Stokes equations under nonlinear slip boundary conditions coupled with the heat equation: A priori error analysis

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ABSTRACT

In this work, we consider the heat equation coupled with Stokes equations under threshold type boundary condition. The conditions for existence and uniqueness of the weak solution are made clear. Next we formulate the finite element problem, recall the conditions of its solvability, and study its convergence by making use of Babuska–Brezzi's conditions for mixed problems. Third we formulate an Uzawa's type iterative algorithm that separates the fluid from heat conduction, study its feasibility, and convergence. Finally the theoretical findings are validated by numerical simulations.

1. INTRODUCTION

In this work, we study the convergence of the finite element solution associated to the system of equations modeled by

$$\begin{aligned} -2\operatorname{div}\nu(\theta)D\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad &\text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} = 0 \quad &\text{in } \Omega,\\ -\kappa\Delta\theta + (\boldsymbol{u}\cdot\nabla)\theta = b \quad &\text{in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^2$, with a Lipschitz-continuous boundary $\partial\Omega$ divided in two parts S and $\Gamma = \partial\Omega \setminus \overline{S}$ with $\overline{\Gamma} \cap \overline{S} = \emptyset$. A more complex model has been derived and studied in 1, 2. In (1.1), $\boldsymbol{u}$ is the velocity and θ the temperature, while p stands for the pressure. b is the external heat source, while $\boldsymbol{f}$ is the external body force per unit volume acting on the fluid. κ is positive and stands for the thermal conductivity. The first equation in (1.1) represents the balance of forces in the system, while the second equation is the incompressibility of the fluid. The third equation is the heat exchange in the system. The force within the fluid is the Cauchy stress tensor $\boldsymbol{T}$ given by the relation

$$\boldsymbol{T} = 2\nu(\theta)D\boldsymbol{u} - p\boldsymbol{I},$$

with ν, positive and representing the viscosity of the fluid and depending on temperature 3. $\boldsymbol{I}$ is the identity tensor, and $D\boldsymbol{u}$ is the symmetric part of the velocity gradient which is defined by

$$2D\boldsymbol{u} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T.$$

The unknowns in the system (1.1) are the velocity $\boldsymbol{u}$, the pressure p, and the temperature of the fluid. From the modeling point of view the first two equations of (1.1) stand for the Boussinesq approximation in a steady approximation. We recall that the Boussinesq approximation means that the density of the fluid is everywhere constant except for the buoyancy term which of course appears in the right-hand side of the first equation. The coupling in (1.1) is represented through; the convective term $(\boldsymbol{u} \cdot \nabla)\theta$ and the expression $\nu(\theta)D\boldsymbol{u}$. The system (1.1) is a simplified model for a number of incompressible fluids when some variations are observed in the temperature and we refer to 4 for one of the first analyses of this simplification. The system of Equation (1.1) is supplemented by the boundary conditions on the velocity and temperature. For that purpose, we assume that

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma \quad \text{and} \quad \theta = \theta_0 \quad \text{on } \partial\Omega, \tag{1.2}$$

θ_0 being given, and nonnegative. On the other part of the boundary S, the velocity is decomposed following its normal and tangential part; that is

$$\boldsymbol{u} = u_n + u_\tau = (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} + (\boldsymbol{u} \cdot \boldsymbol{\tau})\boldsymbol{\tau},$$

where $\boldsymbol{n}$ is the normal outward unit vector to S and $\boldsymbol{\tau}$ is the tangent vector orthogonal to $\boldsymbol{n}$. We first assume the impermeability condition

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } S. \tag{1.3}$$

Just like the velocity, the traction $\boldsymbol{Tn}$ on S is decomposed following its normal and tangential part; that is

$$\begin{aligned} \boldsymbol{Tn} &= (\boldsymbol{Tn} \cdot \boldsymbol{n})\boldsymbol{n} + (\boldsymbol{Tn} \cdot \boldsymbol{\tau})\boldsymbol{\tau} \\ &= (-p + 2\nu\boldsymbol{n} \cdot \boldsymbol{D}(\boldsymbol{u})\boldsymbol{n})\boldsymbol{n} + 2\nu(\boldsymbol{\tau} \cdot \boldsymbol{D}(\boldsymbol{u})\boldsymbol{n})\boldsymbol{\tau} \\ &= (\boldsymbol{Tn})_n + (\boldsymbol{Tn})_\tau. \end{aligned}$$

Let $g : S \to (0, \infty)$ be a nonnegative function called threshold slip or barrier function, the nonlinear slip boundary condition we consider in this work was presented in Leroux 5 and reads as follows:

$$\left.\begin{aligned} &|(\boldsymbol{Tn})_\tau| \le g \Rightarrow \boldsymbol{u}_\tau = \boldsymbol{0}, \\ &|(\boldsymbol{Tn})_\tau| > g \Rightarrow \boldsymbol{u}_\tau \ne \boldsymbol{0}, -(\boldsymbol{Tn})_\tau = g\frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} \end{aligned}\right\} \quad \text{on} \quad S,$$

where $|\boldsymbol{v}|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$ is the Euclidean norm. It can be shown following 6 that the nonlinear slip boundary condition is equivalent to

$$-(\boldsymbol{Tn})_\tau \in g\partial|\boldsymbol{u}_\tau| \qquad \text{on} \quad S, \tag{1.4}$$

where $\partial\,|\cdot|$ is the subdifferential of the real-valued function $|\cdot|$. We recall that if $\mathscr{X}$ is a Hilbert space with scalar product $(\cdot,\cdot)$, and for $x_0 \in \mathscr{X}$, and $y \in \mathscr{X}'$,

$$y \in \partial\Psi(x_0) \text{ means that } \Psi(x) - \Psi(x_0) \geq (y, x - x_0) \quad \forall x \in \mathscr{X}.$$

At this point it is important to note that the motion of a fluid under nonlinear slip boundary condition had been formulated first by Fujita 7, in which slip occurs if $|(\boldsymbol{Tn})_\tau| = g$ and no slip if $|(\boldsymbol{Tn})_\tau| < g$. For the mathematical investigations of (1.1)–(1.4) we assume that $\nu(\cdot)$ is a bounded continuous function defined on $(0, \infty)$ satisfying for some ν_0, ν_1, ν_2

$$\nu \in \mathscr{C}^1(\boldsymbol{R}^+) \quad \text{and for} \quad s \in \boldsymbol{R}^+, \quad 0 < \nu_0 \leq \nu(s) \leq \nu_1 \text{ and } |\nu'(s)| \leq \nu_2. \tag{1.5}$$

It is worth noting that in 8, Navier–Stokes equations coupled with the heat equation under Dirichlet boundary condition are thoroughly analyzed, and the following had been established; construction of a unique weak solution for the continuous and discrete problems, analysis of the convergence of the approximate solution and finally numerical simulations that exhibit qualitative properties of the discretization are highlighted. We have not considered the Navier–Stokes equations in our work because its nonlinearity is similar to the coupling $(\boldsymbol{u}\cdot\nabla)\theta$. Hence from the analytical perspectives the difficulties are the same. Thus our work combined the difficulties of the "Navier–Stokes equations" and the nonclassical boundary conditions (1.4) which is responsible of the inequality symbol in the weak formulation of the problem.

Many research works have dealt with the analysis and computations of fluid flow with Tresca's boundary condition (see 9-13 just to cite a few). But to the best of our knowledge, the researchers in numerical analysis have not yet considered a heat flow driven by nonlinear slip boundary condition.

The literature of heat convection in a liquid medium whose motion is described by the Navier–Stokes or Darcy equations coupled with the heat equation under Dirichlet boundary condition is rich and we refer the reader among others to 8, 14-16.

We address two interesting issues in this work. First, we construct the unique weak solution of the continuous problem and its finite element counterpart. Next, we discuss the convergence of the finite element formulation associated to the weak formulation. In fact we show that the convergence is dominated by the interpolation error for $\boldsymbol{u}_\tau$ on the slip zone S. This result is not a surprise since we are dealing with variational inequality of second type and similar result for different applications had been derived in 17. The technique of proof uses Babuska–Brezzi's tricks for mixed methods 18, 19, even though it is worth mentioning at this juncture that the presence of the temperature equation and the related nonlinear coupling terms bring more complications in the analysis. The second theoretical question we address in this work is the convergence of the algorithm we formulate for the numerical realization of the finite element approximation. Although the algorithm is a Uzawa's type iterative scheme 17, its convergence analysis is complicated due to the nonlinearities in the model. We use the energy method to derive the conditions under which the algorithm converges. The third contribution in this work is the numerical experiments exhibited. Indeed, we demonstrate by means of simulations the reliability of the algorithm (in fact its discrete analog). A companion paper of this work has been accepted for publication in 20, and to the best of our knowledge these are the first contributions toward the mathematical understanding of fluid flow under nonlinear slip boundary condition coupled with the heat. The rest of the paper is organized as follows:

- Section 2 is concerned with the weak formulation, the construction of weak solution, and its finite element approximation.
- Section 3 is devoted to the convergence analysis of the finite element discretization.
- Section 4 is devoted to the formulation of the iterative scheme and its convergence analysis.
- Section 5 is devoted to numerical simulations, discussions, and conclusions.

2. MATHEMATICAL SETTING AND FINITE ELEMENT FORMULATION

2.1 Variational formulation

To write the system (1.1)–(1.4) in a variational form, we need some preliminaries. We adopt the standard definitions 21 for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{H^s(D)}$, and semi-norms $|\cdot|_{H^s(D)}$ for $s \geq 0$, and their subspaces $H_0^s(D)$. For each $s \geq 0$, $H^{-s}(D)$ denotes the dual space of $H_0^s(D)$. The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and inner product are denoted as $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. If $D = \Omega$, we drop D.

Throughout this work, boldface characters denote vector quantities, and $\boldsymbol{H}^1(\Omega) = H^1(\Omega)^2$ and $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^2$.

The following spaces are important in the analysis of (1.1)–(1.5)

$$\begin{aligned} \boldsymbol{V} &= \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \quad : \quad \boldsymbol{v}|_\Gamma = 0, \quad \boldsymbol{v}\cdot\boldsymbol{n}|_S = 0\}, \\ M &= \{q \in L^2(\Omega) \quad : \quad (q,1) = 0\}. \end{aligned} \tag{2.1.}$$

We introduce the following functionals that will be used to write the weak form on the problem in abstract setting.

$$\begin{aligned}
a_1 : \boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega) &\to \mathbb{R} \\
(\boldsymbol{v},\boldsymbol{u}) &\to a_1(\theta : \boldsymbol{v},\boldsymbol{u}) = 2\int_\Omega \nu(\theta) D\boldsymbol{v} : D\boldsymbol{u}\, dx, \\
a_2 : H^1(\Omega)\times H^1(\Omega) &\to \mathbb{R} \\
(\theta,\rho) &\to a_2(\theta,\rho) = \kappa\int_\Omega \nabla\theta\cdot\nabla\rho\, dx \\
b : \boldsymbol{H}^1(\Omega)\times M &\to \mathbb{R} \\
(\boldsymbol{v},q) &\to b(\boldsymbol{v},q) = -\int_\Omega q\,\mathrm{div}\boldsymbol{v}\, dx, \\
j : \boldsymbol{H}^1(\Omega) &\to \mathbb{R} \\
\boldsymbol{v} &\to j(\boldsymbol{v}) = \int_S g|\boldsymbol{v}_\tau|\, d\sigma,
\end{aligned}$$

$$
\begin{aligned}
d: \boldsymbol{H}^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) &\to \mathbb{R} \\
(\boldsymbol{v}, \theta, \rho) &\to d(\boldsymbol{v}, \theta, \rho) = \int_\Omega (\boldsymbol{v} \cdot \nabla)\theta \rho dx,
\end{aligned}
$$

$d\sigma$ being the measure on the surface S. We consider the variational problem: For $\theta_0 \in H^{1/2}(\partial\Omega)$, $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ and $b \in H^{-1}(\Omega)$

$$
\begin{cases}
\text{Find} \quad (\boldsymbol{u}, p, \theta) \in \boldsymbol{V} \times M \times H^1(\Omega), \quad \text{such that} \\
\theta = \theta_0 \quad \text{on} \quad \partial\Omega, \\
\text{and for all} \quad (\boldsymbol{v}, q, \rho) \in \boldsymbol{V} \times M \times H_0^1(\Omega), \\
a_1(\theta; \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + b(\boldsymbol{v} - \boldsymbol{u}, p) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle, \\
b(\boldsymbol{u}, q) = 0, \\
a_2(\theta, \rho) + d(\boldsymbol{u}, \theta, \rho) = \langle b, \rho \rangle,
\end{cases}
\tag{2.2}
$$

with $H^{1/2}(\partial\Omega)$ being the space of trace for elements of $H^1(\Omega)$, $\langle \cdot, \cdot \rangle$ being the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. It can be shown that 6

Proposition 2.1. *Problems (2.2) and (1.1)–(1.4) are equivalent. Indeed, any triplet* $(\boldsymbol{u}, p, \theta) \in \boldsymbol{V} \times M \times H^1(\Omega)$ *is a solution of (1.1)–(1.4) in the sense of distribution if and only if it is a solution of (2.2).*

The following standard results will be used for the analysis of problem (2.2) and its corresponding finite element discretization 6, 21, 22.

The following Poincaré-Friedrich's inequality holds: there is a positive constant c depending on the domain Ω such that

$$
\text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}, \quad \|\boldsymbol{v}\| \leq c|\boldsymbol{v}|_{H^1(\Omega)}, \tag{2.3}
$$

which ensures that the norms $\|\cdot\|_{H^1(\Omega)}$ and $|\cdot|_{H^1(\Omega)}$ are equivalent on $\boldsymbol{V}$.
Given Ω as described, then there exists $c(\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$

$$
\|\boldsymbol{v}\|^4_{L^4(\Omega)} \leq c(\Omega)\|\boldsymbol{v}\|^2_{L^2(\Omega)}\|\boldsymbol{v}\|^2_{H^1(\Omega)}. \tag{2.4}
$$

We also recall Korn's inequality 6: there exists $c(\Omega)$ such that

$$
\|D\boldsymbol{v}\| \geq c(\Omega)\|\nabla \boldsymbol{v}\| \quad \text{for all} \ \boldsymbol{v} \in \boldsymbol{V}. \tag{2.5}
$$

Since ν is bounded from below and above (1.5), we deduce that $a_1(\cdot, \cdot)$ is continuous and elliptic on $\boldsymbol{V}$; this means that for $(\boldsymbol{v}, \boldsymbol{w})$ element of $\boldsymbol{V} \times \boldsymbol{V}$,

$$
a_1(\theta; \boldsymbol{v}, \boldsymbol{w}) \leq \nu_1\|\boldsymbol{v}\|_{H^1(\Omega)}\|\boldsymbol{w}\|_{H^1(\Omega)}, \quad a_1(\theta; \boldsymbol{v}, \boldsymbol{v}) \geq \nu_0 c\|\boldsymbol{v}\|^2_{H^1(\Omega)}. \tag{2.6}
$$

From (2.3), we deduce that $a_2(\cdot, \cdot)$ is continuous and elliptic on $H_0^1(\Omega)$; this means that for (θ, ρ) element of $H_0^1(\Omega) \times H_0^1(\Omega)$,

$$
a_2(\theta, \rho) \leq \kappa c\|\theta\|_{H^1(\Omega)}\|\rho\|_{H^1(\Omega)}, \quad a_2(\rho, \rho) \geq \kappa c\|\rho\|^2_{H^1(\Omega)}. \tag{2.7}
$$

The trilinear form $d(\cdot, \cdot, \cdot)$ enjoys the standard properties 22: for all $(\boldsymbol{v}, \theta, \rho) \in \boldsymbol{H}^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$

$$d(\boldsymbol{v}, \theta, \rho) \leq \|\boldsymbol{v}\rho\|\|\nabla\theta\| \leq \|\boldsymbol{v}\|_{L^4}\|\rho\|_{L^4}\|\nabla\theta\|. \tag{2.8}$$

and if $\boldsymbol{v}$ is such that $\operatorname{div}\boldsymbol{v}|_\Omega = 0$, then

$$d(\boldsymbol{v}, \theta, \rho) = -d(\boldsymbol{v}, \rho, \theta),$$
$$d(\boldsymbol{v}, \rho, \rho) = 0. \tag{2.9}$$

The compatibility condition between the velocity and pressure is very important for the study of (2.1), its proof can be seen in 18, 19: there exists $c(\Omega)$ such that

$$c\|q\| \leq \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{2.10}$$

The kernel of $b(\cdot, \cdot)$ in $\boldsymbol{V}$ is

$$\boldsymbol{V}_{\text{div}} = \{\boldsymbol{v} \in \boldsymbol{V} : \quad b(\boldsymbol{v}, q) = 0 \quad \forall q \in L^2(\Omega)\},$$

which is characterized by

$$\boldsymbol{V}_{\text{div}} = \{\boldsymbol{v} \in \boldsymbol{V} : \quad \operatorname{div}\boldsymbol{v}|_\Omega = 0\}.$$

One easily check that $b(\cdot, \cdot)$ is continuous; that is

$$\text{for all} \quad (\boldsymbol{v}, q) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega), \qquad b(\boldsymbol{v}, q) \leq \|\boldsymbol{v}\|_{H^1(\Omega)}\|q\|. \tag{2.11}$$

The functional $j(\cdot)$ is convex, lower semi continuous (continuous) on $\boldsymbol{V}$ but not differentiable at zero. It can be shown that the solution $(\boldsymbol{u}, p)$ of (2.2) is characterized by

$$\begin{cases} \text{Find} \quad (\boldsymbol{u}, p, \theta, \boldsymbol{\lambda}) \in \boldsymbol{V} \times M \times H^1(\Omega) \times \Lambda, \quad \text{such that} \\ \theta = \theta_0 \quad \text{on} \quad \partial\Omega, \\ \text{and for all} \quad (\boldsymbol{v}, q, \rho) \in \boldsymbol{V} \times M \times H_0^1(\Omega), \\ a_1(\theta; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \int_S g\boldsymbol{\lambda} \cdot \boldsymbol{v}_\tau d\sigma = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\ b(\boldsymbol{u}, q) = 0, \\ \boldsymbol{\lambda} \cdot \boldsymbol{u}_\tau = |\boldsymbol{u}_\tau| \quad \text{a.e. in } S, \\ a_2(\theta, \rho) + d(\boldsymbol{u}, \theta, \rho) = \langle b, \rho \rangle, \end{cases} \tag{2.12}$$

with

$$\Lambda = \{\boldsymbol{\alpha} | \boldsymbol{\alpha} \in \boldsymbol{L}^\infty(S) \ , \quad |\boldsymbol{\alpha}| \leq 1 \text{ a.e. in } S\}.$$

The existence of $\boldsymbol{\lambda}$ in the formulation (2.12) can be proved either by using the Hahn-Banach Theorem (see 17, p. 70, Theorem 5.3), or one can make used of a more constructive approach based on regularization (see 17, p. 82, Theorem 6.3).

2.2 Variational formulation

In what follows, c is a positive constant that may vary from one line to the next, we assume that

$$\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega), \quad g \in L^{\infty}(S), \quad b \in H^{-1}(\Omega) \quad \text{and} \quad \theta_0 \in H^{1/2}(\partial\Omega). \tag{2.13}$$

We claim that

Proposition 2.2. *For any data $(\boldsymbol{f}, b, \theta_0, g)$ satisfying (2.13), there exist three positive constants c_1, c_2, c_3 such that if $(\boldsymbol{u}, \theta, p)$ is given by (2.2), then*

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq \frac{c_1}{\nu_0}\|\boldsymbol{f}\|_{H^{-1}(\Omega)},$$

$$\|\theta\|_{H^1(\Omega)} \leq c_2\left(1 + \frac{1}{\kappa}\right)\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{c_2}{\kappa}\|b\|_{H^{-1}(\Omega)},$$

$$\|p\| \leq \left(1 + c_3\frac{\nu_1}{\nu_0}\right)\|\boldsymbol{f}\|_{H^{-1}(\Omega)}.$$

Proof. We take successively in (2.2) $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} = 2\boldsymbol{u}$. Comparing the two inequalities, we obtain

$$a_1(\theta; \boldsymbol{u}, \boldsymbol{u}) + j(\boldsymbol{u}) = \langle \boldsymbol{f}, \boldsymbol{u}\rangle$$

which after dropping the positive term $j(\boldsymbol{u})$, and application of Holder's inequality and (2.7) yields

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq \frac{c_1}{\nu_0}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}.$$

Next, we state the following result 19 (see Chapter 4, Lemma 2.3): for any $\delta > 0$, there exists a lifting $\widetilde{\theta}_0$ of θ_0 which satisfies

$$\|\widetilde{\theta}_0\|_{L^4(\Omega)} \leq \delta\|\theta_0\|_{H^{1/2}(\partial\Omega)} \quad \text{and} \quad \|\widetilde{\theta}_0\|_{H^1(\Omega)} \leq c\|\theta_0\|_{H^{1/2}(\partial\Omega)}, \tag{2.14}$$

where c is a positive constant independent of δ. We set $\widetilde{\theta} = \theta - \widetilde{\theta}_0$, note that $\widetilde{\theta}|_{\partial\Omega} = 0$ and take $\rho = \widetilde{\theta}$ in (2.2). Noticing that $((\boldsymbol{u}\cdot\nabla)\widetilde{\theta}, \widetilde{\theta}) = 0$, we obtain

$$\begin{aligned} a_2(\widetilde{\theta}, \widetilde{\theta}) &= -a_2(\widetilde{\theta}_0, \widetilde{\theta}) - ((\boldsymbol{u}\cdot\nabla)\widetilde{\theta}_0, \widetilde{\theta}) + \langle b, \widetilde{\theta}\rangle \\ &= -a_2(\widetilde{\theta}_0, \widetilde{\theta}) + ((\boldsymbol{u}\cdot\nabla)\widetilde{\theta}, \widetilde{\theta}_0) + \langle b, \widetilde{\theta}\rangle. \end{aligned}$$

Using Holder's inequality on the right-hand side yields

$$\begin{aligned} \kappa\|\nabla\widetilde{\theta}\|^2 &\leq \kappa\|\nabla\widetilde{\theta}_0\|\|\nabla\widetilde{\theta}\| + \|\boldsymbol{u}\widetilde{\theta}_0\|\|\nabla\widetilde{\theta}\| + \|b\|_{H^{-1}(\Omega)}\|\nabla\widetilde{\theta}\| \\ &\leq \kappa\|\nabla\widetilde{\theta}_0\|\|\nabla\widetilde{\theta}\| + \|\boldsymbol{u}\|_{L^4(\Omega)}\|\widetilde{\theta}_0\|_{L^4(\Omega)}\|\nabla\widetilde{\theta}\| + \|b\|_{H^{-1}(\Omega)}\|\nabla\widetilde{\theta}\| \\ &\leq \kappa c\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\nabla\widetilde{\theta}\| + c\delta\|\boldsymbol{u}\|_{L^4(\Omega)}\|\nabla\widetilde{\theta}\|\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \|b\|_{H^{-1}(\Omega)}\|\nabla\widetilde{\theta}\|, \end{aligned}$$

this is

$$\kappa\|\nabla\widetilde{\theta}\| \leq \kappa c\|\theta_0\|_{H^{1/2}(\partial\Omega)} + c\delta\|\boldsymbol{u}\|_{L^4(\Omega)}\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \|b\|_{H^{-1}(\Omega)}.$$

Using Poincaré Friedrichs's inequality (2.3) and choose $\delta = 1/\|\boldsymbol{u}\|_{L^4(\Omega)}$, we obtain the following inequality

$$\|\widetilde{\theta}\|_{H^1(\Omega)} \le c\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{c}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{c}{\kappa}\|b\|_{H^{-1}(\Omega)}. \tag{2.15}$$

The desired inequality is obtained by combining (2.15) and the triangle's inequality.As for the pressure, we take $v - u = \pm w$ with $w \in H_0^1(\Omega)$. Replacing it in (2.2) and comparing the resulting inequalities, we obtain

$$b(w,p) = (f,w) - a_1(\theta; u, w),$$

which with the inf-sup condition gives

$$\beta\|p\| \le \sup_{v\in H_0^1(\Omega)} \frac{b(v,p)}{\|v\|_{H^1(\Omega)}} \le \|f\|_{H^{-1}(\Omega)} + 2\nu_1\|u\|_{H^1(\Omega)}.$$

Hence the proof is complete after utilization of the bound on the velocity.

The variational problem (2.2) is a mixed variational inequality of second kind and we refer in general to 6, one of the first treatise dealing in a systematic manner of mathematical analysis of variational inequalities. The existence of solution of problem (2.2) will be analyzed by making utilizing; regularization, Galerkin approximation, Brouwer's fixed point, and passage to the limit. For the implementation of the steps mentioned above, we recall that with the lifting $\widetilde{\theta}_0$ introduced earlier, (2.2) is rewritten as follows

$$\begin{cases} \text{Find } (u,p,\theta) \in V \times M \times H_0^1(\Omega), \text{ such that for all } (v,q,\rho) \in V \times M \times H_0^1(\Omega), \\ a_1(\theta+\widetilde{\theta}_0; u, v-u) + b(v-u,p) + j(v) - j(u) \ge \langle f, v-u\rangle, \\ b(u,q) = 0, \\ a_2(\theta+\widetilde{\theta}_0,\rho) + d(u,\theta+\widetilde{\theta}_0,\rho) = \langle b,\rho\rangle. \end{cases} \tag{2.16}$$

We claim that

Proposition 2.3. *The variational problem (2.16) admits at least one solution* $(u,p,\theta) \in V \times M \times H_0^1(\Omega)$

Proof. It is done in several steps.

Regularization. Note that the functional *j* is nondifferentiable at zero. Hence we introduce the parameter $\epsilon > 0$, approaching zero and define the functional $j_\epsilon : V \to \mathbb{R}$ as follows

$$j_\varepsilon(v) = \int_S g\sqrt{|v_\tau|^2 + \varepsilon^2}\; d\sigma.$$

One observes that

$$\lim_{\varepsilon\to 0}(j_\varepsilon(v) - j(v)) = 0.$$

The functional j_ϵ is convex, lower semi-continuous, and twice Gateaux-differentiable with

$$Dj_\varepsilon(u)\cdot v = \int_S g\frac{u_\tau \cdot v_\tau}{\sqrt{|u_\tau|^2+\varepsilon^2}}\; d\sigma. \tag{2.17}$$

Note that Dj_ϵ is monotone, that is

$\langle Dj_\varepsilon(\boldsymbol{u}) - Dj_\varepsilon(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v}\rangle \geq 0 \quad \text{for all} \quad \boldsymbol{v}, \boldsymbol{u} \in \boldsymbol{V}.$

The regularized problem reads:

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon, p_\varepsilon, \theta_\varepsilon) \in \boldsymbol{V} \times M \times H_0^1(\Omega), \text{ such that for all } (\boldsymbol{v}, q, \rho) \in \boldsymbol{V} \times M \times H_0^1(\Omega), \\ a_1(\theta_\varepsilon + \widetilde{\theta}_0; \boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon) + b(\boldsymbol{v} - \boldsymbol{u}_\varepsilon, p_\varepsilon) + j_\varepsilon(\boldsymbol{v}) - j_\varepsilon(\boldsymbol{u}_\varepsilon) \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon\rangle, \\ b(\boldsymbol{u}_\varepsilon, q) = 0, \\ a_2(\theta_\varepsilon + \widetilde{\theta}_0, \rho) + d(\boldsymbol{u}_\varepsilon, \theta_\varepsilon + \widetilde{\theta}_0, \rho) = \langle b, \rho\rangle. \end{cases} \tag{2.18}$$

Since j_ε is differentiable, then (2.18) is equivalent to the variational problem

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon, \theta_\varepsilon) \in \boldsymbol{V}_{\text{div}} \times H_0^1(\Omega) \text{ such that for all } (\boldsymbol{v}, \rho) \in \boldsymbol{V}_{\text{div}} \times H_0^1(\Omega), \\ a_1(\theta_\varepsilon + \widetilde{\theta}_0; \boldsymbol{u}_\varepsilon, \boldsymbol{v}) + \langle Dj_\varepsilon(\boldsymbol{u}_\varepsilon), \boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \\ a_2(\theta_\varepsilon + \widetilde{\theta}_0, \rho) + d(\boldsymbol{u}_\varepsilon, \theta_\varepsilon + \widetilde{\theta}_0, \rho) = \langle b, \rho\rangle. \end{cases} \tag{2.19}$$

Galerkin approximation. First, since $\boldsymbol{V}_{\text{div}}$ is separable, there are $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \dots$ elements of $\boldsymbol{V}_{\text{div}}$, linear independent to each other such that

$$\bigcup_{n=1}^{\infty} \{\boldsymbol{\psi}_n\} \subset \boldsymbol{V}_{\text{div}}, \qquad \overline{\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \dots, \boldsymbol{\psi}_n, \dots\}} = \boldsymbol{V}_{\text{div}}.$$

Let $\boldsymbol{V}_{\text{div}}^n = \{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \dots, \boldsymbol{\psi}_n\}$. Next since $H_0^1(\Omega)$ is separable then there are $\phi_1, \dots, \phi_n$ elements of $H_0^1(\Omega)$, linear independent to each other such that

$$\bigcup_{n=1}^{\infty} \{\phi_n\} \subset H_0^1(\Omega), \qquad \overline{\{\phi_1, \phi_2, \dots, \phi_n, \dots\}} = H_0^1(\Omega).$$

Let $W^n = \{\phi_1, \phi_2, \dots, \phi_n\}$. The Galerkin problem associated to (2.19) reads

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon^n, \theta_\varepsilon^n) \in \boldsymbol{V}_{\text{div}}^n \times W^n \text{ such that for all } (\boldsymbol{v}, \rho) \in \boldsymbol{V}_{\text{div}}^n \times W^n, \\ a_1(\theta_\varepsilon^n + \widetilde{\theta}_0; \boldsymbol{u}_\varepsilon^n, \boldsymbol{v}) + \langle Dj_\varepsilon(\boldsymbol{u}_\varepsilon^n), \boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \\ a_2(\theta_\varepsilon^n + \widetilde{\theta}_0, \rho) + d(\boldsymbol{u}_\varepsilon^n, \theta_\varepsilon^n + \widetilde{\theta}_0, \rho) = \langle b, \rho\rangle. \end{cases} \tag{2.20}$$

To prove the existence of $(\boldsymbol{u}_\varepsilon^n, \theta_\varepsilon^n)$, we will apply the fixed point of Brouwer.

Brouwer's fixed point. Let $(\boldsymbol{v}, \rho) \in \boldsymbol{V}_{\text{div}} \times H_0^1(\Omega)$, we define the mapping $\mathscr{F}$ from $\boldsymbol{V}_{\text{div}} \times H_0^1(\Omega)$ into its dual as follows

$$\begin{aligned} \mathscr{F}(\boldsymbol{u}_\varepsilon, \theta_\varepsilon)(\boldsymbol{v}, \rho) = {} & a_1(\theta_\varepsilon + \widetilde{\theta}_0; \boldsymbol{u}_\varepsilon, \boldsymbol{v}) + \langle Dj_\varepsilon(\boldsymbol{u}_\varepsilon), \boldsymbol{v}\rangle + a_2(\theta_\varepsilon + \widetilde{\theta}_0, \rho) + d(\boldsymbol{u}_\varepsilon, \theta_\varepsilon + \widetilde{\theta}_0, \rho) \\ & - \langle b, \rho\rangle - \langle \boldsymbol{f}, \boldsymbol{v}\rangle. \end{aligned}$$

- $\mathscr{F}$ is continuous. Indeed let $(\boldsymbol{u}_\varepsilon^n, \theta_\varepsilon^n)_{n\geq 1}$ be a sequence of functions in $\boldsymbol{V}_{\text{div}} \times H_0^1(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u}_\varepsilon^n \to \boldsymbol{u}_\varepsilon \quad \text{strongly in} \quad \boldsymbol{V}_{\text{div}}, \\ &\theta_\varepsilon^n \to \theta_\varepsilon \quad \text{strongly in} \quad H^1(\Omega). \end{aligned}$$

We would like to show that $\mathcal{F}(u_\varepsilon^n, \theta_\varepsilon^n)(v,\rho) \to \mathcal{F}(u_\varepsilon, \theta_\varepsilon)(v,\rho)$.

Due to strong convergence of θ_ε^n and the property of $\nu(\cdot)$, for any $v \in V_{\mathrm{div}}$ the sequence $(\nu(\theta_\varepsilon^n + \widetilde{\theta}_0)Dv)_n$ converges to $\nu(\theta_\varepsilon + \widetilde{\theta}_0)Dv$ almost everywhere in Ω and $\|\nu(\theta_\varepsilon^n + \widetilde{\theta}_0)Dv\| \le \nu_1 \|Dv\|$. Thus from the Lebesgue dominated convergence theorem

$$\text{for all } v \in V_{\mathrm{div}}, \quad \lim_{n\to\infty} \nu(\theta_\varepsilon^n + \widetilde{\theta}_0)Dv = \nu(\theta_\varepsilon + \widetilde{\theta}_0)Dv \quad \text{strongly in } L^2(\Omega). \tag{2.21}$$

Thus

$$\text{for all } v \in V_{\mathrm{div}}, \quad \lim_{n\to\infty} a_1(\theta_\varepsilon^n + \widetilde{\theta}_0; u_\varepsilon^n, v) = a_1(\nu(\theta_\varepsilon + \widetilde{\theta}_0); u_\varepsilon, v). \tag{2.22}$$

passing to the limit in $a_2(\cdot,\cdot)$ is direct because it is a linear term. Finally, passing to the limit in the trilinear form $d(\cdot\cdot,\cdot)$ followed from the strong convergence in $L^4(\Omega) \times L^4(\Omega)$ for the terms $(u_\varepsilon^n \cdot \nabla)\theta_\varepsilon^n$. Finally since $Dj_\varepsilon(\cdot)$ is monotone then we have

$$\lim_{n\to\infty} \langle Dj_\varepsilon(u_\varepsilon^n), v\rangle = \langle Dj_\varepsilon(u_\varepsilon), v\rangle.$$

We then conclude that $\mathcal{F}$ is continuous.

- there is a constant r for which $\mathcal{F}(v,\rho)(v,\rho)$ is positive outside the ball $B(0, r)$.

Indeed,

$$\begin{aligned}
\mathcal{F}(v,\rho)(v,\rho) &= a_1(\rho + \widetilde{\theta}_0; v, v) + \langle Dj_\varepsilon(v), v\rangle - \langle f, v\rangle + a_2(\rho + \widetilde{\theta}_0, \rho) - d(v, \rho, \widetilde{\theta}_0) - \langle b, \rho\rangle \\
&\ge 2\nu_0 c\|v\|_1^2 + \kappa c\|\rho\|_1^2 - c\delta\|\theta_0\|_{H^{1/2}(\partial\Omega)}(\|v\|_1^2 + \|\rho\|_1^2) \\
&\quad - c\|f\|_{H^{-1}(\Omega)}\|v\|_1 - \kappa c\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\rho\|_1 - \|b\|_{H^{-1}(\Omega)}\|\rho\|_1 \\
&\ge c[\min(2\nu_0, \kappa) - \delta\|\theta_0\|_{H^{1/2}(\partial\Omega)}](\|v\|_1^2 + \|\rho\|_1^2) \\
&\quad - (c\|f\|_{H^{-1}(\Omega)}^2 + (\kappa c\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \|b\|_{H^{-1}(\Omega)})^2)^{1/2}(\|v\|_1^2 + \|\rho\|_1^2)^{1/2}.
\end{aligned}$$

We take δ and r such that

$$\min(2\nu_0, \kappa) - \delta\|\theta_0\|_{H^{1/2}(\partial\Omega)} > 0, \text{ and}$$
$$c[\min(2\nu_0, \kappa) - \delta\|\theta_0\|_{H^{1/2}(\partial\Omega)}]r - (c\|f\|_{H^{-1}(\Omega)}^2 + (\kappa c\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \|b\|_{H^{-1}(\Omega)})^2)^{1/2} > 0.$$

Hence for any v, ρ elements of $V_{\mathrm{div}} \times H_0^1(\Omega)$, with $\sqrt{\|v\|_1^2 + \|\rho\|_1^2} = r$, we have $\mathcal{F}(v,\rho)(v,\rho) \ge 0$.We recall that $\cup_n V_{\mathrm{div}}^n \times W^n$ is dense in $V_{\mathrm{div}} \times H_0^1(\Omega)$, and the properties established for $\mathcal{F}$ are valid for $V_{\mathrm{div}} \times H_0^1(\Omega)$ replaced by $V_{\mathrm{div}}^n \times W^n$. Thus the Brouwer's fixed point is applicable. Hence there is $u_\varepsilon^n, \theta_\varepsilon^n$ elements of $V_{\mathrm{div}}^n \times W^n$ such that $\mathcal{F}(u_\varepsilon^n, \theta_\varepsilon^n)(v,\rho) = 0$ for all $v, \rho \in V_{\mathrm{div}}^n \times W^n$. This is to say that

$$\begin{cases}
\text{for all } (v,\rho) \in V_{\mathrm{div}}^n \times W^n, \\
a_1(\theta_\varepsilon^n + \widetilde{\theta}_0; u_\varepsilon^n, v) + \langle Dj_\varepsilon(u_\varepsilon^n), v\rangle = \langle f, v\rangle, \\
a_2(\theta_\varepsilon^n + \widetilde{\theta}_0, \rho) + d(u_\varepsilon^n, \theta_\varepsilon^n + \widetilde{\theta}_0, \rho) = \langle b, \rho\rangle.
\end{cases} \tag{2.23}$$

A priori estimates and passage to the limit. The a priori estimates obtained in Proposition will also hold in the discrete setting $V_{\mathrm{div}}^n \times W^n$. These are

$$\|\boldsymbol{u}_\varepsilon^n\|_{H^1(\Omega)} \leq \frac{c_1}{\nu_0}\|\boldsymbol{f}\|_{H^{-1}(\Omega)},$$
$$\|\theta_\varepsilon^n\|_{H^1(\Omega)} \leq c_2\left(1+\frac{1}{\kappa}\right)\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{c_2}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)},$$
$$\|p_\varepsilon^n\| \leq \left(1+c_3\frac{\nu_1}{\nu_0}\right)\|\boldsymbol{f}\|_{H^{-1}(\Omega)}.$$

Then we can find a subsequence, denoted also $(\boldsymbol{u}_{n,\varepsilon}, \theta_{n,\varepsilon})$, such that

$$\begin{aligned}&\boldsymbol{u}_\varepsilon^n \to \boldsymbol{u}_\varepsilon \quad \text{in } \boldsymbol{V}_{\mathrm{div}} \text{ weakly}\\ &\theta_\varepsilon^n \to \theta_\varepsilon \quad \text{in } H_0^1(\Omega) \text{ weakly.}\end{aligned}$$

Now owing to the compactness of the imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, there exits a subsequence, still denoted by $(\boldsymbol{u}_{n,\varepsilon}, \theta_{n,\varepsilon})$, such that

$$(\boldsymbol{u}_\varepsilon^n, \theta_\varepsilon^n) \to (\boldsymbol{u}_\varepsilon, \theta_\varepsilon) \quad \text{weakly in } H^1(\Omega)\times H^1(\Omega)$$
and
$$(\boldsymbol{u}_\varepsilon^n, \theta_\varepsilon^n) \to (\boldsymbol{u}_\varepsilon, \theta_\varepsilon) \quad \text{strongly in } L^4(\Omega)\times L^4(\Omega).$$

Hence one can pass to the limit in (2.23). We note that passing to the limit for linear term is direct and only necessitate the weak convergence, but for the nonlinear terms, we need the arguments used for the continuity of $\mathscr{F}$. We then deduce that for ϵ,

$$\begin{cases}\text{Find} \quad (\boldsymbol{u}_\varepsilon, \theta_\varepsilon) \in \boldsymbol{V}_{\mathrm{div}} \times H_0^1(\Omega), \quad \text{such that}\\ \text{and for all} \quad (\boldsymbol{v}, \rho) \in \boldsymbol{V}_{\mathrm{div}} \times H_0^1(\Omega),\\ a_1(\theta_\varepsilon + \widetilde{\theta}_0; \boldsymbol{u}_\varepsilon, \boldsymbol{v}) + \langle Dj_\varepsilon(\boldsymbol{u}_\varepsilon), \boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle,\\ a_2(\theta_\varepsilon + \widetilde{\theta}_0, \rho) + d(\boldsymbol{u}_\varepsilon, \theta_\varepsilon + \widetilde{\theta}_0, \rho) = \langle b, \rho\rangle,\end{cases} \tag{2.24}$$

which is equivalent to (2.18). Again the a priori estimates obtained in Proposition are valid for the solution of (2.24). Hence we repeat the process of passing to the limit with respect to ϵ. We should re-write (2.18) in the form

$$\begin{cases}\text{for all} \quad (\boldsymbol{v}, \rho) \in \boldsymbol{V}_{\mathrm{div}} \times H_0^1(\Omega),\\ a_1(\theta_\varepsilon + \widetilde{\theta}_0; \boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) + j_\varepsilon(\boldsymbol{u}_\varepsilon) \leq a_1(\theta_\varepsilon + \widetilde{\theta}_0; \boldsymbol{u}_\varepsilon, \boldsymbol{v}) + j_\varepsilon(\boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon\rangle,\\ a_2(\theta_\varepsilon + \widetilde{\theta}_0, \rho) + d(\boldsymbol{u}_\varepsilon, \theta_\varepsilon + \widetilde{\theta}_0, \rho) = \langle b, \rho\rangle.\end{cases} \tag{2.25}$$

Let $\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{div}}$, the application $\boldsymbol{v} \to j_\varepsilon(\boldsymbol{v})$ is continuous and convex, together with the properties of $a_1(\cdot,\cdot)$, $a_2(\cdot,\cdot)$ and $d(\cdot,\cdot,\cdot)$ implies that

$$\begin{cases}\text{for all} \quad (\boldsymbol{v}, \rho) \in \boldsymbol{V}_{\mathrm{div}} \times H_0^1(\Omega),\\ a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{u}) + j(\boldsymbol{u}) \leq \liminf\limits_{\varepsilon}[a_1(\theta_\varepsilon + \widetilde{\theta}_0; \boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) + j_\varepsilon(\boldsymbol{u}_\varepsilon)] \leq a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}) + j(\boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle,\\ a_2(\theta + \widetilde{\theta}_0, \rho) + d(\boldsymbol{u}, \theta + \widetilde{\theta}_0, \rho) = \langle b, \rho\rangle.\end{cases} \tag{2.26}$$

Hence the proof of Proposition is now complete.

The pressure is obtained by regularization and then following the usual procedure described in 19.

About the uniqueness of solutions $(\boldsymbol{u}, \theta)$ of (2.26) we claim that

Proposition 2.4. *Let p_1, p_2 two positive constants greater than one such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$. Assume that $\boldsymbol{u} \in \boldsymbol{W}^{1,p_2}(\Omega)$, and choose either ν_0, ν_2, or κ such that the relation*

$$\left(\kappa - c\left[\left(1 + \frac{1}{\kappa}\right)\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|b\|_{H^{-1}(\Omega)}\right]\frac{c\nu_2}{\nu_0}\|D\boldsymbol{u}\|_{L^{p_2}}\right) \geq 0. \tag{2.27}$$

is satisfied for a positive constant c depending only on Ω. Then the solution of (2.26) is unique.

Proof. Let $(\boldsymbol{u}_1, \theta_1)$ and $(\boldsymbol{u}_2, \theta_2)$ be two solutions of (2.26). We proceed in two steps.

First, we recall that since $\theta_1 - \theta_2|_{\partial\Omega} = 0$, the temperature equation and property of $d(\cdot, \cdot, \cdot)$ give

$$a_2(\theta_1 - \theta_2, \theta_1 - \theta_2) = d(\boldsymbol{u}_1 - \boldsymbol{u}_2, \theta_1, \theta_2 - \theta_1) \leq c\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1\|\theta_1 - \theta_2\|_1\|\theta_1\|_1,$$

which from coercivity on $a_2(\cdot, \cdot)$ and a priori estimates on θ gives

$$\kappa\|\theta_1 - \theta_2\|_1 \leq c\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1\left[\left(1 + \frac{1}{\kappa}\right)\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|b\|_{H^{-1}(\Omega)}\right]. \tag{2.28}$$

Next, the velocity equation leads to

$$\begin{aligned} a_1(\theta_1 + \widetilde{\theta}_0; \boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) &\leq a_1(\theta_1 + \widetilde{\theta}_0; \boldsymbol{u}_2, \boldsymbol{u}_2 - \boldsymbol{u}_1) - a_1(\theta_2 + \widetilde{\theta}_0; \boldsymbol{u}_2, \boldsymbol{u}_2 - \boldsymbol{u}_1) \\ &= \int_\Omega (\nu(\theta_1 + \widetilde{\theta}_0) - \nu(\theta_2 + \widetilde{\theta}_0))D\boldsymbol{u}_2 : D(\boldsymbol{u}_2 - \boldsymbol{u}_1), \end{aligned}$$

from which we deduce that

$$2\nu_0\|D(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|^2 \leq \nu_2\int_\Omega |\theta_1 - \theta_2||D\boldsymbol{u}_2||D(\boldsymbol{u}_2 - \boldsymbol{u}_1)|. \tag{2.29}$$

To estimate the right-hand side of (2.29), we recall or introduce the following facts:Generalized Holder's inequality. Let $1 \leq i \leq n$, $1 < p_i < \infty$ with $f_i \in L^{p_i}(\Omega)$ and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{p} \leq 1.$$

Then $\prod_{i=1}^n f_i \in L^p(\Omega)$ and

$$\left\|\prod_{i=1}^n f_i\right\|_{L^p(\Omega)} \leq \prod_{i=1}^n \|f_i\|_{L^{p_i}(\Omega)}. \tag{2.30}$$

Sobolev inequalities. If Ω is an open set of class C^1 with Γ bounded then

$$W^{1,p}(\Omega) \text{ is embedded in } L^q(\Omega)\begin{cases} \text{for } \frac{1}{q} = \frac{1}{p} - \frac{1}{d} \text{ if } p < d, \\ \text{or} \\ \text{for all } q \in [p, \infty) \text{ if } p = d. \end{cases} \tag{2.31}$$

Making use of (2.30) with $p = 2$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$, (2.29) gives

$$\nu_0 \|D(\boldsymbol{u}_1 - \boldsymbol{u}_2)\| \leq \nu_2 \|\theta_1 - \theta_2\|_{L^{p_1}} \|D\boldsymbol{u}_2\|_{L^{p_2}}. \tag{2.32}$$

We apply Korn's inequality on the left-hand side of (2.18), while on the right-hand side we make use of (2.31) for $d = 2 = p$ and $q = p_1$. We find

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1 \leq \frac{c\nu_2}{\nu_0} \|\theta_1 - \theta_2\|_1 \|D\boldsymbol{u}_2\|_{L^{p_2}}. \tag{2.33}$$

Returning to (2.28) with (2.33), one has

$$\left(\kappa - c\left[\left(1 + \frac{1}{\kappa} \right) \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa} \|b\|_{H^{-1}(\Omega)} \right] \frac{c\nu_2}{\nu_0} \|D\boldsymbol{u}_2\|_{L^{p_2}} \right) \|\theta_1 - \theta_2\|_1 \leq 0,$$

from which we immediately deduce the uniqueness criteria.

Remark 2.1. It is not immediate to obtain the regularity assumption
$\boldsymbol{u} \in \boldsymbol{W}^{1,p_2}(\Omega)$
required just by observing the equations. In fact this condition appears when one analyses the term $\int_\Omega \theta\, D\boldsymbol{u} : D\boldsymbol{v}$. In fact that regularity was already required in the works by 8, 15, 16, where Dirichlet condition is applied on the boundary.

The condition (2.27) required for uniqueness of solutions obtained in Proposition is restrictive, but for nonlinear problems, it is very rare to obtain uniqueness of solutions without restrictions.

The analysis presented can be extended to the following situations:

a. one replaces (1.3) and (1.4) by the leak boundary conditions 23

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad S \quad \text{and} - (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{n}} \in g\partial|\boldsymbol{u}_{\boldsymbol{n}}| \qquad \text{on} \quad S.$$

a. The Dirichlet boundary condition on θ is replaced by the mixed one

$$\theta|_\Gamma = \theta_0 \quad \text{and} \quad \frac{\partial\theta}{\partial\boldsymbol{n}} = \theta_1 \quad \text{on} \quad S.$$

2.3 Finite Element approximation

From now on, we assume that Ω is a polygon. In order to approximate the problem (2.2), we introduce a regular family $(\mathcal{T}_h)_h$ of triangulations of Ω by closed triangles, in the usual sense that

a. For each h, $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_h$.
b. For each h, the intersection of two different elements of $\mathcal{T}_h$ if not empty, is a corner, a whole edge of both elements.
c. The ratio of the diameter h_K of an element K in $\mathcal{T}_h$ to the diameter of its inscribed circle or sphere is bounded by a constant independent of K and h.

As standard, h stands for the maximum of the diameters of the elements of $\mathcal{T}_h$. For each nonnegative integer n and any K in $\mathcal{T}_h$, let $\mathcal{P}_l(K)$ denote the space of restrictions to K of polynomials with two variables and total degree less than or equal to l.

We define then the following finite dimensional spaces which approximate V, M, and $H^1(\Omega)$, respectively;

$$
\begin{aligned}
\boldsymbol{V}_h &= \{\boldsymbol{v}_h \in \mathscr{C}(\overline{\Omega})^2 \cap \boldsymbol{V}, \ \text{for all} \ K \in \mathcal{T}_h, \ \boldsymbol{v}_h|_K \in \mathcal{P}_2(K)^2\},\\
M_h &= \{q_h \in L^2(\Omega) \cap \mathscr{C}(\overline{\Omega}), \quad \text{for all} \ K \in \mathcal{T}_h, \quad q_h|_K \in \mathcal{P}_1(K)\},\\
H_h^1 &= \{v_h \in H^1(\Omega) \cap \mathscr{C}(\overline{\Omega}), \ \text{for all} \ K \in \mathcal{T}_h, \quad v_h|_K \in \mathcal{P}_1(K)\}.
\end{aligned}
$$

We let

$$H_{0h}^1 = H_h^1 \cap H_0^1(\Omega).$$

The finite element approximation of problem (2.2) reads;

$$
\begin{cases}
\text{Find} \quad (\boldsymbol{u}_h, p_h, \theta_h) \in \boldsymbol{V}_h \times M_h \times H_h^1, \quad \text{such that}\\
\theta_h = \theta_0 \quad \text{on} \quad \partial\Omega,\\
\text{and for all} \quad (\boldsymbol{v}_h, q_h, \rho_h) \in \boldsymbol{V}_h \times M_h \times H_{0h}^1,\\
a_1(\theta_h; \boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + b(\boldsymbol{v}_h - \boldsymbol{u}_h, p_h) + j(\boldsymbol{v}_h) - j(\boldsymbol{u}_h) \geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h\rangle,\\
b(\boldsymbol{u}_h, q_h) = 0,\\
a_2(\theta_h, \rho_h) + d_h(\boldsymbol{u}_h, \theta_h, \rho_h) = \langle b, \rho_h\rangle,
\end{cases}
\tag{2.34}
$$

with

$$d_h(\boldsymbol{v}_h, \theta_h, \rho_h) = d(\boldsymbol{v}_h, \theta_h, \rho_h) + \frac{1}{2}((\operatorname{div}\boldsymbol{v}_h)\theta_h, \rho_h). \tag{2.35}$$

The trilinear form $d_h(\cdot, \cdot, \cdot)$ enjoys the properties (2.8) and (2.9) 22, this is to say that for all $(\boldsymbol{v}_h, \theta_h, \rho_h) \in \boldsymbol{V}_h \times H_{0h}^1 \times H_{0h}^1$

$$
\begin{aligned}
&d_h(\boldsymbol{v}_h, \theta_h, \rho_h) = -d_h(\boldsymbol{v}_h, \rho_h, \theta_h),\\
&d_h(\boldsymbol{v}_h, \theta_h, \rho_h) \leq \|\boldsymbol{v}_h\|_{L^4} \|\rho_h\|_{L^4} \|\nabla\theta_h\|.
\end{aligned}
\tag{2.36}
$$

We recall that the discrete version of inf-sup condition (2.11) holds: there exists β independent of h such that

$$\beta\|q_h\| \leq \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{H^1(\Omega)}} \quad \text{for all} \quad q_h \in M_h. \tag{2.37}$$

Remark 2.2. It should be made clear that other choice of elements for the couple velocity/pressure can be adopted as long as the compatibility condition (2.19) is satisfied. The reader may consult 18, 19 for a thorough mathematical discussion of the inf-sup condition (2.19), its implications and elements that satisfied the "test."

The solvability of (2.17) can be obtained by following to the line the proof presented in the continuous version (Propositions and). In fact we state that

Theorem 2.1. *Let ($\boldsymbol{f}$, b, θ_0, g) satisfying (2.13). Let ($\boldsymbol{u}_h$, θ_h, p_h) given by (2.17) then the following holds*

$$\begin{aligned}
&\|\boldsymbol{u}_h\|_{H^1(\Omega)} \le c\|\boldsymbol{f}\|_{H^{-1}(\Omega)},\\
&\|\theta_h\|_{H^1(\Omega)} \le c\left(1+\frac{1}{\kappa}\right)\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{c_2}{\kappa}\|b\|_{H^{-1}(\Omega)},\\
&\|p_h\| \le c\|\boldsymbol{f}\|_{H^{-1}(\Omega)}.
\end{aligned} \tag{2.38}$$

Let p_1, p_2 two positive constants greater than one such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{2}$. Assume that $\boldsymbol{u}_h \in \boldsymbol{W}^{1,p_2}(\Omega)$, and choose either v_0, v_2 or κ such that the relation

$$\left(\kappa - c\left[\left(1+\frac{1}{\kappa}\right)\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|b\|_{H^{-1}(\Omega)}\right]\frac{c\nu_2}{\nu_0}\|D\boldsymbol{u}_h\|_{L^{p_2}}\right) \ge 0. \tag{2.39}$$

holds. Then problem (2.17) admits a unique solution $(\boldsymbol{u}_h, \theta_h, p_h) \in \boldsymbol{V}_h \times H_h^1 \times M_h$.

Similar to the continuous problem, the following characterization of the solution ($\boldsymbol{u}_h$, p_h, θ_h) of the problem (2.17) holds:

Lemma 2.1. *There exists $\lambda_h \in \Lambda$ such that if ($\boldsymbol{u}_h$, p_h, θ_h) is the solution of the problem (2.17) then*

$$\begin{cases}
\text{for all} \quad (\boldsymbol{v}, q, \rho) \in \boldsymbol{V}_h \times M_h \times H_{0h}^1,\\
a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) + \int_S g\boldsymbol{\lambda}_h \cdot \boldsymbol{v}_{\boldsymbol{\tau}} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle,\\
b(\boldsymbol{u}_h, q) = 0,\\
\boldsymbol{u}_{\boldsymbol{\tau},h} \cdot \boldsymbol{\lambda}_h = |\boldsymbol{u}_{\boldsymbol{\tau},h}| \quad \text{a.e. in } S,\\
a_2(\theta_h, \rho) + d_h(\boldsymbol{u}_h, \theta_h + \widetilde{\theta}_0, \rho) = \langle b, \rho\rangle - a_2(\widetilde{\theta}_0, \rho).
\end{cases} \tag{2.40}$$

Proof. We recall that the finite element problem reads;

$$\begin{cases}
\text{Find} \quad (\boldsymbol{u}_h, p_h, \theta_h) \in \boldsymbol{V}_h \times M_h \times H_{0h}^1, \quad \text{such that}\\
\text{and for all} \quad (\boldsymbol{v}_h, q_h, \rho_h) \in \boldsymbol{V}_h \times M_h \times H_{0h}^1,\\
a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + b(\boldsymbol{v}_h - \boldsymbol{u}_h, p_h) + j(\boldsymbol{v}_h) - j(\boldsymbol{u}_h) \ge \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h\rangle,\\
b(\boldsymbol{u}_h, q_h) = 0,\\
a_2(\theta_h, \rho_h) + d_h(\boldsymbol{u}_h, \theta_h, \rho_h) = \langle b, \rho_h\rangle - a_2(\widetilde{\theta}_0, \rho_h) - d_h(\boldsymbol{u}_h, \widetilde{\theta}_0, \rho_h),
\end{cases} \tag{2.41}$$

We need to show that $(\boldsymbol{u}_h, p_h, \theta_h)$ is the solution of (2.41) if and only if there is $\lambda_h \in \Lambda$ such that $(\boldsymbol{u}_h, p_h, \theta_h, \lambda_h)$ solves (2.40).

We follow the proof in 17, p. 70, Theorem 5.3, with the difference that we will be working with the discrete velocity space.

We first assume that $(\boldsymbol{u}_h, p_h, \theta_h)$ is the solution of (2.41) and we would like to construct $\lambda_h \in \Lambda$ with $(\boldsymbol{u}_h, p_h, \theta_h, \lambda_h)$ solution of (2.40).

We take successively $\boldsymbol{v}_h = \boldsymbol{0}$ and $\boldsymbol{v}_h = 2\boldsymbol{u}_h$ in (2.41). Comparing the resulting relations and making use of $(2.41)_2$, we obtain

$$a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{u}_h) + j(\boldsymbol{u}_h) = \langle \boldsymbol{f}, \boldsymbol{u}_h \rangle. \tag{2.42}$$

Now adding (2.42) and $(2.41)_1$ we find

$$\text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle - a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) \le j(\boldsymbol{v}_h),$$

which implies that

$$\forall \ \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad |\langle \boldsymbol{f}, \boldsymbol{v}_h \rangle - a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h)| \le j(\boldsymbol{v}_h) \le \|g\|_{L^\infty(S)} \|\boldsymbol{v}_h\|_{L^1(S)}. \tag{2.43}$$

Hence the mapping $\boldsymbol{v}_h \to \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle - a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h)$ is a linear and continuous functional on $\boldsymbol{V}_h|_S$. But $\boldsymbol{V}_h|_S \subset \boldsymbol{H}^{1/2}(S) \subset \boldsymbol{L}^1(S)$. Thus the Hahn Banach theorem applies, so the above mapping can be extended to $\boldsymbol{L}^1(S)$. Hence from Riesz's representation result there is $\boldsymbol{\lambda}_h \in \boldsymbol{L}^\infty(S)$, $|\boldsymbol{\lambda}_h| \le 1$ a.e on S with

$$a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) + \int_S g\boldsymbol{v}_{\boldsymbol{\tau}} \cdot \boldsymbol{\lambda}_h = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h$$

which is the first relation announced.

Next, for $\lambda_h \in \Lambda$, one has $|\lambda_h| \le 1$, and from Schwarz's inequality one obtains

$$\boldsymbol{u}_{\boldsymbol{\tau},h} \cdot \boldsymbol{\lambda}_h \le |\boldsymbol{u}_{\boldsymbol{\tau},h}| |\boldsymbol{\lambda}_h| \le |\boldsymbol{u}_{\boldsymbol{\tau},h}| \quad \text{a.e on} \quad S. \tag{2.44}$$

For $\boldsymbol{v} = \boldsymbol{u}_h$ in (2.40) one obtains

$$a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{u}_h) + \int_S g\boldsymbol{\lambda}_h \cdot \boldsymbol{u}_{\boldsymbol{\tau},h} = \langle \boldsymbol{f}, \boldsymbol{u}_h \rangle. \tag{2.45}$$

From (2.42) and (2.45) one deduces that

$$\int_S g(|\boldsymbol{u}_{\boldsymbol{\tau},h}| - \boldsymbol{\lambda}_h \cdot \boldsymbol{u}_{\boldsymbol{\tau},h}) = 0. \tag{2.46}$$

From (2.46) and (2.44) it follows that

$$|\boldsymbol{u}_{\boldsymbol{\tau},h}| - \boldsymbol{\lambda}_h \cdot \boldsymbol{u}_{\boldsymbol{\tau},h} = 0 \ \text{ a.e on } \ S$$

which is the second relation announced in (2.40).

Now, we want to show that if $\lambda_h \in \Lambda$ with $(\boldsymbol{u}_h, p_h, \theta_h, \lambda_h)$ solution of (2.40) then $(\boldsymbol{u}_h, p_h, \theta_h)$ is the solution of (2.41).

From (2.40) it follows that for all $\boldsymbol{v} \in \boldsymbol{V}_h$,

$$
\begin{aligned}
a_1(\theta_h+\widetilde{\theta}_0;\boldsymbol{u}_h,\boldsymbol{v}-\boldsymbol{u}_h)+b(\boldsymbol{v}-\boldsymbol{u}_h,p_h) &= \int_S g\boldsymbol{\lambda}_h\cdot\boldsymbol{u}_{\tau,h}-\int_S g\boldsymbol{\lambda}_h\cdot\boldsymbol{v}_\tau+\langle \boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}_h\rangle \\
&= \int_S g|\boldsymbol{u}_{\tau,h}|-\int_S g\boldsymbol{\lambda}_h\cdot\boldsymbol{v}_\tau+\langle \boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}_h\rangle.
\end{aligned}
$$

But since $|\lambda_h|\le 1$ a.e on S, then $\lambda_h\cdot\boldsymbol{v}_\tau\le|\boldsymbol{v}_\tau|$ a.e on S. We then deduce that

$$
\text{for all } \boldsymbol{v}\in\boldsymbol{V}_h \quad a_1(\theta_h+\widetilde{\theta}_0;\boldsymbol{u}_h,\boldsymbol{v}-\boldsymbol{u}_h)+b(\boldsymbol{v}-\boldsymbol{u}_h,p_h)+j(\boldsymbol{v})-j(\boldsymbol{u}_h)\ge\langle \boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}_h\rangle.
$$

Hence ($\boldsymbol{u}_h$, p_h, θ_h) is the solution of (2.41), which ends the proof of the assertion.

Remark 2.3. We are not able to construct $\boldsymbol{\lambda}_h$ in a discrete space Λ_h such that (2.40) holds. Nevertheless, the formulation (2.40) will be very important later for the actual computation of the solution $(\boldsymbol{u},\theta_h,\lambda_h)$.

Because λ_h is not constructed in a finite element space (for example Λ_h), then the formulation (2.40) is not strictly speaking a discrete approximation of (2.12), but on the other hand since the construction of λ_h make use of the discrete quantity $\boldsymbol{u}_h$, then it is not an "abstract element" like $\boldsymbol{\lambda}$.

3. A PRIORI ERROR ANALYSIS

In this section, we would like to find the limit of the sequence ($\boldsymbol{u}_h$, p_h, θ_h) solution of (2.17) when $h\to 0$. We recall that the continuous formulation reads;

$$
\begin{cases}
\text{Find} \quad (\boldsymbol{u},p,\theta)\in\boldsymbol{V}\times M\times H_0^1(\Omega), \quad \text{such that} \\
\text{and for all} \quad (\boldsymbol{v},q,\rho)\in\boldsymbol{V}\times M\times H_0^1(\Omega), \\
a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u})+b(\boldsymbol{v}-\boldsymbol{u},p)+j(\boldsymbol{v})-j(\boldsymbol{u})\ge\langle \boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}\rangle, \\
b(\boldsymbol{u},q)=0, \\
a_2(\theta,\rho)+d(\boldsymbol{u},\theta,\rho)=\langle b,\rho\rangle-a_2(\widetilde{\theta}_0,\rho)-d(\boldsymbol{u},\widetilde{\theta}_0,\rho).
\end{cases}
\tag{3.1}
$$

The main result on this paragraph is stated as follows.

Theorem 3.1. *Let $(\boldsymbol{u},p,\theta)$ be the solution of (3.1) defined in* Proposition *with $\boldsymbol{u}\in\boldsymbol{W}^{1,p_2}(\Omega)$ such that* Proposition *is valid. Let $(\boldsymbol{u}_h,p_h,\theta_h)$ the solution of (2.41) such that $\boldsymbol{u}_h\in\boldsymbol{W}^{1,p_2}(\Omega)$ and (2.39) is valid.*

Then there exists c independent of h such that for all $(\boldsymbol{v}_h,q_h,s_h)\in\boldsymbol{V}_h\times M_h\times H_h^1$,

$$
\begin{aligned}
\|\boldsymbol{u}_h-\boldsymbol{u}\|_{H^1(\Omega)} &\le c\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}+c\|\boldsymbol{v}_{\tau,h}-\boldsymbol{u}_\tau\|_S^{1/2}+c\|p-q_h\|+c\|\theta-s_h\|_{H^1(\Omega)}, \\
\|\theta-\theta_h\|_{H^1(\Omega)} &\le c\|\theta-s_h\|_{H^1(\Omega)}+c\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}+c\|\boldsymbol{v}_{\tau,h}-\boldsymbol{u}_\tau\|_S^{1/2}+c\|p-q_h\|, \\
\|p-p_h\| &\le c\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}+c\|\boldsymbol{v}_{\tau,h}-\boldsymbol{u}_\tau\|_S^{1/2}+c\|p-q_h\|+c\|\theta-s_h\|_{H^1(\Omega)}.
\end{aligned}
$$

Proof. We will use (2.30) and (2.31) with the same parameters as before.

Let $\boldsymbol{w}\in\boldsymbol{H}_0^1(\Omega)$, we take $\boldsymbol{v}-\boldsymbol{u}=\pm\boldsymbol{w}$ in (3.1) and obtain

$$
a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u},\boldsymbol{w})+b(\boldsymbol{w},p)=\langle \boldsymbol{f},\boldsymbol{w}\rangle. \tag{3.2}
$$

We denote by $\boldsymbol{H}^1_{0h}$ a conforming finite element discretization of $\boldsymbol{H}^1_0(\Omega)$. In a similar way using (2.41) we have

$$\text{for all } \boldsymbol{w}_h \in \boldsymbol{H}^1_{0h} \quad a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle. \tag{3.3}$$

Since $\boldsymbol{H}^1_{0h}$ is a subset of $\boldsymbol{H}^1_0(\Omega)$, from (3.2) and (3.3) one has; for all $\boldsymbol{w}_h \in \boldsymbol{H}^1_{0h}$

$$\begin{aligned} b(\boldsymbol{w}_h, p - p_h) &= a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{w}_h) - a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{w}_h) \\ &= a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{w}_h) - a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{w}_h) - a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h), \end{aligned}$$

which by linearity gives; for all $q_h \in M_h$

$$\begin{aligned} b(\boldsymbol{w}_h, q_h - p_h) = {} & a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{w}_h) - a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{w}_h) - a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) \\ & + b(\boldsymbol{w}_h, q_h - p). \end{aligned} \tag{3.4}$$

Now, from the discrete inf-sup condition on $b(\cdot, \cdot)$ and (3.4), we have

$$\begin{aligned} & \beta \|p_h - q_h\| \\ & \quad \le \sup_{\boldsymbol{w}_h \in \boldsymbol{H}^1_{0h}} \frac{b(\boldsymbol{w}_h, q_h - p_h)}{\|\boldsymbol{w}_h\|_{H^1(\Omega)}} \\ & \quad = \sup_{\boldsymbol{w}_h \in \boldsymbol{H}^1_{0h}} \frac{(a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{w}_h) - a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{w}_h)) - a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, q_h - p)}{\|\boldsymbol{w}_h\|_{H^1(\Omega)}} \\ & \quad \le \nu_2 c \|\nabla(\theta - \theta_h)\| \|D\boldsymbol{u}_h\|_{L^{r_2}} + \nu_1 \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\| + c\|p - q_h\| \\ & \quad \le \nu_1 \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\| + \nu_2 c \|\nabla(\theta - \theta_h)\| \|\boldsymbol{u}_h\|_{W^{1,r_2}(\Omega)} + c\|p - q_h\|, \end{aligned}$$

where we have used the generalized Holder's inequality and the Sobolev's inequality. Thus using the triangle inequality yields

$$\|p - p_h\| \le 2\|p - q_h\| + \nu_1 \|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(\Omega)} + \nu_2 c \|\nabla(\theta - \theta_h)\| \|\boldsymbol{u}_h\|_{W^{1,r_2}(\Omega)}. \tag{3.5}$$

Second, we let successively in (3.1) $\boldsymbol{v} = \boldsymbol{u}_h$ and $\boldsymbol{v} = 2\boldsymbol{u} - \boldsymbol{v}_h$ and obtain

$$\begin{aligned} & a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) + b(\boldsymbol{u}_h - \boldsymbol{u}, p) + j(\boldsymbol{u}_h) - j(\boldsymbol{u}) \ge \langle \boldsymbol{f}, \boldsymbol{u}_h - \boldsymbol{u} \rangle, \\ & a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v}_h) + b(\boldsymbol{u} - \boldsymbol{v}_h, p) + j(2\boldsymbol{u} - \boldsymbol{v}_h) - j(\boldsymbol{u}) \ge \langle \boldsymbol{f}, \boldsymbol{u} - \boldsymbol{v}_h \rangle. \end{aligned}$$

Adding these relations gives

$$a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{v}_h) + b(\boldsymbol{u}_h - \boldsymbol{v}_h, p) + j(\boldsymbol{u}_h) - 2j(\boldsymbol{u}) + j(2\boldsymbol{u} - \boldsymbol{v}_h) \ge \langle \boldsymbol{f}, \boldsymbol{u}_h - \boldsymbol{v}_h \rangle. \tag{3.6}$$

Putting together (3.6) and the velocity equation of (2.34), one obtains

$$\begin{aligned} & a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{v}_h) - a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) + b(\boldsymbol{u}_h - \boldsymbol{v}_h, p - p_h) \\ & \quad \ge (j(\boldsymbol{u}) - j(\boldsymbol{v}_h)) + (j(\boldsymbol{u}) - j(2\boldsymbol{u} - \boldsymbol{v}_h)). \end{aligned} \tag{3.7}$$

but

$$a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{v}_h) = a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) + a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h).$$

Hence (3.7) becomes

$$a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)+a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)-a_1(\theta_h+\widetilde{\theta}_0;\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)$$
$$+b(\boldsymbol{u}_h-\boldsymbol{v}_h,p-p_h)\geq(j(\boldsymbol{u})-j(\boldsymbol{v}_h))+(j(\boldsymbol{u})-j(2\boldsymbol{u}-\boldsymbol{v}_h)). \tag{3.8}$$

Now, by linearity one has

$$a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)=a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)-a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}_h-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h).$$

Thus (3.8) becomes

$$a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}_h-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)$$
$$\leq a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)+a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)-a_1(\theta_h+\widetilde{\theta}_0;\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)$$
$$+b(\boldsymbol{u}_h-\boldsymbol{v}_h,p-p_h)+(j(\boldsymbol{v}_h)-j(\boldsymbol{u}))+(j(2\boldsymbol{u}-\boldsymbol{v}_h)-j(\boldsymbol{u})). \tag{3.9}$$

We recall that

$$b(\boldsymbol{u}_h-\boldsymbol{v}_h,p-p_h)=b(\boldsymbol{u}_h-\boldsymbol{u},p-q_h)+b(\boldsymbol{u}_h-\boldsymbol{u},q_h-p_h)+b(\boldsymbol{u}-\boldsymbol{v}_h,p-p_h)$$
$$=b(\boldsymbol{u}_h-\boldsymbol{u},p-q_h)+b(\boldsymbol{u}-\boldsymbol{v}_h,p-p_h).$$

Thus (3.9) yields

$$a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}_h-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)$$
$$\leq a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)+a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)-a_1(\theta_h+\widetilde{\theta}_0;\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)$$
$$+b(\boldsymbol{u}_h-\boldsymbol{u},p-q_h)+b(\boldsymbol{u}-\boldsymbol{v}_h,p-p_h)$$
$$+(j(\boldsymbol{v}_h)-j(\boldsymbol{u}))+(j(2\boldsymbol{u}-\boldsymbol{v}_h)-j(\boldsymbol{u})). \tag{3.10}$$

Third, we have the following bounds

$$a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}_h-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)\geq\nu_0\|D(\boldsymbol{u}_h-\boldsymbol{v}_h)\|^2\geq\nu_0c\|\boldsymbol{u}_h-\boldsymbol{v}_h\|^2_{H^1(\Omega)},$$
$$a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)\leq\nu_1\|D(\boldsymbol{u}-\boldsymbol{v}_h)\|\|D(\boldsymbol{u}_h-\boldsymbol{v}_h)\|\leq\nu_1\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_{H^1(\Omega)},$$
$$a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)-a_1(\theta_h+\widetilde{\theta}_0;\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)\leq\nu_2c\|\nabla(\theta-\theta_h)\|\|\boldsymbol{u}_h\|_{W^{1,r_2}(\Omega)}\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_{H^1(\Omega)},$$
$$b(\boldsymbol{u}_h-\boldsymbol{u},p-q_h)\leq\|\boldsymbol{u}_h-\boldsymbol{u}\|_{H^1(\Omega)}\|p-q_h\|,$$
$$b(\boldsymbol{u}-\boldsymbol{v}_h,p-p_h)\leq\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}\|p-p_h\|,$$
$$(j(\boldsymbol{v}_h)-j(\boldsymbol{u}))+(j(2\boldsymbol{u}-\boldsymbol{v}_h)-j(\boldsymbol{u}))\leq(j(\boldsymbol{v}_h-\boldsymbol{u})+(j(\boldsymbol{u}-\boldsymbol{v}_h)\leq 2c\|g\|_S\|\boldsymbol{v}_{\tau,h}-\boldsymbol{u}_\tau\|_S.$$

So returning to (3.10) one gets

$$\nu_0c\|\boldsymbol{u}_h-\boldsymbol{v}_h\|^2_{H^1(\Omega)}$$
$$\leq\nu_1\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_{H^1(\Omega)}+\nu_2c\|\nabla(\theta-\theta_h)\|\|\boldsymbol{u}_h\|_{W^{1,r_2}(\Omega)}\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_{H^1(\Omega)}$$
$$+\|\boldsymbol{u}_h-\boldsymbol{u}\|_{H^1(\Omega)}\|p-q_h\|+\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}\|p-p_h\|+2c\|g\|_S\|\boldsymbol{v}_{\tau,h}-\boldsymbol{u}_\tau\|_S. \tag{3.11}$$

But application of the Young's inequality yields

$$\nu_0c\|\boldsymbol{u}_h-\boldsymbol{v}_h\|^2_{H^1(\Omega)}$$
$$\leq c\frac{\nu_1^2}{\nu_0}\|\boldsymbol{u}-\boldsymbol{v}_h\|^2_{H^1(\Omega)}+c\frac{\nu_2^2}{\nu_0}\|\nabla(\theta-\theta_h)\|^2\|\boldsymbol{u}_h\|^2_{W^{1,r_2}(\Omega)}+\|\boldsymbol{u}_h-\boldsymbol{u}\|_{H^1(\Omega)}\|p-q_h\|$$
$$+\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}\|p-p_h\|+2c\|g\|_S\|\boldsymbol{v}_{\tau,h}-\boldsymbol{u}_\tau\|_S. \tag{3.12}$$

Applying the triangle inequality on the term $\|u_h - u\|_{H^1(\Omega)}$, one obtains

$$
\begin{aligned}
&\nu_0 c\|u_h - v_h\|^2_{H^1(\Omega)} \\
&\quad \le c\frac{\nu_1^2}{\nu_0}\|u - v_h\|^2_{H^1(\Omega)} + c\frac{\nu_2^2}{\nu_0}\|\nabla(\theta-\theta_h)\|^2\|u_h\|^2_{W^{1,r_2}(\Omega)} + \|u_h - v_h\|_{H^1(\Omega)}\|p - q_h\| \\
&\qquad + \|v_h - u\|_{H^1(\Omega)}\|p - q_h\| + \|u - v_h\|_{H^1(\Omega)}\|p - p_h\| + 2c\|g\|_S\|v_{\tau,h} - u_\tau\|_S.
\end{aligned} \tag{3.13}
$$

Now Young's inequality leads to

$$
\begin{aligned}
&\nu_0 c\|u_h - v_h\|^2_{H^1(\Omega)} \\
&\quad \le \left(c\frac{\nu_1^2}{\nu_0} + \frac{1}{2}\right)\|u - v_h\|^2_{H^1(\Omega)} + c\frac{\nu_2^2}{\nu_0}\|\nabla(\theta-\theta_h)\|^2\|u_h\|^2_{W^{1,r_2}(\Omega)} \\
&\qquad + \left(\frac{c}{\nu_0} + \frac{1}{2}\right)\|p - q_h\|^2 + \|u - v_h\|_{H^1(\Omega)}\|p - p_h\| + 2c\|g\|_S\|v_{\tau,h} - u_\tau\|_S.
\end{aligned} \tag{3.14}
$$

Replacing the relations (3.5) in (3.14), one has

$$
\begin{aligned}
&\nu_0 c\|u_h - v_h\|^2_{H^1(\Omega)} \\
&\quad \le \left(c\frac{\nu_1^2}{\nu_0} + \frac{1}{2}\right)\|u - v_h\|^2_{H^1(\Omega)} + c\frac{\nu_2^2}{\nu_0}\|\nabla(\theta-\theta_h)\|^2\|u_h\|^2_{W^{1,r_2}(\Omega)} + \left(\frac{c}{\nu_0} + \frac{1}{2}\right)\|p - q_h\|^2 \\
&\qquad + 2\|u - v_h\|_{H^1(\Omega)}\|p - q_h\| + \nu_1\|u - u_h\|_{H^1(\Omega)}\|u - v_h\|_{H^1(\Omega)} \\
&\qquad + \nu_2 c\|u - v_h\|_{H^1(\Omega)}\|\nabla(\theta-\theta_h)\|\|u_h\|_{W^{1,r_2}(\Omega)} + 2c\|g\|_S\|v_{\tau,h} - u_\tau\|_S.
\end{aligned} \tag{3.15}
$$

Applying the triangle's inequality followed by the Young's inequality one obtains

$$
\begin{aligned}
&\nu_0 c\|u_h - v_h\|^2_{H^1(\Omega)} \\
&\quad \le \left(\nu_2 + \nu_1 + c\frac{\nu_1^2}{\nu_0} + 1\right)\|u - v_h\|^2_{H^1(\Omega)} + c\left(\frac{\nu_2^2}{\nu_0} + \nu_2\right)\|\nabla(\theta-\theta_h)\|^2\|u_h\|^2_{W^{1,r_2}(\Omega)} \\
&\qquad + 2c\|g\|_S\|v_{\tau,h} - u_\tau\|_S + \left(\frac{c}{\nu_0} + 1\right)\|p - q_h\|^2.
\end{aligned} \tag{3.16}
$$

It is manifest that to close the estimate (3.16) we need to estimate the error on the temperature, and this is what we do next.We take the difference between the temperature equations in (3.1) and (2.41) for $\rho = \rho_h$. One finds

$$
a_2(\theta_h - \theta, \rho_h) + d_h(u_h, \theta_h, \rho_h) - d_h(u, \theta, \rho_h) = d_h(u - u_h, \widetilde{\theta}_0, \rho_h),
$$

which is re-writing as follows; for all $s_h \in H^1_{0h}$ and $v_h \in V_h$

$$
\begin{aligned}
a_2(\theta_h - s_h, \rho_h) + d_h(u_h, \theta_h - s_h, \rho_h) &= a_2(\theta - s_h, \rho_h) + d_h(u_h, \theta - s_h, \rho_h) \\
&\quad + d_h(u - v_h, \theta, \rho_h) + d_h(v_h - u_h, \theta, \rho_h) \\
&\quad + d_h(u - v_h, \widetilde{\theta}_0, \rho_h) + d_h(v_h - u_h, \widetilde{\theta}_0, \rho_h).
\end{aligned} \tag{3.17}
$$

Thus $\theta_h - s_h$ is the solution a the temperature equation of the form (2.41). Thus application of Theorem and Proposition leads to

$$
\|\nabla(\theta_h - s_h)\| \le c\left(\begin{array}{c} \|\nabla(\theta - s_h)\| + \|f\|_{H^{-1}(\Omega)}\|\nabla(\theta - s_h)\| \\ + \|u - v_h\|_{H^1(\Omega)}(\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \|b\|_{H^{-1}(\Omega)}) \\ + \|v_h - u_h\|_{H^1(\Omega)}(\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \|b\|_{H^{-1}(\Omega)}) \end{array}\right). \tag{3.18}
$$

Thus from the inequality of the triangle we obtain

$$\begin{aligned}&\|\nabla(\theta-\theta_h)\|\\&\quad\le c\|\nabla(\theta-s_h)\|+c\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\|\nabla(\theta-s_h)\|+c\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}(\|\theta_0\|_{H^{1/2}(\partial\Omega)}+\|b\|_{H^{-1}(\Omega)})\\&\qquad+c\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_{H^1(\Omega)}(\|\theta_0\|_{H^{1/2}(\partial\Omega)}+\|b\|_{H^{-1}(\Omega)}).\end{aligned}\tag{3.19}$$

We replace (3.19) in (3.16) and apply Young inequality's with appropriate constant leads to

$$\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_{H^1(\Omega)}^2\le c\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}^2+c\|\boldsymbol{v}_{\tau,h}-\boldsymbol{u}_\tau\|_S+c\|p-q_h\|^2+c\|\nabla(\theta-s_h)\|^2,$$

from which we deduce that

$$\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_{H^1(\Omega)}\le c\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}+c\|\boldsymbol{v}_{\tau,h}-\boldsymbol{u}_\tau\|_S^{1/2}+c\|p-q_h\|+c\|\theta-s_h\|_{H^1(\Omega)}.\tag{3.20}$$

Finally application of the triangle inequality lead to the first result announced.

Second from (3.19) and (3.20) one obtains

$$\|\nabla(\theta-\theta_h)\|\le c\|\nabla(\theta-s_h)\|+c\|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)}+c\|\boldsymbol{v}_{\tau,h}-\boldsymbol{u}_\tau\|_S^{1/2}+c\|p-q_h\|.$$

The a priori error on the pressure is obtained by making use of (3.5).

Remark 3.1. In 9, Stokes equations under Tresca's condition is analyzed using mini-element for the couple velocity–pressure and a rate of convergence of order $h^{3/4}$ is derived.

In 10, error estimates for Stokes equations under slip boundary condition using $P_2\times P_1$-element and rate of convergence is derived according to the regularity of the tangential velocity on S.

In 13, penalty finite element are formulated and analyzed for the Stokes equations under Tresca's condition using $P_2\times P_1$. Optimal rate is derived provided that the velocity is H^2-up to the boundary.

One notes that, the error in these studies is dominated by the interpolation error on the slip zone S.

Remark 3.2. From the error estimate obtained it is manifest that the rate of convergence is dominated by the term appearing on the boundary S, and for actual computation of the rate of convergence we recall (see 24, p. 39) that for $1\le p\le\infty$, there exists c such that

$$\|v\|_{L^p(\partial\Omega)}\le c\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W^{1,p}(\Omega)}^{1/p},\quad\text{for all }\ v\in W^{1,p}(\Omega).\tag{3.21}$$

Thus if $(\boldsymbol{u},p,\theta)\in\boldsymbol{H}^2(\Omega)\times H^1(\Omega)\times H^2(\Omega)$, then using (3.21), one has

$$\|\boldsymbol{u}_h-\boldsymbol{u}\|_{H^1(\Omega)}+\|\theta-\theta_h\|_{H^1(\Omega)}+\|p-p_h\|\le ch^{3/4}.$$

If moreover $\boldsymbol{u}|_S\in\boldsymbol{H}^2(S)$, then

$$\|\boldsymbol{u}_h-\boldsymbol{u}\|_{H^1(\Omega)}+\|\theta-\theta_h\|_{H^1(\Omega)}+\|p-p_h\|\le ch.$$

It should be noted that if one replaces in the finite element approximation $j(\boldsymbol{v}_h)$ by its Simpson's approximation $j_h(\boldsymbol{v}_h)$, then it can be shown 10, 17 that

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_{H^1(\Omega)} + \|\theta - \theta_h\|_{H^1(\Omega)} + \|p - p_h\| \leq c\,h.$$

4. ITERATIVE SCHEME

Since the discrete problem (2.41) or (2.40) is nonlinear, it can be solved by an iterative scheme. Considering that we are dealing with two physical processes (fluid flow and convection diffusion), the need to separate these processes for easy computation is very important. Hence our goals in this paragraph are twofold. First, propose an iterative scheme that decoupled these two processes, next study the convergence of the proposed algorithm.

Before addressing these issues, it is worth noting that one of the difficulties in implementing (2.40) is the relation $\lambda_h \cdot \boldsymbol{u}_{\tau,h} = |\boldsymbol{u}_{\tau,h}|$ a.e. in *S* which is difficult to enforce. We provide next an equivalent characterization of that relation for a better derivation of iterative schemes. We claim that

Lemma 4.1. *17. The following propositions are equivalent*

a. $\lambda \in \Lambda$ *and* $\lambda \cdot \boldsymbol{u}_\tau = |\boldsymbol{u}_\tau|$ *a.e. on S,*
b. $\lambda \in \Lambda$ *and* $\int_S \boldsymbol{u}_\tau \cdot (\boldsymbol{\mu} - \lambda) d\sigma \leq 0$ *for all* $\boldsymbol{\mu} \in \Lambda$.

Next, adding and subtracting λ, the relation in (b) is re-written as follows

$$\int_S (\lambda + \gamma \boldsymbol{u}_\tau - \lambda) \cdot (\boldsymbol{\mu} - \lambda) d\sigma \leq 0 \quad \text{for all} \quad \boldsymbol{\mu}, \gamma \in \Lambda \times \mathbb{R}^+,$$

which with the help of projection theorem on Λ (since it is closed and convex) is equivalent to

$$\lambda = \mathscr{P}_\Lambda(\lambda + \gamma \boldsymbol{u}_\tau) \qquad \text{for all} \quad \gamma > 0, \tag{4.1}$$

with $\mathscr{P}_\Lambda : \boldsymbol{L}^2(S) \to \Lambda$ the projection operator defined as follows 17

$$\mathscr{P}_\Lambda(\boldsymbol{\alpha})(x) = \frac{\boldsymbol{\alpha}(x)}{\max(1, |\boldsymbol{\alpha}(x)|)}.$$

We claim that.

Lemma 4.2. *The operator* $\mathscr{P}$ *is a contraction.*

Proof. Indeed, for
$\boldsymbol{\alpha} \in \boldsymbol{L}^2(\Omega)$
, we readily obtain the result if $\max(1, |\boldsymbol{\alpha}|) = 1$, because $\mathscr{P}_\Lambda(\boldsymbol{\alpha})(x) = \boldsymbol{\alpha}(x)$. But if $\max(1, |\boldsymbol{\alpha}|) = |\boldsymbol{\alpha}(x)|$, then $\mathscr{P}_\Lambda(\boldsymbol{\alpha})(x) = \frac{\boldsymbol{\alpha}(x)}{|\boldsymbol{\alpha}(x)|}$. From 17, p. 89 one has

$$\left| \frac{\boldsymbol{a}}{|\boldsymbol{a}|} - \frac{\boldsymbol{b}}{|\boldsymbol{b}|} \right| \leq 2 \frac{|\boldsymbol{a} - \boldsymbol{b}|}{|\boldsymbol{a}| + |\boldsymbol{b}|}.$$

Remark 4.1. It should be noted that since g is nonnegative, the following statements are equivalent

a. Find $\lambda \in \Lambda$ such that $\lambda \cdot u_\tau = |u_\tau|$ a.e. in S,
b. Find $\lambda \in \Lambda$ such that $\int_S g u_\tau \cdot (\mu - \lambda) d\sigma \leq 0$ for all $\mu \in \Lambda$.
c. $\lambda = \mathscr{P}_\Lambda(\lambda + \gamma g u_\tau)$ for all $\gamma > 0$.

From Remark , the following equivalent problem can be formulated (more suitable for the derivation of iterative methods)

$$\begin{cases} \text{Find} \quad (\boldsymbol{u}, p, \theta, \lambda) \in \boldsymbol{V} \times M \times H_0^1(\Omega) \times \Lambda \quad \text{such that,} \\ \text{for all} \quad (\boldsymbol{v}, q, \rho) \in \boldsymbol{V} \times M \times H_0^1(\Omega) \quad \text{and all} \ \ \gamma > 0 \\ a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \int_S g \boldsymbol{\lambda} \cdot \boldsymbol{v}_\tau = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\ b(\boldsymbol{u}, q) = 0, \\ \lambda = \mathscr{P}_\Lambda(\lambda + \gamma g \boldsymbol{u}_\tau) \quad \text{a.e. in} \ \ S, \\ a_2(\theta, \rho) + d(\boldsymbol{u}, \theta + \widetilde{\theta}_0, \rho) = \langle b, \rho \rangle - a_2(\widetilde{\theta}_0, \rho). \end{cases} \tag{4.2}$$

The formulation (4.2) has many numerical advantages and will be used later in the design of the numerical strategy. The new unknown λ is not strictly speaking a Lagrange nor Kuhn Tucker multiplier but has some common properties with such vectors. Hence it is called multiplier by many researchers.We shall consider the following algorithm based on Uzawa iteration.
Initialization: Given $\lambda^0 \in \Lambda$, we compute $(\theta^0, \boldsymbol{u}^0, p^0)$ such that

$$\begin{cases} \text{for all} \quad (\rho, \boldsymbol{v}, q) \in H_0^1(\Omega) \times \boldsymbol{V} \times M, \\ a_2(\theta^0, \rho) = \langle b, \rho \rangle - a_2(\widetilde{\theta}_0, \rho), \\ a_1(\theta^0 + \widetilde{\theta}_0; \boldsymbol{u}^0, \boldsymbol{v}) + b(\boldsymbol{v}, p^0) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\ b(\boldsymbol{u}^0, q) = 0. \end{cases} \tag{4.3}$$

By induction, knowing $\{\boldsymbol{u}^n, p^n, \lambda^n, \theta^n\}$, we compute $\{\theta^{n+1}, \boldsymbol{u}^{n+1}, p^{n+1}, \lambda^{n+1}\}$ by solving.

for all $(\boldsymbol{v}, q) \in \boldsymbol{V} \times M$

$$a_1(\theta^n + \widetilde{\theta}_0; \boldsymbol{u}^{n+1}, \boldsymbol{v}) + b(\boldsymbol{v}, p^{n+1}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle - \int_S g \lambda^n \cdot \boldsymbol{v}_\tau d\sigma,$$
$$b(\boldsymbol{u}^{n+1}, q) = 0. \tag{4.4}$$

for all $\rho \in H_0^1(\Omega)$

$$a_2(\theta^{n+1}, \rho) + d(\boldsymbol{u}^{n+1}, \theta^{n+1} + \widetilde{\theta}_0, \rho) = \langle b, \rho \rangle - a_2(\widetilde{\theta}_0, \rho). \tag{4.5}$$

$$\text{for all} \ \ \gamma > 0, \quad \lambda^{n+1} = \mathscr{P}_\Lambda(\lambda^n + \gamma g \boldsymbol{u}_\tau^{n+1}). \tag{4.6}$$

Remark 4.2. In the algorithm (4.4)–(4.6), the fluid has been decoupled from the heat convection. We shall prove that this Uzawa's type algorithm is convergent by using the kind of energy method presented in Glowinski 17.

The approximate version of (4.4)–(4.6) is described as follows.

Initialization: Given $\lambda_h^0 = (0, 1) \in \Lambda$, we compute $(\theta_h^0, \boldsymbol{u}_h^0, p_h^0)$ such that

$$\begin{cases} \text{for all} \quad (\rho, \boldsymbol{v}, q) \in H_{0h}^1 \times \boldsymbol{V}_h \times M_h, \\ a_2(\theta_h^0, \rho) = \langle b, \rho \rangle - a_2(\widetilde{\theta}_0, \rho) \\ a_1(\theta_h^0 + \widetilde{\theta}_0; \boldsymbol{u}_h^0, \boldsymbol{v}) + b(\boldsymbol{v}, p_h^0) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\ b(\boldsymbol{u}_h^0, q) = 0. \end{cases} \tag{4.7}$$

By induction, knowing $\{\boldsymbol{u}_h^n, p_h^n, \lambda_h^n, \theta_h^n\}$, we compute $\{\theta_h^{n+1}, \boldsymbol{u}_h^{n+1}, p_h^{n+1}, \lambda_h^{n+1}\}$ by solving

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times M_h$

$$\begin{aligned} a_1(\theta_h^n + \widetilde{\theta}_0; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}) + b(\boldsymbol{v}, p_h^{n+1}) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle - \int_S g \boldsymbol{\lambda}_h^n \cdot \boldsymbol{v}_{\boldsymbol{\tau}} d\sigma, \\ b(\boldsymbol{u}_h^{n+1}, q) &= 0. \end{aligned} \tag{4.8}$$

for all $\rho \in H_{0h}^1$

$$a_2(\theta_h^{n+1}, \rho) + d_h(\boldsymbol{u}_h^{n+1}, \theta_h^{n+1} + \widetilde{\theta}_0, \rho) = \langle b, \rho \rangle - a_2(\widetilde{\theta}_0, \rho). \tag{4.9}$$

$$\text{for all} \;\; \gamma > 0, \quad \boldsymbol{\lambda}_h^{n+1} = \mathscr{P}_\Lambda(\boldsymbol{\lambda}_h^n + \gamma g \boldsymbol{u}_{\boldsymbol{\tau}, h}^{n+1}). \tag{4.10}$$

We recall that $d_h(\cdot, \cdot, \cdot)$ has the same properties as $d(\cdot, \cdot, \cdot)$, and because we are dealing with conforming finite element, the study of the convergence of the algorithm (4.8)–(4.10) is not different its continuous counterpart (4.4)–(4.6).

4.1 A priori estimates

In this subsection, we discuss the feasibility of the algorithm (4.4)–(4.6) and establish some a priori bounds.

The first step can be recast as follows; Given $\boldsymbol{u}^n, p^n$ and λ^n, Find $(\boldsymbol{u}^{n+1}, p^{n+1})$ such that for all $(\boldsymbol{v}, q) \in \boldsymbol{V} \times M$

$$\begin{aligned} a_1(\theta^n + \widetilde{\theta}_0; \boldsymbol{u}^{n+1}, \boldsymbol{v}) + b(\boldsymbol{v}, p^{n+1}) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle - \int_S g \boldsymbol{\lambda}^n \cdot \boldsymbol{v}_{\boldsymbol{\tau}} d\sigma, \\ b(\boldsymbol{u}^{n+1}, q) &= 0. \end{aligned} \tag{4.11}$$

The variational problem (4.11) is a perturbed Stokes equations. Hence the existence and uniqueness of solution is obtained from the same conditions needed for the Stokes equations 18, 19. Furthermore we claim that

Proposition 4.1. *Let $(\boldsymbol{u}^{n+1}, p^{n+1})$ be the solution of (4.11). There are c_1, c_2, c_3 independent of n such that the following a priori estimates hold for all $n \geq 1$*

$$\|u^n\|^2 \le \frac{c_1}{\nu_0^2}(\|f\|^2_{H^{-1}(\Omega)} + \|g\|^2_S),$$
$$\|u^n\|^2_{H^1(\Omega)} \le \frac{c_2}{\nu_0^2}(\|f\|^2_{H^{-1}(\Omega)} + \|g\|^2_S),$$
$$\|p^n\| \le c_3\left(1+\frac{1}{\nu_0}\right)(\|f\|_{H^{-1}(\Omega)} + \|g\|_S).$$

The following relation will be used throughout this work

$$ab \le \frac{\varepsilon}{p}a^p + \frac{1}{q\varepsilon^{q/p}}b^q, \quad \text{for all } a,b,\varepsilon > 0, \quad \text{with } \frac{1}{p}+\frac{1}{q} = 1. \tag{4.12}$$

Proof of Proposition 4.1. We take
$v = 2u^{n+1}$
in (4.11) and obtain

$$2a_1(\theta^n + \widetilde{\theta}_0; u^{n+1}, u^{n+1}) = 2\langle f, u^{n+1}\rangle - 2\int_S g\lambda^n \cdot u_\tau^{n+1}\, d\sigma. \tag{4.13}$$

The relation (2.6), the inequality of the trace and (2.5) gives

$$\begin{aligned} 2\langle f, u^{n+1}\rangle &\le 2\|f\|_{H^{-1}(\Omega)}\|\nabla u^{n+1}\| \le 2\|f\|_{H^{-1}(\Omega)}\|Du^{n+1}\| \\ &\le \frac{c}{\nu_0}\|f\|^2_{H^{-1}(\Omega)} + \frac{\nu_0}{2}\|Du^{n+1}\|^2, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} 2\int_S g\lambda^n \cdot u_\tau^{n+1}\, d\sigma \le 2\int_S g|\lambda^n\|u_\tau^{n+1}|\, d\sigma &\le 2\|g\|_S\|u_\tau^{n+1}\|_S \\ &\le 2c\|g\|_S\|\nabla u^{n+1}\| \\ &\le 2c\|g\|_S\|Du^{n+1}\| \\ &\le \frac{c}{\nu_0}\|g\|^2_S + \frac{\nu_0}{2}\|Du^{n+1}\|^2. \end{aligned} \tag{4.15}$$

Hence using the coercivity of $a_1(\cdot,\cdot)$ on the left hand side of (4.13), together with (4.14) and (4.15), one obtains

$$\nu_0\|Du^{n+1}\|^2 \le \frac{c}{\nu_0}\|f\|^2_{H^{-1}(\Omega)} + \frac{c}{\nu_0}\|g\|^2_S. \tag{4.16}$$

We deduce the inequalities on the velocity after utilization of (2.5), (2.3) in (4.16).Next, from the inf-sup condition (2.19) and the first relation in (3.1), one has

$$\begin{aligned} \beta\|p^{n+1}\| &\le \sup_{v\in H_0^1(\Omega)} \frac{b(v,q)}{\|v\|_{H^1(\Omega)}} \\ &\le \sup_{v\in V} \frac{\langle f, v\rangle - a_1(\theta^n + \widetilde{\theta}_0; u^{n+1}, v)}{\|v\|_{H^1(\Omega)}} \\ &\le \|f\|_{H^{-1}(\Omega)} + \nu_1\|u^{n+1}\|_{H^1(\Omega)}, \end{aligned}$$

thus the inequality is obtained by replacing the estimates on $\|\boldsymbol{u}^{n+1}\|_{H^1(\Omega)}$.

The second step within the iterative scheme is re-written as follows:

Find $\theta^{n+1} \in H_0^1(\Omega)$ such that

$$\text{for all } \rho \in H_0^1(\Omega), \qquad a_2^1(\theta^{n+1}, \rho) = \ell_2^1(\rho). \tag{4.17}$$

with

$$\begin{aligned} a_2^1(\theta^{n+1}, \rho) &= a_2(\theta^{n+1}, \rho) + d(\boldsymbol{u}^{n+1}, \theta^{n+1}, \rho), \\ \ell_2^1(\rho) &= \langle b, \rho\rangle - a_2(\widetilde{\theta}_0, \rho) - d(\boldsymbol{u}^{n+1}, \widetilde{\theta}_0, \rho). \end{aligned} \tag{4.18}$$

It is readily checked that problem (4.17) has a unique solution as a consequence of the properties of the trilinear form $d(\cdot, \cdot, \cdot)$ and the bilinear form $a_2(\cdot, \cdot)$.
Proposition 4.2. *Let θ^{n+1} be the solution of (4.17). There are c_1, c_2 (both independent of n) such that for $n \geq 1$ then the following a priori estimates hold*

$$\begin{aligned} \|\theta^n\|^2 &\leq c_1 \left(\frac{1}{\kappa^2} \|b\|^2_{H^{-1}(\Omega)} + \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa^2} \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} \right), \\ \|\theta^n\|^2_{H^1(\Omega)} &\leq c_2 \left(\frac{1}{\kappa^2} \|b\|^2_{H^{-1}(\Omega)} + \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa^2} \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} \right). \end{aligned}$$

Proof. We take $\rho = 2\theta^{n+1}$ in (4.17) and obtain

$$2\kappa\|\nabla\theta^{n+1}\|^2 = 2\langle b, \theta^{n+1}\rangle - 2a_2(\widetilde{\theta}_0, \theta^{n+1}) - 2d(\boldsymbol{u}^{n+1}, \widetilde{\theta}_0, \theta^{n+1}).$$

By using the standard inequalities we have

$$\begin{aligned} 2\langle b, \theta^{n+1}\rangle &\leq 2\|b\|_{H^{-1}(\Omega)}\|\nabla\theta^{n+1}\| \leq \frac{1}{\varepsilon_1}\|b\|^2_{H^{-1}(\Omega)} + \varepsilon_1\|\nabla\theta^{n+1}\|^2, \\ 2a_2(\widetilde{\theta}_0, \theta^{n+1}) &\leq 2\kappa\|\nabla\widetilde{\theta}_0\|\|\nabla\theta^{n+1}\| \\ &\leq 2c\kappa\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\nabla\theta^{n+1}\| \\ &\leq \frac{c\kappa}{\varepsilon_2}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \kappa\varepsilon_2\|\nabla\theta^{n+1}\|^2. \end{aligned}$$

From the properties of $d(\cdot, \cdot, \cdot)$ and (2.14)

$$\begin{aligned} 2d(\boldsymbol{u}^{n+1}, \widetilde{\theta}_0, \theta^{n+1}) &\leq 2\|\boldsymbol{u}^{n+1}\widetilde{\theta}_0\|\|\nabla\theta^{n+1}\| \leq 2\|\boldsymbol{u}^{n+1}\|_{L^4(\Omega)}\|\widetilde{\theta}_0\|_{L^4(\Omega)}\|\nabla\theta^{n+1}\| \\ &\leq 2\delta\|\boldsymbol{u}^{n+1}\|_{L^4(\Omega)}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\nabla\theta^{n+1}\| \\ &\leq \frac{1}{\varepsilon_3}\delta^2\|\boldsymbol{u}^{n+1}\|^2_{L^4(\Omega)}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \varepsilon_3\|\nabla\theta^{n+1}\|^2. \end{aligned}$$

We deduce for $\delta = 1/\|\boldsymbol{u}^{n+1}\|_{L^4(\Omega)}$, $\varepsilon_1 = \varepsilon_3 = \kappa/3$ and $\epsilon_2 = 1/3$ that

$$\kappa\|\nabla\theta^{n+1}\|^2 \leq \frac{3}{\kappa}\|b\|^2_{H^{-1}(\Omega)} + c\kappa\|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}. \tag{4.19}$$

Hence we deduce the estimates on the temperature.

The next result indicate a “consistency” of the algorithm (4.4)–(4.6). In fact we claim that

Lemma 4.3. *The algorithm (4.4)–(4.6) is consistent with the problem (2.12) in the following sense. If* $(\boldsymbol{u},\theta,p,\lambda)$ *is the solution of* (2.12) *and there exists an integer m such that* $(\boldsymbol{u}^m,\theta^m,p^m,\lambda^m)=(\boldsymbol{u},\theta,p,\lambda)$, *then for all* $n\geq m$, $(\boldsymbol{u}^n,\theta^n,p^n,\lambda^n)=(\boldsymbol{u},\theta,p,\lambda)$.

Proof. It is done by induction and it suffice to show that if
$(\boldsymbol{u}^n,\theta^n,p^n,\lambda^n)=(\boldsymbol{u},\theta,p,\lambda)$
then $(\boldsymbol{u}^{n+1},\theta^{n+1},p^{n+1},\lambda^{n+1})=(\boldsymbol{u},\theta,p,\lambda)$.

Using the induction hypothesis in (4.4), one has

$$a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}^{n+1},\boldsymbol{v})+b(\boldsymbol{v},p^{n+1})=\langle \boldsymbol{f},\boldsymbol{v}\rangle-\int_S g\lambda\cdot \boldsymbol{v}_{\boldsymbol{\tau}}d\sigma,$$
$$b(\boldsymbol{u}^{n+1},q)=0. \tag{4.20}$$

By subtracting (4.20) with the first two equations of (2.12) one has

$$a_1(\theta+\widetilde{\theta}_0;\boldsymbol{u}^{n+1}-\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},p^{n+1}-p)=0,$$
$$b(\boldsymbol{u}^{n+1}-\boldsymbol{u},q)=0. \tag{4.21}$$

At this junction we apply Proposition and obtain that $\boldsymbol{u}^{n+1}=\boldsymbol{u}$ and $p^{n+1}=p$.

Next from (3.2) we have

$$a_2(\theta^{n+1},\rho)+d(\boldsymbol{u},\theta^{n+1}+\widetilde{\theta}_0,\rho)=\langle b,\rho\rangle-a_2(\widetilde{\theta}_0,\rho),$$

while from (2.12)

$$a_2(\theta,\rho)+d(\boldsymbol{u},\theta+\widetilde{\theta}_0,\rho)=\langle b,\rho\rangle-a_2(\widetilde{\theta}_0,\rho).$$

Therefore

$$a_2(\theta^{n+1}-\theta,\rho)+d(\boldsymbol{u},\theta^{n+1}-\theta,\rho)=0.$$

For $\rho=\theta^{n+1}-\theta$, we deduce that $\theta^{n+1}=\theta$. We easily deduce that $\lambda^{n+1}=\lambda$.

4.2 Convergence

In this paragraph, we are interested on the convergence analysis of the algorithm (4.4)–(4.6) when *n* tends to infinity. We claim that

Theorem 4.1. *Let* $(\boldsymbol{u}^{n+1},p^{n+1},\theta^{n+1},\lambda^{n+1})$ *be the solution of (4.4)–(4.6) with the regularity* $\boldsymbol{u}^n\in \mathbf{W}^{1,p_2}(\Omega)$.

Let $(\boldsymbol{u},\theta,p,\lambda)$ *be the solution of (2.12) with* $\boldsymbol{u}\in \mathbf{W}^{1,p_2}(\Omega)$.

There exists a positive constant c independent of h such that if $(\gamma,\kappa,\theta_0,b)$ *are taken such that*

$$c\gamma\|g\|^2_{L^\infty(S)}+\frac{v_2^2c\|D\boldsymbol{u}\|^2_{L^{p_2}(\Omega)}}{\kappa}+c\|\theta_0\|_{H^{1/2}(\partial\Omega)}+\frac{c}{\kappa}\|b\|_{H^{-1}(\Omega)}+\frac{c}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\leq 2v_0,$$
$$\frac{c}{\kappa}\|b\|_{H^{-1}(\Omega)}+\frac{c}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}+c\|\theta_0\|_{H^{1/2}(\partial\Omega)}\leq\kappa, \tag{4.22}$$

then the following properties hold

$u^n \to u$ *strongly in* V, $\theta^n \to \theta$ *strongly in* $H^1(\Omega)$,
and the sequence $\{\|\lambda^n - \lambda\|_S\}_n$ *converges.*

Proof. We will adopt the following notation
$\overline{a}^n = a^n - a$
.

We also recall that (2.30) together with (2.31) will be used here.

We let $v = u^{n+1} - u \equiv \overline{u}^{n+1}$ and take the difference between the velocities equations in (3.1) and (2.12). One obtains

$$\int \nu(\theta^n + \widetilde{\theta}_0)|D\overline{u}^{n+1}|^2 = \int (\nu(\theta + \widetilde{\theta}_0) - \nu(\theta^n + \widetilde{\theta}_0))Du : D\overline{u}^{n+1} - \int_S g\overline{\lambda}^n \cdot \overline{u}_\tau^{n+1} d\sigma. \tag{4.23}$$

Next, we take the difference between the temperatures equations in (3.2) and (2.12) for $\rho = \overline{\theta}^{n+1}$. We use the properties of $d(\cdot, \cdot, \cdot)$, Proposition , and (2.14) we find

$$\begin{aligned} \kappa\|\nabla\overline{\theta}^{n+1}\|^2 &= -d(u^{n+1}, \theta^{n+1} + \widetilde{\theta}_0, \overline{\theta}^{n+1}) + d(u, \theta + \widetilde{\theta}_0, \overline{\theta}^{n+1}) \\ &= -d(\overline{u}^{n+1}, \theta^{n+1}, \overline{\theta}^{n+1}) - d(\overline{u}^{n+1}, \widetilde{\theta}_0, \overline{\theta}^{n+1}). \end{aligned} \tag{4.24}$$

Adding (4.23) and (4.24), we find

$$\begin{aligned} \int_S g\overline{\lambda}^n \cdot \overline{u}_\tau^{n+1} = &- \int \nu(\theta^n + \widetilde{\theta}_0)|D\overline{u}^{n+1}|^2 - \kappa\|\nabla\overline{\theta}^{n+1}\|^2 \\ &+ \int (\nu(\theta + \widetilde{\theta}_0) - \nu(\theta^n + \widetilde{\theta}_0))Du : D\overline{u}^{n+1} \\ &- d(\overline{u}^{n+1}, \theta^{n+1}, \overline{\theta}^{n+1}) - d(\overline{u}^{n+1}, \widetilde{\theta}_0, \overline{\theta}^{n+1}). \end{aligned} \tag{4.25}$$

We recall that for $\gamma > 0$, one has

$$\lambda = P_\Lambda(\lambda + \gamma g u_\tau), \qquad \lambda^{n+1} = P_\Lambda(\lambda^n + \gamma g u_\tau^{n+1}).$$

Using the fact that P_Λ is a contraction mapping, we obtain

$$\|\overline{\lambda}^{n+1}\|_S \le \|\overline{\lambda}^n + \gamma g\overline{u}_\tau^{n+1}\|_S.$$

Thus taking the square on both sides and making use of Holder's inequality, trace's inequality and (4.25), one obtains

$$
\begin{aligned}
\|\overline{\lambda}^{n+1}\|_S^2 - \|\overline{\lambda}^n\|_S^2 &\le \gamma^2\|g\overline{u}_\tau^{n+1}\|_S^2 + 2\gamma\int_S g\overline{\lambda}^n\cdot\overline{u}_\tau^{n+1}d\sigma \\
&\le c\gamma^2\|g\|_{L^\infty(S)}^2\|\nabla\overline{u}^{n+1}\|^2 + 2\gamma\int_S g\overline{\lambda}^n\cdot\overline{u}_\tau^{n+1}d\sigma \\
&\le c\gamma^2\|g\|_{L^\infty(S)}^2\|\nabla\overline{u}^{n+1}\|^2 - 2\gamma\int\nu(\theta^n+\widetilde{\theta}_0)|D\overline{u}^{n+1}|^2 - 2\gamma\kappa\|\nabla\overline{\theta}^{n+1}\|^2 \\
&\quad + 2\gamma\int(\nu(\theta+\widetilde{\theta}_0)-\nu(\theta^n+\widetilde{\theta}_0))Du : D\overline{u}^{n+1} \\
&\quad - 2\gamma d(\overline{u}^{n+1},\theta^{n+1},\overline{\theta}^{n+1}) - 2\gamma d(\overline{u}^{n+1},\widetilde{\theta}_0,\overline{\theta}^{n+1}).
\end{aligned} \tag{4.26}
$$

In order to close the inequality (4.26), we need to estimate the terms on the right-hand side of (4.26). First, from the mean value theorem, (1.5), the generalized Holder's inequality (2.30) (with the same parameters as before), one finds

$$
\begin{aligned}
\int_\Omega(\nu(\theta+\widetilde{\theta}_0)-\nu(\theta^n+\widetilde{\theta}_0))Du : D\overline{u}^{n+1} &\le \nu_2\int\overline{\theta}^n\|Du\|D\overline{u}^{n+1}| \\
&\le \nu_2\|\overline{\theta}^n\|_{L^{p_1}(\Omega)}\|Du\|_{L^{p_2}(\Omega)}\|D\overline{u}^{n+1}\|.
\end{aligned}
$$

Now using the Sobolev inequality (2.31) (with the same parameters as before), one has

$$
\|\overline{\theta}^n\|_{L^{p_1}(\Omega)} \le c\|\nabla\overline{\theta}^n\|.
$$

Thus

$$
\int_\Omega(\nu(\theta+\widetilde{\theta}_0)-\nu(\theta^n+\widetilde{\theta}_0))Du : D\overline{u}^{n+1} \le \nu_2 c\|Du\|_{L^{p_2}(\Omega)}\|\nabla\overline{\theta}^n\|\|D\overline{u}^{n+1}\|. \tag{4.27}
$$

Second the properties of $d(\cdot,\cdot,\cdot)$, Proposition , and (2.14) together lead to

$$
\begin{aligned}
&d(\overline{u}^{n+1},\theta^{n+1},\overline{\theta}^{n+1}) + d(\overline{u}^{n+1},\widetilde{\theta}_0,\overline{\theta}^{n+1}) \\
&\quad\le \|\overline{u}^{n+1}\|_{L^4(\Omega)}\|\nabla\theta^{n+1}\|\|\overline{\theta}^{n+1}\|_{L^4(\Omega)} + \|\overline{u}^{n+1}\|_{L^4(\Omega)}\|\nabla\widetilde{\theta}_0\|\|\overline{\theta}^{n+1}\|_{L^4(\Omega)} \\
&\quad\le c\left(\frac{1}{\kappa}\|b\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\|\nabla\overline{u}^{n+1}\|\|\nabla\overline{\theta}^{n+1}\| \\
&\quad + c\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\nabla\overline{u}^{n+1}\|\|\nabla\overline{\theta}^{n+1}\|.
\end{aligned} \tag{4.28}
$$

Returning to (4.26) with (4.27), (4.28) and using (1.5), one has

$$
\begin{aligned}
&\|\overline{\lambda}^{n+1}\|_S^2 - \|\overline{\lambda}^n\|_S^2 \\
&\quad\le \gamma(c\gamma\|g\|_{L^\infty(S)}^2 - 2\nu_0)\|\nabla\overline{u}^{n+1}\|^2 + 2\gamma\nu_2 c\|Du\|_{L^{p_2}(\Omega)}\|\nabla\overline{\theta}^n\|\|\nabla\overline{u}^{n+1}\| \\
&\qquad + \gamma c\left(\frac{1}{\kappa}\|b\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\|\nabla\overline{\theta}^{n+1}\|\|\nabla\overline{u}^{n+1}\| \\
&\qquad + c\gamma\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\nabla\overline{\theta}^{n+1}\|\|\nabla\overline{u}^{n+1}\| - 2\gamma\kappa\|\nabla\overline{\theta}^{n+1}\|^2,
\end{aligned}
$$

which with the help of Young's inequality implies that

$$(\|\overline{\lambda}^{n+1}\|_S^2 + \gamma\kappa\|\nabla\overline{\theta}^{n+1}\|^2) - (\|\overline{\lambda}^{n}\|_S^2 + \gamma\kappa\|\nabla\overline{\theta}^{n}\|^2)$$
$$\leq \gamma\left(c\gamma\|g\|^2_{L^\infty(S)} - 2\nu_0 + \frac{\nu_2^2 c\|Du\|^2_{L^{r_2}(\Omega)}}{\kappa} + c\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{c}{\kappa}\|b\|_{H^{-1}(\Omega)} + \frac{c}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\|\nabla\overline{u}^{n+1}\|^2$$
$$+\gamma\left(\frac{c}{\kappa}\|b\|_{H^{-1}(\Omega)} + \frac{c}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)} + c\|\theta_0\|_{H^{1/2}(\partial\Omega)} - \kappa\right)\|\nabla\overline{\theta}^{n+1}\|^2. \tag{4.29}$$

Having in mind (4.22), it appears that the right hand side of (4.29) is nonpositive. Thus the sequence $(\|\overline{\lambda}^{n+1}\|_S^2 + \gamma\kappa\|\nabla\overline{\theta}^{n+1}\|^2)_n$ is positive and decreasing. Hence the following hold true

$$\lim_{n\to\infty}[(\|\overline{\lambda}^{n+1}\|_S^2 + \gamma\kappa\|\nabla\overline{\theta}^{n+1}\|^2) - (\|\overline{\lambda}^{n}\|_S^2 + \gamma\kappa\|\nabla\overline{\theta}^{n}\|^2)] = 0. \tag{4.30}$$

From (4.30) and (4.29), one deduces that

$$\lim_{n\to\infty}\|\nabla\overline{u}^n\| = 0, \qquad \lim_{n\to\infty}\|\nabla\overline{\theta}^n\| = 0. \tag{4.31}$$

Combining (4.31) and the fact that $(\|\overline{\lambda}^{n}\|_S^2 + \gamma\kappa\|\nabla\overline{\theta}^{n}\|^2)_n$ converges, one deduces that $(\|\overline{\lambda}^{n}\|_S)_n$ converges and

$$\lim_{n\to\infty}[\|\overline{\lambda}^{n+1}\|_S - \|\overline{\lambda}^{n}\|_S] = 0. \tag{4.32}$$

Remark 4.3. Several observations about the convergence of the algorithm (4.4)–(4.6) are in order.

First, choosing ρ through (4.22) is impractical because of the presence of several unknowns expressions such as the constant c, $\|Du\|_{L^{r_2}(\Omega)}$. In our numerical experiments, we show that only limited values of γ are allowed to achieve convergence of the algorithm discussed.

Second, the convergence of $(\lambda_n)_n$ is more complicated and we refer the interested reader to 25 (Chapter 4).

5. NUMERICAL EXPERIMENTS AND CONCLUSION

All computations were performed using Matlab on DELL i3 with 8 GB RAM. The test problems used are designed to illustrate the behavior of the algorithm more than to model an actual phenomenon. The algorithm described will be tested computationally. We stop the computations when the following condition is satisfied

$$\sqrt{\frac{\|u_h^{n+1} - u_h^n\|^2 + \|\theta_h^{n+1} - \theta_h^n\|^2}{\|u_h^n\|^2 + \|\theta_h^n\|^2}} \leq tol \equiv 6.0e-5.$$

For the linear system (4.4) (see Step 1), an incomplete LU factorization (iLU) is performed and the result is used as preconditioner in the GMRES solver. The linear system (4.5) (see Step2) is solved by Gaussian elimination.

5.1 Example 1: choice of γ

The objective is to compute the relative error with respect to γ in order to achieve the convergence of algorithm for a value of g such that $|(\boldsymbol{T}\boldsymbol{n})_\tau| > g$. This will allow us to see a range of admissible values of γ. For that purpose, we consider the unit square $\Omega = (0, 1)^2$ and we assume that its boundary consists of two portions Γ and S defined as follows

$$\Gamma = \{0\} \times (0,1) \cup (0,1) \times \{0\}$$
$$S = S_1 \cup S_2 \text{ and } S_1 = (0,1) \times \{1\}, \quad S_2 = \{1\} \times (0,1).$$

We consider

$$\begin{cases} u_1(x,y) = 20x^2(1-x)^2 y(2-3y), \\ u_2(x,y) = -20(2x - 6x^2 + 4x^3)y^2(1-y), \\ p(x,y) = (2x-1)(2y-1), \\ \theta(x,y) = xy(1-x)(1-y), \end{cases} \tag{5.1}$$

and

$$\nu(\theta) = \frac{1}{2}e^{-\theta} + \frac{1}{4} \quad \text{for which one has} \quad \frac{1}{4} \leq \nu(\theta) \leq \frac{3}{4}.$$

We adjust (f, b) such that

$$-2\operatorname{div}\nu(\theta)D\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega,$$

$$-\kappa\Delta\theta + (\boldsymbol{u} \cdot \nabla)\theta = b \quad \text{in } \Omega.$$

By direct computation and considering $\boldsymbol{n} = (0,1)^T$ and $\boldsymbol{\tau} = (1,0)^T$, one has

$$(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} = -80\nu(\theta)x^2(1-x)^2\boldsymbol{\tau} \quad \text{on} \quad S \quad \text{and} \quad \max_S |(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}| = 3.75.$$

We let $\kappa = 1$, and choose g such that $g < \max_S|(\boldsymbol{T}\boldsymbol{n})_\tau| = 3.75$. Because we do not have the exact solution of boundary value problem (1.1)–(1.5), we assume that the finite element solution obtained for $h = 1/128$ is the reference solution. We compute the relative error

$$RE = \frac{\|\nabla(\boldsymbol{u}_h^n - \boldsymbol{u}_{ref}^n)\|^2 + \|\nabla(\theta_h^n - \theta_{ref}^n)\|^2}{\|\nabla \boldsymbol{u}_{ref}^n\|^2 + \|\nabla\theta_{ref}^n\|^2}$$

for different values of γ. The results reported in Table 1 show the convergence of the proposed algorithm with respect to γ. It is also observable that we do not have convergence for $\gamma = 1, 5, 12$ and $g = 4.7$. Hence the results obtained are in agreement with Theorem .

Table 1. Convergence of the algorithm with respect to γ

γ	10^{-2}	10^{-1}	0.25	0.5	0.75	1	5	12
RE(g = 0.8)	0.0993	0.0993	0.0992	0.0992	0.0992	0.0992	0.0992	0.0992
CPU time	74.15	40.25	14.55	8.75	7.25	6.33	2.89	1.75
Iterations	307	89	55	36	30	26	12	7
RE(g = 4.7)	0.0992	0.0992	0.0992	0.0992	0.0992	–	–	–
CPU time	71.32	17.80	9.75	5.46	3.81	–	–	–
Iterations	295	75	41	23	16	–	–	–

Next, we report in Table 2 the number of iterations and CPU time needed to achieve the convergence for different values of g when γ = 0.25, and for different values of h. It appears that there is no direct correlation between the number of iterations required to rich convergence of the algorithm and the mesh size.

Table 2. Number of iterations and CPU (s)

	$0.8 = g < \max_S \lvert(\boldsymbol{Tn})_\tau\rvert$		$3.75 = g = \max_S \lvert(\boldsymbol{Tn})_\tau\rvert$		$4.7 = g > \max_S \lvert(\boldsymbol{Tn})_\tau\rvert$	
h	**Iter**	**CPU**	**Iter**	**CPU**	**Iter**	**CPU**
1/4	56	0.31	31	0.21	48	0.24
1/8	51	1.28	48	1.16	43	1.54
1/16	55	12.94	45	10.71	41	9.80
1/32	51	139.43	41	113.62	42	116.2
1/64	51	825	45	313	43	284.6

5.2 Example 2: driven cavity flow

This is classical example that has been studied by among others 11, 12 using classical Tresca's condition. The nonlinear slip condition we use is the one formulated by Leroux 5 (1.4).

We consider the problem described in Example 1, with γ = 0.25 and the goal is to identify the slipping/sticking zone depending on the values of g.

Figure 1 is concerned with the situation where $\max_S \lvert(\boldsymbol{Tn})_\tau\rvert > g = 0.8$. It is apparent from the graphs showing streamlines or velocity field that $\boldsymbol{u}_\tau|_S \neq \mathbf{0}$ on $S = S_1 \cup S_2$. Hence the nonlinear slip occurs here.

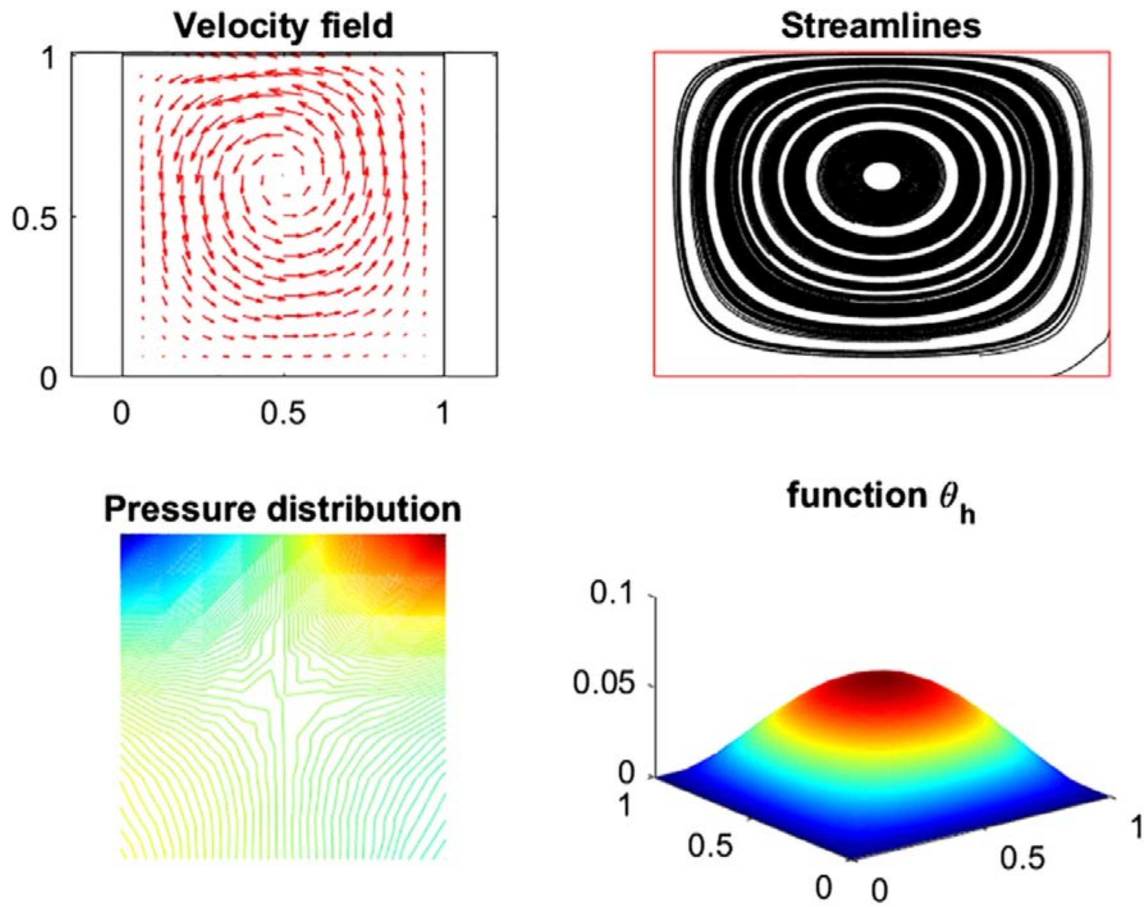

Figure 1. g **= 0.8**

In Figures 2 and 3 one has $\max_S |(Tn)_\tau| \le g$. It is apparent that there are no contact between the fluid and the boundary $\partial\Omega$ of the domain. Hence one can conclude that $u_\tau = 0$ on $S = S_1 \cup S_2$. Therefore no slip is noted. But on the other side, a direct computation reveal that for $u = (u_1, u_2)$ defined in (5.1), $u_\tau|_S \neq 0$. Hence the velocity field given by (1.1)–(1.5) differ from the velocity field defined in (5.1).

One notes through Figures 1-3 that the temperature is nonnegative and bounded from above (this observation is not supported by the theory discussed). Finally, in these three figures, the pressure distribution does not change too much. We believe that small variation on the pressure is due to the fact that the slip condition (1.4) does not take into account the pressure which is by the way only defined in the interior of the domain. Hence the simulations results are in a way in agreement with the slip relation (1.4).

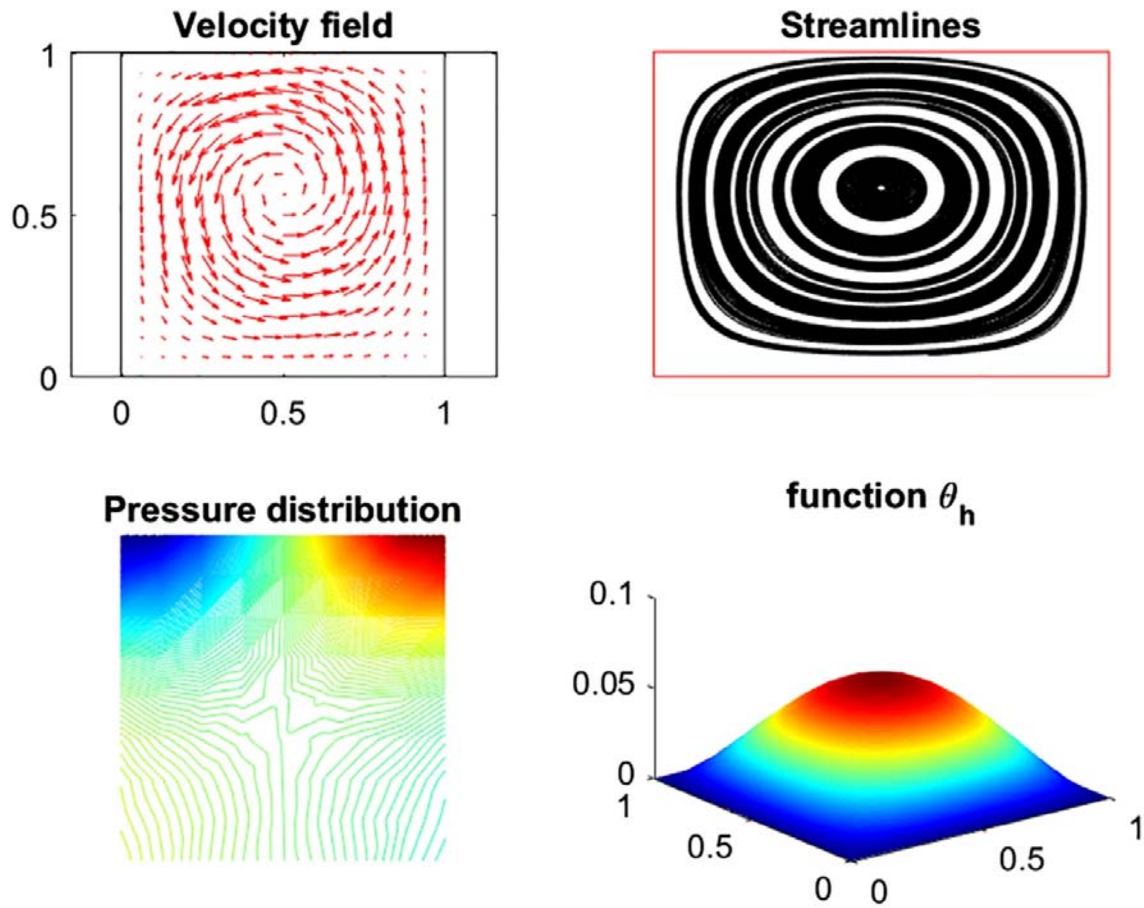

Figure 2. g = 3.75

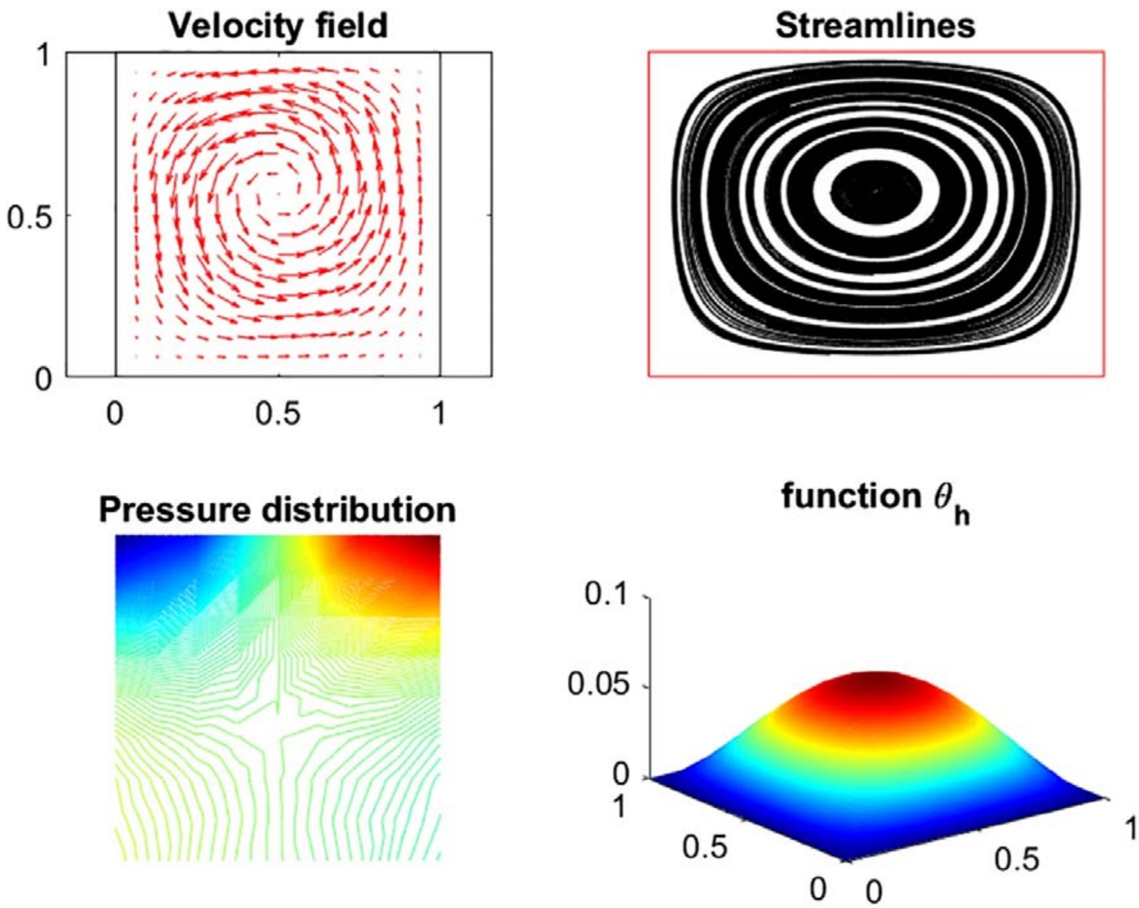

Figure 3. g = 4.7

5.3 Example 3: convergence check

One considers the same problem described in Example 1. Since we do not have the exact solution, then it is assume that the solution obtained for h = 1/128 is the reference solution, and from the simulations in Example 2, we take g = 0.8 to ensure the existence of slip condition on S. We replace $j(v) = (g, v_\tau)_S$ by its Simpson's approximation. It is noted from Tables 3 and 4 that the convergence rate is linear for the quantity $\|u_{ref} - u_h\|_1 + \|\theta_{ref} - \theta_h\|_1 + \|p_{ref} - p_h\|$. Once again these simulations are supported by the theory (Remark).

Table 3. Convergence rates with function g = 0.8

h	$\|u_{ref} - u_h\|$	Rate	$\|u_{ref} - u_h\|_1$	Rate	$\|p_{ref} - p_h\|$	Rate
1/4	7.28e-02		6.14e-01		1.07e-01	
1/8	1.80e-02	2.01	3.02e-01	1.01	4.30e-02	1.31
1/16	4.55e-03	1.98	1.37e-01	1.14	9.76e-02	1.18
1/32	8.55e-04	2.41	2.75e-01	1.00	4.67e-02	1.06
1/64	1.96e-04	2.12	5.51e-01	1.00	1.95e-02	1.26

Table 4. Convergence rates with function g = 0.8

h	$\|\theta_{ref} - \theta_h\|$	Rate	$\|\theta_{ref} - \theta_h\|_1$	Rate
1/4	7.25e-05		2.20e-03	
1/8	9.83e-06	2.48	6.46e-04	1.56
1/16	1.81e-06	2.43	2.53e-04	1.35
1/32	3.92e-07	2.21	1.11e-04	1.18
1/64	7.59e-08	2.36	4.34e-05	1.35

5.4 Concluding remarks

We have investigated the numerical approximation of the Stokes flow under nonlinear slip boundary condition coupled with the heat equation. The conditions under which the weak solution and its finite element counterpart are uniquely defined are highlighted. We have shown convergence of the finite element approximation considered and established that order one rate of convergence can be obtained if greater regularity of the tangential velocity field on the slip zone is imposed. To compute the approximate solution, we have proposed an iterative scheme based on Uzawa-type algorithm and studied its convergence. Finally, we have exhibited some numerical experiments that validate the theoretical findings.

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