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On the risk measures representation and capital allocation in the Backward Stochastic Differential Equation framework

by

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Declaration

I, Lesedi Mabitsela declare that the thesis, which I hereby submit for the degree *Philosophiae Doctor* at the University of Pretoria, is my own independent work and has not previously been submitted by me or any other person for any degree at this or any other university.

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DATE: 08 February 2021

Abstract

In this thesis, we study the representation of dynamic risk measures based on backward stochastic differential equations (BSDEs) and ergodic-BSDEs, and capital allocation. We consider the equations driven by the Brownian motion and the compensated Poisson process. We obtain four results.

Firstly, we consider the representation of dynamic risk measures defined under BSDE, with generators that have quadratic-exponential growth in the control variables. Under this setting, the dynamic capital allocation of the risk measure is obtained via the differentiability of BSDEs with jumps. In this case, we introduce the Malliavin directional derivative that generalises the classical Gâteaux-derivative. Using the capital allocation results and the full allocation property of the Aumann-Shapley, we obtain the representation of the dynamic convex and coherent risk measures. The results are illustrated for the dynamic entropic risk and static coherent risk measures.

Secondly, we consider the representation of dynamic convex risk measure based on the ergodic-BSDEs in the diffusion framework. The maturity-independent risk measure is defined as the first component to the solution of a BSDE whose generator depends on the second component of the solution to the ergodic-BSDE. Using the differentiability results of BSDEs, we determine the capital allocation. Furthermore, we give an example in the form of the forward entropic risk measure and the capital allocation.

Thirdly, we investigate the representation of capital allocation for dynamic risk measures based on BSVIEs from Kromer and Overbeck 2017 and extend it to risk measures based on BSVIEs with jumps. The extension of dynamic risk measure based on BSVIEs with jumps is studied by Agram 2019. In our case, we study capital allocation for dynamic risk measures based on BSVIEs with jumps. In particular, we determine the capital allocation of the dynamic risk measures based on BSVIEs with jumps.

Finally, we study the representation for a forward entropic risk measure using ergodic BSDEs under the jump-diffusion framework. In this case, we notice that when the ergodic BSDE includes jump term the forward entropic risk measure does not satisfy the translation property.

Keywords— Dynamic risk measure, Dynamic entropic risk measure, Capital allocation, Quadratic-exponential BSDE, Ergodic BSDE, Jump-diffusion

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Dedications

To my beloved parents
William Pheya and Tiny Hazel Caroline Mabitsela.

Contents

Glossary

We introduce the notation of the spaces of random variables or processes: Let $p \geq 2$.

- $L^0(\mathcal{F}_t)$ is the space of all finite valued $\mathcal{F}_t$-measurable random variables.
- $L^1(\Omega, \mathcal{F}, P)$ is the space of real valued random variable X such that

$$||X||_1 = \mathbb{E}[|X|] < \infty.$$

- $L^2(\mathcal{F}_T)$ is the space of $\mathcal{F}_T$-measurable, square integrable random variable ξ.
- Let $L^p(\mathcal{F}_t)$ be the space of all real-valued $\mathcal{F}_t$-measurable, p-integrable random variables.
- $\mathbb{S}^p(\mathbb{R})$ is the space of $\mathbb{R}$-valued adapted processes $Y : \Omega \times [0,T]$ with càdlàg paths such that

$$\sup_{t \in [0,T]} |Y(t)|^p] < \infty.$$

- $\mathbb{S}^\infty(\mathbb{R})$ is the space of $\mathbb{R}$-valued essentially bounded càdlàg processes Y such that

$$||Y||_{S^\infty} := || \sup_{t \in [0,T]} |Y(t)| \, ||_\infty < \infty.$$

- $\mathbb{H}^2_W(\mathbb{R})$ is the space of predictable processes $Z : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\mathbb{E}[\int_0^T |Z(s)|^2 ds] < \infty.$$

- $\mathbb{H}^2_N(\mathbb{R})$ denotes the space of predictable processes $\Upsilon : \Omega \times [0,T] \times \mathbb{R}_0 \to \mathbb{R}$, satisfying

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} |\Upsilon(t,\zeta)|^2 \nu(d\zeta) dt \Big] < \infty.$$

- $L^2_\nu(\mathbb{R}_0)$ is the space of $\mathbb{R}$-valued measurable functions satisfying $v(d\zeta)$-almost everywhere (a.e.), which is equipped with the topology of convergence measure (Morlais (2009b)).

- Let $\mathcal{P}$ denote the $\mathbb{F}$-predictable σ-algebra on $\Omega \times [0,T]$.

- Let D_s^i and $D_{s,\zeta}^j$ be the Malliavin derivatives with respect to $W_i(t)$ and $\tilde{N}_j(dt,d\zeta)$ respectively for $i = 1,\dots,d$, $j = 1,\dots,k$ and $0 \leq s \leq t \leq T$.

- We denote by $\mathbb{D}^{1,2}$ the Banach space which is the closure of smooth random variables with norm

$$\|F\|_{1,2}^2 := \mathbb{E}\Big[|F|^2 + \sum_{i=1}^{d}\int_0^T |D_s^i F|^2 ds + \sum_{j=1}^{k}\int_0^T\int_{\mathbb{R}_0} |D_{s,\zeta}^j F|^2 \zeta^2 \nu_j(d\zeta) ds\Big].$$

- The space $L^2_{[BMO]}[0,t]$ is given by

$$\begin{aligned}&L^2_{[BMO]}[0,t] := \{\pi_s, s \in [0,t] : \pi \text{ is } \mathcal{F}_t - \text{progressively measurable}\\ &\text{and } ess\sup_{\tau} \mathbb{E}_{\mathbb{P}}\Big(\int_\tau^t |\pi_s|^2 ds | \mathcal{F}_\tau\Big) < \infty \text{ for any } \mathcal{F} - \text{stopping time}\\ &\tau \in [0,t]\}.\end{aligned}$$

- Consider $\mathcal{L}$ to be the space of all bounded random variables with finite maturity, i.e.

$$\mathcal{L} := \cup_{t \geq 0} L^\infty(\mathcal{F}_t).$$

Chapter 1

Introduction

1.1 Overview

The quantification of financial risk is a crucial task for both researchers and practitioners, as it provides a comprehensive description of the riskiness of a financial position. It is a tool that determines the minimum capital required to be held by a company in order to be financially stable. At the same time, capital allocation is an essential application of risk measures. More so if one is interested in decomposing the overall portfolio risk capital into a sum of the risk contributions by the respective sub-portfolios.

The method of quantifying risk is obtained by assigning a functional to the future payoffs, which are first modelled as a random variable. This functional is called a risk measure. Different types of risk measures are proposed and used in the literature and the finance industry, with Value-at-Risk (VaR) being the most popular. VaR quantifies the maximum possible financial losses over a given time horizon and confidence level. However, Artzner et al. (1999) identified the shortcomings of VaR. It fails to recognise diversification and it is not time consistent. To counter the weaknesses of VaR, Artzner et al. (1999), proposed coherent risk measure and described it as a function that satisfies four properties: translation invariance, monotonicity, sub-additivity and positive homogeneity. (See also Delbaen (2002) for coherent risk measures in the general probability space). Corehent risk measures promote diversification because the risk of a portfolio of assets is less than the risk of holding individual assets. Later, Föllmer and Schied (2002) and independently Frittelli and Gianin (2002) showed that the risk of a portfolio increases nonlinearly with the size of the position because of additional liquidity. As a result, they extended the work of Artzner et al. (1999) by relaxing the properties of positive homogeneity and subadditivity to introduce the concept of convex risk measures. A convex risk measure takes into consideration that the risk of a position may increase in a nonlinear way as a position multiplies by a large factor.

In the abovementioned papers, the authors consider risk measure in a single-period setting. The ideal situation is to measure the risk of a financial position continuously throughout the investment period. Consequently, there is a need for the concept of dynamic risk measures. In a dynamic setting, the risk measure is updated over time according to available information. An important property of dynamic risk measure is time consistency, which describes how risk quantifications at different times are interrelated. Various authors have extended the concept of static risk measure to dynamic risk measure. Peng (1997) introduced g-expectations as nonlinear expectations based on a BSDE

$$\begin{aligned} dY(t) &= -g(t, Y(t), Z(t))dt + Z(t)dW(t), \\ Y(T) &= \xi, \end{aligned} \tag{1.1.1}$$

where the solution is a pair of $\mathcal{F}_t$-adapted processes $(Y(t), Z(t))$. Gianin (2006) showed that conditional g-expectations represents dynamic risk measures under the diffusion BSDE (see also Barrieu and Karoui (2007), Frittelli and Gianin (2002), Peng (2004)). Jiang (2008) proved that g-expectation satisfies the translation invariance property if and only if the generator $g(t, y, z)$ is independent of y and is convex with respect to z for all t. Quenez and Sulem (2013) studied properties of dynamic risk measures based on BSDEs with jumps (see also Øksendal and Sulem (2015) for applications). An extension of quadratic BSDE to jumps was studied by Karoui et al. (2016) and they include an application to entropic risk measures. Part of the thesis considers dynamic risk measures that arise as the solution of quadratic BSDE with jumps.

Further, Zariphopoulou and Žitković (2010) argue that in practice, the maturity associated with a risky position might not be fixed. They introduced maturity-independent risk measure, where the risk measurement of the financial position does not depend on any given time horizon. They incorporate maturity independence and replace its domain by a general domain. As a result, the axioms of the maturity-independent risk measures are the same to the axioms of a replication-invariant convex risk measure. Furthermore, the formulation uses the notion of the forward-performance process. Musiela and Zariphopoulou (2006) was the first to propose the concept of forward-performance (see for instance Musiela and Zariphopoulou (2007), Musiela, Zariphopoulou, et al. (2008), Musiela and Zariphopoulou (2010b)). Zariphopoulou and Žitković (2010) showed that every exponential forward-performance process can be used to construct a dynamic maturity-independent risk measure as the market unfolds over an arbitrary time horizon. Chong et al. (2019) provided a general representation maturity-independent risk measure that satisfies the solution of a BSDE with a generator that depends on the solution of the ergodic BSDE. Liang and Zariphopoulou (2017) proposed the ergodic BSDEs to construct the forward-performance processes.

Fuhrman et al. 2009 introduced the notion of ergodic BSDE and developed further by Debussche et al. 2011. The ergodic BSDEs are an asymptotic limit of the infinite horizon BSDEs (as shown by Fuhrman et al. 2009 and Debussche et al. 2011) and are represented as follows

$$dY_t = (-g(V_t, Z_t) + \lambda)dt + Z_t dW_t,$$

where $\lambda \in \mathbb{R}$ is part of the solution. Cohen and Fedyashov 2014 and

Fedyashov 2016 extended the ergodic BSDE to a jump-diffusion framework and is represented as follows

$$dY_t = (-g(V_t, Z_t, \Psi_t) + \lambda)dt + Z_t dW_t + \int_{\mathbb{R}\backslash 0} \Psi_t \tilde{N}(dt, d\zeta),$$

where $0 \leq t \leq T < \infty$. We adapt this jump model with a different generator. In our analysis, we extend and study with a quadratic growth in the control variable. We further study the behaviour of a forward entropic risk measure as the terminal time of the investment period goes to infinity.

Risk measures are used to determine the amount required to hold as a buffer against unexpected losses for a portfolio. Risk measures can be further used to measure the risk contribution of a subportfolio in a overall portfolio (see for example Cherny (2009), Buch and Dorfleitner (2008), Denault (2001), Kalkbrener (2005) and Tasche (2004)). Capital allocation is the problem of measuring the risk contribution of sub-portfolio in the overall portfolio risk. The methods that are mostly used and studied are the full allocation property of the Aumann-Shapley and Gradient allocation method. Denault (2001) provided the properties of coherent capital allocation. These are the symmetry and riskless allocation, which together justify the gradient allocation principal. The gradient allocation is the Gâteaux derivative of the risk measure of a portfolio in the direction of the subportfolio. Tasche (2004) showed that if the risk measure is smooth, then the partial derivative of the risk measure with respect to the underlying asset is the unique gradient allocation principle. As a result, the risk measure needs to be Gâteaux-differentiable for the gradient allocation to exist. Denault (2001) showed that the Aumman Shapley value is coherent and a practical approach to capital allocation. Kalkbrener (2005) further provided the properties for gradient allocation principle, and shows that the properties are satisfied if and only if the risk measure is positive homogeneous and sub-additive. The gradient allocation properties provided by Denault (2001) are shown to be equivalent to the risk measure axioms of positive homogeneity, sub-additivity and translation invariance respectively (see Buch and Dorfleitner (2008)). For more analysis on the gradient allocation method, see e.g., Tasche (2007).

1.2 Aims and Objectives

The aim of this thesis is to provide and analyse the representation of risk measures and capital allocation. We consider capital allocation because most companies have several subdivisions and would like to allocate their aggregate capital to the underlying subdivisions. There are several motivations for allocate capital. For example, a company would want to redistribute the cost of holding capital equitably across different subdivisions. Moreover, the allocation of capital provides a tool for assessing and comparing subdivisions by determining the return of allocated capital for each division. See Dhaene et al. (2012) for further motivation for studying capital allocation. We further extend the capital allocation to include jumps. The reason for this extension is that jump-diffusion model is essential to capture the extreme movements in a risky asset, for example, caused by the announcement of an important decision made by a company or change in economic policy to the financial market (Rong (2006)).

We first analyse and derive the representation of risk measures and capital allocation constructed using BSDEs under the jump-diffusion setting and apply the results to the entropic risk measure. For the diffusion case, Kromer and Overbeck (2014) derived and analysed the dynamic capital allocation of BSDE based dynamic risk measure. Dynamic risk measures for BSDE with jumps are studied and analysed by Quenez and Sulem (2013) and Øksendal and Sulem (2015). However, the authors did not consider the capital allocation of the risk measure under the jumps framework.

Second, we study the representation of a dynamic maturity-independent risk measure and derive its capital allocation under the diffusion framework.

Maturity-independent risk measures are constructed using BSDEs whose generator depends on the solution of the ergodic-BSDEs. In this case, we extend the works of Chong et al. (2019) to derive the representation of the capital allocation.

We further provided the representation of dynamic capital allocation using dynamic risk measures that occur as a solution to backward stochastic Volterra integral equations (BSVIEs) under the jump setting. These dynamic risk measures allow for the terminal value to be position processes and not only $\mathcal{F}_T$-measurable random variables. The study of dynamic risk measures constructed using BSVIEs with jumps is done by Agram (2019). Furthermore, Kromer and Overbeck (2017) derived the capital allocation of these dynamic risk measures under the diffusion case.

Lastly, we study risk measure representation for the forward entropic risk measure in the jump-diffusion setting. We investigate the behaviour of the forward entropic risk measure when the underlying stock price process is driven by an independent Brownian motion and the Poisson processes. This risk measure is in a category of maturity-independent risk measures introduced by Zariphopoulou and Žitković 2010. The weakness of the classical coherent or dynamic risk measures is that of the fixed time horizon, which is determined at the beginning of the investment period. If not, this presents a challenge to determine whether the risk measure is still the same after the fixed time horizon. This was the focus of the discussion by Chong et al. 2019, and we want to revisit and discuss it in a jump-diffusion framework.

1.3 Structure of the thesis

This thesis constitutes six chapters described as follows:

In this chapter, we outline the introduction and objective of the thesis.

Chapter 2 provides the notations and definitions that we use throughout the thesis. We give a review on concepts of stochastic calculus such as probability spaces, Martingales and Lèvy processes. The chapter also covers the theoretical concepts on BSDEs, risk measures and capital allocation.

Chapter 3, we study the representation of dynamic risk measures under the jump-diffusion process. Moreover, we derive the dynamic risk measure using BSDEs with jumps. By employing Malliavin derivatives of the BSDEs and Gâteaux-derivative, we obtain the capital allocation. We apply the results to derive the representation of the entropic risk measure.

In Chapter 4, we examine the representation of maturity-independent risk measures. We derive the representation from BSDEs whose generator depends on the solution of ergodic BSDEs. We also cover the capital allocation of the maturity-independent risk measures from the Gâteaux-derivative of the underlying risk measure.

Chapter 5 is devoted to the representation of maturity-independent risk measures under the jump-diffusion setting. We use Itô-Ventzell formula to determine the stochastic partial differential equation of the forward performance process to decide the form of the generator of our ergodic BSDE. Hence, we construct the maturity-independent risk measures representation as a solution to the BSDE with a generator that depends on the solution of the ergodic BSDE. We also study the behaviour of a forward entropic risk measure under jumps when we hold a financial position for a longer maturity.

Chapter 6 is another representation of risk measure using BSVIE with jumps. We derive the representation of the capital allocation by differentiating underlying risk measure.

Finally, in Chapter 7, we conclude by providing a summary of our findings.

1.4 Published papers and preprints

The thesis constitutes of four papers on the representation of risk measure and capital allocation listed as follows:

1) Lesedi Mabitsela, Calisto Guambe, and Rodwell Kufakunesu. "A note on representation of BSDE-based dynamic risk measures and dynamic capital allocations". Published in *Communications in Statistics-Theory and Methods (2020): pp 1-20,* doi: 10.1080/03610926.2020.1768405.

2) Lesedi Mabitsela, Rodwell Kufakunesu and Calisto Guambe. "An ergodic BSDE risk representation in a jump-diffusion framework". *Submitted.*

3) Lesedi Mabitsela, Rodwell Kufakunesu and Calisto Guambe. "A note on the ergodic BSDE-based risk representation and dynamic capital allocation". *Submitted.*

4) Lesedi Mabitsela and Rodwell Kufakunesu. "A note on BSVIE-based dynamic capital allocations in a jump framework." *Submitted.*

Chapter 2

Stochastic Calculus and Jump Diffusion Processes

2.1 Introduction

In this chapter, we introduce the concepts and notations that we will use throughout this thesis. We start by giving a review on some of the stochastic calculus concepts that we require for our study. There are great books on Stochastic calculus for example, see Øksendal (2010), Yong and Zhou (1999), Karatzas and Shreve (1998), Cohen and Elliott (2015) and Evans (2012). We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, satisfying the *usual conditions* whenever:

(i) $(\Omega, \mathcal{F}, P)$ is *complete*. A probability space $(\Omega, \mathcal{F}, P)$ is complete when A, $B \in \mathcal{F}$ and A is a P-measure zero event, then $B \subseteq A$.

(ii) $\mathcal{F}_0$ contains all *P-measure zero* events in $\mathcal{F}$. A P-measure zero event is defined as an event $A \in \mathcal{F}$ with $P(A) = 0$; and

(iii) $\{\mathcal{F}\}_{t\geq 0}$ is right continuous, that is $\mathcal{F}_{t^+} = \mathcal{F}_t$.

The set of all events is denoted by Ω, and $\mathcal{F}$ is called a σ-algebra on Ω. It is a family of all subsets of Ω, having the following properties:

(i) $\emptyset \in \mathcal{F}$;

(ii) $A, A^C \in \mathcal{F}$, where A^C is the complement of A in Ω;

(iii) If $A_i \in \mathcal{F}$ for $i = 1, 2, \ldots$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is an increasing collection of sub $\sigma-$algebra of $\mathcal{F}$, that is for $s < t$, $\mathcal{F}_s \subset \mathcal{F}_t$. We denote by $\mathcal{B}$, to be the smallest $\sigma-$algebra of subsets of $\mathbb{R}^n$ containing all open set. The probability measure P on a measure space (Ω, F) is a function $P : \mathcal{F} \to [0, 1]$ such that

(i) $P(\emptyset) = 0$ and $P(\Omega) = 1$.

(ii) If $A_i \in \mathcal{F}$ and $A_i \cap A_j$ for $i \neq j$ then $\cup_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(A_i)$.

The mapping $t \mapsto X(t, \omega)$ is called a sample path of the stochastic process for any $\omega \in \Omega$. A stochastic process $X(t)$ is said to be adapted if it is $\mathcal{F}_t-$measurable for all $t \geq 0$. $\mathcal{F}_t-$measurable implies the values of the process $X(t)$ will be revealed at time t. Furthermore, a stochastic process $X(t)$ is progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, if for all $0 \leq s < t$ the mapping $\omega \mapsto X(s, \omega)$ belongs to the product of $\sigma-$algebra $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$. All available information on an event $X(t)$ up to time t is contained in $\mathbb{F} := \{\mathcal{F}\}_{t\geq 0}$. That is $\mathcal{F}^X \subset \mathcal{F}_t$ for all t, where $\mathcal{F}^X = \sigma(X(t) \,|\, 0 \leq s < t)$ is the natural filtration of $X(t)$. For the rest of the thesis, we consider stochastic processes that are driven by the Brownian motion and Jump process.

2.2 Brownian Motion and Jump Processes

In this section, we present basic concepts relating to the Brownian motion and Jump process. The definitions in this Section are from Protter (2003), Øksendal and Sulem (2005), Applebaum (2009), Tankov 2003 and Delong 2013.

We proceed to define a Lévy process.

Definition 2.2.1. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtrated probability space. An $\mathcal{F}_t$ adapted process $X(t)$ with $X(0) = 0$ a.s. is a Lévy process if

(i) $X(t)$ has increments independent of the past, i.e., $X(t) - X(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$.

(ii) $X(t)$ has stationary increments, i.e., $X(t) - X(s)$ has the same distribution as X_{t-s} for all $0 \leq s < t < \infty$.

(iii) $X(t)$ is continuous in probability, i.e., $\forall \epsilon > 0$,

$$\lim_{t \to s} P(|X(t) - X(s)| > \epsilon) = 0,$$

for all $0 \leq s < t < \infty$.

Important example of Lévy process includes the Brownian motion and the Poisson process, which are both defined below.

Example 2.2.1. The Brownian motion. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtrated probability space. Then an $\mathcal{F}_t$ adapted $\mathbb{R}$-valued process $W(t)$ is called a one-dimensional $\mathbb{F}-$Brownian motion over $[0, \infty]$ if

(i) $W(0) = 0$ a.s.,

(ii) $W(t)$ has continuous sample paths, for any $\omega \in \Omega$,

(iii) for all $0 \leq s < t < \infty$, the increments of $W(t)$ are independent of the past i.e., $W(t) - W(s)$ is independent of $\mathcal{F}_s$,

(iv) for all $0 \leq s < t < \infty$, $W(t) - W(s)$ is normally distributed with mean 0 and covariance $(t - s)$, i.e., $\sim N(0, (t - s))$.

Example 2.2.2. The Poisson process. The Poisson process $N(t)$ of intensity $\lambda > 0$ is a Lévy process taking values in $\mathbb{N} \cup \{0\}$ and so that

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} \, \mathrm{e}^{-\lambda t},$$

for each $n = 0, 1, 2, \ldots,$.

Assume the Lévy process $X(t)$ is càdlàg i.e., rights continuous with left limits. We define the jump of a Lévy process $X(t)$ at $t \geq 0$ as $\Delta X(t) = X(t) - X(t^-)$. Our interest is on the number of jumps of a specific size. Hence, we define a random measure by taking $A \in \mathcal{B}(\mathbb{R}\backslash\{0\})$, then

$$N(t, A) = N(t, A, \omega) = \sum_{0<s\leq t} \mathbf{1}_A(\Delta X(t)).$$

Here, $N(t, A)$ is a Poisson random measure of $X(t)$. It counts the number of jumps of size $\Delta X(t) \in A$ that occur before or at time t. Therefore, $N(t, A)$ is a Poisson process of intensity $\nu(A) = \mathbb{E}[N(1, A)]$, i.e., the intensity gives an average value of the Poison random measure N. The Lévy measure satisfies the following condition:

$$\int_{\mathbb{R}_0} (1 \wedge |\zeta|^2)\nu(d\zeta) < \infty,$$

and it is positive. The function $\nu(A)$ is also called the Lévy measure of $X(t)$, it is a σ-finite measure on $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$. The derivative of $N(t, A)$ is given by $N(dt, d\zeta)$. We denote by $\tilde{N}(dt, d\zeta) := N(dt, d\zeta) - \nu(d\zeta)dt$ the compensated Poisson measure of $N(dt, d\zeta)$. A process $\tilde{N}(dt, d\zeta)$ is a martingale, that is $\mathbb{E}[\tilde{N}_t|\mathcal{F}_s] = \tilde{N}_s$ for all $t > s$.

We now, recall from Applebaum (2009) and Øksendal and Sulem (2005) an essential result in stochastic calculus, that is Itô's formula.

Theorem 2.2.1 (Itô formula)**.** *If X is an Itô-Lèvy process of the form*

$$dX(t) = \mu(t, \omega)dt + \sigma(t, \omega)dW(t) + \int_{\mathbb{R}_0} \Upsilon(t, \zeta)\tilde{N}(dt, d\zeta) \qquad (2.2.1)$$

where $\mu : [0, T] \times \Omega \to \mathbb{R}$, $\sigma : [0, T] \times \Omega \to \mathbb{R}$ and $\Upsilon : [0, T] \times \Omega \times \mathbb{R}_0 \to \mathbb{R}$ are predictable processes with respect to the filtration $\mathbb{F}$ such that the solution $X(t)$ exists. Then for $f \in C^2(\mathbb{R})$, $t \geq 0$ and define $Y(t) = f(t, X(t))$. Then $Y(t)$ is again an Itô-Lèvy process and

$$dY(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))\big[\mu(t, \omega)dt + \sigma(t, \omega)dW(t)\big]$$

$$+\frac{1}{2}\sigma^2(t,\omega)\frac{\partial^2 f}{\partial x^2}(t,X(t))dt + \int_{\mathbb{R}_0}\Big(f(t,X(t^-)+\Upsilon(t,\zeta)) - f(t,X(t^-))$$
$$-\frac{\partial f}{\partial x}(t,X(t^-))\Upsilon(t,\zeta)\Big)\nu(d\zeta)dt$$
$$+\int_{\mathbb{R}_0}\Big(f(t,X(t^-)+\Upsilon(t,\zeta)) - f(t,X(t^-))\Big)\tilde{N}(dt,d\zeta). \qquad (2.2.2)$$

Proof. See Øksendal and Sulem (2005) (Theorem 1.14). □

Remark: The Itô formula for the diffusion is a particular case of Equation (2.2.2), i.e., by removing the jumps from the Itô-Lévy process (2.2.1), we obtain the diffusion case.

We now state the Itô-Ventzell formula for jump-diffusion process. The Itô-Ventzel for diffusion is studied by Ocone and Pardoux (1989) (Theorem 3.1 page 50) and jump process is stated and proved by Øksendal, Zhang, et al. (2007). We apply the Itô-Ventzell formula in Chapter 5 to a forward process U to determine the form of the generator to our ergodic BSDE. Suppose we have two forward processes

$$dX(t) = \mu(t,\omega)dt + \sigma(t,\omega)dW(t) + \int_{\mathbb{R}_0}\Upsilon(t,\zeta)\tilde{N}(dt,d\zeta)$$

and

$$dU(t,x) = b(t,\omega,x)dt + a(t,\omega,x)dW(t) + \int_{\mathbb{R}_0}H(t,\zeta,x)\tilde{N}(dt,d\zeta),$$

where $b : [0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$, $a : [0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$ are progressively-measurable processes and $H : [0,T]\times\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}$ is a stochastic process such that the solution $U(t,x)$ exists.

Theorem 2.2.2 (Itô-Ventzell Formula)**.** *We assume that $U(t,x)$ is C^1 with respect to the space variable $x\in\mathbb{R}$. Then*

$$\begin{aligned} dU(t,X(t)) &= b(t,X(t))dt + a(t,X(t))dW(t) \\ &\quad + \frac{\partial U}{\partial x}(t,X(t))\big[\mu(t)dt + \sigma(t)dW(t)\big] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sigma^2(t)\frac{\partial^2 U}{\partial x^2}(t,X(t))dt+\sigma(t)\frac{\partial}{\partial x}a(t,X(t))dt\\
&+\int_{\mathbb{R}_0}\big[U(t,X(t)+\Upsilon(t,\zeta))\\
&-U(t,X(t))-\frac{\partial U}{\partial x}(t,X(t))\Upsilon(t,\zeta)\big]\nu(d\zeta)\\
&+\int_{\mathbb{R}_0}\big[H(t,X(t)+\Upsilon(t,\zeta))-H(t,X(t))\big]\nu(d\zeta)\\
&+\int_{\mathbb{R}_0}\big[U(t,X(t)+\Upsilon(t,\zeta))-U(t,X(t))\\
&+H(t,X(t)+\Upsilon(t,\zeta))\big]\tilde{N}(dt,d\zeta).
\end{aligned}
\tag{2.2.3}
$$

Proof. See Ocone and Pardoux (1989) (Theorem 3.1) and Øksendal, Zhang, et al. (2007) (Theorem 3.1). □

2.3 BMO martingales and Change of measure

The theory of change of measures and Girsanov transformation are important concept in the application of stochastic process in finance. We recall the concept of uniformly integrability from Cohen and Elliott (2015) (Section 2.5).

Definition 2.3.1. Let $D \subset L^1(\Omega, \mathcal{F}, P)$. Then D is said to be uniformly integrable subset of $L^1(\Omega, \mathcal{F}, P)$ if

$$\int_{\{|X|\geq c\}} |X(\omega)| dP(\omega) \tag{2.3.1}$$

converges to zero uniformly in $X \in D$ as $c \to \infty$.

The concept of uniformly integrable martingales refers to all martingales such that the collection of random variables is uniformly integrable. We recall that a random variable $T : \Omega \to [0, \infty)$ is called a stopping time when an event $(T \leq t) \in \mathcal{F}_t$ for each $t \geq 0$ (Applebaum (2009)).

The Girsanov Theorem illustrates what happens to a stochastic process when the probability measure is replaced by a new equivalent probability measure. We first, recall the Girsanov Theorem for an Itô process from Øksendal and Sulem (2005).

Theorem 2.3.1. *Let $X(t) \in \mathbb{R}$ be an Itô process of the form*

$$dX(t) = \mu(t,\omega)dt + \sigma(t,\omega)dW(t), \quad 0 \leq t \leq T \tag{2.3.2}$$

Assume there exists a process $\theta(t,\omega) \in \mathbb{R}$ such that

$$\sigma(t,\omega)\theta(t,\omega) = \mu(t,\omega) \quad \textit{for a.a. } (t,\omega) \in [0,T] \times \Omega \tag{2.3.3}$$

such that the process $M(t)$ defined for $0 \leq t \leq T$ by

$$M(t) := \exp\{-\int_0^t \theta(s,\omega)dW(s) - \frac{1}{2}\int_0^t \theta^2(t,\omega)ds\} \tag{2.3.4}$$

exists. Define a measure Q on $\mathcal{F}_T$ by

$$dQ(\omega) = M(T)dP(\omega) \quad \textit{on } \mathcal{F}_T. \tag{2.3.5}$$

Assume that

$$\mathbb{E}[M(T)] = 1.$$

Then Q is a probability measure on $\mathcal{F}_T$, Q is equivalent to P and $X(t)$ is a local martingale with respect to Q.

Proof. See Øksendal and Sulem (2005) (Theorem 1.30). □

We now recall the Girsanov Theorem for Lévy process following from Di Nunno et al. (2009) (Theorem 12.21) (see also Øksendal and Sulem (2005) Section 1.4).

Theorem 2.3.2 (Girsanov's Theorem)**.** *Let $\theta(t,\zeta) \leq 1$, $t \in [0,T]$, $\zeta \in \mathbb{R}_0$ and $\mu(t) \in \mathbb{R}$ be $\mathbb{F}-$ predictable processes such that*

$$\int_0^T \int_{\mathbb{R}_0} \{|\ln(1+\theta(s,\zeta))| + \theta^2(s,\zeta)\}\nu(d\zeta)ds < \infty, \qquad P-a.s., \tag{2.3.6}$$

$$\int_0^T \mu^2(s)ds < \infty, \qquad P - a.s. \tag{2.3.7}$$

Let

$$\begin{aligned} M(t) &= \mathcal{E}\Big(\int_0^t \mu(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} (\theta(s,\zeta) - 1)\tilde{N}(ds, d\zeta\Big)_T \\ &= \exp\Big\{ - \int_0^t \mu(s)dW(s) - \int_0^t \mu^2(s)ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \{\ln(1 - \theta(s,\zeta)) + \theta^2(s,\zeta)\}\nu(d\zeta)ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \ln(1 - \theta(s,\zeta))\tilde{N}(ds, d\zeta)\Big\}, \quad t \in [0, T], \end{aligned} \tag{2.3.8}$$

where $\mathcal{E}$ denotes the stochastic exponential or Doléans-Dade exponential. Define a measure Q on $\mathcal{F}_T$ by

$$dQ(\omega) = M(\omega, T)dP(\omega). \tag{2.3.9}$$

Assume that $M(T)$ satisfies the Novikov's Criterion, that is

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T \mu^2(s)ds + \int_0^t \int_{\mathbb{R}_0} \{(1 - \theta(s,\zeta))\ln(1 - \theta(s,\zeta)) \\ &+\theta(s,\zeta)\}\nu(d\zeta)ds\Big)\Big] < \infty. \end{aligned} \tag{2.3.10}$$

Then $\mathbb{E}[M(T)] = 1$ and hence Q is a probability measure on $\mathcal{F}_T$. Define

$$\tilde{N}^Q(dt, d\zeta) = \theta(t,\zeta)\nu(d\zeta)dt + \tilde{N}(dt, d\zeta)$$

and

$$dW^Q(t) = \mu(t)dt + dW(t).$$

Then $\tilde{N}^Q(\cdot,\cdot)$ and $W^Q(\cdot)$ are compensated Poisson random measure of $N(\cdot,\cdot)$ and Brownian motion under Q, respectively.

Proof. see Øksendal and Sulem (2005) (Section 1.4). □

Under the Novikov's Criterion, the stochastic exponential $Z(t)$ is a positive uniformly integrable martingale (Cohen and Elliott (2015) Theorem

15.4.20). This ensures that probability laws applied to the stochastic process still hold under the new probability measure. Next, we introduce the notion of Bounded Mean Oscillation (BMO) martingales. A local martingale M is in the class of BMO-martingales if there exists a constant K, $K > 0$, such that, for all $\mathcal{F}$-stopping times $\mathcal{T}$,

$$ess \sup_{\Omega} \mathbb{E}[\langle M(T)\rangle - \langle M(\mathcal{T})\rangle \mid \mathcal{F}_{\mathcal{T}}] \leq K^2 \qquad \text{and} \qquad |\Delta M(\mathcal{T})|^2 \leq K^2 \,.$$

We say a process $\{M(t)|t \geq 0\}$ is a local martingale if there exists a sequence of stopping times $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ such that as $\mathcal{T}_n \to \infty$, $M(\tau_n)$ is a martingale. For the diffusion case, the BMO-martingale property follows from the first condition, whilst in a jump-diffusion case, we need to ensure the boundedness of the jumps of the local martingale M. Now we recall the Kazamaki's Criterion from Morlais (2009b) (also see Kazamaki (2006) Theorem 2.3 for the continuous case) in the next lemma.

Lemma 2.3.3. *Let δ be such that: $0 < \delta < \infty$ and M a BMO martingale satisfying $\Delta M(t) \geq -1 + \delta$, P-a.s. and for all t, then the Doléans-Dade exponential of M denoted by $\mathcal{E}(M)$ is a uniformly integrable martingale.*

Proof. See Kazamaki (1979). □

2.4 Malliavin Calculus

In this section we provide properties of Malliavin calculus (see Di Nunno et al. (2009), Nualart (2006), Delong and Imkeller (2010) and Fujii and Takahashi (2018) and reference therein on more properties and theory on Malliavin calculus). Malliavin calculus (also known as stochastic calculus of variations) was first introduced by Paul Malliavin (1978) as an infinite dimensional differential calculus on the Wiener space. It uses the theory of integration by parts to study the derivatives of functions on this space. (For an extension of Malliavin calculus to jump process see Bichteler et al. (1987), Carlen and Pardoux (1990) and Di Nunno et al. (2009)). Using Malliavin derivative, we can obtain the representation of the BSDE-based capital allocation of the underlying risk measure under the jump-diffusion framework (this is done in Chapter 3).

We note that a smooth random variable F is Malliavin differentiable if and only if $F \in \mathbb{D}^{1,2} \subset L^2(P)$ (Fujii and Takahashi (2018)). The respective norm is defined as follows

$$\|F\|_{1,2}^2 := \mathbb{E}\Big[|F|^2 + \sum_{i=1}^{d} \int_0^T |D_s^i F|^2 ds + \sum_{j=1}^{k} \int_0^T \int_{\mathbb{R}_0} |D_{s,\zeta}^j F|^2 \zeta^2 \nu_j(d\zeta) ds\Big].$$

We first recall the Itô representation property, for which the diffusion case is provided by Øksendal (2003) (Theorem 4.3.3).

Theorem 2.4.1 (The Itô representation Theorem)**.** *Suppose $F \in L^2(\mathcal{F}_T)$, then F has the following representation*

$$F = \mathbb{E}[F] + \int_0^t \mathcal{Z}(s) dW(s) + \int_0^t \int_{\mathbb{R}_0} \mathcal{K}(s,z) \tilde{N}(ds, d\zeta) \quad 0 \le t \le T, \quad (2.4.1)$$

where $\mathcal{Z}(\cdot)$ and $\mathcal{K}(\cdot,\cdot)$ are predictable processes, integrable with respect to W and $\tilde{N}$.

Proof. See Applebaum (2009) (Theorem 5.3.5). □

By the Itô representation Theorem, we can represent a random variable in terms of stochastic integrals with respect to W and $\tilde{N}$.

We recall from Di Nunno et al. 2009 the Clark-Ocone formula and the chain rule on the Brownian and Poisson probability space $(\Omega, \mathcal{F}, P)$.

Theorem 2.4.2 (The Clark-Ocone Formula)**.** *Let $F \in \mathbb{D}^{1,2}$ be $\mathcal{F}_T$−measurable. Then*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_s F | \mathcal{F}_t] dW(t) + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D_{s,\zeta} F | \mathcal{F}_t] \tilde{N}(ds, d\zeta).$$

Proof. See Di Nunno et al. (2009) (Theorem 4.1 and Theorem 12.16). □

The Clark-Ocone Formula (2.4.2) provides an explicit representation of the processes in the Itô representation theorem (2.4.1), $\mathcal{Z}$ and $\mathcal{K}$ in terms of the Malliavin derivatives as follows

$$\begin{aligned}\mathcal{Z}(s) &= \mathbb{E}[D_s F|\mathcal{F}_t]\,, \\ \mathcal{K}(s,\zeta) &= \mathbb{E}[D_{s,\zeta} F|\mathcal{F}_t].\end{aligned} \tag{2.4.2}$$

The next theorem states the chain rule for evaluating Malliavin derivatives (see Di Nunno et al. (2009) Section 12.2 and Øksendal (1997)).

Theorem 2.4.3 (Chain Rule)**.** *Let $F = F_1, \ldots, F_m \in \mathbb{D}_{1,2}$, and let φ be a real continuous function on $\mathbb{R}$. Suppose $\varphi(F) \in L^2(P \times dt)$ and $\varphi(F + D_{s,z}F) \in L^2(P \times dt \times \nu)$. Then $\varphi(F) \in \mathbb{D}_{1,2}$,*

$$D_s\varphi(F) = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\varphi(F) D_s F_i$$

and

$$D_{s,\zeta}\varphi(F) = \varphi(F + D_{s,\zeta}F) - \varphi(F) \quad t \geq 0 \quad a.s.$$

Proof. See Nualart (2006) (Proposition 1.2.3) and Di Nunno et al. (2009) (Theorem 12.8). □

The following examples are generic and can be found in Øksendal (1997) and Di Nunno et al. (2009).

Example 2.4.1. Let $t_1 \in [0, T]$. The derivative of $\varphi(W(t_1)) = \mathrm{e}^{W(t_1)}$ is

$$D_s\varphi(W(t_1)) = \exp(W(t_1))\, \mathcal{X}_{[0,t_1]}(t).$$

Example 2.4.2. The derivative of $K = \exp\left(\int_0^T \int_{\mathbb{R}_0} h(t)\zeta\tilde{N}(dt, d\zeta)\right)$ is

$$\begin{aligned} D_{s,\zeta}K &= \exp\left(\int_0^T \int_{\mathbb{R}_0} h(t)\zeta\tilde{N}(dt, d\zeta) + h(t)\zeta \right) - K \\ &= K\exp(h(t)\zeta - 1). \end{aligned} \tag{2.4.3}$$

The following definition from Di Nunno et al. (2009) (Definition A.9), we will use in Chapter 3 to determine the representation of the capital allocation under the jump-diffusion framework.

Definition 2.4.1. Let $F : \Omega \to \mathbb{R}$ be random, choose $h \in L^2([0,T])$, and consider

$$\eta(t) = \int_0^t h(s)ds \quad \in \Omega. \tag{2.4.4}$$

Then we define the directional derivative of F at the point $\omega \in \Omega$ in direction $\eta \in \Omega$ by

$$D_\eta F(\omega) = \frac{d}{d\epsilon}[F(\omega + \epsilon\eta)]|_{\epsilon=0},$$

if the derivative exists.

Note that the set of $\eta \in \Omega$ formulated in the form (2.4.4) for some $h \in L^2([0,T])$, is called the *Cameron-Martin* space and is denoted by H.

2.5 Backward Stochastic Differential Equations (BSDEs)

In this section we recall BSDEs driven by two independent stochastic processes - the Brownian motion and Poisson random measure. We consider BSDEs where the generator has a quadratic growth in the Brownian motion component and exponential growth in the jump component.

The BSDE driven by the Brownian motion is an equation of the form

$$Y(t) = \xi + \int_t^T g(s,\omega,Y(s),Z(s))ds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T \tag{2.5.1}$$

where $(W(t))_{0\le t\le T}$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The function $g : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is called the generator or drive and terminal value, ξ, is a real-valued $\mathcal{F}_T-$measurable random variable. The BSDE is expressed in the differential form as

$$\begin{aligned} dY(t) &= -g(t,\omega,Y(t),Z(t))dt + Z(t)dW(t), \\ Y(T) &= \xi. \end{aligned} \tag{2.5.2}$$

In this thesis, we will consider BSDEs studied by Kobylanski et al. (2000) (see also Tevzadze (2008) and Barrieu, Karoui, et al. (2013)) that have a quadratic growth generator, that is, $|g(t,y,z)| \leq K(1+|y|+|z|^2)$. Quadratic growth refers to the quadratic feature of the control term, z, which appears in the generator.

The next definition, recalls (from Cohen and Elliott (2015) and Delong (2013)) the BSDE under a jump-diffusion process.

Definition 2.5.1. Given an $\mathcal{F}_T-$measurable $\mathbb{R}-$valued random variable ξ and function $g : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}_0) \to \mathbb{R}$, a backward stochastic differentiable equation (BSDE) with jumps is an equation of the form

$$\begin{aligned} dY(t) &= -g(t,\omega,Y(t),Z(t),\Upsilon(t,\zeta))dt + Z(t)dW(t) + \int_{\mathbb{R}_0} \Upsilon(t,\zeta)\tilde{N}(dt,d\zeta), \\ Y(T) &= \xi, \end{aligned} \tag{2.5.3}$$

for $t \in [0,T]$ and $g(t,\omega,Y,Z,\Upsilon) := g(t,Y,Z,\Upsilon)$. The integral form on $[t,T]$ is

$$\begin{aligned} Y(t) &= \xi + \int_t^T g(s,\omega,Y(s),Z(s),\Upsilon(s,\zeta))ds - \int_t^T Z(s)dW(s) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} \Upsilon(s,\zeta)\tilde{N}(ds,d\zeta). \end{aligned} \tag{2.5.4}$$

The function g is the generator and the pair (ξ, g) are the data of the BSDE. The solution to the BSDE (2.5.4) with data (ξ, g) is the triple of processes (Y, Z, Υ), with an adapted and càdlàg Y, predictable $\mathbb{R}$-valued process Z and a predictable process Υ taking values from $L^2_\nu(\mathbb{R}_0)$.

The data (ξ, g) of the BSDE (2.5.4) is said to be standard if the following *standard conditions* hold:

(i) $\mathbb{E}[||\xi||^2] < \infty$,

(ii) $\mathbb{E}[\int_0^T ||g(t, \omega, 0, 0, 0)||^2 dt] < \infty$,

(iii) There exists a constant K such that $dP \times dt - a.s.$

$$||g(t, \omega, y, z, \upsilon) - g(t, \omega, y', z', \upsilon')|| \leq K\big(||y - y'||^2 + ||z - z'||^2 + ||\upsilon - \upsilon'||^2\big) \tag{2.5.5}$$

for all $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}$ and $\upsilon, \upsilon' \in L^2_\nu(\mathbb{R}_0)$.

This type of model is introduced by Becherer et al. (2006) and Morlais (2009b) in the context of utility maximization. Other existing results on BSDEs with jumps can be found in Delong (2013), Cohen and Elliott (2015) and Karoui et al. (2016). Becherer et al. (2006) (in Lemma 3.4) showed that the martingale components of the BSDE (2.5.4) are BMO-martingales. The BMO property of the martingale components will be useful later when we apply the Girsanov Theorem. The quadratic-exponential BSDE with jumps that we will be working with in Chapter 3 is from Karoui et al. (2016), Antonelli and Mancini (2016) and Fujii and Takahashi (2018), where the data (ξ, g) of the BSDE in addition to the standard conditions satisfy the following conditions:

(i) The map $(t, \omega) \mapsto g(t, \omega, \cdot, \cdot, \cdot)$ is $\mathbb{F}$-progressively measurable. For every $(y, z, \upsilon) \in \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}_0)$, there exist two constants $\vartheta \geq 0$ and $\gamma > 0$ and a positive $\mathbb{F}$-progressively measurable process $(\ell_t, t \in [0, T])$ such that

$$-\ell_t - \vartheta|y| - \frac{\gamma}{2}|z|^2 - \int_{\mathbb{R}_0} j_\gamma(-\upsilon(t, \zeta))\nu(d\zeta) \leq g(t, y, z, \upsilon) \leq \ell_t + \vartheta|y| + \frac{\gamma}{2}|z|^2 + \int_{\mathbb{R}_0} j_\gamma(\upsilon(t, \zeta))\nu(d\zeta), \tag{2.5.6}$$

$dt \otimes dP$-a.e. $(\omega, t) \in \Omega \times [0, T]$, where $j_\gamma(\upsilon) := \frac{1}{\gamma}(e^{\gamma\upsilon} - 1 - \gamma\upsilon)$.

(ii) $|\xi|$, $(\ell_t, t \in [0,T])$ are essentially bounded i.e., $||\xi||_\infty$, $||\ell||_{S^\infty} < \infty$.

The existence and uniqueness of the solution to the quadratic-exponential BSDE (2.5.4), is proved in Fujii and Takahashi (2018), Antonelli and Mancini (2016) and Morlais (2009b). The exponential refers to the exponential expression appearing in the generator because of the jump coefficient (Karoui et al. (2016)).

A valuable tool in BSDEs is the comparison theorem, which we state here from Cohen and Elliott (2015) and Delong (2013). It lets us compare the solutions of two BSDEs. For example, if the data of two BSDEs satisfy an inequality, then also will their solutions.

Theorem 2.5.1 (The Comparison Theorem)**.** *Let (ξ, g) and (ξ', g') be standard Lipschitz data for two BSDEs, with solutions (Y, Z, Υ) and (Y', Z', Υ') respectively. Suppose*

(1) $\xi \geq \xi'$ *$P-$a.s.,*

(2) $g(\omega, t, y, z, \upsilon) \geq g(\omega, t, y', z', \upsilon')$ $dt \times dP-$ *a.s.*

(3) $g(t, y, z, \upsilon) - g(t, y, z, \upsilon') \leq \int_{\mathbb{R}_0} \Theta^{y,z,\upsilon,\upsilon'}(t,\zeta)[\upsilon(t,\zeta) - \upsilon'(t,\zeta)]\nu(d\zeta)$ *a.s., a.e.,* $(\omega, t) \in \Omega \times [0,T]$, *for all* $y \in \mathbb{R}, z \in \mathbb{R}, \upsilon, \upsilon' \in L^2_\nu(\mathbb{R}_0)$ *and there exists a predictable process* $\Theta^{y,z,\upsilon,\upsilon'} : \Omega \times [0,T] \times \mathbb{R} \to (-1, \infty)$ *such that* $t \mapsto \int_{\mathbb{R}_0} |\Theta^{y,z,\upsilon,\upsilon'}(t,\zeta)|^2 \nu(d\zeta)$ *is uniformly bounded in* $(y, z, \upsilon, \upsilon')$.

Then $Y(t) \geq Y'(t)$ *for* $t \in [0,T]$. *Furthermore, if for some* $A \in \mathcal{F}_t$ *we also have* $(Y(t) - Y'(t))\mathcal{X}_A = 0$, *then* $Y = Y'$ *on* $A \times [t,T]$, *i.e., if* Y *and* Y' *meet, they remain the same from then onwards.*

Proof. See Cohen and Elliott (2015) (Theorem 19.3.4) and Delong (2013) (Theorem 3.2.2). □

2.5.1 Ergodic BSDEs

In this section, we consider ergodic BSDEs, which we use to study forward entropic risk measures. Ergodic BSDEs are an extension to BSDEs, which takes the form

$$Y(t) = Y(T) + \int_t^T (g(X(s), Z(s)) - \lambda)ds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T < \infty. \tag{2.5.7}$$

The solution to the ergodic BSDE (2.5.7) is a triple (Y, Z, λ), where Y and Z are adapted real valued processes and λ is a real number. The process, X, is the stock price process (or the forward process) (this will be explained in Chapter 4). The ergodic BSDE's solution is derived as a limit of solutions to the infinite horizon BSDE. In ergodic BSDEs, there is no terminal condition given at some determined terminal time T. Also, the ergodic BSDE system solves for the backward component by first valuing the forward process over a long enough time instead of a finite deterministic time (Fedyashov (2016)). This long running behaviour is captured in the component λ, which is part of the solution.

The ergodic BSDEs (2.5.7) was first introduced by Fuhrman et al. (2009), where they applied the ergodic BSDE to an optimal control problem. Further research on ergodic BSDE is by Fedyashov (2016), Allan and Cohen (2016) and Debussche et al. (2011).

The ergodic BSDEs are extended to jumps by Cohen and Fedyashov (2014). Given the function g the ergodic BSDE with jumps is given by

$$\begin{aligned} Y(t,x) &= Y(T,x) + \int_t^T [g(X(s,x), Y(s,x), Z(s,x), \Upsilon(s,x,\zeta)) - \lambda]ds \\ &\quad - \int_t^T Z(s,x)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon(s,x,\zeta)\tilde{N}(ds,d\zeta), \end{aligned} \tag{2.5.8}$$

where $0 \leq t \leq T < \infty$, Y is a real valued càdlàg process, Z and Υ are predictable processes. Cohen and Fedyashov (2014) have shown that there exists a Markovian solution to the ergodic BSDE with jumps.

Theorem 2.5.2. *Define*

$$Y(t,x) = v(t, X(t)),$$

there exist processes $Z(x)$ and $\Upsilon(x)$ such that the quadruple $(Y(x), Z(x), \Upsilon(x), \lambda)$ solves the ergodic BSDE

$$\begin{aligned} Y(t,x) &= Y(T,x) + \int_t^T [g(X(s,x), Y(s,x), Z(s,x), \Upsilon(s,x,\zeta)) - \lambda] ds \\ &\quad - \int_t^T Z(s,x) dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon(s,x,\zeta) \tilde{N}(ds, d\zeta), \end{aligned} \tag{2.5.9}$$

for $0 \leq t \leq T < \infty$ Moreover, if there exists any other solution $(Y', Z', \Upsilon', \lambda')$ that satisfies

$$|Y'(t)| < c_x(1 + ||X(t,x)||^2), \tag{2.5.10}$$

for some constant c that may depend on x, then $\lambda = \lambda'$.

Proof. See Cohen and Fedyashov (2014) (Theorem 10). ☐

Cohen and Fedyashov (2014), further proved that there exist a unique Markovian solutions for the ergodic BSDE (2.5.8) with jumps.

Theorem 2.5.3. *Let $(Y, Z, \Upsilon, \lambda)$ and $(Y', Z', \Upsilon', \lambda')$ be two Markovian solutions to the ergodic BSDE. If Y, Y' satisfy the growth condition (2.5.10), v, v' satisfies*

$$v(t,x) = v(t + T^*, x), \quad \forall t > 0$$

and $v'(0,0) = v(0,0)$, then $v = v'$ a.e., for some $T^* > 0$.

Proof. See Cohen and Fedyashov (2014) (Theorem 11). ☐

2.6 Risk Measures

In this section, we define and discuss the properties a risk measure. We also study the connection between risk measures and BSDEs. Risk measures are tools to quantify the riskiness of a financial position held by an investor. We utilise the quantification to decide if the risk is acceptable or not. Risk measures are tools to quantify the riskiness of a financial position held by an investor. We utilise the quantification to decide if the risk is acceptable or not. The risk measure aids the investor to manage the capital required to cushion the risk in unfavourable market conditions. Once, we decide on the required capital, and this then leads us to allocate it to the varies sub-divisions or sub-portfolios.

2.6.1 Definition and properties

We define by $\mathcal{X} \subset L^2(\mathcal{F}_T)$ the space of financial positions ξ.

Definition 2.6.1. (see Artzner et al. (1999), Gianin (2006)) A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is a static risk measure if, for any ξ_1 and ξ_2 in $\mathcal{X}$, it satisfies the following axioms:

A1) Monotonicity: $\rho(\xi_1) \leq \rho(\xi_2)$, $\forall\ \xi_2 \leq \xi_1$;

A2) Translation invariance: $\rho(\xi_1 + m) = \rho(\xi_1) - m$, $m \in \mathbb{R}$;

A3) Subadditivity: $\rho(\xi_1 + \xi_2) \leq \rho(\xi_1) + \rho(\xi_2)$;

A4) Positive homogeneity $\rho(k\xi_1) = k\rho(\xi_1)$, $k \geq 0$;

A5) Convexity: $\rho(\lambda\xi_1 + (1-\lambda)\xi_2) \leq \lambda\rho(\xi_1) + (1-\lambda)\rho(\xi_2)$, $\lambda \in (0,1)$.

The functional $\rho(\xi)$ quantifies the risk of a financial position $\xi \in \mathcal{X}$. The position ξ is acceptable when $\rho(\xi) \leq 0$, and unacceptable otherwise (Artzner et al. (1999)). The functional $\rho(\xi)$ represents the capital amount that an investor can withdraw without changing the acceptability of ξ. Monotonicity implies that ρ is non-increasing with respect to $\xi \in \mathcal{X}$. The financial meaning is that if a position ξ_1 is always higher than ξ_2, then the capital required to support ξ_1 should be less than capital required for ξ_2. Subadditivity allows for risk to be reduced by diversification since the risk of a portfolio $\xi_1 + \xi_2$ is bounded by the sum of individual risk of position ξ_1 and ξ_2. Translation invariance states that if you add a certain amount m to the initial investment position, then the risk of that investment will decrease by that amount m. Note that, if a position ξ is not acceptable, then adding an amount $\rho(\xi)$ to it will make the position acceptable, i.e. $\rho(\xi + \rho(\xi)) = \rho(\xi) - \rho(\xi) = 0$. Positive homogeneity tells us that the capital required to support k identical positions is equal to k times the capital required for one position. The convexity property illustrates how the risk of a position might increase in a nonlinear way as the position is multiplied by a factor, due to liquidity risk of a large financial position.

A convex risk measure ρ whose domain includes $\mathcal{X}$ such that $\rho(\xi) < \infty$ where $\xi \in \mathcal{X}$, satisfies property 5) see Föllmer and Schied (2002) and Frittelli and Gianin (2002), while a coherent risk measure satisfies properties 1) to 4) see Artzner et al. (1999) and Delbaen (2002).

2.6.2 Dynamic Risk Measures

In this subsection, we define a dynamic risk measure, which measures risk at an intermediate time $t \in [0, T]$ while considering all information available up to time t. Firstly, we recall the time-consistent property from Delong (2013)

A6) Time-consistency $\rho_s(\xi) = \rho_s(-\rho_t(\xi))$ $0 \leq s \leq t \leq T$, $\xi \in L^2(\mathcal{F}_T)$.

Put simple, time-consistence means a risk measure is consistent at different times. Measuring risk at time s should be the same as first measuring risk at an intermediate time $t > s$ and then quantify it at time t to time s. We state from (Gianin (2006)) the following definition of a dynamic risk measure:

Definition 2.6.2. A mapping $(\rho_t)_{t\in[0,T]}$ is a dynamic risk measure for all $\xi \in \mathcal{X}$ and $t \in [0, T]$, if the following properties are satisfied:

(a) $\rho_t : \mathcal{X} \to L^0(\mathcal{F}_t)$.

(b) ρ_0 is a static risk measure.

(c) $\rho_T(\xi) = -\xi$ for all $\xi \in \mathcal{X}$.

The risk measure $\rho_t(\xi)$ provides us with the value that is at risk at time t for holding a financial position ξ, which will be liquidated at time T. A dynamic risk measure is called coherent if it satisfies, positive homogeneity, monotonicity, translation invariance and subadditivity. A dynamic convex risk measure satisfies the convexity property and assume $\rho_t(0) = 0$ for any $t \in [0, T]$ (Gianin (2006)).

The following from Delong (2013) defines dynamic risk measures constructed as a solution of BSDEs.

Theorem 2.6.1. *Let $\rho_t^g(\xi) := Y^{-\xi}(t)$, $t \in [0, T]$. Then ρ is monotone, time-consistent dynamic risk measure. In addition,*

(a) if g is sublinear in (z, υ) and independent of y, then ρ is a coherent dynamic risk measure.

(b) If g is convex in (y, z, υ), then ρ is a convex dynamic risk measure

Proof. See Delong (2013) (Theorem 6.2.1 and Proposition 6.2.3). □

The component Y^{ξ} is the solution of the BSDE (2.5.4). The driver g plays an essential role in the construction of risk measures by BSDE. The dynamic entropic risk measure defined by

$$\rho_t(\xi) = \frac{1}{\gamma} \ln \mathbb{E}\Big[e^{-\gamma\xi} \mid \mathcal{F}_t\Big], \quad \gamma > 0,\, t \in [0, T]$$

is a classic example of a time-consistent dynamic risk measure that can be constructed using quadratic BSDE

$$Y(t) = -\xi + \int_t^T \frac{\gamma}{2}|Z|^2 ds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T.$$

See Barrieu and Karoui (2007) (Proposition 6.4).

In the jump-diffusion framework the dynamic entropic risk measure is defined as

$$\rho_t^g(\xi) = Y^{-\xi}(t) \quad \textit{for all } t \in [0, T], \tag{2.6.1}$$

with $Y(t)$ as the first component to the solution $(Y(t), Z(t), \Upsilon(t, \zeta))$ of the BSDE

$$Y(t) \;=\; -\xi + \int_t^T \Big(\frac{\gamma}{2}|Z(s)|^2 + \frac{1}{\gamma}\int_{\mathbb{R}_0} \Big(\exp(\gamma\Upsilon(s,\zeta)) - \gamma\Upsilon(s,\zeta) - 1\Big)\nu(d\zeta)\Big)ds$$

$$
-\int_t^T Z(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon(s,\zeta)\tilde{N}(ds,d\zeta)\,.
$$

The generator

$$
g(t,Z,\Upsilon) = \int_t^T \Big(\frac{\gamma}{2}|Z(s)|^2 + \frac{1}{\gamma}\int_{\mathbb{R}_0}\Big(\exp(\gamma\Upsilon(s,\zeta)) - \gamma\Upsilon(s,\zeta) - 1\Big)\nu(d\zeta)\Big)ds, \tag{2.6.2}
$$

and the terminal value ξ satisfying the standard condition and condition (2.5.6).

Theorem 2.6.2. *Suppose ρ and g are defined as in (2.6.1) and (2.6.2) respectively then ρ is a convex dynamic risk measure, this means the following axioms hold:*

(a) ***Convexity*** *If g is convex, i.e.,*

$$
g(t,\lambda z_1 + (1-\lambda)z_2, \lambda\upsilon_1 + (1-\lambda)\upsilon_2) \leq \lambda g(t,z_1,\upsilon_1) + (1-\lambda)g(t,z_2,\upsilon_2),
$$

$\lambda \in (0,1)$ *and* (t,z_1,υ_1), $(t,z_2,\upsilon_2) \in [0,T]\times\mathbb{R}\times L^2_\nu(\mathbb{R}_0)$

then ρ is convex, i.e.,

$$
\rho^g_t(\lambda\xi_1 + (1-\lambda)\xi_2) \leq \lambda\rho^g_t(\xi_1) + (1-\lambda)\rho^g_t(\xi_2), \quad t\in[0,T].
$$

(b) ***Translation invariance*** *If generator g is independent of y, then $\rho^g_t(\xi)$ is translation invariant, i.e.,*

$$
\rho^g_t(\xi + m) = \rho^g_t(\xi) - m \quad t\in[0,T],\ m\in\mathbb{R}.
$$

(c) ***Monotonicity*** *If $\xi_1 \leq \xi_2$, then $\rho^g_t(\xi_2) \leq \rho^g_t(\xi_1)$, $t\in[0,T]$ and $\xi_1,\xi_2 \in L^2(\mathcal{F}_T)$.*

(d) ***Time-consistency*** *If $\xi \in L^2(\mathcal{F}_T)$, then $\rho_s(\xi) = \rho_s(-\rho_t(\xi))$ for all $0 \leq s \leq t \leq T$.*

Proof. (See Delong (2013) (Chapter 6), Agram (2019) and Barrieu and Karoui (2007) (Section 6)). We provide the sketch of the proof for completeness.

(a) **Convexity**: To prove that

$$\rho_t^g(\lambda\xi_1 + (1-\lambda)\xi_2) \leq \lambda\rho_t^g(\xi_1) + (1-\lambda)\rho_t^g(\xi_2), \quad t \in [0,T],$$

we have to show that

$$Y^{-(\lambda\xi_1+(1-\lambda)\xi_2)}(t) \leq \lambda Y^{-\xi_1}(t) + (1-\lambda)Y^{-\xi_2}(t), \quad t \in [0,T].$$

We fix $\lambda \in (0,1)$ and consider the BSDE

$$\begin{aligned}\hat{Y}^{-(\lambda\xi_1+(1-\lambda)\xi_2)}(t) &= -(\lambda\xi_1 + (1-\lambda)\xi_2) + \int_t^T g(s, \hat{Z}(t), \hat{\Upsilon}(s,\zeta))ds \\ &\quad - \int_t^T \hat{Z}(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \hat{\Upsilon}(s,\zeta)\tilde{N}(ds,d\zeta),\end{aligned}$$

with the solution $(\hat{Y}, \hat{Z}, \hat{\Upsilon}) \in \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}_0)$. We define

$$\begin{aligned}\bar{Y}(t) &= \lambda Y^{-\xi_1}(t) + (1-\lambda)Y^{-\xi_2}(t) \\ \bar{Z}(t) &= \lambda Z^{-\xi_1}(t) + (1-\lambda)Z^{-\xi_2}(t) \\ \bar{\Upsilon}(t) &= \lambda \Upsilon^{-\xi_1}(t) + (1-\lambda)\Upsilon^{-\xi_2}(t)\,.\end{aligned} \tag{2.6.3}$$

Then, we look at the following BSDE

$$\begin{aligned}\bar{Y}(t) &= -(\lambda\xi_1 + (1-\lambda)\xi_2) + \int_t^T \lambda g(s, Z^{-\xi_1}(s), \Upsilon^{-\xi_1}(s,\zeta))ds \\ &\quad + \int_t^T (1-\lambda) g(s, Z^{-\xi_2}(s), \Upsilon^{-\xi_2}(s,\zeta))ds \\ &\quad - \int_t^T \bar{Z}(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \bar{\Upsilon}(s,\zeta)\tilde{N}(ds,d\zeta) \\ &\geq -(\lambda\xi_1 + (1-\lambda)\xi_2) + \int_t^T g(s, \bar{Z}(s), \bar{\Upsilon}(s,\zeta))ds \\ &\quad - \int_t^T \bar{Z}(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \bar{\Upsilon}(s,\zeta)\tilde{N}(ds,d\zeta)\,.\end{aligned}$$

The last inequality is a result of the convexity of g in Z and Υ. Then by the Comparison Theorem (2.5.1) we conclude that

$$\hat{Y}^{-(\lambda\xi_1+(1-\lambda)\xi_2)}(t) \leq \bar{Y}(t),$$

for each $t \in [0,T]$. Thus

$$\rho_t^g(\lambda\xi_1 + (1-\lambda)\xi_2) \leq \lambda\rho_t^g(\xi_1) + (1-\lambda)\rho_t^g(\xi_2), \quad t \in [0,T].$$

(b) **Translation invariance**: We take two BSDEs

$$\begin{aligned} Y^{-\xi+m}(t) &= -\xi + m + \int_t^T g(s, Z^{-\xi+m}(s), \Upsilon^{-\xi+m}(s,\zeta))ds \\ &\quad - \int_t^T Z^{-\xi+m}(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{-\xi+m}(s,\zeta)\tilde{N}(ds,d\zeta) \end{aligned}$$

and

$$\begin{aligned} Y^{-\xi}(t) &= -\xi + \int_t^T g(s, Z^{-\xi}(s), \Upsilon^{-\xi}(s,\zeta))ds \\ &\quad - \int_t^T Z^{\xi}(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\xi}(s,\zeta)\tilde{N}(ds,d\zeta)\,. \end{aligned}$$

We note that

$$Y^{-\xi+m}(t) = Y^{-\xi}(t) + m.$$

Consequently

$$\begin{aligned} \rho_t^g(\xi+m) &= Y^{-\xi+m}(t) \\ &= Y^{-\xi}(t) + m \\ &= \rho_t^g(\xi) - m \quad \text{for each t} \in \text{[0, T]}. \end{aligned} \tag{2.6.4}$$

(c) **Monotonicity**: If $\xi_1 \leq \xi_2$, then by the Comparison Theorem (2.5.1) we have $Y^{-\xi_1}(t) \geq Y^{-\xi_2}(t)$. Hence,

$$\rho_t^g(\xi_2) = Y^{-\xi_2}(t) \leq \rho_t^g(\xi_1) = Y^{-\xi_1}(t).$$

(d) **Time-consistency**: We want to prove that

$$\rho_s(\xi) = \rho_s(-\rho_t(\xi)),$$

i.e.,

$$Y^{-\xi}(s) = Y^{Y^{-\xi}(t)}(s),$$

for three bounded stopping times $0 \leq s \leq t \leq T$. a.s., We note that

$$\begin{aligned}
Y^{Y^{-\xi}(t)}(s) &= Y^{-\xi}(t) + \int_s^T g(u, Z(u), \Upsilon(u,\zeta))du \\
&\quad - \int_s^T Z(u)dW(u) - \int_s^T \int_{\mathbb{R}_0} \Upsilon^{\xi}(u,\zeta)\tilde{N}(du, d\zeta) \\
&= -\xi + \int_t^T g(u, Z(u), \Upsilon(u,\zeta))du \\
&\quad - \int_t^T Z(u)dW(u) - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\xi}(u,\zeta)\tilde{N}(du, d\zeta) \\
&\quad + \int_s^T g(u, Z(u), \Upsilon(u,\zeta))du \\
&\quad - \int_s^T Z(u)dW(u) - \int_s^T \int_{\mathbb{R}_0} \Upsilon^{\xi}(u,\zeta)\tilde{N}(du, d\zeta) \\
&= -\xi + \int_s^T g(u, Z(u), \Upsilon(u,\zeta))du - \int_s^T Z(u)dW(u) \\
&\quad - \int_s^T \int_{\mathbb{R}_0} \Upsilon^{\xi}(u,\zeta)\tilde{N}(du, d\zeta) \\
&= Y^{-\xi}(s), \qquad (2.6.5)
\end{aligned}$$

the process defined by $Y^{Y^{-\xi}(t)}(u)$ on $[0,t]$ and by $Y^{-\xi}(u)$ on $[t,T]$ is the solution of the BSDE (ξ, g, T). Therefore, the uniqueness of the solution implies the time-consistency property (Barrieu and Karoui (2007) Theorem 6.7). Hence,

$$\rho_s(\xi) = \rho_s(-\rho_t(\xi)).$$

□

2.6.3 Capital Allocation

In this subsection we discuss the capital allocation properties. Let $X_1, X_2, \dots, X_n \in \mathcal{X}$ be the financial positions, with the corresponding risk contribution to the overall portfolio denoted by $\rho(X_i|X)$, $i = 1, 2, \dots, n$. Consider a portfolio $X \in \mathcal{X}$, consisting of X_i, subportfolios, that is

$$X = \sum_{i=1}^{n} X_i.$$

The portfolio risk is given by $\rho(X)$. The capital allocation problem is allocating the overall risk $\rho(X)$ of the portfolio X to the individual subportfolios in the portfolio. That is, we require a mapping such that

$$\rho(X) = \sum_{i=1}^{n} \rho(X_i|X). \tag{2.6.6}$$

Such a relation is called the *full allocation property*, since the overall portfolio risk is fully allocated to the individual subportfolios in the portfolio (Tasche (2007)).

Let ρ be a risk measure that is Gâteaux differentiable at X in its domain, then the gradient allocation $\rho(X_i|X)$ is determined by

$$\begin{aligned}\rho(X_i|X) &= \nabla_{X_i}\rho(X) \\ &= \lim_{\epsilon\to 0} \frac{\rho(X+\epsilon X_i) - \rho(X)}{\epsilon} \\ &= \frac{d}{d\epsilon}\rho(X+\epsilon X_i)\Big|_{\epsilon=0}.\end{aligned} \tag{2.6.7}$$

Equation (2.6.7) defines the static gradient allocation principle, which is the Gâteaux-derivative of X in the direction of X_i, for $i = 1, 2, \ldots, n$ (see Kromer and Overbeck (2014) and Kromer and Overbeck (2017)). The static Aumann-Shapley allocation is represented by:

$$\overline{\nabla_{X_i}\rho(X)} = \int_0^1 \nabla_{X_i}\rho(\beta X)d\beta, \qquad i = 1, 2, \ldots, n, \tag{2.6.8}$$

where $\beta \in [0, 1]$ is taken to be portfolio weights. If the risk measure ρ is positive homogeneous, then the Aumann-Shapley allocation reduces to the gradient allocation principle (2.6.7) (Denault (2001)). For the Aumann-Shapley, we do not require the risk measure to be positively homogeneous to satisfy full allocation property. However, the gradient allocation does need the risk measure to be positive homogeneous to satisfy the full allocation property. According to Kromer and Overbeck (2014), the Aumann-Shapley and Gâteaux-derivative can be jointly used to risk measures that do not satisfy the positive homogeneity property. Hence, the combination can be used for convex risk measures that do not satisfy the positive homogeneity property.

Chapter 3

Representation of BSDE-based dynamic risk measures and dynamic capital allocations

3.1 Introduction

In this section, we derive a representation for dynamic capital allocation when the underlying price process includes extreme random price movements. Moreover, we consider the representation of dynamic risk measures defined under BSDEs with generators that grow quadratic-exponentially in the control variables. Dynamic capital allocation is derived from the differentiability of BSDEs with jumps. The results are illustrated by deriving a capital allocation representation for dynamic entropic risk measure and static coherent risk measure.

The remainder of the chapter is organised as follows. In Section 3.2, we present the notations and define concepts that will be used throughout the chapter. Section 3.3, we derive the representation of dynamic risk capital allocation based on the BSDE with jumps. From dynamic risk capital allocation results we derive the representation of the BSDE based dynamic convex and coherent risk measures. We conclude in Section 3.4 with applications of our results to the entropic risk measures.

3.2 BSDE-based dynamic risk representation

In this chapter, we let our source of randomness be modelled by two independent processes: the one-dimensional standard Brownian motion, $W = \{W(t), \mathcal{F}(t); 0 \leq t \leq T\}$, defined on a probability space $(\Omega^W, \mathcal{F}^W, P^W)$, and the independent compensated Poisson random measure, $\tilde{N}(dt, d\zeta) := N(dt, d\zeta) - \nu(d\zeta)dt$ defined on the probability space $(\Omega^{\tilde{N}}, \mathcal{F}^{\tilde{N}}, P^{\tilde{N}})$, with ν on $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$ as the Lévy measure of $N(\cdot, \cdot)$. If we let $\mathcal{B}(\mathbb{R}_0)$ denote the family of Borel sets $A \subset \mathbb{R}$. Then the Poisson random measure $N(A, t)$, counts the number of jumps of size $\Delta X \in A$ that occur on or before time t and its derivative is given by $N(d\zeta, dt)$ (Øksendal and Sulem (2005)).

Let $(\Omega, \mathcal{F}, P)$ be the product of the canonical filtered probability spaces $(\Omega^W \times \Omega^{\tilde{N}}, \mathcal{F}^W \otimes \mathcal{F}^{\tilde{N}}, P^W \otimes P^{\tilde{N}})$ and the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ is the canonical filtration. Let $\mathcal{X}$ be the space of financial positions with maturity T, i.e., $\mathcal{X}$ is consider to be the space $L^\infty(\Omega, \mathcal{F}_T, P)$ with essentially bounded random variables with norm $||X||_\infty = ess \sup |X|$, or the space $L^p(\Omega, \mathcal{F}_T, P)$, for $1 \leq p < \infty$ of p-integrable random variables. The components of $\mathcal{X}$ represent the terminal value or net worth at maturity T of an investment portfolio. The goal is to summaries the financial riskiness of a position $\xi \in \mathcal{X}$ in a single number at any time $t \in [0, T]$, i.e., a dynamic risk measure denoted by $\rho_t(\xi)$.

Definition 3.2.1. We call a mapping $\rho_t : \mathcal{X} \to L^0(\mathcal{F}_t)$, where $\rho_t(\xi) = -\xi$, a dynamic convex risk measure if the properties of monotonicity, translation invariance, convexity and time-consistency are satisfied (see Sub-section 2.6.2).

The quantity $\rho_t(\xi)$ provides us with a single value that summaries the riskiness at time $t \in [0, T]$ of the financial position ξ. The position ξ is said to be acceptable whenever $\rho_t(\xi) \leq 0$, and unacceptable otherwise.

In this work, the dynamic risk measures are constructed using BSDEs. We consider a quadratic-exponential BSDE, (defined as in Karoui et al. (2016)) for $t \in [0,T]$ of the form

$$\begin{aligned} Y(t) &= -\xi + \int_t^T g(s, Y(s), Z(s), \Upsilon(s,\zeta))ds - \int_t^T Z(s)dW(s) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} \Upsilon(s,\zeta)\tilde{N}(ds,d\zeta), \quad 0 \leq t \leq T, \end{aligned} \tag{3.2.1}$$

where $\xi : \Omega \to \mathbb{R}$ and $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}_0) \to \mathbb{R}$. The solution to Equation (3.2.1) is given by the triple $(Y(t), Z(t), \Upsilon(t)) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2_W(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$, where the adapted process $Y(t)$ is controlled by the control processes $Z(t)$ and $\Upsilon(t)$, such that $Y(T) = \xi$. For the existence and uniqueness of such BSDEs, the driver and terminal condition are subject to the standard condition stated in Chapter 2 and the following assumptions. We adapt the assumptions from Fujii and Takahashi (2018) (see also Briand and Hu (2006), Karoui et al. (2016), Royer (2006), Delong (2013)).

Assumption 1. For $m > 0$ and $(y,z,v),(y',z',v') \in \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}_0)$ satisfying

$$|y|, |y'|, ||v||_{L^2_\nu}, ||v'||_{L^2_\nu} \leq m,$$

there exists some positive constant K_m depending on m such that

$$\begin{aligned} |g(t,y,z,v) - g(t,y',z',v')| &\leq K_m(|y-y'| + ||v-v'||_{L^2_\nu}) \\ &\quad + K_m\Big(1 + |z| + |z'| + ||v||_{L^2_\nu} + ||v'||_{L^2_\nu}\Big)|z-z'| \end{aligned} \tag{3.2.2}$$

$dt \otimes dP$ a.e. $(\omega, t) \in \Omega \times [0,T]$.

Assumption 2. For all $t \in [0,T]$, $m > 0$ and $y \in \mathbb{R}, z \in \mathbb{R}, v, v' \in \mathbb{R}$ with $|y|, ||v||_{L^2_\nu}, ||v'||_{L^2_\nu} \leq m$, there exists a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable process $\Theta^{y,z,v,v'}$ satisfying $dt \otimes dP$-a.e.

$$g(t,y,z,v) - g(t,y',z',v') \leq \int_{\mathbb{R}_0} \Theta^{y,z,v,v'}(t,\zeta)[v(t,\zeta) - v'(t,\zeta)]\nu(d\zeta) \tag{3.2.3}$$

and $C_m^1(1 \wedge |\zeta|) \leq \Theta_t^{y,z,\upsilon,\upsilon'}(\zeta) \leq C_m^2(1 \wedge |\zeta|)$. Here C_m^1 and C_m^2 are two constants satisfying the following conditions $C_m^1 > -1$ and $C_m^2 > 0$ and are dependent on m.

Fujii and Takahashi (2018) (in Theorem 3.1) proved the existence of a unique bounded solution $(Y, Z, \Upsilon) \in \mathbb{S}^2 \times \mathbb{H}_W^2 \times \mathbb{H}_N^2$ of the BSDE (3.2.1). Moreover, Z belongs to the set of progressively measurable real valued functions denoted by $\mathbb{H}^2_{BMO(W)}$ satisfying

$$\left|\left| \int_0^{\cdot} Z(s) \right|\right|^2_{BMO(W)} = ess\sup \mathbb{E}\Big[\int_\tau^T |Z(s)|^2 ds \big| \mathcal{F}_\tau \Big] \leq K^2, \quad P - a.s.$$

and Υ belongs to the set of predictable processes, denoted by $\mathbb{H}^2_{BMO(N)}$ satisfying the following

$$\begin{aligned} \left|\left| \int_0^{\cdot} \int_{\mathbb{R}_0} \Upsilon(s,\zeta)\tilde{N}(ds,d\zeta) \right|\right|^2_{BMO(N)} &= ess\sup \mathbb{E}\big[\int_\tau^T \int_{\mathbb{R}_0} |\Upsilon(s,\zeta)|^2 \nu(d\zeta) ds \big| \mathcal{F}_t \big] \\ &\quad + |\Delta M(\mathcal{T})| \leq K^2. \end{aligned}$$

We consider dynamic risk measures constructed using BSDEs with jumps, where the generator g is independent of Y. That is, we define:

$$\rho_t^g(\xi) := Y^{-\xi}(t), \quad for\ t \in [0,T], \tag{3.2.4}$$

where $Y^{-\xi}(t)$ is the first component of the solution $(Y(t), Z(t), \Upsilon(t,\zeta))$ to the BSDE

$$\begin{aligned} Y(t) &= -\xi + \int_t^T g(s, Z(s), \Upsilon(s,\zeta)) ds - \int_t^T Z(s) dW(s) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} \Upsilon(s,\zeta)\tilde{N}(ds,d\zeta), \quad 0 \leq t \leq T, \end{aligned} \tag{3.2.5}$$

where $\xi : \Omega \to \mathbb{R}$ and $g : \Omega \times [0,T] \times \mathbb{R} \times L^2_\nu(\mathbb{R}_0) \to \mathbb{R}$ satisfying the standard conditions and Assumptions 1 - 2, such that

$$g(t, z, \Upsilon(t,\zeta)) := \ell(t, z, \Upsilon) + \frac{1}{2}\gamma|z|^2 + \frac{1}{\gamma}\int_{\mathbb{R}_0} (e^{\gamma\Upsilon} - 1 - \gamma\Upsilon)\nu(dz) \tag{3.2.6}$$

and is a special case of the generator in the standard condition (Equation (2.5.6)), because it is independent of the process $Y(\cdot)$. For the risk measure to satisfy the translation invariance property, the BSDE generator should be independent of $Y(\cdot)$ (Quenez and Sulem (2013)). Similar to Subsection 2.6.2, ρ_t^g defined in (3.2.4) with generator in (3.2.6) is a dynamic convex risk measure satisfies the properties of convexity, translation invariance, monotonicity and time-consistency defined in Chapter 2.

3.3 BSDE differentiability

To define the gradient allocation, we need the differentiability for BSDE with jumps. In the Brownian case, Kromer and Overbeck (2014) used classical differentiability results for BSDEs adopted from Ankirchner et al. (2007). In our case, we use Malliavin's differentiability of the quadratic-exponential BSDE with jumps (see Ankirchner et al. (2007), Fujii and Takahashi (2018)).

As in Fujii and Takahashi (2018), we consider the following quadratic-exponential BSDE:

$$\begin{aligned} Y(t) &= \xi - \int_t^T Z(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon(s,\zeta)\tilde{N}(ds,d\zeta) \\ &\quad + \int_t^T g\Big(s, Y(s), Z(s), \int_{\mathbb{R}_0} p(\zeta)G(s,\Upsilon(s,\zeta))\nu(d\zeta)\Big)ds \,. \end{aligned} \tag{3.3.1}$$

for $t \in [0,T]$ where $\xi : \Omega \to \mathbb{R}$, $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $p^i : \mathbb{R} \to \mathbb{R}$, $G^i : [0,T] \times \mathbb{R} \to \mathbb{R}$ for each $i = 1, \ldots, k$. The driver $g\Big(t, Y(t), Z(t), \int_{\mathbb{R}_0} p(\zeta)G(t,\Upsilon(t,\zeta))\nu(d\zeta)\Big)$, satisfies the standard conditions and Assumption (2), where the last arguments denotes a k-dimensional vector whose i-th element is given by $\int_{\mathbb{R}_0} p^i(\zeta)G^i(s,\Upsilon^i(s,\zeta))\nu^i(d\zeta)$. Fujii and Takahashi (2018) assume that for every $i \in \{1,\ldots,k\}$, the functions p^i and $G^i(t,\Upsilon)$ are continuous, with p^i satisfying $\int_{\mathbb{R}_0} |p^i(\zeta)|^2\nu^i d(\zeta) < \infty$. The function $G^i(t,v)$ is continuous in both arguments and one-time continuously differentiable with respect to v.

Assumption 3. (Fujii and Takahashi (2018)) Let $v_t = \int_{\mathbb{R}_0} p(\zeta)G(t,\Upsilon(t,\zeta))\nu(d\zeta)$ and $v'_t = \int_{\mathbb{R}_0} p(\zeta)G(t,\Upsilon'(t,\zeta))\nu(d\zeta)$.

(i) The terminal value is Malliavin differentiable; $\xi \in \mathbb{D}^{1,2}$.

(ii) For each $m > 0$ and for every $(y,z,\Upsilon) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ satisfying $|y|, ||\Upsilon||_{L^\infty(\nu)} \leq m$, the driver $g(t,y,z,\Upsilon_t)$, $t \in [0,T]$ belongs to $\mathbb{D}^{1,2}$ and its Malliavin derivatives are denoted by $D_s g(t,y,z,v_t)$ and $D_{s,\zeta} g(t,y,z,v_t)$. Furthermore, the driver g is continuously differentiable with respect to its state variables.

(iii) For every $m > 0$ and $(y,z,\Upsilon),(y',z',\Upsilon') \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, satisfying $|y|,|y'|$, $||\Upsilon||_{L^2_\nu}, ||\Upsilon'||_{L^2_\nu} \leq m$, the Malliavin derivative of the driver satisfies the following local Lipschitz conditions;

$$\begin{aligned}|D^i_s g(t,y,z,v_t) - D^i_s g(t,y',z',v'_t)| \;\leq\; & K^{m,i}_s(|y-y'| + |v_t - v'_t| + (1+|z| \\ & +|z'| + |v_t| + |v'_t|)|z-z'|)\end{aligned}$$

for ds-a.e. $s \in [0,T]$ with $i \in 1,\ldots,d$, and

$$\begin{aligned}|D^i_{s,\zeta} g(t,y,z,v_t) - D^i_{s,\zeta} g(t,y',z',v'_t)| \;\leq\; & K^{m,i}_{s,\zeta}(|y-y'| + |v_t - v'_t| + (1+|z| \\ & +|z'| + |v_t| + |v'_t|)|z-z'|)\end{aligned}$$

for ds-a.e. $s \in [0,T]$ with $i \in 1,\ldots,k$. For ever $m > 0$ and (s,ζ), $(K^{m,i}_s(t), t \in [0,T])_{i\in 1,\ldots,d}$ and $(K^{m,i}_{s,\zeta}(t), t \in [0,T])_{i\in 1,\ldots,d}$ are $\mathbb{R}_+$-valued $\mathbb{F}_t$-progressively measurable processes.

(iv) There exists some positive constant $r \geq 2$ such that

$$\int_{[0,T]\times\mathbb{R}^k} \Big(\mathbb{E}\big[|D_{s,\zeta}\xi|^{rq} + \big(\int_0^T |D_{s,\zeta} g(t,0)|dt\big)^{rq} + ||K^m||^{2rq}\big]\Big)^{\frac{1}{q}} \tilde{N}(dt,d\zeta) < \infty$$

hold for $\forall q \geq 1$ and $\forall m > 0$.

Fujii and Takahashi (2018) (in Theorem 5.1), proved that under the above assumptions the solution $(Y,Z,\Upsilon) \in \mathbb{S}^2 \times \mathbb{H}^2_{BMO(W)} \times \mathbb{H}^2_{BMO(N)}$ of the BSDE (3.3.1) is Malliavin differentiable with respect to W and $\tilde{N}$. i.e.

(i) There exists a unique solution $(D_sY, D_sZ, D_s\Upsilon) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$ to the BSDE

$$\begin{aligned}
D_sY(t) &= D_s\xi - \int_t^T D_sZ(u)dW(u) - \int_t^T \int_{\mathbb{R}_0} D_s\Upsilon(u,\zeta)\tilde{N}(du,d\zeta) \\
&+ \int_t^T \Big[D_sg(u,\Theta) + \partial_y g(u,\Theta)D_sY(u) + \partial_z g(u,\Theta)D_sZ(u) \\
&+\partial_v g(u,\Theta)\int_{\mathbb{R}_0} p(\zeta)\partial_\Upsilon G(u,\Upsilon(u,\zeta))D_s(\Upsilon(u,\zeta))\nu(d\zeta)\Big] du,
\end{aligned} \tag{3.3.2}$$

for $0 \le s \le t \le T$ where $\Theta := (Y(t), Z(t), \int_{\mathbb{R}_0} p(\zeta)G(u,\Upsilon(u,\zeta))\nu(d\zeta))$. The solution satisfies $\int_0^T ||D_sY, D_sZ, D_{s,\zeta}\Upsilon||^2 ds < \infty$.

(ii) There exists a unique solution $(D_{s,\zeta}Y, D_{s,\zeta}Z, D_{s,\zeta}\Upsilon) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$ to the BSDE

$$\begin{aligned}
D_{s,\zeta}Y(t) &= D_{s,\zeta}\xi - \int_t^T D_{s,\zeta}Z(u)dW(u) \\
&- \int_t^T \int_{\mathbb{R}_0} D_{s,\zeta}\Upsilon(u,\zeta)\tilde{N}(du,d\zeta) \\
&+\frac{1}{\zeta}\Big[\int_t^T g\Big(u, Y(u) + \zeta D_{s,\zeta}Y(u), Z(u) + \zeta D_{s,\zeta}Z(u), \\
&\int_{\mathbb{R}_0} p(\zeta)G\big(u, \Upsilon(u,\zeta) + \zeta D_{s,\zeta}\Upsilon(u,\zeta)\big)\nu(d\zeta)\Big) \\
&-g(u,\Theta)\Big] du,
\end{aligned} \tag{3.3.3}$$

where $0 \le s \le t \le T$, $\zeta \ne 0$ and for $\zeta^2\nu(\zeta)ds - a.e.$ $(s,\zeta) \in [0.T] \times \mathbb{R}_0$. The solution satisfies $\int_0^T \int_{\mathbb{R}_0} ||D_sY, D_sZ, D_{s,\zeta}\Upsilon||^2 \zeta^2\nu(\zeta)ds < \infty$.

3.4 Capital allocation

For this chapter, we consider the terminal condition ξ of the form $\xi(\epsilon) = \xi + \epsilon\eta$, where $\xi, \eta \in L^\infty(\mathcal{F}_T)$. We will also focus on Malliavin derivative with respect to the Brownian motion W. Hence, there exists a constant $c \in \mathbb{R}$ such that

$$\sup_{\epsilon \in U} ||\xi(\epsilon)||_\infty \leq ||\xi||_\infty + ||\eta||_\infty \sup_{\epsilon \in U} |\epsilon| < c,$$

for every compact set $U \subset \mathbb{R}$. In addition, the functional $\epsilon \mapsto \xi(\epsilon)$ is differentiable and the derivative is given by $D_s \xi(\epsilon) = \eta$. The BSDE version of the dynamic gradient allocation is defined as the directional derivative of the risk measure ρ_t at the point ξ in the direction of η_i, that is:

$$\lim_{\epsilon \to 0} \frac{\rho_t(\xi + \epsilon \eta_i) - \rho_t(\xi)}{\epsilon} := D_{\eta_i} \rho_t(\xi) \quad i = 1, 2, \dots, n\,. \tag{3.4.1}$$

and from Definition 3.2.4 we have that

$$D_{\eta_i} \rho_t(\xi) = D_{\eta_i} Y^\xi(t) \quad i = 1, 2, \dots, n\,.$$

The Malliavin derivative given in Di Nunno et al. (2009) (in Definition A.10) is as follows

$$D_{\eta_i} Y^\xi(t) = \langle D_s Y(t), h \rangle = \int_0^T D_s Y h_i(s) ds$$

for all

$$\eta_i = \int_0^t h_i(s) ds,$$

is a Malliavin directional derivative in the direction of η_i with respect to the Brownian motion, with $h_i \in \mathbb{D} \subseteq L^2([0,T])$, $i = 1, \dots, n$. See the Appendix Section for the meaning of $\mathbb{D}$. We observe that the Malliavin directional derivative generalizes the classical Gâteaux-derivative. The inner product $\langle D_s Y, h \rangle_H$[1] is given by

$$\begin{aligned} \langle D_s Y(t), h \rangle &= \langle D_s \xi, h \rangle - \int_t^T \langle D_s Z(u), h \rangle dW(u) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} \langle D_s \Upsilon(u, \zeta), h \rangle \tilde{N}(du, d\zeta) \\ &\quad + \int_t^T \Big[\partial_z g(u, \hat{\Theta}) \langle D_s Z(u), h \rangle \end{aligned}$$

[1]Note that H is the Cameron-Martin space defined in Chapter 2.

$$+\partial_v g(u,\hat{\Theta})\int_{\mathbb{R}_0} p(\zeta)\partial_\Upsilon G(u,\Upsilon(u,\zeta))\langle D_s\Upsilon(u,\zeta),h\rangle\nu(d\zeta)\Big]du, \quad (3.4.2)$$

for $0 \le t \le T$ where $\hat{\Theta} := (Z(t), \int_{\mathbb{R}_0} p(\zeta)G(u,\Upsilon(u,\zeta))\nu(d\zeta))$. Equation (3.4.1) is the directional derivative of the risk measure ρ_t at the point ξ (the portfolio) in the direction of η_i (subportfolio i). It generalizes the concept given in (2.6.7). We also suppose that $\xi = \sum_{i=1}^n \eta_i$, that is the total sum of the subportfolio should equal to the overall portfolio. Now we are in a position to provide the main result on the representation of the dynamic risk capital allocations as a dynamic gradient allocation.

3.5 Representation of dynamic risk capital allocations

In this section, we derive the dynamic risk capital allocation induced from BSDEs with jumps. We also obtain the representation of BSDE based dynamic convex and dynamic coherent risk measures. We follow the approach of Kromer and Overbeck (2014) in deriving the representation of capital allocation, BSDE based dynamic convex and coherent risk measures.

Theorem 3.5.1. *Let $\xi, \eta_i \in L^\infty(\mathcal{F}_T)$, such that $\xi = \sum_{i=1}^n \eta_i$ for each $i = 1,2,\dots,n$ and $D_{\eta_i}Y(t)$ exists. Suppose that $\partial_z g(t,\hat{\Theta})$ and $\partial_v g(t,\hat{\Theta})p(\zeta)\partial_\Upsilon G(t,\Upsilon(t,\zeta))$ belong to BMO(P). Then the dynamic gradient allocations can be represented by:*

$$D_{\eta_i}Y(t) = D_{\eta_i}\rho_t(\xi) = \mathbb{E}^{\mathbb{Q}^\xi}[-\eta_i \mid \mathcal{F}_t]\,, \quad n = 1,2,\dots,n\,,$$

where $\mathbb{Q}^\xi$ is given by

$$\frac{d\mathbb{Q}^\xi}{dP} := \mathcal{E}\left(\int_0^t \partial_z g(u,\hat{\Theta})dW + \int_0^t\int_{\mathbb{R}_0} p(\zeta)\partial_v g(u,\hat{\Theta})\partial_\Upsilon G(u,\Upsilon(u,\zeta))\tilde{N}(du,d\zeta)\right)(t)\,. \quad (3.5.1)$$

Proof. Since belong to BMO(P), then the stochastic integrals in (3.5.1) are said to be BMO(P)-martingales and the stochastic exponential is a true martingale (Morlais (2009b)). From (Di Nunno et al. (2009) Theorem 12.21,) a new equivalent probability measure $\mathbb{Q}^{\xi}$ is defined by equation (3.5.1). Furthermore, the processes

$$dW^{\mathbb{Q}^{\xi}}(t) = dW(t) - \partial_z g(t, \hat{\Theta})dt$$

and

$$\tilde{N}^{\mathbb{Q}^{\xi}}(dt, d\zeta) = \tilde{N}(dt, d\zeta) - p(\zeta)\partial_v g(t, \hat{\Theta})\partial_{\Upsilon} G(t, \Upsilon(t,\zeta))\nu(d\zeta)dt$$

are the $\mathbb{Q}^{\xi}$-Brownian motion and $\mathbb{Q}^{\xi}$-compensated random measure respectively. We define a function $\Phi_i(t)$ by $\Phi_i(t) := \mathbb{E}^{\mathbb{Q}^{\xi}}[-\eta_i \mid \mathcal{F}_t]$ for each $i = 1, \ldots, n$ and $t \in [0, T]$. Then from the martingale representation property there exists predictable processes $Z^{\eta_i}(t)$ and $\Upsilon^{\eta_i}(t, \zeta)$ integrable with respect to $W^{\mathbb{Q}^{\xi}}$ and $\tilde{N}^{\mathbb{Q}^{\xi}}$ respectively such that

$$\begin{aligned}
\Phi_i(t) &= \Phi_i(T) - \int_t^T Z^{\eta_i}(u)dW(u)^{\mathbb{Q}^{\xi}} \\
&\quad - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\eta_i}(u,\zeta)\tilde{N}^{\mathbb{Q}^{\xi}}(du, d\zeta) \quad 0 \le t \le T \\
&= -\eta_i - \int_t^T Z^{\eta_i}(u)dW(u) + \int_t^T Z^{\eta_i}(u)\partial_z g(u, \hat{\Theta})du \\
&\quad - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\eta_i}(u,\zeta)\tilde{N}(du, d\zeta) \\
&\quad + \int_t^T \int_{\mathbb{R}_0} p(\zeta)\partial_v g(u, \hat{\Theta})\partial_{\Upsilon} G(u, \Upsilon(u,\zeta))\Upsilon^{\eta_i}(u,\zeta)\nu(d\zeta)du \quad 0 \le t \le T\,.
\end{aligned}$$

Comparing the above equation with the BSDE representing the gradient allocation (3.4.2) and we know that under the Assumptions 1 to 4 that (3.4.2) has a unique solution, we can conclude that the dynamic gradient allocation has the representation

$$D_{\eta_i}Y(t) = D_{\eta_i}\rho_t(\xi) = \mathbb{E}^{\mathbb{Q}^{\xi}}[-\eta_i \mid \mathcal{F}_t]\,, \quad i = 1, 2, \ldots, n\,. \tag{3.5.2}$$

□

Remark: This result generalizes Theorem 3.1 in Kromer and Overbeck (2014).

From the above theorem, we can immediately obtain the representation result for BSDE based dynamic convex and dynamic coherent risk measures. The results of the representation of BSDE based dynamic convex and coherent risk measures are established from the full allocation property of the Aumann-Shapley allocation (the static case given in Equation (2.6.8)) (Kromer and Overbeck (2014)).

Corollary 3.5.2. *Let $\xi \in L^\infty(\mathcal{F}_T)$. Suppose that ℓ is convex in z and Υ and $\partial_{Z^{\beta\xi}} g(s, \hat{\Theta})$, $\partial_v g(t, \hat{\Theta}) p(\zeta) \partial_{\Upsilon^{\beta\xi}} G(t, \Upsilon^{\beta\xi}(t, \zeta))$ belong to the class of $BMO(P)$, for any $\beta \in [0, 1]$, where $Z^{\beta\xi}(t)$, $\Upsilon^{\beta\xi}(t, \cdot)$ are the controls to the quadratic-exponential BSDE (3.3.1), with terminal condition $\rho_{t,\beta}(\xi) = -\beta\xi$. Then, the corresponding quadratic-exponential BSDE-based dynamic convex risk measure can be represented by*

$$\rho_t(\xi) = \mathbb{E}[-\Lambda^\xi(T, t)\xi \mid \mathcal{F}_t] ,$$

where

$$\Lambda^\xi(T, t) = \int_0^1 \frac{\mathcal{E}(M^{\beta\xi}(T))}{\mathcal{E}(M^{\beta\xi}(t))} d\beta , \quad \forall t \in [0, T] , \tag{3.5.3}$$

for $M^{\beta\xi}$ defined by

$$M^{\beta\xi}(t) = \int_0^t \partial_{Z^{\beta\xi}} g(s, \hat{\Theta}) dW(s) + \int_0^t \int_{\mathbb{R}_0} \partial_v g(s, \hat{\Theta}) p(\zeta) \partial_{\Upsilon^{\beta\xi}} G(s, \Upsilon^{\beta\xi}(s, \zeta)) \tilde{N}(ds, d\zeta) .$$

Proof. Following Kromer and Overbeck (2014), we consider the following

$$\begin{aligned} \rho_t(\xi) &= \rho_t(1\xi) - \rho_t(0\xi) = \int_0^1 \frac{d}{d\beta} \rho_t(\beta\xi) d\beta \\ &= \int_0^1 \lim_{\epsilon \to 0} \frac{\rho_t((\beta + \epsilon)\xi) - \rho_t(\beta\xi)}{\epsilon} d\beta \\ &= \int_0^1 D_\xi \rho_t(\beta\xi) d\beta . \end{aligned}$$

From the previous theorem, we have

$$\rho_t(\xi) = \int_0^1 \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\xi \mid \mathcal{F}_t] d\beta . \tag{3.5.4}$$

Then since $\xi \in L^\infty(\mathcal{F}_T)$, $\mathbb{Q}^{\beta\xi}$ is an equivalent probability measure, $\forall \beta \in [0,1]$. Hence ξ is $\mathbb{Q}^{\beta\xi}$-a.s. bounded. This implies that $\int_0^1 \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\xi \mid \mathcal{F}_t]d\beta < \infty$. Define $\Lambda^\xi(t) = \mathcal{E}(M^{\beta\xi}(t))$. Then, (3.5.4) can be written by

$$
\begin{aligned}
\rho_t(\xi) &= \int_0^1 \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\xi \mid \mathcal{F}_t]d\beta = \int_0^1 \frac{1}{\Lambda^{\beta\xi}(t)}\mathbb{E}^{P^{\beta\xi}}[-\Lambda^{\beta\xi}(T)\xi \mid \mathcal{F}_t]d\beta \\
&= \mathbb{E}\Big[-\Big(\int_0^1 \frac{\Lambda^{\beta\xi}(T)}{\Lambda^{\beta\xi}(t)}d\beta\Big)\xi \mid \mathcal{F}_t\Big] \\
&= \mathbb{E}[-\Lambda^\xi(T,t)\xi \mid \mathcal{F}_t],
\end{aligned}
$$

which completes the proof. □

Corollary 3.5.3. *Let $\xi \in L^\infty(\mathcal{F}_T)$. Suppose that g is of the form $g(t,z,\Upsilon) = \ell(t,z,\Upsilon)$ is convex and positively homogeneous in both z and Υ. Moreover, suppose that $\partial_z \ell(t, Z^{\beta\xi}(t), \Upsilon^{\beta\xi}(t,\cdot))$, $\partial_\Upsilon \ell(t, Z^{\beta\xi}(t), \Upsilon^{\beta\xi}(t,\cdot))$ belong to the class of $BMO(P)$, for any $\beta \in [0,1]$, which represent the portfolio weights. Then, the corresponding BSDE-based dynamic coherent risk measure can be represented by*

$$\rho_t(\xi) = \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\xi \mid \mathcal{F}_t],$$

where the $\mathbb{Q}^{\beta\xi}$-measure is given by

$$
\begin{aligned}
\frac{d\mathbb{Q}^{\beta\xi}}{dP}\bigg|_{\mathcal{F}_t} = \exp\Big\{ &-\int_0^t \partial_z \ell(t, Z^\xi(s), \Upsilon^\xi(s,\zeta))dW \\
&-\frac{1}{2}\int_0^t \partial_z \ell(s, Z^\xi(t), \Upsilon^\xi(s,\zeta))^2 ds \\
&+\int_0^t \int_{\mathbb{R}_0} \Big(\ln\big(1 - \partial_\Upsilon \ell(s, Z^\xi(s), \Upsilon^\xi(s,\zeta))\big) \\
&+\partial_\Upsilon \ell(s, Z^\xi(s), \Upsilon^\xi(s,\zeta))\Big)\nu(d\zeta)ds \\
&+\int_0^t \int_{\mathbb{R}_0} \ln\big(1 - \partial_\Upsilon \ell(s, Z^\xi(t), \Upsilon^\xi(s,\zeta))\big)\tilde{N}(ds,d\zeta)\Big\}.
\end{aligned}
\tag{3.5.5}
$$

Proof. From Corollary 3.5.2, we have the following representation

$$\rho_t(\xi) = \mathbb{E}[-\Lambda^\xi(T,t)\xi \mid \mathcal{F}_t],$$

with Λ defined in (3.5.3). Given that $g(t,z,\Upsilon) = \ell(t,z,\Upsilon)$ and ℓ is convex and positively homogeneuos, this implies that the corresponding BSDE-based dynamic risk measure $\rho(\cdot)$ satisfies

$$Y^{\beta\xi}(t) = \rho_t(\beta\xi) = \beta\rho_t(\xi) = \beta Y^{\xi}(t) \quad dt \otimes dP - a.s.$$

for $c > 0$ and $0 \leq t \leq T$. We show this by considering two BSDEs given by

$$\begin{aligned} Y^{\beta\xi}(t) &= -\beta\xi - \int_t^T Z^{\beta\xi}(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\beta\xi}(s,\zeta)\tilde{N}(ds,d\zeta) \\ &\quad + \int_t^T g\big(s, Z^{\beta\xi}(s), \Upsilon^{\beta\xi}(s,\zeta)\big)ds\,, \end{aligned}$$

and

$$\begin{aligned} Y^{\xi}(t) &= -\xi - \int_t^T Z^{\xi}(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\xi}(s,\zeta)\tilde{N}(ds,d\zeta) \\ &\quad + \int_t^T g\big(s, Z^{\xi}(s), \Upsilon^{\xi}(s,\zeta)\big)ds\,. \end{aligned}$$

Then, from the proof of Proposition 6.2.3(b) in Delong Delong (2013) we conclude that $Y^{\beta\xi}(t) = \beta Y^{\xi}(t)$, $Z^{\beta\xi}(t) = \beta Z^{\xi}(t)$ and $\Upsilon^{\beta\xi}(t,\zeta) = \beta\Upsilon^{\xi}(t,\zeta)$.

The above results imply that for the representation of the BSDE coherent risk measure, the process $\mathcal{E}(M^{\beta\xi}(t))(\cdot)$ appearing in (3.5.3) becomes

$$\mathcal{E}\bigg(\int_0^t \partial_z g(s, Z^{\beta\xi}(s), \Upsilon^{\beta\xi}(s,\zeta))dW(s) + \int_0^t \int_{\mathbb{R}_0} \partial_\Upsilon g(s, Z^{\beta\xi}(s), \Upsilon^{\beta\xi}(s,\zeta))\tilde{N}(ds,d\zeta)\bigg)(t)$$

$$\begin{aligned} &= \exp\bigg\{ - \int_0^t \partial_z g(s, \beta Z^{\xi}(t), \beta\Upsilon^{\xi}(t,\zeta))dW - \frac{1}{2}\int_0^t \partial_z g(s, \beta Z^{\xi}(s), \beta\Upsilon^{\xi}(s,\zeta))^2 ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \Big(\ln\big(1 - \partial_\Upsilon g(s, \beta Z^{\xi}(s), \beta\Upsilon^{\xi}(s,\zeta))\big) + \partial_\Upsilon g(s, \beta Z^{\xi}(s), \beta\Upsilon^{\xi}(s,\zeta))\Big)\nu(d\zeta)ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \ln\big(1 - \partial_\Upsilon g(s, \beta Z^{\xi}(s), \beta\Upsilon^{\xi}(s,\zeta))\big)\tilde{N}(ds,d\zeta)\bigg\}, \\ &= \exp\bigg\{ - \int_0^t \partial_z g(s, Z^{\xi}(s), \Upsilon^{\xi}(s,\zeta))dW - \frac{1}{2}\int_0^t \partial_z g(s, Z^{\xi}(s), \Upsilon^{\xi}(s,\zeta))^2 ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \Big(\ln\big(1 - \partial_\Upsilon g(s, Z^{\xi}(s), \Upsilon^{\xi}(s,\zeta))\big) + \partial_\Upsilon g(s, Z^{\xi}(t), \Upsilon^{\xi}(s,\zeta))\Big)\nu(d\zeta)ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}_0}\ln\big(1-\partial_\Upsilon g(s,Z^\xi(t),\Upsilon^\xi(s,\zeta))\big)\tilde{N}(ds,d\zeta)\Big\} \\
=\;& \mathcal{E}\bigg(\int_0^t \partial_z g(s,Z^\xi(s),\Upsilon^\xi(s,\zeta))dW(s)+\int_0^t\int_{\mathbb{R}_0}\partial_\Upsilon g(s,Z^\xi(s),\Upsilon^\xi(s,\zeta))\tilde{N}(ds,d\zeta)\bigg)(t) \\
=\;& \mathcal{E}(M^\xi(t))\,,
\end{aligned}
$$

because of the positive homogeneity of g in z and Υ. In the case of dynamic coherent risk measure, $\Lambda^\xi(T,t)$ is given by

$$
\Lambda^\xi(T,t)=\int_0^1\frac{\mathcal{E}(M^{\beta\xi}(T))}{\mathcal{E}(M^{\beta\xi}(t))}d\beta=\frac{\mathcal{E}(M^\xi(T))}{\mathcal{E}(M^\xi(t))}. \tag{3.5.6}
$$

Hence, the BSDE-based coherent risk measure is given by

$$
\rho_t(\xi)=\mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\xi\mid\mathcal{F}_t]\,,
$$

where the $\mathbb{Q}$-measure is defined in (3.5.5).

□

We obtain similar results as Kromer and Overbeck (2014), where the exponential martingale of the BSDE based convex risk measure is dependent on all portfolio weights $\beta\in[0,1]$. The representation of coherent risk measure is dependent only on $\beta=1$. The difference between these two risk representations is emphasized in Equation (3.5.6).

3.6 Example

In this section we apply the results presented early to dynamic entropic risk measure and static coherent entropic risk measures to obtain the gradient capital allocation for each risk measure under the jump framework.

Example 3.6.1. We consider the well known dynamic entropic risk measure given by

$$
\rho_t(\xi)=\frac{1}{\gamma}\ln\mathbb{E}\Big[e^{-\gamma\xi}\mid\mathcal{F}_t\Big]\,,\quad \gamma>0,\,t\in[0,T]\,.
$$

This example was also considered in Kromer and Overbeck (2014). It has been proved that the above entropic measure is a unique solution of the so called canonical quadratic-exponential BSDE (g, ξ) of the form (See Karoui et al. (2016))

$$\rho_t(\xi) = -\xi + \int_t^T \Big(\frac{\gamma}{2}|Z^\xi(s)|^2 + \frac{1}{\gamma}\int_{\mathbb{R}_0}\Big(\exp(\gamma\Upsilon^\xi(s,\zeta)) - \gamma\Upsilon^\xi(s,\zeta) - 1\Big)\nu(d\zeta)\Big)ds - \int_t^T Z^\xi(s)dW(s) - \int_t^T\int_{\mathbb{R}_0}\Upsilon^\xi(s,\zeta)\tilde{N}(ds,d\zeta)\,. \quad (3.6.1)$$

Note that the generator is given by

$$g(t,Z,\Upsilon(\zeta)) = \frac{\gamma}{2}|Z|^2 + \frac{1}{\gamma}\int_{\mathbb{R}_0}\Big(\exp(\gamma\Upsilon(t,\zeta)) - \gamma\Upsilon(t,\zeta) - 1\Big)\nu(d\zeta).$$

From the partial derivatives

$$\partial_z g(t,Z,\Upsilon(\zeta)) = \gamma Z$$

and

$$\partial_\Upsilon g(t,Z,\Upsilon(t,\zeta)) = \int_{\mathbb{R}_0}\Big(\exp(\gamma\Upsilon(t,\zeta)) - 1\Big)\nu(d\zeta)$$

.

Suppose that ξ is from a class of smooth functions such that $D_s^i(\xi)$, $D_{s,\zeta}^i(\xi)$, for $0 \le s \le t \le T$, belong to $BMO(P)$ and $\|\xi\|_{1,2}$ exists and is finite for $i = 1,\ldots,d$ and $t \in [0,T]$. We define a function $\varphi(\xi) = e^{-\gamma\xi}$. Then from the boundedness of ξ and of any $\beta \in [0,1]$, we have that $\varphi(\xi)$ is Malliavin differentiable and the generalized Clark-Ocone formula (Di Nunno et al. (2009), Theorem 12.20)

$$e^{-\gamma\beta\xi} = \mathbb{E}[e^{-\gamma\beta\xi}] + \int_0^T \mathbb{E}[D_s(e^{-\gamma\beta\xi}) \mid \mathcal{F}_t]dW(t) + \int_0^T\int_{\mathbb{R}_0}\mathbb{E}[D_{s,\zeta}(e^{-\gamma\beta\xi}) \mid \mathcal{F}_t]\tilde{N}(dt,d\zeta)\,.$$

Define $\Gamma^{\beta\xi}(t) := \mathbb{E}[e^{-\gamma\beta\xi} \mid \mathcal{F}_t]$ is a positive bounded martingale for $\xi \in \mathcal{F}_T$. Then

$$\Gamma^{\beta\xi}(t) = \Gamma^{\beta\xi}(0) + \int_0^t \mathbb{E}[-\gamma\beta e^{-\gamma\beta\xi}D_s(\xi) \mid \mathcal{F}_s]dW(s) + \int_0^t\int_{\mathbb{R}_0}\mathbb{E}[-\gamma\beta e^{-\gamma\beta\xi}D_{s,\zeta}(\xi) \mid \mathcal{F}_s]\tilde{N}(ds,d\zeta)$$

$$= \quad \Gamma^{\beta\xi}(0) + \gamma \int_0^t \Gamma^{\beta\xi}(s) Z^{\beta\xi}(s) dW(s) + \gamma \int_0^t \int_{\mathbb{R}_0} \Gamma^{\beta\xi}(s) \Upsilon^{\beta\xi}(s,\zeta) \tilde{N}(ds, d\zeta)\,,$$

where

$$Z^{\beta\xi}(t) = \frac{-\beta \mathbb{E}[e^{-\gamma\beta\xi} D_s(\xi) \mid \mathcal{F}_t]}{\mathbb{E}[e^{-\gamma\beta\xi} \mid \mathcal{F}_t]} \quad \text{and} \quad \Upsilon^{\beta\xi}(\mathrm{s},\zeta) = \frac{-\beta \mathbb{E}[\mathrm{e}^{-\gamma\beta\xi} \mathrm{D}_{\mathrm{s},\zeta}(\xi) \mid \mathcal{F}_{\mathrm{t}}]}{\mathbb{E}[\mathrm{e}^{-\gamma\beta\xi} \mid \mathcal{F}_{\mathrm{t}}]}, \tag{3.6.2}$$

are the predictable control processes for the entropic risk measure defined by the BSDE (3.6.1). Furthermore, $Z^{\beta\xi}(\cdot)$ and $\Upsilon^{\beta\xi}(\cdot,\zeta)$ belong to the class of $BMO(P)$, hence $\Gamma^{\beta\xi}(t)$ satisfies the following

$$\begin{aligned} \Gamma^{\beta\xi}(t) \quad = \quad & \Gamma^{\beta\xi}(0) \exp\Big\{ \gamma \int_0^t Z^{\beta\xi}(s) dW(s) - \frac{\gamma^2}{2} \int_0^t |Z^{\beta\xi}(s)|^2 ds \\ & + \int_0^t \int_{\mathbb{R}_0} [\ln(1 + \gamma \Upsilon^{\beta\xi}(s,\zeta)) - \gamma \Upsilon^{\beta\xi}(s,\zeta)] \nu(d\zeta) ds \\ & + \int_0^t \int_{\mathbb{R}_0} \ln(1 + \gamma \Upsilon^{\beta\xi}(s,\zeta)) \tilde{N}(ds, d\zeta) \Big\}. \end{aligned}$$

As a result, the process $\mathcal{N}(t) := \Gamma^{\beta\xi}(t)/\Gamma^{\beta\xi}(0)$ corresponds to the stochastic exponential $\mathcal{E}$ to the process $M^{\beta\xi}(t)$ in (3.5.1) defined by

$$M^{\beta\xi}(t) = \int_0^t \partial_z g(s, Z^{\beta\xi}(s), \Upsilon^{\beta\xi}(s,\zeta)) dW(s) + \int_0^t \int_{\mathbb{R}_0} \partial_\Upsilon g(s, Z^{\beta\xi}(s), \Upsilon^{\beta\xi}(s,\zeta)) \tilde{N}(ds, dz)\,,$$

for $t \in [0,T]$.

Now we define the equivalent probability measure under $\mathbb{Q}^{\beta\xi}$ as

$$\frac{d\mathbb{Q}^{\beta\xi}}{dP}\Big|_{\mathcal{F}_t} = \mathcal{N}(t).$$

Under the new probability measure $\mathbb{Q}^{\beta\xi}$, the processes

$$dW^{\mathbb{Q}^{\beta\xi}}(t) = dW(t) - \gamma Z^{\beta\xi}(t) dW(t)$$

and

$$\tilde{N}^{\mathbb{Q}^{\beta\xi}}(dt, d\zeta) = \tilde{N}(dt, d\zeta) - \gamma \int_{\mathbb{R}_0} \Upsilon^{\beta\xi}(t,\zeta) \nu(d\zeta) dt$$

are the $\mathbb{Q}$- Brownian motion and $\mathbb{Q}$-compensated random measure respectively. Define a function $\Phi_i(t)$ by $\Phi_i(t) := \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\eta_i \mid \mathcal{F}_t]$ for each $i = 1, \ldots, n$.

Let $Z^{\eta_i}(t)$ and $\Upsilon^{\eta_i}(t,\zeta)$ be predictable processes, then from the martingale representation theorem we have,

$$\begin{aligned}\Phi_i(t) &= -\eta_i - \int_t^T Z^{\eta_i}(s)dW^{\mathbb{Q}^{\beta\xi}}(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\eta_i}(s,\zeta)\tilde{N}^{\mathbb{Q}^{\beta\xi}}(ds,d\zeta) \\ &= -\eta_i - \int_t^T Z^{\eta_i}(s)dW(s) + \int_t^T Z^{\eta_i}(s)\gamma Z^{\beta\xi}(s)ds - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\eta_i}(s,\zeta)\tilde{N}(ds,d\zeta) \\ &\quad + \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\eta_i}(s,\zeta)\gamma\Upsilon^{\beta\xi}(s,\zeta)\nu(d\zeta)ds\,.\end{aligned}$$

Moreover,

$$\begin{aligned}\Phi_i(t) = \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\eta_i \mid \mathcal{F}_t] &= \frac{1}{\mathcal{N}(t)}\mathbb{E}^P[-\eta_i\mathcal{N}(T) \mid \mathcal{F}_t] \\ &= \frac{\Gamma(0)}{\Gamma(t)}\mathbb{E}^P\left[-\eta_i\frac{\Gamma(T)}{\Gamma(0)}\middle|\mathcal{F}_t\right] \\ &= \frac{1}{\Gamma(t)}\mathbb{E}^P[-\eta_i\Gamma(T)|\mathcal{F}_t] \\ &= \frac{\mathbb{E}^P[-\eta_i e^{-\gamma\beta\xi}|\mathcal{F}_t]}{\mathbb{E}^P[e^{-\gamma\beta\xi}|\mathcal{F}_t]} \\ &= -\mathbb{E}^P\left[\eta_i\,\frac{e^{-\gamma\beta\xi}}{\mathbb{E}^P[e^{-\gamma\beta\xi}]}\middle|\mathcal{F}_t\right]. \qquad (3.6.3)\end{aligned}$$

Therefore, the gradient capital allocation of the entropic risk measure under the jump framework is given by (3.6.3). If ξ is the portfolio and η_i is the subportfolio. Then Equation (3.6.3) describes the dynamic capital risk contribution of the subportfolio η_i to the risk of portfolio ξ at time t, i.e.,

$$D_{\eta_i}\rho_t(\beta\xi) = -\mathbb{E}^P\left[\eta_i\,\frac{e^{-\gamma\beta\xi}}{\mathbb{E}^P[e^{-\gamma\beta\xi}]}\middle|\mathcal{F}_t\right].$$

In addition, based on the full allocation property of the Aumann-Shapely allocation introduced in Section 2.6.3 holds and applying the results of Corollary 3.5.2, we can represent the BSDE-based dynamic entropic risk measure by

$$\rho_t(\xi) = \int_0^1 D_{\eta_i}\rho_t(\beta\xi)d\beta = \mathbb{E}^P\left[-\int_0^1\left(\frac{e^{-\gamma\beta\xi}}{\mathbb{E}^P[e^{-\gamma\beta\xi}]}\right)d\beta\;\eta_i\middle|\mathcal{F}_t\right] \qquad i = 1, 2, \ldots, n.$$

Example 3.6.2. In the second example we consider the static entropic coherent risk measure at level c defined by (Föllmer and Knispel (2011) in Definition 3.1) as follows

$$\rho(\xi) = \inf_{\gamma>0}\Big(\frac{c}{\gamma} + \frac{1}{\gamma}\ln \mathbb{E}\Big[e^{-\gamma\xi}\Big]\Big) \tag{3.6.4}$$

for $c > 0$ and the risk aversion constant $\gamma > 0$. From Proposition 3.1 by Föllmer and Knispel (2011) there exists a unique $\gamma_c > 0$ such that $c = \mathbb{E}^{\mathbb{Q}}[\int \frac{d\mathbb{Q}}{dP}\ln(\frac{d\mathbb{Q}}{dP})]$ and the infimum of Equation (3.6.4) is attained, i.e.

$$\rho(\xi) = \frac{c}{\gamma_c} + \frac{1}{\gamma_c}\ln \mathbb{E}\Big[e^{-\gamma_c\xi}\Big].$$

The Gâteaux-differentiable of ρ is given by

$$\nabla\rho(\beta\xi) = -\frac{e^{-\gamma_c\beta\xi}}{\mathbb{E}[e^{-\gamma_c\beta\xi}]}\,. \tag{3.6.5}$$

Since the entropic coherent risk measure satisfies the property of positive homogeneity (Föllmer and Knispel (2011)), then the full allocation property of Aumann-Shapley holds. Hence, $\nabla\rho(\beta\xi)$ will reduce to $\nabla\rho(\xi)$ and $\Lambda^{\beta\xi}$ for this case will be given by

$$\Lambda^{\beta\xi} = \int_0^1 \nabla\rho(\xi)d\beta = -\int_0^1 \frac{e^{-\gamma_c\xi}}{\mathbb{E}[e^{-\gamma_c\xi}]}d\beta,$$

and applying the results of Corollary 3.5.2, we have that ρ form Equation (3.6.4) can be represented by

$$\rho(\xi) = \mathbb{E}\Big[-\Big(\int_0^1 \frac{e^{-\gamma_c\xi}}{\mathbb{E}[e^{-\gamma_c\xi}]}d\beta\Big)\xi\Big]\,.$$

3.7 Conclusion

We studied the capital allocation of risk measures constructed from solutions of BSDE with jumps. From the differentiability results of BSDE with jumps and the martingale representation property, we were able to provide the capital allocation representation of the risk measures. We applied the representation obtained in Theorem 3.5.1 to entropic risk measure to achieve the allocation in terms of conditional expectation under the equivalent $\mathbb{Q}$ measure. The current results are based on a fixed time horizon, future work can study capital allocation representation of maturity independent risk measures.

Chapter 4

Ergodic BSDE risk representation and capital allocation

4.1 Introduction

In this chapter, we study the ergodic BSDE-based risk representation and dynamic capital allocation, where we consider the maturity-independent risk measure. We use the differentiability results of BSDEs to determine capital allocation. Moreover, we give an example in the form of a forward entropic risk measure. Chong et al. (2019) applied the ergodic BSDEs to risk measure. In their work, Chong et al. (2019) provided a general representation of maturity independent risk measures using ergodic BSDE.

In this Chapter, we extend the work of Chong et al. (2019) to obtain capital allocation for maturity-independent risk measure. The dynamic capital allocation problems studied in literature are on a fixed time horizon. See, for example, Kromer and Overbeck (2014). They provided a representation of dynamic capital allocation using dynamic risk measures that arise as a solution to BSDEs. Later, Kromer and Overbeck (2017), considered dynamic risk measures that occur as a solution to backward stochastic Volterra integral equations, which allows for position processes and not only $\mathcal{F}_T$-measurable random variables. In both these papers, Kromer and Overbeck extended the capital allocation problem from static to the continuous-time dynamic case. (See, Cherny (2009) for dynamic allocation in discrete-time and Mabitsela et al. (2018) for extension of dynamic allocation to jumps).

We organise this chapter as follows: in Section 4.2, we provide the tools that we will use throughout. We state the market conditions and underlying assumptions we working under, and provide definitions of a forward performance process, maturity-independent risk measure and capital allocation method. In Section 4.3 and 4.4 we give a brief review of ergodic BSDEs and forward entropic risk measures. We present the representation of capital allocation based on maturity-independent risk measures. Finally, apply the maturity-independent capital allocation results to the forward-entropic risk measure and draw up with concluding note.

4.2 Preliminaries

In this section, we state the definitions and notations we use throughout the chapter. We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions (completeness and right-continuity). The filtration $\mathcal{F}_t$, is generated by a d-dimensional Brownian motion denoted by $W = (W(t))_{t\geq 0}$, that is $\mathcal{F}_t := \sigma(W(s) : 0 \leq s \leq t)$. We assume the financial market consists of a risk-free bond earning a zero interest rate and n risky assets. The price process of the n risky assets solves for each $i = 1, \ldots, n$, with $n \leq d$,

$$dS^i(t) = S^i(t)\mu(V(t))dt + S^i(t)\sigma(V(t))dW(t), \quad S^i(0) > 0, \qquad (4.2.1)$$

where $\mu(V(t))$ and $\sigma(V(t))$ are continuous functions on the interval $t \geq 0$, representing the price appreciation rate (or the drift) and the volatility respectively. Both the appreciation rate and the volatility are affected by a stochastic factor, which is modelled by the d-dimensional process V satisfying the following

$$dV(t) = \nu(V(t))dt + \kappa dW(t). \tag{4.2.2}$$

We make the following model assumptions on the coefficients of the assets and factor processes.

Assumption 4. (i) The drift and volatility coefficients, $\mu^i(v) \in \mathbb{R}$ and $\sigma^i(v) \in \mathbb{R}^{1\times d}$ respectively, are uniformly bounded for $v \in \mathbb{R}^d$.

(ii) Let $\sigma(v) := (\sigma^1(v), \dots, \sigma^n(v))^{tr}$, be the volatility matrix with a full row rank n.

(iii) Define the market price of risk as

$$\theta(v) := \sigma(v)^{tr}[\sigma(v)\sigma(v)^{tr}]^{-1}\mu(v), \tag{4.2.3}$$

for $v \in \mathbb{R}^d$, is uniformly bounded and Lipschitz continuous (Chong et al. (2019)).

Assumption 5. (i) The coefficients, $\nu(v) \in \mathbb{R}^d$, of the drift term satisfy a dissipative condition, that is, there exists a large enough constant $C_\nu > 0$, for $v_1, v_2 \in \mathbb{R}^d$ such that the drift coefficient $\nu(v) \in \mathbb{R}^d$ of the factor model satisfies:

$$(\nu(v_1) - \nu(v_2))(v_1 - v_2) \leq -C_\nu |v_1 - v_2|^2.$$

(ii) The volatility matrix $\kappa \in \mathbb{R}^{d\times d}$ is positive definite and normalized to $|\kappa| = 1$.

According to Chong et al. (2019), Assumption 5 allows for the stochastic factor process V to have a unique invariant measure. Hence, making the stochastic factor process to be ergodic and hence, any two paths will converge to each other exponentially fast.

Next, we consider $\tilde{\pi} = (\tilde{\pi}^1, \tilde{\pi}^2, \dots, \tilde{\pi}^n)^{tr}$ to represent the amount of wealth invested in the stocks, where $\tilde{\pi}^i$ is self-financing and represent the amount of wealth invested into asset i. Consider an investor with initial wealth $X_0 = x \in \mathbb{R}$ and current wealth process $X(t)$, satisfying

$$dX(t) = \sum_{i=1}^{n} \tilde{\pi}_t^i \frac{dS^i(t)}{S^i(t)} = \tilde{\pi}_t \sigma(V(t))\theta(V(t))dt + \tilde{\pi}_t \sigma(V(t))dW(t). \quad (4.2.4)$$

Similar to Liang and Zariphopoulou (2017) and Chong et al. (2019), we rescale the investment strategy by the volatility throughout this chapter,

$$\pi_t = \tilde{\pi}_t \sigma(V(t)). \quad (4.2.5)$$

Consequently, the wealth process satisfies

$$dX(t) = \tilde{\pi}_t \big[\theta(V(t))dt + dW(t)\big]. \quad (4.2.6)$$

We define the set of admissible investment strategies for any $t \geq 0$ by

$$\mathcal{A}_{[0,t]} := \{\pi_t \in L^2_{BMO}[0,t] : \pi_s \in \Pi \text{ for } s \in [0,t]\},$$

where Π is a closed and convex set in $\mathbb{R}^d$. Denote by $\mathcal{A} = \cup_{t \geq 0} \mathcal{A}_{[0,t]}$ the set of admissible investment strategies for all $t \geq 0$. In the next subsection we recall the definition and properties of maturity-independent risk measure introduced by Zariphopoulou and Žitković (2010).

4.2.1 Maturity-independent risk measures

At time $t \in [0,T]$, the dynamic risk measure evaluates the risk of a financial position that expires at time T (Delong (2013)). Hence, the investment time horizon is fixed for $t \in [0,T]$. Furthermore, at time $t = 0$, the financial positions considered are both introduced and mature within the pre-defined investment horizon. However, Chong et al. (2019) point out that we frequently have to assess the risk of different financial position without complete knowledge of when they will be introduced, when will they mature, and their sizes. Zariphopoulou and Žitković (2010), introduce the dynamic risk measures which are maturity-independent to accommodate arbitrary upcoming risk positions.

We start with defining the natural domain of the maturity-independent risk measures, that is the space that contains all risk positions that are $\mathcal{F}_t$ measurable and bounded for all times $t > 0$. We recall the definition of maturity-independent convex risk measures from Chong et al. (2019). These properties of maturity-independent risk measures are proven by Zariphopoulou and Žitković (2010) (in Theorem 4.14).

Definition 4.2.1. A functional $\rho : \mathcal{L} \to \mathbb{R}$ is a maturity-independent convex risk measure if it satisfies the following properties, for all $\xi, \bar{\xi} \in \mathcal{L}$ and $\varpi \in [0, 1]$:

(i) Anti-positivity: $\rho(\xi) \leq 0$, $\forall \xi \geq 0$.

(ii) Convexity: $\rho(\varpi\xi + (1-\varpi)\bar{\xi}) \leq \varpi\rho(\xi) + (1-\varpi)\rho(\bar{\xi})$.

(iii) Translation invariance: $\rho(\xi - m) = \rho(\xi) + m$, $\forall m \in \mathbb{R}$.

(iv) Replication and maturity independence: For all $t \geq 0$ and admissible investment strategies $\pi \in \mathcal{A}$,

$$\rho(\xi) = \rho\left(\xi + \int_0^t \pi_s \frac{dS(s)}{S(s)}\right).$$

The convexity property takes into account the nonlinearity of a risk measure associated with the liquidity of a large financial position. Suppose the standard normalization $\rho(0) = 0$, then together with the translation invariance property mean that $\rho(\xi)$ is the minimum capital required to the position in ξ acceptable. As asserted in Zariphopoulou and Žitković 2010, the difference between maturity independent and standard risk measure is the choice of the domain $\mathcal{L}$ and the case that Property 4.2.1(iv) is valid for all maturities $t \geq 0$.

Maturity-independent risk measures are constructed using forward performance processes, developed by Musiela and Zariphopoulou (2006), Musiela and Zariphopoulou (2007), Musiela, Zariphopoulou, et al. (2008). Considering that, the forward performance processes are defined for all $t \in [0, \infty]$. Thus, maturity-independent risk measures are computed for all times.

We recall the definition of the forward performance process found in Zariphopoulou and Žitković (2010) (see also Chong et al. (2019), Musiela and Zariphopoulou (2010b), Musiela and Zariphopoulou (2006), Musiela and Zariphopoulou (2007) and Musiela, Zariphopoulou, et al. (2008)).

Definition 4.2.2. A process $U(t, x)$, $(t, x) \in [0, \infty) \times \mathbb{R}$, is a forward performance process if:

(i) for each $x \in \mathbb{R}$, $U(t, x)$ is $\mathcal{F}_t-$progressively measurable,

(ii) for each $t \geq 0$, the mapping $x \mapsto U(t, x)$ is strictly increasing, strictly concave, continuously differentiable and satisfies the *Inada* conditions, i.e., $\lim_{x\to\infty} U'(x) = 0$ and $\lim_{x\to-\infty} U'(x) = +\infty$.

(iii) for all $\pi \in \mathcal{A}$ and $0 \leq t \leq s$,

$$U(t, X_t^\pi) \geq \mathbb{E}_P[U(s, X_s^\pi)|\mathcal{F}_t],$$

and there exists an optimal $\tilde{\pi} \in \mathcal{A}$ such that,

$$U(t, X_t^{\tilde{\pi}}) = \mathbb{E}_P[U(s, X_s^{\tilde{\pi}})|\mathcal{F}_t],$$

with X^π, $X^{\tilde{\pi}}$ solving Equation (4.2.4).

Unlike the traditional utility function, the forward performance processes are not bound down to a specific time horizon. One can define them at the initial time and generate them throughout the time horizon $t \in [0, \infty)$. These forward performance processes permit one to be able to measure investment positions over $t \in [0, \infty)$.

4.2.2 Capital allocation

In this subsection we recall the capital allocation method and property. Consider a portfolio ξ with the risk of $\rho(\xi)$ and sub-portfolios $\xi_i \in \mathcal{L}$, $i = 1, 2, \ldots, n$ such that $\xi = \sum_{i=1}^{n} \xi_i$. Then the capital allocation problem is to determine the risk contribution of each sub-portfolio ξ_i to the overall portfolio risk. That is we require the following

$$\rho(\xi) = \sum_{i=1}^{n} \rho(\xi_i). \tag{4.2.7}$$

Equation (4.2.7) is known as the capital allocation property, where the portfolio risk is fully distributed to each sub-portfolio (described in Chapter 3 of this thesis). The frequently used method for capital allocation is the gradient allocation. This method is given by the Gâteaux derivative of ρ at ξ in the direction of ξ_i (see also Kromer and Overbeck (2014) and Tsanakas (2009)), that is

$$\nabla_{\xi_i}\rho(\xi;\xi_i) = \lim_{u\to 0}\frac{\rho(\xi+u\xi_i)-\rho(\xi)}{u} = \frac{d}{du}\rho(\xi+u\xi_i)\bigg|_{u=0}. \tag{4.2.8}$$

4.3 The Ergodic BSDE

This section, provide a brief review of ergodic BSDE and the assumptions imposed on its generator. Consider the following ergodic BSDE also considered in Chong et al. (2019)

$$dY_t = (-g(V_t, Z_t) + \lambda)dt + Z_t dW(t), \quad 0 \le t \le T < \infty, \tag{4.3.1}$$

where $\lambda \in \mathbb{R}$ is part of the solution and the generator function $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is defined as

$$g(v, z) := \frac{1}{2}\gamma^2 dist^2\left\{\Pi, \frac{z+\theta(v)}{\gamma}\right\} - \frac{1}{2}|z+\theta(v)|^2 + \frac{1}{2}|z|^2. \tag{4.3.2}$$

The notation $dist\{\Pi, z\}$ represents the distance function from $z \in \mathbb{R}^d$ to Π. The generator g satisfies the following assumption.

Assumption 6. There exist constants $\hat{K} > 0$ $C_v > 0$ and $C_z > 0$ such that

$$|g(v, z)| \leq K\big(1 + |v| + |z|^2\big), \tag{4.3.3}$$

$$|g(t, 0, 0)| \leq \hat{K}\,, \tag{4.3.4}$$

$$|g(v_1, z) - g(v_2, z)| \leq C_v(1 + |z|)|v_1 - v_2|, \tag{4.3.5}$$

and

$$|g(v, z_1) - g(v, z_2)| \leq C_z(1 + |z_1| + |z_2|)|z_1 - z_2| \tag{4.3.6}$$

for any v_1, v_2, z_1, $z_2 \in \mathbb{R}^d$.

Let the Assumptions 4 and 5 hold. Liang and Zariphopoulou (2017) showed in Proposition 10 that Equation (4.3.1) has a unique Markovian solution (Y_t, Z_t, λ), $t \geq 0$. More specifically, there exists a unique $\lambda \in \mathbb{R}$, and functions $y : \mathbb{R}^d \mapsto \mathbb{R}$ and $z : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that

$$(Y_t, Z_t) = (y(V_t), z(V_t)). \tag{4.3.7}$$

The function $y(\cdot)$ has at most linear growth and is unique up to a constant, while the function $z(\cdot)$ is bounded with $|z(\cdot)| \leq \frac{C_v}{C_\nu - C_v}$, where C_ν and C_v appear in Assumption 5 and Assumption 6 respectively.

4.4 Forward entropic risk measures

We recall the definition of forward entropic risk measures introduced by Zariphopoulou and Žitković (2010), Chong et al. (2019).

Definition 4.4.1. Let (Y_t, Z_t, λ) be a solution of ergodic BSDE (4.3.1) and consider the forward exponential performance process given by

$$U(x, t) = -e^{-\gamma x + Y_t - \lambda t}, \tag{4.4.1}$$

with $(t, x) \in [0, \infty) \times \mathbb{R}$ and $\gamma > 0$ is a given constant. In addition, we consider a risk position $\xi_T \in L^\infty(\mathcal{F}_T)$, where $T > 0$ is arbitrary and the risk position is entered into at the initial time $t = 0$. Then, the forward entropic risk measure $\rho_t(\xi_T, T), t \in [0, T]$, is the unique $\mathcal{F}_t$-measurable random variable that satisfies the indifference condition:

$$\begin{aligned} & ess \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}_P \left[U(x + \rho_u(\xi_T; T) + \int_t^T \tilde{\pi}_t \big(\theta(V_t)dt + dW(t)\big) + \xi_T, T) \middle| \mathcal{F}_t \right] \\ = \; & ess \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}_P \left[U(x + \int_t^T \tilde{\pi}_t \big(\theta(V_t)dt + dW(t)\big), T) \middle| \mathcal{F}_t \right], \end{aligned} \tag{4.4.2}$$

for all $(t, x) \in [0, T] \times \mathbb{R}$.

If we take $\xi \in \mathcal{L}$ and let $T_\xi := \inf\{T \geq 0 : \xi \in \mathcal{F}_T\}$, then the forward entropic risk measure of ξ is defined, for $t \in [0, T_\xi]$, as

$$\rho_t(\xi) := \rho_t(\xi; T_\xi).$$

Consequently, for $\xi_T \in L^\infty(\mathcal{F}_T)$, we have $\rho_t(\xi) := \rho_t(\xi_T; T)$.

We emphasise that the forward performance process, U is constructed using the ergodic BSDE (4.3.1). In addition, the forward performance process is defined for all $t \in [0, \infty)$. As a result, the entropic risk measure defined above can be used to measure risk for investment position with arbitrary maturities (Chong et al. (2019)).

Chong et al. (2019) (in Theorem 6), studied the representation of forward entropic risk measures based on BSDE. They showed that if we suppose that Assumptions 4 and 5 hold and the process Z in the ergodic BSDE (4.3.1) is uniformly bounded. Then the forward entropic risk measure of a risk position $\xi_T \in L^\infty(\mathcal{F}_T)$, with an arbitrary maturity $T > 0$, is defined as the first component to the solution of the following BSDE:

$$Y_t^{T,\xi} = -\xi_T + \int_t^T G(V_s, Z_s, Z_s^{T,\xi})ds - \int_t^T Z_s^{T,\xi} dW(s). \tag{4.4.3}$$

The generator $G : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ is defined as $G(v, z, \tilde{z}) := \frac{1}{\gamma}\big(g(v, z + \gamma\tilde{z}) - g(v, z)\big)$, where $g(\cdot, \cdot)$ given by Equation (4.3.2). The BSDE (4.4.3) has a unique solution $(Y_t^{T,\xi}, Z_t^{T,\xi})$, $t \in [0,T]$, where $Y_t^{T,\xi}$ is uniformly bounded and $Z_t^{T,\xi} \in L^2_{BMO}[0,T]$. Hence, the forward entropic risk measure of a position in ξ_T is given by

$$\rho_t(\xi_T) = Y_t^{T,\xi}, \tag{4.4.4}$$

for $t \in [0,T]$.

4.5 Capital Allocation of maturity-independent risk measure

In this Section, we provide our main results. We derive the capital allocation of a maturity-independent risk measure by first considering an exponential forward performance process of the form

$$U(x,t) = -e^{-\gamma x + y(V_t) - \lambda t}, \tag{4.5.1}$$

where $(Y_t, Z_t, \lambda) = (y(V_t), z(V_t), \lambda)$ is the solution to the ergodic BSDE (4.3.1), with the generator g defined in (4.3.2) and the associated optimal portfolio $\pi^* \in \mathcal{A}$ is given by

$$\pi_t^* = \min_{\pi \in \mathcal{A}} \frac{1}{2}\gamma^2 \left[\pi - \frac{\theta(V_t) + Z_t}{\gamma}\right]^2. \tag{4.5.2}$$

Consider the following BSDE

$$Y_t^{T,\xi} = -(\xi_T + u\eta_i) + \int_t^T G(V_s, Z_s, Z_s^{T,\xi})ds - \int_t^T Z_s^{T,\xi} dW(s), \tag{4.5.3}$$

with a unique solution $(Y_t^{T,\xi}, Z_t^{T,\xi})$, where $t \in [0,T]$, $Y_t^{T,\xi}$ is uniformly bounded and $Z_t^{T,\xi} \in L^2_{BMO}[0,t]$. From the definition of the gradient allocation we need the differentiability results to the BSDE (4.5.3).

We use the classical differentiability results of BSDEs from Ankirchner et al. (2007). In their article, Ankirchner et al. (2007), they showed that under the following conditions on the terminal value and the generator.

(C1) Suppose $G : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is an adapted measurable function so that $G(u,v,z,\tilde{z}) = l(\omega,t,u,v,z,\bar{z}) + \gamma^2|z|^2$ where $l(\omega,t,u,v,z,\bar{z}) + \gamma^2|z|^2$ is Lipschitz continuous in $(u,v,z,\tilde{z})$ and continuously differentiable in $(u,v,z,\tilde{z})$. For all $r \geq 1$ and $(u,v,z,\tilde{z})$, the mapping $\mathbb{R}^n \mapsto L^r$, $u \mapsto l(u,v,z,\tilde{z})$ is differentiable for all $u \in \mathbb{R}^n$,

$$\lim_{u' \to u} \mathbb{E}_P \left[\left(\int_0^T (l(u', V_s^{u'}, Z_s^{u'}, \tilde{Z}_s^{u'}) - l(u, V_s^u, Z_s^u, \tilde{Z}_s^u)) ds \right)^r \right] = 0$$

and

$$\lim_{u' \to u} \mathbb{E}_P \left[\left(\int_0^T \Big(\frac{\partial}{\partial u} l(u', V_s^{u'}, Z_s^{u'}, \tilde{Z}_s^{u'}) - \frac{\partial}{\partial u} l(u, V_s^u, Z_s^u, \tilde{Z}_s^u) \Big) ds \right)^r \right] = 0.$$

(C2) the random variables $\xi(u)$ are $\mathcal{F}_T-$adapted and for every compact set $K \subset \mathbb{R}^n$ there exists a constant $c \in \mathbb{R}$ such that $\sup_{u \in K} ||\xi(u)||_\infty < c$. For all $p \geq 1$ the mapping $\mathbb{R}^n \mapsto L^p$, $u \mapsto \xi u$ is differentiable with derivative given by $\nabla \xi$.

With these assumptions and the differentiability of the solution of the ergodic BSDE (Fuhrman et al. (2009)), we have that

$$\begin{aligned} \nabla Y_t^{T,\xi} &= -\eta_i + \int_t^T \Big[\partial_u G(V_s, Z_s, Z_s^{T,\xi}) + \partial_v G(V_s, Z_s, Z_s^{T,\xi}) \nabla V_s \\ &\quad + \partial_z G(V_s, Z_s, Z_s^{T,\xi}) \nabla Z_s + \partial_{\tilde{z}} G(V_s, Z_s, Z_s^{T,\xi}) \nabla Z_s^{T,\xi} \Big] ds \\ &\quad - \int_t^T \nabla Z_s^{T,\xi} dW(s) \,. \end{aligned} \tag{4.5.4}$$

Let $\xi = \sum_{i=1}^n \eta_i$. Then from the above results, the definition of the forward gradient allocation will be

$$\frac{d}{du} \rho_t(\xi + u\eta_i)\big|_{u=0} = \nabla_{\eta_i} Y_t^{T,\xi} \quad i = 1, \ldots, n.$$

Inspired by the steps in Kromer and Overbeck (2014), we derive the next theorem, which is an extension of their results to dynamic maturity independent case.

Theorem 4.5.1. *Suppose Assumption 6 holds and let $\xi_T \in L^\infty(\mathcal{F}_T)$ be the risk position with an arbitrary maturity $T > 0$ and $\eta_i \in L^\infty(\mathcal{F}_T)$ for each $i = 1, 2, \ldots, n$. We assume the processes $(\partial_u G(V_s, Z_s, Z_s^{T,\xi}))_{t\geq 0}$, $(\partial_v G(V_s, Z_s, Z_s^{T,\xi})\nabla V_s)_{t\geq 0}$, $(\partial_z G(V_s, Z_s, Z_s^{T,\xi})\nabla Z_s)_{t\geq 0}$ and*
$(\partial_{\tilde{z}} G(V_s, Z_s, Z_s^{T,\xi})\nabla Z_s^{T,\xi})_{t\geq 0}$ are from BMO(P) and consider $\xi = \sum_{i=1}^n \eta_i$ and the process Z is part of the solution to the ergodic BSDE (4.3.1). Then, the gradient capital allocation exists as a unique solution to (4.5.4) and can be represented by

$$\nabla_{\eta_i} Y_t^{T,\xi} = \nabla_{\eta_i} \rho_t^{T,\xi} = \mathbb{E}_{\mathbb{Q}}[-\eta_i|\mathcal{F}_t] \quad i = 1, \ldots, n, \tag{4.5.5}$$

with the equivalent martingale measure $\mathbb{Q}$ is given by

$$\frac{d\mathbb{Q}}{dP}\bigg|_{\mathcal{F}_t} := \mathcal{E}\left(\int_0^\cdot \left[\frac{\mathcal{D}(V_s, Z_s, Z_s^{T,\xi})}{\nabla Z_s^{T,\xi}} + \partial_{\tilde{z}} G(V_s, Z_s, Z_s^{T,\xi})\right] dW(s)\right)(t), \tag{4.5.6}$$

where

$$\begin{aligned}\mathcal{D}(V_s, Z_s, Z_s^{T,\xi}) &:= \partial_u G(V_s, Z_s, Z_s^{T,\xi}) + \partial_v G(V_s, Z_s, Z_s^{T,\xi})\nabla V_s \\ &\quad + \partial_z G(V_s, Z_s, Z_s^{T,\xi})\nabla Z_s.\end{aligned}$$

Proof. Since we assume that the processes $(\partial_u G(V_s, Z_s, Z_s^{T,\xi}))_{t\geq 0}$, $(\partial_v G(V_s, Z_s, Z_s^{T,\xi})\nabla V_s)_{t\geq 0}$ and $(\partial_z G(V_s, Z_s, Z_s^{T,\xi})\nabla Z_s)_{t\geq 0}$ are from BMO(P), we are able to define a probability measure $\mathbb{Q}$ as in (4.5.6) (see Barrieu and Karoui (2007), Theorem, 7.2 and the Kazamaki's criterion Lemma 2.1). Then if W_t is the Brownian motion under P, we define

$$dW^{\mathbb{Q}}(t) = dW(t) - \int_0^t \left[\frac{\mathcal{D}(V_s, Z_s, Z_s^{T,\xi})}{\nabla Z_s^{T,\xi}} + \partial_{\tilde{z}} G(V_s, Z_s, Z_s^{T,\xi})\right] ds, \tag{4.5.7}$$

to be a Brownian motion under $\mathbb{Q}$, for $t \geq 0$. For each $i = 1, \ldots, n$ we define $N_t^i = \mathbb{E}_{\mathbb{Q}}[\eta_i|\mathcal{F}_t]$. Using the martingale representation theorem, there exists a predictable process Z_t^{T,η_i} so that

$$N_t^i = N_T^i - \int_t^T Z_s^{T,\eta_i} dW^{\mathbb{Q}}(s).$$

Therefore, under the P measure we get

$$N_t^i = -\eta_i + \int_t^T Z_s^{\eta_i}\left[\frac{\mathcal{D}(V_s, Z_s, Z_s^{T,\xi})}{\nabla Z_s^{T,\xi}} + \partial_{\tilde{z}} G(V_s, Z_s, Z_s^{T,\xi})\right] ds - \int_t^T Z_s^{T,\eta_i} dW(s). \tag{4.5.8}$$

Hence, comparing equation (4.5.8) to the gradient allocation BSDE (4.5.4), we can conclude that equation (4.5.8) has unique solution under the assumptions of this theorem. The gradient capital allocation is therefore represented as in (4.5.5).

□

In the next example, we study the capital allocation representation of a forward entropic risk measure.

Example 4.5.1. We consider the explicit expression of the forward entropic risk measure derived by Chong et al. (2019) as

$$\rho_t(\xi_T) = \frac{1}{\gamma|\kappa|^2} \ln \mathbb{E}_{\mathbb{Q}}\left[e^{\gamma|\kappa_2|^2 Y_T^T} | \mathcal{F}_t\right]. \tag{4.5.9}$$

In this case, the financial market is considered to have a single stock whose coefficients depend on a stochastic factor driven by 2-dimensional Brownian motion, that is

$$\begin{aligned} dS_t &= S_t\mu(V_t)dt + S_t\sigma(V_t)dW^1(t) \\ dV_t &= \nu(V_t)dt + \kappa_1 dW^1(t) + \kappa_2 dW^2(t), \end{aligned} \tag{4.5.10}$$

where κ_1 and κ_2 are positive constants. The generator of the ergodic BSDE (4.3.1) is given by

$$g(v, z_1, z_2) = -\frac{1}{2}\left|\theta(v) + z_1\right|^2 + \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2,$$

while the generator of the BSDE (4.5.3) is given by

$$\begin{aligned} G(v, z_1, z_2, \tilde{z}_1, \tilde{z}_2) &= \frac{1}{\gamma}\big(g(v, z_1 + \gamma\tilde{z}_1, z_2 + \gamma\tilde{z}_2) - g(v, z_1, z_2)\big) \\ &= -\theta(v)\tilde{z}_1 + z_2\tilde{z}_2 + \frac{\gamma}{2}|\tilde{z}_2|^2. \end{aligned} \tag{4.5.11}$$

If we define $Y_T^T = \xi_T = h(V_T)$ and let $Z_{1,t}^T = \kappa_1 Z_t^T$ and $Z_{2,t}^T = \kappa_2 Z_t^T$, for some predictable process Z_t^T. Then, $\rho_t(\xi_T)$ for $t \geq 0$ is the first component of the solution to the BSDE

$$\begin{aligned}\rho_t(\xi_T) &= -h(\xi_T) + \int_t^T \left((-\kappa_1 \theta(V_s) + \kappa_2 Z_{2,s}) Z_s^T + \frac{\gamma |\kappa_2|^2}{2} |Z_s^T|^2 \right) dt \\ &\quad - \int_t^T Z_s^T (\kappa_1 dW^1(s) + \kappa_2 dW^2(s)). \end{aligned} \quad (4.5.12)$$

Note that the process Z_t^T is the second component of the solution to the BSDE (4.5.12). The processes Z_1 and Z_2 appear in the ergodic BSDE (4.3.1) representation of the forward performance process (4.5.1).

The generator of the BSDE (4.5.12) satisfies the differentiability condition (C1). Additionally, the terminal condition $\xi_T = h(V_T)$ satisfy condition (C2), we can apply Theorem 4.5.2. Consequently, the derivative of the generator with respect to each variable will be

$$\begin{aligned}\partial_v G(v, z_1, z_2, \tilde{z}_1, \tilde{z}_2) &= -\kappa_1 Z_t^T \partial_v \theta(v) \nabla V, \\ \partial_{z_2} G(v, z_1, z_2, \tilde{z}_1, \tilde{z}_2) &= \kappa_2 Z_t^T, \\ \partial_{\tilde{z}} G(v, z_1, z_2, \tilde{z}_1, \tilde{z}_2) &= -\kappa_1 \theta(V_t) + \kappa_2 Z_{2,t} + \gamma |\kappa_2|^2 Z_t^T. \end{aligned} \quad (4.5.13)$$

Therefore, the forward gradient allocation with respect the forward entropic risk measure is defined as

$$\frac{d}{du} \rho_t(h(V_T) + u\eta)\big|_{u=0} = \nabla_\eta Y_t^{T,\xi},$$

where the component $\nabla_\eta Y_t^{T,\xi}$ is the first part of the solution to the BSDE

$$\begin{aligned}\nabla_\eta Y_t^{T,\xi} &= -(\partial_v h(V_t) \nabla(V_T) + \eta) + \int_t^T \Big[-\kappa_1 Z_s^T \partial_v \theta(v) \nabla V + \kappa_2 Z_s^T \\ &\quad -\kappa_1 \theta(V_s) + \kappa_2 Z_{2,s} + \gamma |\kappa_2|^2 Z_s^T \Big] ds \\ &\quad - \int_t^T Z_s^T (\kappa_1 dW^1(s) + \kappa_2 dW^2(s)). \end{aligned}$$

(4.5.14)

Let $\psi(\xi) = e^{\gamma|\kappa_2|^2\xi}$ and under the assumption that ξ is a smooth functional and is bounded, such that the gradient $D_s\psi(\xi) := \psi'(\xi)D_s(\xi) = \gamma|\kappa_2|^2 e^{\gamma|\kappa_2|^2\xi}D_s\xi$ is from BMO(P). We can use the Clark Ocone formula (see Ocone and Karatzas (1991) and Di Nunno et al. (2009)) and for $\beta \in (0,1)$, so that we have

$$e^{\gamma|\kappa_2|^2\beta\xi} = \mathbb{E}[e^{\gamma|\kappa_2|^2\beta\xi}] + \int_0^T \mathbb{E}[D_s e^{\gamma|\kappa_2|^2\beta\xi}|\mathcal{F}_s](\kappa_1 dW^1(s) + \kappa_2 dW^2(s)).$$

In turn, by taking the conditional expectation of the above equation and applying the chain rule we have

$$\begin{aligned}\Lambda^{\beta\xi}(t) &= \mathbb{E}[e^{\gamma|\kappa_2|^2\beta\xi}|\mathcal{F}_t] \\ &= \mathbb{E}[e^{\gamma|\kappa_2|^2\beta\xi}] + \int_0^T \mathbb{E}[\beta\gamma|\kappa_2|^2 e^{\gamma|\kappa_2|^2\beta\xi}D_s\xi|\mathcal{F}_t](\kappa_1 dW^1(s) + \kappa_2 dW^2(s)),\end{aligned} \tag{4.5.15}$$

where $D_s(\cdot)$ is the Malliavin derivative. We note, that the process $\Lambda_t^{\beta\xi}$ is a positive martingale and can be represented as

$$\Lambda^{\beta\xi}(t) = \Lambda^{\beta\xi}(0) + \gamma|\kappa_2|^2\int_0^t \Lambda^{\beta\xi}(s)\frac{\beta\mathbb{E}[e^{\gamma|\kappa_2|^2\beta\xi}D_s\xi|\mathcal{F}_t]}{\Lambda^{\beta\xi}(s)}(\kappa_1 dW^1(s) + \kappa_2 dW^2(s))$$

$$\Lambda_t^{\beta\xi} = \Lambda^{\beta\xi}(0) + \gamma|\kappa_2|^2\int_0^t \Lambda^{\beta\xi}(s)Z_s^{\beta\xi}(\kappa_1 dW_s^1 + \kappa_2 dW_s^2)$$

with

$$Z_t^{\beta\xi} = \frac{\beta\mathbb{E}[e^{\gamma|\kappa_2|^2\beta\xi}D_s\xi|\mathcal{F}_t]}{\mathbb{E}[e^{\gamma|\kappa_2|^2\beta\xi}|\mathcal{F}_t]}. \tag{4.5.16}$$

Furthermore, the process $\Lambda^{\beta\xi}(\cdot)$ satisfies

$$\begin{aligned}\Lambda^{\beta\xi}(t) &= \Lambda^{\beta\xi}(0)\exp\Bigg(\int_0^t \big(-\kappa_1\theta(V_t) + \kappa_2 Z_{2,t} + \gamma|\kappa_2|^2 Z_t^T\big) \\ &\quad\times(\kappa_1 dW^1(s) + \kappa_2 dW^2(s)) \\ &\quad -\frac{1}{2}\int_0^t \big(-\kappa_1\theta(V_t) + \kappa_2 Z_{2,t} + \gamma|\kappa_2|^2 Z_t^T\big)^2 ds\Bigg).\end{aligned} \tag{4.5.17}$$

Hence, the process $(\Lambda^{\beta\xi}(t)/\Lambda^{\beta\xi}(0))$ corresponds to the stochastic exponential process

$$\mathcal{E}\Bigg(\int_0^t \partial_{\tilde{z}} G(v, z_1, z_2, \tilde{z}_1, \tilde{z}_2)ds\Bigg).$$

4.6 Conclusion

We have shown the representation of the ergodic BSDE-based dynamic risk and capital allocation of a maturity-independent risk measure. We studied maturity-independent risk measures, which generalises the classical dynamic risk measures. We applied our results to the case of the forward entropic risk measure. Our results may be considered as a generalisation of Chong et al. (2019), Kromer and Overbeck (2014), Kromer and Overbeck (2017). In the next chapter, we continue with the maturity-independent risk measure by constructing them using BSDEs with jumps.

Chapter 5

An Ergodic BSDE risk representation in a jump-diffusion framework.

5.1 Introduction

We consider the representation of forward entropic risk measures using the theory of ergodic backward stochastic differential equations in a jump-diffusion framework. Our work can be viewed as an extension of the work considered by Chong et al. 2019 in the diffusion case. We also study the behaviour of a forward entropic risk measure under jumps when a financial position is held for a longer maturity.

Zariphopoulou and Žitković 2010 proposed maturity-independent risk measures to help address how to assess risk positions when the time horizon is not fixed. They formulated the forward entropic risk measures using the forward exponential performance processes. These forward performance processes are introduced and developed by Musiela and Zariphopoulou 2007 (see also Musiela, Zariphopoulou, et al. 2008, Musiela and Zariphopoulou 2009, Musiela and Zariphopoulou 2010a for further improvements) to measure investment performance across all times $t \in [0, \infty)$, which makes the forward entropic risk measures to be defined for all times. Recently, Liang and Zariphopoulou 2017 proposed the use of the ergodic backward stochastic differential equation (ergodic BSDE) to construct the forward performance processes.

Kobylanski et al. 2000 introduced BSDEs with the quadratic growth and random terminal time in and her work was developed by Briand and Confortola 2008. Later, Morlais 2009b proved existence and uniqueness results for the BSDEs with quadratic growth in a jump framework (see also Morlais 2009a, Morlais 2010 for further contributions).

The rest of the chapter is organized as follows. In Section 5.2, we introduce the jump-diffusion model and all the notations that will be using in the rest of the chapter. Section 5.3, we provide the representation of the forward entropic risk measure using the classic BSDE and the ergodic BSDE in a jump model setting. Section 5.4 analyzes the behaviour of a forward entropic risk measure over a long-term horizon. Finally, we conclude the chapter.

5.2 Problem Formulation

Suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is the filtered probability space satisfying the usual conditions [1]. The filtration is generated by two independent processes, d-dimensional standard Brownian motion $\{W_t, t \geq 0\}$ defined on $\Omega \times [0, \infty)$ and the compensated Poisson random measure $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$ defined on $\Omega \times [0, \infty) \times \mathbb{R}_0$. Here, $N(dt, d\zeta)$ counts the number of jumps that occur on or before t, and ν is a positive Lévy measure satisfying

[1] The usual conditions are completeness and right-continuous (Yong and Zhou 1999).

the conditions

$$\int_{|\zeta|\leq 1} |\zeta^2|\nu(d\zeta) < \infty$$

and

$$\int_{|\zeta|\geq 1} \nu(d\zeta) < \infty.$$

The last condition implies that the stock process has finite number of jumps with absolute value greater than one (Tankov 2003). Maturity-independent convex risk measure is defined in Chapter 4, Definition 4.2.1. The only difference is on the dynamics of the stock price $S(t)$. We consider a financial market with n risky investments, with price processes, S_t^i for $i = 1, \dots, n$, satisfying the following stochastic differential equation (SDE)

$$\frac{dS^i(t)}{S^i(t)} = \mu^i(V_t)dt + \sigma^i(V_t)dW(t) + \int_{\mathbb{R}_0} \Upsilon^i(V_t, \zeta)\tilde{N}(dt, d\zeta), \qquad S^i(0) > 0. \tag{5.2.1}$$

The coefficients of the stock price S^i are affected by a stochastic factor, which is modelled by a d-dimensional stochastic process V, with $n \leq d$, satisfying:

$$dV_t = \eta(V_t)dt + \kappa dW(t). \qquad V_0 = v_0 > 0. \tag{5.2.2}$$

We impose the following assumptions to the coefficients so that Equations (5.2.1) and (5.2.2) have solutions.

Assumption 7. The drift $\mu^i(v) \in \mathbb{R}$, volatility $\sigma^i(v) \in \mathbb{R}^{1\times d}$ and jump rate $\Upsilon^i(v, \zeta) > -1$ are $\mathcal{F}_t$-predictable and bounded processes for $v \in \mathbb{R}^d$, satisfying the following condition

$$\int_0^T \big(|\mu(v_t)| + \sigma^2(v_t) + \int_{\mathbb{R}_0} (\Upsilon^i)^2(v_t, \zeta)\nu(\zeta)\big)dt < \infty, \qquad \text{a.s.}$$

Assumption 8. There exists a large enough constant $C_\eta > 0$, for $v_1, v_2 \in \mathbb{R}^d$ such that the drift coefficient $\eta(v) \in \mathbb{R}^d$ of the factor model satisfies:

$$(\eta(v_1) - \eta(v_2))(v_1 - v_2) \leq -C_\eta|v_1 - v_2|^2.$$

Furthermore, the volatility matrix $\kappa \in \mathbb{R}^{d\times d}$ is positive definite and normalized to $|\kappa| = 1$.

Let π_t^i be a self-financing portfolio representing the amount of wealth invested in stock i. The wealth process X solves

$$dX_t = \sum_{i=1}^{n} \frac{\pi_t^i dS^i(t)}{S^i(t)} = \pi_t\big(\mu(V_t)dt + \sigma(V_t)dW(t) + \int_{\mathbb{R}_0} \Upsilon(V_t,\zeta)\tilde{N}(dt,d\zeta)\big), \qquad X_0 > 0, \tag{5.2.3}$$

where the initial wealth is given by $X_0 = x \in \mathbb{R}$. An investment strategy $\pi_t \in \mathbb{R}^n$ is said to be admissible if it is $\mathbb{R}^n$ valued $\mathcal{F}_t$-progressively measurable satisfying $\mathbb{E}(\int_0^t |\pi_t^2| ds < \infty)$. The process X_t is a unique solution of Equation (5.2.3) using π_t, such that $X_t \geq 0$ for all $t \geq 0$, a.s. The set of all admissible strategies is denoted by $\mathcal{A}$.

We now recall from Chong et al. 2019 the notion of forward performance process.

Definition 5.2.1. A process $U(t,x)$, $(t,x) \in [0,\infty) \times \mathbb{R}$, is a forward performance process if:

(i) for each $x \in \mathbb{R}$, $U(t,x)$ is $\mathcal{F}_t-$progressively measurable,

(ii) for each $t \geq 0$, the mapping $x \mapsto U(t,x)$ is strictly increasing, strictly concave, continuously differentiable and satisfies the *Inada* conditions, i.e. $\lim_{x\to\infty} U'(x) = 0$ and $\lim_{x\to-\infty} U'(x) = +\infty$.

(iii) for all $\pi \in \mathcal{A}$ and $0 \leq t \leq s$,

$$U(t, X_t^{\pi}) \geq \mathbb{E}_{\mathbb{P}}[U(s, X_s^{\pi})|\mathcal{F}_t],$$

and there exists an optimal $\tilde{\pi} \in \mathcal{A}$ such that,

$$U(t, X_t^{\tilde{\pi}}) = \mathbb{E}_{\mathbb{P}}[U(s, X_s^{\tilde{\pi}})|\mathcal{F}_t],$$

with X^{π}, $X^{\tilde{\pi}}$ solving Equation (5.2.3).

We derive the associated stochastic partial differential equation (SPDE) for the performance process by applying the Itô-Ventzell formula to $U(t,x)$ for any strategy $\pi \in \mathcal{A}$ (see Musiela and Zariphopoulou 2010b on deriving the SPDE and Øksendal, Zhang, et al. 2007 for the Itô-Ventzell formula for a jump process). We first assume that $U(t,x)$ admits the Lêvy decomposition

$$dU(t,x) = b(t,x)dt + a(t,x)dW(t) + \int_{\mathbb{R}_0} \Phi(t,x,\zeta)\tilde{N}(d^-t,d\zeta),$$

where the processes $b(t,x)$, $a(t,x)$ and $\Phi(t,x,\zeta)$ are $\mathcal{F}_t-$progressively measurable processes and $\tilde{N}(d^-t,d\zeta)$ represents a forward integral. Then we obtain

$$\begin{aligned}
& dU(t,X_t) \\
= \; & b(t,X_t)dt + a(t,X_t)dW(t) + U_x(t,X_t)dX_t + \frac{1}{2}U_{xx}(t,X_t)d\langle X\rangle_t \\
& + a_x(t,X)d\langle W,X\rangle_t + \int_{\mathbb{R}_0}[U(t,X_t+\pi\Upsilon(t,\zeta)) - U(t,X_t) \\
& - U_x(t,X_t)\pi\Upsilon(t,\zeta)]\nu(d\zeta)dt + \int_{\mathbb{R}_0}[\Phi(t,X_t+\pi\Upsilon(t,\zeta)) \\
& - \Phi(t,X_t)]\nu(d\zeta)dt + \int_{\mathbb{R}_0}\Big[U(t^-,X_{t^-}+\pi\Upsilon(t,\zeta)) - U(t^-,X_{t^-}) \\
& + \Phi(t^-,X_{t^-}+\pi\Upsilon(t,\zeta))\Big]\tilde{N}(d^-t,d\zeta) \\
= \; & \Bigg[b(t,X_t) + \pi\mu(V_t)U_x(t,X_t) + \pi\sigma(V_t)a_x(t,X_t) + \frac{1}{2}\pi^2\sigma^2(V_t)U_{xx}(t,X_t) \\
& + \int_{\mathbb{R}_0}\Big([U(t,X_t+\pi\Upsilon(t,\zeta)) - U(t,X_t) - U_x(t,X_t)\pi\Upsilon(t,\zeta)] \\
& + [\Phi(t,X_t+\pi\Upsilon(t,\zeta)) - \Phi(t,X_t)]\nu(d\zeta)\Bigg]dt \\
& + \Big(a(t,X_t) + \pi\sigma(V_t)U_x(t,X_t)\Big)dW(t) + \int_{\mathbb{R}_0}\Big[U(t^-,X_{t^-}+\pi\Upsilon(t,\zeta)) \\
& - U(t^-,X_{t^-}) + \Phi(t^-,X_{t^-}+\pi\Upsilon(t,\zeta))\Big]\tilde{N}(d^-t,d\zeta). \qquad (5.2.4)
\end{aligned}$$

The volatility $a(t,x)$ and the process $\Phi(t,x,\zeta)$ for $t \geq 0$ are model inputs determined by the investor's preference.

From Definition 5.2.1, we know that the process $U(t,X_t^\pi)$ is a supermartingale for any admissible investment strategy π, that is

$$U(t,X^\pi) \geq \mathbb{E}[U(t,x)].$$

Hence, there exists an optimal strategy $\tilde{\pi}$ when the process $U(t, X_t^\pi)$ is a true martingale. The process $U(t, X_t^\pi)$ is a true martingale when the drift term in Equation (5.2.4) is zero. Therefore the optimal strategy is given by

$$\begin{aligned}\tilde{\pi} &= \inf_{\pi\in\mathcal{A}}\Bigg[\pi\mu(V_t)U_x(t,X_t) + \pi\sigma(V_t)a_x(t,X_t) + \frac{1}{2}\pi^2\sigma^2(V_t)U_{xx}(t,X_t) \\ &\quad + \int_{\mathbb{R}_0}\Bigg(\left[U(t,X_t+\pi\Upsilon(t,\zeta)) - U(t,X_t) - U_x(t,X_t)\pi\Upsilon(t,\zeta)\right] \\ &\quad + \left[\Phi(t,X_t+\pi\Upsilon(t,\zeta)) - \Phi(t,X_t)\right]\Bigg)\nu(d\zeta)\Bigg].\end{aligned}$$

We consider an exponential forward performance process given by

$$U(t,x) = -e^{-\gamma x + f(t,V_t)}, \quad (t,x)\in[0,\infty)\times\mathbb{R} \tag{5.2.5}$$

where $\gamma > 0$ and a function $f : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$. By application of Itô to $U(t,x)$ and setting the resulting drift term to zero, we see that the function f solves a semi-linear partial differential equation of the form

$$0 = \frac{\partial}{\partial t}f + \eta(V_t)\nabla f + \frac{1}{2}\kappa^2\nabla^2 f + g(v,\kappa\nabla f,\Upsilon),$$

with g defined as

$$\begin{aligned}g(v,\kappa\nabla f,\Upsilon) &= \frac{1}{2}\gamma^2\sigma^2(v)\left[\pi - \frac{\mu(v)-\frac{1}{2}\sigma(v)\kappa\nabla f}{\gamma\sigma^2(v)}\right]^2 + \frac{1}{2}\big(\mu(V_t) - \frac{1}{2}\sigma(v)\kappa\nabla f\big) \\ &\quad + \frac{1}{2}\kappa^2(\nabla f)^2 + \int_{\mathbb{R}_0}\left[e^{-\gamma\pi\Upsilon(t,\zeta)} - 1 + \gamma\pi\Upsilon(t,\zeta)\right]\nu(d\zeta).\end{aligned} \tag{5.2.6}$$

We consider the following ergodic backward stochastic differential equation

$$dY_t = (-g(V_t,Z_t,\Psi_t)+\lambda)dt + Z_t dW(t) + \int_{\mathbb{R}_0}\Psi(V_t,\zeta)\tilde{N}(dt,d\zeta), \tag{5.2.7}$$

for $0 \le t \le T < \infty$ and a given function $g : \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ and $Z_t \in \mathbb{H}^2_W(\mathbb{R})$, $\Psi(V_t,\zeta)\in\mathbb{H}^2_N(\mathbb{R})$. To ensure the solution to (5.2.7) exists and it is unique we have to impose certain assumptions on g.

Assumption 9. There exist constants $K > 0$, $\hat{K} > 0$ $C_v > 0$ and $C_z > 0$ such that the generator g satisfy

$$|g(t,0,0,0)| \leq \hat{K}. \tag{5.2.8}$$

$$|g(v_1, z, \psi) - g(v_2, z, \psi)| \leq C_v(1 + |z|)|v_1 - v_2|, \tag{5.2.9}$$

and

$$|g(v, z_1, \psi) - g(v, z_2, \psi)| \leq C_z(1 + |z_1| + |z_2|)|z_1 - z_2| \tag{5.2.10}$$

for any v_1, v_2, z_1, $z_2 \in \mathbb{R}$.

Furthermore, there exists $-1 < K_1 \leq 0$ and $K_2 \geq 0$ such that

$$g(v, z, \psi_1) - g(v, z, \psi_2) \leq \int_{\mathbb{R}_0} (\psi_1 - \psi_2)\varphi^{v,z,\psi_1,\psi_2}(\zeta)\nu(d\zeta) \tag{5.2.11}$$

where $\varphi^{v,z,\psi_1,\psi_2} : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_0 \to [-1, \infty)$ is $\mathcal{P} \otimes \mathcal{B}$-measurable and satisfies $K_1(1 \wedge |\zeta|) \leq \varphi(\zeta) \leq K_2(1 \wedge |\zeta|)$ (Royer 2006). With $\mathcal{P}$ denoting the predictable σ-field and $\mathcal{B}$ the Borel σ-field on $\mathbb{R}$.

Theorem 5.2.1. *Suppose Assumption (7), (8) and (9) hold. Then, the ergodic BSDE (5.2.7) with generator given by*

$$\begin{aligned} g(v, z(v_t), \Psi(v_t, \zeta)) &= \frac{\gamma^2}{2}\left[\pi\sigma(v) - \frac{\mu(v_t)/\sigma(v_t) - z(v_t)}{\gamma}\right]^2 \\ &\quad + \frac{1}{2}\big(\mu(v_t)/\sigma(v_t) - z(v_t)\big)^2 + \frac{1}{2}z^2(v_t) \\ &\quad + \int_{\mathbb{R}_0}\Big[e^{-\gamma\pi\Upsilon(v_t,\zeta)+\Psi(v_t,\zeta)} - 1 - \gamma\pi\Upsilon(v,\zeta) \\ &\quad + \Psi(v,\zeta)\Big]\nu(d\zeta), \end{aligned} \tag{5.2.12}$$

has a unique Markovian solution

$$(Y, Z, \Psi, \lambda) = (y(V_t), z(V_t), \psi(V_t), \lambda),$$

for $0 \leq t \leq T < \infty$, with

$$|Y_t| \leq \frac{K}{\alpha}, \quad |Z_t| \leq C_z := \frac{C_v}{C_\eta - C_v} \quad \text{and} \quad |\Psi(v_t, \zeta)| \leq \frac{2K}{\alpha}. \tag{5.2.13}$$

Proof. For the proof, we adapted the method in Liang and Zariphopoulou 2017 to jump framework. We start by establishing that the driver g satisfies Assumptions (9). We consider truncation functions $\tilde{q}:\mathbb{R}^d \to \mathbb{R}^d$, defined as

$$q(z) := \frac{\min(|z|, C_z)}{|z|} z 1_{\{z\neq 0\}}, \quad \text{and} \quad \tilde{q}(\psi) := 1_{|\psi|\leq 1}.$$

and define a truncated ergodic BSDE

$$dY_t = (-g(V_t, q(Z_t), \tilde{q}(\Psi_t)) + \lambda)dt + Z_t dW(t) + \int_{\mathbb{R}_0} \Psi(V_t, \zeta)\tilde{N}(dt, d\zeta), \tag{5.2.14}$$

for $t \geq 0$. We verify that the generator $g(v, q(z), \tilde{q}(\psi))$ satisfies Assumption (9), i.e.

$$|g(v_1, q(z), \tilde{q}(\psi)) - g(v_2, q(z), \tilde{q}(\psi))| \leq C_v(1 + C_z)|v_1 - v_2|, \tag{5.2.15}$$

$$|g(v, q(z_1), \tilde{q}(\psi)) - g(v, q(z_2), \tilde{q}(\psi))| \leq C_z(1 + 2C_z)|z_1 - z_2| \tag{5.2.16}$$

and

$$|g(v, q(z), \tilde{q}(\psi_1)) - g(v, q(z), \tilde{q}(\psi_2))| \leq \int_{\mathbb{R}_0} (\psi_1 - \psi_2)\varphi^{v,z,\psi_1,\psi_2}(\zeta)\nu(d\zeta). \tag{5.2.17}$$

We now, have to prove that there exists a Markovian solution $(Y_t, Z_t, \Psi_t, \lambda)$ to the truncated ergodic BSDE (5.2.14) that satisfies $|Z_t| \leq C_z$ and $|\Psi(v_t, \zeta)| \leq \frac{2K}{\alpha}$ for $t \geq 0$, then $q(Z_t) = Z_t$ and $\tilde{q}(\Psi_t) = \Psi_t$. As a result, this solution $(Y_t, Z_t, \Psi_t, \lambda)$, will also solve the ergodic BSDE (5.2.7). For this part of the proof, we consider a strictly monotonic BSDE with a constant of monotonicity $\alpha > 0$, on a finite horizon $[0, n]$, i.e.

$$\begin{aligned} Y_t^{v,\alpha,n} &= \int_t^n (g(V_u, q(Z_u^{v,\alpha,n}), \tilde{q}(\Psi_u^{v,\alpha,n})) - \alpha Y_u^{v,\alpha,n})du + \int_t^n Z_u^{v,\alpha,n} dW(u) \\ &\quad + \int_t^n \int_{\mathbb{R}_0} \Psi^{v,\alpha,n}(V_u, \zeta)\tilde{N}(du, d\zeta). \end{aligned} \tag{5.2.18}$$

We deduce from Cohen and Fedyashov Cohen and Fedyashov 2014, Theorem 8, (see also Briand and Hu 1998 for the diffusion case), that BSDE

(5.2.18) has a unique solution $(Y_t^{v,\alpha,n}, Z_u^{v,\alpha,n}, \Psi_u^{v,\alpha,n})$ satisfying $|Y_t| \leq \frac{K}{\alpha}$ with $Z_u^{v,\alpha,n} \in \mathbb{H}_W^2(\mathbb{R})$ and $\Psi_u^{v,\alpha,n} \in L_\nu^2(\tilde{N})$. Moreover, we conclude that $(Y_t^{v,\alpha,n}, Z_u^{v,\alpha,n}, \Psi_u^{v,\alpha,n})$, is a unique adapted square integrable solution to the BSDE (5.2.18) for $t \geq 0$. Hence, there exists an adapted square integrable limiting processes $(Y_t^{v,\alpha}, Z_u^{v,\alpha}, \Psi_u^{v,\alpha})$ such that

$$\lim_{n\to\infty} (Y_t^{v,\alpha,n}, Z_u^{v,\alpha,n}, \Psi_u^{v,\alpha,n}) = (Y_t^{v,\alpha}, Z_u^{v,\alpha}, \Psi_u^{v,\alpha}),$$

with $|Y_t| \leq \frac{K}{\alpha}$. Furthermore, the solution is Markovian, that is, there exist functions $y^\alpha(\cdot)$, $z^\alpha(\cdot)$ and $\psi^\alpha(\cdot)$ such that

$$(Y_t^{v,\alpha}, Z_t^{v,\alpha}, \Psi_t^{v,\alpha}) = (y^\alpha(V_t), z^\alpha(V_t), \psi^\alpha(V_t)),$$

is a solution to the infinite horizon BSDE

$$dY_t^{v,\alpha} = (-g(V_t^v, q(Z_t^{v,\alpha}), \tilde{q}(\Psi_t^{v,\alpha})) + \alpha Y_t^{v,\alpha}) + Z_t^{v,\alpha} dW(t) + \int_{\mathbb{R}_0} \Psi_t^{v,\alpha} \tilde{N}(dt, d\zeta). \tag{5.2.19}$$

The next part of the proof is to demonstrate that the Lipschitz continuity property

$$|y^\alpha(V_t^{v_1}) - y^\alpha(V_t^{v_2})| \leq C_z |V_t^{v_1} - V_t^{v_2}|\,,$$

for all $v_1, v_2 \in \mathbb{R}^d$ with the Lipschitz constant C_z. Let $\delta Y_t = Y_t^{\alpha,v_1} - Y_t^{\alpha,v_2}$, $\delta Z_t = Z_t^{\alpha,v_1} - Z_t^{\alpha,v_2}$ and $\delta\Psi_t = \Psi_t^{\alpha,v_1} - \Psi_t^{\alpha,v_2}$, for $t \geq 0$. Subsequently

$$\begin{aligned} d\delta Y_t &= -(g(V_t^{v_1}, q(Z_t^{\alpha,v_1}), \tilde{q}\Psi(V_t^{\alpha,v_1})) - g(V_t^{v_2}, q(Z_t^{\alpha,v_2}), \tilde{q}\Psi(V_t^{\alpha,v_2})))dt \\ &\quad + \alpha\delta Y_t dt + \delta Z_t dW(t) + \int_{\mathbb{R}_0} \delta\Psi_t \tilde{N}(dt, d\zeta) \\ &= -(g(V_t^{v_1}, q(Z_t^{\alpha,v_1}), \tilde{q}\Psi(V_t^{\alpha,v_1})) - g(V_t^{v_2}, q(Z_t^{\alpha,v_2}), \tilde{q}\Psi(V_t^{\alpha,v_2})))dt \\ &\quad + \alpha\delta Y_t dt + \delta Z_t (dW(t) - \beta_t dt) \\ &\quad + \int_{\mathbb{R}_0} \delta\Psi_t (N(dt, d\zeta) - \varphi^{v,z,\psi_1,\psi_2} \nu(d\zeta) dt), \end{aligned} \tag{5.2.20}$$

where

$$\beta_t = \frac{g(V_t^{v_1}, q(Z_t^{\alpha,v_1}), \tilde{q}\Psi(V_t^{\alpha,v_1})) - g(V_t^{v_2}, q(Z_t^{\alpha,v_2}), \tilde{q}\Psi(V_t^{\alpha,v_2}))}{|\delta Z_t|^2} \delta Z_t \mathbf{1}_{\delta \mathbf{Z_t} \neq \mathbf{0}}$$

By Inequality (5.2.15), β is bounded. From the Girsanov's theorem we define $W^{\beta}(t) = W(t) - \int_0^t \beta_u du$ and $\tilde{N}^{\varphi}(dt, d\zeta) = N(dt, d\zeta) - \int_0^t \varphi^{v,z,\psi_1,\psi_2}\nu(d\zeta)du$, for $0 \leq t \leq T$, where $\varphi^{v,z,\psi_1,\psi_2}$ is defined in Assumption (5.2.11). Therefore taking conditional expectation with respect to $\mathbb{Q}$ on $\mathcal{F}_t$ for $0 \leq t < T < \infty$, we get

$$\delta Y_t = e^{-\alpha(T-t)}\mathbb{E}_{\mathbb{Q}}[\delta Y_T|\mathcal{F}_t] + \mathbb{E}_{\mathbb{Q}}\Big[\int_t^T e^{-\alpha(u-t)}\delta g_u du|\mathcal{F}_u\Big].$$

From condition (5.2.13), we note that the first expectation is bounded by $2K/\alpha$, and therefore will converges to zero as $T \to \infty$. We deduce from (5.2.15) that the second expectation is bounded by

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T e^{-\alpha(u-t)}\delta g_u du|\mathcal{F}_u\Big] &\leq C_v(1+C_z)\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T e^{-\alpha(u-t)}|V_u^{v_1} - V_u^{v_2}|du|\mathcal{F}_u\Big] \\ &\leq C_v(1+C_z)\frac{e^{\alpha t}(e^{-(\alpha+C_\eta)t} - e^{-(\alpha+C_\eta)T})}{\alpha + C_\eta}|v_1 - v_2|.\end{aligned} \tag{5.2.21}$$

The last inequality is based on the Grownwall Inequality. Hence, as $T \to \infty$ yields

$$|y^{\alpha}(V_t^{v_1}) - y^{\alpha}(V_t^{v_2})| \leq C_z|V_t^{v_1} - V_t^{v_2}|. \tag{5.2.22}$$

To obtain the third inequality in Condition (5.2.15), we consider a stochastic factor with a jump term [2], this yields

$$dV_t = \eta(V_t)dt + \kappa dW(t) + \int_{\mathbb{R}_0} \zeta\tilde{N}(dt, d\zeta)\,,$$

where the coefficients satisfy Assumptions (7) and (8). Suppose that $y^{\alpha}(\cdot) \in C^2(\mathbb{R}^d)$. By Itô's formula to $y^{\alpha}(V_t^v)$ we get

$$dy^{\alpha}(V_t^v) = \nabla y^{\alpha}(V_t^v)\eta(V)dt + \nabla y^{\alpha}(V_t^v)\kappa dW(t) + \frac{1}{2}\nabla^2 y^{\alpha}(V_t^v)\kappa^2 dt$$

[2]Note that for this work we consider a stochastic factor in the diffusion case throughout the paper. If we include a jump term in the stochastic factor then our generator will be dependent on the Y variable. Hence, the stochastic factor with jumps will not be ideal for risk representation using BSDE, because the translation invariance property will not hold.

$$\int_{\mathbb{R}_0} (y^\alpha(V_t^v + \zeta) - y^\alpha(V_t^v) - \nabla y^\alpha(V_t^v)\zeta)\nu(d\zeta)dt$$
$$+ \int_{\mathbb{R}_0} (y^\alpha(V_t^v + \zeta) - y^\alpha(V_t^v))\tilde{N}(dt, d\zeta). \tag{5.2.23}$$

Comparing terms in the infinite horizon BSDE (5.2.19) and Equation (5.2.23), we deduce that

$$Z_t^{\alpha,v} = \nabla y^\alpha(V_t^v)\kappa, \tag{5.2.24}$$

$$\begin{aligned} \alpha Y_t^{v,\alpha} &= \nabla y^\alpha(V_t^v)\eta(V) + \frac{1}{2}\nabla^2 y^\alpha(V_t^v)\kappa^2 + \int_{\mathbb{R}_0} (y^\alpha(V_t^v + \zeta) - y^\alpha(V_t^v) \\ &\quad - \nabla y^\alpha(V_t^v)\zeta)\nu(d\zeta) + g(V_t^v, q(Z_t^{v,\alpha}), \tilde{q}(\Psi_t^{v,\alpha})) \end{aligned} \tag{5.2.25}$$

and

$$\Psi(V_t, \zeta) = y^\alpha(V_t^v + \zeta) - y^\alpha(V_t^v), \tag{5.2.26}$$

for $v \in \mathbb{R}^d$. Equation (5.2.25) is a Partial Integro-Differential Equation (PIDE) with a unique bounded solution, $y^\alpha(\cdot) \in C^2(\mathbb{R}^d)$. We conclude that $|y^\alpha(v)| \leq \frac{K}{\alpha}$. Furthermore, using Assumption (8) and Equation (5.2.24) and from condition (5.2.22), we conclude that for $t \leq 0$, $|Z_t^{\alpha,v}| \leq C_z$. From Equation (5.2.26), we have that $|\Psi(V_t, \zeta)| \leq \frac{2K}{\alpha}$.

To show that λ is a constant, the proof follows similarly as in Liang and Zariphopoulou 2017. □

In the following theorem, we connect the solution of the ergodic BSDE with jumps (5.2.7) to the exponential forward performance process (5.2.5). To do this, we adopt the procedure by Liang and Zariphopoulou 2017 in Theorem 3, where they made the same connection under the diffusion case.

Theorem 5.2.2. *Suppose that Assumptions 7 and 8 hold, and let $(Y_t, Z_t, \Psi_t, \lambda)$, $t \geq 0$ be a unique Markovian solution to Equation (5.2.7). Then,* *(i) the process $U(t,x)$, $(t,x) \in [0,\infty) \times \mathbb{R}$, is an exponential forward performance process defined as*

$$U(t,x) = -e^{-\gamma x + Y_t - \lambda t}, \tag{5.2.27}$$

with volatility

$$a(t,x) = -e^{-\gamma x + Y_t - \lambda t} Z_t$$

and jump rate

$$\Phi(t,x,\zeta) = -e^{-\gamma x + Y_t - \lambda t}(e^{-\Psi} - 1).$$

(ii) The optimal investment strategy is given by

$$\begin{aligned} \tilde{\pi} &= \inf_{\pi \in \mathcal{A}} \Bigg(\frac{\gamma^2}{2} \Big[\pi\sigma(v) - \frac{\mu(v_t)/\sigma(v_t) - z(v_t)}{\gamma} \Big]^2 + \frac{1}{2}\big(\mu(v_t)/\sigma(v_t) - z(v_t)\big)^2 \\ &\quad + \frac{1}{2} z^2(v_t) + \int_{\mathbb{R}_0} \Big[e^{-\gamma\pi\Upsilon(v_t,\zeta) + \Psi(v_t,\zeta)} - 1 - \gamma\pi\Upsilon(v,\zeta) \\ &\quad + \Psi(v,\zeta) \Big] \nu(d\zeta) \Bigg). \end{aligned} \tag{5.2.28}$$

Proof. We start by first showing that $U(t,x)$ satisfies the super-martingale property for any admissible investment strategy $\pi \in \mathcal{A}$ for all $0 \le t \le s$, that is

$$\mathbb{E}_{\mathbb{P}}[-e^{-\gamma x + Y_s - \lambda s} | \mathcal{F}_t] \le -e^{-\gamma x + Y_t - \lambda t},$$

and for an optimal investment strategy $\tilde{\pi}$, $U(t,x)$ is a martingale, that is,

$$\mathbb{E}_{\mathbb{P}}[-e^{-\gamma X^{\tilde{\pi}} + Y_s - \lambda s} | \mathcal{F}_t] = -e^{-\gamma X^{\tilde{\pi}} + Y_t - \lambda t}.$$

Based on the wealth process (5.2.3), $e^{-\gamma X}$ can be written as

$$\begin{aligned} e^{-\gamma X_s} &= e^{-\gamma X_u} \exp\Big\{ -\int_t^s \gamma\pi\mu(V_u)du - \int_t^s \gamma\pi\sigma(V_u)dW(u) \\ &\quad - \int_u^s \int_{\mathbb{R}_0} \gamma\pi\Upsilon(u,\zeta)\tilde{N}(du,d\zeta) \Big\}. \end{aligned} \tag{5.2.29}$$

On the other hand, the ergodic BSDE (5.2.7) is given by

$$\begin{aligned} Y_s - \lambda s &= Y_t - \lambda t - \int_t^s g(V_u, Z_u, \Psi_u)du + \int_t^s Z_u dW(u) \\ &\quad + \int_t^s \int_{\mathbb{R}_0} \Psi(u,\zeta)\tilde{N}(du, d\zeta). \end{aligned} \tag{5.2.30}$$

Combining the above expressions yields

$$\begin{aligned} e^{-\gamma X_s + Y_s - \lambda s} &= e^{-\gamma X_t + Y_t - \lambda t} \exp\Big\{ - \int_t^s \big(\gamma\mu(V_u)\pi + g(V_u, Z_u, \Psi_u)\big)du \\ &\quad - \int_t^s (\gamma\pi\sigma(V_u) - Z_u)dW(u) - \int_t^s \int_{\mathbb{R}_0} (\gamma\pi\Upsilon(v,\zeta) \\ &\quad - \Psi(v,\zeta))\tilde{N}(du, d\zeta)\Big\}. \end{aligned} \tag{5.2.31}$$

Then we take expectation under the probability measure $\mathbb{P}$, given $\mathcal{F}_t$, i.e.,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[e^{-\gamma X_s + Y_s - \lambda s} \mid \mathcal{F}_t] &= e^{-\gamma X_t + Y_t - \lambda t} \mathbb{E}_{\mathbb{P}}\Big[\exp\Big\{ - \int_t^s \big(\gamma\mu(V_u)\pi \\ &\quad + g(V_u, Z_u, \Psi_u)\big)du - \int_t^s (\gamma\pi\sigma(V_u) - Z_u)dW(u) \\ &\quad - \int_t^s \int_{\mathbb{R}_0} (\gamma\pi\Upsilon(v,\zeta) - \Psi(v,\zeta))\tilde{N}(du, d\zeta)\Big\} \Big| \mathcal{F}_t\Big]. \end{aligned} \tag{5.2.32}$$

We define a new probability measure $\mathbb{Q}$, for $s \geq 0$ and $\pi \in \mathcal{A}$ using the process $\widetilde{M}_u$, $u \in [0, s]$ defined as the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, therefore

$$\widetilde{M}_u = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{E}(M)_u,$$

where

$$M_u = \exp\Big\{ - \int_t^s (\gamma\pi\sigma(V_u) - Z_u)dW(u) - \int_t^s \int_{\mathbb{R}_0} (\gamma\pi\Upsilon(v,\zeta) - \Psi(v,\zeta))\tilde{N}(du, d\zeta)\Big\}.$$

Since the processes Z_u, π_u and Ψ_u belong to $BMO(\mathbb{P})$, the process M_u is a BMO-martingale, and consequently the stochastic exponential $\mathcal{E}(M)_u$ is a true martingale (Morlais 2009b). Hence

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\bigg[\exp\bigg(\int_t^s\big(g^{\pi}(V_u,Z_u,\Psi_u)-g(V_u,Z_u,\Psi_u)\big)du\bigg)\frac{\widetilde{M}_s}{\widetilde{M}_t}\bigg|\mathcal{F}_t\bigg]\\
&=\mathbb{E}_{\mathbb{Q}}\bigg[\exp\bigg(\int_t^s\big(g^{\pi}(V_u,Z_u,\Psi_u)-g(V_u,Z_u,\Psi_u)\big)du\bigg)\bigg|\mathcal{F}_t\bigg], \qquad (5.2.33)
\end{aligned}$$

with

$$\begin{aligned}
g^{\pi}(v,z(v_t),\psi(v_t,\zeta)) \;:=\;& \frac{\gamma^2}{2}\bigg[\pi\sigma(v)-\frac{\mu(v_t)/\sigma(v_t)-z(v_t)}{\gamma}\bigg]^2\\
&+\frac{1}{2}\big(\mu(v_t)/\sigma(v_t)-z(v_t)\big)^2+\frac{1}{2}z^2(v_t)\\
&+\int_{\mathbb{R}_0}\bigg[e^{-\gamma\pi\Upsilon(v_t,\zeta)+\psi(v_t,\zeta)}-1-\gamma\pi\Upsilon(v,\zeta)\\
&+\psi(v,\zeta)\bigg]\nu(d\zeta).
\end{aligned} \qquad (5.2.34)$$

Since $g^{\pi}(v,z(V_t),\psi(v,\zeta))\leq g(v,z(V_t),\psi(v,\zeta))$, we can conclude that

$$\mathbb{E}_{\mathbb{P}}[-e^{-\gamma X^{\pi}+Y_s-\lambda s}|\mathcal{F}_t]\leq -e^{-\gamma X+Y_t-\lambda t}.$$

Further, for $\pi=\tilde{\pi}$ defined in (5.2.28), we have

$$g^{\tilde{\pi}}(v,z(V_t),\psi(v_t))=g(v,z(V_t),\psi(v_t))$$

and hence

$$\mathbb{E}_{\mathbb{P}}[-e^{-\gamma X^{\tilde{\pi}}+Y_s-\lambda s}|\mathcal{F}_t]=-e^{-\gamma X^{\tilde{\pi}}+Y_t-\lambda t}.$$

To show the second part of the theorem, we apply Itô's formula to Equation (5.2.27) that yields,

$$dU(t,x)=(\cdots)dt+U(Z_t-\gamma\pi\sigma(V_t))dW(t)+U\int_{\mathbb{R}_0}(e^{-\gamma\pi\Upsilon(v,\zeta)+\Psi(v,\zeta)}-1)\tilde{N}(dt,d\zeta).$$

We then, compare the above equation to Equation (5.2.4) and obtain the following

$$a(t,x)=-e^{-\gamma x+Y_t-\lambda t}Z_t,$$

and

$$\Phi(t, x, \zeta) = -e^{-\gamma x + Y_t - \lambda t}(e^{-\Psi} - 1).$$

It is not difficult to see that the infimum function in Equation (5.2.28) is convex with respect to π that is the second derivative respect to π of the infimum function is positive. Therefore the minimum in Equation (5.2.28) exists. □

5.3 Forward entropic risk measure and ergodic BSDE with jumps

In this section, we recall the definition of forward entropic risk measure. We then provide the representation of a forward entropic risk measure as the solution of a BSDE and ergodic BSDE.

Definition 5.3.1. Consider the forward exponential performance process $U(x, t) = -e^{-\gamma x + Y_t - \lambda t}$, with $(t, x) \in [0, \infty) \times \mathbb{R}$. Consider a risk position $\xi_T \in L^\infty(\mathcal{F}_T)$, where $T > 0$ is arbitrary and the risk position is entered into at the initial time $t = 0$. Then, the forward entropic risk measure $\rho_t(\xi_T, T), t \in [0, T]$, is the unique $\mathcal{F}_t$-measurable random variable that satisfies the indifference condition

$$ess \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}_{\mathbb{P}}\left[U(X_u^\pi + \rho_u(\xi_T; T) + \xi_T, T)\middle|\mathcal{F}_t\right] = \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}_{\mathbb{P}}\left[U(X_u^\pi, T)\middle|\mathcal{F}_t\right] \tag{5.3.1}$$

for all $(t, x) \in [0, T] \times \mathbb{R}$.

If we let $\xi \in \mathcal{L}$ and consider $T_\xi := \inf\{T \geq 0 : \xi \in \mathcal{F}_T\}$, then the forward entropic risk measure of ξ is defined, for $t \in [0, T_\xi]$, as

$$\rho_t(\xi) := \rho_t(\xi; T_\xi).$$

Therefore, for $\xi_T \in L^\infty(\mathcal{F}_T)$, we have $\rho_t(\xi) := \rho_t(\xi_T; T)$.

The next theorem gives a representation of the forward entropic risk measure as a solution of an associated BSDE, with a generator that depends on a solution of the ergodic BSDE.

Theorem 5.3.1. *Let $\xi_T \in L^\infty(\mathcal{F}_T)$ be a risk position with an arbitrary maturity $T > 0$. Supposes that Assumptions 7, 8 and 9 hold, and the processes Z and Ψ in the ergodic BSDE (5.2.7) are uniformly bounded. Consider, the BSDE*

$$\begin{aligned} Y_t^{T,\xi} &= -\xi_T + \int_t^T G(V_u, Z_u, Z_u^{T,\xi}, \Psi_u, \Psi_u^{T,\xi})du - \int_t^T Z_u^{T,\xi} dW(u) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} \Psi_u^{T,\xi} \tilde{N}(du, d\zeta), \end{aligned} \tag{5.3.2}$$

where the generator $G(v, z, \tilde{z}, \psi, \tilde{\psi}) = \frac{1}{\gamma}\big(g(v, z + \gamma\tilde{z}, \psi + \gamma\tilde{\psi}) - g(v, z, \psi)\big)$, with $g(\cdot, \cdot, \cdot)$ given by (5.2.34). Then the following statements hold:

(i) The BSDE (5.3.2) has a unique solution $(Y_t^{T,\xi}, Z_t^{T,\xi}, \Psi_t^{T,\xi}) \in \mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}_W^2(\mathbb{R}) \times L_\nu^2(\tilde{N})$, for $t \in [0, T]$.

(ii) The forward entropic risk measure of a position in ξ_T is given by

$$\rho_t(\xi_T) = Y_t^{T,\xi},$$

for $t \in [0, T]$.

Proof. Since the associated parameters are bounded and Lipschitz continuous (Assumption (7) and (8)), and the generator g in (5.2.34) satisfies Assumption (9). These assumptions imply that g is Lipschitz continuous in z and υ, a.s.. Therefore, we know from Morlais 2009b (see also Royer 2006 and Guambe and Kufakunesu 2018) that there exists a unique solution to the BSDE (5.3.2) with a generator given by g in (5.2.34) and the risk position $\xi_T \in L^\infty(\mathcal{F}_T)$.

(i) For $t \in [0,T]$, the generator $G(v, Z, \tilde{z}, \Psi, \tilde{\psi})$ is Lipschitz continuous in z and ψ, that is,

$$|G(v, Z_t, \tilde{z}_1, \Psi_t, \tilde{\psi}) - G(v, Z_t, \tilde{z}_2, \Psi_t, \tilde{\psi})| \leq C_z(1+2Z_t+\gamma|\tilde{z}_1|+\gamma|\tilde{z}_2|)|\tilde{z}_1-\bar{z}_2|$$

and

$$|G(v, Z_t, \tilde{z}, \Psi_t, \tilde{\psi}_1) - G(v, Z_t, \tilde{z}, \Psi_t, \tilde{\psi}_2)| \leq \int_{\mathbb{R}_0} |\tilde{\psi}_1 - \tilde{\psi}_2| \varphi^{v,z,\psi_1,\psi_2} \nu(d\zeta)$$

where Z_t and Ψ_t are uniformly bounded in $\mathbb{H}^2_W(\mathbb{R}) \times L^2_\nu(\tilde{N})$. Considering that G is a linear combination of g, we deduce that G has the same form as g in (5.2.34). Therefore, using the fact that $\xi_T \in L^\infty(\mathcal{F}_t)$, we conclude (following Morlais 2009b, Royer 2006 and Guambe and Kufakunesu 2018) that Equation (5.3.2) has a unique solution for $t \in [0,T]$.

(ii) We consider the forward performance process in (5.2.27) and that $\rho_t(\xi_T) \in \mathcal{F}_t$, $t \in [0,T]$. Then we have

$$\begin{aligned}
& ess \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}_\mathbb{P}\Big[U(X^\pi_u + \rho_u(\xi_T;T) + \xi_T, T)\Big|\mathcal{F}_t\Big] \\
= \;& e^{-\gamma\rho_t(\xi_T)} ess \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}_\mathbb{P}\Big[-\exp\Big\{-\gamma\Big(x + \int_t^T \pi_u \mu(V_u) dt \\
& + \int_t^T \pi\sigma(V_u) dW(u) + \int_t^T \int_{\mathbb{R}_0} \pi_u \Upsilon_u \tilde{N}(du, d\zeta)\Big) + Y_T \\
& -\lambda T - \gamma\xi_T\Big\}\Big|\mathcal{F}_t\Big].
\end{aligned} \tag{5.3.3}$$

In order to prove the second part of the theorem, we define for $s \in [t,T]$, the process

$$P^\pi_s \;:=\; -\exp\Big\{-\gamma\Big(x + \int_t^s \pi_s \mu(V_s) dt + \int_t^s \pi\sigma(V_s) dW(s)$$

$$+\int_t^s \int_{\mathbb{R}_0} \pi_s \Upsilon_s \tilde{N}(ds, d\zeta)\Big) + Y_s - \lambda s + \gamma Y_s^T \Big\}. \qquad (5.3.4)$$

As in Chong et al. 2019, we will show that the process P_s^{π} is a supermartingale for all $\pi \in \mathcal{A}_{[t,T]}$ and that there exists $\tilde{\pi} \in \mathcal{A}_{[t,T]}$ such that $P_s^{\tilde{\pi}}$ is a martingale.

For $0 \le t \le r \le s \le T$, the exponent of P_s^{π} satisfies

$$\begin{aligned}
& -\gamma\Big(x + \int_t^s \pi_u \mu(V_u)du + \int_t^s \pi\sigma(V_u)dW(u) \\
& + \int_t^s \int_{\mathbb{R}_0} \pi_u \Upsilon_u \tilde{N}(du, d\zeta)\Big) + Y_s - \lambda s + \gamma Y_s^T \\
= \; & -\gamma\Big(x + \int_t^r \pi_u \mu(V_u)du + \int_t^r \pi_u\sigma(V_u)dW(u) \\
& + \int_t^r \int_{\mathbb{R}_0} \pi_u \Upsilon_u \tilde{N}(du, d\zeta)\Big) + Y_r - \lambda r + \gamma Y_r^T \\
& -\gamma\Big(x + \int_r^s \pi_u \mu(V_u)du + \int_r^s \pi_u\sigma(V_u)dW(u) \\
& + \int_r^s \int_{\mathbb{R}_0} \pi_u \Upsilon_u \tilde{N}(du, d\zeta)\Big) + (Y_s - Y_r) \\
& -(\lambda s - \lambda r) + \gamma(Y_s^T - Y_r^T).
\end{aligned} \qquad (5.3.5)$$

Furthermore, from the ergodic BSDE (5.2.7) and BSDE (5.3.2), we have that

$$\begin{aligned}
(Y_s - Y_r) - \lambda(s - r) \; &= \; -\int_r^s g(V_u, Z_u, \Psi_u)du + \int_r^s Z_u dW(u) \\
& \quad + \int_r^s \int_{\mathbb{R}_0} \Psi_u \tilde{N}(du, d\zeta),
\end{aligned}$$

and

$$\begin{aligned}
Y_s^T - Y_r^T \; &= \; -\frac{1}{\gamma}\int_r^s \big(g(V_u, Z_u + \gamma Z_u^T, \Psi_u + \gamma \Psi_u^T) - g(V_u, Z_u, \Psi_u)\big)du \\
& \quad + \int_r^s Z_u^T dW(u) + \int_r^s \int_{\mathbb{R}_0} \Psi_u^T \tilde{N}(du, d\zeta).
\end{aligned}$$

Combining the above three equations and applying the conditional expectation, yields

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{P}}\Big[-\exp\Big\{-\gamma\Big(x+\int_t^s \pi_u\mu(V_u)du+\int_t^s \pi\sigma(V_u)dW(u) \\
& +\int_t^s\int_{\mathbb{R}_0}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big)+Y_s-\lambda s+\gamma Y_s^T\Big\}\Big|\mathcal{F}_r\Big] \\
= \; & -\exp\Big\{-\gamma\Big(x+\int_t^r \pi_u\mu(V_u)du+\int_t^r \pi_u\sigma(V_u)dW(u) \\
& +\int_t^r\int_{\mathbb{R}_0}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big)+Y_r-\lambda r+\gamma Y_r^T\Big\} \\
& \times\mathbb{E}_{\mathbb{P}}\Big[\exp\Big\{\int_r^s\big(-\gamma\pi_u\mu(V_u)-g(V_u,Z_u+\gamma Z_u^T,\Psi_u+\gamma\Psi_u^T)\big)du \\
& +\int_r^s\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)dW(u) \\
& +\int_r^s\int_{\mathbb{R}_0}\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\tilde{N}(du,d\zeta)\Big\}\Big|\mathcal{F}_r\Big].
\end{aligned}
\tag{5.3.6}
$$

We consider a process $\mathcal{M}_s := \exp\big\{\int_r^s\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)dW(u) + \int_r^s\int_{\mathbb{R}_0}\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\tilde{N}(du,d\zeta)\big\}$, with $-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T > -1$ for a.s. (ω,t,ζ). From Assumptions (7)-(8) and the fact that $(Z_t^{T,\xi},\Psi_t^{T,\xi}) \in \mathbb{H}_W^2(\mathbb{R})\times L_\nu^2(\tilde{N}_p)$, we conclude that the process $\mathcal{M}_s$ is a BMO-martingale. Define a probability measure $\mathbb{Q}^\pi$ by

$$
\frac{d\mathbb{Q}^\pi}{d\mathbb{P}} = \mathcal{E}(\mathcal{M})_T,
$$

on $\mathcal{F}_T$, where

$$
\begin{aligned}
& \mathcal{E}(\mathcal{M})_T \\
= \; & \exp\Big\{\int_0^T\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)dW(u)-\frac{1}{2}\int_0^T\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2du \\
& +\int_0^T\int_{\mathbb{R}_0}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)du
\end{aligned}
$$

$$
+\int_0^T\int_{\mathbb{R}_0}\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\tilde{N}(du,d\zeta)\Big\},
\tag{5.3.7}
$$

provided that $\int_0^T\int_{\mathbb{R}_0}(e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1)^2\nu(d\zeta)du<\infty$ {for more on exponential martingale see Applebaum 2009, Øksendal and Sulem 2005 and Papapantoleon 2008}. Therefore, $\frac{d\mathbb{Q}^\pi}{d\mathbb{P}}|_{\mathcal{F}_T}=\mathcal{E}(\mathcal{M})_T$ is uniformly integrable, given that the process $\mathcal{M}_s$ is a BMO-martingale. Now, we have that

$$
\begin{aligned}
&\exp\Big\{\int_r^s\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)dW(u)\\
&+\int_r^s\int_{\mathbb{R}_0}\big(e^{-\gamma\pi_u\Upsilon_u}+\Psi_u+\gamma\Psi_u^T\big)\tilde{N}(du,d\zeta)\Big\}\\
=\;&\exp\Big\{\frac{1}{2}\int_r^s\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2du\\
&-\int_r^s\int_{\mathbb{R}_0}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1+\big(-\gamma\pi_u\Upsilon_u\\
&+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)du\Big\}\frac{\mathcal{E}(\mathcal{M})_s}{\mathcal{E}(\mathcal{M})_r}.
\end{aligned}
\tag{5.3.8}
$$

Hence, from (5.3.6),

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[-\exp\Big\{-\gamma\Big(x+\int_t^s\pi_u\mu(V_u)du+\int_t^s\pi\sigma(V_u)dW(u)\\
&+\int_t^s\int_{\mathbb{R}_0}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big)+Y_s-\lambda s+\gamma Y_s^T\Big\}\Big|\mathcal{F}_r\Big]\\
=\;&-\exp\Big\{-\gamma\Big(x+\int_t^r\pi_u\mu(V_u)du+\int_t^r\pi_u\sigma(V_u)dW(u)\\
&+\int_t^r\int_{\mathbb{R}_0}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\Big)+Y_r-\lambda r+\gamma Y_r^T\Big\}\\
&\times\mathbb{E}_{\mathbb{P}}\Big[\exp\Big\{\int_r^s\big(-\gamma\pi_u\mu(V_u)-g(V_u,Z_u+\gamma Z_u^T,\Psi_u+\gamma\Psi_u^T)\big)du\\
&+\frac{1}{2}\int_r^s\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2du-\int_r^s\int_{\mathbb{R}_0}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}\\
&-1+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)du\Big\}\frac{\mathcal{E}(N)_s}{\mathcal{E}(N)_r}\Big|\mathcal{F}_r\Big]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\exp\bigg\{-\gamma\bigg(x+\int_t^r \pi_u\mu(V_u)du+\int_t^r \pi_u\sigma(V_u)dW(u) \\
& +\int_t^r\int_{\mathbb{R}_0} \pi_u\Upsilon_u\tilde{N}(du,d\zeta)\bigg)+Y_r-\lambda r+\gamma Y_r^T\bigg\} \\
& \times\mathbb{E}_{\mathbb{Q}^\pi}\bigg[\exp\bigg\{\int_r^s \big(-\gamma\pi_u\mu(V_u)-g(V_u,Z_u+\gamma Z_u^T,\Psi_u+\gamma\Psi_u^T)\big)du \\
& +\frac{1}{2}\int_r^s \big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2du \\
& -\int_r^s\int_{\mathbb{R}_0} \big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)du\bigg\}\bigg|\mathcal{F}_r\bigg].
\end{aligned}
\tag{5.3.9}
$$

Following the same procedure as in Chong et al. 2019, we show that for any $u\in[r,s]$,

$$
\begin{aligned}
& -\gamma\pi_u\mu(V_u)+\frac{1}{2}\big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2-\int_r^s\int_{\mathbb{R}_0}\big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1 \\
& +\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)\ \geq\ g(V_u,Z_u+\gamma Z_u^T,\Psi_u+\gamma\Psi_u^T),
\end{aligned}
\tag{5.3.10}
$$

then

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}^\pi}\bigg[\exp\bigg\{\int_r^s \big(-\gamma\pi_u\mu(V_u)-g(V_u,Z_u+\gamma Z_u^T,\Psi_u+\gamma\Psi_u^T)\big)du \\
& +\frac{1}{2}\int_r^s \big(-\gamma\pi_u\sigma(V_u)+Z_u+\gamma Z_u^T\big)^2du \\
& -\int_r^s\int_{\mathbb{R}_0} \big[e^{(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T)}-1+\big(-\gamma\pi_u\Upsilon_u+\Psi_u+\gamma\Psi_u^T\big)\big]\nu(d\zeta)du\bigg\}\bigg|\mathcal{F}_r\bigg]\geq 1.
\end{aligned}
\tag{5.3.11}
$$

As a result the super-martingale property

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{P}}\bigg[-\exp\bigg\{-\gamma\bigg(x+\int_t^s \pi_u\mu(V_u)du+\int_t^s \pi\sigma(V_u)dW(u) \\
& +\int_t^s\int_{\mathbb{R}_0} \pi_u\Upsilon_u\tilde{N}(du,d\zeta)\bigg)+Y_s-\lambda s+\gamma Y_s^T\bigg\}\bigg|\mathcal{F}_r\bigg] \\
\leq \quad & -\exp\bigg\{-\gamma\bigg(x+\int_t^r \pi_u\mu(V_u)du+\int_t^r \pi_u\sigma(V_u)dW(u)
\end{aligned}
$$

$$+\int_t^r \int_{\mathbb{R}_0} \pi_u \Upsilon_u \tilde{N}(du, d\zeta)\Big) + Y_r - \lambda r + \gamma Y_r^T\Big\}$$

will hold. Note that the left hand side of the equation (5.3.10) can be written as

$$\begin{aligned}
& -\gamma\pi_u\mu(V_u) + \frac{1}{2}\big(-\gamma\pi_u\sigma(V_u) + Z_u + \gamma Z_u^T\big)^2 - \int_r^s \int_{\mathbb{R}_0} \big[e^{(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi_u^T)} \\
& -1 + \big(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi_u^T\big)\big]\nu(d\zeta) \\
= \; & \frac{\gamma^2}{2}\big|\pi\sigma - (Z^T + \frac{Z_u + \mu(V_u)/\sigma(V_u)}{\gamma})\big|^2 - \frac{1}{2}\big|Z_u + \gamma Z_u^T + \mu(V_u)/\sigma(V_u)\big|^2 \\
& + \frac{1}{2}|Z_u + \gamma Z_u^T|^2 - \int_r^s \int_{\mathbb{R}_0} \big[e^{(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi_u^T)} - 1 \\
& + \big(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi_u^T\big)\big]\nu(d\zeta).
\end{aligned} \tag{5.3.12}$$

In particular, for any $\pi_u \in \mathcal{A}_{[t,T]}$,

$$\begin{aligned}
& \frac{\gamma^2}{2}\big|\pi\sigma - (Z^T + \frac{Z_u + \mu(V_u)/\sigma(V_u)}{\gamma})\big|^2 - \frac{1}{2}\big|Z_u + \gamma Z_u^T + \mu(V_u)/\sigma(V_u)\big|^2 + \frac{1}{2}|Z_u \\
& + \gamma Z_u^T|^2 - \int_r^s \int_{\mathbb{R}_0} \big[e^{(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi_u^T)} - 1 + \big(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi_u^T\big)\big]\nu(d\zeta) \\
\geq \; & \inf_{\pi_u \in \mathcal{A}_{[t,T]}} \Big\{\frac{\gamma^2}{2}\big|\pi\sigma - (Z^T + \frac{Z_u + \mu(V_u)/\sigma(V_u)}{\gamma})\big|^2 \\
& \quad - \frac{1}{2}\big|Z_u + \gamma Z_u^T + \mu(V_u)/\sigma(V_u)\big|^2 + \frac{1}{2}|Z_u + \gamma Z_u^T|^2 \\
& \quad - \int_r^s \int_{\mathbb{R}_0} \big[e^{(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi_u^T)} - 1 + \big(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi_u^T\big)\big]\nu(d\zeta)\Big\},
\end{aligned}$$

and using $g(V_u, Z_u + \gamma Z_u^T, \Psi_u + \gamma\Psi_u^T)$ as in (5.2.6), we conclude that the super-martingale property holds true.

The martingale property of the process $P^{\tilde{\pi}}$, holds true if $\tilde{\pi} \in \mathcal{A}_{[t,T]}$ and

$$\begin{aligned}
\tilde{\pi} \; = \; & \inf_{\pi_u \in \mathcal{A}_{[t,T]}} \Big\{\frac{\gamma^2}{2}\big|\pi\sigma - (Z^T + \frac{Z_u + \mu(V_u)/\sigma(V_u)}{\gamma})\big|^2 \\
& \quad - \frac{1}{2}\big|Z_u + \gamma Z_u^T + \mu(V_u)/\sigma(V_u)\big|^2 + \frac{1}{2}|Z_u + \gamma Z_u^T|^2 \\
& \quad - \int_r^s \int_{\mathbb{R}_0} \big[e^{(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi_u^T)} - 1 + \big(-\gamma\pi_u\Upsilon_u + \Psi_u + \gamma\Psi_u^T\big)\big]\nu(d\zeta)\Big\}.
\end{aligned}$$

Combining the results from above, we obtain that $\mathbb{E}_{\mathbb{P}}[P_T^{\pi}|\mathcal{F}_t] \leq P_t^{\pi}$, and hence, for any $\pi \in \mathcal{A}_{[t,T]}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\bigg[- e^{-\gamma\left(x+\int_t^T \pi_u\mu(V_u)du+\int_t^T \pi\sigma(V_u)dW_u+\int_t^T\int_{\mathbb{R}_0}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\right)+Y_T-\lambda T-\gamma\xi_T}\bigg|\mathcal{F}_t\bigg] \\ \leq -e^{-\gamma x+Y_t-\lambda t+\gamma Y_t^T},\end{aligned} \tag{5.3.13}$$

and for $\pi = \tilde{\pi} \in \mathcal{A}_{[t,T]}$, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\bigg[- e^{-\gamma\left(x+\int_t^T \pi_u\mu(V_u)du+\int_t^T \pi\sigma(V_u)dW_u+\int_t^T\int_{\mathbb{R}_0}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\right)+Y_T-\lambda T-\gamma\xi_T}\bigg|\mathcal{F}_t\bigg] \\ = -e^{-\gamma x+Y_t-\lambda t+\gamma Y_t^T}.\end{aligned} \tag{5.3.14}$$

Subsequently,

$$\begin{aligned} ess \sup_{\pi\in\mathcal{A}_{[t,T]}} \mathbb{E}_{\mathbb{P}}\bigg[- e^{-\gamma\left(x+\int_t^T \pi_u\mu(V_u)dt+\int_t^T \pi\sigma(V_u)dW_u+\int_t^T\int_{\mathbb{R}_0}\pi_u\Upsilon_u\tilde{N}(du,d\zeta)\right)+Y_T-\lambda T-\gamma\xi_T}\bigg|\mathcal{F}_t\bigg]. \\ = -e^{-\gamma x+Y_t-\lambda t+\gamma Y_t^T},\end{aligned} \tag{5.3.15}$$

and using condition (5.3.1), we obtain

$$-e^{-\gamma\rho_t(\xi_T)-\gamma x+Y_t-\lambda t+\gamma Y_t^T} = -e^{-\gamma x+Y_t-\lambda t},$$

and hence,

$$\rho_t(\xi_T) = Y_t^T,$$

which concludes the proof. □

Similar to Chong et al. 2019, the above representation also satisfies the time-consistent property, which means any risk position defined at time T can be evaluated indifferently at any intermediary time u for any $0 \leq t \leq u \leq T < \infty$. See also Bion Bion-Nadal 2008 for construction of risk measures using BSDE with jumps. Chong Chong et al. 2019, highlights the difference between the traditional entropic risk measure and the forward entropic risk measure. The first difference is that the forward entropic risk measure is defined for all time $t \leq 0$, while the traditional entropic risk measure is determined for a finite time $t \in [0, T]$. The second difference is that the generator of the BSDE (5.3.2), depends on the process Z, which is part of the solution of the ergodic BSDE (5.2.7) that gives the forward exponential process in (5.2.27).

5.4 Long-term maturity behaviour of the forward entropic risk measure

We consider a contingent claim written on the stochastic factor, this position is represented as follow

$$\xi_T = -h(V_T), \tag{5.4.1}$$

where $h : \mathbb{R} \mapsto \mathbb{R}$ is uniformly bounded and is Lipschitz continuous function with a Lipschitz constant C_h. From Theorem 5.3.1, we know that the risk of the position is represented as the solution for the BSDE (5.3.2), that is $\rho(\xi_T) = Y_t^{T,\xi}$, where $Y_t^{T,\xi}$ satisfies

$$\begin{aligned} Y_t^{T,h} &= h(V_T) + \int_t^T G(V_u, Z_u, Z_u^{T,h}, \Psi_u, \Psi_u^{T,h})du - \int_t^T Z_u^{T,h} dW(u) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} \Psi_u^{T,h} \tilde{N}(du, d\zeta). \end{aligned} \tag{5.4.2}$$

To analyse the long term behaviour of the forward risk measure, we associate the above BSDE to the ergodic BSDE given as

$$\begin{aligned} \hat{Y}_t &= \hat{Y}_{T'} + \int_t^{T'} \big(G(V_u, Z_u, \hat{Z}_u, \Psi_u, \hat{\Psi}_u) - \lambda\big)du - \int_t^{T'} \hat{Z}_u dW(u) \\ &\quad - \int_t^{T'} \int_{\mathbb{R}_0} \hat{\Psi}_u \tilde{N}(du, d\zeta), \end{aligned} \tag{5.4.3}$$

for $0 \leq t \leq T' < \infty$. We analyze the approximation of $Y_0^{T,h}$ by $\hat{Y}_0 + \hat{\lambda}T$ for large T. In Chong et al. Chong et al. 2019 the driver of the ergodic BSDE (5.4.3) depends only on the solution Z_t of the ergodic BSDE (5.2.7) of the forward performance process. In our case, the driver of the ergodic BSDE (5.4.3) will depend on the solution Z and Ψ of the ergodic BSDE (5.2.7). As was pointed by Chong et al. Chong et al. 2019, this creates technical issues, which results in examining the Markovian and non-Markovian forward processes separately. Following a similar route, we analyze the long-term maturity behaviour in the Markovian case. The non-Markovian case follows closely as in Chong et al. Chong et al. 2019.

5.4.1 Markovian forward performance process

Let us consider the case

$$U(t,x) = -e^{-\gamma x + y(V_t) - \lambda t} \tag{5.4.4}$$

where $(Y(V_t), Z(V_T), \Psi(V_t)) = (y(V_t), z(V_t), \psi(V_t), \lambda)$, is the solution of the ergodic BSDE (5.2.7). The driver $G(V_u, Z_u, \hat{Z}_u, \Psi_u, \hat{\Psi}_u)$ of the ergodic BSDE (5.4.3) depends on $z(V_t)$ and $\psi(V_t)$. The functions $z(\cdot)$ and $\psi(\cdot)$ are bounded and hence the driver G is Lipschitz continuous in $\hat{z}$ and $\hat{\psi}$ as in (5.2.10) and (5.2.11). However, the generator G may not be Lipschitz continuous in v, which affects the existence and uniqueness of the solution to the ergodic BSDE (5.4.3). To overcome this problem, we consider an auxiliary quadratic BSDE defined by,

$$\begin{aligned}\hat{Y}_t^{T,h} &= h(V_T) + \frac{Y_T - \lambda T}{\gamma} + \int_t^T \frac{1}{\gamma} g(V_u, \gamma \hat{Z}_u^{T,h}, \gamma \hat{\Psi}_u^{T,h}) du - \int_t^T \hat{Z}_u^{T,h} dW(u) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} \hat{\Psi}_u^{T,h} \tilde{N}(du, d\zeta),\end{aligned} \tag{5.4.5}$$

with $(\hat{Y}_t^{T,h}, \hat{Z}_t^{T,h}, \hat{\Psi}_t^{T,h})$ given as

$$(\hat{Y}_t^{T,h}, \hat{Z}_t^{T,h}, \hat{\Psi}_t^{T,h}) := \left(Y_t^{T,h} + \frac{Y_t - \lambda t}{\gamma}, Z_t^{T,h} + \frac{Z_t}{\gamma}, \Psi_t^{T,h} + \frac{\Psi_t}{\gamma} \right),$$

and g is given in (5.2.34).

We now recall from Chong et al. 2019 the following proposition with some results for the stochastic factor model.

Proposition 5.4.1. *(Chong et al. 2019) If Assumption 8 holds, then for all* $t \geq 0$*,*

(i) the stochastic factor process satisfies $|V_t^{v_1} - V_t^{v_2}|^2 \leq e^{-2C_\eta t}|v_1 - v_2|^2$ *where* $v_1, v_2 \in \mathbb{R}^d$*.*

(ii) If we assume that the process V^v *satisfies the following SDE*

$$dV_t^v = (\eta(V_t^v) + H(V_t^v))dt + \kappa dW^H(t),$$

where $H : \mathbb{R} \mapsto \mathbb{R}$ *is a measurable bounded function,* $\mathbb{Q}^H$ *and* $\mathbb{P}$ *are equivalent probability measures, and* W^H *is a* $\mathbb{Q}^H$*-Brownian motion. Then, for some constant* $C > 0$, $\mathbb{E}_{\mathbb{Q}^H}[|V_t^v|^2] \leq C(1 + |v|^2)$*.*

(iii) For any measurable function $\phi : \mathbb{R}^d \to \mathbb{R}$ *with polynomial growth rate* $\vartheta > 0$*, and* $v_1, v_2 \in \mathbb{R}^d$*,*

$$|\mathbb{E}_{\mathbb{Q}^H}[|\phi(V_t^{v_1}) - \phi(V_t^{v_2})|] \leq C(1 + |v_1|^{1+\vartheta} + |v_2|^{1+\vartheta})e^{-\hat{C}_\eta t},$$

where the constants C *and* $\hat{C}_\eta$ *depend on the function* H *only through* $sup_{v\in\mathbb{R}^d}|H(v)|$*.*

The proof of (i) and (ii) follows from the Gronwall's inequality and application of the Lyapunov argument respectively (see Chong et al. 2019 and Fleming and McEneaney 1995 Lemma 3.1). For the proof to the third part of the proposition (*basic coupling estimate*) is given in Lemma 3.4 of Hu, Madec, et al. 2015 and also see Theorem 2.4 from Debussche et al. 2011 and Theorem 5 from Cohen and Fedyashov 2014.

Theorem 5.4.2. *Let Assumption 7 and 8 hold, and assume that the forward performance process* $U(t, x)$ *is given by* (5.2.27)*. Then* *(i) there exists a unique solution* $(\hat{Y}_t^{T,h}, \hat{Z}_t^{T,h}, \hat{\Psi}_t^{T,h}) = (\hat{y}^{T,h}(V_t), \hat{z}^{T,h}(V_t), \hat{\psi}^{T,h}(V_t))$ *of the quadratic BSDE* (5.4.5) *for* $t \in [0, T]$.

(ii) For $(t, v) \in [0, \infty) \times \mathbb{R}^d$, *we have that*

$$|\hat{y}^{T,h}(t, v)| \leq C_T(1 + |v|)$$

and $\hat{z}^{T,h}, \hat{\psi}^{T,g}$ *are uniformly bounded such that,*

$$|\hat{z}^{T,h}(v, t)| \leq C_z, \qquad |\psi(v_t, \zeta)| \leq \frac{2K}{\alpha}.$$

Proof. The existence and uniqueness of the solution to the quadratic BSDE (5.4.5) follows from Morlais 2009b (Section 3.2, Theorem 1 and 2). Analogous to Chong et al. 2019, the linear growth condition of the function $\hat{y}^{T,g}(t, v)$ follows from the boundedness of $Y_t^{T,h}$ and the linear growth condition of $y(\cdot)$. Note that $(Y_u^T, Z_u^T, \Psi_u^T) = (Y_u^{T,t,v}, Z_u^{T,t,v}, \Psi_u^{T,t,v}) = (y^T(V_u^{t,v}, u), z^T(V_u^{t,v}, u), \Psi^T(V_u^{t,v}, u))$ for $u \in [t, T]$ with $V_t^{t,v} = v$ and some measurable functions $y^T(\cdot, \cdot)$, $z^T(\cdot, \cdot)$ and $\Psi^T(\cdot, \cdot)$. We consider a truncated BSDE version of (5.4.5)

$$\begin{aligned} \hat{Y}_t^{T,h} &= h(V_T) + \frac{Y_T - \lambda T}{\gamma} + \int_t^T \frac{1}{\gamma} g(V_u, \gamma q(\hat{Z}_u^{T,h}), \gamma \tilde{q}(\hat{\Psi}_u^{T,h})) du \\ &\quad - \int_t^T \hat{Z}_u^{T,h} dW(u) - \int_t^T \int_{\mathbb{R}_0} \hat{\Psi}_u^{T,h} \tilde{N}(du, d\zeta), \end{aligned} \tag{5.4.6}$$

where the truncation functions $q(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde{q} : \mathbb{R}^d \to \mathbb{R}^d$ are defined as

$$q(z) := \frac{\min(|z|, C_z)}{|z|} z 1_{\{z \neq 0\}}, \quad \text{and} \quad \tilde{q}(\psi) := 1_{|\psi| \leq 1}.$$

Now, it then follows that the generator g of the truncated BSDE (5.4.6) is Lipschitz i.e.

$$|g(v_1, \gamma q(z), \gamma \tilde{q}(\psi)) - g(v_2, \gamma q(z), \gamma \tilde{q}(\psi))| \leq C_v |v_1 - v_2|, \tag{5.4.7}$$

$$|g(v, \gamma q(z_1), \gamma \tilde{q}(\psi)) - g(v, \gamma q(z_2), \gamma \tilde{q}(\psi))| \leq C_z |z_1 - z_2| \tag{5.4.8}$$

and

$$|g(v, \gamma q(z), \gamma \tilde{q}(\psi_1)) - g(v, \gamma q(z), \gamma \tilde{q}(\psi_2))| \leq \int_{\mathbb{R}_0} |\psi_1 - \psi_2| \varphi^{v,z,\psi_1,\psi_2} \nu(d\zeta), \tag{5.4.9}$$

for any v_1, v_2, z_1, z_2, ψ_1, $\psi_2 \in \mathbb{R}^d$. Consequently, we have

$$\begin{aligned}
& \hat{Y}_t^{T,t,v_1} - \hat{Y}_t^{T,t,v_2} \\
= \ & h(V_T^{t,v_1}) - h(V_T^{t,v_2}) + \frac{1}{\gamma}(Y_T^{t,v_1} - Y_T^{t,v_2}) \\
& + \int_t^T \frac{1}{\gamma}\big[g\big(V_u^{t,v_1}, \gamma q(\hat{Z}_u^{T,t,v_1}), \gamma q(\hat{\Psi}_u^{T,t,v_1})\big) - g\big(V_u^{t,v_2}, \gamma q(\hat{Z}_u^{T,t,v_2}), \gamma q(\hat{\Psi}_u^{T,t,v_2})\big)\big] du \\
& - \int_t^T \big(\hat{Z}_u^{T,t,v_1} - \hat{Z}_u^{T,t,v_2}\big) dW(u) - \int_t^T \int_{\mathbb{R}_0} \big(\hat{\Psi}_u^{T,t,v_1} - \hat{\Psi}_u^{T,t,v_2}\big) \tilde{N}(du, d\zeta) \\
= \ & h(V_T^{t,v_1}) - h(V_T^{t,v_2}) + \frac{1}{\gamma}(y(V_T^{t,v_1}) - y(V_T^{t,v_2})) \\
& + \int_t^T \frac{1}{\gamma}\big[g\big(V_u^{t,v_1}, \gamma q(\hat{Z}_u^{T,t,v_1}), \gamma q(\hat{\Psi}_u^{T,t,v_1})\big) - g\big(V_u^{t,v_2}, \gamma q(\hat{Z}_u^{T,t,v_2}), \gamma q(\hat{\Psi}_u^{T,t,v_2})\big)\big] du \\
& - \int_t^T \big(\hat{Z}_u^{T,t,v_1} - \hat{Z}_u^{T,t,v_2}\big)(dW(u) - \beta du) \\
& - \int_t^T \int_{\mathbb{R}_0} \big(\hat{\Psi}_u^{T,t,v_1} - \hat{\Psi}_u^{T,t,v_2}\big)(\tilde{N}(du, d\zeta) - \varphi^{v,z,\psi_1,\psi_2}\nu(d\zeta)du)
\end{aligned} \tag{5.4.10}$$

and denote

$$\begin{aligned}
\beta_t \ := \ & \frac{\big[g\big(V_u^{t,v_1}, \gamma q(\hat{Z}_u^{T,t,v_1}), \gamma q(\hat{\Psi}_u^{T,t,v_1})\big) - g\big(V_u^{t,v_2}, \gamma q(\hat{Z}_u^{T,t,v_2}), \gamma q(\hat{\Psi}_u^{T,t,v_2})\big)\big]}{\gamma |\hat{Z}_u^{T,t,v_1} - \hat{Z}_u^{T,t,v_2}|^2} \times \\
& \times |\hat{Z}_u^{T,t,v_1} - \hat{Z}_u^{T,t,v_2}| 1_{\{\hat{Z}_u^{T,t,v_1} \neq \hat{Z}_u^{T,t,v_2}\}}
\end{aligned}$$

Using the Girsanov's theorem we can define $W^\beta(t) := W(t) - \int_0^t \beta du$ and $\tilde{N}^\varphi(dt, d\zeta) := \tilde{N}(dt, d\zeta) - \int_0^t \varphi^{v,z,\psi_1,\psi_2}\nu(d\zeta)du$ for $0 \leq t \leq T$, where $\varphi^{v,z,\psi_1,\psi_2}$ is defined in Assumption (5.2.11). For all t we define $\delta Z_t := \hat{Z}_t^{T,t,v_1} - \hat{Z}_t^{T,t,v_2}$ and $\delta \Psi_t := \hat{\Psi}_t^{T,t,v_1} - \hat{\Psi}_t^{T,t,v_2}$ and introduce

$$\mathcal{M}_t = \int_0^t \delta Z_u dW^\beta(u) + \int_0^t \int_{\mathbb{R}_0} \delta \Psi_u \tilde{N}^\varphi(du, d\zeta),$$

which is a local martingale under the measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, defined on $\mathcal{F}_T$. Thus, taking conditional expectation under the $\mathbb{Q}$ measure on $\mathcal{F}_t$ and using the Lipschitz condition of $h(v)$, in (5.4.1), $y(v)$ in (5.2.13) and

$g(v, \gamma q(z), \gamma\tilde{q}(\psi))$ in (5.4.7) to (5.4.9), we obtain the following results

$$\begin{aligned} |\hat{Y}_t^{T,t,v_1} - \hat{Y}_t^{T,t,v_2}| &= |\hat{y}_t^{T,t,v_1} - \hat{y}_t^{T,t,v_2}| \\ &\leq C_h \mathbb{E}_{\mathbb{Q}}[|V_T^{t,v_1} - V_T^{t,v_2}||\mathcal{F}_t] + \frac{K}{\gamma}\mathbb{E}_{\mathbb{Q}}[|V_T^{t,v_1} - V_T^{t,v_2}||\mathcal{F}_t] \\ &\quad + \frac{C_v}{\gamma}\mathbb{E}_{\mathbb{Q}}\left[\int_t^T |V_u^{t,v_1} - V_u^{t,v_2}||\mathcal{F}_t\right]. \end{aligned} \tag{5.4.11}$$

Furthermore, using the results from Proposition 5.4.1 we conclude that

$$|\hat{Y}_t^{T,t,v_1} - \hat{Y}_t^{T,t,v_2}| \leq \left(C_h + \frac{K}{\gamma} + \frac{C_v}{\gamma}\right)|v_1 - v_2|.$$

□

The proof of the asymptotic behaviour of the forward entropic risk measure is the same as the diffusion case in Theorem 10 from Chong et al. 2019, where they show that the forward entropic risk measure converges to a constant as the time horizon increases.

5.5 Conclusion

In this chapter, we introduced jumps into the ergodic BSDE with quadratic growth in the control variable. We proved, under certain conditions, that there exists a unique Markovian for a quadratic-exponential ergodic BSDE with bounded jumps. Afterwards, we derived the representation of forward entropic risk measure, which depends on the results from the ergodic BSDE. We derived the connection between the ergodic BSDEs with jumps and the PIDE. We noticed that when the stochastic factor includes jumps, the corresponding generator of the ergodic BSDE contains Y_t and consequently the translation invariance property is not satisfied.

Chapter 6

BSVIE-based dynamic capital allocations in a jump framework

6.1 Introduction

In this chapter, we study the notion of capital allocation derived from dynamic risk measures constructed using Backward Stochastic Volterra Integral Equations (BSVIEs) with jumps. We prove the differentiability of the BSVIEs with jumps. The capital allocation of the dynamic risk measure is derived and an example is given.

Yong 2007 extended the work of risk measures of BSDEs to a specific type of BSDE called backward stochastic volterra integral equations (BSVIEs). BSVIEs are more general as compared to BSDEs because they consider position processes instead of $\mathcal{F}_T-$measurable random variables (see Section 6.2 for notation). Agram 2019, studied dynamic risk measures based on BSVIEs with jumps. For more theory and applications of BSVIEs see amongst other Yong 2013 and Kromer and Overbeck 2017, and BSVIEs with jumps see Agram, Øksendal, et al. 2016, Agram, Øksendal, et al. 2018.

This chapter is arranged as follows: in Section 6.2, we outline all the relevant notations and concepts that we use throughout the Chapter. In Section 6.3, we provide the main results. Finally, we apply the results from Section 6.3 to an example and conclude. In Section 6.4, we conclude.

6.2 Preliminaries

Let $T < \infty$ be a finite time horizon. The source of randomness is modelled through a probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ generated by the one-dimensional Brownian motion W_t and an independent Poisson random measure $N(dt, d\zeta)$. Let the compensated Poisson random measure of N be defined by $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - v(d\zeta)dt$, with $v(d\zeta)dt$ denoting the Lévy measure of N on $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ so that

$$\int_{\mathbb{R}_0} \min\{1, |\zeta|^2\}\nu(d\zeta) < \infty.$$

Taking $\Delta := \{(t, s) \in [0, T]^2 : t \leq s\}$. We define the following spaces of variables or processes.

- $L^2(\mathcal{F}_T)$ is the space of $\mathcal{F}_T$-measurable processes $\psi : [0, T] \times \Omega \to \mathbb{R}$ for all $t \in [0, T]$, with norm
$$||\psi||^2_{L^2_{\mathcal{F}_T}[0,T]} = \mathbb{E}\bigg[\int_0^T |\psi(t)|^2 dt\bigg] < \infty.$$
- Denote by H^2_y the space of $\mathbb{F}$-measurable cádlág processes $Y : [0, T] \times \Omega \to \mathbb{R}$, with the norm
$$||Y||^2_{H^2_y} := \mathbb{E}\bigg[\int_0^T |Y(t)|^2 dt\bigg] < \infty.$$
- Let H^2_z be the space of $\mathbb{F}$-predictable processes $Z : \Delta \times \Omega \to \mathbb{R}$ such that
$$||Z||^2_{H^2_Z} := \mathbb{E}\bigg[\int_0^T \int_t^T |Z(t, s)|^2 ds dt\bigg],$$
where $s \mapsto Z(t, s)$ is $\mathbb{F}$-predictable on $[t, T]$.

- L^2_ν is the space of all Borel functions $\Upsilon : \mathbb{R}_0 \to \mathbb{R}$ equipped with the norm
$$||\Upsilon||^2_{L^2_\nu} := \int_{\mathbb{R}_0} \Upsilon(t,s,\zeta)^2 \nu(d\zeta) < \infty.$$

- Let H^2_ν denote the space of $\mathbb{F}$-predictable processes $\Upsilon : \Delta \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$, such that
$$\mathbb{E}\Big[\int_0^T \int_t^T \int_{\mathbb{R}_0} |\Upsilon(t,s,\zeta)|^2 \nu(d\zeta) ds dt\Big] < \infty,$$
where $s \mapsto \Upsilon(t,s,\cdot)$ is $\mathbb{F}$-predictable on $[t,T]$. The space H^2_ν has the norm
$$||\Upsilon||^2_{H_\nu} := \mathbb{E}\Big[\int_0^T \int_t^T \int_{\mathbb{R}_0} |\Upsilon(t,s,\zeta)|^2 \nu(d\zeta) ds dt\Big].$$

We consider the following BSVIE with unknowns Y, Z and Υ

$$\begin{aligned} Y(t) &= -\psi(t) + \int_t^T g(t,s,Y(s),Z(t,s),\Upsilon(t,s,\cdot))ds - \int_t^T Z(t,s)dW(s) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} \Upsilon(t,s,\zeta)\tilde{N}(ds,d\zeta), \quad t \in [0,T], \end{aligned} \tag{6.2.1}$$

where $\psi \in L^2(\mathcal{F}_T)$ is a position (wealth) process instead of a random variable and $g : \Delta \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ is the generator. From Yong 2007, we know that $\psi(\cdot)$ can represent the total wealth process of a portfolio consisting of a combination of contingency claims, some current cash flows (dividends), some positions of shares and bonds at time t. Here $\psi(\cdot)$ is a stochastic processes that is $\mathcal{F}_T$-measurable. We also consider the following Backward Stochastic Differential Equation (BSDE)

$$\begin{aligned} Y(t) &= -\xi + \int_t^T g(s,Y(s),Z(s),\Upsilon(s,\cdot))ds - \int_t^T Z(s)dW(s) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} \Upsilon(s,\zeta)\tilde{N}(ds,d\zeta), \quad t \in [0,T], \end{aligned} \tag{6.2.2}$$

where the terminal condition, ξ, is a square integrable random variable. The difference between BSDE (6.2.2) and BSVIE (6.2.1) as stated in Yong 2007 (see page 4) are as follows:

(i) The generator, g is dependent on both s and t.

(ii) The generator does depend on $Z(t,s)$ and also on $Z(s,t)$. Under the jump BSVIE, the generator will depend on both $\Upsilon(t,s,\cdot)$ and $\Upsilon(s,t,\cdot)$ (see Agram 2019).

(iii) The process $\psi(\cdot)$ is allowed to just $\mathcal{B}[0,T] \otimes \mathcal{F}_T$-measurable (not necessarily $\mathbb{F}$-adapted), where $\mathcal{B}[0,T]$ is the Borel σ-field on [0,T].

Next, we state the conditions that the generator and the position process must satisfy in order for the solution of the BSVIE (6.2.1) to exist.

(H.1) (i) The function $g : \Delta \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies the following integrability condition

$$\mathbb{E}\Big[\int_0^T \Big(\int_t^T g(t,s,0,0,0)ds\Big)^2 dt\Big] < \infty.$$

(ii) The function g satisfies the following Lipschitz condition. There exists a constant $C > 0$ such that, for all $t, s \in [0,T]$

$$\begin{aligned} |g(t,s,y,z,\upsilon(\cdot)) - g(t,s,y',z',\upsilon'(\cdot))| \;&\leq\; C\Big(|y-y'| + |z-z'| \\ &+ \Big(\int_{\mathbb{R}_0} |\upsilon(\zeta) - \upsilon'(\zeta)|^2 \nu(d\zeta)\Big)^{1/2}\Big) \end{aligned}$$

for all $(t,s) \in \Delta$, $y, y', z, z' \in \mathbb{R}$, $\upsilon(\cdot), \upsilon'(\cdot) \in L^2_\nu$.

(H.2) The terminal condition $\psi(\cdot) \in L^2(\mathcal{F}_T)$.

Under the assumptions (H.1) and (H.2), Agram, Øksendal, et al. 2016 showed that there exists a unique solution $(Y, Z, \Upsilon) \in H_y \times H_z \times H_\upsilon$ of the BSVIE (6.2.1) and the following estimate holds

$$||(Y,Z,\Upsilon)||^2_{H_y \times H_z \times H_\upsilon} \leq C\mathbb{E}\Big[|\psi(t)|^2 + \Big(\int_t^T g(t,s,0,0,0)ds\Big)^2\Big].$$

We refer to Hu and Øksendal 2019 for an explicit solution formula for a special case of linear BSVIEs with jumps.

The concept of dynamic risk measures constructed using BSVIE with jumps is introduced by Agram 2019. Define

$$\rho(t, \psi(\cdot)) = Y^{-\psi(\cdot)}(t), \qquad \text{for all} \quad t \in [0, T], \tag{6.2.3}$$

where Y is the first component to the solution of the BSVIE given by:

$$\begin{aligned} Y(t) &= -\psi(t) + \int_t^T g(t, s, Z(t,s), \Upsilon(t, s, \cdot))ds - \int_t^T Z(t,s)dW(s) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} \Upsilon(t, s, \zeta)\tilde{N}(ds, d\zeta), \quad t \in [0, T], \end{aligned} \tag{6.2.4}$$

with the terminal condition $\psi(\cdot) \in L^2(\mathcal{F}_T)$ and generator $g : \Delta \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ satisfying assumptions (H.1) and (H.2) respectively. The approach of Agram, Øksendal, et al. 2018 is motivated by the work of Yong 2007 who studied dynamic risk measures induced by BSVIE in the continuous framework. Agram 2019 (in Theorem 4.1 page 11) (also see Yong 2007 for the diffusion case), they define a dynamic risk measure $\rho : L^2(\mathcal{F}_T) \to L^2(\mathbb{F})$ of the return process $\psi(\cdot)$ induced by BSVIE with jumps if the following holds:

(i) **Convexity**: For any $\psi_1(\cdot)$, $\psi_2(\cdot) \in L^2(\mathcal{F}_T)$ and $\lambda \in [0, 1]$

$$\rho(t, \lambda\psi_1(\cdot) + (1-\lambda)\psi_2(\cdot)) \leq \lambda\rho(t, \psi_1(\cdot)) + (1-\lambda)\rho(t, \psi_2(\cdot)).$$

(ii) **Monotonicity**: If $\psi_1(\cdot) \leq \psi_2(\cdot)$, then $\rho(t, \psi_1(\cdot)) \geq \rho(t, \psi_2(\cdot))$.

(iii) **Translation invariance**: If $\psi(\cdot) \in L^2(\mathcal{F}_T)$ and for any $c \in \mathbb{R}$, then

$$\rho(t, \psi(\cdot) + c) = \rho(t, \psi(\cdot)) - c, \quad \text{a.s.} \quad t \in [0, T].$$

(iv) **Past independence**: If $\psi_1(\cdot)$, $\psi_2(\cdot) \in L^2(\mathcal{F}_T)$ and $\psi_1(s) = \psi_2(s)$ for all $s \in [t, T]$ then $\rho(t, \psi_1(\cdot)) = \rho(t, \psi_2(\cdot))$.

Convexity demonstrates that the dynamic risk measure can increases in a nonlinear way as the position process is multiplied by a factor, due to liquidity risk of the large position. Monotonicity states that the capital required to support a large position process is smaller than the capital required to support a small position. Translation invariance means that if you increase a position by some constant then the capital required to maintain the new position will decrease by that constant. Lastly, Past independence means that the dynamic risk measure is only dependent on the current and future information of the position process.

Now consider $\psi(t)$ to be a portfolio process with the risk $\rho(\psi(t))$ and $\phi(t)$ the sub-portfolio process such that $\sum_{i=1}^{n}\phi_i(t) = \psi(t)$. Then the capital allocation problem is to determine the risk contribution of sub-portfolio process $\phi_i(t)$ to risk of $\psi(t)$. If the risk measure, ρ is Gâteaux-differentiable, then the gradient allocation is given by

$$\frac{d}{du}\rho(t;\psi(t)+u\phi_i(t))\bigg|_{u=0} := \lim_{u\to 0}\frac{\rho(\psi(t)+u\phi_i(t))}{u} = \nabla Y(t), \tag{6.2.5}$$

The above expression represents the risk capital that the sub-portfolio process $\phi(\cdot)$ contributes to the portfolio process $\psi(\cdot)$. It describes the continuous-time dynamic capital that should be allocated to the sub-portfolio process $\phi(\cdot)$ (see Kromer and Overbeck 2017).

In the next section we study dynamic capital allocation where the risk measure is constructed using BSVIE with jumps.

6.3 Representation of dynamic risk capital allocations

For the capital allocation, we need the differentiability results of the BSVIE with jumps. The differentiability of the BSVIE with jumps will allow us to define the gradient allocation. We consider the following BSVIE

$$Y^u(t) = -(\psi(t)+u\phi(t)) + \int_t^T g(t,s,u,Y^u(s),Z^u(t,s),\Upsilon^u(t,s,\zeta))ds$$

$$-\int_t^T Z^u(t,s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon^u(t,s,\zeta)\tilde{N}(ds,d\zeta), \qquad (6.3.1)$$

for any $u \in \mathbb{R}$.

6.3.1 Differentiability of BSVIEs with jumps

To derive the differentiability results of BSVIE (6.3.1) we impose the following set of assumptions.

(D1) The terminal process $\psi : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is continuous, $\psi(t,u) \in L^2(\mathcal{F}_T)$, the map $u \mapsto \psi(t,u)$ is differentiable for all $t \in [0,T]$ and the derivative is given by $\nabla\psi(t,x) \in L^2(\mathcal{F}_T)$.

(D2) The generator $g : \Delta \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ is $\mathcal{B}(\Delta \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}) \otimes \mathcal{F}_T$ measurable such that $s \mapsto g(t,s,u,y,z,\upsilon)$ is $\mathbb{F}$-progressively measurable for all $(t,u,z,\upsilon) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with the following integrability condition holds

$$\mathbb{E}\left[\int_0^T \left(\int_t^T |g(t,s,u,0,0,0)|ds\right)^2 dt\right] \leq \infty \qquad \forall u \in \mathbb{R}.$$

Furthermore, the generator g is such that the map $(t,s,u,y,z,\upsilon) \mapsto g(t,s,u,y,z,\upsilon)$ is continuously differentiable. In addition g is Lipschitz continuous that is there exists a deterministic function $L : \Delta \to [0,\infty)$ satisfying

$$\sup_{t\in[0,T]} \int_t^T L(t,s)^{2+\epsilon} ds < \infty$$

for some $\epsilon > 0$ such that

$$|g(t,s,u,y,z,\upsilon) - g(t,s,u,y',z',\upsilon')| \leq L(t,s)(|y-y'| + ||z-'z|| + |\upsilon-\upsilon'|),$$

for all $(t,s) \in \Delta$, $(y,z,\upsilon),(y',z',\upsilon') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Moreover, the map $u \mapsto g(t,s,u,y,z,\upsilon)$ is differentiable and for all $(u,u') \in \mathbb{R}$

$$\lim_{u\to u'} \mathbb{E}\left[\int_0^T \left(\int_t^T \left|\frac{\partial}{\partial u} g(t,s,u',Y^{u'},Z^{u'}(s),\Upsilon^{u'}(s,\zeta)) - \frac{\partial}{\partial u} g(t,s,u,Y^u,Z^u(s),\Upsilon^u(s,\zeta))\right| ds\right)^2 dt\right] = 0. \qquad (6.3.2)$$

(D3) Let $\delta : \Delta \times \Omega \times \mathbb{R}_0 \to \mathbb{R}$ be a measurable process satisfying

$$\int_{\mathbb{R}_0} |\delta(t,s,\zeta)|^2 \nu(d\zeta) < \infty,$$

for any $(t,s) \in \Delta$.

The results of the differentiability of BSVIE (6.3.1) under the above assumptions is given by the following corollary. Our approach to the proof of the corollary closely follows that of Overbeck and Röder 2018, (Theorem 3 page 21), differentiability results of the path-dependent BSVIEs with jumps.

Proposition 6.3.1. *Let (D1) and (D2) hold, then the function*

$$\mathbb{R} \to H_y^2 \times \mathcal{H}_z^2 \times H_\nu^2, \qquad u \mapsto (Y^u(\cdot), Z^u(\cdot,\cdot), \Upsilon^u(\cdot,\cdot,\cdot)),$$

is differentiable and the derivative is a unique adapted solution to the BSVIE

$$\begin{aligned}
\nabla Y^u(t) &= \nabla\psi(t) - \int_t^T \nabla Z^u(t,s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \nabla\Upsilon^u(t,s,\zeta)\tilde{N}(ds,d\zeta) \\
&+ \int_t^T \partial_u g\Big(t,s,Y^u(s),Z^u(t,s),\int_{\mathbb{R}_0} \Upsilon^u(t,s,\zeta)\delta(t,s,\zeta)\nu(d\zeta)\Big)\nabla Y^u(s)ds \\
&+ \int_t^T \partial_z g\Big(t,s,Y^u(s),Z^u(t,s),\int_{\mathbb{R}_0} \Upsilon^u(t,s,\zeta)\delta(t,s,\zeta)\nu(d\zeta)\Big)\nabla Z^u(t,s)ds \\
&+ \int_t^T \int_{\mathbb{R}_0} \partial_\nu g\Big(t,s,Y^u(s),Z^u(t,s),\int_{\mathbb{R}_0} \Upsilon^u(t,s,\zeta)\delta(t,s,\zeta)\nu(d\zeta)\Big) \\
&\times\nabla\Upsilon^u(t,s,\zeta)\nu(d\zeta)ds. \qquad (6.3.3)
\end{aligned}$$

Proof. We define for all $(t,s) \in \Delta$, $u \in \mathbb{R}$ and $h \in \mathbb{R}_0$ the following

$$\mathcal{Y}^h(t) = \frac{1}{h}(Y^{u+h}(t) - Y^u(t)), \qquad \mathcal{Z}^h(t,s) = \frac{1}{h}(Z^{u+h}(t,s) - Z^u(t,s)),$$

$$\mathcal{U}^h(t,s,\zeta) = \frac{1}{h}(\Upsilon^{u+h}(t,s,\zeta) - \Upsilon^u(t,s,\zeta)) \quad \text{and} \quad \Psi^h(t) = \frac{1}{h}(\psi^{u+h}(t) - \psi^u(t)).$$

We then have

$$\mathcal{Y}^h(t) = \Psi^h(t) - \int_t^T \mathcal{Z}^h(t)dW(s) - \int_t^T \int_{\mathbb{R}_0} \mathcal{U}^h(t,s,\zeta)\tilde{N}(ds,d\zeta)$$

$$+\int_t^T \frac{1}{h}\Big(g\big(t,s,u+h,Y^{u+h}(s),Z^{u+h}(t,s),\int_{\mathbb{R}_0}\Upsilon^{u+h}(t,s,\zeta)\delta(t,s,\zeta)\nu(d\zeta)\big)$$
$$-g\big(t,s,u,Y^{u}(s),Z^{u}(t,s),\int_{\mathbb{R}_0}\Upsilon^{u}(t,s,\zeta)\delta(t,s,\zeta)\nu(d\zeta)\big)\Big)ds. \tag{6.3.4}$$

Define a mapping $l_{u,h}:[0,1]\to\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ by

$$\begin{aligned} l_{u,h}(\theta) &= \Big(u+\theta h, Y^u(t)+\theta h\mathcal{Y}^h(t), Z^u(t,s)+\theta h\mathcal{Z}^h(t,s),\\ &\int_{\mathbb{R}_0}\big(\Upsilon(t,s,\zeta)+\theta h\mathcal{U}^h(t,s,\zeta)\delta(t,s,\zeta)\nu(d\zeta)\big)\Big) \end{aligned} \tag{6.3.5}$$

From the above notation if follows that

$$\frac{1}{h}\frac{\partial}{\partial\theta}l_{u,h}(\theta)=\Big(1,\mathcal{Y}^h(t),\mathcal{Z}^h(t,s),\int_{\mathbb{R}_0}\mathcal{U}^h(t,s,\zeta)\delta(t,s,\zeta)\nu(d\zeta)\Big)$$

Equation (6.3.4) can be rewritten as

$$\begin{aligned} \mathcal{Y}^h(t) &= \Psi^h(t)-\int_t^T\mathcal{Z}^h(t)dW(s)-\int_t^T\int_{\mathbb{R}_0}\mathcal{U}^h(t,s,\zeta)\tilde{N}(ds,d\zeta)\\ &+\int_t^T\Big(m^{s,t}_{u,h}\big(Y^h(s),Z^h(t,s),\int_{\mathbb{R}_0}\Upsilon^h(t,s,\zeta)\delta(t,s,\zeta)\nu(d\zeta)\big)\\ &+A_{u,h}(t,s)\Big)ds, \end{aligned} \tag{6.3.6}$$

where $m^{s,t}_{u,h}:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a linear function defined by

$$m^{s,t}_{u,h}=B_{u,h}(t,s)y+C_{u,h}(t,s)z+D_{u,h}(t,s)\upsilon,$$

with

$$A_{u,h}(t,s)=\int_0^1\frac{\partial}{\partial u}g(l_{u,h}(\theta))d\theta,$$
$$B_{u,h}(t,s)=\int_0^1\frac{\partial}{\partial y}g(l_{u,h}(\theta))d\theta,$$
$$C_{u,h}(t,s)=\int_0^1\frac{\partial}{\partial z}g(l_{u,h}(\theta))d\theta,$$

$$D_{u,h}(t,s) = \int_0^1 \frac{\partial}{\partial v} g(l_{u,h}(\theta))d\theta.$$

From Hu and Øksendal 2019, there exists a unique solution $(\mathcal{Y}, \mathcal{Z}, \mathcal{U})$ to the linear BSVIEs (6.3.6). It then follows that

$$\begin{aligned} \mathbb{E}\Big[\int_0^T |\mathcal{Y}^h(t) - \mathcal{Y}^{h'}(t)|^2 dt\Big] \;&\leq\; C\Big(\mathbb{E}\Big[\int_0^T |\Psi^h(t) - \Psi^{h'}(t)|^2 dt\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T \Big(\int_t^T \big(A_{u,h}(t,s) - A_{u,h'}(t,s)\big)^2 ds\Big) dt\Big]\Big). \end{aligned} \tag{6.3.7}$$

Consequently, for $h, h' \in \mathbb{R}_0$ and (D1), we have

$$\lim_{h \to h'} \mathbb{E}\Big[\int_0^T |\Psi^h(t) - \Psi^{h'}(t)|^2 dt\Big] = 0,$$

and (D2) yields

$$\lim_{h \to h'} \mathbb{E}\Big[\int_0^T \Big(\int_t^T \big(A_{u,h}(t,s) - A_{u,h'}(t,s)\big)^2 ds\Big) dt\Big] = 0.$$

Hence, if follows

$$\lim_{h \to h'} \mathbb{E}\Big[\int_0^T |\mathcal{Y}^h(t) - \mathcal{Y}^{h'}(t)|^2 dt\Big] = 0,$$

From the same argument, it follows that

$$\lim_{h \to h'} \mathbb{E}\Big[\int_0^T \int_t^T |\mathcal{Z}^h(t,s) - \mathcal{Z}^{h'}(t,s)|^2 ds dt\Big] = 0,$$

and

$$\lim_{h \to h'} \mathbb{E}\Big[\int_0^T \int_t^T \int_{\mathbb{R}} |\mathcal{U}^h(t,s,\zeta) - \mathcal{U}^{h'}(t,s,\zeta)|^2 \nu(d\zeta) ds dt\Big] = 0.$$

Therefore, we conclude that

$$\lim_{h \to h'} ||(\mathcal{Y}^h, \mathcal{Z}^h, \mathcal{U}^h) - (\mathcal{Y}^{h'}, \mathcal{Z}^{h'}, \mathcal{U}^{h'})||^2_{H_y^2 \times H_z^2 \times H_v^2} = 0.$$

Considering that H_y^2, H_z^2 and H_v^2 are Banach spaces the sequences $\mathcal{Y}^h$ converges to $\frac{\partial}{\partial u} Y^u(t)$, $\mathcal{Z}^h$ converges to $\frac{\partial}{\partial u} Z^u(t,s)$ and $\mp^h$ converges to $\frac{\partial}{\partial u} \Upsilon^u(t,s,\zeta)$. Under the above term by term convergence, $(\frac{\partial}{\partial u} Y^u(t), \frac{\partial}{\partial u} Z^u(t,s), \frac{\partial}{\partial u} \Upsilon^u(t,s,\zeta))$ is a solution to the BSVIE with jumps (6.3.3). □

6.3.2 Gradient capital allocation

In this section we derive the dynamic capital allocation and it is given as the first component of solution to the BSVIE with jumps.

Corollary 6.3.2. *The dynamic capital allocation corresponding the dynamic risk measure $\rho(t, \psi(t))$ in (6.2.3) exists and is given $\nabla Y(t)$, which is a unique solution to Equation (6.3.3).*

Proof. We can deduce that $\rho(t; \psi(t) + u\phi(t))$ is the dynamic risk measure for the position process $\psi(t) + u\phi(t)$. It follows from Equations (6.2.3) and (6.2.4) that

$$\rho(t; \psi(t) + u\phi(t)) = Y^u(t) \quad \text{for all} \quad t \geq 0$$

where the process $Y(t)$ is the first component of the unique solution of following BSVIE

$$\begin{aligned} Y^u(t) &= -(\psi(t) + u\phi(t)) + \int_t^T g(t, s, u, Y^u(s), Z^u(t, s), \Upsilon^u(t, s, \zeta))ds \\ &\quad - \int_t^T Z^u(t, s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon^u(t, s, \zeta)\tilde{N}(ds, d\zeta), \end{aligned} \tag{6.3.8}$$

for any $u \in \mathbb{R}$. It then follows from Corollary (6.3.1) that the dynamic gradient allocation defined as

$$\frac{d}{du}\rho(t; \psi(\cdot) + u\phi(\cdot))\bigg|_{u=0} = \nabla Y(t),$$

exists, with $\nabla Y(t)$ being the first component of the unique solution to Equation (6.3.3). ☐

If the jump term is removed the results of the Corollary (6.3.2) will be the same as Kromer and Overbeck 2017.

Example 6.3.1. We consider the following BSVIE

$$Y(t) = -\psi(t) - \int_t^T Z(t, s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon(t, s, \zeta)\tilde{N}(ds, d\zeta) + \int_t^T \Big(Y(s)r(s)$$

$$+ \quad g_0(t,s)\sqrt{1+|Z(t,s)|^2+\int_{\mathbb{R}_0}\left|\Upsilon(t,s,\zeta)\right|^2\delta(t,s,\zeta)\nu(d\zeta)}\Bigg)ds, \qquad (6.3.9)$$

where the generator is given by

$$g(t,s,y,z,v) = Y(s)r(s)+g_0(t,s)\sqrt{1+|Z(t,s)|^2+\int_{\mathbb{R}_0}\left|\Upsilon(t,s,\zeta)\right|^2\delta(t,s,\zeta)\nu(d\zeta)\right|^2}.$$

According to Yong 2007, the choice of the generator depends on the agent's appetite towards risk, the greater the generator the more conservative the agent is towards the risk (see also Delong 2013 page 240 on the economic interpretation of the generator). The generator satisfies condition (H.1) and also condition (D2). By taking $r(t)=0$, then the generator is independent of the Y process. Hence, the dynamic risk measure $\rho(\cdot)$, is given by the first component of the solution to the BSVIE (6.3.9). We apply Corollary (6.3.1) to obtain the corresponding dynamic gradient allocation of $\psi(\cdot)$, which is given by ∇Y, the first component of the solution to the following BSVIE

$$\begin{aligned}
\nabla Y^u(t) &= \nabla\psi(t) - \int_t^T \nabla Z^u(t,s)dW(s) - \int_t^T\int_{\mathbb{R}_0}\nabla\Upsilon^u(t,s,\zeta)\tilde{N}(ds,d\zeta) \\
&\quad + \int_t^T r(s)\nabla Y^u(s)ds \\
&\quad + \int_t^T g_0(t,s)\frac{Z(s,t)}{\sqrt{1+|Z(t,s)|^2+\int_{\mathbb{R}_0}|\Upsilon(t,s,\zeta)|^2\delta(t,s,\zeta)\nu(d\zeta)}}\nabla Z^u(t,s)ds \\
&+ \int_t^T\int_{\mathbb{R}_0} g_0(t,s)\frac{\Upsilon(t,s,\zeta)\delta(t,s,\zeta)\nu(d\zeta)}{\sqrt{1+|Z(t,s)|^2+\int_{\mathbb{R}_0}|\Upsilon(t,s,\zeta)|^2\delta(t,s,\zeta)\nu(d\zeta)}} \\
&\quad \times\nabla\Upsilon^u(t,s,\zeta)\nu(d\zeta)ds.
\end{aligned} \qquad (6.3.10)$$

In general, the explicit solution of BSVIEs is hard to get. We would have to use numerical methods to find the solution of the above BSVIE (6.3.10), this is a subject of further research.

6.4 Conclusion

In this chapter, we studied the capital allocation problem for dynamic risk measures constructed using the solution of BSVIEs with jumps. We derived the differentiability results of BSVIEs with jumps. Hence, the dynamic capital allocation is determined as the first component of the solution to the BSVIE derivative. To obtain an approximation of solution to the BSVIE derivative, we would need to employ numerical methods. This can be considered as a further research topic.

Chapter 7

Conclusion

Summary and future research

In this research, we established the representation of dynamic risk measures and capital allocation defined using BSDEs and their generalization (BSVIEs and ergodic-BSDEs). Our framework was under the diffusion and jump-diffusion case.

In addressing the main research question, we were able to determine the representation of the capital allocation where the underlying risk measure is:

- represented using BSDE(s) with quadratic-exponential growth in the control processes, that allowed us to determine the risk associated with a final random variable investment outcome at any time during the investment period;
- represented using ergodic-BSDE(s) that allowed us to determine the risk associated with the final investment outcome where the final investment payout date is flexible;
- determine from the BSVIE with a generator that depends on (s, t) and the control processes rely on both s and t for $s \in [t, T]$, the represen-

tation of a dynamic risk measure when the terminal value is a position (wealth) process.

The capital allocation was determined using the gradient allocation method, (i.e. Gateaux Derivative and Malliavin directional derivative), as it allows one to differentiate the risk measure of the portfolio in the direction of the sub-portfolio. This is critical in ensuring that the overall portfolio risk has been fully allocated to each sub-portfolio.

Our work determined the representation of capital allocation under the full-allocation property, and we did not consider sub-allocation property (introduced by Centrone and Rosazza (2018)). For further research work, we will consider sub-allocation. Another area of interest is in determining an optimal BSDE-based capital allocation where we formulate and solve an optimization problem. (Dhaene et al. (2012) investigated optimal capital allocation based on well-known risk measures (such as conditional tail expectation) but not on BSDE-based risk measures). The practical implementation of the results herein is a possible future interest.

Appendix A

Axioms of Capital Allocation

We provide the reader with some of the capital allocation axioms in literature (see Kalkbrener (2005), Denault (2001) and Centrone and Rosazza (2018)).

We recall the three axioms for capital allocation proposed by Kalkbrener (2005) and we will use similar notation from therein. Let $\Lambda : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ denote the capital allocation, then $\Lambda(X_i, X)$ defines the risk capital allocation of X_i whenever X_i is considered a subportfolio of portfolio X. We say that Λ is a capital allocation associated with the risk measure ρ if it satisfies $\Lambda(X, X) = \rho(X)$ this means the capital allocated to X, is its risk $\rho(X)$, whenever X is a portfolio that contains itself.

Definition A.1. Let $\rho : \mathcal{X} \to \mathbb{R}$. A capital allocation (with respect to ρ) is a function $\Lambda : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that for every $X \in \mathcal{X}$

$$\Lambda(X, X) = \rho(X).$$

The capital allocation Λ is called

(i) *linear:* $\Lambda(aX+bY, Z) = a\Lambda(X, Z)+b\Lambda(Y, Z) \quad \forall a, b \in \mathbb{R}, X, Y, Z \in \mathcal{X}$.

(ii) diversifying: $\Lambda(X, Y) \leq \Lambda(X, X) \quad \forall X, Y \in \mathcal{X}$.

(iii) continuous at Y*:* $\lim_{\epsilon \to 0} \Lambda(X, Y + \epsilon X) = \Lambda(X, X) \quad \forall X, Y \in \mathcal{X}$.

The linear axiom states that the risk capital allocated to the subportfolios should add up to the overall portfolio risk, that is, if $Z = aX + bY$, then

$$\rho(Z) = \Lambda(Z, Z) = a\Lambda(X, Z) + b\Lambda(Y, Z).$$

Diversifying axiom says that the risk capital of a portfolio allocated to the sub-portfolio should not exceed the risk capital of a portfolio allocated to itself. The last axiom, says that a small change to the portfolio has a minimal effect on the risk capital allocated to the subportfolio.

The next theorem we recall from Kalkbrener (2005). The risk capital allocated is uniquely determined by the directional derivative of the underlying risk measure at the portfolio in the direction of sub-portfolio Kalkbrener (2005).

Theorem A.1. Let Λ be a linear, diversifying capital allocation with respect to ρ. If Λ is continuous at $Y \in \mathcal{X}$ then for all $X \in \mathcal{X}$

$$\Lambda(X, Y) = \lim_{\epsilon \to 0} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon}.$$

Proof. See Kalkbrener (2005) (Theorem 3.1). □

We recall the definition of Aumann-Shapley capital allocation method proposed by Tsanakas (2009) for convex risk measures.

Definition A.2. For a aggregate portfolio $X \in \mathcal{X}$ and a risk measure ρ that is Gateaux differentiable at βX, $\beta \in [0, 1]$, the Aumann-Shapley capital allocation Λ^{AS} is define by

$$\Lambda^{AS}(Y, X) = \int_0^1 \Lambda(Y, \beta X) d\beta.$$

Appendix B

Malliavin Calculus

In this section, we recall the two definition of the derivatives of F presented in Di Nunno et al. (2009). The first definition is the stochastic derivative $D_t F$ of $F \in \mathcal{D}_{1,2}$.

Definition B.1. Assume that $F : \Omega \to \mathbb{R}$ has a directional derivative in all directions of η of the form $\eta \in H$ in the strong sense, that is

$$D_\eta F(\omega) := \lim_{\epsilon \to 0} \frac{F(\omega + \epsilon\eta) - F(\omega)}{\epsilon},$$

exists in $L^2(P)$. Assume in addition that there exists $\phi(t,\omega) \in L^2(P \times dt)$ such that

$$D_\eta F(\omega) = \int_0^T \psi(t,\omega)(t)dt \qquad for\ all\ \eta \in H.$$

Then we say that F is differentiable and we set

$$D_t F(\omega) := \psi(t,w).$$

We call $D.F \in L^2(P \times dt)$ the stochastic derivative of F. The set of all differentiable random variables is denoted by $\mathcal{D}_{1,2}$.

The set of $\eta \in \Omega$, that is written in the above form for some $h \in L^2([0,T])$, is called the $Cameron - Martin\ space$ and it is denoted by H (Di Nunno et al. (2009)).

The second definition provides the Malliavin derivative $D_t F$ of $F \in D_{1,2}$.

Definition B.1. Let $F \in D_{1,2}$ so that there exists $\{F_n\}_{n=1}^{\infty} \subset \mathbb{P}$ such that

$$F_n \to F \qquad in\ L^2(P)$$

and $\{D_t F_n\}_{n=1}^{\infty}$ is convergent in $L^2(P \times dt)$. Then we define

$$D_t F = \lim_{n\to\infty} D_t F_n \qquad in\ L^2(P \times dt)$$

and

$$D_\eta F = \int_0^T D_t F \cdot h(t) dt$$

for all $\eta(t) = \int_0^t h(s)ds \in H$, with $h \in L^2([0,T])$. We call $D_t F$ the Malliavin derivative of F.

Di Nunno et al. (2009) shows in Lemma A.18 page 358 that if $F \in \mathcal{D}_{1,2} \cap D_{1,2}$, then the two derivatives coincide. In view of this, we use the same symbol $D_t F$ for the derivative and $D_\eta F$ for the directional derivative of all elements $F \in \mathcal{D}_{1,2} \cap D_{1,2}$.

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