{1}{(2-1)!}\frac{d^{2-1}}{d(ku)^{2-1}}\left((ku)^2-(ku)\right)^{2-1}\\
&= \frac{1}{1!}\frac{d^1}{d(ku)^1}\left((ku)^2-(ku)\right)^{1}\\
&= \frac{d^1}{d(ku)^1}\left((ku)^2-(ku)\right)^{1}\\
&= \frac{d}{k\ du}\left((ku)^2-(ku)\right)\\
&= \frac{1}{k}\left(2k^2u-k\right)\\
&= 2(ku) - 1.
\end{aligned}$$

The third polynomial is obtained when $r = 3$ as

$$\begin{aligned}
P_2^*(ku) &= \frac{1}{(3-1)!}\frac{d^{3-1}}{d(ku)^{3-1}}\Big((ku)^2 - (ku)\Big)^{3-1} \\
&= \frac{1}{2!}\frac{d^2}{d(ku)^2}\Big((ku)^2 - (ku)\Big)^2 \\
&= \frac{1}{2k^2}\frac{d^2}{du^2}\Big((ku)^2 - (ku)\Big)^2 \\
&= \frac{1}{2k^2}\frac{d}{du}\left(2\left(k^2u^2 - ku\right)\left(2k^2u - k\right)\right) \\
&= \frac{1}{k^2}\Big(\left(2k^2u - k\right)^2 + \left(2k^2\right)\left(k^2u^2 - ku\right)\Big) \\
&= \frac{1}{k^2}\left(4k^4u^2 - 4k^3u + k^2 + 2k^4u^2 - 2k^3u\right) \\
&= 6\,(ku)^2 - 6\,(ku) + 1.
\end{aligned}$$

Finally, to obtain the fourth polynomial, $r = 4$ is substituted into Eq.(3.20) to obtain

$$\begin{aligned}
P_3^*(ku) &= \frac{1}{(4-1)!}\frac{d^{4-1}}{d(ku)^{4-1}}\Big((ku)^2 - (ku)\Big)^{4-1} \\
&= \frac{1}{3!}\frac{d^3}{d(ku)^3}\Big((ku)^2 - (ku)\Big)^3 \\
&= \frac{1}{6k^3}\frac{d^3}{du^3}k^3\left(ku^2 - u\right)^3 \\
&= \frac{1}{6}\frac{d^2}{du^2}\Big(3\left(ku^2 - u\right)^2(2ku - 1)\Big) \\
&= \frac{1}{2}\frac{d}{du}\Big(2\left(ku^2 - u\right)(2ku - 1)^2 + (2k)\left(ku^2 - u\right)^2\Big) \\
&= (2ku - 1)^3 + 2\,(2k)\left(ku^2 - u\right)(2ku - 1) + (2k)\left(ku^2 - u\right)(2ku - 1) \\
&= (2ku - 1)\left(10k^2u^2 - 10ku + 1\right) \\
&= 20\,(ku)^3 - 30\,(ku)^2 + 12\,(ku) - 1.
\end{aligned}$$

$\blacksquare$

3.3.4 r^{th} order L-moments for Two-piece Distributions

Theorem 3.3.4. *Let T be a random variable from a two-piece distribution, denoted as $T \sim TP(\mu, \sigma, \alpha)$, and defined by the quantile function, $Q_T(s)$. The general expression for the r^{th} order L-moment is given as*

$$L_{T:r} = \mu^* + \sigma\Big(L_{X:r} - 0.5(1-\alpha) \times \sum_{j=1}^{r} c_{j-1}^{(r-1)}\frac{\mu_{j:j}}{j}\Big), \tag{3.21}$$

where μ^* is a location parameter that takes on the value of $-\infty < \mu < \infty$ if $r = 1$, and zero for all values of $r > 1$. $c_{j-1}^{(r-1)}$ is the $(j-1)^{th}$ coefficient of the r^{th} order shifted Legendre polynomial, and $\mu_{j:j}$ is the expectation of the j^{th} order statistic from a sample of size n from a half distribution.

Proof. The expressions for the r^{th} order L-moments of T can be obtained by using Hosking (1990)'s definition, where the product of the quantile function and shifted Legendre polynomials is integrated over its piecewise domain. Therefore,

$$\begin{aligned}
L_{T:r} &= \int_0^{\frac{1}{2}} (\mu + \alpha\sigma Q_{X:0}(s))P_{r-1}^*(s)ds + \int_{\frac{1}{2}}^1 (\mu + \sigma Q_{X:0}(s))P_{r-1}^*(s)ds \\
&= \int_0^{\frac{1}{2}} (\mu + \alpha\sigma Q_{X:0}(s))P_{r-1}^*(s)ds + \left\{\int_0^1 (\mu + \sigma Q_{X:0}(s))P_{r-1}^*(s)ds - \int_0^{\frac{1}{2}} (\mu + \sigma Q_{X:0}(s))P_{r-1}^*(s)ds\right\} \\
&= \int_0^1 (\mu + \sigma Q_{X:0}(s))P_{r-1}^*(s)ds - \sigma(1-\alpha)\int_0^{\frac{1}{2}} Q_{X:0}(s)P_{r-1}^*(s)ds \\
&= \int_0^1 \mu P_{r-1}^*(s)ds + \sigma\left(\int_0^1 Q_{X:0}(s)P_{r-1}^*(s)ds - (1-\alpha)\int_0^{\frac{1}{2}} Q_{X:0}(s)P_{r-1}^*(s)ds\right) \\
&= \mu^* + \sigma\left(L_{X:r} - 0.5(1-\alpha)\int_0^1 Q_{Z:0}\left(p\right)P_{r-1}^*\left(p\right)dp\right),
\end{aligned} \tag{3.22}$$

such that $P_{r-1}^*(p)$ is a polynomial of degree $(r-1)$ expressed as $P_{r-1}^*(p) = c_0^{(r-1)} + c_1^{(r-1)}p + c_2^{(r-1)}p^2 + \ldots + c_{r-1}^{(r-1)}p^{r-1}$. $Q_{Z:0}(p)$ is the standard quantile function of the half parent distribution in Eq.(3.2) and $L_{X:r}$ is the r^{th} order L-moment of the parent distribution, X. The final step is attained by using the results in Lemma 3.3.2, where the scaling factor for the integral $k = \frac{1}{2}$ and Eq.(3.2) where $s = \frac{p}{2}$.

In order to further simplify the results in Eq.(3.22), let I_{r-1} be defined as

$$I_{r-1} = \int_0^1 Q_Z\left(p\right)P_{r-1}^*\left(p\right)dp = \int_0^1 Q_Z\left(p\right)\left(c_0^{(r-1)} + c_1^{(r-1)}p + c_2^{(r-1)}p^2 + \ldots + c_{r-1}^{(r-1)}p^{r-1}\right)dp. \tag{3.23}$$

The polynomial coefficients of $P_{r-1}^*(p)$ are summarised in Table 3.1, for $j = 1, 2, \ldots, r-1$.

	0	1	2	3	$\cdots$	j
0	$c_0^{(0)}$					
1	$c_0^{(1)}$	$c_1^{(1)}$				
2	$c_0^{(2)}$	$c_1^{(2)}$	$c_2^{(2)}$			
3	$c_0^{(3)}$	$c_1^{(3)}$	$c_2^{(3)}$	$c_3^{(3)}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
$r-1$	$c_0^{(r-1)}$	$c_1^{(r-1)}$	$c_2^{(r-1)}$	$c_3^{(r-1)}$		$c_j^{(r-1)}$

Table 3.1: The coefficients of a polynomial of degree $(r-1)$.

Consider $\int_0^1 Q_Z(p)p^r dp$ where $r \in \mathbb{Z}^+$. By setting $p = F_Z(z)$, $Q_Z(p) = z$ and subsequently $dp = f_Z(z)dz$. Therefore,

$$\begin{aligned}
\int_0^1 Q_Z(p)p^r dp &= \int_0^\infty z(F_Z(z))^r f_Z(z)dz \\
&= (r+1)\int_0^\infty \frac{z(F_Z(z))^r f_Z(z)}{r+1}dz \\
&= \frac{E(Z_{r+1:r+1})}{r+1} \\
&= \frac{1}{r+1}\mu_{r+1:r+1},
\end{aligned} \tag{3.24}$$

where $\mu_{r+1:r+1}$ is the expected value of the $(r+1)^{th}$ largest observation in a sample of size $(r+1)$ from the half distribution, Z, of the parent distribution, X.

Hence by making use of the result in Eq.(3.24) and Eq.(3.26), Eq.(3.22) can be rewritten as

$$\begin{aligned}
L_{T:r} &= \mu^* + \sigma\left(L_{X:r} - 0.5(1-\alpha)\int_0^1 Q_Z(p)P^*_{r-1}(p)dp\right) \\
&= \mu^* + \sigma\left(L_{X:r} - 0.5(1-\alpha)\int_0^1 \left(Q_Z(p)c_0^{(r-1)} + Q_Z(p)c_1^{(r-1)}p + ... + Q_Z(p)c_{r-1}^{(r-1)}p^{r-1}\right)dp\right) \\
&= \mu^* + \sigma\Bigg(L_{X:r} - 0.5(1-\alpha) \\
&\quad \times \left\{\int_0^1 Q_Z(p)c_0^{(r-1)}dp + \int_0^1 Q_Z(p)c_1^{(r-1)}p\ dp + \int_0^1 Q_Z(p)c_2^{(r-1)}p^2dp + ... + \int_0^1 Q_Z(p)c_{r-1}^{(r-1)}p^{r-1}dp\right\}\Bigg) \\
&= \mu^* + \sigma\left(L_{X:r} - 0.5(1-\alpha)\left\{c_0^{(r-1)}E(Z_{1:1}) + c_1^{(r-1)}\frac{E(Z_{2:2})}{2} + c_2^{(r-1)}\frac{E(Z_{3:3})}{3} + ... + c_{r-1}^{(r-1)}\frac{E(Z_{r:r})}{r}\right\}\right) \\
&= \mu^* + \sigma\left(L_{X:r} - 0.5(1-\alpha) \times \sum_{j=1}^{r} c_{j-1}^{(r-1)}\frac{E(Z_{j:j})}{j}\right) \\
&= \mu^* + \sigma\left(L_{X:r} - 0.5(1-\alpha) \times \sum_{j=1}^{r} c_{j-1}^{(r-1)}\frac{\mu_{j:j}}{j}\right)
\end{aligned} \tag{3.25}$$

■

Lemma 3.3.5. *Let $Z_{r:n}$ denote the r^{th} order statistic in a random sample of size n from a half distribution. Then $E(Z_{r:n})$ can be expressed as*

$$E(Z_{r:n}) = \frac{1}{B(r, n-r+1)}\int_0^1 Q_X\left(\frac{p}{2}\right)\left(\frac{p}{2}\right)^{r-1}\left(1-\frac{p}{2}\right)^{n-r}dp \tag{3.26}$$

where $Q_X\left(\frac{p}{2}\right)$ is the quantile function of Z. This result is adopted from Eq.(3.5), which was first defined by Hosking (1990).

■

Theorem 3.3.6. *The expressions for the first 4 L-moments for T, a real-valued random variable from the two-piece family of distributions, are*

$$L_{T:1} = \mu + \sigma\left(L_{X:1} - 0.5(1-\alpha)c_0^{(0)} L_{Z:1}\right)$$
$$L_{T:2} = \sigma\left(L_{X:2} - 0.5(1-\alpha) \times \left(c_0^{(1)} L_{Z:1} + \frac{c_1^{(1)}}{2}(L_{Z:1} + L_{Z:2})\right)\right)$$
$$L_{T:3} = \sigma\left(L_{X:3} - 0.5(1-\alpha) \times \left(L_{Z:1}\left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\right) + L_{Z:2}\left(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2}\right) + L_{Z:3}\frac{c_2^{(2)}}{6}\right)\right)$$

and

$$L_{T:4} = \sigma\left(L_{X:4} - 0.5(1-\alpha) \times \left(L_{Z:1}\left(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\right) + \frac{L_{Z:2}}{2}(c_1^{(3)} + c_2^{(3)} + \frac{9}{10}c_3^{(3)}) + \frac{L_{Z:3}}{2}(\frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{2}) + \frac{L_{Z:4}}{20}c_3^{(3)}\right)\right), \tag{3.27}$$

respectively.

Proof. The values of $r = 1, 2, 3$ and 4 are substituted into Eq.(3.25), respectively. This results in the first 4 L-moments of a two-piece distribution being defined in terms of order statistics as

$$L_{T:1} = \mu + \sigma\Big(L_{X:1} - 0.5(1-\alpha) \times c_0^{(0)}\mu_{1:1}\Big)$$
$$L_{T:2} = \sigma\Big(L_{X:2} - 0.5(1-\alpha) \times \Big\{c_0^{(1)}\mu_{1:1} + c_1^{(1)}\frac{\mu_{2:2}}{2}\Big\}\Big)$$
$$L_{T:3} = \sigma\Big(L_{X:3} - 0.5(1-\alpha) \times \Big\{c_0^{(2)}\mu_{1:1} + c_1^{(2)}\frac{\mu_{2:2}}{2} + c_2^{(2)}\frac{\mu_{3:3}}{3}\Big\}\Big)$$

and

$$L_{T:4} = \sigma\Big(L_{X:4} - 0.5(1-\alpha) \times \Big\{c_0^{(3)}\mu_{1:1} + c_1^{(3)}\frac{\mu_{2:2}}{2} + c_2^{(3)}\frac{\mu_{3:3}}{3} + c_3^{(3)}\frac{\mu_{4:4}}{4}\Big\}\Big), \tag{3.28}$$

respectively.

The expected values of the order statistics of the half distribution, Z, in Eq.(3.28) can be expressed in terms of its L-moments. Thus

$$\begin{aligned} L_{T:1} &= \mu + \sigma(L_{X:1} - 0.5(1-\alpha)c_0^{(0)}\mu_{1:1}) \\ &= \mu + \sigma(L_{X:1} - 0.5(1-\alpha)c_0^{(0)}L_{Z:1}), \end{aligned} \tag{3.29}$$

where $L_{Z:1}$, L-location, is the first L-moment of Z.

In the case of $r = 2$, $L_{T:2}$ in Eq.(3.28) can be transformed and presented in terms of Hosking's results. By making use of these results, the second L-moment of Z, referred to as L-scale can be given as

$$L_{Z:2} = \frac{1}{2}E(Z_{2:2} - Z_{1:2}) = \frac{1}{2}(\mu_{2:2} - \mu_{1:2}). \tag{3.30}$$

To further simplify Eq.(3.30), consider $L_{Z:r}$ to be the r^{th} row in an infinite triangular array, where $\mu_{i:r}$ is a point in the row, for $1 \leq i \leq r$, with $r > 1$. The triangle rule for order statistics, developed by Arnold and Meeden (1975), indicates that the expected values and moments of order statistics of samples from an arbitrary distribution are known to satisfy the recursive relationship $i\mu_{i+1:r} + (r-i)\mu_{i:r} = r\mu_{i:r-1}$. This enables the expected values of the order statistics in the form $\mu_{i:r}$ to be represented in terms of $\mu_{i:i}, \mu_{i+1:i+1}, ..., \mu_{r:r}$, ultimately resulting in the results in Eq.(3.28) being fully represented in terms of the L-moments of X and Z.

Therefore $\mu_{1:2}$ in Eq.(3.30) can be solved by setting $i = 1$ and $r = 2$ to give

$$\begin{aligned} &\mu_{2:2} + \mu_{1:2} = 2\mu_{1:1} \\ &\Rightarrow \mu_{1:2} = 2\mu_{1:1} - \mu_{2:2}. \end{aligned} \tag{3.31}$$

From Eq.(3.31), $L_{Z:2}$ can be expressed as

$$\begin{aligned} L_{Z:2} &= \frac{1}{2}(\mu_{2:2} - \mu_{1:2}) \\ &= \frac{1}{2}(\mu_{2:2} - 2\mu_{1:1} + \mu_{2:2}) \\ &= \mu_{2:2} - \mu_{1:1} \\ \Rightarrow &\mu_{2:2} = L_{Z:2} + \mu_{1:1} = L_{Z:2} + L_{Z:1}. \end{aligned} \tag{3.32}$$

Consequently,

$$\begin{aligned} L_{T:2} &= \sigma\Big(L_{X:2} - 0.5(1-\alpha) \times \Big\{c_0^{(1)}\mu_{1:1} + c_1^{(1)}\frac{\mu_{2:2}}{2}\Big\}\Big) \\ &= \sigma\Big(L_{X:2} - 0.5(1-\alpha) \times \Big\{c_0^{(1)}L_{Z:1} + \frac{c_1^{(1)}}{2}(L_{Z:1} + L_{Z:2})\Big\}\Big). \end{aligned} \tag{3.33}$$

For $r = 3$, the expression for $L_{T:3}$ from Eq.(3.28) can be given in terms of the L-moments of Z, by using the triangle rule to evaluate

$$\begin{aligned} I_2 &= c_0^{(2)}E(Z_{1:1}) + \frac{c_1^{(2)}}{2}E(Z_{2:2}) + \frac{c_2^{(2)}}{3}E(Z_{3:3}) \\ &= c_0^{(2)}\mu_{1:1} + \frac{c_1^{(2)}}{2}\mu_{2:2} + \frac{c_2^{(2)}}{3}\mu_{3:3}. \end{aligned}$$

Hosking's rules are used to present the 3^{rd} L-moment of Z as

$$\begin{aligned} L_{Z:3} &= \frac{1}{3}E(Z_{3:3} - 2X_{2:3} + Z_{1:3}) \\ &= \frac{1}{2}(\mu_{3:3} - 2\mu_{2:3} + \mu_{1:3}). \end{aligned} \tag{3.34}$$

In order to represent $\mu_{3:3}$ as a function of the L-moments, the triangle rule dictates, with $r = 2$ and $n = 3$, that

$$\begin{aligned} &2\mu_{3:3} + \mu_{2:3} = 3\mu_{2:2} \\ &\Rightarrow \mu_{2:3} = 3\mu_{2:2} - 2\mu_{3:3}. \end{aligned} \tag{3.35}$$

Similarly, with $r = 1$ and $n = 3$, the triangle rule gives

$$\begin{aligned}
&\mu_{2:3} + 2\mu_{1:3} = 3\mu_{1:2} \\
&\Rightarrow \mu_{1:3} = \frac{1}{2}\Big(3\mu_{1:2} - \mu_{2:3}\Big) \\
&= \frac{1}{2}\Big(3\Big(2\mu_{1:1} - \mu_{2:2}\Big) - \Big(3\mu_{2:2} - 2\mu_{3:3}\Big)\Big) \\
&= \frac{1}{2}\Big(6\mu_{1:1} - 3\mu_{2:2} - 3\mu_{2:2} + 2\mu_{3:3}\Big) \\
&= 3\mu_{1:1} - 3\mu_{2:2} + \mu_{3:3}.
\end{aligned} \tag{3.36}$$

By taking both Eqs.(3.35 and 3.36) and substituting in Eq.(3.34), the 3^{rd} L-moment for Z can be denoted as

$$\begin{aligned}
L_{Z:3} &= \frac{1}{3}\Big(\mu_{3:3} - 2(3\mu_{2:2} - 2\mu_{3:3}) + 3\mu_{1:1} - 3\mu_{2:2} + \mu_{3:3}\Big) \\
&= \frac{1}{3}\Big(6\mu_{3:3} - 9\mu_{2:2} + 3\mu_{1:1}\Big) \\
&= 2\mu_{3:3} - 3\mu_{2:2} + \mu_{1:1}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mu_{3:3} &= \frac{1}{2}\Big(L_{Z:3} + 3\mu_{2:2} - \mu_{1:1}\Big) \\
&= \frac{1}{2}\Big(L_{Z:3} + 3L_{Z:1} + 3L_{Z:2} - L_{Z:1}\Big) \\
&= \frac{1}{2}\Big(L_{Z:3} + 3L_{Z:2} + 2L_{Z:1}\Big).
\end{aligned} \tag{3.37}$$

Therefore $L_{T:3}$ can be rewritten as

$$\begin{aligned}
L_{T:3} &= \sigma\Big\{L_{X:3} - 0.5(1-\alpha) \times \Big(c_0^{(2)} L_{Z:1} + \frac{c_1^{(2)}}{2}(L_{Z:1} + L_{Z:2}) + \frac{c_2^{(2)}}{3}\frac{1}{2}(L_{Z:3} + 3L_{Z:2} + 2L_{Z:1})\Big)\Big\} \\
&= \sigma\Big\{L_{X:3} - 0.5(1-\alpha) \times \Big(L_{Z:1}\Big(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\Big) + L_{Z:2}\Big(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2}\Big) + L_{Z:3}\frac{c_2^{(2)}}{6}\Big)\Big\}.
\end{aligned} \tag{3.38}$$

In the case of $L_{T:4}$ the same procedure is considered. For $r = 4$, I_3 is defined as

$$\begin{aligned}
I_3 &= \sum_{j=1}^{4} c_{j-1}^{(3)} \frac{E(Z_{j:j})}{j} \\
&= c_0^{(3)} E(Z_{1:1}) + \frac{c_1^{(3)}}{2} E(Z_{2:2}) + \frac{c_2^{(3)}}{3} E(Z_{3:3}) + \frac{c_3^{(3)}}{4} E(Z_{4:4}).
\end{aligned} \tag{3.39}$$

The 4^{th} L-moment of Z in terms of order statistics is given as

$$\begin{aligned}
L_{Z:4} &= \frac{1}{4} E(Z_{4:4} - 3Z_{3:4} + 3Z_{2:4} - Z_{1:4}) \\
&= \frac{1}{4}(\mu_{4:4} - 3\mu_{3:4} + 3\mu_{2:4} - \mu_{1:4}).
\end{aligned} \tag{3.40}$$

In order to represent $\mu_{i:4}$, for $1 \leq i < 4$, in terms of L-moments, the triangle rule is used once again. Hence, with $r = 3$ and $n = 4$,

$$\begin{aligned} &3\mu_{4:4} + \mu_{3:4} = 4\mu_{3:3} \\ &\Rightarrow \mu_{3:4} = 4\mu_{3:3} - 3\mu_{4:4}. \end{aligned} \tag{3.41}$$

Next, with $r = 2$ and $n = 4$, it follows that

$$\begin{aligned} &2\mu_{3:4}{+}2\mu_{2:4} = 4\mu_{2:3} \\ &\Rightarrow \mu_{2:4} = 2\mu_{2:3} - \mu_{3:4} \\ &\qquad = 2(3\mu_{2:2} - 2\mu_{3:3}) - 4(\mu_{3:3} - 3\mu_{4:4}) \\ &\qquad = 6\mu_{2:2} - 4\mu_{3:3} - 4\mu_{3:3} + 3\mu_{4:4} \\ &\qquad = 6\mu_{2:2} - 8\mu_{3:3} + 3\mu_{4:4}. \end{aligned} \tag{3.42}$$

Finally, with $r = 1$ and $n = 4$, we obtain

$$\begin{aligned} &\mu_{2:4} + 3\mu_{1:4} = 4\mu_{1:3} \\ &\Rightarrow \mu_{1:4} = \frac{1}{3}(4\mu_{1:3} - \mu_{2:4}) \\ &\qquad = \frac{1}{3}(4(3\mu_{1:1} - 3\mu_{2:2} + \mu_{3:3}) - 6\mu_{2:2} + 8\mu_{3:3} - 3\mu_{4:4}) \\ &\qquad = \frac{1}{3}(12\mu_{1:1} - 18\mu_{2:2} + 12\mu_{3:3} - 3\mu_{4:4}) \\ &\qquad = 4\mu_{1:1} - 6\mu_{2:2} + 4\mu_{3:3} - \mu_{4:4}. \end{aligned} \tag{3.43}$$

Through the substitution of Eqs.(3.41-3.43) into Eq.(3.40), $L_{Z:4}$ is obtained as

$$\begin{aligned} L_{Z:4} &= \frac{1}{4}(\mu_{4:4} - 3(4\mu_{3:3} - 3\mu_{4:4}) + 3(6\mu_{2:2} - 8\mu_{3:3} + 3\mu_{4:4}) - (4\mu_{1:1} - 6\mu_{2:2} + 4\mu_{3:3} - 4\mu_{4:4})) \\ &= \frac{1}{4}(20\mu_{4:4} - 40\mu_{3:3} + 24\mu_{2:2} - 4\mu_{1:1}) \\ &= 5\mu_{4:4} - 10\mu_{3:3} + 6\mu_{2:2} - \mu_{1:1}, \end{aligned}$$

which yields $\mu_{4:4}$ as

$$\begin{aligned} \mu_{4:4} &= \frac{1}{5}(L_{Z:4} + 10\mu_{3:3} - 6\mu_{2:2} + \mu_{1:1}) \\ &= \frac{1}{5}(L_{Z:4} + 5(L_{Z:3} + 3L_{Z:2} + 2L_{Z:1}) - 6(L_{Z:1} + L_{Z:2}) + \lambda_1) \\ &= \frac{1}{5}(L_{Z:4} + 5L_{Z:3} + 9L_{Z:2} + 5L_{Z:1}). \end{aligned} \tag{3.44}$$

In conclusion, $L_{T:4}$ in Eq.(3.28) is rewritten by making use of Eqs.(3.32), (3.37) and (3.44) to give the final result as

$$
\begin{aligned}
L_{T:4} &= \sigma\Big\{L_{X:4} - 0.5(1-\alpha)\times\Big(c_0^{(3)}L_{Z:1} + \frac{c_1^{(3)}}{2}(L_{Z:1}+L_{Z:2}) + \frac{c_2^{(3)}}{3}\frac{1}{2}(L_{Z:3}+3L_{Z:2}+2L_{Z:1}) \\
&+ \frac{c_3^{(3)}}{4}\cdot\frac{1}{5}(L_{Z:4}+5L_{Z:3}+9L_{Z:2}+5L_{Z:1})\Big)\Big\} \\
&= \sigma\Big\{L_{X:4} - 0.5(1-\alpha)\times\Big(L_{Z:1}\Big(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\Big) + \frac{L_{Z:2}}{2}\left(c_1^{(3)} + c_2^{(3)} + \frac{9}{10}c_3^{(3)}\right) \\
&+ \frac{L_{Z:3}}{2}\left(\frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{2}\right) + \frac{L_{Z:4}}{20}c_3^{(3)}\Big)\Big\}.
\end{aligned} \tag{3.45}
$$

■

Considering that the expressions of the two-piece L-moments, $L_{T:r}$, are in terms of the L-moments of the parent and half distributions, Z and X, respectively, the coefficients in Table 3.1 can then be given for $r = 1, 2, 3$ and 4, in Table 3.2. These will be equivalent to the coefficients of the r^{th} order shifted Legendre polynomials in Eq.(3.12).

$r-1$ \ $j-1$	0	1	2	3
0	1			
1	-1	2		
2	1	-6	6	
3	-1	12	-30	20

Table 3.2: The coefficients of a polynomial of degree $(r-1)$ for $r = 1, 2, 3$ and 4.

3.4 Quantile Measures of Distributional Form

The methodology in Section3.2 is used to generate two-piece families of distributions whose quantile functions can be expressed in closed-form. This permits the construction of quantile measures of distributional form to explore their properties with respect to the location, scale and shape of a two-piece distribution. The measures that will be used to explore the properties of these distributions exist for all parameter values of a distribution.

3.4.1 Location

The median is well-known for its ability to uphold the property of robustness. As such, it is the chosen measure of location. It is defined as

$$me = Q\left(\frac{1}{2}\right). \tag{3.46}$$

3.4.2 Spread

The spread function, by MacGillivray and Balanda (1988), is the measure of choice to summarise the spread of the two-piece families of distributions that have been generated. It is location-invariant and a strictly increasing function. It is defined as

$$S(s) = Q(s) - Q(1-s), \qquad \frac{1}{2} < s < 1. \tag{3.47}$$

It can be noted $Q(s) > Q(1-s)$ for all values of $\frac{1}{2} < s < 1$, therefore $S(s) > 0$. This implies it meets the requirements for it to be a valid spread function. Special cases of the spread function include the inter-quartile range, (IQR), and the inter-decile range, (IDR), for which the values of s are set as $\frac{3}{4}$ and $\frac{9}{10}$, respectively.

3.4.3 Shape

γ-functional

The γ-functional is an asymmetry functional that was defined by MacGillivray (1986) as

$$\gamma(s) = \frac{Q(s) + Q(1-s) - 2Q(\frac{1}{2})}{Q(s) - Q(1-s)} = \frac{Q(s) + Q(1-s) - 2me}{S(s)}, \qquad \frac{1}{2} < s < 1. \tag{3.48}$$

As can be seen, the γ-functional is a function of the difference between the quantile function evaluated at s and $(1-s)$, and twice the median in the numerator. It is however scaled by the spread function in Eq.(3.47) in the denominator. As the numerator increases, the functional value increases and vice versa. The functional is bounded by -1 and 1. A special case is Bowley's quartile-based measure of skewness proposed by Bowley (1902), which is obtained by setting $s = \frac{3}{4}$.

Ratio-of-spread function

MacGillivray and Balanda (1988) introduced the ratio-of-spread functions as an additional measure of kurtosis. It describes the position of the probability mass in the tails of the distribution and is measured for any pairs of values u and v. This function is denoted as

$$R(u,v) = \frac{S(u)}{S(v)}, \qquad \frac{1}{2} < v < u < 1. \tag{3.49}$$

Since $S(u) > S(v)$, for $\frac{1}{2} < v < u < 1$, it then follows that $R(u,v) > 1$.

3.4.4 Skewness-invariant Measures of Kurtosis

Kurtosis, in statistical terms, is used to measure the heaviness of the tails of a probability density curve of a random variable from an existing distribution. In most cases, it has been summarised in terms of its conventional standardized fourth central moment, known as the kurtosis moment-ratio. It is commonly denoted as

$$\alpha_4 = \frac{E(X-\mu)^4}{\sigma^4},$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are the location and scale parameters, respectively, of the random variable, X.

The normal distribution, with $\alpha_4 = 3$ as the theoretical kurtosis moment-ratio value, is used as the standard against which other distributions are compared. Distributions with $\alpha_4 > 3$ are called leptokurtic or heavy-tailed, $\alpha_4 < 3$ are termed platykurtic or light-tailed, whilst those with $\alpha_4 = 3$ are referred to as mesokurtic as they resemble the normal distribution. These terms were first used by Pearson (1905).

This measure has its own drawbacks that limits its use in comparison to other measures of kurtosis. It is a difficult measure to define in cases where any of the first 4 central moments of a random variable do not exist. Furthermore, since the peakedness attains clarity when it is relative to the weights of the tails in a distribution, it implies that α_4 cannot be used to accurately measure the kurtosis of leptokurtic distributions.

Pearson (1916) showed that $\alpha_4 \geq \alpha_3^2 + 1$, where α_3 is the skewness moment-ratio of a distribution. The implication of an increase in the value of α_3 is that α_4 also increases. This becomes progressively difficult to elaborate on the kurtosis of an asymmetric distribution. The solution proposed by Jones et al. (2011) was to define kurtosis measures that were independent of skewness parameters of a random variable. The objective was to better the description of the kurtosis of both symmetric and asymmetric distributions regardless of whether or not they were heavy-tailed. In effect, this measure would be termed skewness-invariant.

Moreover, in order to deal with the dilemma of the inexistence of any of the first 4 central moments of a random variable, this measure would make use of the quantile function, $Q(s)$, where $0 < s < 1$. The measure proposed would be a ratio of linear combinations of differences between quantile functions, of the form $Q(s) - Q(1-s)$, for values $0 < s < 1$. The result is a scale-invariant measure which reduces the variability of the overall kurtosis measure.

Definition 3.4.4.1. *A skewness-invariant kurtosis measure will then be identified if it takes on the general form*

$$\frac{\sum_{i=0}^{n_1} g_i(Q(s_i) - Q(1-s_i)}{\sum_{j=0}^{n_2} h_j(Q(s_j) - Q(1-s_j))}, \tag{3.50}$$

where $n_1,\ n_2 \in \mathbb{Z}^+$*, whereas* $g_i = 1, 2, ..., n_1$ *and* $h_j = 1, 2, ..., n_2$ *are constants.*

■

From Eq.(3.47), $S(s) = Q(s) - Q(1-s)$, culminating in Eq.(3.50) being redefined as

$$\frac{\sum_{i=0}^{n_1} g_i S(s_i)}{\sum_{j=0}^{n_2} h_j S(s_j)} = \frac{\sum_{i=0}^{n_1} g_i(Q(s_i) - Q(1-s_i))}{\sum_{j=0}^{n_2} h_j(Q(s_j) - Q(1-s_j))}. \tag{3.51}$$

Special cases of Eq.(3.51) arise for specific values of $0 < s < 1$.

- The p-indexed measure was mentioned by MacGillivray and Balanda (1988) as one of the early occuring measures of kurtosis. It is given as

$$t(s) = \frac{Q(0.5+s) - Q(0.5-s)}{Q(0.75) - Q(0.25)}, \qquad 0 < s < \frac{1}{2}.$$

- Moor's kurtosis (Moors (1988)) which is based on octiles is defined as

$$M = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}.$$

- The quintile-based measure is seen to be an extension of Bowley (1902)'s measure of skewness. It is presented by Jones et al. (2011) as

$$J = \frac{Q(\frac{4}{5}) - 3Q(\frac{3}{5}) + 3Q(\frac{2}{5}) - Q(\frac{1}{5})}{Q(\frac{4}{5}) - Q(\frac{1}{5})}.$$

Theorem 3.4.1. *Let T be a two-piece distribution denoted by $T \sim TP(\mu,\ \sigma,\ \alpha)$, where $-\infty < \mu < \infty$, $\sigma > 0$ and $\alpha > 0$ are the location, spread and skewness parameters, respectively. The median, spread function, γ-functional and ratio-of-spread functions for T are*

$$me = Q_T\left(\frac{1}{2}\right) = \mu, \tag{3.52}$$

$$S_T(s) = \sigma(1+\alpha)Q_{X:0}(s), \tag{3.53}$$

$$\gamma_T(s) = \frac{(1-\alpha)}{(1+\alpha)}, \tag{3.54}$$

and

$$R_T(u,v) = \frac{Q_{X:0}(u)}{Q_{X:0}(v)}, \tag{3.55}$$

respectively. The skewness-invariant quantile-based measures of kurtosis, with respect the α, takes on the general form

$$\frac{\sum_{i=1}^{n_1} g_i S_T(u_i)}{\sum_{j=1}^{n_2} h_j S_T(u_j)} = \frac{\sum_{i=1}^{n_1} g_i (Q_{X:0}(u_i))}{\sum_{j=1}^{n_2} h_j (Q_{X:0}(u_j))}, \tag{3.56}$$

where $n_1,\ n_2 \in \mathbb{Z}^+$, whereas $g_i : i = 1, 2, ..., n_1$ and $h_j : j = 1, 2, ..., n_2$ are constants.

Proof. Through the substitution of the quantile functions from Eq.(3.3) into Eqs.(3.52 - 3.56), the median is attained as

$$\begin{aligned} me &= Q_T\left(\frac{1}{2}\right) \\ &= \mu + \sigma Q_{X:0}\left(\frac{1}{2}\right) \\ &= \mu + \sigma \cdot 0 \\ &= \mu, \end{aligned}$$

since the quantile function of a symmetric distribution is zero when evaluated at $\frac{1}{2}$.

The spread function $S_T(s)$ in Eq.(3.53) is obtained by replacing Eq.(3.3) into Eq.(3.54), to give rise to

$$\begin{aligned} S_T(s) &= Q_T(s) - Q_T(1-s) \\ &= \mu + \sigma Q_{X:0}(s) - (\mu + \sigma\alpha Q_{X:0}(1-s)) \\ &= \sigma Q_{X:0}(s) + \sigma\alpha Q_{X:0}(s) \\ &= \sigma(1+\alpha)Q_{X:0}(s), \qquad \frac{1}{2} < s < 1. \end{aligned}$$

The γ-functional will make use of Eq.(3.48) to obtain

$$\begin{aligned}\gamma_T(s) &= \frac{Q_T(s) + Q_T(1-s) - 2Q_T(\frac{1}{2})}{Q_T(s) - Q_T(1-s)} \\ &= \frac{\mu + \sigma Q_{X:0}(s) + (\mu + \sigma\alpha Q_X(1-s)) - 2\mu}{\sigma(1+\alpha)Q_{X:0}(s)} \\ &= \frac{2\mu + \sigma(1-\alpha)Q_{X:0}(s) - 2\mu}{\sigma(1+\alpha)Q_{X:0}(s)} \\ &= \frac{(1-\alpha)}{(1+\alpha)}, \qquad \frac{1}{2} < s < 1.\end{aligned}$$

The value of the γ-functional tends to 1 when α approaches 0, while it tends to -1 when α tends to ∞. The results of the spread function will be used to obtain the ratio-of-spread functions as

$$\begin{aligned}R(u,v) &= \frac{S_T(u)}{S_T(v)} \\ &= \frac{\sigma(1+\alpha)Q_{X:0}(u)}{\sigma(1+\alpha)Q_{X:0}(v)} \\ &= \frac{Q_{X:0}(u)}{Q_{X:0}(v)}, \qquad \frac{1}{2} < v < u < 1\end{aligned}$$

The general form of a skewness-invariant quantile-based measure of kurtosis follows directly from Eq.(3.56) as

$$\frac{\sum_{i=1}^{n_1} g_i S_T(u_i)}{\sum_{j=1}^{n_2} h_j S_T(u_j)} = \frac{\sum_{i=1}^{n_1} g_i(\sigma(1-\alpha)Q_{X:0}(u_i))}{\sum_{j=1}^{n_2} h_j(\sigma(1-\alpha)Q_{X:0}(u_j))} = \frac{\sum_{i=1}^{n_1} g_i(Q_{X:0}(u_i))}{\sum_{j=1}^{n_2} h_j(Q_{X:0}(u_j))}.$$

■

Remark. *The ratio-of-spread functions is independent of the skewness parameter, α, hence deemed skewness-invariant. This implies that the kurtosis value of the two-piece distribution is constant for all values of α, according to van Zwet's ordering $\leq_s$ (Zwet (1964)).*

3.5 Parameter Estimation

Gilchrist (2000) discussed the essence of parameter estimation and its intended outcome. He described it as the process of matching the fitted model to a data set in order to obtain parameter estimates that gave a good fit or a good match, before proceeding to document the most commonly used methods.

The method of moments developed by Pafnuty Chebychev in 1887, yields estimators that are biased, in as much as they are simple to construct. The method of likelihood estimation yields aims to obtain estimators from the maximisation of a likelihood function. Both methods can be used in cases where the CDF or the PDF of a model exists.

The challenge of matching a data set to a model with no closed-form expressions for neither the CDF or PDF presented a platform for other methods to be considered. In order to define and generalize these methods, consider $H(\underline{\theta})$ to be a set of functions that describes the population's properties. $H(\underline{\theta})$ will depend on the quantile function, $Q(s;\theta)$, of the model as well as its parameters, $\underline{\theta}$. Innately, the number of functions in $H(\underline{\theta})$ will correspond to the number of sample quantities needed to be matched to it.

The advantage of using quantile functions is that they become very handy in cases where the model is a quantile-based distribution. This has led to the development and use of methods such as the method of percentiles or quantiles for selected percentiles (Bury (1975)), where $H(\underline{\theta}) =$ median (me), interquartile range (IQR) or the quantile function $(Q(s;\theta))$ itself.

Another technique is the method of probability weighted moments (Landwehr et al. (1979)), where the set of functions are weights,

$$w_{k,j} \text{ for } k = i \text{ and } j = 0 \text{ or } k = 0 \text{ and } j = I,$$

with i=the number of sample quantities.

In the case of the method of L-moments (Hosking (1990)), the function $(H(\underline{\theta}))$ to be matched to the sample quantities will correspond to the L-moments of the model while the sample quantities will be the sample L-moments. This method of estimation will be used to estimate the parameters for two-piece families of distributions because of the closed-form expression for the L-moments in Eq.(3.25). Moreover, these estimators are not susceptible to large variability where higher orders of polynomial functions are used in the matching procedure.

The procedure will consider four sample quantities that will be used to match the population functions that are to be estimated. These will be depicted by L_i, where $i = 1, 2, 3$ and 4. The first two, L_1 and L_2, will be matched for the location and scale functions, respectively, while L_3 and L_4 will be used to give matches for the L-skewness (τ_3) and L-kurtosis (τ_4) ratios, respectively. These quantities are defined in terms of order statistics of a sample of size n. That being said, U-statistics (Hoeffding (1948)) will be used to estimate these quantities. U-statistics entail defining functions as the average of the combinatorial sub-samples of size r, which are obtained from the observed data of size n.

Lemma 3.5.1. *If $X_1, X_2, X_3, \ldots, X_n$ is a sample of size n, and $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ is the ordered sample, then the r^{th} sample ℓ-moment as defined by Hosking (1990) is given as*

$$\ell_r = \binom{n}{r}^{-1} \sum_{1<i_1<} \sum_{i_2<} \sum_{i_3<} \cdots \sum_{<i_r} \sum_{<n} r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} x_{i_{r-k:n}}, \tag{3.57}$$

for $r = 1, 2, 3, \cdots, n$.

■

Lemma 3.5.2. *The first four sample ℓ-moments are obtained from Eq.(3.57), as*

$$\ell_1 = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x},$$

$$\ell_2 = \frac{1}{2} \binom{n}{2}^{-1} \sum_{i>j}^{n} \sum (x_i - x_j),$$

$$\ell_3 = \frac{1}{3} \binom{n}{3}^{-1} \sum_{i>j>k}^{n} \sum \sum (x_i - 2x_j + x_k)$$

and

$$\ell_4 = \frac{1}{4} \binom{n}{4}^{-1} \sum_{i>j>k>l}^{n} \sum \sum \sum (x_i - 3x_j + 3x_k - x_l), \tag{3.58}$$

respectively.

The sample L-skewness and L-kurtosis ratios are defined as

$$t_3 = \frac{\ell_3}{\ell_2} \quad \text{and} \quad t_4 = \frac{\ell_4}{\ell_2}, \tag{3.59}$$

respectively.

■

The sample estimates in Lemma 3.5.2 will be matched to population quantities to give parameter estimates for a univariate two-piece distribution in the following steps:

STEP 1:

The results in Eq.(3.58) are used to obtain the first four sample L-moments from the observed data set. Thereafter, the sample L-moment ratios, t_3 and t_4, are obtained using Eq.(3.59). In order to determine if the proposed two-piece distribution can be fit to the data set, the values of t_3 and t_4 are verified if they lie in the $(\tau_{T:3}, \tau_{T:4})$-space of the two-piece distribution.

In the case of a two-piece distribution whose symmetric parent distribution has no shape parameter, the $(\tau_{T:3}, \tau_{T:4})$-space will consist of a horizontal line transversing the theoretical L-kurtosis ratio value at τ_4. The limits of t_3 will be the corresponding $|\tau_3|$ of the two-piece distribution. The value of t_4 should be equivalent to τ_4. If the values are found to lie on the line, then the estimation procedures can continue, else the two-piece distribution cannot be fitted to the data.

In the event that the proposed two-piece distribution has a parent distribution with an additional shape parameter, the $(\tau_{T:3}, \tau_{T:4})$-space will be a region covered by the possible combinations obtained by the presence of the two shape parameters. Similarly, if the values of t_3 and t_4 lie in this region, then the next estimation step can follow.

STEP 2:

The two-piece distributional shape exhibits the presence of skewness. However, the L-kurtosis ratio value of the two-piece distribution is expected to remain the same as that of the parent distribution. If the theoretical expression of τ_4 of the parent distribution is dependent on another shape parameter, then set the value of t_4 to be equivalent to the theoretical result of τ_4 and solve for the unknown shape parameter estimate. The solutions for the estimates should be checked against the range of possible values they can assume to see if they are valid.

STEP 3:

Using the result from Step 2, solve for $\hat{\alpha}$, the asymmetry parameter estimate. Set t_3 equal to the theoretical expression of $\tau_{T:3}$.

STEP 4:

The solution for the scale parameter estimate, $\hat{\sigma}$, can be found by substituting the parameter estimate results from Steps 2 and 3 into the theoretical expression of $L_{T:2}$. Equate this function to ℓ_2 and solving for it accordingly.

STEP 5:

The final estimate, $\hat{\mu}$, the location parameter estimate, is similarly obtained by substituting the parameter estimates from Steps $2-4$ into the theoretical expression of $L_{T:1}$ from the two-piece distribution, then equating that to ℓ_1 and solving for $\hat{\mu}$.

STEP 6:

The standard errors for the parameter estimates obtained in Steps $2-5$ are then computed using the parametric bootstrap procedure. In this thesis, $N = 10000$ samples will be used.

3.6 Model Validation

This section presents the various methods that can be used to substantiate the fit of a model to a data set. Though there are numerous aspects of validation that can be considered, only those with respect to quantile modelling will be taken into consideration.

3.6.1 Graphical displays

Histograms

The primary form of visual validation that will be used to address the fit of a model are histograms with superimposed probability density curves. The probability density curves are obtained by plotting $\hat{f}_P(p)$ against $\hat{Q}_X(p)$, which are the fitted density quantile and quantile function values, for $0 < p < 1$.

Q-Q Plots

Q-Q (quantile-quantile) plots graphically illustrate the observed data points plotted against the empirical observations. In this context, $x_{i:n}$, which is the i^{th} order statistic from a sample of size n, is plotted against $\hat{Q}(s_{i:n})$, the empirical quantile function. The plotting points, $s_{i:n}$, are defined as

$$p_{i:n} = \frac{i - c}{n + 1 - 2c}. \tag{3.60}$$

.

In this thesis, $c = \frac{1}{3}$ is used as it provides a plotting point that closely approaches the median value. This enables the ordered statistic to be compared to any quantile value.

3.6.2 Goodness-of-fit measures

Castillo and Hadi (1997) introduced the average scaled absolute error (ASAE) as a measure that can be used to compare the fit of various models to a data set. It is defined as

$$\text{ASAE} = \frac{1}{n} \sum_{i=1}^{n} \frac{|x_{i:n} - \hat{Q}(S_{i:n})|}{(x_{n:n} - x_{1:n})}, \quad i = 1, \cdots, n, \tag{3.61}$$

where $\hat{Q}(S_{i:n})$ is the empirical quantile function of the fitted distribution.

This measure can either be used for models that are quantile-based or that are defined through their CDF or PDF. The smaller the ASAE value, the better the fit of distribution to the data. In this thesis, $p_{i:n}$ from Eq.(3.60) will be used, as opposed to $p_{i:n} = \frac{i}{n+1}$ which was proposed by Castillo and Hadi (1997).

3.7 Tail Behaviour

The tail behaviour of the probability density curve of a continuous distribution is classically evaluated through its probability density function, $f_X(x)$, should it exist. However, in the case of quantile-based distributions, the tail behaviour is evaluated through the density quantile function, $f_S(s)$, since no closed-form expression of the probability density function exists.

The tail behaviour investigation involves determining the value that the probability density curve approaches at the endpoints. This is explored through computing $\lim_{s\to 0} f_S(s)$ for the left tail, and $\lim_{s\to 1} f_S(s)$ for the right tail. Depending on the results obtained for the two-piece distribution, either the two-piece PDF or two-piece density quantile functions will be used for evaluation.

Furthermore, the slope of the density curve at these two tails is also evaluated to further determine the behaviour of this distribution. This can be done through obtaining expressions for the slope and taking its limits for each tail respectively.

King (1999) derived a formula for obtaining the slope of the density curves for quantile-based distributions. This measure is defined as

$$\xi = -\frac{dq_S(s)}{ds} \cdot \frac{1}{(q_S(s))^3}, \quad 0 < s < 1. \tag{3.62}$$

3.8 Conclusion

The method of quantile splicing was proposed in Section 3.2 with the aim of presenting techniques that can be used to introduce asymmetry to univariate symmetric distributions. This technique makes use of the quantile functions of half distributions as the kernels to generate families of two-piece distributions. It was shown that the method can be used on univariate distributions that are either quantile-based or defined through their CDF or PDF.

A general expression for the r^{th} order L-moments of two-piece distributions was derived in

Section 3.3.4. This expression results in the r^{th} order L-moments of two-piece distributions being expressed in terms of the L-moments of both the half distributions and the parent symmetric distributions, Z and X, respectively. The L-moments derived proved to be closed-form and simplistic in nature.

The results in Section 3.4.4 indicate that the level of kurtosis for the two-piece distributions will be the same as that of the parent distribution, but extended levels of skewness are attained. Through the various measures of kurtosis presented, it can be seen that they are indeed skewness-invariant.

The method of L-moments estimation in Section 3.5 is developed for parameter estimation since the r^{th} order L-moments obtained take on closed forms. The parametric boostrap procedure is used to obtain the asymptotic standard errors for the parameter estimates.

UNIVARIATE TWO-PIECE DISTRIBUTIONS

4.1 Introduction

Univariate symmetric distributions have been used in literature as building blocks to generate skewed families of distributions. Various skewing techniques have been discussed in Chapter 3, where distributional differences and advantages have been highlighted. These methods have made use of CDFs or PDFs as the building blocks of the univariate distributions to obtain the skewed families, with the intent of increasing their flexibility with regards to distributional form.

In this chapter, the quantile splicing technique proposed in Section 3.2 is applied to univariate symmetric distributions that do not have any shape parameter. The asymmetry parameter introduced will increase the flexibility of the density curves with respect to distributional form. The arcsine, uniform, cosine, logistic, normal, hyperbolic secant and Student's $t(2)$ distributions are selected as the parent distributions that will be generalized to obtain two-piece families of distributions. They have different levels of kurtosis, with the arcsine distribution exhibiting the lowest level whilst the Student's $t(2)$ distribution has the highest level.

Since the quantile functions of the above-mentioned distributions are used as the building blocks to obtain the two-piece distributions, they will require standardizing first. This will be achieved when the first two L-moments, that is the L-location and L-scale are set to 0 and 1, respectively. Subsequently, the location and scale parameters are then solved under these new specifications. For the cosine, uniform and arcsine distributions, the reparametrization of these distributions with respect to their lower boundaries will ensure the condition of $L_1 = 0$ and $L_2 = 1$ are met. The proposed two-piece distributions are then derived, as well as their quantile measures of distributional form for location, spread and shape. The results in Eq.(3.27) will be used to derive the r^{th} order L-moments for the distributions.

4.2 Two-piece Student's $t(2)$ Distribution

The Student's t-distribution was presented by Student (1908) as the model to be used in the event that underlying data is not normally distributed. In such a case, the population standard deviation has to be estimated by the sample standard deviation.

In addition to having a location and scale parameter, it has an additional parameter which determines the shape of the distribution. This parameter, termed the degrees of freedom, is denoted by $\nu > 0$. When $\nu \to \infty$, the t-distribution tends to the normal distribution which is its limiting case. This distribution has been widely used in robust statistical modeling of data sets (Lange et al. (1989)), where the errors have extended tails.

Definition 4.2.0.1. *A random variable, X, from the Student's t-distribution is defined by the following probability density function, presented by Johnson et al. (1995), as*

$$f_X(x) = \frac{\Gamma(\frac{1}{2}(\nu+1))}{\sqrt{(\pi\nu)}\Gamma(\frac{1}{2}\nu)} \frac{1}{(1+\frac{x^2}{\nu})^{(\frac{\nu+1}{2})}}, \qquad -\infty < x < \infty,\ \nu > 0. \tag{4.1}$$

■

The concern that arises with this distribution is in the complexity of the general form, which makes it difficult and less suited to depict basic calculations. Therefore, a need arises for the use of one of its special cases. The Student's $t(2)$ distribution, first introduced by Jones (2002b), highlights the tractability of this distribution due to its more simplistic form and nature. Its PDF, CDF and quantile functions are simpler in structure as compared to the Student's t-distribution.

Lemma 4.2.1. *The probability density function of a Student's $t(2)$ distribution, with $\mu \in \mathbb{R}$ and $\sigma > 0$, as given by Jones (2002b), is obtained by substituting $\nu = 2$ into Eq.(4.1) to obtain*

$$f_X(x) = \frac{1}{\sigma}\left(2 + \left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{3}{2}}, \qquad -\infty < x < \infty. \tag{4.2}$$

The CDF and quantile function attained in closed form, respectively, as

$$F_X(x) = \frac{1}{2}\left(1 + \frac{\frac{x-\mu}{\sigma}}{\sqrt{2+(\frac{x-\mu}{\sigma})^2}}\right), \qquad -\infty < x < \infty, \tag{4.3}$$

and

$$Q_X(p) = F^{-1}(p) = \mu + \sigma\left(\frac{2p-1}{(2p(1-p))^{\frac{1}{2}}}\right), \qquad 0 < p < 1, \tag{4.4}$$

respectively.

■

4.2.1 Distributional properties of the Student's $t(2)$ distribution

Shape

The probability density curve of the Student's $t(2)$ distribution exhibits a unimodal shape, with the mode at μ. Furthermore, the mean value of the distribution is μ, whilst higher order moments do not exist.

Variability

Various measures of variability such as the mean absolute deviation, the median absolute deviation, the interquartile range and the Gini's mean difference can be used to summarise the spread. Jones (2002b) was able to obtain the values of these measures as:

- Mean absolute deviation $\Rightarrow E(|T|) = \sqrt{2} \simeq 1.414$.
- Median absolute deviation $\Rightarrow Q_X(\frac{3}{4}) = \sqrt{\frac{2}{3}} \simeq 0.816$.
- Interquartile range $\Rightarrow 2Q_X\left(\frac{3}{4}\right) = 2\sqrt{\frac{2}{3}} \simeq 1.633$.
- Gini's mean difference $\Rightarrow E(|T_1 - T_2|) = \frac{\pi}{\sqrt{2}} \simeq 2.221$.

T, T_1 and T_2 are independent random variables from the Student's $t(2)$ distribution, whereas $Q_X\left(\frac{3}{4}\right)$ is the quantile function in Eq.(4.4), evaluated at $p = \frac{3}{4}$.

Kurtosis

The quantile function in Eq.(4.4) is used to obtain quantile-based measures of kurtosis since the moment-based measures cannot be obtained apart from the mean. Jones (2002b) studied Groeneveld (1998)'s measure, Moors' kurtosis (Moors (1988)), as well as the L-kurtosis ratio of Hosking (1990), τ_4, which is obtained through the use of order statistics. These results were found to be easier to obtain since the quantile function took on a simpler nature.

The results obtained are as follows:

- Groeneveld's kurtosis measure $\Rightarrow \frac{Q_X(\frac{7}{8})-2Q_X(\frac{3}{4})+Q_X(\frac{5}{8})}{Q_X(\frac{7}{8})-Q_X(\frac{5}{8})} = 0.271$.
- Moors' kurtosis measure $\Rightarrow \frac{Q_X(\frac{7}{8})-Q_X(\frac{5}{8})}{Q_X(\frac{3}{4})} = 1.517$.
- L-kurtosis ratio $\Rightarrow \tau_4 = \frac{3}{8} = 0.375$.

These results were compared to similar results for the normal and logistic distributions. The Student's $t(2)$ distribution had higher values than the normal and logistic distributions for all 3 measures. This is an indication that it is more leptokurtic (heavier-tailed) than the other two distributions.

Student's $t(2)$ and other distributions

There are numerous relationships that can be drawn between the Student's $t(2)$ distribution and other univariate distributions.

- Since $t = \frac{Z}{\sqrt{X_v/v}}$, where $Z \sim N(0,1)$ and $X_v \sim \chi^2(v)$, then $t(2) \simeq \frac{Z}{\sqrt{X_2/2}}$, where $Z \sim N(0,1)$ and $\frac{X_2}{2} \sim \exp(1)$.

- The standard uniform distribution is used to generate random variables from the Student's $t(2)$ through the probability integral transform. A single uniform variate, denoted by $U \sim UNIF(0,1)$, is substituted in Eq.(4.4) to generate the variables as
$$t(2) = \frac{2U-1}{(2U(1-U))^{\frac{1}{2}}}.$$

- Let S and T be standard exponential distribution random variables. Then
$$t(2) \simeq \frac{S-T}{\sqrt{2ST}}.$$

Order statistics and r^{th} order L-moments

Lemma 4.2.2. *Suppose $X_{i:n},\ i = 1, ..., n$ is the i^{th} order statistic of a Student's $t(2)$ random variable from a sample of size n. Then the probability density function of $X_{i:n}$ is defined as*
$$f_{X_{i:n}}(x) = \frac{1}{B(i, n-i+1)2^{n+1/2}} \left(1 + \frac{x}{\sqrt{2+x^2}}\right)^{i+1/2} \left(1 - \frac{x}{\sqrt{2+x^2}}\right)^{n-i+3/2} \tag{4.5}$$

Lemma 4.2.3. *The r^{th} order L-moments function for the Student's $t(2)$ distribution, for $r \geq 2$, is given by Jones (2002b) as*
$$L_r = \frac{1}{2^{\frac{3}{2}}} \sum_{j=0}^{r-2} (-1)^j \frac{\Gamma(r)\Gamma(j+\frac{1}{2})\Gamma(r-j-\frac{3}{2})}{\Gamma(j+1)\Gamma(j+2)\Gamma(r-j-1)\Gamma(r-j)}, \tag{4.6}$$
with the first four corresponding L-moments as
$$L_{X:1} = 0, \quad L_{X:2} = \frac{\pi}{2\sqrt{2}}, \quad L_{X:3} = 0, \quad \text{and} \quad L_{X:4} = \frac{3\pi}{16\sqrt{2}}, \tag{4.7}$$

respectively.

The subsequent L*-skewness and* L*-kurtosis ratios are*

$$\tau_{X:3} = 0 \quad and \quad \tau_{X:4} = \frac{3}{8},$$

respectively.

The standard Student's t(2) distribution is obtained when $\mu = 0$ and $\sigma = \frac{2\sqrt{2}}{\pi}$.

■

4.2.2 Proposed two-piece Student's $t(2)$ distribution.

Definition 4.2.2.1. *A real-valued random variable is said to have a two-piece Student's $t(2)$ distribution if its quantile function is defined as*

$$Q_T(s) = \begin{cases} \mu + \alpha\sigma \frac{2s-1}{(2s(1-s))^{\frac{1}{2}}}, & s \leq \frac{1}{2}, \\ \mu + \sigma \frac{2s-1}{(2s(1-s))^{\frac{1}{2}}}, & s > \frac{1}{2}, \end{cases} \tag{4.8}$$

where $-\infty < \mu < \infty$, $\sigma > 0$ and $\alpha > 0$ are the location, scale and shape parameters, respectively.

Its CDF and PDF are given as

$$F_T(x) = \begin{cases} \frac{1}{2}\left(1 + \frac{\frac{x-\mu}{\alpha\sigma}}{\sqrt{2+\left(\frac{x-\mu}{\alpha\sigma}\right)^2}}\right), & x \leq \mu, \\ \frac{1}{2}\left(1 + \frac{\frac{x-\mu}{\sigma}}{\sqrt{2+\left(\frac{x-\mu}{\sigma}\right)^2}}\right), & x > \mu, \end{cases}$$

and

$$f_T(x) = \begin{cases} \frac{1}{\alpha\sigma}\left(2 + \left(\frac{x-\mu}{\alpha\sigma}\right)^2\right)^{-\frac{3}{2}}, & x \leq \mu, \\ \frac{1}{\sigma}\left(2 + \left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{3}{2}}, & x > \mu, \end{cases} ,$$

respectively.

■

It can be seen from the probability density curves from Fig.4.1 that the distribution exhibits negative skewness when $\alpha > 1$ (dotted curve), symmetry when $\alpha = 1$ (solid curve) and subsequently positive skewness when $0 < \alpha < 1$ (dot-dashed).

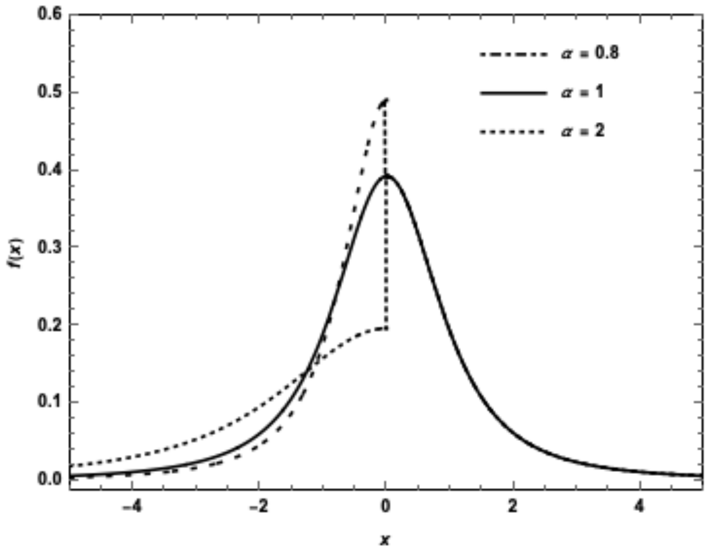


Figure 4.1: The probability density curves for the two-piece Student's $t(2)$ distribution with $L_1 = 0$ and $L_2 = 1$, for varying values of $\alpha > 0$.

Quantile measures of distributional form

The quantile measures of distributional form for location, shape and spread for the two-piece Student's $t(2)$ distribution are obtained by substituting Eq.(4.8) into Eqs.(3.46-3.49), respectively.

- **Location**

  The median is obtained as

$$\begin{aligned} me &= Q_T\left(\frac{1}{2}\right) \\ &= \mu + \sigma\left(\frac{2\cdot\frac{1}{2} - 1}{(2\cdot\frac{1}{2}(1-\frac{1}{2}))^{\frac{1}{2}}}\right) \\ &= \mu. \end{aligned}$$

- **Spread**

  The spread function is derived as

$$\begin{aligned} S_T(s) &= Q_T(s) - Q_T(1-s) \\ &= \left\{\mu + \sigma\left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}}\right)\right\} - \left\{\mu + \alpha\sigma\left(\frac{(2(1-s)-1}{(2(1-s)(1-(1-s)))^{\frac{1}{2}}}\right)\right\} \\ &= \sigma\left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}}\right) + \alpha\sigma\left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}}\right) \\ &= \sigma(1+\alpha)\left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}}\right), \qquad \frac{1}{2} < s < 1. \end{aligned}$$

- **Shape**

The γ-functional is obtained as

$$
\begin{aligned}
\gamma_T(s) &= \frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)} \\
&= \frac{\mu + \sigma\left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}}\right) + \mu + \alpha\sigma\left(\frac{2(1-s)-1}{(2(1-s)(1-(1-s)))^{\frac{1}{2}}}\right) - 2\mu}{\sigma(1+\alpha)\left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}}\right)} \\
&= \frac{1-\alpha}{1+\alpha}, \qquad \frac{1}{2} < s < 1.
\end{aligned}
$$

- **Ratio-of-spread functions**

The ratio-of-spread functions, for $\frac{1}{2} < v < u < 1$, is derived as

$$
R_T(u,v) = \frac{Q_T(u)}{Q_T(v)} = \frac{\left(\frac{2u-1}{(2u(1-u))^{\frac{1}{2}}}\right)}{\left(\frac{2v-1}{(2v(1-v))^{\frac{1}{2}}}\right)} = \frac{(2u-1)\cdot(2v(1-v))^{\frac{1}{2}}}{(2v-1)\cdot(2u(1-u))^{\frac{1}{2}}}.
$$

r^{th} order L-moments

Theorem 4.2.4. *The first four L-moments for the standard half Student's $t(2)$ distribution are*

$$
\begin{aligned}
L_{Z:1} &= -\sqrt{2}, \\
L_{Z:2} &= \frac{\pi}{2\sqrt{2}}, \\
L_{Z:3} &= -\frac{1}{\sqrt{2}}, \\
&\textit{and} \\
L_{Z:4} &= \frac{3\pi}{16\sqrt{2}}, \qquad (4.9)
\end{aligned}
$$

respectively.

Proof. See Section 4.9.1 for detailed proofs. ∎

Theorem 4.2.5. *The first four L-moments for the standard two-piece Student's $t(2)$ distribution* *are given as*

$$L_{T:1} = \frac{2}{\pi}(1-\alpha),$$
$$L_{T:2} = \frac{1}{2}(1+\alpha),$$
$$L_{T:3} = \frac{1}{\pi}(1-\alpha),$$
and
$$L_{T:4} = \frac{3}{16}(1+\alpha), \tag{4.10}$$

respectively.

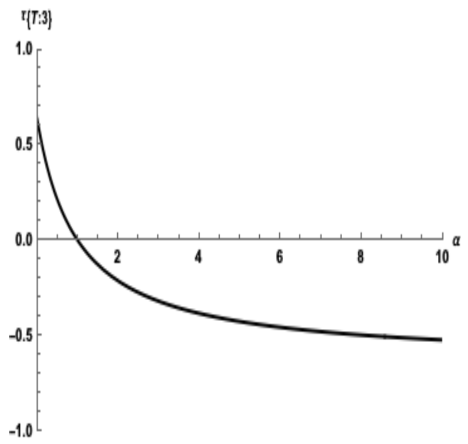
Proof. See Section 4.9.1 for detailed proofs. ■

It can be noted that the results above are simplistic in nature, keeping in line with the advantage of the Student's $t(2)$ L-moments functions which are simple in form. The resulting L-skewness and L-kurtosis ratios are

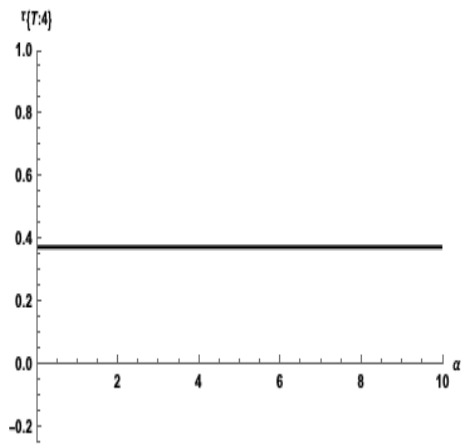
$$\tau_{T:3} = \frac{L_{T:3}}{L_{T:2}} = \frac{2}{\pi}\frac{(1-\alpha)}{(1+\alpha)} = 0.63662\frac{(1-\alpha)}{(1+\alpha)} \quad \text{and} \quad \tau_{T:4} = \frac{L_{T:4}}{L_{T:2}} = \frac{3}{8} = 0.375,$$

respectively.

Fig.4.2(a) shows the skewness range of the two-piece Student's $t(2)$ distribution which is $(-\frac{2}{\pi}; \frac{2}{\pi})$ for $\alpha > 0$, whereas Fig.4.2(b) shows a constant level of kurtosis for all values of α at $\tau_{T:4} = \frac{3}{8}$.



(a) L-skewness ratio plot



(b) L-kurtosis ratio plot

Figure 4.2: The L-skewness and L-kurtosis ratio plots for the two-piece Student's $t(2)$ distribution.

The space covered by the two-piece Student's $t(2)$ distribution is indicated on the L-moment ratio diagram in Fig.4.3 by the horizontal line. The symmetric Student's $t(2)$ distribution is obtained at $(\tau_{T:3}, \tau_{T:4})=(0, \frac{3}{8})$ when $\alpha = 1$.

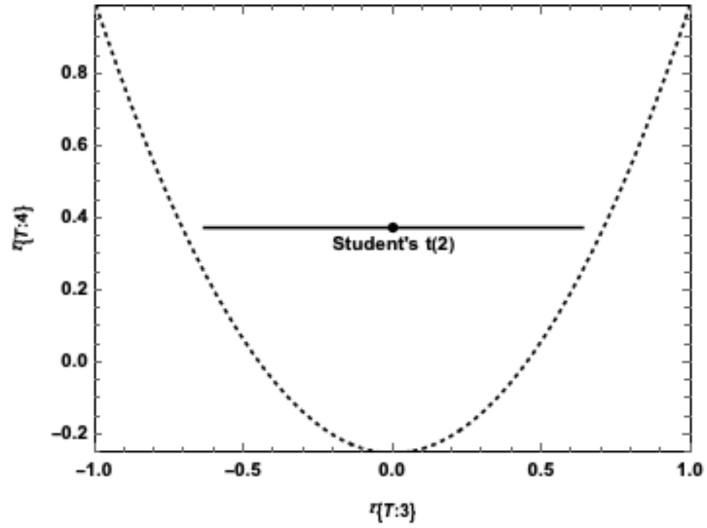


Figure 4.3: The L-moment ratio diagram for the two-piece Student's $t(2)$ distribution.

The dotted curve at $\tau_{T:4} = \frac{1}{4}(5\tau_{T:3}^2 - 1)$ is the lower boundary for all probability distributions.

4.3 Two-piece Hyperbolic Secant Distribution

The hyperbolic secant distribution (HSD) is a heavy-tailed, bell-shaped distribution that was first studied by Baten (1934) and later on by Talacko (1956). It is a generator distribution of the sixth natural exponential family (Morris (1982)), and it possesses a quadratic variance function (NEF-QVF). This means that the variance is at most a quadratic function of the mean. The HSD does not receive the same amount of attention as the other symmetric distributions due to its lack of connectivity to other commonly known distributions.

Perks (1932) derived a family of generalized HSD which could fit the data in mortality statistics. Talacko (1956) derived the properties of the HSD, as well as showed its role in the theory of Wiener's stochastic function. Since the CDF exhibits a closed-form, it can be used extensively in the financial sector to obtain option prices quickly and precisely. Furthermore, Vaughan (2002) presented two studies which revealed the HSD fitted data better than the normal distribution due to its heavier tails.

The solid curve in Fig.4.4 indicates that the HSD exhibits heavier tails than both the normal (dashed curve) and logistic (dot-dashed curve) distributions.

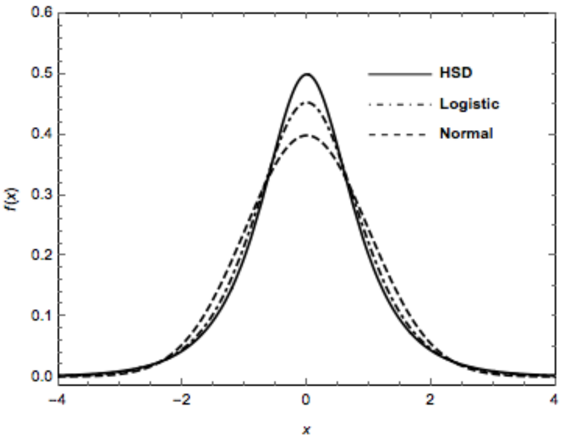


Figure 4.4: The probability density curves of the HSD, logistic and normal distribution.

Definition 4.3.0.1. *A real-valued random variable X is said to have a standard hyperbolic secant distribution, denoted $X \sim HSD(0,1)$, if it's CDF, PDF and quantile function are respectively defined as*

$$F_X(x) = \frac{2}{\pi}\arctan(e^x), \qquad x \in (-\infty, \infty), \tag{4.11}$$

$$f_X(x) = \frac{2}{\pi}\frac{e^x}{1+e^{2x}}, \qquad x \in (-\infty, \infty), \tag{4.12}$$

and

$$Q(p) = \log\left(\tan\left(\frac{\pi p}{2}\right)\right), \qquad p \in (0,1). \tag{4.13}$$

■

4.3.1 Distributional properties of the HSD

Moments

The moment generating function of the HSD is given as

$$M_X(t) = \sec(t), \qquad |t| < \frac{\pi}{2}. \tag{4.14}$$

The mean and variance follow from Eq.(4.14) as

$$\mu = E(X) = M_X'(0) = 0 \text{ and } \sigma = E(X)^2 - (E(X))^2 = M_X''(0) - (M_X'(0))^2 = 1,$$

respectively.

Shape

The probability density curve of the HSD is unimodal and symmetric around 0. The skewness moment-ratio $\alpha_3 = 0$ and kurtosis-moment ratio is $\alpha_4 = 5$ Since $\alpha_4 > 3$, it indicates the HSD is more leptokurtic than the normal distribution .

HSD and other distributions

- The HSD arises from the Cauchy distribution or the ratio of two independent Gaussian distributions.
- Vaughan (2002) studied a symmetric family of distributions with varying levels of kurtosis ranging from 1 to ∞. They include thick and thin-tailed members, expanding the versatility of their use in modelling various data. Moreover, all the moments of these distributions are finite.
- Esscher's transformation, by Esscher (1932), was applied to Vaughan (2002)'s generalized secant hyperbolic (GSH) distribution, giving rise to the skew generalized secant distribution (SGSH) which was proposed by Fischer (2006).
- The sin-arcsinh (SAS) transformation of Jones and Pewsey (2009) was used to develop asymmetric families of distributions that have the HSD as a special case.

Order statistics and r^{th} order L-moments

Definition 4.3.1.1. *Suppose $X_{i:n},\ i = 1, ..., n$ is the i^{th} order statistic of a HSD random variable from a sample of size n. The probability density function of $X_{i:n}$ is defined as*

$$f_{X_{i:n}}(x) = \frac{1}{B(i, n-i+1)2^{n+1/2}} \left(1 + \frac{x}{\sqrt{2+x^2}}\right)^{i+1/2} \left(1 - \frac{x}{\sqrt{2+x^2}}\right)^{n-i+3/2} \tag{4.15}$$

■

Lemma 4.3.1. *Suppose X is characterized by the functions in Eqs.(4.11-4.13). The r^{th}-order*

L-moments are

$$L_{X:r} = 0 \qquad \textit{for odd values of } r,$$
$$L_{X:2} = \frac{7\zeta[3]}{\pi^2},$$
$$\textit{and}$$
$$L_{X:4} = \frac{42\pi^2\zeta[3] - 465\zeta[5]}{\pi^4}, \tag{4.16}$$

with the L-skewness and L-kurtosis ratios as

$$\tau_{X:3} = 0 \quad \textit{and} \quad \tau_{X:4} = \frac{42\pi^2\zeta[3] - 465\zeta[5]}{\pi^2 7\zeta[3]} = 0.193977,$$

respectively.

The standard HSD is obtained when $\mu = 0$ and $\sigma = \frac{\pi^2}{7\zeta[3]}$.

■

4.3.2 Proposed two-piece hyperbolic secant distribution

Definition 4.3.2.1. *A real-valued random variable is said to have a two-piece hyperbolic secant distribution if its quantile function, CDF and PDF are defined as*

$$Q_T(s) = \begin{cases} \mu + \sigma\alpha\log(\tan(\frac{\pi s}{2})), & s \le \frac{1}{2}, \\ \mu + \sigma\log(\tan(\frac{\pi s}{2})), & s > \frac{1}{2}, \end{cases} \tag{4.17}$$

$$F_T(x) = \begin{cases} \frac{2}{\pi}\arctan(e^{\frac{x-\mu}{\alpha\sigma}}), & x \le \mu, \\ \frac{2}{\pi}\arctan(e^{\frac{x-\mu}{\sigma}}), & x > \mu, \end{cases} \tag{4.18}$$

and

$$f_T(x) = \begin{cases} \frac{2}{\pi\alpha\sigma}\frac{e^{(\frac{x-\mu}{\alpha\sigma})}}{1+e^{2(\frac{x-\mu}{\alpha\sigma})}}, & x \le \mu, \\ \frac{2}{\pi\sigma}\frac{e^{(\frac{x-\mu}{\sigma})}}{1+e^{2(\frac{x-\mu}{\sigma})}}, & x > \mu, \end{cases} \tag{4.19}$$

respectively.

■

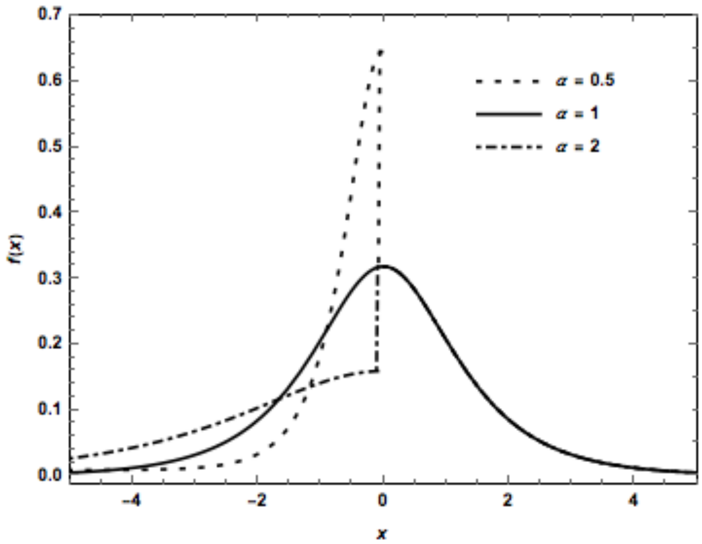


Figure 4.5: The probability density curves for the two-piece hyperbolic secant distribution with $L_1 = 0$ and $L_2 = 1$, for varying values of $\alpha > 0$.

The probability density curves for the two-piece distribution with varying values of $\alpha > 0$ is displayed in Fig.4.5. When $\alpha < 1$, the two-piece HSD exhibits positive skewness as depicted by the dashed-curve. In this case, the values of $\tau_{T:3}$ and $\gamma_T(s)$ are positive. The distribution is negatively skewed when $\alpha > 1$, as shown by the dot-dashed curve, with the corresponding values for $\tau_{T:3}$ and $\gamma_T(s)$ negative.

The symmetric HSD, when $\alpha = 1$, is represented by the solid curve. It is obtained and its values for the L-skewness ratio and the $\gamma_T(s)$ are zero.

Quantile measures of distributional form

The quantile measures of distributional form for location, spread and shape for the two-piece HSD are obtained by substituting Eq.(4.17) into Eqs.(3.46-3.49), respectively.

- **Location**

The median is obtained as

$$\begin{aligned} me &= Q_T\Big(\frac{1}{2}\Big) \\ &= \mu + \sigma \log\Big(\tan\Big(\frac{\pi}{4}\Big)\Big) \\ &= \mu + \sigma \log(1) \\ &= \mu. \end{aligned}$$

- **Spread**

The spread function is

$$\begin{aligned}
S_T(s) &= Q_T(s) - Q_T(1-s) \\
&= \Big\{\mu + \sigma \log\Big(\tan\Big(\frac{\pi s}{2}\Big)\Big)\Big\} - \Big\{\mu + \alpha\sigma \log\Big(\tan\Big(\frac{\pi(1-s)}{2}\Big)\Big)\Big\} \\
&= \sigma \log\Big(\tan\Big(\frac{\pi s}{2}\Big)\Big) + \alpha\sigma \log\Big(\tan\Big(\frac{\pi s}{2}\Big)\Big) \\
&= \sigma(1+\alpha) \log\Big(\tan\Big(\frac{\pi s}{2}\Big)\Big), \qquad \frac{1}{2} < s < 1.
\end{aligned}$$

- **Shape**

Through the substitution of Eq.(4.17) into Eq.(3.48), the γ-functional is attained as

$$\begin{aligned}
\gamma_T(s) &= \frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)} \\
&= \frac{\mu + \sigma \log\Big(\tan\Big(\frac{\pi s}{2}\Big)\Big) + \mu + \alpha\sigma \log\Big(\tan\Big(\frac{\pi(1-s)}{2}\Big)\Big) - 2\mu}{\sigma(1+\alpha)\log\Big(\tan\Big(\frac{\pi s}{2}\Big)\Big)} \\
&= \frac{\sigma \log\Big(\tan\Big(\frac{\pi s}{2}\Big)\Big) - \alpha\sigma \log\Big(\tan\Big(\frac{\pi s}{2}\Big)\Big)}{\sigma(1+\alpha)\log\Big(\tan\Big(\frac{\pi s}{2}\Big)\Big)} \\
&= \frac{1-\alpha}{1+\alpha}, \qquad \frac{1}{2} < s < 1.
\end{aligned}$$

- **Ratio-of-spread functions**

The ratio-of-spread functions is given as

$$R_T(u,v) = \frac{S(u)}{S(v)} = \frac{\sigma(1+\alpha)\log\Big(\tan(\frac{\pi u}{2})\Big)}{\sigma(1+\alpha)\log\Big(\tan(\frac{\pi v}{2})\Big)} = \frac{\log\Big(\tan(\frac{\pi u}{2})\Big)}{\log\Big(\tan(\frac{\pi v}{2})\Big)}, \qquad \frac{1}{2} < v < u < 1.$$

Order statistics and r^{th} order L-moments

Theorem 4.3.2. *The first four L-moments of a standard half HSD random variable are obtained as*

$$\begin{aligned}
L_{Z:1} &= -\frac{4}{\pi}\ \boldsymbol{G}, \\
L_{Z:2} &= \frac{7\zeta(3)}{\pi^2}, \\
L_{Z:3} &= \frac{1}{32\pi^3}\left(64\boldsymbol{G}\pi^2 + \psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right) - 672\zeta(3)\pi\right)
\end{aligned}$$

and

$$L_{Z:4} = \frac{1}{\pi^4}\left(42\zeta(3)\pi^2 - 465\zeta(5)\right). \tag{4.20}$$

respectively.

Proof. See Section 4.9.2 for detailed proofs. ■

Theorem 4.3.3. *The first four* L*-moments of a two-piece HSD random variable are obtained as*

$$L_{T:1} = \frac{2\pi}{7\zeta(3)}\boldsymbol{G}(1-\alpha),$$
$$L_{T:2} = \frac{1}{2}(1+\alpha),$$
$$L_{T:3} = \frac{1}{448\zeta(3)\pi}\left(64\boldsymbol{G}\pi^2 + \psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right) - 672\zeta(3)\pi\right)(1-\alpha),$$

and

$$L_{T:4} = \frac{1}{14\zeta(3)\pi^2}\left(42\zeta(3)\pi^2 - 465\zeta(5)\right)(1+\alpha), \tag{4.21}$$

respectively.

Proof. See Section 4.9.2 for detailed proofs. ■

Subsequently, the L-skewness and L-kurtosis ratios are

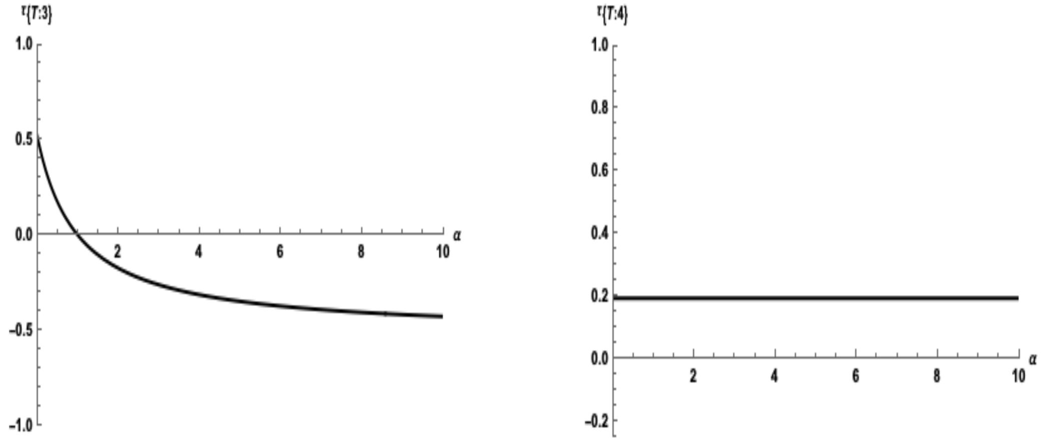
$$\tau_{T:3} = \frac{L_{T:3}}{L_{T:2}} = \frac{1}{224\zeta(3)\pi}\left(64\boldsymbol{G}\pi^2 + \psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right) - 672\zeta(3)\pi\right)\frac{(1-\alpha)}{(1+\alpha)} = 0.520306\frac{(1-\alpha)}{(1+\alpha)}$$

and

$$\tau_{T:4} = \frac{L_{T:4}}{L_{T:2}} = \frac{1}{7\zeta(3)\pi^2}\left(42\zeta(3)\pi^2 - 465\zeta(5)\right) = 0.193977,$$

respectively.

The graphs depicting the L-skewness and L-kurtosis ratios are represented in Fig.4.6. It can be noted from Fig.4.6(a) that $\tau_{T:3}$ is bounded on $(-0.52; 0.52)$ for $\alpha > 0$, whilst $\tau_{T:4}$ is constant for $\alpha > 0$ at 0.193977. The L-moment ratio diagram, represented by Fig 4.7, shows the extended level of skewness that is possessed by the new distribution. The level of kurtosis remains constant at $\tau_{T:4} = 0.193977$



(a) The L-skewness ratio diagram

(b) The L-kurtosis ratio diagram

Figure 4.6: The L-skewness and L-kurtosis ratio plots for the two-piece HSD.

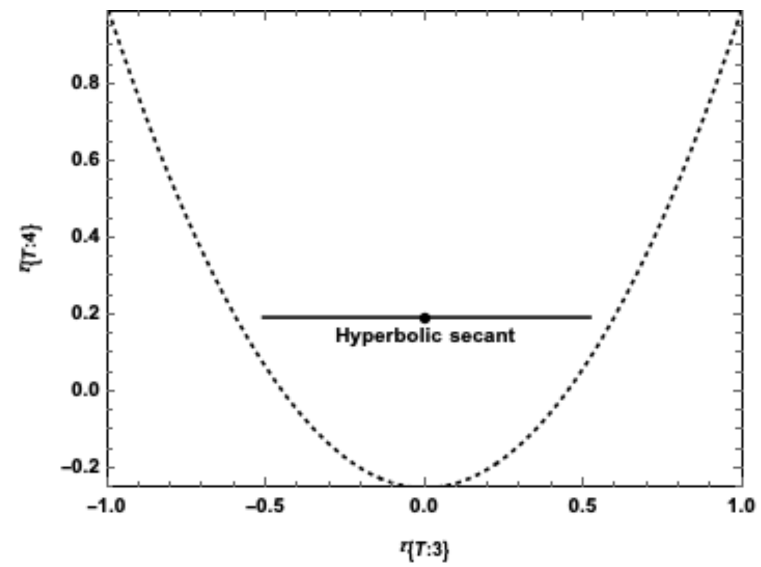


Figure 4.7: The L-moment ratio diagram for the two-piece hyperbolic secant distribution.

4.4 Two-piece Logistic Distribution

The logistic function was first proposed in population demographic modeling where Verhulst (1838) showed that the population growth rate is a function of the transient population and a proportion of the resources available. When those properties are incorporated into the growth model, the population exhibits sigmoid growth patterns. The logistic function has been used in population growth modeling (Pearl and Reed (1920)), survival data analysis (Plackett (1959)) as well as income distribution studies (Fisk (1961)).

Definition 4.4.0.1. *A real-valued random variable X is said to have a logistic distribution, denoted by $X\sim L(\mu,\sigma)$, if its CDF, PDF, quantile function are given as*

$$F_X(x) = \frac{e^{(\frac{x-\mu}{\sigma})}}{1+e^{(\frac{x-\mu}{\sigma})}}, \qquad -\infty < x < \infty, \tag{4.22}$$

$$f_X(x) = \frac{e^{\frac{x-\mu}{\sigma}}}{\sigma(1+e^{\frac{x-\mu}{\sigma}})^2}, \qquad -\infty < x < \infty, \tag{4.23}$$

and

$$Q_X(p) = \mu + \sigma \log\left(\frac{p}{1-p}\right), \qquad 0 < p < 1, \tag{4.24}$$

respectively, where $-\infty < \mu < \infty$ and $\sigma > 0$ are the location and scale parameters, respectively.

Since Eq.(4.23) can be rewritten as $f(x) = \frac{1}{4}\text{sech}^2\left(\frac{x}{2}\right)$, the logistic distribution can also be referred to as the sech-squared distribution.

4.4.1 Distributional properties of the logistic distribution

Moments

The moment generating function of the logistic distribution is defined as

$$M_X(t) = \mathrm{e}^{\mu t}\mathrm{B}(1-\sigma t)(1+\sigma t), \quad |t| < \frac{1}{\sigma}. \tag{4.25}$$

The mean and variance can be derived from Eq.(4.25) as

$$\mu = E(X) = M_X'(0) = 0 \text{ and } \sigma = E(X)^2 - (E(X))^2 = M_X''(0) - (M_X'(0))^2 = \frac{\sigma^3\pi^2}{3},$$

Shape

The logistic distribution is symmetric about $-\infty < \mu < \infty$, exhibiting a unimodal shape. The skewness moment-ratio, $\alpha_3 = 0$, whilst the kurtosis moment-ratio $\alpha_4 = 4.2$. This shows that the logistic distribution is more leptokurtic than the normal distribution since $\alpha_4 > 3$.

Logistic and other distributions

The logistic distribution has shown its relationship to other univariate continuous distributions.

- If $X \sim L(\mu, \sigma)$, then $kX + d \sim L(k\mu + d, k\sigma)$.
- If $X \sim UNIF(0, 1)$ then $\mu + \sigma \frac{\log(X)}{\log(1-X)} \sim L(\mu, \sigma)$.
- Let $X \sim \exp(1)$. Then $\mu + \sigma \log\left(\exp^X - 1\right) \sim L(\mu, \sigma)$.
- If $X, Y \sim \exp(1)$, then $\mu - \sigma \log\left(\frac{X}{Y}\right) \sim L(\mu, \sigma)$.
- The logistic distribution closely approximates the Student's t-distribution with 9 degrees of freedom (Mudholkar and George (1978)).
- The logistic distribution is a special case of both Tukey's lambda distribution and the generalised hyperbolic secant family of distributions (Perks (1932)).

r^{th} order L-moments

Lemma 4.4.1. *The r^{th} order L-moments, $L_{X:r}$, for a standard random variable X from the logistic distribution are presented by Hosking (1986) as*

$$L_{X:r} = \begin{cases} 0, & \text{for odd values of } r \\ \frac{2}{r(r-1)}, & \text{for even values of } r. \end{cases} \tag{4.26}$$

with the L-skewness and L-kurtosis ratios as

$$\tau_{X:3} = 0 \quad \text{and} \quad \tau_{X:4} = \frac{1}{6},$$

respectively.

The standard logistic distribution is obtained when $\mu = 0$ and $\sigma = 1$.

■

4.4.2 Proposed two-piece logistic distribution

Definition 4.4.2.1. *A real-valued random variable is said to have a two-piece logistic distribution if its quantile function, CDF and PDF are defined as*

$$Q_T(s) = \begin{cases} \mu + \alpha\sigma \log(\frac{s}{1-s}), & s \leq \frac{1}{2}, \\ \mu + \sigma \log(\frac{s}{1-s}), & s > \frac{1}{2}, \end{cases} \tag{4.27}$$

$$F_T(X) = \begin{cases} \frac{e^{(\frac{x-\mu}{\alpha\sigma})}}{1+e^{(\frac{x-\mu}{\alpha\sigma})}}, & x \leq \mu \\ \frac{e^{(\frac{x-\mu}{\sigma})}}{1+e^{(\frac{x-\mu}{\sigma})}}, & x > \mu \end{cases} \tag{4.28}$$

and

$$f_T(X) = \begin{cases} \frac{e^{\frac{x-\mu}{\alpha\sigma}}}{\alpha\sigma(1+e^{\frac{x-\mu}{\alpha\sigma}})^2}, & x \leq \mu, \\ \frac{e^{\frac{x-\mu}{\sigma}}}{\sigma(1+e^{\frac{x-\mu}{\sigma}})^2}, & x > \mu \end{cases} \tag{4.29}$$

respectively.

■

Fig.4.8 displays the probability density curves for the two-piece logistic distribution with varying values of $\alpha > 0$. When $\alpha < 1$, the distribution is positively skewed as illustrated by the dashed-curve, negatively skewed when $\alpha > 1$, as shown by the dotted curve, and symmetric as represented by the solid curve when $\alpha = 1$.

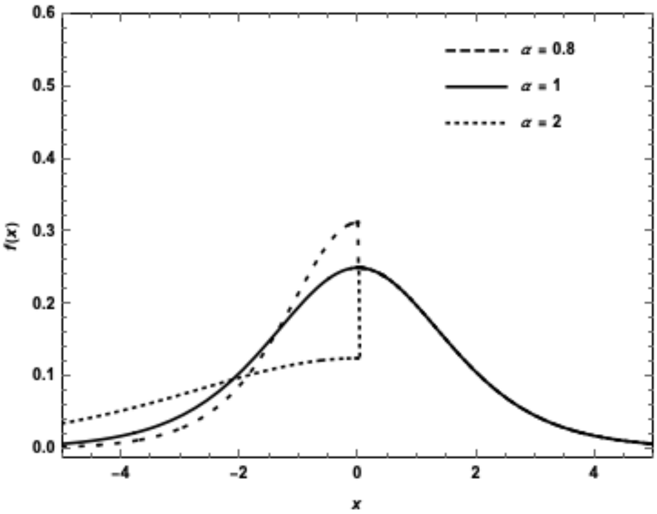


Figure 4.8: The probability density curves for the two-piece logistic distribution with $L_1 = 0$ and $L_2 = 1$, for varying values of $\alpha > 0$.

Quantile measures of distributional form

The quantile measures of distributional form for location, spread and shape for the two-piece logistic distribution are obtained by substituting Eq.(4.27) into Eqs.(3.46-3.49), respectively.

- **Location**

  The median is obtained as

$$\begin{aligned}
me &= Q_T\left(\frac{1}{2}\right) \\
&= \mu + \sigma \log\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right) \\
&= \mu + \sigma \log(1) \\
&= \mu
\end{aligned}$$

- **Spread**

  The spread function is

$$\begin{aligned}
S_T(s) &= Q_T(s) - Q_T(1-s) \\
&= \left\{\mu + \sigma \log\left(\frac{s}{1-s}\right)\right\} - \left\{\mu + \alpha\sigma \log\left(\frac{1-s}{1-(1-s)}\right)\right\} \\
&= \sigma(1+\alpha)\log\left(\frac{s}{1-s}\right), \qquad \frac{1}{2} < s < 1.
\end{aligned}$$

- **Shape**

  The γ-functional is obtained by substituting Eq.(4.27) into Eq.(3.48) such that

$$\begin{aligned}
\gamma_T(s) &= \frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)} \\
&= \frac{\mu + \sigma \log\left(\frac{s}{1-s}\right) + \mu + \alpha\sigma \log\left(\frac{1-s}{1-(1-s)}\right) - 2\mu}{\sigma(1+\alpha)\log\left(\frac{s}{1-s}\right)} \\
&= \frac{\sigma \log\left(\frac{s}{1-s}\right) - \alpha\sigma \log\left(\frac{s}{1-s}\right)}{\sigma(1+\alpha)\log\left(\frac{s}{1-s}\right)} \\
&= \frac{1-\alpha}{1+\alpha}, \qquad \frac{1}{2} < v < u < 1.
\end{aligned}$$

- **Ratio-of-spread functions**

  The ratio-of-spread functions is given as

$$R_T(u,v) = \frac{S_T(u)}{S_T(v)} = \frac{\sigma(1+\alpha)\log\left(\frac{u}{1-u}\right)}{\sigma(1+\alpha)\log\left(\frac{v}{1-v}\right)} = \frac{\log\left(\frac{u}{1-u}\right)}{\log\left(\frac{v}{1-v}\right)},$$

where $\frac{1}{2} < v < u < 1$. Note that, akin to the L-kurtosis ratio, the ratio-of-spread functions is skewness-invariant with respect to α.

r^{th} **order L-moments**

Theorem 4.4.2. *The first four L-moments of a standard half logistic random variable are given as*

$$\begin{aligned} L_{Z:1} &= -2\log(2), \\ L_{Z:2} &= 1, \\ L_{Z:3} &= -\frac{1}{2}, \\ &\textit{and} \\ L_{Z:4} &= \frac{1}{6}, \end{aligned} \tag{4.30}$$

respectively.

Proof. See Section 4.9.3 for detailed proofs. ∎

Theorem 4.4.3. *The first four L-moments of a standard two-piece logistic random variable are given as*

$$\begin{aligned} L_{T:1} &= \log(2)(1-\alpha) \\ L_{T:2} &= \frac{1}{2}(1+\alpha) \\ L_{T:3} &= \frac{1}{4}(1-\alpha) \\ &\textit{and} \\ L_{T:4} &= \frac{1}{12}(1+\alpha). \end{aligned} \tag{4.31}$$

respectively.

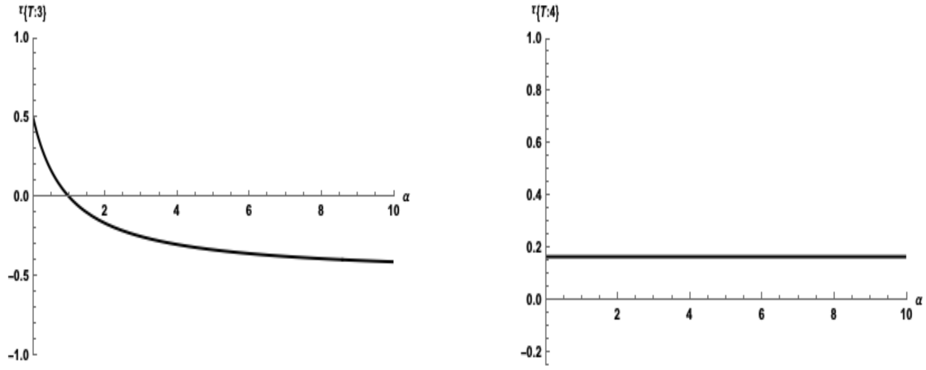
Proof. See Section 4.9.3 for detailed proofs. ∎

Therefore, the L-skewness and L-kurtosis ratio ratios are

$$\tau_{T:3} = \frac{L_{T:3}}{L_{T:2}} = \frac{1}{2}\frac{(1-\alpha)}{(1+\alpha)} \quad \text{and} \quad \tau_{T:4} = \frac{L_{T:4}}{L_{T:2}} = \frac{1}{6},$$

respectively. These expressions for the L-moments and L-moment ratios correspond to those obtained by Balakrishnan et al. (2017), who made use of expectations of order statistics. The value of $\tau_{T:4}$ is equivalent to the L-kurtosis ratio of the logistic distribution. The special case of the two-piece logistic is the logistic distribution which is obtained when $\alpha = 1$.

In Fig.4.9(a), it can be seen that $0 < \tau_{T:3} < 0.5$ when $\alpha < 1$, and it tapers off at -0.5 as α tends to ∞. Fig.4.9(b) indicates that the level of kurtosis which is depicted by $\tau_{T:4}$ is a constant at $\frac{1}{6}$.



(a) The L-skewness ratio plot **(b)** The L-kurtosis ratio plot

Figure 4.9: The L-skewness and L-kurtosis ratio plots for the two-piece logistic distribution.

The L-skewness and L-kurtosis ratio values are used to obtain an L-moment ratio diagram, represented by Fig. 4.10, will show the increased level of skewness that is acquired by the new distribution, as compared to the logistic distribution, for a given level of kurtosis. The logistic distribution is obtained as a special case when $(\tau_{T:3}, \tau_{T:4}) = (0, \frac{1}{6})$. The dotted curve is the boundary for all distributions.

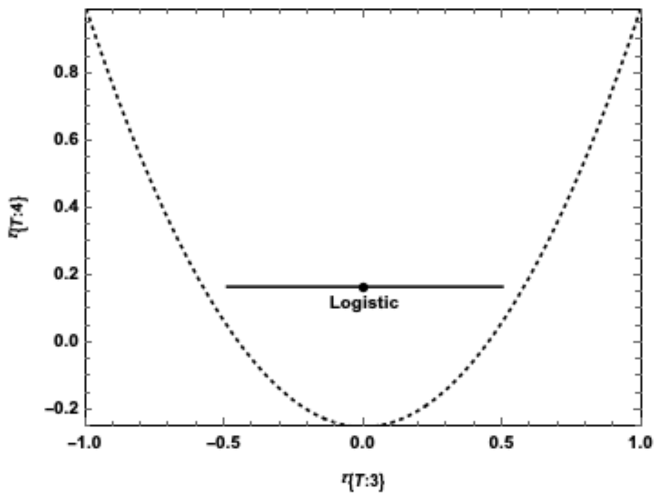


Figure 4.10: The L-moment ratio diagram for the two-piece logistic distribution with $\alpha > 0$.

4.5 Two-piece Normal Distribution

The discovery of the normal distribution, also referred to as the bell-curve or the Gaussian distribution, was first attributed to De Moivre during his early studies on the coefficients of the binomial expansion in 1738. The distribution was introduced by Gauss (1809) whilst presenting the concepts of the method of maximum likelihood and the method of least squares. In addition, Laplace proved the central limit theorem in 1810 which further solidified the importance of the normal distribution and its applications. The normal distribution has been widely used in natural and social sciences, where the underlying distribution of data may be unknown.

Definition 4.5.0.1. *Let X be a real-valued random variable from the normal distribution, denoted $X \sim N(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma > 0$ are the location and spread parameters, respectively. The CDF, PDF and quantile function are defined as*

$$F_X(x) = \frac{1}{2}\left(1 + \text{erf}^{-1}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right), \qquad (-\infty < x < \infty), \tag{4.32}$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \qquad (-\infty < x < \infty)$$

and

$$Q_X(p) = \mu + \sigma\sqrt{2}\text{erf}^{-1}(2p-1), \qquad 0 < p < 1, \tag{4.33}$$

respectively.

■

4.5.1 Distributional properties of the normal distribution

Symmetry

- The normal distribution is symmetric about the location parameter, μ, which is the mean. The mean is equivalent to the median and the mode.
- The probability density curve exhibits unimodality and has two inflection points at $|x-\mu| = \sigma$. It is also log-concave.

Moments

The moment generating function of the normal distribution is given as

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}, \qquad t \in \mathbb{R}. \tag{4.34}$$

All the moments for the normal distribution exist and are finite. Therefore, the mean and variance can be found by taking the first and second derivatives of Eq.(4.34) and setting $t = 0$, such that

$$E(X) = M_X'(0) = \mu \qquad \text{and} \qquad Var(X) = M_X''(0) - (M_X'(0))^2 = \sigma^2.$$

The central moments about the mean are given as

$$\mu_k = E((X-\mu)^k) = \begin{cases} 0 & \text{if k is odd} \\ \sigma^k (k-1)!! & \text{if k is even.} \end{cases} \tag{4.35}$$

From Eq.(4.35), the skewness and kurtosis moment-ratios can be obtained as $\alpha_3 = 0$ and $\alpha_4 = 3$, respectively.

Maximum entropy

The normal distribution has maximum entropy over all probability distributions in its class with known mean and variance. It is obtained as

$$H(X) = \frac{1}{2}\left(1 + \log\left(2\sigma^2\pi\right)\right).$$

Infinite divisibility

A random variable from the normal distribution, with mean μ and variance σ^2, can be seen as the sum of n random variables with mean $\frac{\mu}{n}$ and variance $\frac{\sigma^2}{n}$, so long as $n \in \mathbb{Z}^+$. This implies that the normal distribution is infinitely divisible.

Central limit theorem

Suppose that $X_1, X_2, ..., X_n$ is a sample of i.i.d. random variables from the same distribution, with known mean, μ, and variance σ^2. The distribution of the sample mean scaled by $\sqrt{n}$ is approximately normally distributed as the sample size becomes larger. This property allows for other distributions to be approximated by the normal distribution, as presented in Table 4.1.

Distribution	Parameter(s)	Normal approximation
Binomial $Bin(n, p)$	n, p	$\mu = np \qquad \sigma^2 = np(1-p)$ $n \to \infty$
Poisson $Poi(\lambda)$	λ	$\mu = \lambda \qquad \sigma^2 = \lambda$ $\lambda \to \infty$
Chi-Square χ^2_k	k	$\mu = k \qquad \sigma^2 = 2k$ $k \to \infty$
Student's t $t(\nu)$	ν	$\mu = 0 \qquad \sigma^2 = 1$ $\nu \to \infty$

Table 4.1: Univariate distributions approximated by the normal distribution.

Normal and other distributions

If $X \sim N\left(\mu, \sigma^2\right)$, then:

- $e^X \sim LN\left(\mu, \sigma^2\right)$.
- $|X| \sim N_f\left(\mu, \sigma^2\right)$. If $\mu = 0$, then $|X|$ has a half normal distribution.
- $\frac{|X-\mu|}{\sigma} \sim \chi^2_1$.
- $\left(\frac{X}{\sigma}\right)^2 \sim \chi^2_1\left(\frac{\mu^2}{\sigma^2}\right)$. It reduces to χ^2_1 when $\mu = 0$.
- X has a truncated normal distribution if it bounded on the interval $[a, b]$.

If $X_1, X_2 \sim N(0, 1)$, then:

- $\frac{X_1}{X_2} \sim Cauchy\,(0, 1)$.
- $X_1 \pm X_2 \sim N(0, 2)$.
- $\sum_{i=1}^{n} X_i \sim \chi^2_n$.

If $X_1, ..., X_n \sim N(0, 1)$ and $Y_1, ..., Y_n \sim N(0, 1)$, then $F = \frac{\sum_{i=1}^{n} X_i^2/n}{\sum_{i=1}^{m} Y_i^2/m} \sim F_{n,m}$.

r^{th} **order L-moments**

Lemma 4.5.1. *The first four corresponding L-moments for a standard normal random variable are*

$$L_{X:1} = 0, \quad L_{X:2} = \frac{1}{\sqrt{\pi}}, \quad L_{X:3} = 0, \quad \text{and} \quad L_{X:4} = \frac{1}{\sqrt{\pi}}\left(\frac{30}{\pi}\arctan(\sqrt{2}) - 9\right) \tag{4.36}$$

The L-skewness and L-kurtosis ratios are

$$\tau_3 = 0 \qquad \textit{and} \qquad \tau_4 = \frac{30}{\pi}\arctan(\sqrt{2}) - 9, \tag{4.37}$$

respectively.

The standard normal distribution is obtained when $\mu = 0$ and $\sigma = \sqrt{2}$.

■

4.5.2 Proposed two-piece normal distribution

Definition 4.5.2.1. *A random variable is said to have a two-piece normal distribution if its quantile function, PDF and CDF are defined as*

$$Q_T(s) = \begin{cases} \mu + \alpha\sigma\sqrt{2}\mathrm{erf}^{-1}(2s-1), & s \leq \frac{1}{2}, \\ \mu + \sigma\sqrt{2}\mathrm{erf}^{-1}(2s-1), & s > \frac{1}{2}, \end{cases} \tag{4.38}$$

$$f_T(s) = \begin{cases} \frac{1}{\alpha\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\alpha^2\sigma^2}} & x \leq \mu, \\ \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} & x > \mu, \end{cases}$$

and

$$F_T(s) = \begin{cases} \frac{1}{2}\left(1 + \mathrm{erf}^{-1}\left(\frac{x-\mu}{\alpha\sigma\sqrt{2}}\right)\right) & x \leq \mu, \\ \frac{1}{2}\left(1 + \mathrm{erf}^{-1}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right) & x > \mu, \end{cases}$$

respectively.

■

The probability density curves for the two-piece normal distribution, with varying values of $\alpha > 0$, are illustrated in Fig.4.11. The distribution is positively skewed as shown by the dashed-curve when $\alpha < 1$, negatively skewed when $\alpha > 1$ as shown by the dotted curve, and symmetric as represented by the solid curve when $\alpha = 1$.

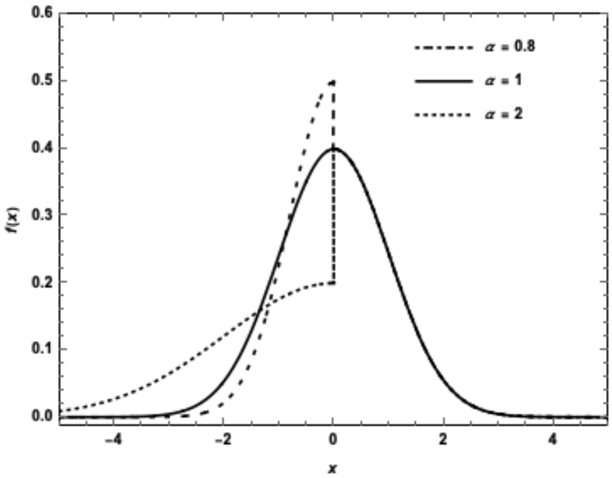


Figure 4.11: The probability density curves for the two-piece normal distribution with $L_1 = 0$ and $L_2 = 1$, for varying values of $\alpha > 0$.

Quantile measures of distributional form

The quantile measures of distributional form for of location, spread and shape for the two-piece normal distribution are obtained by substituting Eq.(4.38) into Eqs.(3.46-3.49), respectively.

- **Location**

  The median is obtained as

$$
\begin{aligned}
me &= Q_T\Big(\frac{1}{2}\Big) \\
&= \mu + \sigma\sqrt{2}\mathrm{erf}^{-1}(2 \cdot \frac{1}{2} - 1)\Big) \\
&= \mu.
\end{aligned}
$$

- **Spread**

  The spread function is derived

$$
\begin{aligned}
S_T(s) &= Q_T(s) - Q_T(1-s) \\
&= \Big\{\mu + \sigma\sqrt{2}\mathrm{erf}^{-1}(2s-1)\Big\} - \Big\{\mu + \alpha\sigma\sqrt{2}\mathrm{erf}^{-1}(2s-1)\Big\} \\
&= \sqrt{2}\sigma(1-\alpha)\Big(\mathrm{erf}^{-1}(2s-1)\Big), \qquad \frac{1}{2} < s < 1.
\end{aligned}
$$

- **Shape**

The shape functional is obtained as

$$\begin{aligned}
\gamma_T(s) &= \frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)} \\
&= \frac{\mu + \sigma\sqrt{2}\mathrm{erf}^{-1}(2s-1) + \mu + \alpha\sigma\sqrt{2}\mathrm{erf}^{-1}(2s-1) - 2\mu}{\sigma(1-\alpha)\mathrm{erf}^{-1}(2s-1)} \\
&= \frac{1+\alpha}{1-\alpha}, \qquad \frac{1}{2} < v < u < 1.
\end{aligned}$$

- **Ratio-of-spread functions**

The ratio-of-spread functions, for $\frac{1}{2} < v < u < 1$, is derived as

$$R_T(u,v) = \frac{Q_T(u)}{Q_T(v)} = \frac{\sqrt{2}\sigma(1-\alpha)\Big(\mathrm{erf}^{-1}(2u-1)\Big)}{\sqrt{2}\sigma(1-\alpha)\Big(\mathrm{erf}^{-1}(2v-1)\Big)} = \frac{\mathrm{erf}^{-1}(2u-1)}{\mathrm{erf}^{-1}(2v-1)}.$$

r^{th} order L-moments

The order statistics from the half normal distribution will be derived and be used to obtain the L-moments of the two-piece normal distribution.

Theorem 4.5.2. *The first four L-moments for the standard half normal distribution are*

$$\begin{aligned}
L_{Z:1} &= -\sqrt{\frac{2}{\pi}} \\
L_{Z:2} &= \frac{1}{\sqrt{\pi}} \\
L_{Z:3} &= 0.264252 \\
L_{Z:4} &= \frac{1}{\sqrt{\pi}}\left(\frac{30}{\pi}\arctan(\sqrt{2}) - 9\right),
\end{aligned} \tag{4.39}$$

respectively.

Proof. See Section 4.9.4 for detailed proofs. ∎

Theorem 4.5.3. *The first four L-moments of the two-piece normal distribution is given by*

$$\begin{aligned}
L_{T:1} &= \frac{1}{\sqrt{2}}(1-\alpha) \\
L_{T:2} &= \frac{1}{2}(1+\alpha) \\
L_{T:3} &= 0.2341872(1-\alpha) \\
L_{T:4} &= \frac{1}{2}\left(\frac{30}{\pi}\arctan(\sqrt{2}) - 9\right)(1+\alpha),
\end{aligned} \tag{4.40}$$

respectively.

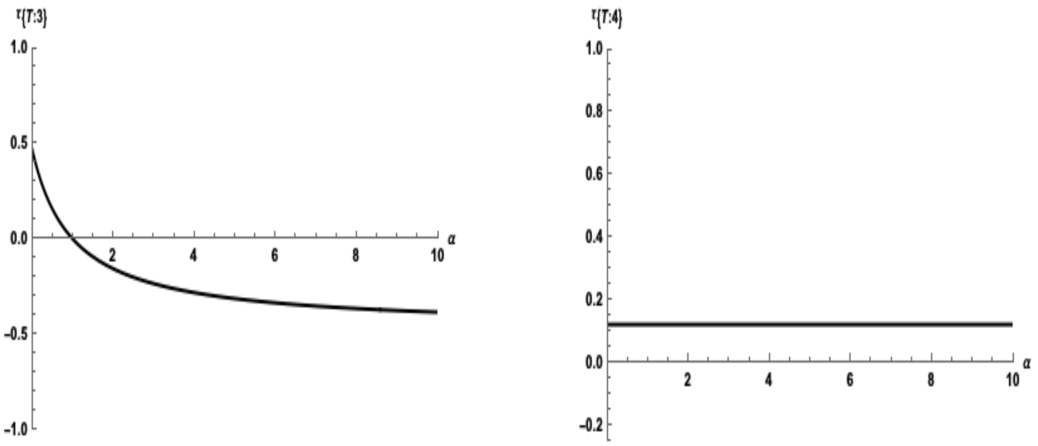
Proof. See Section 4.9.4 for detailed proofs. ■

In effect, the L-skewness and L-kurtosis ratios are given as

$$\tau_{T:3} = 0.468373\frac{(1-\alpha)}{(1+\alpha)} \quad \text{and} \quad \tau_{T:4} = \frac{30}{\pi}\arctan(\sqrt{2}) - 9 = 0.122602,$$

respectively.

Fig.4.12(a) shows the L-skewness range of the two-piece normal distribution which is $(-0.468373; 0.468373)$ for $\alpha > 0$, whereas Fig.4.12(b) shows a constant level of L-kurtosis, for $\alpha > 0$, at 0.122602.



(a) The L-skewness ratio plot **(b)** The L-kurtosis ratio plot

Figure 4.12: The L-skewness and L-kurtosis plots for the two-piece normal distribution

The $(\tau_{T:3}, \tau_{T:4})$-space covered by the two-piece normal distribution is indicated on Fig.4.13 below by the solid horizontal line.

The symmetric normal distribution is obtained at $(\tau_{T:3}, \tau_{T:4})$=$(0, 0.122602)$, when $\alpha = 1$. When $\alpha > 1$, the distribution is less heavy-tailed, and subsequently more heavy-tailed when $0 < \alpha < 1$. The dashed curve at $\tau_{T:4} = \frac{1}{4}(5\tau_{T:3}^2 - 1)$ is the lower boundary for all probability distributions.

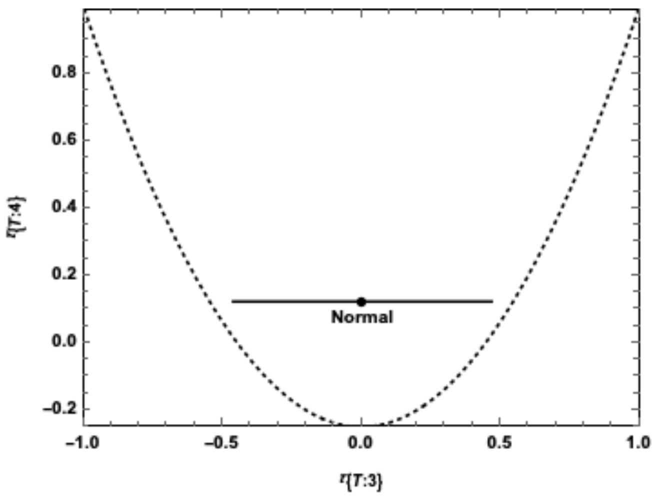


Figure 4.13: The L-moment ratio diagram for the two-piece normal distribution with $\alpha > 0$.

4.6 Two-piece Cosine Distribution

The cosine distribution is a special case of the incomplete beta distribution which was proposed by Jones (2002a). The complementary beta distribution exhibits more tractable computational results with respect to the expectations of order statistics and L-moments. As a result, it may be used to model data where the order statistics may be of primary interest.

Definition 4.6.0.1. *Suppose X is a real-valued random variable from the cosine distribution denoted $X \sim COS(\mu - \sigma, \mu + \sigma)$. Its CDF, PDF and quantile functions are respectively defined as*

$$F_X(x) = \begin{cases} 0, & x \leq \mu - \sigma \\ \sin^2\left(\frac{\pi}{2}\left(\frac{x-(\mu-\sigma)}{2\sigma}\right)\right), & \mu - \sigma < x < \mu + \sigma, \\ 1, & x \geq \mu + \sigma, \end{cases} \tag{4.41}$$

$$f_X(x) = \begin{cases} \frac{\pi}{4\sigma}\sin\left(\pi\left(\frac{x-(\mu-\sigma)}{2\sigma}\right)\right), & \mu - \sigma < x < \mu + \sigma, \\ 0, & elsewhere \end{cases} \tag{4.42}$$

and

$$Q_X(p) = \mu + \sigma\left(\frac{4}{\pi}\arcsin\left(\sqrt{p}\right) - 1\right), \qquad 0 < p < 1, \tag{4.43}$$

where $\mu, \sigma > 0$ are the location and spread parameters, respectively.

■

4.6.1 Distributional properties of the cosine distribution

Symmetry

The cosine distribution is symmetric for $\mu = \sigma > 0$.

Shape

The proability density curve of the cosine distribution is unimodal if $\mu, \sigma < 1$ and has a bath-tub shape when $\mu, \sigma > 1$. The density curve is J-shaped when $\mu < 1$ and $\sigma > 1$, whereas it is reversed J-shaped when $\mu > 1$ and $\sigma < 1$.

r^{th} order L-moments

Lemma 4.6.1. *The r^{th} order L-moments function for the cosine distribution, for $r = 1, ..., 4$, are*

$$L_{X:1} = 0, \quad L_{X:2} = \frac{1}{4}, \quad L_{X:3} = 0, \quad \text{and} \quad L_{X:4} = \frac{1}{64}, \tag{4.44}$$

respectively.

The subsequent L-moment skewness and kurtosis ratios are

$$\tau_{X:3} = 0 \quad \textit{and } \tau_{X:4} = \frac{1}{16},$$

respectively.

The standard cosine distribution is obtained when $\mu = 0$ and $\sigma = 4$.

■

4.6.2 Proposed two-piece cosine distribution.

Definition 4.6.2.1. *A real-valued random variable is said to have a two-piece cosine distribution if its quantile function is defined as*

$$Q_T(s) = \begin{cases} \mu + \alpha\sigma \left(\frac{4}{\pi} \arcsin(\sqrt{s}) - 1\right), & s \leq \frac{1}{2}, \\ \mu + \sigma \left(\frac{4}{\pi} \arcsin(\sqrt{s}) - 1\right), & s > \frac{1}{2}, \end{cases} \tag{4.45}$$

where μ, σ and $\alpha > 0$ are the location, scale and shape parameters, respectively, and its CDF and PDF are given as

$$F_T(x) = \begin{cases} 0, & x \leq \mu - \alpha\sigma \\ \sin^2\left(\frac{\pi}{2}\left(\frac{x-(\mu-\alpha\sigma)}{2\alpha\sigma}\right)\right), & \mu - \alpha\sigma < x < \mu, \\ \sin^2\left(\frac{\pi}{2}\left(\frac{x-(\mu-\sigma)}{2\sigma}\right)\right), & \mu < x < \mu + \sigma, \\ 1, & x \geq \mu + \sigma \end{cases}$$

and

$$f_T(x) = \begin{cases} \frac{\pi}{4\alpha\sigma}\sin\left(\pi\left(\frac{x-(\mu-\alpha\sigma)}{2\alpha\sigma}\right)\right), & \mu - \alpha\sigma < x < \mu, \\ \frac{\pi}{4\sigma}\sin\left(\pi\left(\frac{x-(\mu-\sigma)}{2\sigma}\right)\right), & \mu < x < \mu + \sigma, \\ 0, & elsewhere \end{cases},$$

respectively.

■

The two-piece cosine probability density curves are illustrated in Fig.4.14. The distribution exhibits negative skewness when $\alpha > 1$ as indicated by the dotted curve. The solid curve is indicative of symmetry when $\alpha = 1$. Positive skewness is obtained when $0 < \alpha < 1$ as can be seen on the dashed curve where $\alpha = 0.8$.

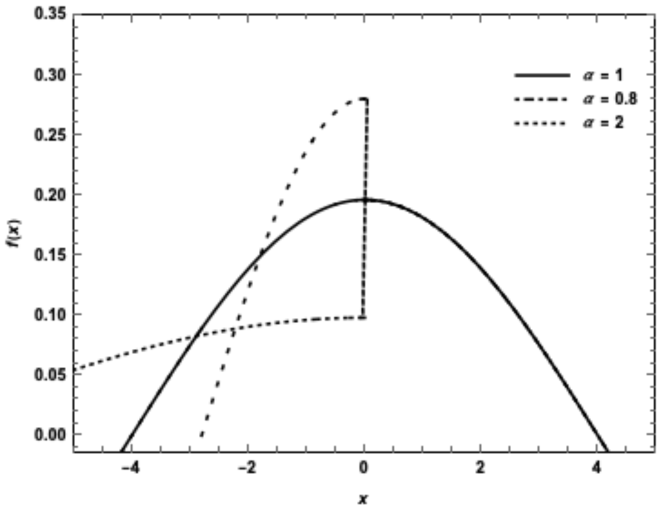


Figure 4.14: The probability density curves for the two-piece cosine distribution with $L_1 = 0$ and $L_2 = 1$, for varying values of $\alpha > 0$.

Quantile measures of distributional form

The quantile measures of distributional form for location, shape and spread for the two-piece cosine distribution are obtained by substituting Eq.(4.8) into Eqs.(3.46-3.49), respectively.

- **Location**

The median is obtained as

$$\begin{aligned}
me &= Q_T\left(\frac{1}{2}\right) \\
&= \mu + \sigma\left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{1}{2}}\right) - 1\right) \\
&= \mu + \sigma\left(\frac{4}{\pi}\cdot\frac{\pi}{4} - 1\right) \\
&= \mu.
\end{aligned}$$

- **Spread**

The spread function is derived as

$$\begin{aligned}
S_T(s) &= Q_T(s) - Q_T(1-s) \\
&= \left(\mu + \sigma\left(\frac{4}{\pi}\arcsin\left(\sqrt{s}\right) - 1\right)\right) - \left(\mu + \alpha\sigma\left(\frac{4}{\pi}\arcsin\left(\sqrt{1-s}\right) - 1\right)\right) \\
&= \frac{4}{\pi}\sigma\left(\left(\arcsin\left(\sqrt{s}\right) - \alpha\left(\frac{\pi}{2} - \arcsin\left(\sqrt{s}\right)\right)\right)\right) - \sigma(1-\alpha) \\
&= \frac{4}{\pi}\sigma\left(1+\alpha\right)\arcsin\left(\sqrt{s}\right) - \sigma(1+\alpha) \\
&= \sigma\left(\frac{4}{\pi}\arcsin\left(\sqrt{s}\right) - 1\right)\left(1+\alpha\right), \qquad\qquad \frac{1}{2} < s < 1.
\end{aligned}$$

- **Shape**

The γ-functional is obtained as

$$\begin{aligned}
\gamma_T(s) &= \frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)} \\
&= \frac{\left(\mu + \sigma\left(\frac{4}{\pi}\arcsin\left(\sqrt{s}\right) - 1\right)\right) + \left(\mu + \alpha\sigma\left(\frac{4}{\pi}\arcsin\left(\sqrt{1-s}\right) - 1\right)\right) - 2\mu}{\sigma\left(\frac{4}{\pi}\arcsin(\sqrt{s}) - 1\right)\left(1+\alpha\right)} \\
&= \frac{\frac{4}{\pi}\sigma\left(\left(\arcsin\left(\sqrt{s}\right) + \alpha\left(\frac{\pi}{2} - \arcsin\left(\sqrt{s}\right)\right)\right)\right) - \sigma(1-\alpha)}{\sigma\left(\frac{4}{\pi}\arcsin(\sqrt{s}) - 1\right)\left(1+\alpha\right)} \\
&= \frac{\frac{4}{\pi}\left(\arcsin\left(\sqrt{s}\right) + \alpha\arccos\left(\sqrt{s}\right)\right) - (1+\alpha)}{\sigma\left(\frac{4}{\pi}\arcsin(\sqrt{s}) - 1\right)\left(1+\alpha\right)} \\
&= \frac{\sigma\left(\frac{4}{\pi}\arcsin(\sqrt{s}) - 1\right)\left(1-\alpha\right)}{\sigma\left(\frac{4}{\pi}\arcsin(\sqrt{s}) - 1\right)\left(1+\alpha\right)} \\
&= \frac{1-\alpha}{1+\alpha}, \qquad \frac{1}{2} < v < u < 1.
\end{aligned}$$

- **Ratio-of-spread functions**

The ratio-of-spread functions, for $\frac{1}{2} < v < u < 1$, is derived as

$$R_T(u,v) = \frac{S_T(u)}{S_T(v)} = \frac{\sigma\left(\frac{4}{\pi}\arcsin\left(\sqrt{u}\right) - 1\right)(1+\alpha)}{\sigma\left(\frac{4}{\pi}\arcsin\left(\sqrt{v}\right) - 1\right)(1+\alpha)} = \frac{\frac{4}{\pi}\arcsin\left(\sqrt{u}\right) - 1}{\frac{4}{\pi}\arcsin\left(\sqrt{v}\right) - 1}.$$

Order statistics and r^{th} order L-moments

Theorem 4.6.2. *The first four L-moments for the standard half cosine distribution are*

$$\begin{aligned} L_{Z:1} &= \frac{2}{\pi} - 1, \\ L_{Z:2} &= \frac{1}{4}, \\ L_{Z:3} &= -\frac{1}{3\pi}, \\ &\textit{and} \\ L_{Z:4} &= \frac{1}{64}, \end{aligned} \tag{4.46}$$

respectively.

Proof. See Section 4.9.5 for detailed proofs. ■

Theorem 4.6.3. *The first four L-moments for the two-piece cosine distribution are given as*

$$\begin{aligned} L_{T:1} &= \left(2 - \frac{4}{\pi}\right)(1-\alpha), \\ L_{T:2} &= \frac{1}{2}(1+\alpha), \\ L_{T:3} &= \frac{2}{3\pi}(1-\alpha), \\ &\textit{and} \\ L_{T:4} &= \frac{1}{32}(1+\alpha), \end{aligned} \tag{4.47}$$

respectively.

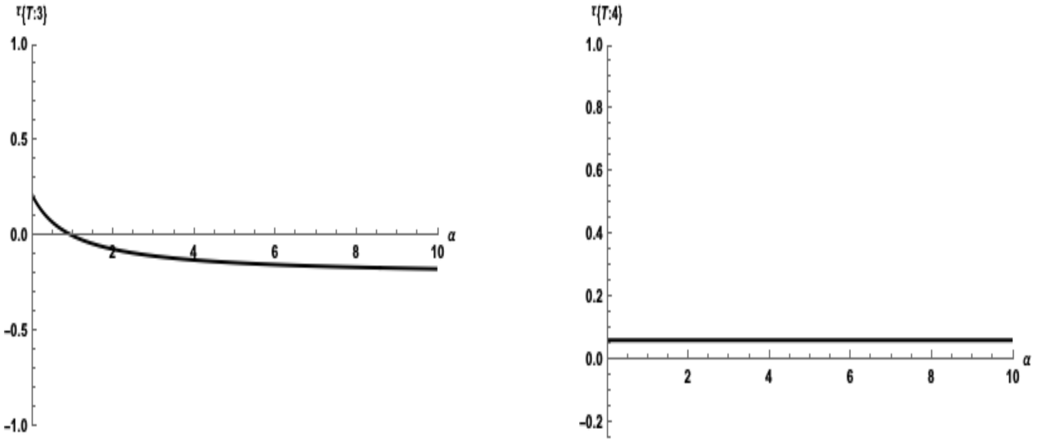
Proof. See Section 4.9.5 for detailed proofs. ■

The resulting L-skewness and L-kurtosis ratios are

$$\tau_{T:3} = \frac{L_{T:3}}{L_{T:2}} = \frac{4}{3\pi}\frac{(1-\alpha)}{(1+\alpha)} = 0.42441\frac{(1-\alpha)}{(1+\alpha)} \quad \text{and} \quad \tau_{T:4} = \frac{L_{T:4}}{L_{T:2}} = \frac{1}{16},$$

respectively.

Fig.4.15(a) shows the L-skewness range of the two-piece cosine distribution which is $(-\frac{4}{3\pi}; \frac{4}{3\pi})$ for $\alpha > 0$, whereas Fig.4.15(b) shows a constant level of L-kurtosis for all values of $\alpha > 0$ at $\tau_{T:4} = \frac{1}{16}$.



(a) L-skewness ratio plot **(b)** L-kurtosis ratio plot

Figure 4.15: The L-skewness and L-kurtosis ratio plots for the two-piece cosine distribution.

The $(\tau_{T:3}, \tau_{T:4})$-space covered by the two-piece cosine distribution is indicated on the L-moment ratio diagram in Fig.4.16 by the solid horizontal line. The symmetric cosine distribution is obtained when $\alpha = 1$ at $(\tau_{T:3}, \tau_{T:4})$=$(0, \frac{1}{16})$.

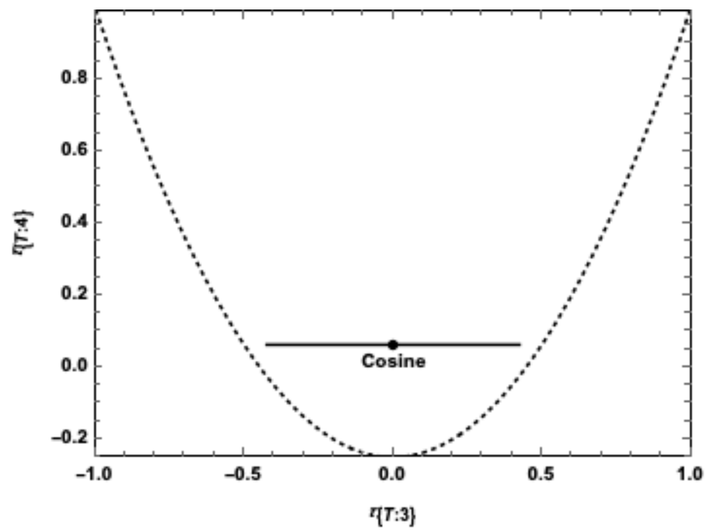


Figure 4.16: The L-moment ratio diagram for the two-piece cosine distribution.

The dotted curve at $\tau_4 = \frac{1}{4}(5\tau_3^2 - 1)$ is the lower boundary for all probability distributions.

4.7 Two-piece Uniform Distribution

The uniform distribution, also referred to as the rectangular distribution (Balakrishnan and Nevzorov (2003)), is the simplest of all continuous distributions. It arises as the limiting case of the discrete uniform distribution (Johnson et al. (1995)). This distribution has been used in hypothesis testing, random sampling from arbitrary distributions and also in finance.

Definition 4.7.0.1. *A random variable X is said to follow the uniform distribution, denoted $X \sim UNIF(\mu - \sigma, \mu + \sigma)$, if its PDF is defined as*

$$f_X(x) = \begin{cases} \frac{1}{2\sigma}, & \mu - \sigma \leq x \leq \sigma + \mu \\ 0, & elsewhere, \end{cases} \tag{4.48}$$

where $\mu > 0$ and $\sigma > 0$ are the location and scale parameters, respectively.

■

The CDF and quantile function of the uniform distribution follow from Eq.(4.48) as

$$F_X(x) = \begin{cases} 0, & x < \mu - \sigma \\ \frac{x-(\mu-\sigma)}{2\sigma}, & \mu - \sigma \leq x \leq \mu + \sigma \\ 1, & x > \mu + \sigma. \end{cases} \tag{4.49}$$

and

$$Q_X(p) = \mu + \sigma(2p - 1), \qquad 0 < p < 1, \tag{4.50}$$

respectively.

4.7.1 Distributional properties of the uniform distribution

Balakrishnan and Nevzorov (2003) documented the distributional properties for the uniform distribution.

Moments

- The central moments about the origin are given as

$$\mu_n^{'} = E(X^n) = \frac{(\sigma + \mu)^{n+1} - (\mu - \sigma)^{n+1}}{2\sigma(n+1)}, \quad n \in \mathbb{N}. \tag{4.51}$$

When $\mu = 0$ and $\sigma = 1$, then $E(X^n) = \frac{1}{2(n+1)}$.

- The central moments about the mean are given as

$$\mu_n = E(X - E(X))^n = \frac{\sigma^n (1 - (-1)^{n+1})}{(n+1)2^{n+1}}, \quad n \in \mathbb{N}. \tag{4.52}$$

- The mean and variance follow directly from Eqs.(4.51) and (4.52), by setting $n = 1$ and $n = 2$, respectively to obtain

$$\mu_1^{'} = E(X) = \mu \quad \text{and} \quad \mathrm{Var}(X) = \mu_2 = \frac{\sigma^2}{3}.$$

Shape

The Pearson coefficient of skewness, α_3, is equal to zero, which implies that the uniform distribution is symmetric. The Pearson coefficient of kurtosis, α_4, is equal to 1.8. Therefore, the uniform distribution is platykurtic (lighter-tailed) in comparison to the normal distribution, whose theoretical value for α_4 is 3.

Probability integral transform

Suppose that F and G are the CDF and inverse CDF of any continuous distribution. Then the transformation $Y = G(X)$, where $X \sim UNIF(0,1)$, yields a random variable Y whose CDF is F. This property is used to simulate random samples from any continuous probability distribution.

Uniform and other distributions

There are numerous relationships that can be drawn between the uniform distribution and other univariate distributions.

- If $U \sim UNIF(0,1)$, then $1 - U \sim UNIF(0,1)$.

- If $X \sim UNIF(0,1)$, then by the probability integral transform property, $Y = -\frac{1}{\lambda}\ln(X) \sim \exp(\lambda)$.

- Let $X \sim UNIF(0,1)$. It follows that $Y = X^n \sim \text{Beta}\left(\frac{1}{n}, 1\right)$. When $n = 1$, then $X \sim UNIF(0,1)$ is the special case of the standard Beta$(1,1)$.

- The distribution of the sum of n independent, identical, standard uniform random variables is the Irwin-Hall distribution, (Irwin (1927), Hall (1927)), a name coined by Johnson et al. (1995).

Order statistics and r^{th} order L-moments

Definition 4.7.1.1. *Suppose $X_{i:n}$, $i = 1, ..., n$, is the i^{th} order statistic of a standard uniform random variable from a sample of size n. The PDF of $X_{i:n}$ is defined as*

$$f_{X_{i:n}(x)} = \frac{n!}{(i-1)!(n-i)!} x^{i-1}(1-x)^{n-i} \tag{4.53}$$

while the expected value of $X_{i:n}$, which follows from Eq.(4.53) is

$$E(X_{i:n}) = \frac{i}{n+1}.$$

∎

Lemma 4.7.1. *The first 4 L-moments for the uniform distribution are*

$$L_{X:1} = 0, \quad L_{X:2} = \frac{1}{3}, \quad L_{X:3} = 0, \quad \text{and} \quad L_{X:4} = 0. \tag{4.54}$$

The subsequent L-skewness and L-kurtosis ratios are

$$\tau_{X:3} = 0 \quad \textit{and} \quad \tau_{X:4} = 0, \tag{4.55}$$

respectively.

The standard uniform distribution is obtained when $\mu = 0$ and $\sigma = 3$.

∎

4.7.2 Proposed two-piece uniform distribution

Definition 4.7.2.1. *A real-valued random variable is said to have a two-piece uniform distribution if its quantile function, CDF and PDF are defined as*

$$Q_T(s) = \begin{cases} \mu + \alpha\sigma(2s-1), & s \leq \frac{1}{2}, \\ \mu + \sigma(2s-1), & s > \frac{1}{2}, \end{cases} \tag{4.56}$$

$$F_T(x) = \begin{cases} 0, & x \leq \mu - \alpha\sigma, \\ \frac{x-(\mu-\alpha\sigma)}{2\alpha\sigma}, & \mu - \alpha\sigma \leq x < \mu, \\ \frac{x-(\mu-\alpha\sigma)}{2\alpha\sigma}, & \mu \leq x \leq \mu - \alpha\sigma, \\ 1, & x > \mu + \sigma, \end{cases} \tag{4.57}$$

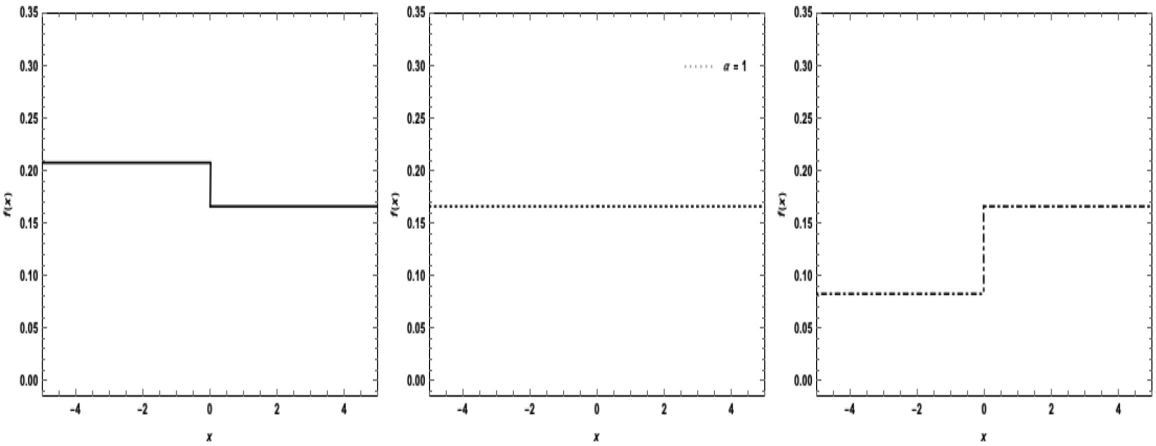
and

$$f_T(x) = \begin{cases} \frac{1}{2\alpha\sigma}, & \mu - \alpha\sigma \leq x < \mu, \\ \frac{1}{2\sigma}, & \mu \leq x < \mu + \sigma, \end{cases} \tag{4.58}$$

where $-\infty < \mu < \infty$*,* $\sigma > 0$ *and* $\alpha > 0$ *are the location, scale and asymmetry parameters, respectively.*

■

The probability density curves for the two-piece uniform distribution are shown in Fig.4.17. Symmetry is obtained when $\alpha = 1$ as indicated on Fig.4.17(b).



(a) $\alpha = 0.8$ **(b)** $\alpha = 1.$ **(c)** $\alpha = 2$

Figure 4.17: The probability density curves for the two-piece uniform distribution with $L_1 = 0$ and $L_2 = 1$, for $\alpha > 0$.

Quantile measures of distributional form

The quantile measures of distributional form for location, shape and spread for the two-piece uniform distribution are obtained by substituting Eq.(4.56) into Eqs.(3.46-3.49), respectively.

- **Location**

The median is obtained as

$$
\begin{aligned}
me &= Q_T\left(\frac{1}{2}\right) \\
&= \mu + \sigma\left(2 \cdot \frac{1}{2} - 1\right) \\
&= \mu
\end{aligned}
$$

- **Spread**

The spread function is derived as

$$
\begin{aligned}
S_T(s) &= Q_T(s) - Q_T(1-s) \\
&= (\mu + \sigma(2s-1)) - (\mu + \alpha\sigma(2(1-s)-1)) \\
&= \sigma(2s - 1 - \alpha(2 - 2s - 1)) \\
&= \sigma(2s(1+\alpha) - (1+\alpha)) \\
&= \sigma(2s-1)(1+\alpha), \qquad\qquad \frac{1}{2} < s < 1.
\end{aligned}
$$

- **Shape**

The γ-functional is obtained as

$$
\begin{aligned}
\gamma_T(s) &= \frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)} \\
&= \frac{(\mu + \sigma(2s-1)) + (\mu + \alpha\sigma(2(1-s)-1)) - 2\mu}{\sigma(2s-1)(1+\alpha)} \\
&= \frac{\sigma(2s - 1 + \alpha(1-2s))}{\sigma(2s-1)(1+\alpha)} \\
&= \frac{\sigma(2s-1)(1-\alpha)}{\sigma(2s-1)(1+\alpha)} \\
&= \frac{(1-\alpha)}{(1+\alpha)}, \qquad \frac{1}{2} < v < u < 1.
\end{aligned}
$$

- **Ratio-of-spread functions**

The ratio-of-spread functions, for $\frac{1}{2} < v < u < 1$, is derived as

$$
R_T(u,v) = \frac{S_T(u)}{S_T(v)} = \frac{\sigma(2u-1)(1+\alpha)}{\sigma(2v-1)(1+\alpha)} = \frac{2u-1}{2v-1}.
$$

r^{th} **order L-moments**

Theorem 4.7.2. *The first four L-moments of a standard half uniform random variable are obtained as*

$$\begin{aligned} L_{Z:1} &= -\frac{1}{2}, \\ L_{Z:2} &= \frac{1}{3}, \\ L_{Z:3} &= -\frac{1}{8}, \\ &\textit{and} \\ L_{Z:4} &= 0, \end{aligned} \tag{4.59}$$

respectively.

Proof. See Section 4.9.6 for detailed proofs. ∎

Theorem 4.7.3. *The first four L-moments of a two-piece uniform random variable are given by*

$$\begin{aligned} L_{T:1} &= \frac{3}{4}(1-\alpha), \\ L_{T:2} &= \frac{1}{2}(1+\alpha), \\ L_{T:3} &= \frac{3}{16}(1-\alpha), \\ &\textit{and} \\ L_{T:4} &= 0, \end{aligned} \tag{4.60}$$

respectively.

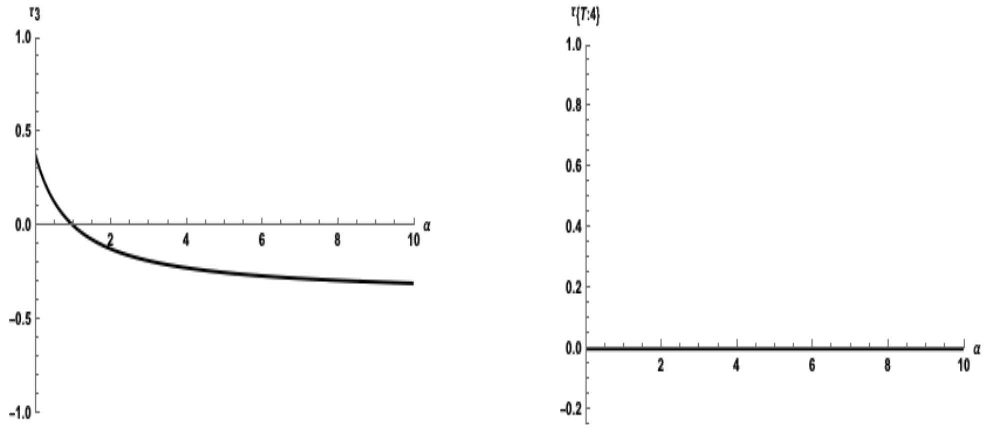
Proof. See Section 4.9.6 for detailed proofs. ∎

The resulting L-skewness and L-kurtosis ratios are

$$\tau_{T:3} = \frac{L_{T:3}}{L_{T:2}} = \frac{3}{8}\frac{(1-\alpha)}{(1+\alpha)} \quad \text{and} \quad \tau_{T:4} = \frac{L_{T:4}}{L_{T:2}} = 0,$$

respectively.

Fig.4.18(a) shows the L-skewness range of the two-piece uniform distribution which is $(-\frac{3}{8}; \frac{3}{8})$ for $\alpha > 0$, whereas Fig.4.18(b) shows a constant level of L-kurtosis at $\tau_{T:4} = 0$.



(a) The L-skewness ratio plot **(b)** The L-kurtosis ratio plot

Figure 4.18: The L-skewness and L-kurtosis ratio plot for the two-piece uniform distribution.

The $(\tau_{T:3}, \tau_{T:4})$-space covered by the two-piece uniform distribution is indicated on Fig.4.19 by the solid horizontal line. The uniform distribution is obtained at $(\tau_{T:3}, \tau_{T:4})$=$(0, 0)$ when $\alpha = 1$.

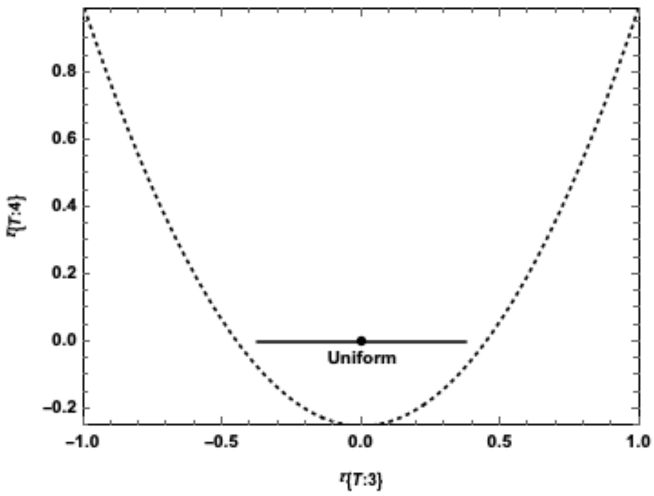


Figure 4.19: The L-moment ratio diagram for the two-piece uniform distribution with $\alpha > 0$.

The dotted curve at $\tau_{T:4} = \frac{1}{4}(5\tau_{T:3}^2 - 1)$ is the lower boundary for all probability distributions.

4.8 Two-piece Arcsine Distribution

A random variable from the arcsine distribution, introduced by Balakrishnan and Nevzorov (2003), is a bounded distribution whose support is $[\mu, \mu + \sigma]$, where $\mu, \sigma > 0$. The distribution has been used in Brownian motion as well as the Jeffrey's prior in probability of success of the Bernoulli trial.

Definition 4.8.0.1. *Let X be a random variable from the arcsine distribution, denoted $X \sim Arcsine(\mu-\sigma, \mu+\sigma)$. The CDF, PDF and quantile function are given as*

$$F_X(x) = \frac{2}{\pi}\arcsin\left(\sqrt{\frac{x-(\mu-\sigma)}{2\sigma}}\right) \qquad \mu-\sigma < x < \mu+\sigma, \tag{4.61}$$

$$f_X(x) = \frac{1}{2\pi\sigma}\frac{1}{\sqrt{\left(\frac{x-(\mu-\sigma)}{2\sigma}\right)\left(1-\frac{x-(\mu-\sigma)}{2\sigma}\right)}} \qquad \mu-\sigma < x < \mu+\sigma, \tag{4.62}$$

and

$$Q_X(p) = \mu + \sigma\left(2\sin^2\left(\frac{\pi p}{2}\right) - 1\right) \qquad 0 < p < 1, \tag{4.63}$$

respectively.

■

4.8.1 Distributional properties of the arcsine distribution

Moments

The moment generating function of the arcsine distribution is

$$M_X(t) = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{2r+1}{2r+2}\right)\frac{t^k}{k!}, \qquad t \in \mathbb{R}. \tag{4.64}$$

From Eq.(4.64), the mean and the variance of the standard arcsine distribution are obtained as $\frac{1}{2}$ and $\frac{1}{8}$, respectively. For the general arcsine distribution, the mean and variance are $\mu + \frac{\sigma}{2}$ and $\frac{\sigma^2}{8}$, respectively.

Shape

- The probability density curve of the arcsine distribution exhibits a bath-tub shape.
- The skewness and kurtosis moment-ratios are $\alpha_3 = 0$ and $\alpha_4 = \frac{3}{2}$, respectively. Since $\alpha_3 = 0$, the distribution is symmetric, whereas $\alpha_4 = \frac{3}{2}$ indicates that the distribution has lighter tails than the normal distribution ($\alpha_4 = 3$).

Arcsine and other distributions

- The standard arcsine distribution, $X \sim \text{Arcsine}(0, 1)$, is a special case of the beta distribution such that $X \sim \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right)$.
- If $X \sim \text{Arcsine}(-1, 1)$, then $X^2 \sim \text{Arcsine}(0, 1)$.
- Suppose that a and b are positive values. Then if $X \sim \text{Arcsine}(\theta, \beta)$, it follows that $aX + b \sim \text{Arcsine}(a\theta + b, a\beta + b)$.
- If X and Y are identical and independent uniform random variables on the interval $(-\pi, \pi)$, then $\sin(X)$, $\sin(2X)$, $-\cos(2X)$, $\sin(X+Y)$ and $\sin(X-Y)$ all follow the $\text{Arcsine}(-1, 1)$ distribution (Arnold and Groeneveld (1980)).

r^{th} order L-moments

Lemma 4.8.1. *Suppose X is a random variable from the arcsine distribution. The first 4 L-moments are given as*

$$L_{X:1} = 0, \quad L_{X:2} = \frac{4}{\pi^2}, \quad L_{X:3} = 0, \quad \textit{and} \quad L_{X:4} = \frac{24}{\pi^2}\left(1 - \frac{10}{\pi^2}\right), \tag{4.65}$$

respectively.

The L-skewness and L-kurtosis ratios are obtained as

$$\tau_3 = 0 \qquad \textit{and} \qquad \tau_4 = 6 - \frac{60}{\pi^2},$$

respectively.

The standard arcsine distribution is obtained when $\mu = 0$ and $\sigma = \frac{\pi^2}{4}$.

■

4.8.2 Proposed two-piece arcsine distribution

Definition 4.8.2.1. *A real-valued random variable is said to be from the two-piece cosine distribution if its CDF, PDF and quantile function are given as*

$$F_T(X) = \begin{cases} \frac{2}{\pi} \arcsin\left(\sqrt{\frac{x-(\mu-\alpha\sigma)}{2\alpha\sigma}}\right), & \mu - \alpha\sigma < x < \mu, \\ \frac{2}{\pi} \arcsin\left(\sqrt{\frac{x-(\mu-\sigma)}{2\sigma}}\right), & \mu < x < \mu + \sigma, \end{cases}$$

$$f_T(X) = \begin{cases} \frac{1}{2\pi\alpha\sigma} \frac{1}{\sqrt{\left(\frac{x-(\mu-\alpha\sigma)}{2\alpha\sigma}\right)\left(1-\frac{x-(\mu-\alpha\sigma)}{2\alpha\sigma}\right)}}, & \mu-\alpha\sigma < x < \mu, \\ \frac{1}{2\pi\sigma} \frac{1}{\sqrt{\left(\frac{x-(\mu-\sigma)}{2\sigma}\right)\left(1-\frac{x-(\mu-\sigma)}{2\sigma}\right)}}, & \mu < x < \mu+\sigma, \end{cases}$$

and

$$Q_T(s) = \begin{cases} \mu + \alpha\sigma\left(2\sin^2\left(\frac{\pi p}{2}\right) - 1\right), & s \leq \frac{1}{2}, \\ \mu + \sigma\left(2\sin^2\left(\frac{\pi p}{2}\right) - 1\right), & s > \frac{1}{2}, \end{cases} \tag{4.66}$$

respectively.

■

The probability density curves from Fig.4.20 are illustrated for $\alpha > 0$. Symmetry is attained when $\alpha = 1$ as depicted by the solid curve.

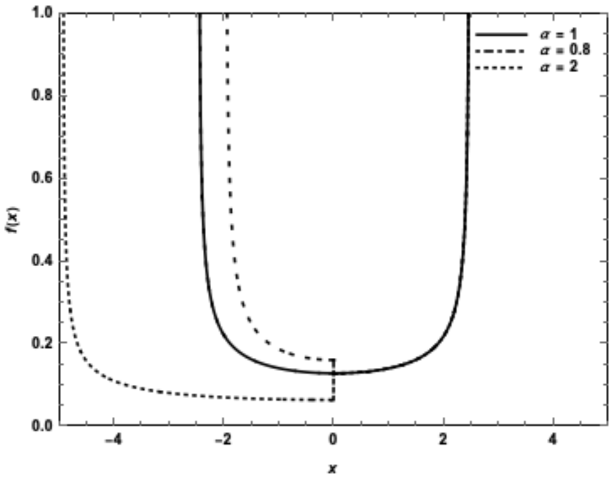


Figure 4.20: The probability density curves for the two-piece arcsine distribution with $L_1 = 0$ and $L_2 = 1$, for varying values of $\alpha > 0$.

Quantile measures of distributional form

The quantile measures of distributional form for location, shape and spread for the two-piece arcsine distribution are obtained by substituting Eq.(4.66) into Eqs.(3.46-3.49), respectively.

- **Location**

The median is obtained as

$$\begin{aligned} me &= Q\Big(\frac{1}{2}\Big) \\ &= \mu + \sigma \left(2\sin^2\left(\frac{\pi \cdot \frac{1}{2}}{2}\right) - 1\right) \\ &= \mu. \end{aligned}$$

- **Spread**

The spread function is derived as

$$\begin{aligned} S_T(s) &= Q_T(s) - Q_T(1-s) \\ &= \left(\mu + \sigma\left(2\sin^2\left(\frac{\pi s}{2}\right) - 1\right)\right) - \left(\mu + \alpha\sigma\left(2\sin^2\left(\frac{\pi(1-s)}{2}\right) - 1\right)\right) \\ &= \sigma\left(2\sin^2\left(\frac{\pi s}{2}\right) - 1\right) - \alpha\sigma\left(2 - 2\sin^2\left(\frac{\pi s}{2}\right) - 1\right) \\ &= \sigma\left(2\sin^2\left(\frac{\pi s}{2}\right)(1+\alpha) - (1+\alpha)\right), \\ &= \sigma(1+\alpha)\left(2\sin^2\left(\frac{\pi s}{2}\right) - 1\right), \qquad \frac{1}{2} < s < 1. \end{aligned}$$

- **Shape**

The γ-functional is obtained as

$$\begin{aligned} \gamma_T(s) &= \frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)} \\ &= \frac{\left(\mu + \sigma\left(2\sin^2\left(\frac{\pi s}{2}\right) - 1\right)\right) + \left(\mu + \alpha\sigma\left(2\sin^2\left(\frac{\pi(1-s)}{2}\right) - 1\right)\right) - 2\mu}{\sigma(1+\alpha)\left(2\sin^2\left(\frac{\pi s}{2}\right) - 1\right)} \\ &= \frac{\sigma\left(2\sin^2\left(\frac{\pi s}{2}\right) - 1\right) + \alpha\sigma\left(2 - 2\sin^2\left(\frac{\pi s}{2}\right) - 1\right)}{\sigma(1+\alpha)\left(2\sin^2\left(\frac{\pi s}{2}\right) - 1\right)} \\ &= \frac{\sigma\left(2\sin^2\left(\frac{\pi s}{2}\right)(1+\alpha) - (1+\alpha)\right)}{\sigma(1+\alpha)\left(2\sin^2\left(\frac{\pi s}{2}\right) - 1\right)} \\ &= \frac{\sigma(1-\alpha)\left(2\sin^2\left(\frac{\pi s}{2}\right) - 1\right)}{\sigma(1+\alpha)\left(2\sin^2\left(\frac{\pi s}{2}\right) - 1\right)} \\ &= \frac{(1-\alpha)}{(1+\alpha)}, \qquad \frac{1}{2} < v < u < 1. \end{aligned}$$

- **Ratio-of-spread functions**

The ratio-of-spread functions, for $\frac{1}{2} < v < u < 1$, is derived as

$$R_T(u,v) = \frac{S_T(u)}{S_T(v)} = \frac{\sigma(1+\alpha)\left(2\sin^2\left(\frac{\pi u}{2}\right) - 1\right)}{\sigma(1+\alpha)\left(2\sin^2\left(\frac{\pi v}{2}\right) - 1\right)} = \frac{\left(2\sin^2\left(\frac{\pi u}{2}\right) - 1\right)}{\left(2\sin^2\left(\frac{\pi v}{2}\right) - 1\right)}.$$

r^{th} **order L-moments**

Theorem 4.8.2. *The first four L-moments of a standard half arcsine random variable are obtained as*

$$L_{Z:1} = -\frac{2}{\pi},$$
$$L_{Z:2} = \frac{4}{\pi^2},$$
$$L_{Z:3} = \frac{1}{\pi^3}(\pi^2 - 12\pi + 24),$$

and

$$L_{Z:4} = \frac{24}{\pi^2}(\pi^2 - 10), \tag{4.67}$$

respectively.

Proof. See Section 4.9.7 for detailed proofs. ■

Theorem 4.8.3. *The first four L-moments of a standard two-piece arcsine random variable are given as*

$$L_{T:1} = \frac{\pi}{4}(1 - \alpha),$$
$$L_{T:2} = \frac{1}{2}(1 + \alpha),$$
$$L_{T:3} = \frac{1}{8\pi}\left(\pi^2 - 12\pi + 24\right)(1 - \alpha),$$

and

$$L_{T:4} = \left(3 - \frac{30}{\pi^2}\right)(1 + \alpha), \tag{4.68}$$

respectively.

Proof. See Section 4.9.7 for detailed proofs. ■

Therefore, it follows that the L-skewness and L-kurtosis ratios are

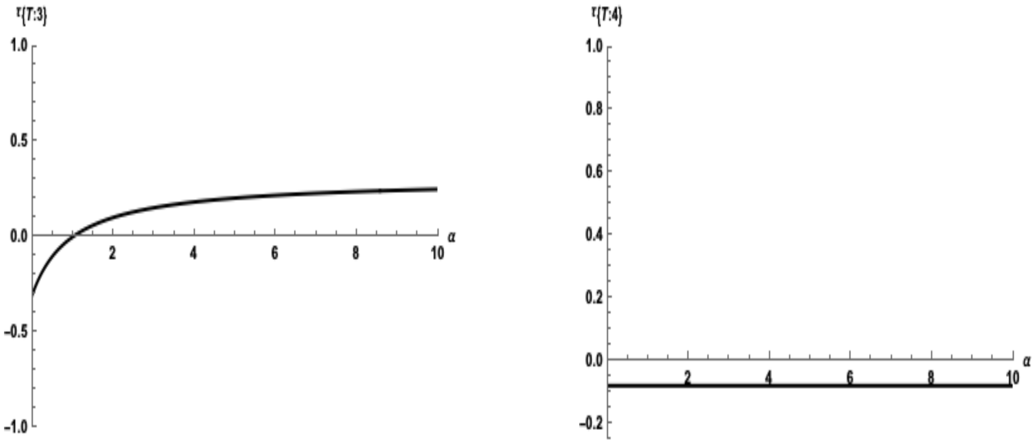
$$\tau_{T:3} = \frac{L_{T:3}}{L_{T:2}} = \frac{1}{4\pi}\left(\pi^2 - 12\pi + 24\right)\frac{(1-\alpha)}{(1+\alpha)} = 0.3048\frac{(1-\alpha)}{(1+\alpha)} \tag{4.69}$$

and

$$\tau_{T:4} = \frac{L_{T:4}}{L_{T:2}} = 6 - \frac{60}{\pi^2} = -0.07927, \tag{4.70}$$

respectively.

Fig.4.21(a) shows the L-skewness range of the two-piece arcsine distribution which is $(-0.3048; 0.3048)$ for $\alpha > 0$, whereas Fig.4.21(b) shows a constant level of L-kurtosis is achieved for $\alpha > 0$ at $\tau_{T:4} = -0.07927$.



(a) L-skewness ratio plot

(b) L-kurtosis ratio plot

Figure 4.21: The L-skewness and L-kurtosis ratio plots for the two-piece Student's $t(2)$ distribution.

The $(\tau_{T:3}, \tau_{T:4})$-space covered by the two-piece arcsine distribution is indicated by the horizontal line on the L-moment ratio diagram in Fig.4.22. The symmetric arcsine distribution is obtained at (τ_3, τ_4)=$(0, -0.07927)$ when $\alpha = 1$.

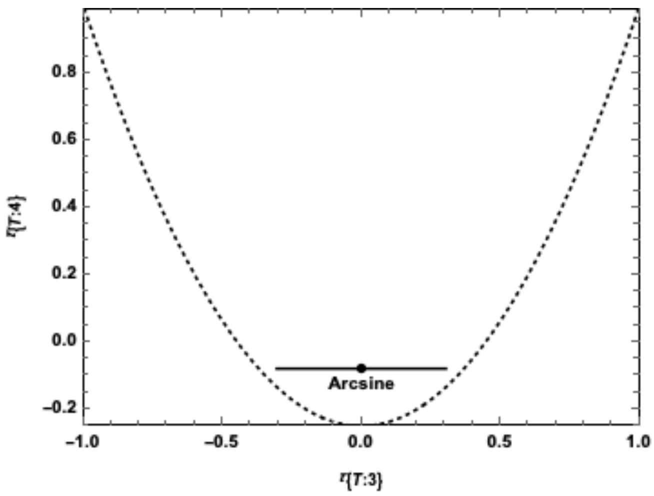


Figure 4.22: The L-moment ratio diagram for the two-piece arcsine distribution with $\alpha > 0$.

The dotted curve at $(\tau_{T:3}, \tau_{T:4}) = \frac{1}{4}(5\tau_{T:3}^2 - 1)$ is the lower boundary for all probability distributions.

4.9 Appendix

This section contains the derivations of the results for the standard two-piece univariate distributions generated in Sections 4.2-4.8. Foremost, the L-moments for the half univariate parent distributions are obtained using the results in Eq.(3.17) of Lemma 3.3.2 and the r^{th} order shifted scaled Legendre polynomials in Eq.(3.19) of Theorem 3.3.3. The scaling factor is set to $k = \frac{1}{2}$. Subsequently, the results for the L-moments of the two-piece distributions are obtained by substituting the results of the L-moments for the half distributions into Eq.(3.27), for the respective values of $r = 1, ..., 4$.

4.9.1 Two-piece Student's $t(2)$ distribution

Theorem 4.9.1. *Suppose X is a real-valued Student's $t(2)$ random variable with $\mu \in \mathbb{R}$ and $\sigma > 0$ as the location and scale parameters, respectively. Let Z be a real-valued standardized random variable from the standard half Student's $t(2)$ distribution. The first 4 L-moments of Z are given by Eq.(4.9).*

Proof. The depth in the standard quantile function from Eq.(4.4) is set as $\frac{p}{2}$, and substituted into Eq.(3.17). The first L-moment of Z, where $r = 1$ and $P_0(\frac{1}{2}) = 1$, is obtained as

$$\begin{aligned}
L_{Z:1} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right) dp \\
&= \int_0^1 \frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}} \, dp \\
&= -\int_0^1 (1-p) \, p^{-\frac{1}{2}} \left(1-\frac{p}{2}\right)^{-\frac{1}{2}} dp \\
&= -\mathrm{B}\left(\frac{1}{2}, 2\right) \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{5}{2}, \frac{1}{2}\right) \\
&= -\frac{4}{3} \times \frac{3}{2\sqrt{2}} \\
&= -\sqrt{2}.
\end{aligned}$$

When $r = 2$ and $P_1(\frac{1}{2}) = p - 1$, the second L-moment is obtained as

$$\begin{aligned}
L_{Z:2} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)(p-1)\ dp \\
&= \int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right)(p-1)\ dp \\
&= \int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right)p\ dp - \int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right)\ dp \\
&= \int_0^1 (1-p)\ p^{\frac{1}{2}}\left(1-\frac{p}{2}\right)^{-\frac{1}{2}} dp - \left(-\sqrt{2}\right) \\
&= \left(\mathrm{B}\left(\frac{3}{2},2\right)\ {}_2F_1\left(\frac{1}{2},\frac{3}{2};\frac{7}{2},\frac{1}{2}\right)\right) - \left(-\sqrt{2}\right) \\
&= \left(\frac{4}{15}\times\frac{15}{8\sqrt{2}}(\pi-4)\right) - \left(-\sqrt{2}\right) \\
&= \frac{\pi}{2\sqrt{2}}.
\end{aligned}$$

For the third L-moment, $r = 3$ and $P_2(\frac{1}{2}) = 6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1$. Therefore,

$$\begin{aligned}
L_{Z:3} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right)+1\right) dp \\
&= \int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right)+1\right) dp \\
&= \frac{6}{4}\int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right)p^2 dp - 3\int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right)p\ dp + \int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right) dp \\
&= \frac{6}{4}\left(\int_0^1 (1-p)\ p^{\frac{3}{2}}\left(1-\frac{p}{2}\right)^{-\frac{1}{2}} dp\right) - 3\left(-\left(\frac{4}{15}\times\frac{15}{8\sqrt{2}}(\pi-4)\right)\right) + \left(-\sqrt{2}\right) \\
&= \frac{6}{4}\left(-\mathrm{B}\left(\frac{5}{2},2\right)\ {}_2F_1\left(\frac{1}{2},\frac{5}{2};\frac{9}{2},\frac{1}{2}\right)\right) - 3\left(-\left(\frac{4}{15}\times\frac{15}{8\sqrt{2}}(\pi-4)\right)\right) + \left(-\sqrt{2}\right) \\
&= \frac{6}{4}\left(-\sqrt{2}\left(\frac{\pi}{2}-\frac{5}{3}\right)\right) - 3\left(-\left(\frac{4}{15}\times\frac{15}{8\sqrt{2}}(\pi-4)\right)\right) + \left(-\sqrt{2}\right) \\
&= -\frac{1}{\sqrt{2}}.
\end{aligned}$$

The fourth L-moment is obtained when $r=4$ and $P_3(\frac{1}{2}) = 20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1$, as

$$\begin{aligned}
L_{Z:4} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right)\, dp \\
&= \int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right)\, dp \\
&= \frac{20}{8}\int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right) p^3 dp - \frac{30}{4}\int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right) p^2 dp + 6\int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right) p\, dp \\
&\quad - \int_0^1 \left(\frac{p-1}{\sqrt{p\left(1-\frac{p}{2}\right)}}\right) dp \\
&= \frac{20}{8}\left(-\int_0^1 (1-p)\, p^{\frac{5}{2}}\left(1-\frac{p}{2}\right)^{-\frac{1}{2}} dp\right) - \frac{30}{4}\left(-\sqrt{2}\left(\frac{\pi}{2} - \frac{5}{3}\right)\right) + 6\left(-\left(\frac{4}{15} \times \frac{15}{8\sqrt{2}}(\pi - 4)\right)\right) + \sqrt{2} \\
&= \frac{20}{8}\left(-\mathrm{B}\left(\frac{7}{2}, 2\right)\, {}_2F_1\left(\frac{1}{2}, \frac{7}{2}; \frac{11}{2}, \frac{1}{2}\right)\right) - \frac{30}{4}\left(-\sqrt{2}\left(\frac{\pi}{2} - \frac{5}{3}\right)\right) + 6\left(-\left(\frac{4}{15} \times \frac{15}{8\sqrt{2}}(\pi - 4)\right)\right) + \sqrt{2} \\
&= \frac{20}{8}\left(-\sqrt{2}\left(\frac{15}{16}\pi - 3\right)\right) - \frac{30}{4}\left(-\sqrt{2}\left(\frac{\pi}{2} - \frac{5}{3}\right)\right) + 6\left(-\left(\frac{4}{15} \times \frac{15}{8\sqrt{2}}(\pi - 4)\right)\right) + \sqrt{2} \\
&= \frac{3\pi}{16\sqrt{2}}.
\end{aligned}$$

The final results were obtained using Gradshteyn and Ryzhik (2007, 3.197.3). ■

Theorem 4.9.2. *Suppose T is a real-valued standardized random variable from the two-piece Student's $t(2)$ distribution, denoted as $T \sim t(2)_{TP}(0, \frac{2\sqrt{2}}{\pi}, \alpha)$, where 0 is the location parameter, $\frac{2\sqrt{2}}{\pi}$ is the scale parameter and $\alpha > 0$ is the asymmetry parameter. The first 4 L-moments of T are given in Eq.(4.10).*

Proof. The results in Eq.(4.10) are obtained by substituting the L-moments of X in Eq.(4.7) and the results of the L-moments of Z in Eq.(4.9) into Eq.(3.27), for $r = 1, ..., 4$.

For $r = 1$, the first L-moment of T is

$$\begin{aligned}
L_{T:1} &= \mu + \sigma\left(L_{X:1} - 0.5(1-\alpha)c_0^{(0)} L_{Z:1}\right) \\
&= \frac{2\sqrt{2}}{\pi}\left(0 - 0.5(1-\alpha)\left(-\sqrt{2}\right)\right) \\
&= \frac{2}{\pi}(1-\alpha).
\end{aligned}$$

The second L-moment of T, obtained when $r = 2$, is

$$\begin{aligned}
L_{T:2} &= \sigma\left(L_{X:2} - 0.5(1-\alpha)\left(c_0^{(1)} L_{Z:1} + \frac{c_1^{(1)}}{2}(L_{Z:1} + L_{Z:2})\right)\right) \\
&= \frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{2\sqrt{2}} - 0.5(1-\alpha)\left((-1)\left(-\sqrt{2}\right) + \frac{2}{2}\left(-\sqrt{2} + \frac{\pi}{2\sqrt{2}}\right)\right)\right) \\
&= \frac{1}{2}(1+\alpha).
\end{aligned}$$

When $r = 3$, the third L-moment of T is

$$
\begin{aligned}
L_{T:3} &= \left(L_{X:3} - 0.5(1-\alpha) \left(\left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3} \right) L_{Z:1} + \left(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2} \right) L_{Z:2} + \frac{c_2^{(2)}}{6} L_{Z:3} \right) \right) \\
&= \frac{2\sqrt{2}}{\pi} \left(0 - 0.5(1-\alpha) \left(\left(1 - \frac{6}{2} + \frac{6}{3} \right) \left(-\sqrt{2} \right) + \left(-\frac{6}{2} + \frac{6}{2} \right) \left(\frac{\pi}{2\sqrt{2}} \right) + \frac{6}{6} \left(-\frac{1}{\sqrt{2}} \right) \right) \right) \\
&= \frac{1}{\pi}(1-\alpha).
\end{aligned}
$$

The fourth L-moment of T, when $r = 4$, is

$$
\begin{aligned}
L_{T:4} = \sigma \Bigg(& L_{X:4} - 0.5(1-\alpha) \Bigg(\left(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4} \right) L_{Z:1} + \left(c_1^{(3)} + c_2^{(3)} + \frac{9}{10} c_3^{(3)} \right) \frac{L_{Z:2}}{2} \\
& + \left(\frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{2} \right) \frac{L_{Z:3}}{2} + \frac{L_{Z:4}}{20} c_3^{(3)} \Bigg) \Bigg) \\
= \frac{2\sqrt{2}}{\pi} \Bigg(& \frac{3\pi}{16\sqrt{2}} - 0.5(1-\alpha) \Bigg(\left(-1 + \frac{12}{2} - \frac{30}{3} + \frac{20}{4} \right) \left(-\sqrt{2} \right) + \frac{1}{2} \left(12 - 30 + \frac{9}{10} \cdot 20 \right) \left(\frac{\pi}{2\sqrt{2}} \right) \\
& - \frac{1}{2} \left(-\frac{30}{3} + \frac{20}{2} \right) \left(-\frac{1}{\sqrt{2}} \right) + \frac{20}{20} \cdot \left(\frac{3\pi}{16\sqrt{2}} \right) \Bigg) \Bigg) \\
= \frac{3}{16} & (1+\alpha).
\end{aligned}
$$

■

4.9.2 Two-piece hyperbolic secant distribution

Theorem 4.9.3. *Let X be a real-valued random variable from the hyperbolic secant distribution, with $\mu \in \mathbb{R}$ and $\sigma > 0$ as the location and scale parameters, respectively. Let Z be a real-valued standardized random variable from the standard half hyperbolic secant distribution. The first 4 L-moments of Z are given by Eq.(4.20).*

Proof. The depth in the standard quantile function from Eq.(4.13) is set as $\frac{p}{2}$, and substituted into Eq.(3.17). For the first L-moment of Z, $r = 1$ and $P_0(\frac{1}{2}) = 1$. Therefore,

$$
\begin{aligned}
L_{Z:1} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right) dp \\
&= \int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right) \, dp \\
&= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \log\left(\tan\left(t\right)\right) dt \\
&= -\frac{4}{\pi} \, \boldsymbol{G}.
\end{aligned}
$$

When $r = 2$ and $P_1(\frac{1}{2}) = p - 1$, the second L-moment of Z is obtained as

$$\begin{aligned}
L_{Z:2} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)(p-1)\ dp \\
&= \int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)(p-1)\ dp \\
&= \int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)p\ dp - \int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)\ dp \\
&= \left(\frac{4}{\pi}\right)^2 \int_0^{\frac{\pi}{4}} t\ \log(\tan(t))\,dt - \left(-\frac{4}{\pi}\boldsymbol{G}\right) \\
&= \left(\frac{4}{\pi}\right)^2 \left(\frac{1}{16}(7\zeta(3) - 4\pi\boldsymbol{G})\right) - \left(-\frac{4}{\pi}\boldsymbol{G}\right) \\
&= \frac{7\zeta(3)}{\pi^2}.
\end{aligned}$$

The third L-moment of Z is obtained by setting $r = 3$ and $P_2(\frac{1}{2}) = 6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1$, such that

$$\begin{aligned}
L_{Z:3} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1\right)dp \\
&= \int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1\right)\ dp \\
&= \frac{6}{4}\int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)p^2 dp - 3\int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)p\ dp + \int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)dp \\
&= \frac{6}{4}\left(-\int_0^1 p^2 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)dp\right) - 3\left(\frac{16}{\pi^2}\left(\frac{1}{16}(7\zeta(3) - 4\pi\boldsymbol{G})\right)\right) + \left(-\frac{4}{\pi}\boldsymbol{G}\right) \\
&= \frac{6}{4}\left(\left(\frac{4}{\pi}\boldsymbol{G} - \frac{\psi^{(3)}\left(\frac{1}{4}\right)}{48\pi^3} + \frac{\psi^{(3)}\left(\frac{3}{4}\right)}{48\pi^3}\right)\right) - 3\left(\frac{16}{\pi^2}\left(\frac{1}{16}(7\zeta(3) - 4\pi\boldsymbol{G})\right)\right) + \left(-\frac{4}{\pi}\boldsymbol{G}\right) \\
&= \frac{1}{32\pi^3}\left(64\boldsymbol{G}\pi^2 + \psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right) - 672\zeta(3)\pi\right).
\end{aligned}$$

For the fourth L-moment of Z, $r = 4$ and $P_3(\frac{1}{2}) = 20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1$. Therefore,

$$\begin{aligned}
L_{Z:4} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right)\ dp \\
&= \int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right)\ dp \\
&= \frac{20}{8}\int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)p^3 dp - \frac{30}{4}\int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)p^2 dp + 6\int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)p\ dp \\
&\quad - \int_0^1 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)dp \\
&= \frac{20}{8}\left(-\int_0^1 p^3 \log\left(\tan\left(\frac{\pi p}{4}\right)\right)\right)dp - \frac{30}{4}\left(-\left(\frac{4}{\pi}\boldsymbol{G} - \frac{\psi^{(3)}\left(\frac{1}{4}\right)}{48\pi^3} + \frac{\psi^{(3)}\left(\frac{3}{4}\right)}{48\pi^3}\right)\right) \\
&\quad + 6\left(\frac{16}{\pi^2}\left(\frac{1}{16}(7\zeta(3) - 4\pi\boldsymbol{G})\right)\right) - \left(-\frac{4}{\pi}\boldsymbol{G}\right) \\
&= \frac{20}{8}\left(-\left(\frac{4\boldsymbol{G}}{\pi} - \frac{\psi^{(3)}\left(\frac{1}{4}\right)}{16\pi^3} + \frac{\psi^{(3)}\left(\frac{3}{4}\right)}{16\pi^3} + 186\zeta(5)\right)\right) - \frac{30}{4}\left(-\left(\frac{4}{\pi}\boldsymbol{G} - \frac{\psi^{(3)}\left(\frac{1}{4}\right)}{48\pi^3} + \frac{\psi^{(3)}\left(\frac{3}{4}\right)}{48\pi^3}\right)\right) \\
&\quad + 6\left(\frac{16}{\pi^2}\left(\frac{1}{16}(7\zeta(3) - 4\pi\boldsymbol{G})\right)\right) - \left(-\frac{4}{\pi}\boldsymbol{G}\right) \\
&= \frac{1}{\pi^4}\left(42\zeta(3)\pi^2 - 465\zeta(5)\right).
\end{aligned}$$

The final results were obtained using Gradshteyn and Ryzhik (2007, 4.227.2) and some computed from Wolfram Research, Inc. (2020). ■

Theorem 4.9.4. *Let T be a real-valued standardized random variable from the two-piece hyperbolic secant distribution, denoted as $T\sim HSD_{TP}(0,\ \frac{\pi^2}{7\zeta(3)},\ \alpha)$, where 0, 1 and $\alpha > 0$ are the location, scale and asymmetry parameters, respectively. The first 4 L-moments of T are given in Eq.(4.21).*

Proof. The results in Eq.(4.21) are obtained by substituting the L-moments of X in Eq.(4.16) and the results of the L-moments of Z in Eq.(4.20) into Eq.(3.27), for $r = 1, \ldots, 4$.
The first L-moment of T, for $r = 1$, is

$$\begin{aligned} L_{T:1} &= \mu + \sigma\left(L_{X:1} - 0.5(1-\alpha)c_0^{(0)} L_{Z:1}\right) \\ &= \frac{\pi^2}{7\zeta(3)}\left(0 - 0.5(1-\alpha)\left(-\frac{4}{\pi}\ \boldsymbol{G}\right)\right) \\ &= \frac{2\pi}{7\zeta(3)}\boldsymbol{G}(1-\alpha). \end{aligned}$$

The second L-moment of T, when $r = 2$, is

$$\begin{aligned} L_{T:2} &= \sigma\left(L_{X:2} - 0.5(1-\alpha)\left(c_0^{(1)} L_{Z:1} + \frac{c_1^{(1)}}{2}(L_{Z:1} + L_{Z:2})\right)\right) \\ &= \frac{\pi^2}{7\zeta(3)}\left(\frac{7\zeta(3)}{\pi^2} - 0.5(1-\alpha)\left((-1)\left(-\frac{4}{\pi}\ \boldsymbol{G}\right) + \frac{2}{2}\left(\left(-\frac{4}{\pi}\ \boldsymbol{G}\right) + \left(\frac{7\zeta(3)}{\pi^2}\right)\right)\right)\right) \\ &= \frac{1}{2}(1+\alpha). \end{aligned}$$

When $r = 3$, the third L-moment of T is

$$\begin{aligned} L_{T:3} &= \sigma\left(L_{X:3} - 0.5(1-\alpha)\left(\left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\right)L_{Z:1} + \left(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2}\right)L_{Z:2} + \frac{c_2^{(2)}}{6}L_{Z:3}\right)\right) \\ &= \frac{\pi^2}{7\zeta(3)}\left(0 - 0.5(1-\alpha)\left(\left(1 - \frac{6}{2} + \frac{6}{3}\right)\left(-\frac{4}{\pi}\ \boldsymbol{G}\right) + \left(-\frac{6}{2} + \frac{6}{2}\right)\left(\frac{7\zeta(3)}{\pi^2}\right)\right.\right. \\ &\quad \left.\left. + \frac{6}{6}\left(\frac{1}{32\pi^3}\left(64\boldsymbol{G}\pi^2 + \psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right) - 672\zeta(3)\pi\right)\right)\right)\right) \\ &= \frac{\pi^2}{7\zeta(3)}\left(0 - 0.5(1-\alpha)\left(\frac{1}{32\pi^3}\left(64\boldsymbol{G}\pi^2 + \psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right) - 672\zeta(3)\pi\right)\right)\right) \\ &= \frac{1}{448\zeta(3)\pi}\left(64\boldsymbol{G}\pi^2 + \psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right) - 672\zeta(3)\pi\right)(1-\alpha). \end{aligned}$$

The fourth L-moment of T is obtained when $r = 4$ as

$$
\begin{aligned}
L_{T:4} &= \sigma\left(L_{X:4} - 0.5(1-\alpha)\left(\left(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\right)L_{Z:1} + \left(c_1^{(3)} + c_2^{(3)} + \frac{9}{10}c_3^{(3)}\right)\frac{L_{Z:2}}{2}\right.\right.\\
&\quad \left.\left. + \left(\frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{2}\right)\frac{L_{Z:3}}{2} + \frac{L_{Z:4}}{20}c_3^{(3)}\right)\right)\\
&= \frac{\pi^2}{7\zeta(3)}\left(\frac{1}{\pi^4}\left(42\zeta(3)\pi^2 - 465\zeta(5)\right) - 0.5(1-\alpha)\left(\left(-1 + \frac{12}{2} - \frac{30}{3} + \frac{20}{4}\right)\left(-\frac{4}{\pi}\,\boldsymbol{G}\right)\right.\right.\\
&\quad + \frac{1}{2}\left(12 - 30 + \frac{9}{10}\cdot 20\right)\left(\frac{7\zeta(3)}{\pi^2}\right) + \frac{20}{20}\cdot\left(\frac{1}{\pi^4}\left(42\zeta(3)\pi^2 - 465\zeta(5)\right)\right)\\
&\quad \left.\left. + \frac{1}{2}\left(-\frac{30}{3} + \frac{20}{2}\right)\left(\frac{\sigma}{32\pi^3}\left(64\boldsymbol{G}\pi^2 + \psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right) - 672\zeta(3)\pi\right)\right)\right)\right)\\
&= \frac{\pi^2}{7\zeta(3)}\left(\frac{1}{\pi^4}\left(42\zeta(3)\pi^2 - 465\zeta(5)\right) - 0.5(1-\alpha)\left(\frac{1}{\pi^4}\left(42\zeta(3)\pi^2 - 465\zeta(5)\right)\right)\right)\\
&= \frac{1}{14\zeta(3)\pi^2}\left(42\zeta(3)\pi^2 - 465\zeta(5)\right)(1+\alpha).
\end{aligned}
$$

■

4.9.3 Two-piece logistic distribution

Theorem 4.9.5. *Assume X is a real-valued logistic random variable, with $\mu \in \mathbb{R}$ and $\sigma > 0$ as the location and scale parameters, respectively. Let Z be a real-valued standardized random variable from the standard half logistic distribution. The first 4 L-moments of Z are given by Eq.(4.30).*

Proof. The depth in the standard quantile function from Eq.(4.24) is set as $\frac{p}{2}$, and substituted into Eq.(3.17). The first L-moment of Z, where $r = 1$ and $P_0(\frac{1}{2}) = 1$, is obtained as

$$
\begin{aligned}
L_{Z:1} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right) dp\\
&= \int_0^1 \log\left(\frac{p}{2-p}\right)\; dp\\
&= \left(\int_0^1 \log(p)\; dp - \int_0^1 \log(2-p)\; dp\right)\\
&= p(\log(p) - 1) - ((p-2)\log(2-p) - p)\Big|_0^1\\
&= -1 + 1 - 2\log 2\\
&= -2\log(2).
\end{aligned}
$$

For $r = 2$ and $P_1(\frac{1}{2}) = p - 1$, the second L-moment of Z is

$$\begin{aligned}
L_{Z:2}L_{Z:2} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)(p-1)\ dp \\
&= \int_0^1 \log\left(\frac{p}{2-p}\right)(p-1)\ dp \\
&= \int_0^1 \log\left(\frac{p}{2-p}\right)p\ dp - \int_0^1 \log\left(\frac{p}{2-p}\right)\ dp \\
&= \int_0^1 p\log(p)\ dp - \int_0^1 p\log(2-p)\ dp - (\mu - 2\sigma\log(2)) \\
&= p^2\left(\frac{\log(p)}{2} - \frac{1}{4}\right) - \frac{1}{2}\left(\left(p^2+4\right)\log(2-p) - \left(\frac{p^2}{2} + 2p\right)\right)\Bigg|_0^1 - (-2\log(2)) \\
&= -\frac{1}{4} + \frac{5}{4} - 2\log(2) - (\mu - 2\sigma\log(2)) \\
&= 1.
\end{aligned}$$

The third L-moment of Z is obtained when $r = 3$ and $P_2(\frac{1}{2}) = 6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1$ as

$$\begin{aligned}
L_{Z:3} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1\right)dp \\
&= \int_0^1 \log\left(\frac{p}{2-p}\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1\right)\ dp \\
&= \frac{6}{4}\int_0^1 \log\left(\frac{p}{2-p}\right)p^2 dp - 3\int_0^1 \log\left(\frac{p}{2-p}\right)p\ dp + \int_0^1 \log\left(\frac{p}{2-p}\right)dp \\
&= \frac{6}{4}\left(\int_0^1 p^2\log(p)\ dp - \int_0^1 p^2\log(2-p)\ dp\right) - 3\left((1 - 2\log(2))\right) + (-2\log(2)) \\
&= \frac{6}{4}\left(p^3\left(\frac{\log(p)}{3} - \frac{1}{9}\right) - \frac{1}{3}\left(\left(p^3 - 8\right)\log(2-p) - \left(\frac{p^3}{3} + p^2 + 4p\right)\right)\right)\Bigg|_0^1 - 3\left((1 - 2\log(2))\right) \\
&\quad + (-2\log(2)) \\
&= \frac{6}{4}\left(\left(-\frac{1}{9} + \frac{16}{9} - \frac{8}{3}\log(2)\right)\right) - 3\left(1 - 2\log(2)\right) + (-2\log(2)) \\
&= -\frac{1}{2}.
\end{aligned}$$

The fourth L-moment of Z, for $r = 4$ and $P_3(\frac{1}{2}) = 20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1$, is derived as

$$\begin{aligned}
L_{Z:4} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right)\, dp \\
&= \int_0^1 \log\left(\frac{p}{2-p}\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right)\, dp \\
&= \frac{20}{8}\int_0^1 \log\left(\frac{p}{2-p}\right) p^3 dp - \frac{30}{4}\int_0^1 \log\left(\frac{p}{2-p}\right) p^2 dp + 6\int_0^1 \log\left(\frac{p}{2-p}\right) p\ dp \\
&\quad - \int_0^1 \log\left(\frac{p}{2-p}\right) dp \\
&= \frac{20}{8}\left(\int_0^1 p^3 \log(p)\ dp - \int_0^1 p^3 \log(2-p)\ dp\right) - \frac{30}{4}\left(-\frac{1}{9} + \frac{16}{9} - \frac{8}{3}\log(2)\right) + 6\left(1 - 2\log(2)\right) \\
&\quad - \left(-2\log(2)\right) \\
&= \frac{20}{8}\left(p^4\left(\frac{\log(p)}{4} - \frac{1}{16}\right) - \frac{1}{4}\left(\left(p^4 - 16\right)\log(2-p) - \left(\frac{p^4}{4} + \frac{2}{3}p^3 + 2p^2 + 8p\right)\right)\right)\Bigg|_0^1 \\
&\quad - \frac{30}{4}\left(\frac{15}{9} - \frac{8}{3}\log(2)\right) + 6\left(1 - 2\log(2)\right) - \left(-2\log(2)\right) \\
&= \frac{20}{8}\left(\frac{128}{48} - 4\log(2)\right) - \frac{30}{4}\left(\frac{15}{9} - \frac{8}{3}\log(2)\right) + 6\left(1 - 2\log(2)\right) - \left(-2\log(2)\right) \\
&= \frac{1}{6}.
\end{aligned}$$

The final results were obtained using Gradshteyn and Ryzhik (2007, 2.723.1, 2.729.1-2.729.4).

■

Theorem 4.9.6. *Assume T is a real-valued standardized random variable from the two-piece logistic distribution, denoted as $T \sim L_{TP}(0,\ 1,\ \alpha)$, where 0 is the location parameter, 1 is the scale parameter and $\alpha > 0$ is the asymmetry parameter. The first 4 L-moments of T are given in Eq.(4.31).*

Proof. The results in Eq.(4.31) are obtained by substituting the L-moments of X in Eq.(4.26) and the results of the L-moments of Z in Eq.(4.30) into Eq.(3.27), for $r = 1, \ldots, 4$.
For $r = 1$, the first L-moment of T is

$$\begin{aligned}
L_{T:1} &= \mu + \sigma\left(L_{X:1} - 0.5(1-\alpha)c_0^{(0)} L_{Z:1}\right) \\
&= 0 - 0.5(1-\alpha)\left(-2\log 2\right) \\
&= (1-\alpha)\log 2.
\end{aligned}$$

The second L-moment of T, when $r = 2$, is given as

$$\begin{aligned}
L_{T:2} &= \sigma\left(L_{X:2} - 0.5(1-\alpha)\left(c_0^{(1)}L_{Z:1} + \frac{c_1^{(1)}}{2}(L_{Z:1} + L_{Z:2})\right)\right)\\
&= 1 - 0.5(1-\alpha)\Big((-1)(-2\log 2) + \frac{2}{2}(-2\log 2 + 1)\Big)\\
&= \frac{1}{2}(1+\alpha).
\end{aligned}$$

For $r = 3$, the third L-moment of T is

$$\begin{aligned}
L_{T:3} &= \sigma\left(L_{X:3} - 0.5(1-\alpha)\left(\left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\right)L_{Z:1} + \left(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2}\right)L_{Z:2} + \frac{c_2^{(2)}}{6}L_{Z:3}\right)\right)\\
&= 0 - 0.5(1-\alpha)\left(\left(1 - \frac{6}{2} + \frac{6}{3}\right)(-2\log 2) + \left(-\frac{6}{2} + \frac{6}{2}\right)\cdot 1 - \frac{6}{6}\cdot\frac{1}{2}\right)\\
&= \frac{1}{4}(1-\alpha).
\end{aligned}$$

The fourth L-moment of T, for $r = 4$, is

$$\begin{aligned}
L_{T:4} &= \sigma\Bigg(L_{X:4} - 0.5(1-\alpha)\Bigg(\left(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\right)L_{Z:1} + \left(c_1^{(3)} + c_2^{(3)} + \frac{9}{10}c_3^{(3)}\right)\frac{L_{Z:2}}{2}\\
&\quad + \left(\frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{2}\right)\frac{L_{Z:3}}{2} + \frac{L_{Z:4}}{20}c_3^{(3)}\Bigg)\Bigg)\\
&= \frac{1}{6} - \frac{1}{2}(1-\alpha)\Bigg(\left(-1 + \frac{12}{2} - \frac{30}{3} + \frac{20}{4}\right)(-2\log 2) + \frac{1}{2}\left(12 - 30 + \frac{9}{10}\cdot 20\right)\cdot 1 - \frac{1}{2}\cdot\frac{1}{2}\left(-\frac{30}{3} + \frac{20}{2}\right)\\
&\quad + \frac{20}{20}\left(\frac{1}{6}\right)\Bigg)\\
&= \frac{1}{12}(1+\alpha).
\end{aligned}$$

■

4.9.4 Two-piece normal distribution

Theorem 4.9.7. *Let X be a real-valued random variable from the normal distribution, with $\mu \in \mathbb{R}$ and $\sigma > 0$ as the location and scale parameters, respectively, and Z be a real-valued standardized random variable from the standard half normal distribution. The first 4 L-moments of Z are given by Eq.(4.39).*

Proof. The depth in the the standard quantile function from Eq.(4.33) is set as $\frac{p}{2}$, and substituted into Eq.(3.17). The first L-moment of Z, where $r = 1$ and $P_0(\frac{1}{2}) = 1$, is obtained as

$$\begin{aligned} L_{Z:1} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right) dp \\ &= \int_0^1 \sqrt{2}\ \mathrm{erf}^{-1}(p-1)\ dp \\ &= -\sqrt{\frac{2}{\pi}} \\ &= -\sqrt{\frac{2}{\pi}}. \end{aligned}$$

For the second L-moment of Z, $r = 2$ and $P_1(\frac{1}{2}) = p - 1$. Hence

$$\begin{aligned} L_{Z:2}L_{Z:2} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)(p-1)\ dp \\ &= \int_0^1 \sqrt{2}\ \mathrm{erf}^{-1}(p-1)(p-1)\ dp \\ &= \int_0^1 \sqrt{2}\ \mathrm{erf}^{-1}(p-1)p\ dp - \int_0^1 \sqrt{2}\ \mathrm{erf}^{-1}(p-1)\ dp \\ &= \sqrt{2}\cdot\frac{\sqrt{2}-2}{2\sqrt{\pi}} - \left(-\sqrt{\frac{2}{\pi}}\right) \\ &= \frac{1}{\sqrt{\pi}}. \end{aligned}$$

When $r = 3$ and $P_2(\frac{1}{2}) = 6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1$, then the third L-moment of Z is

$$\begin{aligned} L_{Z:3} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1\right) dp \\ &= \int_0^1 \sqrt{2}\ \mathrm{erf}^{-1}(p-1)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1\right)\ dp \\ &= \frac{6}{4}\int_0^1 \sqrt{2}\ \mathrm{erf}^{-1}(p-1)p^2\ dp - 3\int_0^1 \sqrt{2}\ \mathrm{erf}^{-1}(p-1)p\ dp + \int_0^1 \sqrt{2}\ \mathrm{erf}^{-1}(p-1)\ dp \\ &= \frac{6}{4}(\sqrt{2}\cdot 0.078938) - 3\left(\frac{1}{\sqrt{2\pi}}\left(\sqrt{2}-2\right)\right) + \left(-\sqrt{\frac{2}{\pi}}\right) \\ &= 0.264252. \end{aligned}$$

The fourth L-moment is obtained when $r = 4$ and $P_3(\frac{1}{2}) = 20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1$, as

$$\begin{aligned} L_{Z:4} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right)\ dp \\ &= \int_0^1 sqrt2\mathrm{erf}^{-1}(p-1)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right)\ dp \\ &= \frac{20}{8}\int_0^1 \sqrt{2}\mathrm{erf}^{-1}(p-1)p^3\ dp - \frac{30}{4}\int_0^1 \sqrt{2}\mathrm{erf}^{-1}(p-1)p^2\ dp + 6\int_0^1 \sqrt{2}\mathrm{erf}^{-1}(p-1)p\ dp \\ &\quad - \int_0^1 \sqrt{2}\mathrm{erf}^{-1}(p-1)\ dp \\ &= \frac{20}{8}\left(\sqrt{2}(0.427296)\right) - \frac{30}{4}\times 0.111635 + 6\frac{1}{\sqrt{2\pi}}(\sqrt{2}-2) - \left(-\sqrt{\frac{2}{\pi}}\right) \\ &= \frac{1}{\sqrt{\pi}}\left(\frac{30}{\pi}\arctan(\sqrt{2}) - 9\right). \end{aligned}$$

Final results were computed using Wolfram Research, Inc. (2020). The results were obtained using Wolfram Research, Inc. (2020). ■

Theorem 4.9.8. *Suppose T is a real-valued standardized random variable from the two-piece normal distribution, denoted as $T \sim N_{TP}(0, \sqrt{\pi}, \alpha)$, where 0 is the location parameter, 1 is the scale parameter and $\alpha > 0$ is the asymmetry parameter. The first 4 L-moments of T are given in Eq.(4.40).*

Proof. The results in Eq.(4.40) are obtained by substituting the L-moments of X in Eq.(4.36) and the results of the L-moments of Z in Eq.(4.39) into Eq.(3.27), for the respective values of $r > 0$.

For $r = 1$, the first L-moment of T is

$$\begin{aligned}
L_{T:1} &= \mu + \sigma \left(L_{X:1} - 0.5(1-\alpha) c_0^{(0)} L_{Z:1} \right) \\
&= \sqrt{\pi} \left(0 - 0.5(1-\alpha) \left(-\sqrt{\frac{2}{\pi}} \right) \right) \\
&= \frac{1}{\sqrt{2}} (1-\alpha).
\end{aligned}$$

The second L-moment of T, for $r = 2$, is

$$\begin{aligned}
L_{T:2} &= \sigma \left(L_{X:2} - 0.5(1-\alpha) \left(c_0^{(1)} L_{Z:1} + \frac{c_1^{(1)}}{2} (L_{Z:1} + L_{Z:2}) \right) \right) \\
&= \sqrt{\pi} \left(\frac{1}{\sqrt{\pi}} - 0.5(1-\alpha) \left((-1) \left(\sqrt{\frac{2}{\pi}} \right) + \frac{2}{2} \left(\sqrt{\frac{2}{\pi}} + \frac{1}{\sqrt{\pi}} \right) \right) \right) \\
&= \frac{1}{2} (1+\alpha).
\end{aligned}$$

The third L-moment fo T is obtained when $r = 3$ as

$$\begin{aligned}
L_{T:3} &= \sigma \left(L_{X:3} - 0.5(1-\alpha) \left(\left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3} \right) L_{Z:1} + \left(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2} \right) L_{Z:2} + \frac{c_2^{(2)}}{6} L_{Z:3} \right) \right) \\
&= \sqrt{\pi} \left(0 - 0.5(1-\alpha) \left(\left(1 - \frac{6}{2} + \frac{6}{3} \right) \left(-\sqrt{\frac{2}{\pi}} \right) + \left(-\frac{6}{2} + \frac{6}{2} \right) \left(\frac{1}{\sqrt{\pi}} \right) - \frac{6}{6} \cdot (0.264252) \right) \right) \\
&= 0.2341872(1-\alpha).
\end{aligned}$$

Finally, the fourth L-moment is obtained when $r=4$ as

$$\begin{aligned}
L_{T:4} &= \sigma\Bigg(L_{X:4} - 0.5(1-\alpha)\Bigg(\left(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\right) L_{Z:1} + \left(c_1^{(3)} + c_2^{(3)} + \frac{9}{10}c_3^{(3)}\right)\frac{L_{Z:2}}{2} \\
&\quad + \left(\frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{2}\right)\frac{L_{Z:3}}{2} + \frac{L_{Z:4}}{20}c_3^{(3)}\Bigg)\Bigg) \\
&= \sqrt{\pi}\Bigg(\frac{1}{\sqrt{\pi}}\left(\frac{30}{\pi}\arctan\left(\sqrt{\pi}\right) - 9\right) - 0.5(1-\alpha)\Bigg(\left(-1 + \frac{12}{2} - \frac{30}{3} + \frac{20}{4}\right)\left(-\sqrt{\frac{2}{\pi}}\right) \\
&\quad + \frac{1}{2}\left(12 - 30 + \frac{9}{10}\cdot 20\right)\left(\frac{1}{\sqrt{\pi}}\right) - 0.264252\left(-\frac{30}{3} + \frac{20}{2}\right) + 1\cdot\frac{1}{\sqrt{\pi}}\left(\frac{30}{\pi}\arctan\left(\sqrt{\pi}\right) - 9\right)\Bigg)\Bigg) \\
&= \frac{1}{2}\left(\frac{30}{\pi}\arctan\left(\sqrt{2}\right) - 9\right)(1+\alpha).
\end{aligned}$$

■

4.9.5 Two-piece cosine distribution

Theorem 4.9.9. *Let X be a real-valued cosine random variable, with $\mu \in \mathbb{R}$ and $\sigma > 0$ as the location and scale parameters, respectively, and Z be a real-valued standardized random variable from the standard half cosine distribution. The first 4 L-moments of Z are given by Eq.(4.46).*

Proof. The depth in the standard quantile function from Eq.(4.43) is set as $\frac{p}{2}$, and substituted into Eq.(3.17). Using the transformation of variables, let $t = \arcsin\left(\sqrt{\frac{p}{2}}\right)$ be defined such that $p = 2\sin^2(t)$ and $dp = 2\sin(2t)dt$. Then it follows that the interval will be evaluated from $(0; \frac{\pi}{4})$. The first L-moment of Z, where $r=1$ and $P_0(\frac{1}{2}) = 1$, is obtained as

$$\begin{aligned}
L_{Z:1} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right) dp \\
&= \int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right) - 1\right)\, dp \\
&= \frac{4}{\pi}\int_0^1 \arcsin\left(\sqrt{\frac{p}{2}}\right) dp - \int_0^1 dp \\
&= \frac{8}{\pi}\int_0^{\frac{\pi}{4}} t\cdot\sin(2t)dt - p\Big|_0^1 \\
&= \frac{8}{\pi}\cdot\frac{1}{4} - 1 \\
&= \frac{2}{\pi} - 1.
\end{aligned}$$

When $r = 2$ and $P_1(\frac{1}{2}) = p - 1$, the second L-moment is obtained as

$$\begin{aligned}
L_{Z:2}L_{Z:2} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)(p-1)\ dp \\
&= \int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)(p-1)\ dp \\
&= \int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)p\ dp - \int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)\ dp \\
&= \frac{16}{\pi}\int_0^{\frac{\pi}{4}} t\cdot\sin^2(t)\cdot\sin(2t)dt - \frac{p^2}{2}\Big|_0^1 - \left(\frac{2}{\pi}-1\right) \\
&= \frac{16}{\pi}\left(\frac{1}{64}(8-\pi)\right) - \frac{1}{2} - \left(\frac{2}{\pi}-1\right) \\
&= \frac{1}{4}.
\end{aligned}$$

For the third L-moment, $r = 3$ and $P_2(\frac{1}{2}) = 6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1$. Therefore,

$$\begin{aligned}
L_{Z:3} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right)+1\right)dp \\
&= \int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right)+1\right)\ dp \\
&= \frac{3}{2}\int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)p^2dp - 3\int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)p\ dp + \int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)dp \\
&= \frac{3}{2}\left(\mu\frac{p^3}{3}\Big|_0^1 + \frac{32\sigma}{\pi}\int_0^{\frac{\pi}{4}} t\cdot\sin^4(t)\cdot\sin(2t)dt - \sigma\frac{p^3}{3}\Big|_0^1\right) - 3\left(\frac{\mu}{2}+\frac{2}{\pi}\sigma-\frac{3}{4}\right) + \left(\mu+\sigma\left(\frac{2}{\pi}-1\right)\right) \\
&= \frac{3}{2}\left(\frac{32}{\pi}\left(\frac{1}{576}(44-9\pi)\right)-\frac{\sigma}{3}\right) - 3\left(\frac{2}{\pi}-\frac{3}{4}\right) + \left(\frac{2}{\pi}-1\right) \\
&= -\frac{1}{3\pi}.
\end{aligned}$$

The fourth L-moment is obtained when $r = 4$ and $P_3(\frac{1}{2}) = 20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1$, as

$$\begin{aligned}
L_{Z:4} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right)-1\right)\ dp \\
&= \int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right)-1\right)\ dp \\
&= \frac{20}{8}\int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)p^3dp - \frac{30}{4}\int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)p^2dp \\
&\quad + 6\int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)p\ dp - \int_0^1 \left(\frac{4}{\pi}\arcsin\left(\sqrt{\frac{p}{2}}\right)-1\right)dp \\
&= \frac{20}{8}\left(\frac{64}{\pi}\int_0^{\frac{\pi}{4}} t\cdot\sin^6(t)\cdot\sin(2t)dt - \frac{p^4}{4}\Big|_0^1\right) - \frac{30}{4}\left(\frac{22}{9\pi}-\frac{5}{6}\right) + 6\left(\frac{2}{\pi}-\frac{3}{4}\right) - \left(\frac{2}{\pi}-1\right) \\
&= \frac{20}{8}\left(\frac{64}{\pi}\left(\frac{5}{96}-\frac{27}{2048}\pi\right)-\frac{1}{4}\right) - \frac{25}{3\pi} + \frac{11}{4} \\
&= \frac{1}{64}.
\end{aligned}$$

Final results were obtained using Gradshteyn and Ryzhik (2007, 1.321.1, 1.321.2, 1.323.2, 1.335.1, 2.633.1). ■

Theorem 4.9.10. *Suppose T is a real-valued standardized random variable from the two-piece cosine distribution, denoted as $T\sim COS_{TP}(0,\ 4,\ \alpha)$, where 0 is the location parameter, 4 is the scale parameter and $\alpha > 0$ is the asymmetry parameter. The first 4 L-moments of T are given in Eq.(4.47).*

Proof. The results in Eq.(4.47) are obtained by substituting the L-moments of X in Eq.(4.44) and the L-moments of Z in Eq.(4.46) into Eq.(3.27), for the respective values of $r > 0$.
For $r = 1$, the first L-moment of T is

$$\begin{aligned}
L_{T:1} &= \mu + \sigma\left(L_{X:1} - 0.5(1-\alpha)c_0^{(0)}L_{Z:1}\right)\\
&= 4\left(0 - 0.5(1-\alpha)\left(\frac{2}{\pi} - 1\right)\right)\\
&= \left(2 - \frac{4}{\pi}\right)(1-\alpha).
\end{aligned}$$

The second L-moment of T, obtained when $r = 2$, is

$$\begin{aligned}
L_{T:2} &= \sigma\left(L_{X:2} - 0.5(1-\alpha)\left(c_0^{(1)}L_{Z:1} + \frac{c_1^{(1)}}{2}(L_{Z:1} + L_{Z:2})\right)\right)\\
&= 4\left(\frac{1}{4} - 0.5(1-\alpha)\left((-1)\left(\frac{2}{\pi} - 1\right) + \frac{2}{2}\left(\left(\frac{2}{\pi} - 1\right) + \left(\frac{1}{4}\right)\right)\right)\right)\\
&= \frac{1}{2}(1+\alpha).
\end{aligned}$$

When $r = 3$, the third L-moment of T is

$$\begin{aligned}
L_{T:3} &= \sigma\left(L_{X:3} - 0.5(1-\alpha)\left(\left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\right)L_{Z:1} + \left(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2}\right)L_{Z:2} + \frac{c_2^{(2)}}{6}L_{Z:3}\right)\right)\\
&= 4\left(0 - 0.5(1-\alpha)\left(\left(1 - \frac{6}{2} + \frac{6}{3}\right)\left(\frac{2}{\pi} - 1\right) + \left(-\frac{6}{2} + \frac{6}{2}\right)\left(\frac{1}{4}\right) + \frac{6}{6}\left(-\frac{1}{3\pi}\right)\right)\right)\\
&= \frac{2}{3\pi}(1-\alpha).
\end{aligned}$$

The fourth L-moment of T, when $r = 4$, is

$$\begin{aligned}
L_{T:4} &= \sigma\Bigg(L_{X:4} - 0.5(1-\alpha)\Bigg(\left(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\right)L_{Z:1} + \left(c_1^{(3)} + c_2^{(3)} + \frac{9}{10}c_3^{(3)}\right)\frac{L_{Z:2}}{2}\\
&\quad + \left(\frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{2}\right)\frac{L_{Z:3}}{2} + \frac{L_{Z:4}}{20}c_3^{(3)}\Bigg)\Bigg)\\
&= 4\Bigg(\frac{1}{64} - 0.5(1-\alpha)\Bigg(\left(-1 + \frac{12}{2} - \frac{30}{3} + \frac{20}{4}\right)\left(\frac{2}{\pi} - 1\right) + \frac{1}{2}\left(12 - 30 + \frac{9}{10}\cdot 20\right)\left(\frac{1}{4}\right)\\
&\quad - \frac{1}{2}\left(-\frac{30}{3} + \frac{20}{2}\right)\left(\frac{1}{3\pi}\right) + \frac{20}{20}\left(\frac{1}{64}\right)\Bigg)\\
&= \frac{1}{32}(1+\alpha).
\end{aligned}$$

■

4.9.6 Two-piece uniform distribution

Theorem 4.9.11. *Let X be a real-valued random variable from the uniform distribution, with $\mu \in \mathbb{R}$ and $\sigma > 0$ as the location and scale parameters, respectively, and Z be a real-valued standardized random variable from the standard half uniform distribution. The first 4 L-moments of Z are given by Eq.(4.59).*

Proof. The depth in the standard quantile function from Eq.(4.50) is set as $\frac{p}{2}$, and substituted into Eq.(3.17). The first L-moment of Z, where $r = 1$ and $P_0(\frac{1}{2}) = 1$, is obtained as

$$\begin{aligned} L_{Z:1} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right) dp \\ &= \int_0^1 (p-1)\ dp \\ &= \frac{p^2}{2}\bigg|_0^1 - p\bigg|_0^1 \\ &= -\frac{1}{2}. \end{aligned}$$

When $r = 2$ and $P_1(\frac{1}{2}) = p - 1$, the second L-moment of Z is obtained as

$$\begin{aligned} L_{Z:2}L_{Z:2} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)(p-1)\ dp \\ &= \int_0^1 (p-1)(p-1)\ dp \\ &= \int_0^1 (p-1)p\ dp - \int_0^1 (p-1)\ dp \\ &= \frac{p^3}{3}\bigg|_0^1 - \frac{p^2}{2}\bigg|_0^1 - \left(-\frac{1}{2}\right) \\ &= \frac{1}{3}. \end{aligned}$$

The third L-moment of Z is obtained by setting $r = 3$ and $P_2(\frac{1}{2}) = 6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1$, such that

$$\begin{aligned} L_{Z:3} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1\right) dp \\ &= \int_0^1 (p-1)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1\right)\ dp \\ &= \frac{6}{4}\int_0^1 (p-1)p^2\ dp - 3\int_0^1 (p-1)p\ dp + \int_0^1 (p-1)\ dp \\ &= \frac{6}{4}\left(\frac{p^4}{4}\bigg|_0^1 - \frac{p^3}{3}\bigg|_0^1\right) - 3\left(-\frac{1}{6}\right) + \left(-\frac{1}{2}\right) \\ &= -\frac{1}{8}. \end{aligned}$$

For the fourth L-moment of Z, $r = 4$ and $P_3(\frac{1}{2}) = 20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1$. Therefore,

$$\begin{aligned}
L_{Z:4} &= \int_0^1 Q_{X:0}\left(\frac{1}{2}\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right)\ dp \\
&= \int_0^1 (p-1)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right)\ dp \\
&= \frac{20}{8}\int_0^1 (p-1)p^3\ dp - \frac{30}{4}\int_0^1 (p-1)p^2\ dp + 6\int_0^1 (p-1)p\ dp \\
&\quad - \int_0^1 (p-1)\ dp \\
&= \frac{20}{8}\left(\frac{p^5}{5}\Big|_0^1 - \frac{p^4}{4}\Big|_0^1\right) - \frac{30}{4}\left(-\frac{1}{12}\right) + 6\left(-\frac{1}{6}\right) - \left(-\frac{1}{2}\right) \\
&= 0.
\end{aligned}$$

■

Theorem 4.9.12. *Suppose T is a real-valued standardized random variable from the two-piece uniform distribution, denoted as $T \sim UNIF_{TP}(0,\ 3,\ \alpha)$, where 0 is the location parameter, 3 is the scale parameter and $\alpha > 0$ is the asymmetry parameter. The first 4 L-moments of T are given in Eq.(4.60).*

Proof. The results in Eq.(4.60) are obtained by substituting the L-moments of X in Eq.(4.54) and the results of the L-moments of Z in Eq.(4.59) into Eq.(3.27), for the respective values of $r > 0$.

For $r = 1$, the first L-moment of T is

$$\begin{aligned}
L_{T:1} &= \mu + \sigma\left(L_{X:1} - 0.5(1-\alpha)c_0^{(0)}L_{Z:1}\right) \\
&= 3\left(0 - 0.5(1-\alpha)\left(-\frac{1}{2}\right)\right) \\
&= \frac{3}{4}(1-\alpha).
\end{aligned}$$

The second L-moment of T, when $r = 2$, is given as

$$\begin{aligned}
L_{T:2} &= \sigma\left(L_{X:2} - 0.5(1-\alpha)\left(c_0^{(1)}L_{Z:1} + \frac{c_1^{(1)}}{2}(L_{Z:1} + L_{Z:2})\right)\right) \\
&= 3\left(\frac{1}{3} - 0.5(1-\alpha)\left((-1)\left(-\frac{1}{2}\right) + \frac{2}{2}\left(-\frac{1}{2} + \frac{1}{3}\right)\right)\right) \\
&= \frac{1}{2}(1+\alpha).
\end{aligned}$$

For $r = 3$, the third L-moment of T is

$$\begin{aligned}
L_{T:3} &= \sigma\left(L_{X:3} - 0.5(1-\alpha)\left(\left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\right)L_{Z:1} + \left(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2}\right)L_{Z:2} + \frac{c_2^{(2)}}{6}L_{Z:3}\right)\right) \\
&= 3\left(0 - 0.5(1-\alpha)\left(\left(1 - \frac{6}{2} + \frac{6}{3}\right)\left(-\frac{1}{2}\right) + \left(-\frac{6}{2} + \frac{6}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{6}{6}\right)\left(-\frac{1}{8}\right)\right)\right) \\
&= \frac{3}{16}(1-\alpha).
\end{aligned}$$

The fourth L-moment of T, for $r = 4$, is

$$\begin{aligned}
L_{T:4} &= \sigma\Bigg(L_{X:4} - 0.5(1-\alpha)\Bigg(\left(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\right) L_{Z:1} + \left(c_1^{(3)} + c_2^{(3)} + \frac{9}{10}c_3^{(3)}\right)\frac{L_{Z:2}}{2} \\
&\quad + \left(\frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{2}\right)\frac{L_{Z:3}}{2} + \frac{L_{Z:4}}{20}c_3^{(3)}\Bigg)\Bigg) \\
&= 3\Bigg(0 - 0.5(1-\alpha)\Bigg(\left(-1 + \frac{12}{2} - \frac{30}{3} + \frac{20}{4}\right)\left(-\frac{1}{2}\right) + \frac{1}{2}\left(12 - 30 + \frac{9}{10}\cdot 20\right)\left(\frac{1}{3}\right) \\
&\quad - \frac{1}{8}\cdot\frac{1}{2}\left(-\frac{30}{3} + \frac{20}{2}\right) + \frac{20}{20}\cdot(0)\Bigg)\Bigg) \\
&= 0.
\end{aligned}$$

■

4.9.7 Two-piece arcsine distribution

Theorem 4.9.13. *Suppose X is a real-valued arcsine random variable, with $\mu \in \mathbb{R}$ and $\sigma > 0$ as the location and scale parameters, respectively. Let Z be a real-valued standardized random variable from the standard half arcsine distribution. The first 4 L-moments of Z are given by Eq.(4.67).*

Proof. The depth in the standard quantile function from Eq.(4.63) is set as $\frac{p}{2}$, and substituted into Eq.(3.17). The first L-moment of Z, where $r = 1$ and $P_0(\frac{1}{2}) = 1$, is obtained as

$$\begin{aligned}
L_{Z:1} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right) dp \\
&= \int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right) dp \\
&= 2\int_0^1 \sin^2\left(\frac{\pi p}{4}\right)\, dp - \int_0^1 \, dp \\
&= \int_0^1 \left(1 - \cos\left(\frac{\pi p}{2}\right)\right)\, dp - p\Big|_0^1 \\
&= p - \sin\left(\frac{\pi p}{2}\right)\cdot\frac{2}{\pi}\Big|_0^1 - 1 \\
&= -\frac{2}{\pi}.
\end{aligned}$$

When $r = 2$ and $P_1(\frac{1}{2}) = p - 1$, the second L-moment of Z is

$$\begin{aligned}
L_{Z:2} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)(p-1)\ dp \\
&= \int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right)\left(2\left(\frac{p}{2}\right) - 1\right) dp \\
&= \int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right) p\ dp - \int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right) dp \\
&= 2\int_0^1 p\sin\left(\frac{\pi p}{4}\right)^2 dp - \int_0^1 p\ dp - \left(-\frac{2}{\pi}\right) \\
&= \int_0^1 p\left(1 - \cos\left(\frac{\pi p}{2}\right)\right)\ dp - \frac{p^2}{2}\bigg|_0^1 - \left(-\frac{2}{\pi}\right) \\
&= \frac{1}{2} - \int_0^1 p\cos\left(\frac{\pi p}{2}\right)\ dp - \frac{1}{2} - \left(-\frac{2}{\pi}\right) \\
&= -\left(\frac{2p}{\pi}\sin\left(\frac{\pi p}{2}\right) + \frac{4}{\pi^2}\sin\left(\frac{\pi p}{2} + \frac{\pi}{2}\right)\bigg|_0^1\right) - \left(-\frac{2}{\pi}\right) \\
&= -\left(\frac{2}{\pi} - \frac{4}{\pi^2}\right) - \left(-\frac{2}{\pi}\right) \\
&= \frac{4}{\pi^2}.
\end{aligned}$$

The third L-moment of Z is obtained when $r = 3$ and $P_2(\frac{1}{2}) = 6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1$, such that

$$\begin{aligned}
L_{Z:3} &= \int_0^1 Q_{X:0}\left(\frac{p}{2}\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1\right) dp \\
&= \int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right)\left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1\right) dp \\
&= \frac{3}{2}\int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right) p^2\ dp - 3\int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right) p\ dp + \int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right) dp \\
&= \frac{3}{2}\left(2\int_0^1 p^2\sin^2\left(\frac{\pi p}{4}\right) dp - \int_0^1 p^2\ dp\right) - 3\left(-\frac{2}{\pi} + \frac{4}{\pi^2}\right) + \left(-\frac{2}{\pi}\right) \\
&= \frac{3}{2}\left(\int_0^1 p^2\left(1 - \cos\left(\frac{\pi p}{2}\right)\right)\ dp - \frac{p^3}{3}\bigg|_0^1\right) + \frac{4}{\pi} - \frac{12}{\pi^2} \\
&= \frac{3}{2}\left(\frac{1}{3} - \int_0^1 p^2\cos\left(\frac{\pi p}{2}\right)\ dp - \frac{1}{3}\right) + \frac{4}{\pi} - \frac{12}{\pi^2} \\
&= \frac{3}{2}\left(-\left(\sum_{k=0}^{2} k!\binom{2}{k}\frac{p^{2-k}}{(\pi/2)^{k+1}}\sin\left(\frac{\pi p}{2} + \frac{\pi k}{2}\right)\right)\bigg|_0^1\right) + \frac{4}{\pi} - \frac{12}{\pi^2} \\
&= \frac{3}{2}\left(-\left(\frac{2p^2}{\pi}\sin\left(\frac{\pi p}{2}\right) + \frac{8p}{\pi^2}\sin\left(\frac{\pi p}{2} + \frac{\pi}{2}\right) + \frac{16}{\pi^3}\sin\left(\frac{\pi p}{2} + \pi\right)\right)\bigg|_0^1\right) + \frac{4}{\pi} - \frac{12}{\pi^2} \\
&= \frac{3}{2}\left(-\frac{2}{\pi} + \frac{16}{\pi^3}\right) + \frac{4}{\pi} - \frac{12}{\pi^2} \\
&= \frac{1}{\pi^3}\left(\pi^2 - 12\pi + 24\right).
\end{aligned}$$

The fourth L-moment of Z is obtained when $r = 4$ and $P_3(\frac{1}{2}) = 20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1$.

Therefore,

$$\begin{aligned}
L_{Z:4} &= \int_0^1 Q_{X:0}\left(\frac{1}{2}\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right)\ dp \\
&= \int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right)\left(20\left(\frac{p}{2}\right)^3 - 30\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1\right) dp \\
&= \frac{20}{8}\int_0^1 2\left(\sin^2\left(\frac{\pi p}{4}\right) - 1\right) p^3\ dp - \frac{30}{4}\int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right) p^2\ dp + 6\int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right) p\ dp \\
&\quad - \int_0^1 \left(2\sin^2\left(\frac{\pi p}{4}\right) - 1\right) dp \\
&= \frac{20}{8}\left(2\int_0^1 p^3 \sin\left(\frac{\pi p}{4}\right)^2 dp - \int_0^1 p^3\ dp\right) - \frac{30}{4}\left(-\frac{2}{\pi} + \frac{16}{\pi^3}\right) + 6\left(-\frac{2}{\pi} + \frac{4}{\pi^2}\right) - \left(-\frac{2}{\pi}\right) \\
&= \frac{20}{8}\left(\int_0^1 p^3\left(1 - \cos\left(\frac{\pi p}{2}\right)\right)\ dp - \left.\frac{p^4}{4}\right|_0^1\right) + \frac{5}{\pi} + \frac{24}{\pi^2} - \frac{120}{\pi^3} \\
&= \frac{20}{8}\left(\frac{1}{4} - \int_0^1 p^3\cos\left(\frac{\pi p}{2}\right)\ dp - \frac{1}{4}\right) + \frac{5}{\pi} + \frac{24}{\pi^2} - \frac{120}{\pi^3} \\
&= \frac{20}{8}\left(-\left.\left(\sum_{k=0}^{3} k!\binom{3}{k}\frac{p^{3-k}}{(\pi/2)^{k+1}}\sin\left(\frac{\pi p}{2} + \frac{\pi k}{2}\right)\right)\right|_0^1\right) + \frac{5}{\pi} + \frac{24}{\pi^2} - \frac{120}{\pi^3}\sigma \\
&= \frac{20}{8}\left(-\left.\left(\frac{2p^3}{\pi}\sin\left(\frac{\pi p}{2}\right) + \frac{12p^2}{\pi^2}\sin\left(\frac{\pi p}{2} + \frac{\pi}{2}\right) + \frac{48p}{\pi^3}\sin\left(\frac{\pi p}{2} + \pi\right) + \frac{96}{\pi^4}\sin\left(\frac{\pi p}{2} + \frac{3}{2}\pi\right)\right)\right|_0^1\right) \\
&\quad + \frac{5}{\pi} + \frac{24}{\pi^2} - \frac{120}{\pi^3} \\
&= -\left(\frac{5}{\pi}\sin\left(\frac{\pi}{2}\right) + \frac{60}{\pi^2}\sin(\pi) + \frac{120}{\pi^3}\sin\left(\frac{3}{2}\pi\right) + \frac{240}{\pi^4}\left(\sin(2\pi) - \sin\left(\frac{3}{2}\pi\right)\right)\right) + \frac{5}{\pi} + \frac{24}{\pi^2} - \frac{120}{\pi^3} \\
&= \frac{24}{\pi^4}\left(\pi^2 - 10\right).
\end{aligned}$$

The final results were obtained using Gradshteyn and Ryzhik (2007, 1.321.1, 2.633.1). ■

Theorem 4.9.14. *Suppose T is a real-valued standardized random variable that follows the two-piece arcsine distribution, denoted as $T \sim Arcsine_{TP}\left(0, \frac{\pi^2}{4}, \alpha\right)$, where 0, $\frac{\pi^2}{4}$ and $\alpha > 0$ are the location, scale and asymmetry parameters, respectively. The first 4 L-moments of T are given in Eq.(4.68).*

Proof. The results in Eq.(4.68) are obtained by substituting the L-moments of X in Eq.(4.65) and the results of the L-moments of Z in Eq.(4.67) into Eq.(3.27), for the respective values of $r > 0$. The first L-moment of T, for $r = 1$, is

$$\begin{aligned}
L_{T:1} &= \mu + \sigma\left(L_{X:1} - 0.5(1-\alpha)c_0^{(0)} L_{Z:1}\right) \\
&= \frac{\pi^2}{4}\left(0 - 0.5(1-\alpha)\left(-\frac{2}{\pi}\right)\right) \\
&= \frac{\pi}{4}(1-\alpha).
\end{aligned}$$

For $r = 2$, the second L-moment of T is obtained as

$$\begin{aligned}
L_{T:2} &= \sigma\left(L_{X:2} - 0.5(1-\alpha)\left(c_0^{(1)}L_{Z:1} + \frac{c_1^{(1)}}{2}(L_{Z:1} + L_{Z:2})\right)\right)\\
&= \frac{\pi^2}{4}\left(\frac{4}{\pi^2} - 0.5(1-\alpha)\left((-1)\left(-\frac{2}{\pi}\right) + \frac{2}{2}\left(-\frac{2}{\pi}\right) + \frac{4}{\pi^2}\right)\right)\\
&= 1 - 0.5(1-\alpha)\\
&= \frac{1}{2}(1+\alpha).
\end{aligned}$$

The third L-moment of T is obtained when $r = 3$ as

$$\begin{aligned}
L_{T:3} &= \sigma\left(L_{X:3} - 0.5(1-\alpha)\left(\left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\right)L_{Z:1} + \left(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2}\right)L_{Z:2} + \frac{c_2^{(2)}}{6}L_{Z:3}\right)\right)\\
&= \frac{\pi^2}{4}\left(0 - 0.5(1-\alpha)\left(\left(1 - \frac{6}{2} + \frac{6}{3}\right)\left(-\frac{2}{\pi}\right) + \left(-\frac{6}{2} + \frac{6}{2}\right)\left(\frac{4}{\pi^2}\right) - \frac{6}{6}\left(\frac{1}{\pi^3}\left(\pi^2 - 12\pi + 24\right)\right)\right)\right)\\
&= 0.5(1-\alpha)\left(\frac{1}{4\pi}\left(\pi^2 - 12\pi + 24\right)\right)\\
&= \frac{1}{8\pi}\left(\pi^2 - 12\pi + 24\right)(1-\alpha).
\end{aligned}$$

Finally, the fourth L-moment of T is obtained when $r = 4$ as

$$\begin{aligned}
L_{T:4} &= \sigma\Bigg(L_{X:4} - 0.5(1-\alpha)\Bigg(\left(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\right)L_{Z:1} + \left(c_1^{(3)} + c_2^{(3)} + \frac{9}{10}c_3^{(3)}\right)\frac{L_{Z:2}}{2}\\
&\quad + \left(\frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{2}\right)\frac{L_{Z:3}}{2} + \frac{L_{Z:4}}{20}c_3^{(3)}\Bigg)\Bigg)\\
&= \frac{\pi^2}{4}\Bigg(\left(6 - \frac{60}{\pi^2}\right) - \frac{1}{2}(1-\alpha)\Bigg(\left(-1 + \frac{12}{2} - \frac{30}{3} + \frac{20}{4}\right)\left(-\frac{2}{\pi}\right) + \frac{1}{2}\left(12 - 30 + \frac{9}{10}\cdot 20\right)\left(\frac{4}{\pi^2}\right)\\
&\quad + \frac{1}{2}\left(-\frac{30}{3} + \frac{20}{2}\right)\left(\frac{1}{\pi^3}\left(\pi^2 - 12\pi + 24\right)\right) + \left(\frac{24}{\pi^2} - \frac{240}{\pi^4}\right)\Bigg)\Bigg)\\
&= \frac{\pi^2}{4}\left(\left(\frac{24}{\pi^2} - \frac{240}{\pi^4}\right) - \frac{1}{2}(1-\alpha)\left(\frac{24}{\pi^2} - \frac{240}{\pi^4}\right)\right)\\
&= 3\left(1 - \frac{10}{\pi^2}\right)(1+\alpha).
\end{aligned}$$

■

TWO-PIECE TUKEY LAMBDA DISTRIBUTION

5.1 Introduction

The skewing methodology developed in Section 3.2 as well as the expressions for the r^{th} order L-moments in Section 3.3.4, were applied to univariate symmetric distributions without a shape parameter as shown in Chapter 4. The results obtained were synonymous with those derived in Chapter 2, in that the two-piece distributions exhibit skewness-invariant measures of kurtosis. Notably, the L-kurtosis ratio value of the proposed two-piece distributions would be the same as that of their respective parent distributions.

In this chapter, the exploration of an extension of the results to a family of symmetric distributions with a single shape parameter is considered. The Tukey lambda distribution, introduced by Tukey (1960), will be used as the parent distribution in this chapter. It is a symmetric quantile-based distribution with a single shape parameter that governs its kurtosis levels. This distribution will be presented and discussed in detail in Section 5.2.

The results from Theorem 3.2.1 will be used to construct the two-piece Tukey lambda distribution. Since this parent Tukey lambda distribution does not have closed-form expressions for the CDF of PDF, the two-piece distribution will be characterized through its quantile function.

The quantile measures of distributional form for location, shape and spread will be derived for the proposed two-piece family of distributions in Section 5.4. These results will highlight the consistency governed by the methodology through the skewness-invariant measures of kurtosis that are obtained.

The parameter space of the two-piece Tukey lambda distribution as well as the support for the distribution is presented in Section 5.5. This is with regards to the extensive levels of flexibility achieved through the different combinations of the two shape parameters that are present.

The classes and regions of the generalization are discussed in more detail in Section 5.6. The probability density curves of the distribution are also presented for different combinations of the shape parameters.

In Section 5.7, the expression for the r^{th} order L-moments of the two-piece Tukey lambda distribution are derived by making use of the relationships between order statistics, and essentially the L-moments of the Tukey lambda distribution and its corresponding half distribution. The L-moment ratio diagram for each class as well as the two-piece generalization as a whole are also illustrated in this section.

Section 5.8 will present the analysis of the tail behaviour of the distribution, by investigating the density of each tail, as well as the slope of the density on both the right and left tails. The estimation algorithm for obtaining the method of L-moments estimates for the two-piece generalization is presented in Section 5.9.

In conclusion, Section 5.10 will illustrate the fitting of the proposed two-piece Tukey lambda distribution to two real data sets. The results obtained will be compared to those of the GPD Type of the generalized lambda distribution (GLD$_{GPD}$) of van Staden (2014).

5.2 Tukey Lambda Distribution

Tukey (1960) proposed a family of distributions known as the lambda distributions. These are defined as distributions of the function $p^{\lambda} - (1-p)^{\lambda}$, where p is uniformly distributed on $(0,1)$ and $\lambda \in \mathbb{R}$ is a shape parameter that controls the level of kurtosis, resulting in the probability density curves exhibiting extensive distributional shapes.

This symmetric family of distributions is defined entirely through its quantile function and hence referred to as quantile-based, since it's CDF and PDF do not exhibit closed forms. This makes the use of conventional estimation procedures tedious and complicated to use.

5.2.1 Definition and special cases

Definition 5.2.1.1. *The quantile function, quantile density and density quantile functions of a real-valued random variable X, from the Tukey lambda distribution, are given as*

$$Q_X(p) = \begin{cases} \mu + \frac{\sigma}{\lambda}\left(p^{\lambda} - (1-p)^{\lambda}\right) & \lambda \neq 0,\ 0 < p < 1, \\ \mu + \sigma \log\left(\frac{p}{1-p}\right) & \lambda = 0,\ 0 < p < 1, \end{cases} \tag{5.1}$$

$$q_X(p) = \sigma\Big(p^{\lambda-1} + (1-p)^{\lambda-1}\Big), \quad 0 < p < 1,$$

and

$$f_P(p) = \frac{1}{\sigma\Big(p^{\lambda-1} + (1-p)^{\lambda-1}\Big)}, \quad 0 < p < 1,$$

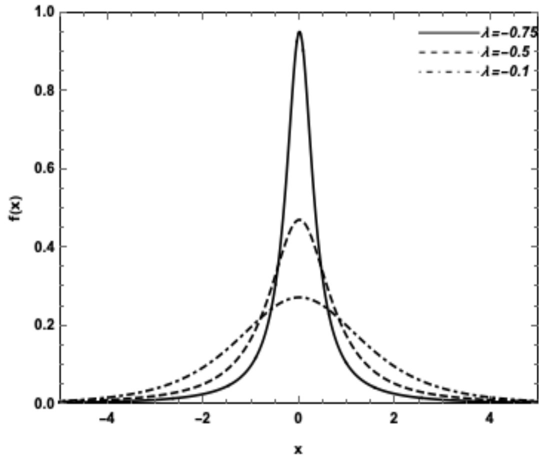
respectively, where $-\infty < \mu < \infty$*,* $\sigma > 0$ *and* $\lambda \in \mathbb{R}$ *are the location, scale and shape parameters, respectively.*

■

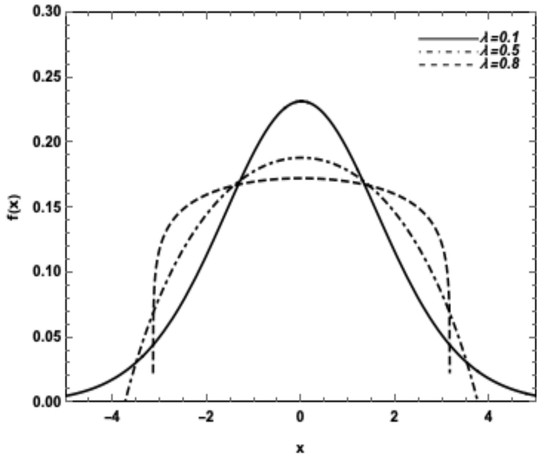
5.2.2 Distributional properties

The probability density curves from the Tukey lambda family of distributions exhibit various distributional shapes as a result of the kurtosis parameter, $\lambda \in \mathbb{R}$.

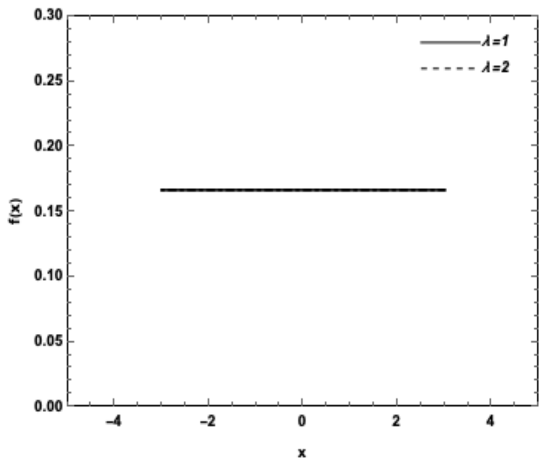
Fig.5.1(a) indicates the curves are unimodal bell-shaped with infinite support when $\lambda \leq 0$, and have bounded support when $0 < \lambda < 1$ as illustrated by Fig.5.1(b). The uniform distribution is obtained when $\lambda = 1$ or $\lambda = 2$ as seen in Fig.5.1(c). When $1 < \lambda < 2$, the U-shaped or bath-tub distributions are obtained in Fig.5.1(d), while Fig.5.1(e) shows unimodal truncated distributions are obtained when $\lambda > 2$. The logistic distribution is a special case that is obtained when $\lambda = 0$.



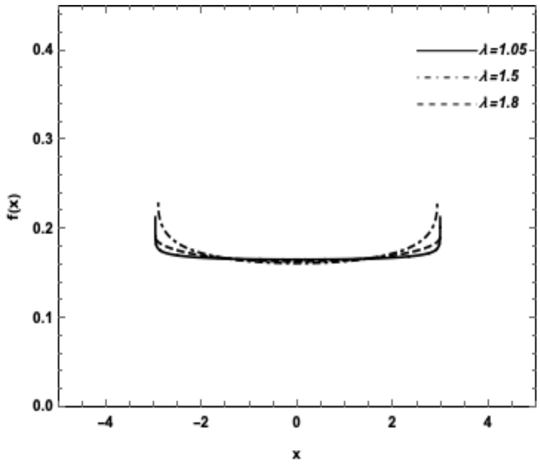
(a) $\lambda = -0.75, -0.5, \text{and } -0.1.$



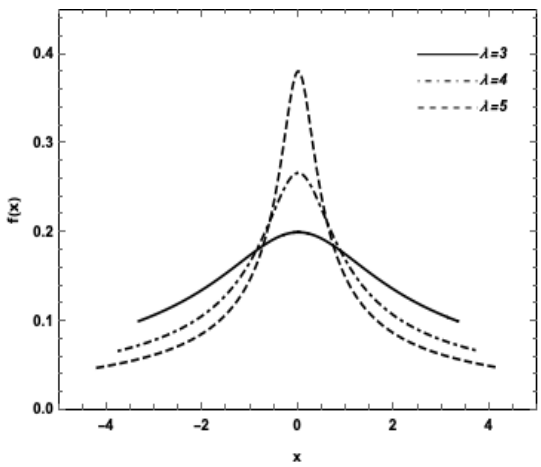
(b) $\lambda = 0.1,\ 0.5$ and $0.8.$



(c) $\lambda = 1$ and $2.$



(d) $\lambda = 1.05,\ 1.5$ and $1.8.$



(e) $\lambda = 3,\ 4$ and $5.$

Figure 5.1: The probability density curves for the Tukey lambda distribution with $L_1 = 0$ and $L_2 = 1$, for varying values of $\lambda \in \mathbb{R}$.

5.2.3 Generalizations of the lambda distribution

Ramberg-Schmeiser Type (GLD$_{RS}$)

Ramberg and Schmeiser (1972) wished to simplify the difficulty in generating symmetric random variables for univariate probability distributions. This was achieved through the use of the inverse transformation method since some distributions have CDFs that did not have a closed-form expressions or that were complicated in nature to use, such as the normal distribution.

Their proposed method would simplify this through the use of a simple form of the inverse function that would approximate the inverse functions of many continuous distributions, even when they do not exist in closed-form. The inverse CDF of the lambda distribution given in Eq.(5.1), with parameters, was used to generate the values of the random variables from the values of p, which are obtained from a uniform source of pseudo-random numbers. Ramberg and Schmeiser (1974) subsequently created an algorithm to generate asymmetric unimodal random variables by using the Tukey lambda distribution.

They introduced a generalized lambda distribution that included an additional shape parameter to accommodate the flexibility in distributional shape that these asymmetric random variables would possess. This generalization, denoted as GLD$_{RS}$, is defined through its quantile function as

$$Q_X(p) = \mu + \frac{1}{\sigma}\Big(p^{\lambda_1} - (1-p)^{\lambda_2}\Big), \quad \text{for } 0 \leq p \leq 1, \tag{5.2}$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are the location and scale parameters, respectively, whereas λ_1, $\lambda_2 \in \mathbb{R}$ are the shape parameters. If the random variable is symmetric, then $\lambda_1 = \lambda_2$ and the mean of the GLD is equal to μ. In the case of asymmetry, $\lambda_1 \neq \lambda_2$.

Karian and Dudewicz (2000) and Karian (2010) presented an in-depth study on this family of distributions, including its various functions, probabilistic properties and parameter estimation. The doctoral theses of King (1999) and van Staden (2014) present additional results for the GLD$_{RS}$.

Some of the areas where the GLD$_{RS}$ has been applied to are in actuarial science (Balasooriya and Low (2008)), biochemistry (Ramos-Fernández et al. (2008)), computer science (Gautama and van Gemund (2006)), economics (Pacáková and Sipková (2007)), queuing theory (Robinson and Chen (2003)) and signal processing (Karvanen et al. (2002)).

Freimer-Mudholkar-Kollia-Lin Type (GLD$_{FMKL}$)

The Freimer-Mudholkar-Kollia-Lin (FMKL) Type of the GLD was introduced by Freimer et al. (1988) and is denoted as GLD$_{FMKL}$. As with the GLD$_{RS}$, this distribution is quantile-based and defined through its quantile function given as

$$Q_X(p) = \mu + \frac{1}{\sigma}\left(\frac{p^{\lambda_1}-1}{\lambda_1} - \frac{(1-p)^{\lambda_2}-1}{\lambda_2}\right), \qquad 0 < p < 1,$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are the location and scale parameters, respectively, while λ_1, $\lambda_2 \in \mathbb{R}$ are the shape parameters.

The quantile density function and density quantile function of the GLD$_{FMKL}$ are respectively

$$q_X(p) = \frac{p^{\lambda_1-1} + (1-p)^{\lambda_2-1}}{\sigma}, \qquad 0 < p < 1,$$

and

$$f_P(p) = \frac{\sigma}{p^{\lambda_1-1} + (1-p)^{\lambda_2-1}}, \qquad 0 < p < 1.$$

The support of the distribution is bounded below by $\mu - \frac{1}{\sigma\lambda_1}$ when $\lambda_1 > 0$, bounded above by $\mu + \frac{1}{\sigma\lambda_2}$ when $\lambda_2 > 0$ and has infinite support when $\lambda_1 < 0$ and $\lambda_2 < 0$. Although not shown by Freimer et al. (1988), it is noted that the quantile function of the standard generalized Pareto distribution (GPD) is used as the building block of the quantile function of the GLD$_{FMKL}$.

Generalized Pareto Distribution Type (GLD$_{GPD}$)

van Staden (2014) developed a methodology that was used to develop quantile functions of symmetric quantile-based distributions by making use of asymmetric quantile functions that either has a bounded or half-infinite support. In the methodology, the quantile functions were obtained by taking the weighted sum of the standard and reflected quantile functions of an asymmetric distribution. In the case of the GLD$_{GPD}$ type, the quantile function of the generalized Pareto distribution is used as the kernel.

Definition 5.2.3.1. *A real-valued random variable X, denoted as $X \sim GLD_{GPD}(\mu, \sigma, \eta, \kappa)$, is said to have the GLD_{GPD} type distribution if its quantile function is defined as*

$$Q_X(p) = \begin{cases} \mu + \sigma\left((1-\eta)\left(\frac{p^\kappa-1}{\kappa}\right) + \eta\left(\frac{(1-p)^\kappa-1}{\kappa}\right)\right), & \kappa \neq 0 \\ \mu + \sigma\left((1-\eta)\log(p) + \eta\log(1-p)\right), & \kappa = 0, \end{cases} \tag{5.3}$$

where $-\infty < \mu < \infty$, $\sigma > 0$, $0 \leq \eta \leq 1$ and $\kappa \in \mathbb{R}$ are the location, scale and shape parameters, respectively.

■

The GLD$_{GPD}$ is symmetric when $\eta = \frac{1}{2}$, else it is asymmetric. The skew logistic distribution of van Staden and King (2015) is the limiting distribution of the GLD$_{GPD}$, when $\kappa = 0$.
The quantile density and density quantile functions follow from Eq.(5.3) as

$$q_X(p) = \sigma\left((1-\eta)p^{\kappa-1} + \eta(1-p)^{\kappa-1}\right)$$

and

$$f_P(p) = \frac{1}{\sigma\left((1-\eta)p^{\kappa-1} + \eta(1-p)^{\kappa-1}\right)},$$

respectively.

This resulting generalization is highly flexible with respect to distributional shape. Akin to the GLD$_{RS}$ and GLD$_{FMKL}$, it exhibits uniform, unimodal, J-shaped, U-shaped and truncated density curves. Some special cases of this distribution are the exponential, logistic, generalized Pareto, uniform and the Tukey lambda distributions. van Staden (2014) has presented the parameter space and support, as well as the regions and classes of this distribution in great depth.

Lemma 5.2.1. *If $X \sim GLD_{GPD}(\mu, \sigma, \eta, \kappa)$, then the L-location, L-scale, L-skewness and L-kurtosis ratios as derived by van Staden (2014), are*

$$\begin{aligned}
L_{X:1} &= \frac{(2\eta-1)}{\kappa+1} \\
L_{X:2} &= \frac{1}{(\kappa+1)(\kappa+2)} \\
\tau_{X:3} &= \frac{(2\eta-1)(1-\kappa)}{\kappa+3} \\
&\textit{and} \\
\tau_{X:4} &= \frac{(\kappa-1)(\kappa-2)}{(\kappa+3)(\kappa+4)},
\end{aligned} \tag{5.4}$$

respectively.

■

Notably, the L-kurtosis ratio is only influenced by κ, whilst the L-skewness ratio is governed by both shape parameters. This indicates that the L-kurtosis ratio of the GLD$_{GPD}$ is skewness-invariant. Since the L-moments take on simple closed-form expressions, and there is evidence of a skewness-invariant measure of L-kurtosis, the method of L-moments estimation can be used for parameter estimation.

5.3 Two-piece Tukey Lambda Distribution

Consider the first case of the Tukey lambda distribution when $\lambda \neq 0$ in Eq.(5.1). By employing the results of the quantile function in Eq.(3.3), the quantile function of the two-piece Tukey lambda distribution is defined as

$$Q_T(s) = \begin{cases} \mu + \frac{\alpha\sigma}{\lambda}\Big(s^\lambda - (1-s)^\lambda\Big), & s \leq \frac{1}{2}, \\ \mu + \frac{\sigma}{\lambda}\Big(s^\lambda - (1-s)^\lambda\Big), & s > \frac{1}{2}. \end{cases} \tag{5.5}$$

Lemma 5.3.1. *The quantile density and density quantile functions of the two-piece Tukey lambda distribution are defined as*

$$q_T(s) = \begin{cases} \alpha\sigma\Big(s^{\lambda-1} + (1-s)^{\lambda-1}\Big), & s \leq \frac{1}{2}, \\ \sigma\Big(s^{\lambda-1} + (1-s)^{\lambda-1}\Big), & s > \frac{1}{2}, \end{cases} \tag{5.6}$$

and

$$f_S(s) = \begin{cases} \frac{1}{\alpha\sigma\Big(s^{\lambda-1}+(1-s)^{\lambda-1}\Big)}, & s \leq \frac{1}{2}, \\ \frac{1}{\sigma\Big(s^{\lambda-1}+(1-s)^{\lambda-1}\Big)}, & s > \frac{1}{2}, \end{cases} \tag{5.7}$$

respectively.

Proof. The results follow directly from Eq.(5.5) since $q_T(s) = Q'_T(s)$ and $f_S(s) = q_T^{-1}(s)$. ■

5.4 Quantile Measures of Distributional Form

The quantile measures of distributional form for location, spread and shape for the two-piece Tukey lambda distribution are obtained by substituting Eq.(5.5) into Eqs.(3.52-3.56) respectively.

5.4.1 Location

The median is obtained when $s = \frac{1}{2}$. Therefore

$$\begin{aligned} me &= Q\Big(\frac{1}{2}\Big) \\ &= \mu + \frac{\sigma}{\lambda}\Big(\frac{1}{2}^\lambda - \Big(1 - \frac{1}{2}\Big)^\lambda\Big) \\ &= \mu + \frac{\sigma}{\lambda}(0) \\ &= \mu. \end{aligned}$$

5.4.2 Spread

The spread function, for $\frac{1}{2} < s < 1$, is

$$\begin{aligned}
S_T(s) &= Q_T(s) - Q_T(1-s) \\
&= \left\{\mu + \frac{\sigma}{\lambda}\Big(s^\lambda - \Big(1-s\Big)^\lambda\Big) - \Big(\mu + \frac{\alpha\sigma}{\lambda}\Big((1-s)^\lambda - \Big(1-(1-s)\Big)^\lambda\Big)\Big)\right\} \\
&= \frac{\sigma}{\lambda}\Big(s^\lambda - \Big(1-s\Big)^\lambda\Big) - \frac{\alpha\sigma}{\lambda}\Big((1-s)^\lambda - \Big(s\Big)^\lambda\Big) \\
&= \frac{\sigma}{\lambda}(1+\alpha)\Big(s^\lambda - \Big(1-s\Big)^\lambda\Big).
\end{aligned}$$

5.4.3 Shape

γ-functional

Following the substitution of Eq.(5.5) into Eq.(3.48), the γ-functional is attained as

$$\begin{aligned}
\gamma_T(s) &= \frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)} \\
&= \frac{\mu + \frac{\sigma}{\lambda}\Big(s^\lambda - \Big(1-s\Big)^\lambda\Big) + \mu + \frac{\alpha\sigma}{\lambda}\Big((1-s)^\lambda - \Big(s\Big)^\lambda\Big) - 2\mu}{\frac{\sigma}{\lambda}(1+\alpha)\Big(s^\lambda - \Big(1-s\Big)^\lambda\Big)} \\
&= \frac{1-\alpha}{1+\alpha}, \qquad \frac{1}{2} < s < 1.
\end{aligned}$$

Ratio-of-spread function

The ratio-of-spread function, for $\frac{1}{2} < v < u < 1$, is derived as

$$\begin{aligned}
R_T(u,v) &= \frac{S_T(u)}{S_T(v)} \\
&= \frac{\frac{\sigma}{\lambda}(1+\alpha)\Big(u^\lambda - \Big(1-u\Big)^\lambda\Big)}{\frac{\sigma}{\lambda}(1+\alpha)\Big(v^\lambda - \Big(1-v\Big)^\lambda\Big)} \\
&= \frac{u^\lambda - (1-u)^\lambda}{v^\lambda - (1-v)^\lambda}.
\end{aligned}$$

5.4.4 Skewness-Invariant Measure of Kurtosis

The skewness-invariant measure of kurtosis for the two-piece Tukey lambda distribution will follow from Eq.(3.56) if it takes the general form of

$$\frac{\sum_{i=1}^{n_1} g_i(Q_T(u_i))}{\sum_{j=1}^{n_2} h_j(Q_T(u_j))} = \frac{\sum_{i=1}^{n_1} g_i(u_i^\lambda - (1-u_i)^\lambda)}{\sum_{j=1}^{n_2} h_j(u_j^\lambda - (1-u_j)^\lambda)},$$

where n_1, $n_2 \in \mathbb{Z}^+$, and $g_i = 1, 2, ..., n_1$ and $h_j = 1, 2, ..., n_2$ are constants.

5.5 Parameter Space and Support

The parameter space of the two-piece Tukey lambda distribution can be divided into four distinct regions that are based on the combinations of the two shape parameters, $\alpha > 0$ and $\lambda \neq 0$, can take. The support of the distribution in each region is also presented. Table 5.1 indicates the different combinations of α and λ, as well as the corresponding parameter support.

Region	Shape parameter values	Support
I	$\alpha > 0, \lambda > 2$	$\left(-\frac{\alpha(\lambda+1)(\lambda+2)}{2\lambda}, \frac{(\lambda+1)(\lambda+2)}{2\lambda}\right)$
II	$\alpha > 0, 1 < \lambda < 2$	$\left(-\frac{\alpha(\lambda+1)(\lambda+2)}{2\lambda}, \frac{(\lambda+1)(\lambda+2)}{2\lambda}\right)$
	$\alpha > 0, \lambda = 1, 2$	$(-3\alpha; 3)$
III	$\alpha > 0, 0 < \lambda < 1$	$\left(-\frac{\alpha(\lambda+1)(\lambda+2)}{2\lambda}, \frac{(\lambda+1)(\lambda+2)}{2\lambda}\right)$
IV	$\alpha > 0, \lambda < 0$	$(-\infty, \infty)$

Table 5.1: Parameter space and support of the two-piece Tukey lambda distribution, in terms of Regions I, II, III and IV.

5.6 Classes

An alternative classification scheme can be used for the two-piece Tukey lambda distribution, in which the (α, λ)-space is divided into four classes based on the distributional shape obtained by the probability density curve of the distribution. The four classes are presented and graphical examples of density curves from each class are given.

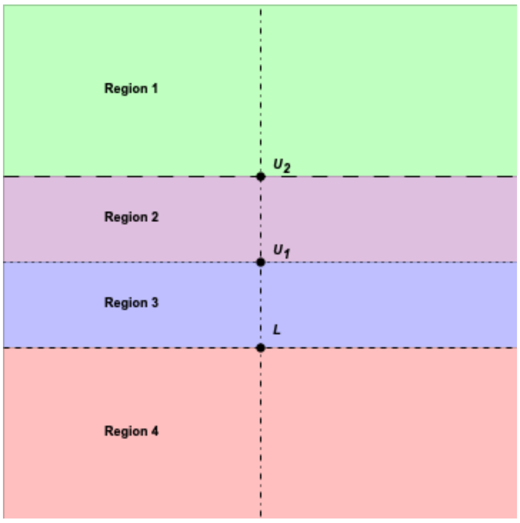


Figure 5.2: The parameter space of the two-piece Tukey lambda distribution in terms of Regions 1, 2, 3 and 4. The dot-dashed line, $\alpha = 1$, indicates symmetric distributions. The logistic distribution (L) is attained when $\lambda = 0$ whilst the uniform distribution (U_1 and U_2) is obtained when $\lambda = 1$ or 2, respectively.

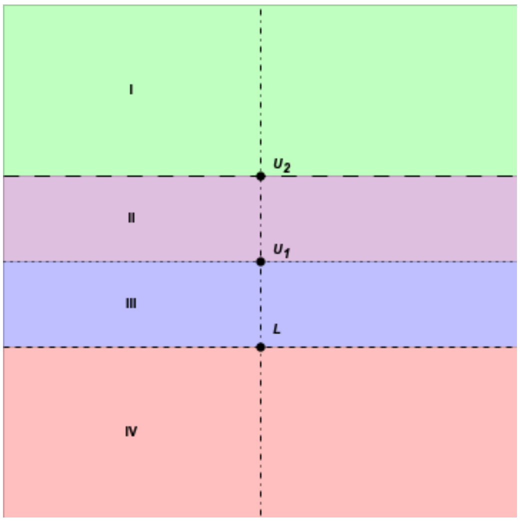
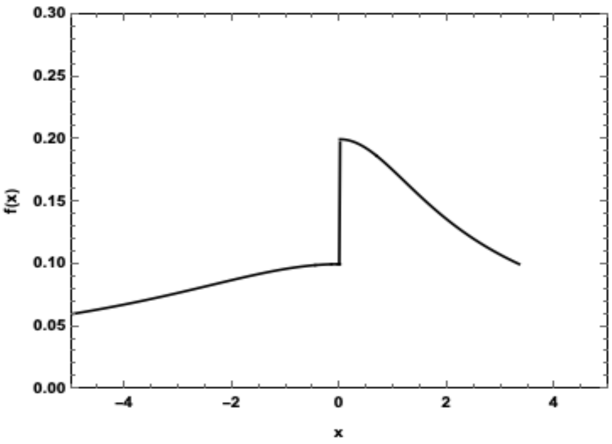


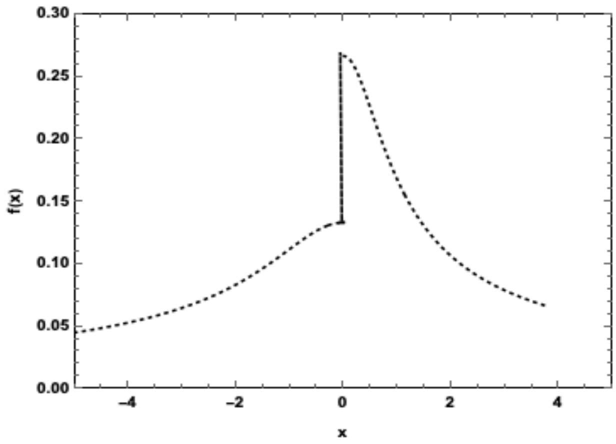
Figure 5.3: The parameter space of the two-piece Tukey lambda distribution in terms of Classes I, II, III and IV. The dot-dashed line, $\alpha = 1$, indicates symmetric distributions. The logistic distribution (L) is attained when $\lambda = 0$ whilst the uniform distribution (U_1 and U_2) is obtained when $\lambda = 1$ or 2, respectively.

5.6.1 Class I

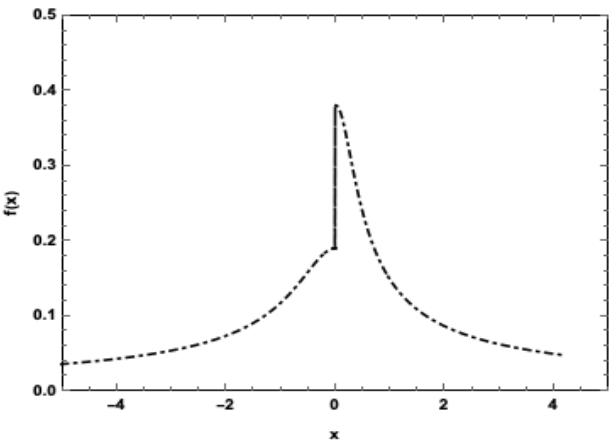
In this class, the probability density curves are unimodal truncated since $\lambda > 2$. The distributional shapes of the two-piece Tukey lambda distribution are illustrated in Figure 5.4. The values of α are fixed while the values of λ are changed in Fig. 5.4(a)-5.4(c). In Fig.5.4(d), the values of $\alpha > 0$ are varied and $\lambda = 4$.



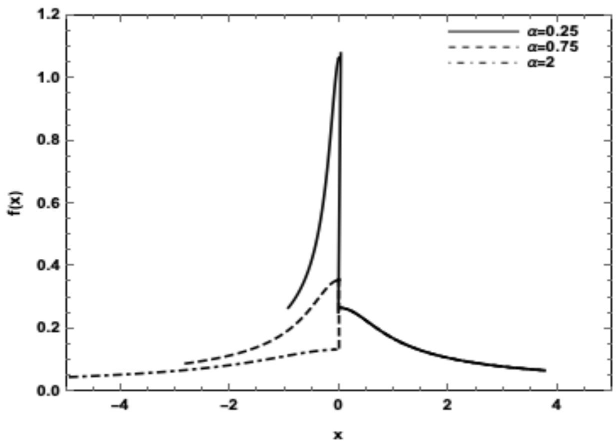
(a) $\alpha = 2$ and $\lambda = 3$.



(b) $\alpha = 2$ and $\lambda = 4$.



(c) $\alpha = 2$ and $\lambda = 5$.



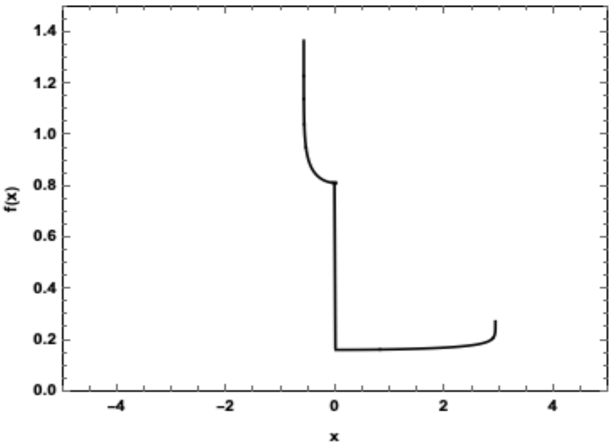
(d) $\lambda = 4, \alpha = 0.25,\ 0.75$ and 2.

Figure 5.4: The probability density curves for the two-piece Tukey lambda distribution with $L_1 = 0$ and $L_2 = 1$, for varying values of $\alpha > 0$ and $\lambda > 2$.

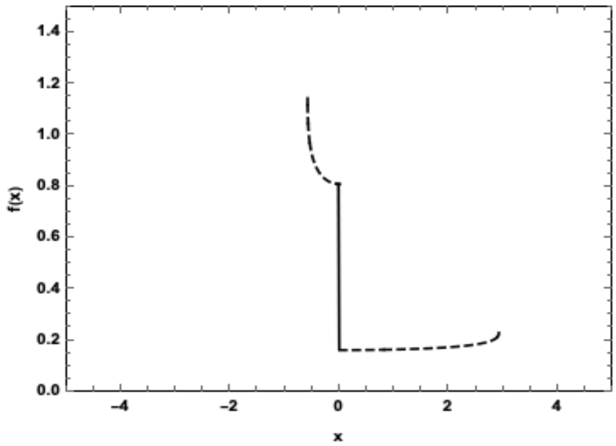
The curves in Fig.5.4(d) indicate that the distribution is negatively skewed when $\alpha > 1$ and positively skewed when $\alpha < 1$.

5.6.2 Class II

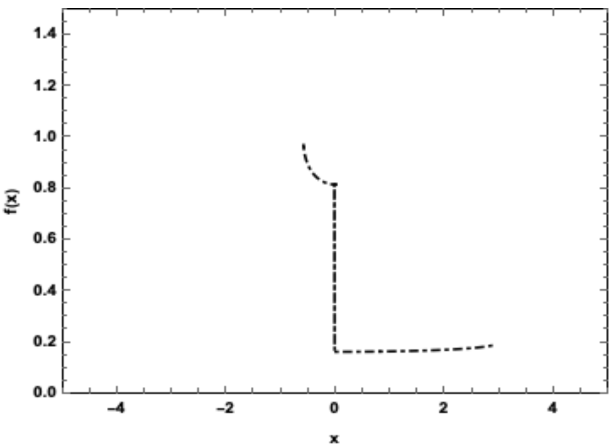
Class II considers probability density curves obtained when $1 \leq \lambda \leq 2$, thereby having a bounded support.. It is characterized by distributions that exhibit probability density functions termed as U-shaped, as well as the uniform distribution. The curves in Fig.5.5(a)-5.5(c) represent combinations of $\alpha = 0.2$ with $1 < \lambda < 2$, whereas Fig.5.5(d) illustrates the probability density curves when $\lambda = 1.5$ and $\alpha > 0$ is varied.



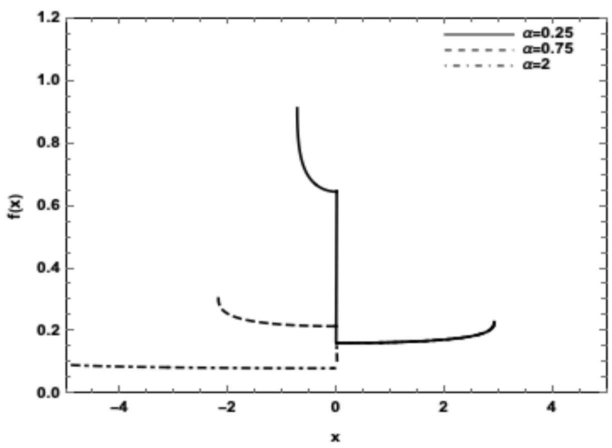
(a) $\alpha = 0.2$ and $\lambda = 1.25$.



(b) $\alpha = 0.2$ and $\lambda = 1.5$.



(c) $\alpha = 0.2$ and $\lambda = 1.75$.



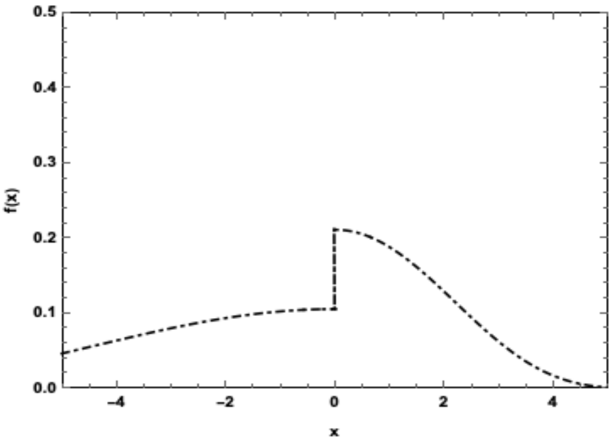
(d) $\lambda = 1.5, \alpha = 0.25,\ 0.75$ and 2.

Figure 5.5: The probability density curves for the two-piece Tukey lambda distribution with $L_1 = 0$ and $L_2 = 1$, for varying values of $\alpha > 0$ and $1 < \lambda < 2$.

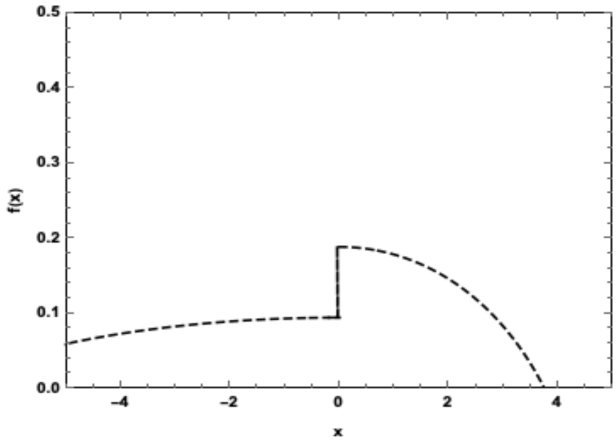
The two-piece uniform distribution is obtained when $\lambda = 1$ or 2 for varying values of $\alpha > 0$. The probability density curves can be seen in Fig.(4.17) in Chapter 4.

5.6.3 Class III

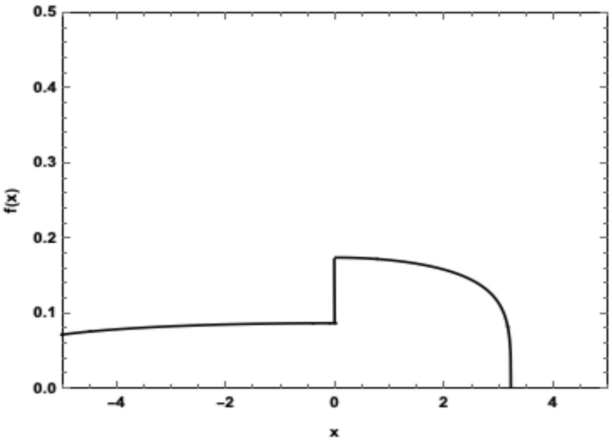
The probability density curves in this class are unimodal with a bounded support, as illustrated in Fig.5.6. This arises when $0 < \lambda < 1$ and $\alpha > 0$. In Fig.5.6(a)-5.6(c), $\alpha = 2$ while $0 < \lambda < 1$ is varied, whereas Fig.5.6(d) the value of $\alpha > 0$ is varied while $\lambda = 0.5$



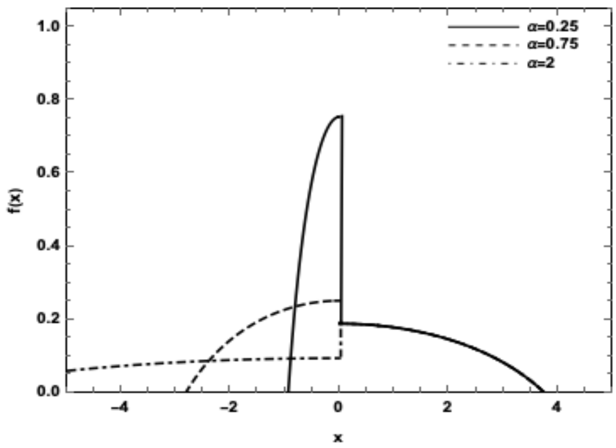
(a) $\alpha = 2$ and $\lambda = 0.25$.



(b) $\alpha = 2$ and $\lambda = 0.5$.



(c) $\alpha = 2$ and $\lambda = 0.75$.



(d) $\lambda = 0.5, \alpha = 0.25,\ 0.75$ and 2.

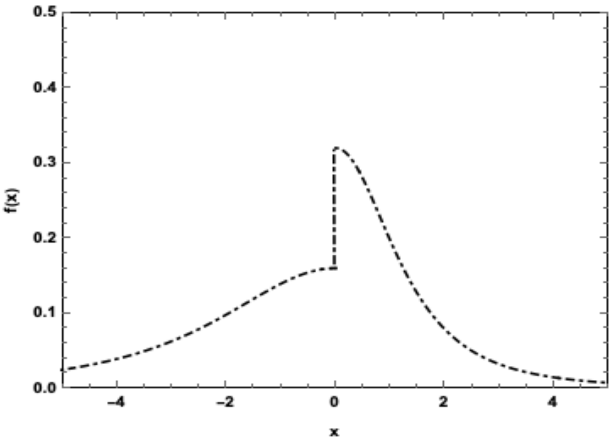
Figure 5.6: The probability density curves for the two-piece Tukey lambda distribution with $L_1 = 0$ and $L_2 = 1$, for varying values of $\alpha > 0$ and $0 < \lambda 1 <$.

Furthermore, the distribution is negatively skewed when $\alpha > 1$ and positively skewed when $\alpha < 1$.

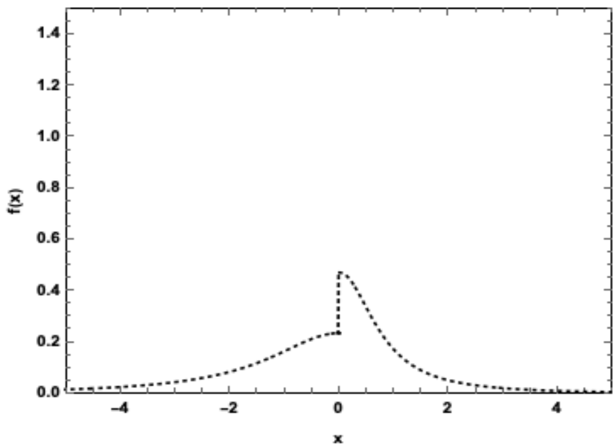
5.6.4 Class IV

The distributional shapes of the probability density curves in Class IV are unimodal bell-shaped with infinite support, occurring when $\lambda \leq 0$ and $\alpha > 0$. Examples of density curves from Class IV are shown in Figure 5.7, for selected values of α and $\lambda \leq 0$.

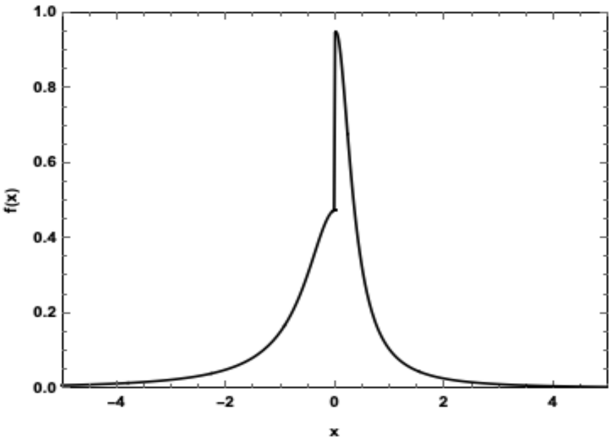
In Fig.5.7(a)-5.7(c), $\alpha = 2$ while $\lambda < 0$ is varied, whereas in Fig.5.6(d) the value of $\alpha > 0$ is varied while $\lambda = -0.5$ The two-piece logistic distribution is obtained as a special case when $\lambda = 0$.



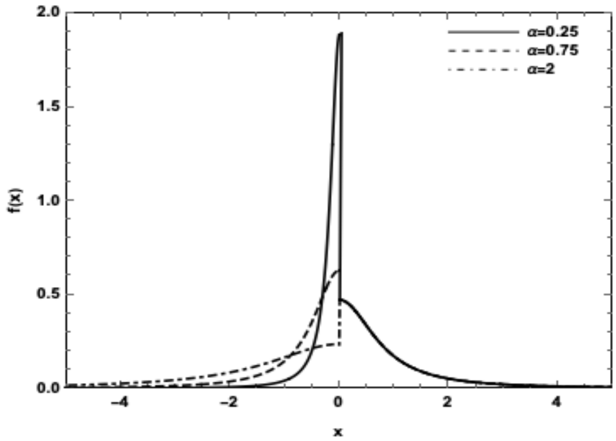
(a) $\alpha = 2$ and $\lambda = -0.25$.



(b) $\alpha = 2$ and $\lambda = -0.5$.



(c) $\alpha = 2$ and $\lambda = -0.75$.



(d) $\lambda = -0.5, \alpha = 0.25,\ 0.75$ and 2.

Figure 5.7: The probability density curves for the two-piece Tukey lambda distribution with $L_1 = 0$ and $L_2 = 1$, for varying values of $\alpha > 0$ and $\lambda < 0$.

5.7 r^{th} Order L-moments

In order to characterize the two-piece Tukey lambda distribution through the r^{th} order L-moments, the results in Section 3.3.4 will be used. These results will make use of the L-moments from the Tukey lambda distribution and its corresponding half distribution.

Lemma 5.7.1. *The first* 4 L*-moments of a real-valued random variable X, from the Tukey lambda distribution exist if* $\lambda > -1$*. They are given as*

$$L_{X:1} = 0, \quad L_{X:2} = \frac{2}{(\lambda+1)(\lambda+2)}, \quad L_{X:3} = 0 \quad \text{and} \quad L_{X:4} = \frac{2(\lambda-1)(\lambda-2)}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}, \tag{5.8}$$

where $-\infty < \mu < \infty$, $\sigma > 0$ *and* $\lambda > -1$ *are the location, spread and shape parameters, respectively.*

The L*-skewness and* L*-kurtosis ratios are obtained as*

$$\tau_{X:3} = \frac{L_3}{L_2} = 0 \quad \text{and} \quad \tau_{X:4} = \frac{L_4}{L_2} = \frac{(\lambda-1)(\lambda-2)}{(\lambda+3)(\lambda+4)}, \tag{5.9}$$

respectively.

■

The distribution is standardized when $\mu = 0$ and $\sigma = \frac{(\lambda+1)(\lambda+2)}{2}$.

Theorem 5.7.2. *The first 4* L*-moments of a standard half Tukey lambda random variable Z, denoted as* $L_{Z:1}$ *to* $L_{Z:4}$*, respectively, are*

$$\begin{aligned}
L_{Z:1} &= \frac{(1-2^{\lambda})}{\lambda 2^{\lambda}},\\
L_{Z:2} &= \frac{2}{(\lambda+1)(\lambda+2)},\\
L_{Z:3} &= \frac{-\lambda^2 - 5\lambda + \lambda 2^{\lambda+1}(1-\lambda)}{2^{\lambda}\lambda(\lambda+1)(\lambda+2)(\lambda+3)}\\
&\quad \text{and}\\
L_{Z:4} &= \frac{2(\lambda-1)(\lambda-2)}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}.
\end{aligned} \tag{5.10}$$

Proof. The first 4 L-moments of Z are obtained by making use of the results in Lemma 3.3.2 and Theorem 3.3.3, where $k = 2$.

For $r = 1$, $P_0^*(ku) = P_0^*(\frac{p}{2}) = 1$. Therefore, $L_{Z:1}$ is derived as

$$\begin{aligned}
L_{Z:1} &= \int_0^1 \frac{1}{\lambda}\left(\left(\frac{p}{2}\right)^{\lambda} - \left(1-\frac{p}{2}\right)^{\lambda}\right) \times 1 \; dp \\
&= \frac{1}{\lambda}\int_0^1 \left(\frac{p}{2}\right)^{\lambda} dp - \frac{1}{\lambda}\int_0^1 \left(1-\frac{p}{2}\right)^{\lambda} dp \\
&= \frac{1}{\lambda}\cdot\frac{p^{\lambda}}{2^{\lambda}(\lambda+1)}\bigg|_0^1 - (-2)\frac{1}{\lambda}\cdot\frac{\left(1-\frac{p}{2}\right)^{\lambda+1}}{(\lambda+1)}\bigg|_0^1 \\
&= \frac{1}{\lambda(\lambda+1)}\left(\frac{1}{2^{\lambda}} + \frac{1}{2^{\lambda}} - 2\right) \\
&= \frac{\left(1-2^{\lambda}\right)}{\lambda 2^{\lambda-1}(\lambda+1)}.
\end{aligned} \tag{5.11}$$

For $r = 2$, $P_1^*(ku) = P_1^*(\frac{p}{2}) = 2\left(\frac{p}{2}\right) - 1 = p - 1$. Therefore, $L_{Z:2}$ is derived as

$$\begin{aligned}
L_{Z:2} &= \int_0^1 \frac{1}{\lambda}\left(\left(\frac{p}{2}\right)^{\lambda} - \left(1-\frac{p}{2}\right)^{\lambda}\right) \times (p-1) \; dp \\
&= \int_0^1 \frac{p}{\lambda}\left(\left(\frac{p}{2}\right)^{\lambda} - \left(1-\frac{p}{2}\right)^{\lambda}\right) \; dp - \int_0^1 \frac{1}{\lambda}\left(\left(\frac{p}{2}\right)^{\lambda} - \left(1-\frac{p}{2}\right)^{\lambda}\right) \; dp \\
&= \frac{1}{\lambda}\left(\int_0^1 \frac{p^{\lambda+1}}{2^{\lambda}} dp - \int_0^1 p\left(1-\frac{p}{2}\right)^{\lambda} dp\right) - \frac{1}{\lambda(\lambda+1)}\left(\frac{1-2^{\lambda-1}}{2^{\lambda-1}}\right) \\
&= \frac{1}{\lambda}\left(\frac{p^{\lambda+2}}{2^{\lambda}(\lambda+2)}\bigg|_0^1 - \frac{2^{\lambda+2}-\lambda-3}{2^{\lambda}(\lambda+1)(\lambda+2)}\right) - \frac{1}{\lambda(\lambda+1)}\left(\frac{1-2^{\lambda-1}}{2^{\lambda-1}}\right) \\
&= \frac{1}{\lambda 2^{\lambda}(\lambda+2)}\left(1 - \frac{2^{\lambda+2}-\lambda-3}{(\lambda+1)}\right) - \frac{1}{\lambda(\lambda+1)}\left(\frac{1-2^{\lambda}}{2^{\lambda-1}}\right) \\
&= \frac{1}{\lambda 2^{\lambda}(\lambda+1)(\lambda+2)}\left(2\lambda+4-2^{\lambda+2}\right) - \frac{1}{\lambda 2^{\lambda}(\lambda+1)}\left(2-2^{\lambda+1}\right) \\
&= \frac{1}{\lambda 2^{\lambda}(\lambda+1)}\left(\frac{2\lambda+4-2^{\lambda+2}-(\lambda+2)\left(2-2^{\lambda+1}\right)}{\lambda+2}\right) \\
&= \frac{1}{\lambda 2^{\lambda}(\lambda+1)(\lambda+2)}\cdot\left(\lambda 2^{\lambda+1}\right) \\
&= \frac{2}{(\lambda+1)(\lambda+2)}.
\end{aligned} \tag{5.12}$$

Similarly, for $r = 3$, $P_2^*(ku) = P_2^*(\frac{p}{2}) = 6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1$. Therefore,

$$\begin{aligned}
L_{Z:3} &= \int_0^1 \frac{1}{\lambda}\left(\left(\frac{p}{2}\right)^\lambda - \left(1-\frac{p}{2}\right)^\lambda\right) \times \left(6\left(\frac{p}{2}\right)^2 - 6\left(\frac{p}{2}\right) + 1\right)\, dp \\
&= \frac{6}{\lambda}\int_0^1 \frac{p^2}{4}\left(\left(\frac{p}{2}\right)^\lambda - \left(1-\frac{p}{2}\right)^\lambda\right)\, dp - \frac{6}{\lambda}\int_0^1 \frac{p}{2}\left(\left(\frac{p}{2}\right)^\lambda - \left(1-\frac{p}{2}\right)^\lambda\right)\, dp \\
&+ \int_0^1 \left(\frac{1}{\lambda}\left(\left(\frac{p}{2}\right)^\lambda - \left(1-\frac{p}{2}\right)^\lambda\right)\right)\, dp \\
&= \frac{6}{\lambda}\left(\int_0^1 \frac{p^{\lambda+2}}{2^{\lambda+2}}dp - \int_0^1 \frac{p^2}{4}\left(1-\frac{p}{2}\right)^\lambda dp\right) - 3\left(\frac{(2\lambda+4-2^{\lambda+2})}{\lambda 2^\lambda(\lambda+1)(\lambda+2)}\right) + \frac{(2-2^{\lambda+1})}{\lambda 2^\lambda(\lambda+1)} \\
&= \frac{6}{2^{\lambda+2}\lambda}\left(\frac{p^{\lambda+3}}{(\lambda+3)}\Big|_0^1 - \frac{2^{\lambda+4} - 7\lambda - \lambda^2 - 14}{(\lambda+1)(\lambda+2)(\lambda+3)}\right) + \frac{-4\lambda - 8 + 8\cdot 2^\lambda - 2\lambda\cdot 2^\lambda}{\lambda 2^\lambda(\lambda+1)(\lambda+2)} \\
&= \frac{6\left((\lambda+1)(\lambda+2) - 2^{\lambda+4} + 7\lambda + \lambda^2 + 14\right)}{2^{\lambda+2}\lambda(\lambda+1)(\lambda+2)(\lambda+3)} + \frac{-4\lambda - 8 + 8\cdot 2^\lambda - 2\lambda\cdot 2^\lambda}{\lambda 2^\lambda(\lambda+1)(\lambda+2)} \\
&= \frac{6(2\lambda^2 + 10\lambda + 16 - 2^{\lambda+4})}{2^{\lambda+2}\lambda(\lambda+1)(\lambda+2)(\lambda+3)} + \frac{-4\lambda - 8 + 8\cdot 2^\lambda - 2\lambda\cdot 2^\lambda}{\lambda 2^\lambda(\lambda+1)(\lambda+2)} \\
&= \frac{3\lambda^2 + 15\lambda + 24 - 24\cdot 2^\lambda + (\lambda+3)\left(-4\lambda - 8 + 8\cdot 2^\lambda - 2\lambda\cdot 2^\lambda\right)}{2^\lambda\lambda(\lambda+1)(\lambda+2)(\lambda+3)} \\
&= \frac{3\lambda^2 + 15\lambda + 24 - 24\cdot 2^\lambda - 4\lambda^2 - 20\lambda - 24 + 2\lambda\cdot 2^\lambda - 2\lambda^2\cdot 2^\lambda + 24\cdot 2^\lambda}{2^\lambda\lambda(\lambda+1)(\lambda+2)(\lambda+3)} \\
&= \frac{-\lambda^2 - 5\lambda + 2\lambda\cdot 2^\lambda - 2\lambda^2\cdot 2^\lambda}{2^\lambda\lambda(\lambda+1)(\lambda+2)(\lambda+3)}
\end{aligned} \tag{5.13}$$

Finally, in order to obtain $L_{Z:4}$ when $r = 4$, $P_3^*(ku) = P_3^*(\frac{p}{2}) = 20\left(\frac{p}{2}\right)^3 - 20\left(\frac{p}{2}\right)^2 + 12\left(\frac{p}{2}\right) - 1$. Hence

$L_{Z:4}$ is derived as

$$
\begin{aligned}
L_{Z:4} &= \int_0^1 \frac{1}{\lambda}\left(\left(\frac{p}{2}\right)^{\lambda} - \left(1-\frac{p}{2}\right)^{\lambda}\right) \times \left(20\left(\frac{p}{2}\right)^{3} - 20\left(\frac{p}{2}\right)^{2} + 12\left(\frac{p}{2}\right) - 1\right)\, dp \\
&= \frac{20}{\lambda}\int_0^1 \frac{p^3}{8}\left(\left(\frac{p}{2}\right)^{\lambda} - \left(1-\frac{p}{2}\right)^{\lambda}\right)\, dp - \frac{30}{\lambda}\int_0^1 \frac{p^2}{4}\left(\left(\frac{p}{2}\right)^{\lambda} - \left(1-\frac{p}{2}\right)^{\lambda}\right)\, dp \\
&\quad + \frac{12}{\lambda}\int_0^1 \frac{p}{2}\left(\left(\frac{p}{2}\right)^{\lambda} - \left(1-\frac{p}{2}\right)^{\lambda}\right)\, dp - \int_0^1 \left(\frac{1}{\lambda}\left(\left(\frac{p}{2}\right)^{\lambda} - \left(1-\frac{p}{2}\right)^{\lambda}\right)\right)\, dp \\
&= \frac{20}{\lambda}\left(\int_0^1 \frac{p^{\lambda+3}}{2^{\lambda+3}}dp + \int_0^1 \frac{p^3}{8}\left(1-\frac{p}{2}\right)^{\lambda} dp\right) - 30\left(\frac{2\lambda^2+10\lambda+16-2^{\lambda+4}}{2^{\lambda+2}\lambda(\lambda+1)(\lambda+2)(\lambda+3)}\right) + 12\left(\frac{\left(2\lambda+4-2^{\lambda+2}\right)}{\lambda 2^{\lambda+1}(\lambda+1)(\lambda+2)}\right) \\
&\quad - \frac{\left(2-2^{\lambda+1}\right)}{2^{\lambda}\lambda(\lambda+1)} \\
&= \frac{20}{2^{\lambda+3}\lambda}\left(\left.\frac{p^{\lambda+4}}{(\lambda+4)}\right|_0^1 - \frac{96\cdot 2^{\lambda} - 90 - 53\lambda - 12\lambda^2 - \lambda^3}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}\right) - 30\left(\frac{2\lambda^2+10\lambda+16-2^{\lambda+4}}{2^{\lambda+2}\lambda(\lambda+1)(\lambda+2)(\lambda+3)}\right) \\
&\quad + 12\left(\frac{\left(2\lambda+4-2^{\lambda+2}\right)}{\lambda 2^{\lambda+1}(\lambda+1)(\lambda+2)}\right) - \frac{\left(2-2^{\lambda+1}\right)}{2^{\lambda}\lambda(\lambda+1)} \\
&= \frac{20}{2^{\lambda+3}\lambda}\left(\frac{(\lambda+1)(\lambda+2)(\lambda+3) - \left(96\cdot 2^{\lambda} - 90 - 53\lambda - 12\lambda^2 - \lambda^3\right)}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}\right) - 15\left(\frac{\lambda^2+5\lambda+8-2^{\lambda+3}}{2^{\lambda}\lambda(\lambda+1)(\lambda+2)(\lambda+3)}\right) \\
&\quad + 12\left(\frac{\left(\lambda+2-2^{\lambda}\right)}{\lambda 2^{\lambda+1}(\lambda+1)(\lambda+2)}\right) - \frac{\left(2-2^{\lambda+1}\right)}{2^{\lambda}\lambda(\lambda+1)} \\
&= \frac{5(\lambda^3+9\lambda^2+32\lambda+48-48\cdot 2^{\lambda})}{2^{\lambda}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} - \frac{15(\lambda+4)\left(\lambda^2+5\lambda+8-2^{\lambda+3}\right)}{2^{\lambda}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} + \frac{12(\lambda+3)(\lambda+4)\left(\lambda+2-2^{\lambda+1}\right)}{\lambda 2^{\lambda}(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} \\
&\quad - \frac{(\lambda+2)(\lambda+3)(\lambda+4)\left(2-2^{\lambda+1}\right)}{2^{\lambda}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} \\
&= \frac{1}{2^{\lambda}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} \times \left(4\lambda 2^{\lambda} - 6\lambda^2 2^{\lambda} + 2\lambda^3 2^{\lambda}\right) \\
&= \frac{2(\lambda-1)(\lambda-2)}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}.
\end{aligned}
\tag{5.14}
$$

The final results in Eqs.(5.12), (5.13) and (5.14) were obtained using Gradshteyn and Ryzhik (2007, 3.197.3). ■

Theorem 5.7.3. *The r^{th} order L-moments for a standard two-piece Tukey lambda random variable, for $1 \leq r \leq 4$, are given as*

$$
\begin{aligned}
L_{T:1} &= \frac{(\lambda+2)(2^{\lambda}-1)}{\lambda 2^{\lambda+1}}(1-\alpha), \\
L_{T:2} &= \frac{1}{2}(1+\alpha), \\
L_{T:3} &= \frac{\lambda+5+2^{\lambda+1}(\lambda-1)}{(\lambda+3)2^{\lambda+2}}(1-\alpha)
\end{aligned}
$$

and

$$
L_{T:4} = \frac{(\lambda-1)(\lambda-2)}{2(\lambda+3)(\lambda+4)}(1+\alpha), \tag{5.15}
$$

whilst the L-skewness and L-kurtosis ratios are subsequently obtained as

$$\tau_{T:3} = \frac{(1-\alpha)}{(1+\alpha)} \frac{\lambda + 5 + 2^{\lambda+1}(\lambda-1)}{(\lambda+3)2^{\lambda+1}} \qquad \textit{and} \qquad \tau_{T:4} = \frac{(\lambda-1)(\lambda-2)}{(\lambda+3)(\lambda+4)}, \tag{5.16}$$

respectively.

Proof. The first 4 L-moments in Eq.(5.15) are obtained by making use of $L_{X:1}$ to $L_{X:4}$, and L-moments $L_{Z:1}$ to $L_{Z:4}$ from Eqs.(5.8) and (5.10), respectively. To obtain $L_{T:1}$, $L_{X:1}$ and $L_{Z:1}$, as well as the polynomial coefficient $c_0^{(0)}$in the first row of Table 3.2 are substituted into Eq.(3.29) to obtain

$$\begin{aligned} L_{T:1} &= \mu + \sigma\left(L_{X:1} - 0.5(1-\alpha) \times c_0^{(0)} L_{Z:1}\right) \\ &= \frac{(\lambda+1)(\lambda+2)}{2}\left(0 - 0.5(1-\alpha)\left(\frac{(1-2^\lambda)}{\lambda 2^{\lambda-1}(\lambda+1)}\right)\right) \\ &= (1-\alpha)\ \frac{(\lambda+2)(2^\lambda-1)}{\lambda 2^{\lambda+1}}. \end{aligned}$$

$L_{T:2}$ is obtained when $L_{X:2}$, $L_{Z:1}$ and $L_{Z:2}$, as well as $c_0^{(1)}$ and $c_1^{(1)}$ from the second row of Table 3.2 are substituted into Eq.(3.33) to give

$$\begin{aligned} L_{T:2} &= \sigma\Big(L_{X:2} - 0.5(1-\alpha) \times \Big\{c_0^{(1)} L_{Z:1} + \frac{c_1^{(1)}}{2}(L_{Z:1} + L_{Z:2})\Big\}\Big) \\ &= \frac{(\lambda+1)(\lambda+2)}{2}\left(\frac{2}{(\lambda+1)(\lambda+2)} - 0.5(1-\alpha) \times \left\{(-1)\frac{(1-2^\lambda)}{\lambda 2^\lambda(\lambda+1)} + \frac{2}{2}\left(\frac{(1-2^\lambda)}{\lambda 2^\lambda(\lambda+1)}\right.\right.\right. \\ &\quad \left.\left.\left. + \frac{2}{(\lambda+1)(\lambda+2)}\right)\right\}\right) \\ &= \frac{(\lambda+1)(\lambda+2)}{2}\Big(1 - 0.5(1-\alpha)\Big) \\ &= \frac{1}{2}(1+\alpha). \end{aligned}$$

$L_{T:3}$ is derived by using $L_{X:3}$, $L_{Z:1}$, $L_{Z:2}$ and $L_{Z:3}$, as well as $c_0^{(2)}$, $c_1^{(2)}$ and $c_2^{(2)}$ from the third row of Table 3.2. This yields

$$\begin{aligned} L_{T:3} &= \sigma\left(L_{X:3} - 0.5(1-\alpha) \times \left\{L_{Z:1}\Big(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\Big) + L_{Z:2}\Big(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\Big) + L_{Z:3}\frac{c_2^{(2)}}{6}\right\}\right) \\ &= \frac{(\lambda+1)(\lambda+2)}{2}\left(0 - 0.5(1-\alpha) \times \left\{\frac{(1-2^\lambda)}{\lambda 2^\lambda(\lambda+1)} \times \left(1 + -\frac{6}{2} + \frac{6}{3}\right) + \left(-\frac{6}{2} + \frac{6}{2}\right)\frac{2}{(\lambda+1)(\lambda+2)}\right.\right. \\ &\quad \left.\left. + \frac{-\lambda^2 - 5\lambda + \lambda 2^{\lambda+1} - \lambda^2 2^{\lambda+1}}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)2^\lambda}\left(\frac{6}{6}\right)\right\}\right) \\ &= \frac{(\lambda+1)(\lambda+2)}{2}\left(-0.5(1-\alpha) \times \frac{-\lambda^2 - 5\lambda + \lambda 2^{\lambda+1} - \lambda^2 2^{\lambda+1}}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)2^\lambda}\right) \\ &= \frac{\lambda + 5 + 2^{\lambda+1}(\lambda-1)}{(\lambda+3)2^{\lambda+2}}(1-\alpha). \end{aligned}$$

The last L-moment derived, $L_{T:4}$, will result from the substitution of $L_{X:4}$, $L_{Z:1}$, $L_{Z:2}$, $L_{Z:3}$ and $L_{Z:4}$, as well as $c_0^{(3)}$, $c_1^{(3)}$, $c_2^{(3)}$ and $c_3^{(3)}$ from the fourth row of Table 3.2. Therefore

$$\begin{aligned}
L_{T:4} &= \sigma\Big(L_{X:4} - 0.5(1-\alpha)\times\Big\{L_{Z:1}\Big(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\Big) + L_{Z:2}\Big(\frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{2} + \frac{9c_3^{(3)}}{20}\Big) \\
&\quad + L_{Z:3}\Big(\frac{c_2^{(3)}}{6} + \frac{c_3^{(3)}}{4}\Big) + L_{Z:4}\frac{c_3^{(3)}}{20}\Big\}\Big) \\
&= \frac{(\lambda+1)(\lambda+2)}{2}\left(\frac{2(\lambda-1)(\lambda-2)}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} - 0.5(1-\alpha)\times\left\{\frac{(1-2^\lambda)}{\lambda 2^\lambda(\lambda+1)}\times\Big(-1+\frac{12}{2}-\frac{30}{3}+\frac{20}{4}\Big)\right.\right. \\
&\quad + \frac{2}{(\lambda+1)(\lambda+2)}\cdot\Big(\frac{12}{2}-\frac{30}{2}+\frac{9\cdot 20}{20}\Big) - \frac{\lambda^2+5\lambda+\lambda 2^{\lambda+1}(1-\lambda)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)2^\lambda}\times\Big(-\frac{30}{6}+\frac{20}{4}\Big) \\
&\quad \left.\left. + \frac{2(\lambda-1)(\lambda-2)}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}\times\Big(\frac{20}{20}\Big)\right\}\right) \\
&= \frac{(\lambda+1)(\lambda+2)}{2}\left(\frac{2(\lambda-1)(\lambda-2)}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} - 0.5(1-\alpha)\times\frac{2(\lambda-1)(\lambda-2)}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}\right) \\
&= \frac{(\lambda-1)(\lambda-2)}{2(\lambda+3)(\lambda+4)}(1+\alpha).
\end{aligned}$$

The subsequent L-skewness and L-kurtosis ratios are then obtained as

$$\begin{aligned}
\tau_{T:3} &= \frac{L_{T:3}}{L_{T:2}} \\
&= \frac{(1-\alpha)\ \frac{\lambda+5+2^{\lambda+1}(\lambda-1)}{(\lambda+3)2^{\lambda+2}}}{(1+\alpha)\ 0.5} \\
&= \frac{(1-\alpha)}{(1+\alpha)}\cdot\frac{\lambda+5+2^{\lambda+1}(\lambda-1)}{(\lambda+3)2^{\lambda+1}}
\end{aligned}$$

and

$$\begin{aligned}
\tau_{T:4} &= \frac{L_{T:4}}{L_{T:2}} \\
&= \frac{(1+\alpha)\ 0.5\ \frac{(\lambda-1)(\lambda-2)}{(\lambda+3)(\lambda+4)}}{(1+\alpha)\ 0.5} \\
&= \frac{(\lambda-1)(\lambda-2)}{(\lambda+3)(\lambda+4)},
\end{aligned}$$

respectively. ■

It can be noted that the L-kurtosis in Eq.(5.16) is skewness-invariant with respect to α, despite being a function of λ. This implies the two-piece Tukey lambda distribution will have varying levels of kurtosis with varying levels of skewness introduced by α. The special case of this distribution, which is the two-piece logistic distribution of Balakrishnan et al. (2017), is obtained when $\lambda = 0$.

5.7.1 L-moment ratio diagram

The L-skewness and L-kurtosis ratios in Eq.(5.16) are used to obtain an L-moment ratio diagram, illustrated by Fig.5.8. It depicts the extended level of skewness that is acquired by the new distribution, as compared to the parent Tukey lambda distribution, for varying levels of kurtosis. Since the distribution has two shape parameters, the L-moment ratio disgram will consist of a region as a result of the possible combinations of $\alpha > 0$ and $\lambda > -1$.

The solid line represents the two-piece logistic distribution whose L-skewness values range from -0.5 to 0.5, with a constant level of kurtosis at $\tau_{T:4} = 0.1667$. The dot-dashed line is representative of the two-piece uniform distribution whose L-skewness range of values is $(-0.375; 0.375)$, with $\tau_{T:4} = 0$. The dashed line represents the boundary for all distributions defined as $\tau_{T:4} = 0.25(5\tau_{T:3}^2 - 1)$, where $-1 < \tau_{T:3} < 1$.

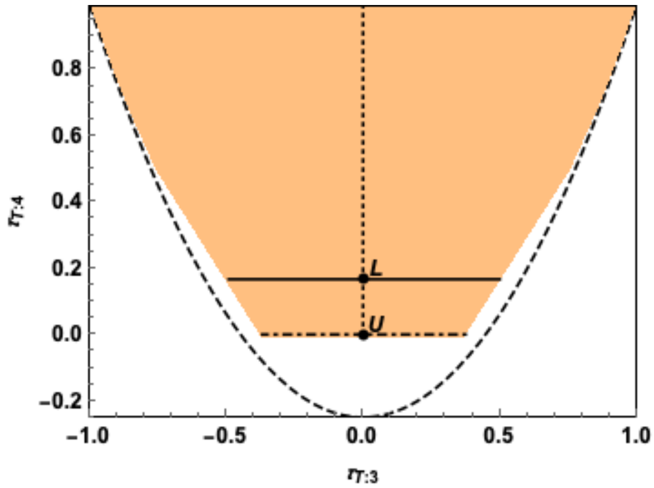


Figure 5.8: The L-moment ratio diagram for the two-piece Tukey lambda distribution with $\alpha > 0$ and $\lambda > -1$. The dashed line represents the boundary for all distributions. Note that the vertical dotted line, where $\alpha = 1$, indicates symmetric distributions. The logistic and uniform distributions are represented by L and U respectively.

In order to find the minimum point of λ, the first derivative of $\tau_{T:4}$ will be obtained, set to

0 and solved for λ as follows:

$$\begin{aligned}
\frac{d\tau_{T:4}}{d\lambda} &\Rightarrow \frac{(\lambda-1)+(\lambda-2)(\lambda+3)(\lambda+4)-(\lambda+3)+(\lambda+4)(\lambda-1)(\lambda-2)}{(\lambda+3)^2(\lambda+4)^2}=0 \\
&\Rightarrow \frac{(2\lambda-3)(\lambda+3)(\lambda+4)-(2\lambda+7)(\lambda-1)(\lambda-2)}{(\lambda+3)^2(\lambda+4)^2}=0 \\
&\Rightarrow \frac{2\lambda^3+14\lambda^2+24\lambda-3\lambda^2-21\lambda-36-2\lambda^3-6\lambda^2+4\lambda+7\lambda^2-21\lambda+14}{(\lambda+3)^2(\lambda+4)^2}=0 \\
&\Rightarrow 10\lambda^2+20\lambda-50=0. \qquad (5.17)
\end{aligned}$$

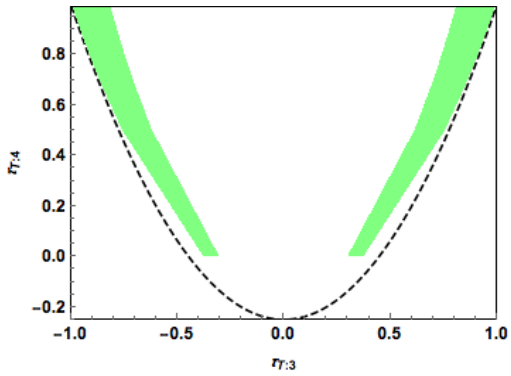
In order to solve for the minimum value of λ, the quadratic formula, will be used. Therefore

$$\begin{aligned}
\lambda &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\
&= \frac{-20\pm\sqrt{20^2-4\cdot 10\cdot -50}}{2\cdot 10} \\
&= \frac{-20\pm\sqrt{2400}}{20} \\
&\Rightarrow \lambda = -3.44195 \text{ or } 1.4495. \qquad (5.18)
\end{aligned}$$

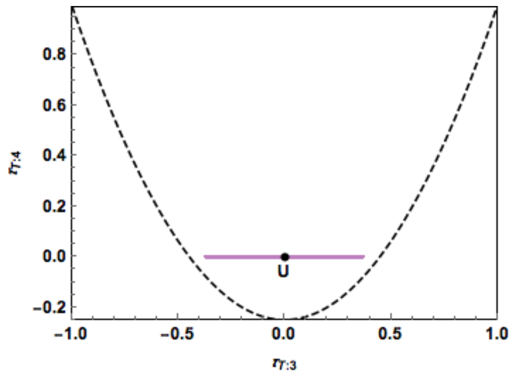
Since τ_4 is obtained when $\lambda > -1$, then $\lambda = 1.4495$ is chosen as the value which when substituted gives the minimum point of $\tau_{T:4}$ as -0.0102.

5.7.2 L-moment ratio diagrams for the two-piece Tukey lambda classes

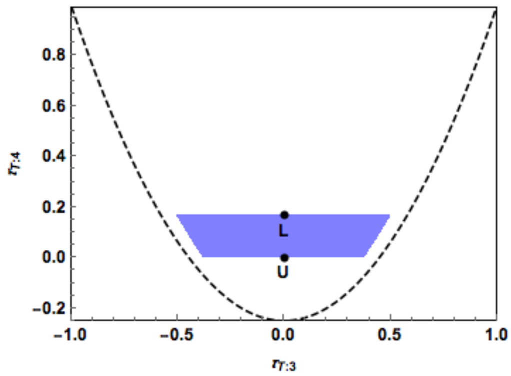
The L-moment ratio diagrams for the four classes are illustrated in Fig.5.9. The L-moments exist in Class I, II, III, Class IV only if $\lambda > -1$.



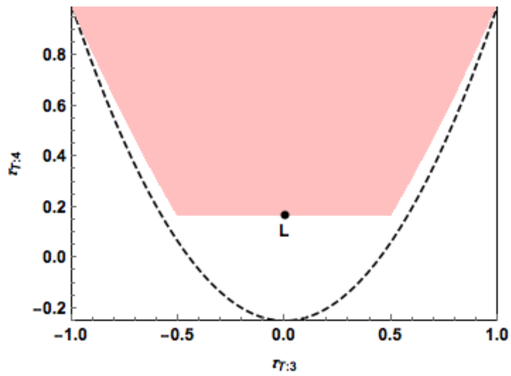
(a) CLASS I



(b) CLASS II



(c) CLASS III



(d) CLASS IV

Figure 5.9: The L-moment ratio diagrams for Class I, II, III and IV for the two-piece Tukey lambda distribution. The green, purple, blue and pink-shaded areas are the $(\tau_{T:3}, \tau_{T:4})$ regions covered by Class I, II, III and IV respectively. The logistic and uniform distributions are indicated by L and U, respectively.

5.8 Tail Behaviour

The tail behaviour of the two-piece Tukey lambda distribution is evaluated through the density quantile function $f_S(s)$ in Eq.(5.7), in order to determine the value that the probability density curve as it approaches the endpoints. This is explored through computing $\lim_{s\to 0} f_S(s)$ for the left tail, and $\lim_{s\to 1} f_S(s)$ for the right tail.

The slope of the probability density curve at these two tails is also evaluated by using Eq.(3.62). In the case of the two-piece Tukey lambda distribution,

$$\xi = \begin{cases} -\frac{(\lambda-1)(p^{\lambda-2}-(1-p)^{\lambda-2})}{\alpha\sigma^2(p^{\lambda-1}+(1-p)^{\lambda-1})^3}, & s \leq \frac{1}{2} \\ -\frac{(\lambda-1)(p^{\lambda-2}-(1-p)^{\lambda-2})}{\sigma^2(p^{\lambda-1}+(1-p)^{\lambda-1})^3}, & s > \frac{1}{2}. \end{cases} \tag{5.19}$$

The values obtained for the density and the slope of the density curve are summarized in Table 5.2.

Class	Shape parameter values	Density (Left)	Slope (Left)	Density (Right)	Slope (Right)
I	$\alpha > 0, \lambda > 2$	$\frac{1}{\alpha\sigma}$	$\frac{\lambda-1}{\alpha\sigma^2}$	$\frac{1}{\sigma}$	$-\frac{\lambda-1}{\sigma^2}$
II	$\alpha > 0, 1 < \lambda < 2$	$\frac{1}{\alpha\sigma}$	$-\infty$	$\frac{1}{\sigma}$	∞
	$\alpha > 0, \lambda = 1$	$\frac{1}{2\alpha\sigma}$	0	$\frac{1}{2\sigma}$	0
	$\alpha > 0, \lambda = 2$	$\frac{1}{\alpha\sigma}$	0	$\frac{1}{\sigma}$	0
III	$\alpha > 0, 0 < \lambda < 0.5$	0	0	0	0
	$\alpha > 0, \lambda = 0.5$	0	∞	0	$-\infty$
	$\alpha > 0, 0.5 < \lambda < 1$	0	$\frac{1}{2\alpha\sigma^2}$	0	$-\frac{1}{2\sigma^2}$
IV	$\alpha > 0, \lambda < 0$	0	0	0	0

Table 5.2: The values approached by the density curve and the slope of the density curve of the two-piece Tukey lambda distribution at the end-points of the tails.

5.9 Parameter Estimation

The steps outlined in Section 3.5 will be used to estimate the location, spread and two shape parameters from a sample, while adhering to any restrictions concerning the parameter estimates that need to be met.

STEP 1:

The first four sample L-moments from the observed data set will be derived using the results in Eq.(3.58). The sample L-moment ratios, t_3 and t_4, are obtained using Eq.(3.59). The two values are then verified if they lie in the ($\tau_{T:3}$,$\tau_{T:4}$)-space of the two-piece Tukey lambda distribution illustrated in Fig.(5.8). Should these values be found to lie in that space, then the estimation procedures can continue, else the two-piece Tukey lambda distribution cannot be fitted to the data set.

STEP 2:

Since the two-piece Tukey lambda distribution results indicate it is skewness-invariant with respect to $\alpha > 0$, then $\tau_{T:4}$ remains unchanged. Therefore, t_4 will be used in the theoretical result obtained for $\tau_{T:4}$ in Eq.(5.16) to solve for $\hat{\lambda}$. The minimum value for $\hat{\lambda}$ should be 1.4495, shown in Eq.(5.18), in order for the estimate to be valid in the model fit. The *Solve* function in Wolfram Research, Inc. (2020) or the quadratic formula can be used to obtain the valid solution for $\hat{\lambda}$ by solving

$$\hat{\lambda} = \frac{-(7t_4+3) \pm \sqrt{(7t_4+3)^2 - 4(t_4-1)(6t_4-1)}}{2(t_4-1)}. \tag{5.20}$$

STEP 3:

An estimate for $\hat{\alpha}$ will be derived from the theoretical L-skewness ratio result in Eq.(5.16) as well as the value of t_3 from the sample. Let g be the coefficient for $\tau_{T:3}$, obtained when the value of the valid $\hat{\lambda}$ estimated is substituted into the equation. The estimate is then obtained when

$$\hat{\alpha} = \frac{1-\frac{t_3}{g}}{1+\frac{t_3}{g}}, \tag{5.21}$$

is solved.

STEP 4:

Similarly, the value of ℓ_2, the theoretical result of L_2 in Eq.(5.15) as well as the value of $\hat{\alpha}$ from Step 3 will be required to obtain an estimate for $\hat{\sigma}$.

Therefore

$$\hat{\sigma} = \frac{\ell_2}{0.5(1+\hat{\alpha})}. \tag{5.22}$$

STEP 5:

The final estimate, $\hat{\mu}$, will be calculated when $\hat{\alpha}$, $\hat{\sigma}$, $\hat{\lambda}$ and ℓ_1 are substituted into the theoretical result for $L_{T:1}$ in Eq.(5.15) and the estimate is then solved for.

Hence,

$$\hat{\mu} = \ell_1 - \hat{\sigma}(1-\hat{\alpha})\frac{(\hat{\lambda}+2)(2^{\hat{\lambda}-1})}{\hat{\lambda}2^{\hat{\lambda}+1}}. \tag{5.23}$$

5.10 Application to Data Sets

The Tukey lambda distribution has been widely used in fitting various data sets due to the wide range of distributional forms the probability density curves can take. These include the fields of biochemistry, economics, computer science, among others. This section will present two data set fittings of the two-piece Tukey lambda distribution. The fit of the model is compared to that of the GLD_{GPD} of van Staden (2014) and the perfomance of each discussed. The first data set consists of information on the albumin levels of cirrhosis patients from the Mayo Clinic, while the second data set investigates the body mass index (BMI) levels of Australian athletes.

5.10.1 Albumin Levels of Cirrhosis Patients

Albumin is a protein produced by the liver that prevents the fluid in the bloodstream from seeping into other tissues in the body. It also transports hormones, vitamins and enzymes around the body. Low albumin levels are an indicator of possible liver or kidney failure. The albumin (gm/dl) levels of 419 patients were obtained from the primary biliary cirrhosis data of Mayo Clinic (Fleming and Harrington (1991)).

The first two sample L-moments, i.e. the L-location and L-scale, as well as the sample L-skewness and L-kurtosis ratios, are calculated as $\ell_1 = 3.4974$, $\ell_2 = 0.2365$, $t_3 = -0.07395$ and $t_4 = 0.1504$, respectively. It can be noted that the values of t_3 and t_4 lie in Class III of Region III.

Fig.5.10(a) gives a histogram of the albumin levels in the patients.

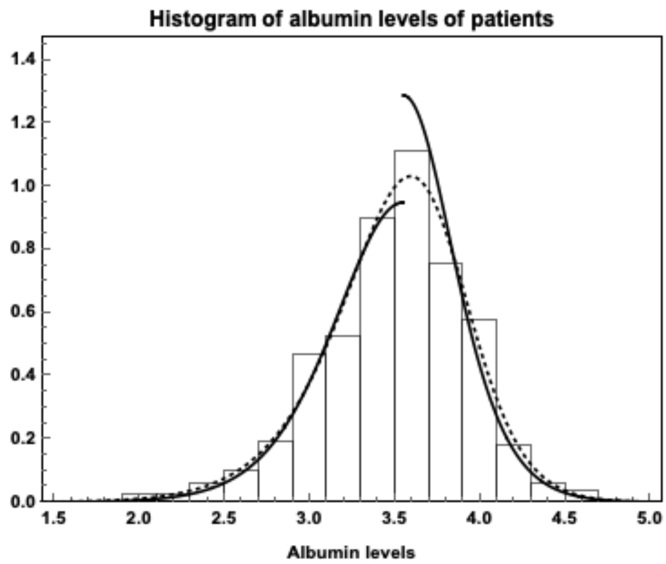


Figure 5.10: A histogram of the albumin levels of patients together with the probability density curves of the fitted two-piece Tukey lambda (solid curve) and GLD_{GPD} (dashed curve) distributions.

The values of t_3 and t_4 fall in the $(\tau_{T:3}, \tau_{T:4})$-space of the two-piece Tukey's lambda distribution illustrated in Fig.5.8, and so the proposed two-piece distribution can be used to fit this data set.

Since $t_3 < 0$, the data is negatively skewed. The data also exhibits heavy tails since $t_4 > 0.122602$, the theoretical L-kurtosis ratio value of the normal distribution.

The parameter estimates of the fitted two-piece Tukey lambda distribution are obtained using the method of L-moments. The estimates, together with their asymptotic standard errors, are shown in Table 5.3.

$\hat{\mu}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\lambda}$
3.54743	0.200693	1.35678	0.048657
(0.02345)	(0.01155)	(0.12205)	(0.05189)

Table 5.3: Parameter estimates with asymptotic standard errors (in parentheses) for the two-piece Tukey lambda distribution fitted to the albumin levels of patients.

Since $0 < \hat{\lambda} < 1$, the distribution of the data will have a bounded support $(-29.9525; 22.0762)$. The parameter estimates and corresponding asymptotic standard errors for the GLD_{GPD} fit are displayed in Table 5.4.

$\hat{\mu}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\lambda}$
3.61226	0.508071	0.381509	0.048657
(0.0371525)	(0.0445206)	(0.035559)	(0.052089)

Table 5.4: Parameter estimates with asymptotic standard errors (in parentheses) for the GLD_{GPD} fitted to the albumin levels of patients.

The standard errors for the location, scale and kurtosis parameter seem to be larger for the GLD_{GPD} estimates than for those of the two-piece Tukey lambda distribution. The Q-Q plots for the fits of the two-piece Tukey lambda distribution and GLD_{GPD} are illustrated in Fig.5.11.

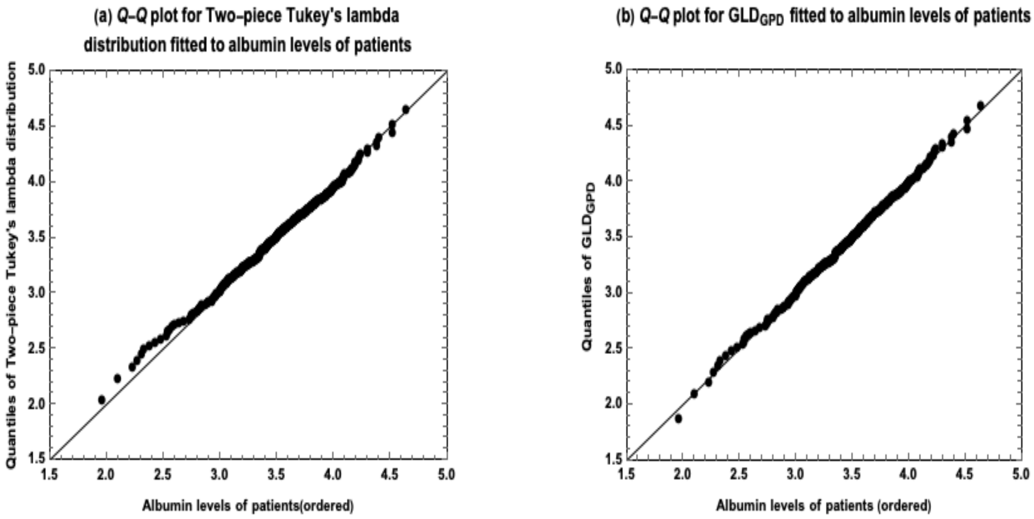


Figure 5.11: Q-Q plots for the two-piece Tukey lambda distribution and the GLD_{GPD} fitted to the albumin levels of patients.

It is evident that the comparison between the two Q-Q plots indicates the fitted GLD_{GPD} provides a slightly better fit, specifically in the lower tail. The ASAE value for the GLD_{GPD} is 0.004142, which is lower than that of the two-piece Tukey lambda distribution which is 0.0096217. It therefore seems the GLD_{GPD} provides a better fit to the data set.

5.10.2 Body Mass Index Levels of Australian Athletes

The data set `ais` consists of various measurements on 202 Australian athletes (Cook and Weisberg, 2009). It is available in the `R`-package `sn` (Azzalini, 2014). Here we only consider the variable measuring the athletes' body mass index, BMI. The sample L-location and L-scale values from the data are $\ell_1 = 22.9559$ and $\ell_2 = 1.54562$, respectively. The L-skewness ratio and L-kurtosis ratio values are $t_3 = 0.110844$ and $t_4 = 0.183603$, respectively. These lie in Class IV of Region IV as can be seen from Fig.5.9(c).

Fig.5.12 gives a histogram of the BMI, together with the probability density curves of the two-piece Tukey lambda distribution (solid curve) and the GLD_{GPD} (dotted curve).

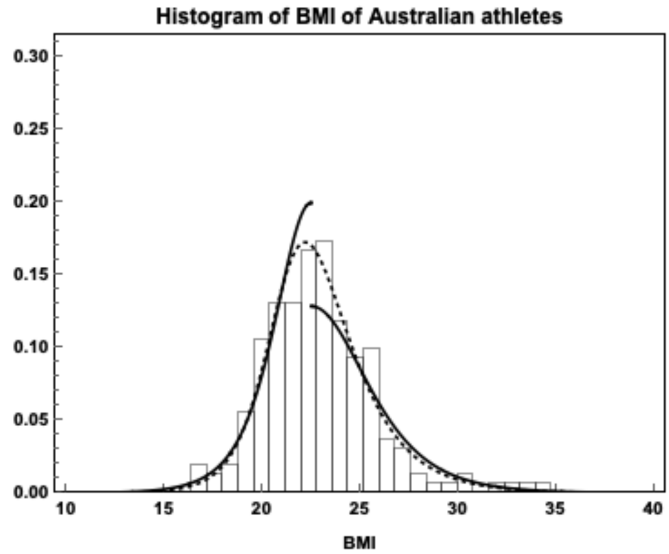


Figure 5.12: A histogram of the body mass index of Australian athletes together with the probability density curves of the fitted two-piece Tukey lambda (solid curve) and GLD_{GPD} distributions.

Since $t_3 > 0$, the data is positively skewed. The data exhibits heavy tails since $t_4 > 0.122602$, the theoretical L-kurtosis ratio value of the normal distribution. The parameter estimates of the fitted two-piece Tukey lambda distribution obtained using the method of L-moments, and their corresponding asymptotic standard errors, are summarised in Table 5.5, whilst the results for GLD_{GPD} are presented in Table 5.6. Since $\hat{\lambda} < 0$, the distribution of the data will have infinite support for both fits.

$\hat{\mu}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\lambda}$
22.4953	1.88033	0.64398	-0.04701
(0.20591)	(0.20751)	(0.10682)	(0.07365)

Table 5.5: Parameter estimates with asymptotic standard errors (in parentheses) for the two-piece Tukey lambda distribution fitted to the BMI of Australian athletes.

$\hat{\mu}$	$\hat{\sigma}$	$\hat{\delta}$	$\hat{\lambda}$
22.0122	2.8767	0.6563	-0.04701
(0.33605)	(0.33333)	(0.05307)	(0.07382)

Table 5.6: Parameter estimates with asymptotic standard errors (in parentheses) for the GLD_{GPD} fitted to the BMI of Australian athletes.

The two Q-Q plots are similar with the fitted GLD_{GPD} providing a slightly better fit especially in the upper tail.

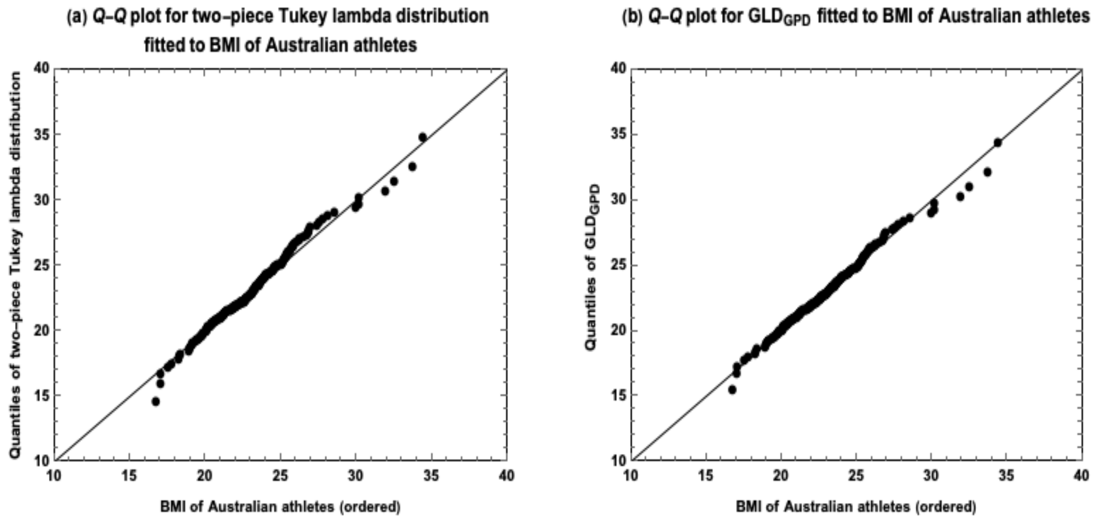


Figure 5.13: Q-Q plots for the two-piece Tukey lambda distribution and the GLD_{GPD} fitted to the body mass index (BMI) of Australian athletes.

The ASAE value for the GLD_{GPD} is 0.008346, which is lower than the two-piece Tukey lambda distribution's ASAE value of 0.014028. It therefore seems the GLD_{GPD} provides a better fit to the data set.

5.11 Conclusion

The results in Section 3.2 are applied to the two-piece Tukey lambda distribution. The quantile function, quantile density function as well as the density quantile functions are derived. The quantile measures of distributional form are also derived, indicating that the kurtosis measures are skewness-invariant.

The parameter space and support of the distribution is obtained for the classes of the distribution, based on the combinations of $\alpha > 0$ and $\lambda \in \mathbb{R}$. Using the results in Section 3.3.4, the r^{th} order L-moments of the two-piece Tukey lambda distribution are derived in terms of the L-moments of both the parent Tukey lambda distribution and the half Tukey's lambda distribution.

The tail behaviour of the two-piece distribution is also presented, as well as the parameter estimation procedure. Finally, two data sets are used to compare the fit of the two-piece Tukey lambda distribution to that of the GLD$_{GPD}$.

CONCLUSION

This thesis is centered around a methodology that makes use of the quantile functions of half distributions from symmetric univariate distributions, to develop asymmetric univariate distributions.

6.1 Construction of two-piece families of distributions

Quantile splicing, as developed in Chapter 3, is aimed at generalizing symmetric distributions, using the quantile functions of half distributions as the building blocks. The quantile functions are spliced at the median point and an asymmetry parameter is introduced to the half of the distribution whose domain is below the median point. This skewing mechanism can be implemented for distributions defined through their CDF, PDF or quantile functions, as illustrated in the examples in Chapters 3 and 4. Furthermore, the quantile measures of distributional form for shape indicate that the level of kurtosis of the parent distribution remains constant. The advantage is that the analysis of the distributional shape of the generalizations obtained, in terms of skewness and kurtosis, can be done seperately.

6.2 Construction of the r^{th} order L-moments

Quantile splicing is used to derive the general results for the r^{th} order L-moments of these two-piece families of distributions. This general form reveals a relationship between the L-moments of the symmetric parent distribution and the half distribution. The order statistics are used to achieve this relationship.

The results give rise to a simple relationship between the parameters of the distribution and the L-moments. The benefit is that computational difficulty in parameter estimation is reduced when the method of L-moments estimation is used.

6.3 An Extension of the Quantile Splicing Technique

An extension of the quantile splicing technique in Section 3.3 is proposed, where a general method of introducing asymmetry to univariate distributions through splicing the quantile functions of symmetric distributions at location points other than the median is investigated. The general forms of the CDF, PDF and quantile functions are also derived. Likewise, a general form for the r^{th} order L-moments could be derived in the same fashion as the results in Section 3.3.4, along with the first 4 L-moments.

6.3.1 Extended Piecewise Distributions

Suppose X is a continuous random variable from a symmetric distribution, defined on an interval $(-\infty;\infty)$, with the CDF, PDF and quantile function defined accordingly. Let $0 < k < 1$ be defined. From Eqs.(3.1) and (3.2), the quantile functions for the piecewise distribution, for any k, can be defined as

$$Q_T(p) = \begin{cases} \mu + \alpha\sigma\left(Q_X(p) - Q_X(k)\right) & \text{for } p \leq k, \\ \mu + \sigma\left(Q_X(p) - Q_X(k)\right) & \text{for } p > k. \end{cases} \tag{6.1}$$

where $-\infty < \mu < \infty$, $\sigma > 0$ and $\alpha > 0$ are the location, scale and shape parameters, respectively.

It follows that for $p \leq k$, then

$$\begin{aligned} &\Rightarrow x = \mu + \alpha\sigma\left(Q_X(p) - Q_X(k)\right) \\ &\Rightarrow \frac{x-\mu}{\alpha\sigma} + Q_X(k) = Q_X(p) \\ &\Rightarrow F_X\left(\frac{x-\mu}{\alpha\sigma} + Q_X(k)\right) = p. \end{aligned}$$

Similarly, for $p > k$, then

$$\begin{aligned} &\Rightarrow x = \mu + \sigma\left(Q_X(p) - Q_X(k)\right) \\ &\Rightarrow \frac{x-\mu}{\sigma} + Q_X(k) = Q_X(p) \\ &\Rightarrow F_X\left(\frac{x-\mu}{\sigma} + Q_X(k)\right) = p. \end{aligned}$$

Hence the CDF of X follows as

$$F_T(x) = \begin{cases} F_X\left(\frac{x-\mu}{\alpha\sigma} + Q_X(k)\right) & \text{for } x \leq \mu, \\ F_X\left(\frac{x-\mu}{\sigma} + Q_X(k)\right) & \text{for } x > \mu. \end{cases} \tag{6.2}$$

The probability density function is obtained from Eq.(6.2) as

$$f_T(x) = \begin{cases} \frac{1}{\alpha\sigma} f_X\left(\frac{x-\mu}{\alpha\sigma} + Q_X(k)\right) & \text{for } x \leq \mu, \\ \frac{1}{\sigma} f_X\left(\frac{x-\mu}{\sigma} + Q_X(k)\right) & \text{for } x > \mu. \end{cases} \tag{6.3}$$

6.3.2 r^{th} Order L-moments

The r^{th} order L-moments will be derived for the piecewise families of distributions. They will be used to further explore the features of the distributions in terms of the quantile measure of distributional form. Moreover, estimation procedures can also be created for the parameters. The shifted scaled polynomials to be used will first be derived, since both a scaling and shifting factor are introduced to the quantile functions.

r^{th} Order Shifted Scaled Polynomials

The results of the shifted scaled polynomials in Eq.(3.18) take into account that a scaling factor is introduced into the Legendre polynomial. The polynomials obtained indicate that their coefficients remain the same as those in Eq.(3.12). In the case of the piecewise distributions obtained from the extended quantile splicing technique, the shifted Legendre polynomial has both a scaling and a shifting factor introduced. The outcome is a factor that is a polynomial of the variable of interest, p.

In order for the L-moments to be obtained, the shifted scaled polynomials will be obtained by making use of Eq.(3.18).

Lemma 6.3.1. *The r^{th} order shifted Legendre polynomials, with constants $-\infty < a, b < \infty$, are denoted as*

$$P^*_{r-1}(ap+b) = P_{r-1}(2(ap+b)-1) = \frac{1}{(r-1)!}\frac{d^{r-1}}{a^{r-1}d(p)^{r-1}}\left((ap+b)(ap+b-1)\right)^{r-1}, \tag{6.4}$$

where $r > 1$, *such that the first* 4 *shifted scaled polynomials are obtained as*

$$
\begin{aligned}
&P_0^*\,(ap+b) = 1\\
&P_1^*\,(ap+b) = 2\,(ap+b) - 1\\
&P_2^*\,(ap+b) = 6\,(ap+b)^2 - 6\,(ap+b) + 1\\
&P_3^*\,(ap+b) = 20\,(ap+b)^3 - 30\,(ap+b)^2 + 12\,(ap+b) - 1. \qquad (6.5)
\end{aligned}
$$

Proof. The results in Eq.(6.4) follows suit from Eq.(3.11) by replacing p with $(ap+b)$. Therefore,

$$
\begin{aligned}
P_{r-1}^*\,(ap+b) &= \frac{1}{(r-1)!}\frac{d^{r-1}}{d(ap+b)^{r-1}}\left((ap+b)^2-(ap+b)\right)^{r-1}\\
&= \frac{1}{(r-1)!}\frac{d^{r-1}}{a^{r-1}d(p)^{r-1}}\left((ap+b)((ap+b)-1)\right)^{r-1}.
\end{aligned}
$$

By recursively substituting $r = 1, 2, 3$ and 4 into Eq.(6.4) and taking the respective derivative, the first 4 shifted scaled polynomials will be obtained. For $r = 1$, the first polynomial is obtained as

$$
\begin{aligned}
P_0^*\,(ap+b) &= \frac{1}{(1-1)!}\frac{d^{1-1}}{a^{1-1}d(p)^{1-1}}\left((ap+b)((ap+b)-1)\right)^{1-1}\\
&= \frac{1}{0!}\frac{d^0}{a^0d(p)^0}\left((ap+b)((ap+b)-1)\right)^0\\
&= 1.
\end{aligned}
$$

The second polynomial is obtained when $r = 2$ as

$$
\begin{aligned}
P_1^*\,(ap+b) &= \frac{1}{(2-1)!}\frac{d^{2-1}}{a^{2-1}d(p)^{2-1}}\left((ap+b)((ap+b)-1)\right)^{2-1}\\
&= \frac{d}{a\ d(p)}\left((ap+b)((ap+b)-1)\right)\\
&= \frac{1}{a}\left(a(ap+b-1)+a(ap+b)\right)\\
&= 2(ap+b)-1.
\end{aligned}
$$

By substituting $r = 3$ into Eq.(6.4), the third polynomial can be obtained as

$$
\begin{aligned}
P_2^*\,(ap+b) &= \frac{1}{(3-1)!}\frac{d^{3-1}}{a^{3-1}d(p)^{3-1}}\left((ap+b)((ap+b)-1)\right)^{3-1}\\
&= \frac{1}{2}\frac{d^2}{a^2dp^2}\left((ap+b)((ap+b)-1)\right)^2\\
&= \frac{1}{2a^2}\frac{d}{dp}2\left((ap+b)((ap+b)-1)\right)\left(a\left(2(ap+b)-1\right)\right)\\
&= \frac{1}{a}\frac{d}{dp}\left((ap+b)((ap+b)-1)(2(ap+b)-1)\right)\\
&= \frac{1}{a}\left(a((ap+b)-1)(2(ap+b)-1)+a(ap+b)(2(ap+b)-1)+2a(ap+b)((ap+b)-1)\right)\\
&= 2(ap+b)^2-3(ap+b)+1+2(ap+b)^2-(ap+b)+2(ap+b)^2-2(ap+b)\\
&= 6(ap+b)^2-6(ap+b)+1.
\end{aligned}
$$

Finally, the fourth polynomial, obtained when r is set to 4 in Eq.(6.4) is

$$\begin{aligned}
P_3^*(ap+b) &= \frac{1}{(4-1)!}\frac{d^{4-1}}{a^{4-1}dp^{4-1}}\left((ap+b)((ap+b)-1)\right)^{4-1}\\
&= \frac{1}{6}\frac{d^3}{a^3dp^3}\left((ap+b)^2-(ap+b)\right)^3\\
&= \frac{1}{2a^2}\frac{d^2}{dp^2}\left(\left((ap+b)^2-(ap+b)\right)^2(2(ap+b)-1)\right)\\
&= \frac{1}{a}\frac{d}{dp}\left(((ap+b)^2-(ap+b))(2(ap+b)-1)^2+((ap+b)^2-(ap+b))^2\right)\\
&= (2(ap+b)-1)\left[(2(ap+b)-1)^2+((ap+b)^2-(ap+b))\right]+(((ap+b)^2-(ap+b))\\
&\times\left[4(((ap+b)^2-(ap+b))+(2(ap+b)-1))\right]\\
&= (2(ap+b)-1))^3+6(2(ap+b)-1)((ap+b)^2-(ap+b))\\
&= 8(ap+b)^3-12(ap+b)^2+6(ap+b)-1+12(ap+b)^3-12(ap+b)^2\\
&-6(ap+b)^2+6(ap+b)\\
&= 20(ap+b)^3-30(ap+b)^2+12(ap+b)-1.
\end{aligned}$$

■

Remark. *The results in Eq.(6.5) satisfy the recursive relationship for any higher order polynomial of p.*

r^{th} Order L-moments for Piecewise Distributions

Eq.(3.25) in Section 3.3.4 makes use of the expectation of the r^{th} largest order statistic, as defined in Eq.(3.24), to derive the general form of r^{th} order L-moments function for the two-piece distribution. Similarly, the general formula for the L-moments for families of distributions obtained using the extended quantile splicing methodology, for any point $0 < k < 1$, will be derived using the same principle. The first 4 L-moments will also be derived by making use of the triangle rule (Arnold and Meeden (1975)) and the relationship of order statistics in Eq.(3.6), from Hosking (1990).

Theorem 6.3.2. *Suppose T is a random variable from a piecewise distribution obtained from the extended quantile splicing methodology denoted by $T \sim TP(\mu,\sigma,\alpha)$, where $-\infty < \mu < \infty$, $\sigma > 0$ and $\alpha > 0$ are the location, spread and asymmetry parameters, respectively, and $0 < k < 1$. The general form of the L-moments is*

$$L_{T:r} = \mu^* + \sigma\left(L_{X:r} - Q_X(k)\right) - k\sigma(1-\alpha)\left(\frac{\sum_{j=1}^{r} c_{j-1}^{(r-1)}\mu_{j:j}}{j} - Q_X(k)\frac{\sum_{j=1}^{r} c_{j-1}^{(r-1)}}{j}\right), \tag{6.6}$$

where $Q_X(k)$ is the quantile function of X evaluated at $0 < k < 1$, $\mu_{j:j}$ is the expectation of the r^{th} largest observation in a sample of size r from the k^{th} piece distribution obtained from the parent distribution, X, and $c_{j-1}^{(r-1)}$ for $j = 1, \ldots, r$ and $r \geq 1$, are equal to the coefficients of the shifted scaled Legendre polynomials captured in Table (3.2).

Proof. The L-moments formula will be derived from the piecewise quantile function of T given in Eq.(6.1) as follows:

$$
\begin{aligned}
L_{T:r} &= \int_0^k (\mu + \alpha\sigma\,(Q_X(p) - Q_X(k)))P_{r-1}^*(p)dp + \int_k^1 (\mu + \sigma\,(Q_X(p) - Q_X(k)))P_{r-1}^*(p)dp \\
&= \int_0^k (\mu + \alpha\sigma Q_X(p))\,P_{r-1}^*(p)dp - \alpha\sigma\int_0^k Q_X(k)P_{r-1}^*(p)dp + \int_k^1 (\mu + \sigma Q_X(p))\,P_{r-1}^*(p)dp \\
&\quad - \sigma\int_k^1 Q_X(k)P_{r-1}^*(p)dp \\
&= \int_0^1 \mu P_{r-1}^*(p)dp + \alpha\sigma\int_0^k Q_X(p)P_{r-1}^*(p)dp + \sigma\left[\int_0^1 Q_X(p)P_{r-1}^*(p)dp - \int_0^k Q_X(p)P_{r-1}^*(p)dp\right] \\
&\quad - \alpha\sigma\int_0^k Q_X(k)P_{r-1}^*(p)dp - \sigma\left(\int_0^1 Q_X(k)P_{r-1}^*(p)dp - \int_0^k Q_X(k)P_{r-1}^*(p)dp\right) \\
&= \mu^* + k\alpha\sigma\int_0^1 Q_X(u)P_{r-1}^*(u)du + \sigma\left(L_{X:r} - k\int_0^1 Q_X(u)P_{r-1}^*(u)du\right) - k\alpha\sigma\int_0^1 Q_X(k)P_{r-1}^*(u)du \\
&\quad - \sigma Q_X(k)\left(\int_0^1 P_{r-1}^*(p)dp - k\int_0^1 P_{r-1}^*(u)du\right) \\
&= \mu^* + \sigma\,(L_{X:r} - Q_X(k)) - k\sigma(1-\alpha)\left(\int_0^1 Q_X(u)P_{r-1}^*(u)du - Q_X(k)\int_0^1 P_{r-1}^*(u)du\right) \\
&= \mu^* + \sigma\,(L_{X:r} - Q_X(k)) - k\sigma(1-\alpha)\Bigg(\int_0^1 Q_X(u)\left(c_0^{(r-1)} + c_1^{(r-1)}u + c_2^{(r-1)}u^2 \cdots + c_{r-1}^{(r-1)}u^{(r-1)}\right)du \\
&\quad - Q_X(k)\int_0^1\left(c_0^{(r-1)} + c_1^{(r-1)}u + c_2^{(r-1)}u^2 \cdots + c_{r-1}^{(r-1)}u^{(r-1)}\right)du\Bigg) \\
&= \mu^* + \sigma\,(L_{X:r} - Q_X(k)) - k\sigma(1-\alpha)\Bigg(\int_0^1 Q_X(u)c_0^{(r-1)}du + \int_0^1 Q_X(u)c_1^{(r-1)}u\ du + \ldots \\
&\quad + \int_0^1 Q_X(u)c_{r-1}^{(r-1)}u^{(r-1)}du - Q_X(k)\left(\int_0^1 c_0^{(r-1)} + \int_0^1 c_1^{(r-1)}u\ du + \cdots + \int_0^1 c_{r-1}^{(r-1)}u^{(r-1)}du\right)\Bigg) \\
&= \mu^* + \sigma\,(L_{X:r} - Q_X(k)) - k\sigma(1-\alpha)\Bigg(c_0^{(r-1)}E\,(S_{1:1}) + \frac{c_1^{(r-1)}E\,(S_{2:2})}{2} + \cdots + \frac{c_{r-1}^{(r-1)}E\,(S_{r:r})}{r} \\
&\quad - Q_X(k)\left(c_0^{(r-1)} + \frac{c_1^{(r-1)}}{2} + \frac{c_2^{(r-1)}}{3} + \cdots + \frac{c_{r-1}^{(r-1)}}{r}\right)\Bigg) \\
&= \mu^* + \sigma\,(L_{X:r} - Q_X(k)) - k\sigma(1-\alpha)\left(\frac{\sum_{j=1}^r c_{j-1}^{(r-1)}E\,(S_{j:j})}{j} - Q_X(k)\frac{\sum_{j=1}^r c_{j-1}^{(r-1)}}{j}\right) \\
&= \mu^* + \sigma\left((L_{X:r} - Q_X(k)) - k(1-\alpha)\left(\frac{\sum_{j=1}^r c_{j-1}^{(r-1)}\mu_{j:j}}{j} - Q_X(k)\frac{\sum_{j=1}^r c_{j-1}^{(r-1)}}{j}\right)\right).
\end{aligned}
$$

■

Remark. *The location parameter* $\mu^* = \int_0^1 \mu P^*_{r-1}(p)dp$ *is equal to* μ *when* $r = 1$*, and* 0 *for* $r > 1$*. The coefficients* $c_{j-1}^{(r-1)}$*, for* $j = 1, \ldots, 4$ *and* $r \geq 1$*, are equal to those captured in Table 3.2.*

Theorem 6.3.3. *The first* 4 L*-moments of a piecewise distribution constructed using the extended quantile splicing technique are*

$$L_{T:1} = \mu + \sigma\left((L_{X:1} - Q_X(k)) - k(1-\alpha)c_0^{(0)}\left(L_{S:1} - Q_X(k)\right)\right)$$

$$L_{T:2} = \sigma\left((L_{X:2} - Q_X(k)) - k(1-\alpha)\left(\left(c_0^{(1)} + \frac{c_1^{(1)}}{2}\right)L_{S:1} + \frac{c_1^{(1)}}{2}L_{S:2} - \left(c_0^{(1)} + \frac{c_1^{(1)}}{2}\right)Q_X(k)\right)\right)$$

$$\begin{aligned} L_{T:3} = \sigma\Bigg(& (L_{X:3} - Q_X(k)) - k(1-\alpha)\Bigg(\left(c_0^{(2)} + \frac{c_1^{(1)}}{2} + \frac{c_2^{(2)}}{3}\right)L_{S:1} + \left(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2}\right)L_{S:2} + \frac{c_2^{(2)}}{6}L_{S:3} \\ & - \left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\right)Q_X(k)\Bigg)\Bigg) \end{aligned}$$

and

$$\begin{aligned} L_{T:4} = \sigma\Bigg(& (L_{X:4} - Q_X(k)) - k(1-\alpha)\Bigg(\left(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\right)L_{S:1} + \left(\frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{2} + \frac{9c_3^{(3)}}{20}\right)L_{S:2} \\ & + \left(\frac{c_2^{(3)}}{6} + \frac{c_3^{(3)}}{4}\right)L_{S:3} + \frac{c_3^{(3)}}{20}L_{S:4} - \left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3} + \frac{c_3^{(3)}}{4}\right)Q_X(k)\Bigg)\Bigg), \end{aligned} \tag{6.7}$$

respectively, where $L_{S:r}$ *is the* r^{th} *order* L*-moments for the* k^{th} *piece distribution.*

Proof. The steps used in Section 3.3.4, where both (Arnold and Meeden (1975))'s triangle rule and the relationship of order statistics in Eq.(3.6) are used in tandem, will be implemented to obtain the results in Eq.(6.7) in the same manner.

For $r = 1$, then

$$\begin{aligned} L_{T:1} &= \mu^* + \sigma\left((L_{X:1} - Q_X(k)) - k(1-\alpha)\left(\frac{\sum_{j=1}^{1} c_{j-1}^{(1-1)}\mu_{j:j}}{j} - Q_X(k)\frac{\sum_{j=1}^{1} c_{j-1}^{(1-1)}}{j}\right)\right) \\ &= \mu + \sigma\left((L_{X:1} - Q_X(k)) - k(1-\alpha)\left(c_0^{(0)}\mu_{1:1} - Q_X(k)c_0^{(0)}\right)\right) \\ &= \mu + \sigma\left((L_{X:1} - Q_X(k)) - k(1-\alpha)c_0^{(0)}\left(L_{S:1} - Q_X(k)\right)\right). \end{aligned}$$

In the case $r = 2$, Eqs.(3.31) and (3.32) are used, such that

$$\begin{aligned}
L_{T:2} &= \sigma\left((L_{X:2} - Q_X(k)) - k(1-\alpha)\left(\frac{\sum_{j=1}^{2} c_{j-1}^{(2-1)}\mu_{j:j}}{j} - Q_X(k)\frac{\sum_{j=1}^{2} c_{j-1}^{(2-1)}}{j}\right)\right)\\
&= \sigma\left((L_{X:2} - Q_X(k)) - k(1-\alpha)\left(c_0^{(1)}\mu_{1:1} + \frac{c_1^{(1)}}{2}\mu_{2:2} - \left(c_0^{(1)} + \frac{c_1^{(1)}}{2}\right)Q_X(k)\right)\right)\\
&= \sigma\left((L_{X:2} - Q_X(k)) - k(1-\alpha)\left(c_0^{(1)}L_{S:1} + \frac{c_1^{(1)}}{2}\left(L_{S:2} + L_{S:1}\right) - \left(c_0^{(1)} + \frac{c_1^{(1)}}{2}\right)Q_X(k)\right)\right)\\
&= \sigma\left((L_{X:2} - Q_X(k)) - k(1-\alpha)\left(\left(c_0^{(1)} + \frac{c_1^{(1)}}{2}\right)L_{S:1} + \frac{c_1^{(1)}}{2}L_{S:2} - \left(c_0^{(1)} + \frac{c_1^{(1)}}{2}\right)Q_X(k)\right)\right).
\end{aligned}$$

For $r = 3$, the results in Eqs.(3.35-3.37) are made use of to derive the third L-moment as

$$\begin{aligned}
L_{T:3} &= \sigma\left((L_{X:3} - Q_X(k)) - k(1-\alpha)\left(\frac{\sum_{j=1}^{3} c_{j-1}^{(3-1)}\mu_{j:j}}{j} - Q_X(k)\frac{\sum_{j=1}^{3} c_{j-1}^{(3-1)}}{j}\right)\right)\\
&= \sigma\left((L_{X:3} - Q_X(k)) - k(1-\alpha)\left(c_0^{(2)}\mu_{1:1} + \frac{c_1^{(2)}}{2}\mu_{2:2} + \frac{c_2^{(2)}}{3}\mu_{3:3} - \left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\right)Q_X(k)\right)\right)\\
&= \sigma\Bigg((L_{X:3} - Q_X(k)) - k(1-\alpha)\Bigg(c_0^{(2)}L_{S:1} + \frac{c_1^{(2)}}{2}\left(L_{S:1} + L_{S:2}\right) + \frac{c_2^{(2)}}{6}\left(L_{S:3} + 3L_{S:2} + 2L_{S:1}\right)\\
&\quad - \left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\right)Q_X(k)\Bigg)\Bigg)\\
&= \sigma\Bigg((L_{X:3} - Q_X(k)) - k(1-\alpha)\Bigg(\left(c_0^{(2)} + \frac{c_1^{(1)}}{2} + \frac{c_2^{(2)}}{3}\right)L_{S:1} + \left(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2}\right)L_{S:2} + \frac{c_2^{(2)}}{6}L_{S:3}\\
&\quad - \left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\right)Q_X(k)\Bigg)\Bigg).
\end{aligned}$$

Finally, the fourth L-moment is obtained when Eqs.(3.41-3.44) are used such that

$$\begin{aligned}
L_{T:4} &= \sigma\left((L_{X:4} - Q_X(k)) - k(1-\alpha)\left(\frac{\sum_{j=1}^{4} c_{j-1}^{(4-1)}\mu_{j:j}}{j} - Q_X(k)\frac{\sum_{j=1}^{4} c_{j-1}^{(4-1)}}{j}\right)\right)\\
&= \sigma\left((L_{X:4} - Q_X(k)) - k(1-\alpha)\left(c_0^{(3)}\mu_{1:1} + \frac{c_1^{(3)}}{2}\mu_{2:2} + \frac{c_2^{(3)}}{3}\mu_{3:3} + \frac{c_3^{(3)}}{4}\mu_{4:4} - \left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3} + \frac{c_3^{(3)}}{4}\right)Q_X(k)\right)\right)\\
&= \sigma\Bigg((L_{X:4} - Q_X(k)) - k(1-\alpha)\Bigg(c_0^{(3)}L_{S:1} + \frac{c_1^{(3)}}{2}\left(L_{S:1} + L_{S:2}\right) + \frac{c_2^{(3)}}{6}\left(L_{S:3} + 3L_{S:2} + 2L_{S:1}\right)\\
&\quad + \frac{c_3^{(3)}}{20}\left(L_{S:4} + 5L_{S:3} + 9L_{S:2} + 5L_{S:1}\right) - \left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3} + \frac{c_3^{(3)}}{4}\right)Q_X(k)\Bigg)\Bigg)\\
&= \sigma\Bigg((L_{X:4} - Q_X(k)) - k(1-\alpha)\Bigg(\left(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\right)L_{S:1} + \left(\frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{2} + \frac{9c_3^{(3)}}{20}\right)L_{S:2}\\
&\quad + \left(\frac{c_2^{(3)}}{6} + \frac{c_3^{(3)}}{4}\right)L_{S:3} + \frac{c_3^{(3)}}{20}L_{S:4} - \left(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3} + \frac{c_3^{(3)}}{4}\right)Q_X(k)\Bigg)\Bigg).
\end{aligned}$$

■

REFERENCES

Abtahi, A., Behboodian, J., and Sharafi, M. (2012). A general class of univariate skew distributions considering Stein's lemma and infinite divisibility. *Metrika*, 75(2):193–206.

Akinsete, A., Famoye, F., and Lee, C. (2008). The beta-Pareto distribution. *Statistics*, 42(6):547–563.

Alexander, C., Cordeiro, G. M., Ortega, E. M. M., and Sarabia, J. M. (2012). Generalized beta-generated distributions. *Computational Statistics & Data Analysis*, 56(6):1880–1897.

Alshawarbeh, E., Famoye, F., and Lee, C. (2013). Beta-Cauchy distribution: some properties and applications. *Journal of Statistical Theory and Applications*, 12(4):378–391.

Alshawarbeh, E., Lee, C., and Famoye, F. (2012). The beta-Cauchy distribution. *Journal of Probability and Statistical Science*, 10(1):41–57.

Arellano-Valle, R. B., Gómez, H. W., and Quintana, F. A. (2005). Statistical inference for a general class of asymmetric distributions. *Journal of Statistical Planning and Inference*, 128(2):427 – 443.

Arnold, B. C. and Beaver, R. J. (2000). Hidden truncation models. *Sankhyā: The Indian Journal of Statistics, Series A*, 46(1):23–35.

Arnold, B. C., Beaver, R. J., Azzalini, A., Balakrishnan, N., Bhaumik, A., Dey, D. K., Cuadras, C. M., and Sarabia, J. (2002). Skewed multivariate models related to hidden truncation and/or selective reporting. *Test*, 11(1):7–54.

Arnold, B. C. and Groeneveld, R. A. (1980). Some properties of the arcsine distribution. *Journal of the American Statistical Association*, 75(369):173–175.

Arnold, B. C. and Meeden, G. D. (1975). Characterizations of distributions by sets of moments of order statistics. *The Annals of Statistics*, 3(3):754–758.

Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*, 12(2):171–178.

Azzalini, A. (1986). Further results on a class of distributions which includes the normal ones. *Rivistica Statistica*, 46(2):199–208.

Azzalini, A. (2005). The skew-normal distribution and related multivariate families. *Scandinavian Journal of Statistics*, 32(2):159–188.

Azzalini, A. (2014). The *R* 'sn' package: The skew-normal and skew-t distributions (version 1.0.0).

Balakrishnan, N. (1985). Order statistics from the half logistic distribution. *Journal of Statistical Computation and Simulation*, 20(4):287–309.

Balakrishnan, N., Dai, Q., and Liu, K. (2017). A skew logistic distribution as an alternative to the model of van Staden and King. *Communications in Statistics: Simulation and Computation*, 46(5):4082–4097.

Balakrishnan, N. and Nevzorov, V. B. (2003). *A Primer on Statistical Distributions*. John Wiley & Sons, Inc., New York.

Balakrishnan, N. and So, H. Y. (2015). A generalization of quantile-based skew logistic distribution of van Staden and King. *Statistics & Probability Letters*, 107:44–51.

Balasooriya, U. and Low, C.-K. (2008). Modeling insurance claims with extreme observations: Transformed kernel density and generalized lambda distribution. *North American Actuarial Journal*, 12(2):129–142.

Baten, W. D. (1934). The probability law for the sum of n independent variables, each subject to the law (1/2h)sech(πx/2h). *Bulletin of the American Mathematical Society*, 40(4):284–290.

Bowley, A. L. (1902). *Elements of Statistics* (2nd Ed). P.S. King, London.

Box, G. E. P. and Tiao, G. C. (1973). *Bayesian Inference in Statistical Analysis*. Addison-Wesley Publishing Company, Reading, Massachusetts.

Burr, I. W. (1942). Cumulative frequency functions. *The Annals of Mathematical Statistics*, 13(2):215–232.

Burr, I. W. (1968). On a general system of distributions III. The sample range. *Journal of the American Statistical Association*, 63(322):636–643.

Burr, I. W. (1973). Parameters for a general system of distributions to match a grid of α_3 and α_4. *Communications in Statistics: Theory and Methods*, 2(1):1–21.

Bury, K. V. (1975). *Statistical Models in Applied Science*. John Wiley & Sons, Inc., New York.

Castillo, E. and Hadi, A. S. (1997). Fitting the generalized Pareto distribution to data. *Journal of the American Statistical Association*, 92(440):1609–1620.

Castillo, N. O., Gómez, H. W., Leiva, V., and Sanhueza, A. (2011). On the Fernández–Steel distribution: Inference and application. *Computational Statistics & Data Analysis*, 55(11):2951–2961.

Cook, R. D. and Weisberg, S. (2009). *An Introduction to Regression Graphics*. John Wiley & Sons, New York.

Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81(7):883–898.

David, H. A. (1981). *Order Statistics*. (2nd ed), John Wiley & Sons, Inc., New York.

Esscher, F. (1932). On the probability function in the collective theory of risk. *Scandinavian Actuarial Journal*, 15(3):175–195.

Eugene, N., Lee, C., and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics: Theory and Methods*, 31(4):497–512.

Famoye, F., Lee, C., and Eugene, N. (2004). Beta-normal distribution: Bimodality properties and application. *Journal of Modern Applied Statistical Methods*, 3(1):85–103.

Famoye, F., Lee, C., and Olumolade, O. (2005). The beta-Weibull distribution. *Journal of Statistical Theory and Applications*, 4(2):121–136.

Fernández, C., Osiewalski, J., and Steel, M. F. J. (1995). Modeling and inference with ν-spherical distributions. *Journal of the American Statistical Association*, 90(432):1331–1340.

Fernández, C. and Steel, M. F. J. (1998). On bayesian modeling of fat tails and skewness. *Journal of the American Statistical Association*, 93(441):359–371.

Fischer, M. J. (2006). The skew generalized secant hyperbolic family. *Austrian Journal of Statistics*, 35(4):437–443.

Fischer, M. J. and Vaughan, D. (2002). Classes of skewed generalized secant hyperbolic distributions. *Technical Report 45, Universität Erlangen-Nürnberg: Lehrstuhl für Statistik und Ökonometrie.*

Fisk, P. (1961). The graduation of income distributions. *Econometrica: journal of the Econometric Society*, 29(2):171–185.

Fleming, T. R. and Harrington, D. P. (1991). *Counting Processes and Survival Analysis.* (Vol 1), John Wiley & Sons, Inc., New York.

Freimer, M.and Kollia, G., Mudholkar, G., and Lin, C. (1988). A study of the generalized Tukey lambda family. *Communications in Statistics: Theory and Methods*, 17(10):3547–3567.

Fry, T. R. L. (1993). Univariate and multivariate Burr distributions: A survey. *Pakistan Journal of Statistics*, 9(1):1–24.

Gail, M. H. and Gastwirth, J. L. (1978). A scale-free goodness-of-fit test for the exponential distribution based on the Gini statistic. *Journal of the Royal Statistical Society. Series B (Methodological)*, 40(3):350–357.

Gauss, C. F. (1809). *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium.* (Vol 7), Perthes et Besser, Hamburg.

Gautama, H. and van Gemund, A. J. C. (2006). Low-cost static performance prediction of parallel stochastic task compositions. *IEEE Transactions on Parallel and Distributed Systems*, 17(1):78–91.

Gilchrist, W. G. (2000). *Statistical Modelling with Quantile Functions.* Chapman & Hall/CRC Press, Boca Raton, Florida.

Gini, C. (1912). Variabilità e mutabilità. contributi allo studio dele relazioni e delle distribuzioni statistiche. *Studi Economico-Giuridici della Università di Cagliari*, 3.

Gottschalk, V. (1948). Symmetrical bi-modal frequency curves. *Journal of the Franklin Institute*, 245(3):245–252.

Gradshteyn, I. S. and Ryzhik, I. M. (2007). *Table of Integrals, Series, and Products.* (7th ed), Academic press, San Diego, California.

Groeneveld, R. A. (1998). A class of quantile measures for kurtosis. *The American Statistician*, 52(4):325–329.

Hall, P. (1927). The distribution of means for samples of size n drawn from a population in which the variate takes values between 0 and 1, all such values being equally probable. *Biometrika*, 19(3-4):240–245.

Hankin, R. K. S. and Lee, A. (2006). A new family of non-negative distributions. *Australian & New Zealand Journal of Statistics*, 48(1):67–78.

Hastings Jr, C., Mosteller, F., Tukey, J. W., and Winsor, C. P. (1947). Low moments for small samples: a comparative study of order statistics. *The Annals of Mathematical Statistics*, 18(3):413–426.

Hoeffding, W. (1948). A class of statistics with asymptotically normal distribution. *The Annals of Mathematical Statistics*, 19(3):293–325.

Hosking, J. R. M. (1986). The theory of probability weighted moments. *Research report RC12210, IBM Research Division, Yorktown Heights.*

Hosking, J. R. M. (1989). *Some Theoretical Results Concerning L-moments.* IBM Research Division, Yorktown Heights, New York.

Hosking, J. R. M. (1990). *L*-moments: Analysis and estimation of distributions using linear combinations of order statistics. *Journal of the Royal Statistical Society, Series B (Methodological)*, 52(1):105–124.

Irwin, J. O. (1927). On the frequency distribution of the means of samples from a population having any law of frequency with finite moments, with special reference to Pearson's Type II. *Biometrika*, 19(3-4):225–239.

Jamalizadeh, A., Arabpour, A. R., and Balakrishnan, N. (2011). A generalized skew two-piece skew-normal distribution. *Statistical Papers*, 52(2):431–446.

Johnson, N. L. (1949). Systems of frequency curves generated by methods of translation. *Biometrika*, 36(1-2):149–176.

Johnson, N. L., Kotz, S., and Balakrishnan, N. (1994). *Continuous Univariate Distributions.* (Vol 1, 2nd ed.), John Wiley & Sons, Inc., New York.

Johnson, N. L., Kotz, S., and Balakrishnan, N. (1995). *Continuous Univariate Distributions.* (Vol 2, 2nd Ed.), John Wiley & Sons, Inc., New York.

Jones, M. C. (2002a). The complementary beta distribution. *Journal of Statistical Planning and Inference*, 104(2):329–337.

Jones, M. C. (2002b). Student's simplest distribution. *Journal of the Royal Statistical Society: Series D (The Statistician)*, 51(1):41–49.

Jones, M. C. (2004). On some expressions for variance, covariance, skewness and L-moments. *Journal of Statistical Planning and Inference*, 126(1):97–106.

Jones, M. C. and Faddy, M. J. (2003). A skew extension of the t-distribution, with applications. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 65(1):159–174.

Jones, M. C. and Pewsey, A. (2009). Sinh-arcsinh distributions. *Biometrika*, 96(4):761–780.

Jones, M. C., Rosco, J. F., and Pewsey, A. (2011). Skewness-invariant measures of kurtosis. *The American Statistician*, 65(2):89–95.

Karian, Z. A. (2010). *Handbook of Fitting Statistical Distributions with R.* Chapman & Hall/CRC Press, Boca Raton, Florida.

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods*. Chapman & Hall/CRC press, Boca Raton, Florida.

Karvanen, J., Eriksson, J., and Koivunen, V. (2002). Adaptive score functions for maximum likelihood ICA. *Journal of VLSI signal processing systems for signal, image and video technology*, 32(1-2):83–92.

Kim, H. J. (2005). On a class of two-piece skew-normal distributions. *Statistics*, 39(6):537–553.

King, R. (1999). *New distributional fitting methods applied to the generalised λ distribution*. PhD thesis, Queensland University of Technology, Queensland, Australia.

Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of Hydrology*, 46(1-2):79–88.

Landwehr, J. M., Matalas, N. C., and Wallis, J. R. (1979). Probability weighted moments compared with some traditional techniques in estimating Gumbel parameters and quantiles. *Water Resources Research*, 15(5):1055–1064.

Lange, K. L., Little, R. J. A., and Taylor, J. M. G. (1989). Robust statistical modeling using the t-distribution. *Journal of the American Statistical Association*, 84(408):881–896.

Lee, C., Famoye, F., and Olumolade, O. (2007). Beta-weibull distribution: Some properties and applications to censored data. *Journal of Modern Applied Statistical Methods*, 6(1):173–186.

Legendre, A. M. (1786). 'Recherches sur l'attraction des spheroides homogenes'. Mémoires de Mathématiques et de Physique, présentés à l'Académie Royale des Sciences, par divers savans, et lus dans ses Assemblées. *De l'Imprimerie Royale*, 10:411–435.

MacGillivray, H. L. (1986). Skewness and asymmetry: Measures and orderings. *The Annals of Statistics*, 14(3):994–1011.

MacGillivray, H. L. and Balanda, K. P. (1988). The relationships between skewness and kurtosis. *Australian & New Zealand Journal of Statistics*, 30(3):319–337.

Moors, J. J. A. (1988). A quantile alternative for kurtosis. *Journal of the Royal Statistical Society: Series D (The Statistician)*, 37(1):25–32.

Morais, A. L., Cordeiro, G. M., and Cysneiros, A. H. M. A. (2013). The beta generalized logistic distribution. *Brazilian Journal of Probability and Statistics*, 27(2):185–200.

Morris, C. N. (1982). Natural exponential families with quadratic variance functions. *The Annals of Statistics*, 10(1):65–80.

Mudholkar, G. S. and George, E. O. (1978). A remark on the shape of the logistic distribution. *Biometrika*, 65(3):667–668.

Nadarajah, S., Cordeiro, G. M., and Ortega, E. M. M. (2012). General results for the Kumaraswamy-G distribution. *Journal of Statistical Computation and Simulation*, 82(7):951–979.

Nadarajah, S. and Kotz, S. (2004). The beta Gumbel distribution. *Mathematical Problems in Engineering*, 2004(4):323–332.

Nassiri, V. and Loris, I. (2013). A generalized quantile regression model. *Journal of Applied Statistics*, 40(5):1090–1105.

Pacáková, V. and Sipková, L. (2007). Generalized lambda distributions of households incomes. *E&M Ekonomie a Management*, 10(1):98–107.

Pearl, R. and Reed, L. J. (1920). On the rate of growth of the population of the United States since 1790 and its mathematical representation. *Proceedings of the National Academy of Sciences of the United States of America*, 6(6):275–288.

Pearson, K. (1895). Contributions to the mathematical theory of evolution. II. Skew variation in homogeneous material. *Philosophical Transactions of the Royal Society of London, Series A*, 186:343–424.

Pearson, K. (1905). 'Das Fehlergesetz und Seine Verallgemeinerungen Durch Fechner und Pearson.' A Rejoinder. *Biometrika*, 4(1-2):169–212.

Pearson, K. (1916). Mathematical contributions to the theory of evolution. XIX. Second supplement to a memoir on skew variation. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, 216:429–457.

Perks, W. (1932). On some experiments in the graduation of mortality statistics. *Journal of the Institute of Actuaries*, 63(1):12–57.

Plackett, R. L. (1959). The analysis of life test data. *Technometrics*, 1(1):9–19.

Ramberg, J. S. and Schmeiser, B. W. (1972). An approximate method for generating symmetric random variables. *Communications of the Association for Computing Machinery*, 15(11):987–990.

Ramberg, J. S. and Schmeiser, B. W. (1974). An approximate method for generating asymmetric random variables. *Communications of the Association for Computing Machinery*, 17(2):78–82.

Ramos-Fernández, A., Paradela, A., Navajas, R., and Albar, J. P. (2008). Generalized method for probability-based peptide and protein identification from tandem mass spectrometry data and sequence database searching. *Molecular & Cellular Proteomics*, 7(9):1748–1754.

Robinson, L. W. and Chen, R. R. (2003). Scheduling doctors' appointments: optimal and empirically-based heuristic policies. *IIE Transactions*, 35(3):295–307.

Rodrigues, O. (1816). *De l'attraction des sphéroïdes, Correspondence sur l'É-cole Impériale Polytechnique*. PhD thesis, Thesis for the Faculty of Science of the University of Paris, Paris, France.

Sillitto, G. P. (1951). Interrelations between certain linear systematic statistics of samples from any continuous population. *Biometrika*, 38(3-4):377–382.

Student (1908). The probable error of a mean. *Biometrika*, 6(1):1–25.

Talacko, J. (1956). Perks' distributions and their role in the theory of Wiener's stochastic variables. *Trabajos de Estadística y de Investigación Operativa*, 7(2):159–174.

Tukey, J. W. (1960). The practical relationship between the common transformations of percentages of counts and of amounts. *Technical Report 36, Statistical Techniques Research Group, Princeton University, Princeton, New Jersey.*

Tukey, J. W. (1965). Which part of the sample contains the information? *Proceedings of the National Academy of Sciences of the United States of America*, 53(1):127–134.

van Staden, P. J. (2014). *Modeling of generalized families of probability distribution in the quantile statistical universe*. PhD thesis, Department of Statistics, University of Pretoria, Pretoria, South Africa.

van Staden, P. J. and King, R. A. R. (2015). The quantile-based skew logistic distribution. *Statistics & Probability Letters*, 96:109–116.

Vaughan, D. C. (2002). The generalized secant hyperbolic distribution and its properties. *Communications in Statistics: Theory and Methods*, 31(2):219–238.

Verhulst, P. F. (1838). Notice sur la loi que la population suit dans son accroissement. *Correspondance Mathématique et Physique*, 10:113–126.

Vicari, S. K. D. and Kotz, S. (2005). Survey of developments in the theory of continuous skewed distributions. *Metron*, 63(2):225–261.

Wang, J., Boyer, J., and Genton, M. G. (2004). A skew-symmetric representation of multivariate distributions. *Statistica Sinica*, 14(4):1259–1270.

Wolfram Research, Inc. (2020). Mathematica, Version 12.1. Champaign, Illinois.

Zwet, W. R. (1964). *Convex transformations of random variables, Mathematical Centre tracts, No. 7.* Mathematisch Centrum, The Netherlands.