Metric aspects of noncommutative geometry

by

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Submitted in partial fulfilment of the requirements for the degree
Magister Scientiae in Physics
in the Faculty of Natural and Agricultural Sciences
University of Pretoria
Pretoria
May 2019

Abstract

We study noncommutative geometry from a metric point of view by constructing examples of spectral triples and explicitly calculating Connes's spectral distance between certain associated pure states. After considering instructive finite-dimensional spectral triples, the noncommutative geometry of the infinite-dimensional Moyal plane is studied. The corresponding spectral triple is based on the Moyal deformation of the algebra of Schwartz functions on the Euclidean plane.

Declaration

I, Werndly Jakobus van Staden, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

SIGNATURE: ....

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Acknowledgements

I am particularly grateful for the guidance and supervision offered by Professor Rocco Duvenhage. The elegance that accompanies his profound mathematical intuition continues to reassure and inspire me. I wish to extend my gratitude to NITheP for their financial assistance.

Contents

Introduction

Even though particle experiments are delving deeper into the structure of matter, the geometric structure of spacetime is still unknown. Classical geometry cannot account for both general relativity and quantum mechanics, since the former grounds its description of gravitation on purely geometric concepts while the latter renounces intuitive geometric concepts. Noncommutative geometry attempts to bridge this gap by providing a mathematical framework for a geometric understanding of fundamental interactions and thus opening a path toward quantising gravity. [21]

In classical Riemannian geometry, the pointwise multiplication of functions on a manifold makes the space of functions an infinite-dimensional commutative algebra such that $f_1 f_2 = f_2 f_1$. Such an algebra, together with a Dirac operator, characterises the geometry completely. Noncommutative geometry generalises this algebraic conception of geometry to a case where the analogues of functions or "coordinates" on a space no longer commute. Ergo, the enterprise of noncommutative geometry is to reformulate, as much as possible, the geometry of a manifold in terms of an algebra of functions defined on it and then to generalise the corresponding results of differential geometry to the case of a noncommutative algebra [24]. Such noncommutative spaces are present, for example, in quantum mechanics for the phase space of a particle, where the functions on phase space are replaced by noncommuting operators in Hilbert space. The advantage of passing from the commutative to the noncommutative case is that we drop the notion of a localised point, while still being able to measure distances [5].

An essential step in this generalisation to noncommutative geometry is finding a noncommutative counterpart for the notion of a Riemannian metric. Also, since matter is fermionic, one has to extend the notion of spin structure on a Riemannian manifold to noncommutative geometry. This has been achieved by Connes via *spectral triples*. A compact noncommutative spin manifold is characterised by a *spectral triple* subject to a list of axioms laid out in [8]. According to [5, Definition 1.120], a *spectral triple* is a triple (A, H, D), where A is a unital involutive algebra, which stands for the algebra of coordinates, represented by π on the fermionic Hilbert space H, and D is a self-adjoint not necessarily bounded operator on H, which is the generalisation of the Dirac operator, such that the commutator $[D, \pi(a)]$ is bounded for every $a \in A$ and the resolvent of D is compact. A key result is the reconstruction theorem in [7], which recovers the classical geometry of a compact spin manifold M from the noncommutative setup by assuming that the algebra of coordinates is commutative, and by showing that it is isomorphic to the algebra of smooth functions on M. See [19] for details of this reconstruction.

The framework described above has recently been generalised to include noncompact noncommutative spin manifolds [17]. Noncompactness of a manifold corresponds to non-unitality of the algebra A in the definition of a spectral triple. Moreover, in the noncompact case, i.e. for non-unital A, the resolvent condition is replaced by asking instead that the operators $\pi(a)(D - \lambda I)^{-1}$ are compact for any λ in the resolvent set of D. This defines a *noncompact spectral triple*.

Another generalisation has been undertaken in [9], where noncommutative geometry for symmetric non-self-adjoint operators is explored. Instead of asking the Dirac operator D to be self-adjoint, a *pre-spectral triple* only requires D to be closed and symmetric. Naturally, an accompanying relaxation of the condition on the commutator is necessary: The commutator $[D, \pi(a)]$ is only required to have bounded extension for every $a \in A$.

A *finite spectral triple* (A, H, D) is defined as a spectral triple of dimension zero such that both A and H are finite-dimensional. As explained in [1], if we replace the algebra of functions on a Riemannian manifold by a finite-dimensional matrix algebra, we are left with a geometry called a matrix geometry. Specifically, as per [1], the matrix geometries where the algebra is simple are called fuzzy spaces. Matrix geometries can be seen as noncommutative finite-dimensional approximations to Riemannian manifolds. A *finite spectral triple* endowed with a *real structure* is called a *finite real spectral triple*. The axioms for a *finite real spectral triple* can be found in [1]. The *real structure* is necessary if one is interested in constructing a real geometry.

Some of the metric information encoded in a spectral triple can be extracted via Connes's algebraic formulation of distance, which he introduced in the framework of noncommutative geometry in [6]. It is defined on the space $S(A)$ of states of a non-necessarily commutative C^*-algebra A. If $A = C_0(\Omega)$ is the commutative algebra of continuous functions vanishing at infinity on a locally compact topological space Ω, then the pure states correspond to characters. The Gelfand theorem allows us to interpret these characters as points, with A the algebra of functions over these points. Formally, the pure states of A are in one-to-one correspondence with the points of Ω, viewed as the evaluation

$$\delta_x (f) := f (x) \tag{.1}$$

for all $x \in \Omega$ and $f \in A$. Since the construction of a noncommutative geometry is initiated by choosing a noncommutative algebra of functions, the pure states of such a noncommutative algebra appear as natural candidates to play the role of points in a noncommutative framework [25].

Given a spectral triple (A, H, D), Connes's *spectral distance* is defined between any two states $\tau_1, \tau_2 \in S (A)$ by

$$d_D (\tau_1, \tau_2) := \sup_{a \in A} \left\{ |\tau_1 (a) - \tau_2 (a)| : \| [D, \pi (a)] \|_{op} \leq 1 \right\} \tag{.2}$$

where $\|\cdot\|_{op}$ is the operator norm for the representation of A in $\mathfrak{B} (H)$ [5, Chapter 10 (1.468)]. As shown in [4, Proposition 1.119], if $A = C_0^\infty (M)$ is the commutative algebra of smooth functions vanishing at infinity on a locally compact and complete Riemannian manifold M, acting on the Hilbert space H of square-integrable differential forms via the multiplicative representation

$$(f \psi) (x) = f (x) \psi (x) \qquad \forall x \in M, \psi \in H \tag{.3}$$

and $D = d + d^\dagger$ is the signature operator (d is the exterior derivative and $d^\dagger$ its adjoint), then the spectral distance between the pure states δ_x of A, as per (.1), returns the geodesic distance on M:

$$d_D (\delta_x, \delta_y) = d_{geo} (x, y) \tag{.4}$$

Therefore, the spectral distance is a generalisation of the Riemannian geodesic distance that also makes sense in a noncommutative context. Note that the spectral distance does not rely on any notion ill-defined in a quantum context, such as points or paths between points. In this sense, this distance displays potential compatibility with a description of spacetime at the Planck scale [26].

The meaning of the spectral distance in a noncommutative context remains obscure. In the spirit of exploration and generalisation, one may relax some of the conditions on the constituents of a spectral triple in order to study specific metric aspects of noncommutative geometry. In particular, the conditions involving the resolvent and the self-adjointness of the Dirac operator do not impose any constraints on the spectral distance. Therefore, we will present a slightly relaxed definition of a spectral triple, which will be sufficient for extracting metric information via the spectral distance formula. Furthermore, pursuant to the above-mentioned suggestion concerning the role of pure states in noncommutative geometry, we choose to restrict to such states. Even though purity of state is not necessarily an adequate criteria for characterising points in a noncommutative context, a study of the pure states, owing to their prominence within the space of states, not only serves as an appropriate point of departure but also lays the groundwork for studying states in general.

This exploratory dissertation pursues an inceptive understanding of the metric aspects of noncommutative geometry. Specifically, it aims at illuminating the notion of distance in a noncommutative context. To this end, we will construct spectral triples for certain prototypical examples of noncommutative spaces and explicitly calculate the spectral distance between certain associated pure states. Our investigation will commence with instructive finite-dimensional spectral triples in Part I and culminate in an archetypal infinite-dimensional example, namely the Moyal plane, in Part II.

Moyal spaces have their origin in the study of quantum mechanics in phase space [30]. Since then, they have become paradigmatic examples of noncommutative geometries by deformation. Most recently, Moyal spaces can be seen in attemps at developing quantum field theory on noncommutative spacetime. However, their metric aspects have received limited attention. We study the noncommutative geometry of the Moyal plane from a metric point of view. Following the outline in [12], we construct a spectral triple based on the Moyal deformation of the algebra of Schwartz functions on the Euclidean plane $\mathbb{R}^2$ and calculate the spectral distance between certain pure states.

Part I

Definitions and instructive examples

After expounding the concepts relevant to a metric study of noncommutative geometry, we study certain instructive finite-dimensional examples. Chapter 1 introduces the notion of a spectral triple and presents Connes's spectral distance formula as a mechanism for extracting metric information from a spectral triple. In Chapter 2, as a first didactic example, we construct an elementary commutative finite spectral triple and calculate the spectral distance in the associated discrete space of pure states. We advance into noncommutativity by examining a finite spectral triple built around a matrix algebra in Chapter 3.

Chapter 1

Spectral triples and the spectral distance

We present here the two fundamental definitions relevant to our study - that of a spectral triple and that of the spectral distance. To elucidate these definitions, we include in Sections 1.1 and 1.2 some auxiliary material on C^*-algebras, representations, and states. Section 1.3 is devoted to properties of the spectral distance formula: First, we show that the spectral distance conforms, up to an interesting anomaly, with the traditional notion of a distance function. Then, we reproduce a result from [21, Lemma 1] that simplifies the search for a supremum when calculating the spectral distance. In Section 1.4, we specialise our definition of a spectral triple to the finite-dimensional case and briefly discuss the resulting simplifications pertaining to the spectral distance.

1.1 Basic definitions

As promised in the introduction, we present the following relaxed definition of a spectral triple, which is sufficient for our metric study of noncommutative geometry. The subsequent definitions describe the innards of a spectral triple and guide us towards the definition of the spectral distance.

Definition 1.1.1. *A* ***spectral triple*** (A, H, D) *is given by an involutive algebra* A*, together with a faithful representation* π *on a Hilbert space* H*, and a symmetric not necessarily bounded operator* $D : \mathfrak{D}(D) \to H$ *(called the Dirac operator) defined on the domain* $\mathfrak{D}(D)$ *of* D*, such that the commutator* $[D, \pi(a)] : \mathfrak{D}(D) \to H$ *is bounded for every* $a \in A$*.*

Note that this definition deviates from the standard definition in [5, Definition 1.120]: Instead of asking D to be self-adjoint, we only require it to be symmetric, thus resembling the pre-spectral triples in [9]. Note that a symmetric operator is necessarily densely defined; in other words, $\mathfrak{D}(D)$

has to be dense in H. Unlike the pre-spectral triples in [9], we consider neither the closures of symmetric Dirac operators nor the extensions of the commutators. Furthermore, we do not require unitality of A, similar to the noncompact spectral triples in [17]. Lastly, we do not impose any conditions on the resolvent of D. For the remainder of this chapter, we will assume some background regarding unbounded operators but will review a few concepts in Section 7.1.

Our focus will be on algebras that have C^*-closures; therefore, we include the following definitions from [31, Chapters 1-3] involving C^*-algebras and their representations.

Definition 1.1.2. *A **Banach $*$-algebra** is an involutive algebra V, together with a complete submultiplicative norm such that*

$$\|v^*\| = \|v\| \qquad \text{for all } v \in V$$

*Moreover, a **C^*-algebra** is a Banach $*$-algebra V such that*

$$\|v^*v\| = \|v\|^2 \qquad \text{for all } v \in V$$

Definition 1.1.3. *Let V be a $*$-algebra (involutive algebra). A **C^*-norm** on V is a submultiplicative norm such that*

$$\|v^*\| = \|v\| \quad \text{and} \quad \|v^*v\| = \|v\|^2 \qquad \text{for all } v \in V$$

*A **pre-C^*-algebra** is a $*$-algebra V, together with a C^*-norm $\|\cdot\|$. Completing V with respect to $\|\cdot\|$ yields a C^*-algebra.*

Definition 1.1.4. *A **representation** of a C^*-algebra V is a pair (H, φ), where H is a Hilbert space and $\varphi : V \to \mathfrak{B}(H)$ is a $*$-homomorphism. We also say that $\varphi(V)$ is a representation of V on H.*
*A representation (H, φ) is called **faithful** if φ is injective.*

In all of this dissertation, $\mathfrak{B}(H)$ denotes the space of all bounded linear operators on the Hilbert space H. It is a C^*-algebra when equipped with the operator norm.

Now we state our second central definition as found in [5, Chapter 10 (1.468)], namely that of the spectral distance. Since this distance is defined on the space of states of an algebra, we include a definition and a discussion of states in the context of C^*-algebras [31, Chapter 3 and 5].

Definition 1.1.5. *Let (A, H, D) be a spectral triple as per Definition 1.1.1. The **spectral distance** between any two states τ_1, τ_2 of A is*

$$d_D(\tau_1, \tau_2) = \sup_{a \in A} \left\{ |\tau_1(a) - \tau_2(a)| : \|[D, \pi(a)]\|_{op} \leq 1 \right\}$$

where $\|\cdot\|_{op}$ is the operator norm for the representation of A in $\mathfrak{B}(H)$.

Definition 1.1.6. *A **state** on a C^*-algebra A is a positive linear functional on A of norm one. A state τ on a C^*-algebra A is a **pure state** if it has the property that whenever ρ is a positive linear functional on A such that $\rho \leq \tau$, necessarily there is a number $t \in [0,1]$ such that $\rho = t\tau$. In other words, a state is pure if it cannot be written as a convex combination of two other states.*

If A is unital, the condition that τ be of norm one in Definition 1.1.6 is equivalent to $\tau(1) = 1$ [31, Corollary 3.3.4]. We let $S(A)$ and $PS(A)$ denote the set of states and the set of pure states of A respectively.

Strictly speaking, the notion of a state is reserved for C^*-algebras. However, it is legitimate to talk about states for pre-C^*-algebras: Consider a pre-C^*-algebra A with representation π on some Hilbert space H. Then a state on the C^*-closure of $\pi(A)$, denoted $\overline{\pi(A)}$, defines by restriction a unique positive linear map of norm 1 from A to $\mathbb{C}$. The continuity in the C^*-norm ensures that any state on $\overline{\pi(A)}$ is uniquely determined by its restriction to A. Therefore, it is not necessary to distinguish between a state on $\overline{\pi(A)}$ and its restriction. In fact, $S(A)$ and $PS(A)$ can be identified with $S\left(\overline{\pi(A)}\right)$ and $PS\left(\overline{\pi(A)}\right)$ respectively if π is faithful.

Following the intuition, as mentioned in the introduction, that the pure states are reasonable contenders to inherit the role of points in noncommutative geometry, we will, throughout this dissertation, restrict our calculations of the spectral distance to such states.

1.2 Representations and states

The definitions and theorems below, as quoted from [31, Chapters 4 and 5], exhibit the inherent relationship between representations and states, specifically in the context of C^*-algebras. Moreover, these theorems will be used to determine explicit expressions for the pure states between which we intend to calculate spectral distances. We remark that these definitions differ from those in [31] only inasmuch as we accommodate the convention of choosing inner products to be linear in the second argument and conjugate linear in the first.

Definition 1.2.1. *If (H, φ) is a representation of a C^*-algebra A, then we let $\varphi(A)H$ denote the linear span of the set*

$$\{\varphi(a)h : a \in A, h \in H\}$$

*and let $[\varphi(A)H]$ denote the closure of $\varphi(A)H$. We say $\varphi(A)$ acts **non-degenerately** on H if*

$$[\varphi(A)H] = H$$

*In this case, we call the representation (H, φ) **non-degenerate**.*

Definition 1.2.2. *Let (H, φ) be a representation of a C^*-algebra A. We call $x \in H$ a **cyclic vector** for (H, φ) if*

$$[\varphi(A)\,x] = H$$

*If (H, φ) admits a cyclic vector, then we say it is a **cyclic representation**.*

Theorem 1.2.3. *If (H, φ) is a non-degenerate representation of a C^*-algebra A, then it is a direct sum of cyclic representations of A.*

Definition 1.2.4. *Two representations (H_1, φ_1) and (H_2, φ_2) of a C^*-algebra A are **unitarily equivalent** if there is a unitary $u : H_1 \to H_2$ such that*

$$\varphi_2(a) = u\varphi_1(a)\,u^* \qquad \text{for all} \quad a \in A$$

Theorem 1.2.5. *Let (H_1, φ_1) and (H_2, φ_2) be representations of a C^*-algebra A with cyclic vectors x_1 and x_2 respectively. They are unitarily equivalent, with $x_2 = u(x_1)$, if and only if*

$$\langle x_1, \varphi_1(a)\,x_1\rangle = \langle x_2, \varphi_2(a)\,x_2\rangle \qquad \text{for all} \quad a \in A$$

Definition 1.2.6. *Let B be a subset of an algebra A. The **commutant** of B, denoted B', is defined as the set of all elements in A that commute with every element in B. That is,*

$$B' := \{a \in A \;:\; ab = ba \;\; \text{for all} \;\; b \in B\}$$

Note that B' is a subalgebra of A.

Definition 1.2.7. *A representation (H, φ) of a C^*-algebra A is **irreducible** if the algebra $\varphi(A)$ acts irreducibly on H.*

Theorem 1.2.8. *Let (H, φ) be a nonzero representation of a C^*-algebra A. (H, φ) is **irreducible** if and only if $\varphi(A)' = \mathbb{C}I$, where I is the identity operator on H.*
If (H, φ) is irreducible, then every nonzero vector of H is cyclic for (H, φ).

Theorem 1.2.9. *The **Gelfand-Naimark-Segal (GNS) theorem** asserts that for each state τ on a C^*-algebra A, there exists a representation (H_τ, φ_τ) of A, called the GNS representation for τ, and a cyclic vector $\Omega_\tau \in H_\tau$ such that*

$$\tau(a) = \langle \Omega_\tau, \varphi_\tau(a)\,\Omega_\tau\rangle$$

for all $a \in A$. The GNS representation for a state τ is unique in the sense that any other representation (H, φ) of A containing a cyclic vector corresponding to τ is unitarily equivalent to (H_τ, φ_τ).

Theorem 1.2.10. *Let τ be a state on a C^*-algebra A. It is a pure state if and only if the GNS representation (H_τ, φ_τ) is irreducible.*

Theorem 1.2.11. *Let (H, φ) be a representation of a C^*-algebra A and let x be a unit cyclic vector for (H, φ). Then the functional*

$$\tau : A \to \mathbb{C} : a \mapsto \langle x, \varphi(a)(x) \rangle$$

is a state of A and (H, φ) is unitarily equivalent to (H_τ, φ_τ). Moreover, if (H, φ) is irreducible, then τ is pure.

Theorem 1.2.12. *Let τ be a state on a C^*-algebra A. If A is abelian, then τ is pure if and only if it is a character on A.*

1.3 Properties of the spectral distance

Here we examine the spectral distance formula in two distinct ways: The first serves as justification for calling it a distance and the second makes it more amenable to calculations under an assumption on the algebra that forms part of the spectral triple. The following lemma shows that the spectral distance, as per Definition 1.1.5, satisfies the conditions of a distance function in the usual sense, except that it admits infinite distance between states.

Lemma 1.3.1. *The spectral distance d_D, as per Definition 1.1.5, defines a distance (possibly infinite) on $S(A)$, i.e. for all $\tau_1, \tau_2, \tau_3 \in S(A)$, it holds that*

(1) $d_D(\tau_1, \tau_2) \geq 0$
(2) $d_D(\tau_1, \tau_2) = 0$ if and only if $\tau_1 = \tau_2$
(3) $d_D(\tau_1, \tau_2) = d_D(\tau_2, \tau_1)$
(4) $d_D(\tau_1, \tau_3) \leq d_D(\tau_1, \tau_2) + d_D(\tau_2, \tau_3)$

Moreover, if there exists an element $a \in A$ such that $\|[D, \pi(a)]\|_{op} = 0$ and $\tau_1(a) \neq \tau_2(a)$, then $d_D(\tau_1, \tau_2) = +\infty$.

Proof. The non-negativity in (1) and symmetry in (3) follow immediately from the non-negativity and symmetry of the absolute value function as it appears in d_D.

To prove positive-definiteness, consider the following: If $\tau_1 = \tau_2$, then clearly $d_D(\tau_1, \tau_2) = 0$. Conversely, let $d_D(\tau_1, \tau_2) = 0$ and suppose that $\tau_1 \neq \tau_2$. Then there exists some $a_\infty \in A$ such that $\tau_1(a_\infty) \neq \tau_2(a_\infty)$. If $[D, \pi(a_\infty)]$ is bounded, then consider the element

$$a_0 := \frac{a_\infty}{\|[D, \pi(a_\infty)]\|_{op} + 1} \in A$$

Clearly $\tau_1(a_0) \neq \tau_2(a_0)$ with $\|[D, \pi(a_0)]\|_{op} < 1$. This implies $d_D(\tau_1, \tau_2) \neq 0$, which contradicts the assumption. If $[D, \pi(a_\infty)]$ is unbounded, it contradicts the fact that (A, H, D) is a spectral triple, since Definition 1.1.1

requires $[D, \pi(a)]$ to be bounded for all $a \in A$. Hence, $\tau_1 = \tau_2$.

Next, we prove the triangle inequality in (4):

$$\begin{aligned} d_D(\tau_1, \tau_3) &= \sup_{a \in A} \left\{ |\tau_1(a) - \tau_3(a)| : \|[D, \pi(a)]\|_{op} \leq 1 \right\} \\ &= \sup_{a \in A} \left\{ |\tau_1(a) - \tau_2(a) + \tau_2(a) - \tau_3(a)| : \|[D, \pi(a)]\|_{op} \leq 1 \right\} \\ &\leq \sup_{a \in A} \left\{ |\tau_1(a) - \tau_2(a)| + |\tau_2(a) - \tau_3(a)| : \|[D, \pi(a)]\|_{op} \leq 1 \right\} \\ &\leq d_D(\tau_1, \tau_2) + d_D(\tau_2, \tau_3) \end{aligned}$$

for all $\tau_1, \tau_2, \tau_3 \in S(A)$.

Finally, let $a \in A$ such that $\|[D, \pi(a)]\|_{op} = 0$ and $\tau_1(a) \neq \tau_2(a)$. Then the supremum in d_D can also be searched over elements $a_n := na \in A$ for $n \in \mathbb{N}$, since $\|[D, \pi(a_n)]\|_{op} = 0$. For such elements, the subject in the supremum is

$$|\tau_1(a_n) - \tau_2(a_n)| = n\,|\tau_1(a) - \tau_2(a)| \neq 0$$

Since n can be made arbitrarily large, it follows that $d_D(\tau_1, \tau_2) = +\infty$. □

The following lemma, as per [21, Lemma 1], will show that the supremum in the spectral distance can be searched equivalently on the self-adjoint elements of a C^*-algebra A. Let A^{sa} denote the set of self-adjoint elements of A. In the proof, we use standard results from the theory of unbounded operators, which can be found, for example, in [23, Chapter 10].

Lemma 1.3.2. *Let (A, H, D) be a spectral triple and $\tau_1, \tau_2 \in S(A)$, where A is a C^*-algebra. The supremum in the spectral distance, as per Definition 1.1.5, can be searched equivalently on A^{sa}, such that*

$$d_D(\tau_1, \tau_2) = \sup_{a \in A^{sa}} \left\{ |\tau_1(a) - \tau_2(a)| : \|[D, \pi(a)]\|_{op} \leq 1 \right\} \tag{1.3.2.1}$$

Proof. Let

$$Q := \left\{ a \in A : \|[D, \pi(a)]\|_{op} \leq 1 \right\}$$

There exists a sequence $(a_n)_{n=0}^{\infty} \in Q$ such that

$$|\tau_1(a_n) - \tau_2(a_n)| \to d_D(\tau_1, \tau_2)$$

Let $\theta_n := \arg(\tau_1(a_n) - \tau_2(a_n))$ and consider the self-adjoint element

$$b_n := \frac{1}{2}\left(a_n e^{-i\theta_n} + a_n^* e^{i\theta_n} \right) \in A^{sa}$$

Then it holds that

$$\begin{aligned} e^{-i\theta_n}\left(\tau_1(a_n)-\tau_2(a_n)\right) &= e^{-i\theta_n}\left|\tau_1(a_n)-\tau_2(a_n)\right|e^{i\theta_n} \\ &= \left|\tau_1(a_n)-\tau_2(a_n)\right| \\ &\to d_D\left(\tau_1,\tau_2\right) \end{aligned} \tag{1.3.2.2}$$

Even though it seems obvious that $\left\|\left[D,\pi\left(a_n^*\right)\right]\right\|_{op}=\left\|\left[D,\pi\left(a_n\right)\right]\right\|_{op}$, the unboundedness of D forces us to consider this claim carefully. Since A is a C^*-algebra, it has a representation in the bounded linear operators, i.e. $\pi\left(a_n\right)\in\mathfrak{B}\left(H\right)$. Therefore, its adjoint $\pi\left(a_n\right)^*$ exists and is defined on H. Also note that D is densely defined in H, which implies that the product $\pi\left(a_n\right)D$ is densely defined. It then follows from the boundedness of $\pi\left(a_n\right)$ and the symmetry of D that

$$\left(\pi\left(a_n\right)D\right)^*=D^*\pi\left(a_n\right)^*\supset D\pi\left(a_n\right)^* \tag{1.3.2.3}$$

$D\pi\left(a_n\right)$ is also densely defined. In this case, we can say

$$\pi\left(a_n\right)^*D\subset\pi\left(a_n\right)^*D^*\subset\left(D\pi\left(a_n\right)\right)^* \tag{1.3.2.4}$$

In other words, $\left(D\pi\left(a_n\right)\right)^*$ is an extension of the operator $\pi\left(a_n\right)^*D$. From (1.3.2.3) and (1.3.2.4), we can deduce that

$$\left[D,\pi\left(a_n^*\right)\right]:=D\pi\left(a_n\right)^*-\pi\left(a_n\right)^*D\subset\left(\pi\left(a_n\right)D\right)^*-\left(D\pi\left(a_n\right)\right)^* \tag{1.3.2.5}$$

where the extension on the right is densely defined. Since $\pi\left(a_n\right)D$, $D\pi\left(a_n\right)$, and $\pi\left(a_n\right)D-D\pi\left(a_n\right)$ are all densely defined, it holds that

$$\left(\pi\left(a_n\right)D\right)^*-\left(D\pi\left(a_n\right)\right)^*\subset\left(\pi\left(a_n\right)D-D\pi\left(a_n\right)\right)^* \tag{1.3.2.6}$$

Together, (1.3.2.5) and (1.3.2.6) imply that

$$\left[D,\pi\left(a_n^*\right)\right]\subset\left(\pi\left(a_n\right)D-D\pi\left(a_n\right)\right)^*=\left(-\left[D,\pi\left(a_n\right)\right]\right)^* \tag{1.3.2.7}$$

Since $\left[D,\pi\left(a_n\right)\right]$ is bounded, it follows that $\left[D,\pi\left(a_n^*\right)\right]$ is bounded. Therefore, the latter has a unique extension to H with the same norm. Hence, we can conclude from (1.3.2.7) that

$$\left\|\left[D,\pi\left(a_n^*\right)\right]\right\|_{op}=\left\|\left[D,\pi\left(a_n\right)\right]\right\|_{op} \tag{1.3.2.8}$$

Applying (1.3.2.8), we find that

$$\begin{aligned} \left\|\left[D,\pi(b_n)\right]\right\|_{op} &= \frac{1}{2}\left\|\left[D,\pi(a_ne^{-i\theta_n})\right]+\left[D,\pi(a_n^*e^{i\theta_n})\right]\right\|_{op} \\ &\leq 1/2\left\|\left[D,\pi(a_n)\right]\right\|_{op}+1/2\left\|\left[D,\pi(a_n^*)\right]\right\|_{op} \\ &= 1/2\left\|\left[D,\pi(a_n)\right]\right\|_{op}+1/2\left\|\left[D,\pi(a_n)\right]\right\|_{op} \\ &\leq 1/2+1/2 \\ &= 1 \end{aligned} \tag{1.3.2.9}$$

Note that $\phi(a^*) = \overline{\phi(a)}$ for any positive linear functional ϕ [3, II.6.2]. This holds, in particular, for any state τ. Using this fact, and applying (1.3.2.2), we observe the following convergence:

$$\begin{aligned} |(\tau_1 - \tau_2)(b_n)| &= \frac{1}{2}(\tau_1 - \tau_2)\left(a_n e^{-i\theta_n}\right) + \frac{1}{2}\overline{(\tau_1 - \tau_2)\left(a_n e^{-i\theta_n}\right)} \\ &\to d_D(\tau_1, \tau_2) \end{aligned} \tag{1.3.2.10}$$

Since every a_n determines a self-adjoint b_n, the required result follows from (1.3.2.9) and (1.3.2.10). □

1.4 Finite spectral triples

We restrict our attention now to finite dimensions. The purpose of limiting to this more accommodating environment is to identify beacons that might guide and instruct our later attempts at understanding the less tractable infinite-dimensional case. The following definition will be used in the instructive examples that occupy Part I of this dissertation.

Definition 1.4.1. *A* ***finite spectral triple*** (A, H, D) *is a spectral triple, as per Definition 1.1.1, where* H *is finite-dimensional.*

Note that, since H is finite-dimensional, $\pi(A)$ is a C^*-subalgebra of $\mathfrak{B}(H)$. Therefore, A is a C^*-algebra that inherits the structure of $\mathfrak{B}(H)$ by means of π. Also, since every operator in finite dimensions is bounded and the domain is the whole Hilbert space, symmetry of D is equivalent to self-adjointness.

One expects some simplification in the spectral distance formula when applied to finite spectral triples; indeed, the operator norm adopts an expression more amenable to calculations in finite dimensions. Even though all norms on finite-dimensional algebras are not only equivalent, in the sense that they generate the same topology, but also complete, we take this opportunity to briefly introduce the relevant spaces and notations on our way towards this equivalent expression for the operator norm called the spectral norm (the missing details appear in most introductory functional analysis textbooks).

Let $M_n(\mathbb{C})$ denote the space of all $n \times n$ complex matrices. It becomes a unital C^*-algebra when equipped with matrix multiplication and an involution defined by conjugate transposition (the unit is given by the $n \times n$ identity matrix). Note that $M_n(\mathbb{C}) = \mathfrak{B}(H_n)$, where $H_n = \mathbb{C}^n$ with the usual inner product. The spectral norm is defined, for example in [20, 5.6.6], by

$$\|T\|_{spec} := \max\left\{\sqrt{\lambda} \,:\, \lambda \text{ is an eigenvalue of } T^*T\right\} \tag{1.1}$$

for all $T \in M_n(\mathbb{C})$, where T^* denotes the conjugate transpose of T. The following theorem can be seen as a special case of [31, Thm 2.1.1]; therefore, we omit the proof.

Theorem 1.4.2. *The operator norm on $\mathfrak{B}(H_n)$ coincides with the spectral norm on $M_n(\mathbb{C})$:*

$$\|\cdot\|_{op} = \|\cdot\|_{spec}$$

Chapter 2

Discrete space

In this chapter, we perform a humble, yet instructive, inaugural probe into the modus operandi of the spectral distance formula. We consider an elementary commutative finite spectral triple that has a discrete space of pure states. In Section 2.1, we assemble the spectral triple and determine the associated pure states. Sections 2.2 and 2.3 are dedicated to the calculation of the spectral distance in two low-dimensional examples of the discrete space constructed in Section 2.1. Specifically, we consider a two-point space and a three-point space.

2.1 The spectral triple and pure states

Let $A_n = \mathbb{C}^n$, $H_n = \mathbb{C}^n$, and define the mapping

$$\pi : A_n \to M_n(\mathbb{C}) : a \mapsto \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \qquad \text{for all} \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in A_n$$

Let D be a self-adjoint $n \times n$ matrix with complex entries D_{ij} such that

$$D_{ij} = \overline{D_{ji}} \in \mathbb{C}$$

for all $i, j \in \{1, \ldots, n\}$, where the overbar denotes the complex conjugate of a complex number.

Proposition 2.1.1. (A_n, H_n, D) *is a finite spectral triple.*

Proof. Note that $A_n = \mathbb{C}^n$, equipped with componentwise addition, scalar multiplication, pointwise multiplication, and an involution defined by componentwise complex conjugation, is a finite-dimensional commutative unital C^*-algebra, where the unit is given by the n-dimensional vector with every component 1, and $\|a\| = \max\{|a_1|, \ldots, |a_n|\}$.

Since $\pi(a)$ acts on elements of H_n by matrix multiplication and returns elements in H_n for all $a \in A_n$, we consider $\pi : A_n \to \mathfrak{B}(H_n)$ as a map into the space of linear operators on H_n, which is clearly a faithful representation of A_n. It follows from Definition 1.4.1 that (A_n, H_n, D) is a finite spectral triple. □

Note that the positive elements of A_n are exactly

$$\{a^*a : a \in A_n\} = \{a : a \in A_n \text{ and } a_1, \ldots, a_n \geq 0\}$$

Proposition 2.1.2. *The pure states of A_n are given by*

$$\delta_i(a) = a_i \qquad \textit{for} \quad i = 1, \ldots, n$$

where $a \in A_n$. The space of pure states is the discrete n-point space

$$PS(A_n) = \{\delta_1, \ldots, \delta_n\}$$

Proof. For all $i \in \{1, \ldots, n\}$, δ_i is a state on the C^*-algebra A_n because $\delta_i(1) = 1$ and $\delta_i(a) = a_i \geq 0$ if $a \geq 0$. Also, for $a, b \in A_n$, it holds that

$$\delta_i(ab) = a_i b_i = \delta_i(a)\,\delta_i(b)$$

Therefore, δ_i is a character on A_n. Since A_n is abelian, it follows from Theorem 1.2.12 that δ_i is a pure state.

In order to show that the δ_i constitute all the pure states of A_n, let τ be an arbitrary pure state on A_n. Since τ is a linear functional on A_n, we can represent it as a row vector $\tau = (\tau_1, \ldots, \tau_n)$, where $\tau(a) = \tau a$ is given by usual matrix multiplication. Since τ is pure, Theorem 1.2.12 confirms that it is a character. Therefore, for all $a, b \in A_n$

$$\begin{aligned} \tau_1 a_1 b_1 + \ldots + \tau_n a_n b_n &= \tau(ab) \\ &= \tau(a)\,\tau(b) \\ &= (\tau_1 a_1 + \ldots + \tau_n a_n)(\tau_1 b_1 + \ldots + \tau_n b_n) \qquad (2.1.2.1) \end{aligned}$$

If $a = b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, then (2.1.2.1) implies that $\tau_1 = \tau_1^2$. So, either $\tau_1 = 0$ or $\tau_1 = 1$. Note that $\tau_1 \geq 0$ because τ is a state and thus $\tau \geq 0$. Similarly, $\tau_j \in \{0, 1\}$ for all $j \in \{1, \ldots, n\}$. Since τ is a state, it has to hold that

$$1 = \tau(1) = \tau_1 + \ldots + \tau_n$$

Accordingly, exactly one τ_j has to be 1 and the rest 0. In other words, $\tau = \delta_i$ for some $i \in \{1, \ldots, n\}$. Hence, the δ_i constitute all the pure states and

$$PS(A_n) = \{\delta_1, \ldots, \delta_n\}$$

□

We proceed now to calculate the spectral distance between the pure states of discrete spaces as above for two low-dimensional examples - the two-point space ($n = 2$) and the three-point space ($n = 3$).

2.2 Two-point space

We consider the finite spectral triple (A_2, H_2, D), which is the spectral triple in Proposition 2.1.1 with $n = 2$. Observe that in the spectral distance formula, as per Definition 1.1.5, D only appears in the commutator with elements of the representation $\pi(A_2)$. Since the diagonal of D clearly commutes with $\pi(A_2)$, we might as well choose D with null-diagonal when computing the spectral distance. Therefore, we use

$$D := \begin{pmatrix} 0 & m \\ \overline{m} & 0 \end{pmatrix}$$

where $m \in \mathbb{C}$ is nonzero. The space of pure states of A_2 is the two-point space

$$PS(A_2) = \{\delta_1, \delta_2\}$$

as per Proposition 2.1.2.

Proposition 2.2.1. *For the finite spectral triple (A_2, H_2, D), the spectral distance between the two pure states δ_1 and δ_2 of A_2 is*

$$d_D(\delta_1, \delta_2) = \frac{1}{|m|}$$

Proof. We begin by evaluating the norm constraint as featured in the spectral distance formula. To this end, let $a \in A_2$ be arbitrary and consider that

$$[D, \pi(a)] = \begin{pmatrix} 0 & m(a_2 - a_1) \\ \overline{m}(a_1 - a_2) & 0 \end{pmatrix}$$

and

$$[D, \pi(a)]^* [D, \pi(a)] = \begin{pmatrix} |m|^2 |a_1 - a_2|^2 & 0 \\ 0 & |m|^2 |a_2 - a_1|^2 \end{pmatrix}$$

The spectral norm, as defined in (1.1), is evaluated as

$$\|[D, \pi(a)]\|_{spec} = |m| |a_1 - a_2|$$

Now we find the spectral distance as

$$\begin{aligned} d_D(\delta_1, \delta_2) &= \sup_{a \in A_2} \left\{ |\delta_1(a) - \delta_2(a)| : \|[D, \pi(a)]\|_{spec} \leq 1 \right\} \\ &= \sup_{a \in A_2} \left\{ |a_1 - a_2| : |a_1 - a_2| \leq \frac{1}{|m|} \right\} \\ &= \frac{1}{|m|} \end{aligned}$$

□

We offer a geometric interpretation of this result by comparing it to the classical case: In the classical case, the discrete two-point space can be viewed as a 0-dimensional topological manifold with the discrete topology. Note that any function on a discrete space into a topological space is continuous. The length of the discrete curve between the two points is just the distance between the appropriate functions evaluated at those points. Since there is only this one curve, it follows that the geodesic distance is solely determined by the positive constant that defines the Riemannian metric. Therefore, since the proposition above shows the same result, yet still makes sense in a noncommutative context, we conclude the following: The spectral distance equips the discrete two-point space $\{\delta_1, \delta_2\}$ with a generalisation of the geodesic distance. We should add that the interpretation of a geodesic as a minimal path length does not make sense here inasmuch as there are no points (pure states) between δ_1 and δ_2 to constitute a path.

2.3 Three-point space

We consider the finite spectral triple (A_3, H_3, D), which is the spectral triple in Proposition 2.1.1 with $n = 3$. Once again, we may choose D with null-diagonal for the purpose of computing the spectral distance. It is not surprising that, even for this simple finite spectral triple, calculation of the spectral norm involves laborious calculations. We moderate the technical grind by restricting to Dirac operators with real entries. Let

$$D := \begin{pmatrix} 0 & D_{12} & D_{13} \\ D_{12} & 0 & D_{23} \\ D_{13} & D_{23} & 0 \end{pmatrix} \tag{2.3.1}$$

with every $D_{ij} \in \mathbb{R}$ nonzero. The space of pure states of A_3 is the three-point space

$$PS\,(A_3) = \{\delta_1, \delta_2, \delta_3\}$$

as per Proposition 2.1.2.

Proposition 2.3.1. *For the finite spectral triple (A_3, H_3, D), the spectral distances among the elements of $PS\,(A_3)$ are*

$$d_D\,(\delta_1, \delta_2) = \sqrt{\frac{D_{23}^2 + D_{13}^2}{D_{12}^2 D_{23}^2 + D_{23}^2 D_{13}^2 + D_{13}^2 D_{12}^2}}$$

$$d_D\,(\delta_2, \delta_3) = \sqrt{\frac{D_{13}^2 + D_{12}^2}{D_{23}^2 D_{13}^2 + D_{13}^2 D_{12}^2 + D_{12}^2 D_{23}^2}}$$

$$d_D\,(\delta_3, \delta_1) = \sqrt{\frac{D_{12}^2 + D_{23}^2}{D_{13}^2 D_{12}^2 + D_{12}^2 D_{23}^2 + D_{23}^2 D_{13}^2}}$$

Proof. We begin by evaluating the norm constraint. Let $a \in A_3$ be arbitrary and consider that

$$[D, \pi(a)]^* [D, \pi(a)] = \begin{pmatrix} D_{12}^2 |a_1 - a_2|^2 + D_{13}^2 |a_1 - a_3|^2 & D_{13} D_{23} \overline{(a_1 - a_3)} (a_2 - a_3) & D_{12} D_{23} \overline{(a_1 - a_2)} (a_3 - a_2) \\ D_{23} D_{13} \overline{(a_2 - a_3)} (a_1 - a_3) & D_{12}^2 |a_2 - a_1|^2 + D_{23}^2 |a_2 - a_3|^2 & D_{12} D_{13} \overline{(a_2 - a_1)} (a_3 - a_1) \\ D_{23} D_{12} \overline{(a_3 - a_2)} (a_1 - a_2) & D_{13} D_{12} \overline{(a_3 - a_1)} (a_2 - a_1) & D_{13}^2 |a_3 - a_1|^2 + D_{23}^2 |a_3 - a_2|^2 \end{pmatrix}$$

The spectral norm, as defined in (1.1), becomes

$$\| [D, \pi(a)] \|_{spec} = \sqrt{D_{12}^2 |a_1 - a_2|^2 + D_{13}^2 |a_1 - a_3|^2 + D_{23}^2 |a_2 - a_3|^2}$$

We substitute this expression into Definition 1.1.5 for the spectral distance. Also, we restrict our search for the supremum to the self-adjoint elements in A_3 thanks to Lemma 1.3.2. For $j, k \in \{1, 2, 3\}$,

$$\begin{aligned} d_D (\delta_j, \delta_k) &= \sup_{a \in A_3} \left\{ |\delta_j (a) - \delta_k (a)| : \| [D, \pi(a)] \|_{spec} \leq 1 \right\} \\ &= \sup_{a \in A_3^{sa}} \left\{ |a_j - a_k| : \sqrt{D_{12}^2 |a_1 - a_2|^2 + D_{13}^2 |a_1 - a_3|^2 + D_{23}^2 |a_2 - a_3|^2} \leq 1 \right\} \end{aligned} \tag{2.3.1.1}$$

Let

$$d_1 := D_{12}^2 \qquad d_2 := D_{23}^2 \qquad d_3 := D_{13}^2 \tag{2.3.1.2}$$

and

$$x := a_1 - a_2 \qquad y := a_2 - a_3 \qquad z := a_3 - a_1 \tag{2.3.1.3}$$

Clearly $d_1, d_2, d_3 > 0$. Also, since we consider $a \in A_3^{sa}$ when searching for the supremum, we have that $a_1, a_2, a_3 \in \mathbb{R}$ and thus $x, y, z \in \mathbb{R}$. Furthermore, define the functions

$$f (x, y, z) := d_1 x^2 + d_2 y^2 + d_3 z^2 \tag{2.3.1.4}$$

and

$$g (x, y, z) := x + y + z = 0 \tag{2.3.1.5}$$

Consider the spectral distance $d_D (\delta_1, \delta_2)$. In terms of the variables and functions defined above, the supremum in (2.3.1.1) would be attained if one were to find a maximal x subject to the constraint $f (x, y, z) \leq 1$. In fact, we may assume, without loss of generality, that $f (x, y, z) = 1$ when searching for such a maximum, since whenever $f (x, y, z) < 1$, x can simply be increased until $f (x, y, z) = 1$ is reached. Additionally, note that (2.3.1.5) contributes $x = - (y + z)$, which implies that maximising x is equivalent to minimising $y + z$. Accordingly, we will minimise $y + z$ subject to the constraint

$$d_1 (y + z)^2 + d_2 y^2 + d_3 z^2 = 1 \tag{2.3.1.6}$$

which describes an ellipse in the yz-plane, namely the intersection of the ellipsoid $f(x,y,z)=1$ and the plane $g(x,y,z)=0$.

The extrema of $y+z$ will occur at those points (y,z) in the yz-plane where the contour lines of $y+z$, which are lines with slope $\frac{dz}{dy}=-1$, are tangent to the ellipse (2.3.1.6). The minimum will obviously be in the third quadrant of the yz-plane where y and z are negative (maximum in the first quadrant). Therefore, we set $\frac{dz}{dy}=-1$ in the derivative of (2.3.1.6) and solve for the negative values of y and z:

$$0=2d_1(y+z)\left(1+\frac{dz}{dy}\right)+2d_2y+2d_3z\frac{dz}{dy}=2d_2y-2d_3z$$

This implies that

$$z=\frac{d_2}{d_3}y \tag{2.3.1.7}$$

Substituting (2.3.1.7) into (2.3.1.6) and solving for y gives

$$y=-\frac{1}{\sqrt{d_1\left(1+\frac{d_2}{d_3}\right)^2+d_2+\frac{d_2^2}{d_3}}} \tag{2.3.1.8}$$

The minimum occurs at the point (y,z) given by (2.3.1.8) and (2.3.1.7); its value is

$$\begin{aligned}
y+z&=y+\frac{d_2}{d_3}y\\
&=(d_2+d_3)\frac{y}{d_3}\\
&=-\frac{d_2+d_3}{\sqrt{d_1(d_2+d_3)^2+d_2d_3^2+d_2^2d_3}}\\
&=-\sqrt{\frac{d_2+d_3}{d_1d_2+d_2d_3+d_3d_1}}
\end{aligned}$$

Thus x has maximum

$$x=-(y+z)=\sqrt{\frac{d_2+d_3}{d_1d_2+d_2d_3+d_3d_1}} \tag{2.3.1.9}$$

Since we have found a maximal x that satisfies the constraint equation, the supremum in (2.3.1.1) is attained. After returning to the original parameters by substituting (2.3.1.2) into (2.3.1.9), we find that the spectral distance is

$$d_D(\delta_1,\delta_2)=\sqrt{\frac{D_{23}^2+D_{13}^2}{D_{12}^2D_{23}^2+D_{23}^2D_{13}^2+D_{13}^2D_{12}^2}}$$

The distances $d_D(\delta_2,\delta_3)$ and $d_D(\delta_3,\delta_1)$ are found similarly by maximising y and z respectively; the maxima of y and z are obtained by rotating cyclically through $1,2,3$ in (2.3.1.9). $\square$

The following corollary shows that the spectral distances among the elements of $PS(A_3)$ satisfy the triangle inequality to the square. Interpreted classically, this means that the three points (pure states) of $PS(A_3)$ lie on the corners of a triangle with no obtuse angles, i.e. all the angles are less than or equal to $\pi/2$.

Corollary 2.3.2.

$$d_D(\delta_1,\delta_3)^2 \le d_D(\delta_1,\delta_2)^2 + d_D(\delta_2,\delta_3)^2$$
$$d_D(\delta_1,\delta_2)^2 \le d_D(\delta_1,\delta_3)^2 + d_D(\delta_2,\delta_3)^2$$
$$d_D(\delta_2,\delta_3)^2 \le d_D(\delta_1,\delta_2)^2 + d_D(\delta_1,\delta_3)^2$$

Proof. Using the distances as per Proposition 2.3.1, we find that

$$\begin{aligned} d(\delta_1,\delta_2)^2 + d(\delta_2,\delta_3)^2 &= \frac{D_{13}^2 + D_{23}^2 + D_{12}^2 + D_{13}^2}{D_{12}^2 D_{13}^2 + D_{12}^2 D_{23}^2 + D_{23}^2 D_{13}^2} \\ &\ge \frac{D_{12}^2 + D_{23}^2}{D_{12}^2 D_{13}^2 + D_{12}^2 D_{23}^2 + D_{23}^2 D_{13}^2} \\ &= d(\delta_1,\delta_3)^2 \end{aligned}$$

The remaining inequalities follow similarly. □

We have shown that for a given Dirac operator D as per (2.3.1), the spectral distances in the three-point space are determined by the coefficients of D and satisfy the triangle inequality to the square. Now we show that this process is invertible: If three positive numbers a, b, c satisfy the triangle inequality to the square, they determine a Dirac operator D such that a, b, c are the spectral distances in the three-point space. The proof reveals a surprising analogy to electric circuits; we apply a well-known result that equates a triangular circuit to a stellar circuit. The point of this exercise, although a simple example, is to see whether one can in general build spectral triples to fit desired metrics.

Proposition 2.3.3. *Let a, b, c be three positive numbers such that*

$$a^2 + b^2 \ge c^2 \qquad b^2 + c^2 \ge a^2 \qquad a^2 + c^2 \ge b^2$$

Then there exists an operator D such that

$$d_D(\delta_1,\delta_2) = a \qquad d_D(\delta_1,\delta_3) = b \qquad d_D(\delta_2,\delta_3) = c$$

with coefficients given by

$$\begin{cases} D_{12} = \sqrt{\frac{2(b^2+c^2-a^2)}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \\ D_{13} = \sqrt{\frac{2(a^2+c^2-b^2)}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \\ D_{23} = \sqrt{\frac{2(a^2+b^2-c^2)}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \end{cases}$$

Proof. The first step is to generate a procedure that takes three spectral distances and returns the coefficients of the Dirac operator. Let

$$R_{12} := \frac{1}{D_{12}^2} \qquad R_{23} := \frac{1}{D_{23}^2} \qquad R_{13} := \frac{1}{D_{13}^2} \tag{2.3.3.1}$$

Using Proposition 2.3.1, we write

$$\begin{aligned} \frac{1}{d_D(\delta_1, \delta_2)^2} &= \frac{D_{12}^2 D_{13}^2 + D_{12}^2 D_{23}^2 + D_{23}^2 D_{13}^2}{D_{13}^2 + D_{23}^2} \\ &= \frac{1}{R_{12}} + \frac{1}{R_{23} + R_{13}} \end{aligned} \tag{2.3.3.2}$$

Analogously, we have

$$\frac{1}{d_D(\delta_1, \delta_3)^2} = \frac{1}{R_{13}} + \frac{1}{R_{23} + R_{12}} \qquad\qquad \frac{1}{d_D(\delta_2, \delta_3)^2} = \frac{1}{R_{23}} + \frac{1}{R_{12} + R_{13}}$$

These equations bring a certain electric circuit to mind - a triangular circuit with resistances R_{12}, R_{23}, R_{13}:

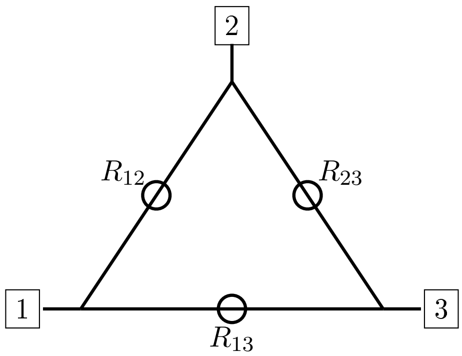


Figure 2.1: Triangular circuit

Figure 2.1, together with (2.3.3.2), implies that $d_D(\delta_1, \delta_2)^2$ is the resistance measured between points 1 and 2 of the triangular circuit. In the same way, $d_D(\delta_1, \delta_3)^2$ is the resistance measured between points 1 and 3, and $d_D(\delta_2, \delta_3)^2$ is the resistance measured between points 2 and 3. Accordingly, if one can find three resistances R_{12}, R_{23}, R_{13} that induce resistances $d_D(\delta_i, \delta_j)^2$ between points i and j, then the coefficients D_{ij} will follow from (2.3.3.1).

A standard result in the analysis of electric circuits, see for example [33, Chapter 5], is that a triangular circuit is equivalent to a stellar circuit. Consider a stellar circuit with resistances r_1, r_2, r_3:

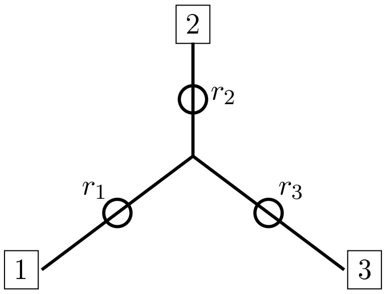


Figure 2.2: Stellar circuit

The resistances in the equivalent triangular and stellar circuits (as set up in Figures 2.1 and 2.2) are related via the formulas

$$\begin{cases} R_{12} = \frac{1}{r_3}\left(r_1 r_2 + r_1 r_3 + r_2 r_3\right) \\ R_{13} = \frac{1}{r_2}\left(r_1 r_2 + r_1 r_3 + r_2 r_3\right) \\ R_{23} = \frac{1}{r_1}\left(r_1 r_2 + r_1 r_3 + r_2 r_3\right) \end{cases} \tag{2.3.3.3}$$

Note that the resistances in the stellar circuit are all set in series. Therefore, the resistances $d_D\left(\delta_i, \delta_j\right)^2$ measured between points i and j are given by

$$\begin{cases} d_D\left(\delta_1, \delta_2\right)^2 = r_1 + r_2 \\ d_D\left(\delta_1, \delta_3\right)^2 = r_1 + r_3 \\ d_D\left(\delta_2, \delta_3\right)^2 = r_2 + r_3 \end{cases} \tag{2.3.3.4}$$

Rewriting in terms of each r_i separately gives

$$\begin{cases} 2r_1 = d_D\left(\delta_1, \delta_2\right)^2 + d_D\left(\delta_1, \delta_3\right)^2 - d_D\left(\delta_2, \delta_3\right)^2 \\ 2r_2 = d_D\left(\delta_1, \delta_2\right)^2 + d_D\left(\delta_2, \delta_3\right)^2 - d_D\left(\delta_1, \delta_3\right)^2 \\ 2r_3 = d_D\left(\delta_1, \delta_3\right)^2 + d_D\left(\delta_2, \delta_3\right)^2 - d_D\left(\delta_1, \delta_2\right)^2 \end{cases} \tag{2.3.3.5}$$

If we substitute these expressions into (2.3.3.3), we can phrase the result in terms of the coefficients of D via (2.3.3.1) as follows:

$$\begin{cases} D_{12} = \sqrt{\frac{2\left(d(1,3)^2+d(2,3)^2-d(1,2)^2\right)}{2\left(d(1,2)^2 d(1,3)^2+d(1,2)^2 d(2,3)^2+d(1,3)^2 d(2,3)^2\right)-d(2,3)^4-d(1,3)^4-d(1,2)^4}} \\ D_{13} = \sqrt{\frac{2\left(d(1,2)^2+d(2,3)^2-d(1,3)^2\right)}{2\left(d(1,2)^2 d(1,3)^2+d(1,2)^2 d(2,3)^2+d(1,3)^2 d(2,3)^2\right)-d(2,3)^4-d(1,3)^4-d(1,2)^4}} \\ D_{23} = \sqrt{\frac{2\left(d(1,2)^2+d(1,3)^2-d(2,3)^2\right)}{2\left(d(1,2)^2 d(1,3)^2+d(1,2)^2 d(2,3)^2+d(1,3)^2 d(2,3)^2\right)-d(2,3)^4-d(1,3)^4-d(1,2)^4}} \end{cases} \tag{2.3.3.6}$$

where $d\left(i, j\right)$ is shorthand for $d_D\left(\delta_i, \delta_j\right)$. The formulas in (2.3.3.6) describe the passage from spectral distances to the coefficients of the Dirac operator.

Now we are ready to prove the required result: Let a, b, c be three positive numbers such that

$$a^2 + b^2 \geq c^2 \qquad b^2 + c^2 \geq a^2 \qquad a^2 + c^2 \geq b^2$$

Then define

$$r_1 := \frac{1}{2}\left(a^2 + b^2 - c^2\right) \qquad r_2 := \frac{1}{2}\left(a^2 + c^2 - b^2\right) \qquad r_3 := \frac{1}{2}\left(b^2 + c^2 - a^2\right)$$

These are positive numbers on account of the assumed inequalities between a, b, c. Therefore, we can consider them as resistances in a stellar circuit. They define spectral distances via (2.3.3.4) as

$$d_D(\delta_1, \delta_2) = a \qquad d_D(\delta_1, \delta_3) = b \qquad d_D(\delta_2, \delta_3) = c$$

and an associated Dirac operator via (2.3.3.6), where the coefficients are given, after factorisation, by

$$\begin{cases} D_{12} = \sqrt{\frac{2(b^2+c^2-a^2)}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \\ D_{13} = \sqrt{\frac{2(a^2+c^2-b^2)}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \\ D_{23} = \sqrt{\frac{2(a^2+b^2-c^2)}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \end{cases}$$

□

Unfortunately, calculation of the spectral distance in higher dimensions becomes very arduous. In fact, already for the four-point space, the spectral distance is only explicitly computable when additional restrictions are placed on the Dirac operator [25]. Moreover, the spectral distance cannot be inverted as is the case for the three-point space. There are, however, ways to overcome this issue. In [21], for example, by allowing the dimension of the Hilbert space to increase, the passage from Dirac operator to spectral distance can once again be inverted. Despite the restrictions on the Dirac operator, there are interesting cases that are explicitly computable in arbitrary dimensions. One such case, as found in [21], is where the Dirac operator D is chosen as the incidence matrix of the n-point graph associated to the n pure states. When the graph is maximally connected, i.e. all coefficients of D are equal to a fixed constant, the spectral distance between any two pure states can be explicitly calculated.

Chapter 3

Matrix space

In this chapter, we expand our survey to noncommutative spaces. We restrict once again to finite dimensions, in keeping with our strategy of systematically broadening the scope of our inquiry. Specifically, we consider a finite spectral triple built around the noncommutative algebra of $n \times n$ complex matrices. After determining the associated space of pure states, we calculate the spectral distance and venture a geometric interpretation for a low-dimensional example.

3.1 The spectral triple and pure states

Let $A_n = M_n(\mathbb{C})$, $H_n = \mathbb{C}^n$, and define the mapping

$$\pi : A_n \to M_n(\mathbb{C}) : a \mapsto a \qquad \text{for all } a \in A_n$$

as the usual representation of matrices. Let D be a self-adjoint element of $M_n(\mathbb{C})$ with complex entries D_{ij} such that

$$D_{ij} = \overline{D_{ji}} \in \mathbb{C}$$

for all $i, j \in \{1, \ldots, n\}$, where the overbar denotes the complex conjugate of a complex number. It follows from Definition 1.4.1 that (A_n, H_n, D) is a finite spectral triple.

Proposition 3.1.1. *The pure states of A_n are given by*

$$\omega_\xi(a) = \langle \xi, a\xi \rangle = \xi^* a \xi$$

for every normalised vector $\xi \in H_n$, where $a \in A_n$. Moreover, two normalised vectors determine the same pure state if and only if they are equal up to a phase. In other words, the space of pure states is isomorphic to the complex projective space:

$$PS(A_n) \simeq \mathbb{C}P^{n-1}$$

Proof. Since $\pi(A_n)$ is isomorphic to $\mathfrak{B}(H_n)$, we have $\pi(A_n)' = \mathbb{C}I_n$. Thus (H_n, π) is an irreducible representation of A_n. Theorem 1.2.8 shows that every nonzero vector in H_n is cyclic for (H_n, π). As per Theorem 1.2.11, every normalised vector $\xi \in H_n$ determines a pure state of A_n by

$$\omega_\xi(a) = \langle \xi, \pi(a)\xi \rangle = \langle \xi, a\xi \rangle = \xi^* a \xi$$

In fact, all the pure states of A_n are determined by such normalised vectors (see Example 5.1.1 in [31]).

Let $\xi, \zeta \in H_n$ be two normalised vectors. Suppose that they determine the same pure state; that is, for all $a \in A_n$

$$\omega_\xi(a) = \langle \xi, \pi(a)\xi \rangle = \langle \zeta, \pi(a)\zeta \rangle = \omega_\zeta(a)$$

Then there exists a unitary matrix u such that $a = uau^*$ for all $a \in A_n$ and $\xi = u\zeta$. This implies that there exists a constant $\theta \in [0, 2\pi]$ such that

$$u = \begin{pmatrix} e^{i\theta} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta} \end{pmatrix} = e^{i\theta} I_n$$

which gives

$$\xi = u\zeta = e^{i\theta} I_n \zeta = e^{i\theta}\zeta$$

Hence, ξ and ζ are equal up to a phase.

Conversely, suppose that ξ and ζ are equal up to a phase, i.e.

$$\xi = e^{i\theta}\zeta$$

for some $\theta \in [0, 2\pi]$. For all $a \in A_n$, it holds that

$$\begin{aligned} \omega_\xi(a) &= \xi^* a \xi \\ &= e^{-i\theta}\zeta^* a e^{i\theta}\zeta \\ &= \zeta^* a \zeta \\ &= \omega_\zeta(a) \end{aligned}$$

Hence, ξ and ζ determine the same pure state.

Equality up to a nonzero complex number defines an equivalence relation on H_n: For all nonzero $\psi, \phi \in H_n$, let

$$\psi \sim \phi \qquad \text{when} \qquad \psi = \lambda\phi \quad \text{for some nonzero} \quad \lambda \in \mathbb{C}$$

The equivalence classes in H_n are called the projective rays. The set of all equivalence classes with respect to $\sim$, i.e. the quotient space ${}^{H_n \setminus \{0\}}/_{\sim}$, is the space of complex lines through the origin of H_n and is called the complex projective space $\mathbb{C}P^{n-1}$. The set of projective rays is a homogeneous space for the unitary group $U(n)$ of $n \times n$ unitary matrices, with the group operation of matrix multiplication. Note that a unit vector does not uniquely determine a representative of a ray, because it retains its normalisation when multiplied by any λ with absolute value 1. The equivalence relation on the normalised vectors in H_n is given by equality up to a phase. Explicitly, for all normalised vectors $\xi, \zeta \in H_n$

$$\xi \sim \zeta \qquad \text{when} \qquad \xi = e^{i\theta}\zeta \quad \text{for some} \quad \theta \in [0, 2\pi]$$

Since we have shown a bijective correspondence between the pure states and the normalised vectors equal up to a phase, we may identify each pure state with a representative of a projective ray. Each representative may of course be identified with the entire projective ray. Hence, the space of pure states is isomorphic to the complex projective space $\mathbb{C}P^{n-1}$:

$$PS(A_n) \simeq \mathbb{C}P^{n-1}$$

□

For the finite spectral triple (A_n, H_n, D), as above, with the most general Dirac operator (an arbitrary self-adjoint $n \times n$ matrix), the spectral distance cannot be calculated in arbitrary dimensions n. However, such computations are possible for $n = 2$. Before we evaluate this case, consider the following lemma that holds in arbitrary dimensions and shows the invariance of the spectral distance under simultaneous unitary transformation of the Dirac operator and the pure states.

Lemma 3.1.2. *Let (A_n, H_n, D) be the finite spectral triple with associated pure states ω_ξ as above. For any unitary $U \in A_n$ such that*

$$\tilde{D} := U^* D U \qquad \text{and} \qquad \tilde{\omega}_\xi(a) := \omega_\xi(UaU^*)$$

where $a \in A_n$, the spectral distance satisfies

$$d_{\tilde{D}}(\tilde{\omega}_\xi, \tilde{\omega}_\zeta) = d_D(\omega_\xi, \omega_\zeta)$$

Proof.

$$\begin{aligned}
d_{\tilde{D}}(\tilde{\omega}_\xi, \tilde{\omega}_\zeta) &= \sup_{a \in A_n} \left\{ |(\tilde{\omega}_\xi - \tilde{\omega}_\zeta)(a)| : \left\| \left[\tilde{D}, a\right] \right\|_{spec} \le 1 \right\} \\
&= \sup_{a \in A_n} \left\{ |(\omega_\xi - \omega_\zeta)(UaU^*)| : \|U\left[U^*DU, a\right]U^*\|_{spec} \le 1 \right\} \\
&= \sup_{UaU^* \in A_n} \left\{ |(\omega_\xi - \omega_\zeta)(UaU^*)| : \|[D, UaU^*]\|_{spec} \le 1 \right\} \\
&= \sup_{a \in A_n} \left\{ |(\omega_\xi - \omega_\zeta)(a)| : \|[D, a]\|_{spec} \le 1 \right\} \\
&= d_D(\omega_\xi, \omega_\zeta)
\end{aligned}$$

□

3.2 The two-sphere

We consider the finite spectral triple (A_2, H_2, D), which is the spectral triple in Section 3.1 with $n = 2$. Since D is self-adjoint, it is diagonalisable by a unitary transformation. Let $U \in A_n$ be the unitary that diagonalises D such that

$$\tilde{D} := U^* D U = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

where D_1, D_2 are the strictly positive eigenvalues of D. As per Proposition 3.1.1, the pure states of A_2 are determined by normalised vectors $\xi \in H_2$ as

$$\omega_\xi(a) = \xi^* a \xi$$

The associated space of pure states is the complex projective line $\mathbb{C}P^1$:

$$PS(A_2) \simeq \mathbb{C}P^1$$

Proposition 3.2.1. *For the finite spectral triple (A_2, H_2, D), the spectral distance between any two pure states ω_ξ and ω_ζ of A_2, where*

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in H_2$$

such that $\|\xi\| = \|\zeta\| = 1$, is

$$d_D(\omega_\xi, \omega_\zeta) = \begin{cases} \frac{2}{|D_1 - D_2|} \sqrt{1 - |\langle \xi, \zeta \rangle|^2} & \text{if } \; |\xi_1| = |\zeta_1| \\ +\infty & \text{otherwise} \end{cases}$$

if D has distinct eigenvalues. If $D_1 = D_2$, then $d_D(\omega_\xi, \omega_\zeta) = +\infty$.

Proof. We sidestep some of the computational difficulties in calculating $d_D(\omega_\xi, \omega_\zeta)$ by first considering the distance between the same two pure states but according to the diagonalised operator $\tilde{D}$, i.e.

$$\begin{aligned} d_{\tilde{D}}(\omega_\xi, \omega_\zeta) &= \sup_{a \in A_2} \left\{ |(\omega_\xi - \omega_\zeta)(a)| : \left\| \left[\tilde{D}, \pi(a)\right] \right\|_{spec} \leq 1 \right\} \\ &= \sup_{a \in A_2^{sa}} \left\{ |(\omega_\xi - \omega_\zeta)(a)| : \left\| \left[\tilde{D}, \pi(a)\right] \right\|_{spec} \leq 1 \right\} \end{aligned} \tag{3.2.1.1}$$

After expressing (3.2.1.1) in terms of an inner product, Lemma 3.1.2 will allow us to easily convert to the desired spectral distance; the point is that the inner product is preserved by unitary operators. Note that we search the supremum in (3.2.1.1) on A_2^{sa}, as allowed by Lemma 1.3.2. It is worth mentioning that the map $a \mapsto UaU^*$ is an automorphism on $\mathfrak{B}(H_n)$. Therefore, searching for a supremum over all $a \in A_2$ is equivalent to searching over all $UaU^* \in A_2$. In particular, since UaU^* is self-adjoint whenever a is self-adjoint, Lemma 1.3.2 is still applicable when considering the spectral distance for unitarily transformed operators and pure states. Keeping this in mind, we consider arbitrary $a \in A_2^{sa}$ henceforth. The fact that $a^* = a$ has components that satisfy $a_{11}, a_{22} \in \mathbb{R}$, $a_{12} = \overline{a_{21}}$, and $|a_{12}| = |a_{21}|$ will be used without further mention.

Let us start by evaluating the norm constraint in (3.2.1.1): We find that

$$\left[\tilde{D}, \pi(a)\right] = \begin{pmatrix} 0 & (D_1 - D_2)\, a_{12} \\ (D_2 - D_1)\, a_{21} & 0 \end{pmatrix}$$

and

$$\left[\tilde{D}, \pi(a)\right]^* \left[\tilde{D}, \pi(a)\right] = \begin{pmatrix} |D_2 - D_1|^2 \, |a_{21}|^2 & 0 \\ 0 & |D_1 - D_2|^2 \, |a_{12}|^2 \end{pmatrix}$$

The norm, therefore, evaluates to

$$\left\| \left[\tilde{D}, \pi(a)\right] \right\|_{spec} = |a_{12}| \, |D_1 - D_2| \tag{3.2.1.2}$$

At this point, we take the opportunity to consider the trivial case where $D_1 = D_2$. The intermediary distance in (3.2.1.1) is unnecessary in this case. Directly from Lemma 3.1.2, with the norm in (3.2.1.2) zero, it follows that

$$\begin{aligned} d_D(\omega_\xi, \omega_\zeta) &= d_{\tilde{D}}(\tilde{\omega}_\xi, \tilde{\omega}_\zeta) \\ &= \sup_{a \in A_2^{sa}} \{ |\tilde{\omega}_\xi(a) - \tilde{\omega}_\zeta(a)| : 0 \le 1 \} \\ &= +\infty \end{aligned}$$

Henceforth, we let $D_1 \neq D_2$ and consider them distinct eigenvalues of D. Substituting (3.2.1.2) into (3.2.1.1) gives

$$d_{\tilde{D}}(\omega_\xi, \omega_\zeta) = \sup_{a \in A_2^{sa}} \left\{ |(\omega_\xi - \omega_\zeta)(a)| : |a_{12}| \le \frac{1}{|D_1 - D_2|} \right\} \tag{3.2.1.3}$$

The difference between the two pure states can be written as

$$\begin{aligned} &(\omega_\xi - \omega_\zeta)(a) \\ &= \xi^* a \xi - \zeta^* a \zeta \\ &= \sum_{i,j=1}^{2} a_{ij} \left(\overline{\xi_i} \xi_j - \overline{\zeta_i} \zeta_j \right) \\ &= a_{11} \left(|\xi_1|^2 - |\zeta_1|^2 \right) + a_{22} \left(|\xi_2|^2 - |\zeta_2|^2 \right) + a_{12} \left(\overline{\xi_1} \xi_2 - \overline{\zeta_1} \zeta_2 \right) + a_{21} \left(\overline{\xi_2} \xi_1 - \overline{\zeta_2} \zeta_1 \right) \end{aligned} \tag{3.2.1.4}$$

The expanded form is included seeing as it parades the appropriate choice of components when searching for the supremum. We are now ready to evaluate the different cases.

Case 1: $|\xi_1| \neq |\zeta_1|$
Let $a_L \in A_2^{sa}$ such that

$$a_L = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{where} \qquad L \in \mathbb{R}^+$$

For a_L, (3.2.1.4) becomes

$$(\omega_\xi - \omega_\zeta)(a_L) = L\left(|\xi_1|^2 - |\zeta_1|^2\right)$$

Also, it satisfies the constraint so that (3.2.1.3) is solved as

$$\begin{aligned} d_{\tilde{D}}(\omega_\xi, \omega_\zeta) &= \sup_{a \in A_2^{sa}} \left\{ |(\omega_\xi - \omega_\zeta)(a)| : |a_{12}| \leq \frac{1}{|D_1 - D_2|} \right\} \\ &\geq \sup_{L \in \mathbb{R}^+} \left\{ \left| L\left(|\xi_1|^2 - |\zeta_1|^2\right) \right| \right\} \\ &= +\infty \end{aligned}$$

It follows that $d_D(\omega_\xi, \omega_\zeta) = +\infty$.

Case 2: $|\xi_1| = |\zeta_1|$
Since ξ, ζ are normalised, we have

$$|\xi_1|^2 + |\xi_2|^2 = \xi^*\xi = 1 = \zeta^*\zeta = |\zeta_1|^2 + |\zeta_2|^2$$

which implies that

$$|\xi_2| = |\zeta_2|$$

Then (3.2.1.4) becomes

$$\begin{aligned} |(\omega_\xi - \omega_\zeta)(a)| &= \left| a_{12}\left(\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right) + a_{21}\left(\overline{\xi_2}\xi_1 - \overline{\zeta_2}\zeta_1\right) \right| \\ &= \left| 2\Re\left(a_{12}\left(\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right)\right) \right| \\ &\leq 2\left| a_{12}\left(\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right) \right| \\ &= 2|a_{12}|\left|\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right| \end{aligned} \tag{3.2.1.5}$$

If we can find an element $a \in A_2^{sa}$ that reaches the upper bound in (3.2.1.5), then the supremum in (3.2.1.3) will be attained by choosing $|a_{12}| = \frac{1}{|D_1 - D_2|}$.
Let $a_\theta \in A_2^{sa}$ such that

$$a_\theta = \begin{pmatrix} 0 & |a_{12}| e^{-i\theta} \\ |a_{12}| e^{i\theta} & 0 \end{pmatrix} \qquad \text{where} \qquad \theta = \arg\left(\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right)$$

For a_θ, (3.2.1.5) becomes

$$\begin{aligned}
|(\omega_\xi - \omega_\zeta)(a_\theta)| &= \left|2\Re\left(|a_{12}|\, e^{-i\theta}\left(\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right)\right)\right| \\
&= \left|2\Re\left(|a_{12}|\, e^{-i\theta}\left|\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right| e^{i\theta}\right)\right| \\
&= \left|2\Re\left(|a_{12}|\left|\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right|\right)\right| \\
&= 2\,|a_{12}|\left|\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right|
\end{aligned}$$

Hence, the upper bound is reached for $a_\theta \in A_2^{sa}$. Substituting into (3.2.1.3) gives

$$\begin{aligned}
d_{\tilde{D}}(\omega_\xi, \omega_\zeta) &= \sup_{a_{12}\in\mathbb{C}}\left\{2\,|a_{12}|\left|\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right| : |a_{12}| \leq \frac{1}{|D_1 - D_2|}\right\} \\
&= \frac{2\left|\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right|}{|D_1 - D_2|} \qquad (3.2.1.6)
\end{aligned}$$

We proceed to find an intrinsic formulation of (3.2.1.6), in terms of the normalised vectors that determine the pure states. Besides its aesthetic value, the eventual expression will provide an easy way of converting from our intermediary distance to the distance that is the actual intent of this proposition. Recall that we are still dealing with the case where $|\xi_1| = |\zeta_1|$ and $|\xi_2| = |\zeta_2|$. We can write

$$\begin{aligned}
\left|\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right|^2 &= |\xi_1|^2|\xi_2|^2 + |\zeta_1|^2|\zeta_2|^2 - \overline{\zeta_1}\zeta_2\xi_1\overline{\xi_2} - \overline{\xi_1}\xi_2\zeta_1\overline{\zeta_2} \\
&= |\xi_1|^2\left(1 - |\zeta_1|^2\right) + |\zeta_2|^2\left(1 - |\xi_2|^2\right) - \overline{\zeta_1}\zeta_2\xi_1\overline{\xi_2} - \overline{\xi_1}\xi_2\zeta_1\overline{\zeta_2} \\
&= \left(|\xi_1|^2 + |\zeta_2|^2\right) - |\xi_1|^2|\zeta_1|^2 - |\zeta_2|^2|\xi_2|^2 - \overline{\zeta_1}\zeta_2\xi_1\overline{\xi_2} - \overline{\xi_1}\xi_2\zeta_1\overline{\zeta_2} \\
&= 1 - \left(\overline{\xi_1}\zeta_1 + \overline{\xi_2}\zeta_2\right)\left(\xi_1\overline{\zeta_1} + \xi_2\overline{\zeta_2}\right) \\
&= 1 - (\xi^*\zeta)(\xi^*\zeta)^* \\
&= 1 - |\langle\xi, \zeta\rangle|^2
\end{aligned}$$

Substituting this expression into (3.2.1.6) gives

$$d_{\tilde{D}}(\omega_\xi, \omega_\zeta) = \frac{2}{|D_1 - D_2|}\sqrt{1 - |\langle\xi, \zeta\rangle|^2} \qquad (3.2.1.7)$$

If ξ is a normalised vector, i.e. $\xi^*\xi = 1$, that determines a pure state ω_ξ, then

$$(U^*\xi)^*(U^*\xi) = \xi^*UU^*\xi = 1$$

shows that $U^*\xi$ is normalised and determines the pure state

$$\begin{aligned}
\omega_{U^*\xi}(a) &= (U^*\xi)^*\, a\, (U^*\xi) \\
&= \xi^* U a U^* \xi \\
&= \omega_\xi(UaU^*) \\
&= \tilde{\omega}_\xi(a)
\end{aligned}$$

Using this formulation in (3.2.1.7), together with Lemma 3.1.2, gives

$$\begin{aligned} d_D(\omega_\xi, \omega_\zeta) &= d_{\tilde{D}}(\tilde{\omega}_\xi, \tilde{\omega}_\zeta) \\ &= d_{\tilde{D}}(\omega_{U^*\xi}, \omega_{U^*\zeta}) \\ &= \frac{2}{|D_1 - D_2|}\sqrt{1 - |\langle U^*\xi, U^*\zeta\rangle|^2} \\ &= \frac{2}{|D_1 - D_2|}\sqrt{1 - |\langle \xi, \zeta\rangle|^2} \end{aligned}$$

as required. □

It is possible to formulate the distances in Proposition 3.2.1 in terms of Euclidean coordinates on the two-sphere S^2. This follows from the fact that the complex projective line $\mathbb{C}P^1$, and thus the space of pure states $PS(A_2)$, is isomorphic to the two-sphere S^2. We briefly exhibit this correspondence; details can be found in [32, Chapters 1 and 4].

Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the canonical basis for $\mathbb{C}^2$. For a normalised vector $\psi \in H_2$, it holds that

$$\langle \psi, \psi \rangle = |\psi_1|^2 + |\psi_2|^2 = 1$$

This constraint on the components allows us to write

$$\begin{aligned} \psi &= \psi_1 e_1 + \psi_2 e_2 \\ &= e^{i\gamma}\Big(\cos(\theta/2)e_1 + e^{i\varphi}\sin(\theta/2)e_2\Big) \end{aligned}$$

where $\varphi, \gamma \in [0, 2\pi]$ and $\theta \in [0, \pi]$. Since normalised vectors equal up to a phase determine the same ray in $\mathbb{C}P^1$, the factor $e^{i\gamma}$ can be dropped. In other words, we may write each $\psi \in \mathbb{C}P^1$ as

$$\psi = \cos(\theta/2)e_1 + e^{i\varphi}\sin(\theta/2)e_2$$

where $\varphi \in [0, 2\pi]$ and $\theta \in [0, \pi]$. The parameters θ and φ may be reinterpreted as spherical coordinates in $\mathbb{R}^3$. We let θ be the colatitude with respect to the z-axis and φ the longitude with respect to the x-axis. They specify a point $\overline{p}$ on the unit two-sphere by

$$\overline{p} = (x, y, z) = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$$

such that

$$\|\overline{p}\|^2 = x^2 + y^2 + z^2 = 1$$

Proposition 3.2.2. *Every $\xi \in \mathbb{C}P^1$ is associated to a point $p_\xi = (x_\xi, y_\xi, z_\xi)$ on the unit two-sphere via*

$$x_\xi = 2\Re\left(\xi_1\overline{\xi_2}\right) \qquad y_\xi = 2\Im\left(\overline{\xi_1}\xi_2\right) \qquad z_\xi = |\xi_1|^2 - |\xi_2|^2$$

Proof. Every $\xi \in \mathbb{C}P^1$ can be written in terms of the canonical basis as $\xi = \xi_1 e_1 + \xi_2 e_2$, where the coefficients are given by

$$\xi_1 = \cos(\theta/2) \quad \text{and} \quad \xi_2 = e^{i\varphi}\sin(\theta/2)$$

Simple substitutions give

$$\begin{aligned} 2\Re\left(\xi_1\overline{\xi_2}\right) &= 2\Re\left(\cos(\theta/2)e^{-i\varphi}\sin(\theta/2)\right) \\ &= 2\cos(\theta/2)\cos(\varphi)\sin(\theta/2) \\ &= \cos\varphi\sin\theta \\ &= x_\xi \end{aligned}$$

$$\begin{aligned} 2\Im\left(\overline{\xi_1}\xi_2\right) &= 2\Im\left(\cos(\theta/2)e^{i\varphi}\sin(\theta/2)\right) \\ &= 2\sin(\varphi)\sin(\theta/2)\cos(\theta/2) \\ &= \sin\varphi\sin\theta \\ &= y_\xi \end{aligned}$$

$$\begin{aligned} |\xi_1|^2 - |\xi_2|^2 &= \cos^2(\theta/2) - \sin^2(\theta/2) \\ &= \cos\theta \\ &= z_\xi \end{aligned}$$

where $p_\xi = (x_\xi, y_\xi, z_\xi) = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$ specifies a point on the unit two-sphere. □

This sphere, considered isomorphic to the complex projective line $\mathbb{C}P^1$, is sometimes called the Bloch sphere and denoted S^2. Note that the basis vectors correspond to the poles of the Bloch sphere. Since we have already shown that the space of pure states $PS(A_2)$ is isomorphic to $\mathbb{C}P^1$, it follows that the pure states correspond to points on the Bloch sphere and

$$PS(A_2) \simeq S^2$$

Consequently, we can write the spectral distance between pure states in Proposition 3.2.1 in terms of Euclidean coordinates, i.e. as distances on S^2.

Proposition 3.2.3. *For the finite spectral triple (A_2, H_2, D), the spectral distance between any two pure states ω_ξ and ω_ζ of A_2, where*

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in H_2$$

such that $\|\xi\| = \|\zeta\| = 1$, is

$$d_D(\omega_\xi, \omega_\zeta) = \begin{cases} \frac{1}{|D_1 - D_2|}\sqrt{(x_\xi - x_\zeta)^2 + (y_\xi - y_\zeta)^2} & \text{if} \quad z_\xi = z_\zeta \\ +\infty & \text{otherwise} \end{cases}$$

if D has distinct eigenvalues. If $D_1 = D_2$, then $d_D(\omega_\xi, \omega_\zeta) = +\infty$.

Proof. The pure states ω_ξ and ω_ζ correspond to points $p_\xi = (x_\xi, y_\xi, z_\xi) \in S^2$ and $p_\zeta = (x_\zeta, y_\zeta, z_\zeta) \in S^2$ respectively, as treated in Proposition 3.2.2 and the ensuing discussion. If we focus for a moment on the components

$$z_\xi = |\xi_1|^2 - |\xi_2|^2 \quad \text{and} \quad z_\zeta = |\zeta_1|^2 - |\zeta_2|^2$$

it follows easily from the fact that ξ and ζ are normalised that

$$z_\xi = z_\zeta \quad \text{if and only if} \quad |\xi_1| = |\zeta_1|$$

Therefore, the conditions for finiteness of the spectral distance remain the same as in Proposition 3.2.1. Specifically, the distance is infinite whenever $z_\xi \neq z_\zeta$. All we are left to do is rewrite the expression for the case $|\xi_1| = |\zeta_1|$ in terms of the new variables. We use the relations in Proposition 3.2.2, together with the fact that

$$\overline{\alpha}\beta + \overline{\beta}\alpha = 2\Big(\Re(\alpha)\Re(\beta) + \Im(\alpha)\Im(\beta)\Big) \qquad \text{for all} \qquad \alpha, \beta \in \mathbb{C}$$

to find

$$\begin{aligned}
\left(2\left|\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right|\right)^2 &= 4\left(\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right)\left(\xi_1\overline{\xi_2} - \zeta_1\overline{\zeta_2}\right) \\
&= 4\left(\left|\xi_1\overline{\xi_2}\right|^2 + \left|\zeta_1\overline{\zeta_2}\right|^2 - \overline{\zeta_1}\zeta_2\xi_1\overline{\xi_2} - \overline{\xi_1}\xi_2\zeta_1\overline{\zeta_2}\right) \\
&= 4\left(\Re\left(\xi_1\overline{\xi_2}\right)^2 + \Im\left(\xi_1\overline{\xi_2}\right)^2 + \Re\left(\zeta_1\overline{\zeta_2}\right)^2 + \Im\left(\zeta_1\overline{\zeta_2}\right)^2\right) \\
&\quad - 8\left(\Re\left(\zeta_1\overline{\zeta_2}\right)\Re\left(\xi_1\overline{\xi_2}\right) + \Im\left(\zeta_1\overline{\zeta_2}\right)\Im\left(\xi_1\overline{\xi_2}\right)\right) \\
&= x_\xi^2 + y_\xi^2 + x_\zeta^2 + y_\zeta^2 - 2x_\zeta x_\xi - 2y_\zeta y_\xi \\
&= (x_\xi - x_\zeta)^2 + (y_\xi - y_\zeta)^2
\end{aligned}$$

Substituting this expression into (3.2.1.6) gives

$$\begin{aligned}
d_{\tilde{D}}(\omega_\xi, \omega_\zeta) &= \frac{2\left|\overline{\xi_1}\xi_2 - \overline{\zeta_1}\zeta_2\right|}{|D_1 - D_2|} \\
&= \frac{1}{|D_1 - D_2|}\sqrt{(x_\xi - x_\zeta)^2 + (y_\xi - y_\zeta)^2}
\end{aligned}$$

Since $d_D(\omega_\xi, \omega_\zeta) = d_{\tilde{D}}(\omega_\xi, \omega_\zeta)$ as shown in Proposition 3.2.1, the required result follows. □

Proposition 3.2.3 allows for geometric interpretation: The spectral distance between pure states associated to the spectral triple (A_2, H_2, D) is, up to a constant factor, the Euclidean distance on the two-sphere S^2 restricted to planes of constant altitude (equal z-components), where the distance between two planes of different altitude is infinite. In other words, the spectral distance equips the two-sphere with a metric that slices the sphere into circles at infinite distance from one another, where the distance on each circle is proportional to the Euclidean distance. In particular, the poles of S^2 are at infinite distance from any other pure state.

Part II

The Moyal plane

Here we study the nonommutative geometry of the Moyal plane from a metric point of view. The Moyal plane is an isospectral deformation of the Euclidean plane $\mathbb{R}^2$; that is, the Moyal plane is an infinite-dimensional spectral triple in which the algebra is a noncommutative deformation of a commutative algebra of functions on the Euclidean plane, while the Dirac operator keeps the same spectrum as in the commutative case [39]. Following the outline in [12], we construct a spectral triple based on the Moyal deformation of the algebra of Schwartz functions on $\mathbb{R}^2$ and calculate the spectral distance between some of the associated pure states.

In Chapter 4, we define and characterise the algebra of the Moyal plane A. Chapters 5 through 8 are concerned with finding equivalent representations of A; this not only paves the way towards assembling the spectral triple of the Moyal plane but also provides a context within which calculation of the spectral distance becomes tractable. After extending the Moyal product to larger spaces of tempered distributions in Chapter 5, we construct a basis for $L^2\left(\mathbb{R}^2\right)$ consisting of Schwartz functions in Chapter 6. The ensuing basis expansions allow us to represent the Schwartz space as a sequence space in Chapter 7. Moreover, we use this sequence representation to find a matricial form of the Moyal product, which we extend to $L^2\left(\mathbb{R}^2\right)$. The foundations laid in Chapters 5 through 7 support the equivalent representation of A that we define in Chapter 8. Finally, after determining the pure states of A in Chapter 9 and assembling the spectral triple of the Moyal plane in Chapter 10, we calculate the spectral distance between certain pure states in Chapter 11.

Chapter 4

The algebra A of the Moyal plane

The first ingredient in the spectral triple of the Moyal plane is the non-commutative algebra called the algebra of the Moyal plane. This algebra is formed by equipping the space of Schwartz functions on $\mathbb{R}^2$ with a Moyal product. In Section 4.1, we introduce the Schwartz space and its natural topology. The Moyal product is defined in Section 4.2 and used to define and characterise the algebra of the Moyal plane.

4.1 Schwartz space S

Here we define the Schwartz space and characterise it as a Fréchet space. First, we introduce a few notational conventions. Then, we state the relevant definitions and theorems without proof. The proofs and more detailed accounts of the concepts can be found in [34, Chapter V].

The Euclidean plane $\mathbb{R}^2$ is parametrised by Cartesian coordinates x_j for $j = 1, 2$. For any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, we let

$$x \cdot y \tag{4.1.1}$$

denote the usual dot product between vectors. Also, for $j = 1, 2$, we let

$$\partial_j = \frac{\partial}{\partial x_j} \tag{4.1.2}$$

denote the usual partial derivatives. If $\alpha_i \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, then

$$\alpha = (\alpha_1, \alpha_2) \tag{4.1.3}$$

denotes a two-dimensional multi-index such that

$$|\alpha| = \alpha_1 + \alpha_2$$

Note that for two multi-indices $\alpha, \beta \in \mathbb{N}_0^2$,

$$\alpha \leq \beta \quad \text{if and only if} \quad \alpha_i \leq \beta_i \ \text{ for } \ i = 1, 2$$

Furthermore, we write

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \tag{4.1.4}$$

and

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \tag{4.1.5}$$

where $\partial^\alpha f = f$ if $|\alpha| = 0$ for any function f.

Definition 4.1.1. *A* ***seminorm*** *on a vector space X is a map $\rho : X \to [0, \infty)$ such that for all $x, y \in X$ and $\lambda \in \mathbb{C}$:*
(i) $\rho(x + y) \leq \rho(x) + \rho(y)$
(ii) $\rho(\lambda x) = |\lambda| \rho(x)$
A family of seminorms $\{\rho_\alpha\}_{\alpha \in I}$, where I is some index set, is said to ***separate points*** *if*
(iii) $\rho_\alpha(x) = 0$ for all $\alpha \in I$ implies $x = 0$

Definition 4.1.2. *A* ***locally convex space*** *is a vector space X with a family $\{\rho_\alpha\}_{\alpha \in I}$ of seminorms separating points. The* ***natural topology*** *on a locally convex space is the weakest topology in which all the ρ_α are continuous and in which the operations of addition and scalar multiplication are continuous.*

Definition 4.1.3. *A net $\{x_\beta\}$ in a locally convex space X is called* ***Cauchy*** *if and only if, for all $\epsilon > 0$, and for each seminorm ρ_α there is a β_0 so that $\rho_\alpha(x_\beta - x_\gamma) < \epsilon$ if $\beta, \gamma > \beta_0$. X is called* ***complete*** *if every Cauchy net converges.*

The important structure on a locally convex space is the natural topology rather than the particular seminorms used to generate the topology. We call two families of seminorms $\{\rho_\alpha\}_{\alpha \in A}$ and $\{d_\beta\}_{\beta \in B}$ on a vector space X **equivalent** if they generate the same natural topology.

Theorem 4.1.4. *Let $\{\rho_\alpha\}_{\alpha \in A}$ and $\{d_\beta\}_{\beta \in B}$ be two families of seminorms. They are equivalent families of seminorms if and only if for each $\alpha \in A$, there are $\beta_1, \ldots, \beta_n \in B$ and $C > 0$ so that for all $x \in X$*

$$\rho_\alpha(x) \leq C \left(d_{\beta_1}(x) + \ldots + d_{\beta_n}(x) \right)$$

and for each $\beta \in B$, there are $\alpha_1, \ldots, \alpha_m \in A$ and $D > 0$ so that for all $x \in X$

$$d_\beta(x) \leq D \left(\rho_{\alpha_1}(x) + \ldots + \rho_{\alpha_m}(x) \right)$$

Theorem 4.1.5. *A locally convex space X is metrisable (has topology generated by a metric) if and only if the topology on X is generated by some countable family of seminorms.*

Theorem 4.1.6. *Let $\{\rho_n\}_{n=1}^{\infty}$ be a countable family of seminorms generating the topology on X. The topology is given by the metric defined by*

$$\rho(x,y) = \sum_{n=1}^{\infty} 2^{-n} \left(\frac{\rho_n(x-y)}{1+\rho_n(x-y)} \right)$$

and a net $\{x_\alpha\}$ in X is Cauchy in this metric if and only if it is Cauchy in each ρ_n. Thus a metrisable locally convex space X is complete as a metric space if and only if it is complete as a locally convex space.

Definition 4.1.7. *A **Fréchet space** is a complete metrisable locally convex space.*

Definition 4.1.8. *The **Schwartz space** of smooth rapidly decreasing complex-valued functions on $\mathbb{R}^2$, denoted by $S(\mathbb{R}^2)$, is defined as the set*

$$S(\mathbb{R}^2) := \left\{ f \in C^\infty(\mathbb{R}^2) : \|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^2} \left| x^\alpha \partial^\beta f(x) \right| < \infty \text{ for all } \alpha,\beta \in \mathbb{N}_0^2 \right\}$$

where $C^\infty(\mathbb{R}^2)$ denotes the space of infinitely differentiable (smooth) complex-valued functions on $\mathbb{R}^2$.

The functions in $S(\mathbb{R}^2)$ are those functions which, together with their derivatives, fall off more quickly than the inverse of any polynomial. In other words, $x^\alpha \partial^\beta f(x)$ goes to zero as $|x| \to \infty$ for all $\alpha, \beta \in \mathbb{N}_0^2$. Therefore, the functions in $S(\mathbb{R}^2)$ are said to be rapidly decreasing.

Henceforth, we let

$$S := S(\mathbb{R}^2) \tag{4.1.6}$$

Note that for each seminorm $\|\cdot\|_{\alpha,\beta}$, an open ball of radius r centered at some $f \in S$ is given by

$$B_{\|\cdot\|_{\alpha,\beta}}(f,r) = \left\{ g \in S : \|g-f\|_{\alpha,\beta} < r \right\} \tag{4.1.7}$$

Thus, each $\|\cdot\|_{\alpha,\beta}$ specifies a topology $\tau_{\alpha\beta}$ on S. A set is open according to $\tau_{\alpha\beta}$ if it can be expressed as a union of open balls. The topologies $\tau_{\alpha\beta}$ put together generate the natural topology τ on S, as per Definition 4.1.2, and is called the Schwartz topology; in other words, the Schwartz topology is the smallest topology containing all sets of $\bigcup_{\alpha,\beta} \tau_{\alpha\beta}$, thus making each seminorm $\|\cdot\|_{\alpha,\beta}$ continuous.

Theorem 4.1.9. *The Schwartz space S with the natural topology given by the seminorms $\|\cdot\|_{\alpha,\beta}$ is a Fréchet space.*

4.2 Moyal product $\star$

We equip the Schwartz space S with a Moyal product $\star$, also called the twisted product, and show that this produces a noncommutative, non-unital, involutive algebra. Further properties of the Moyal product are explicated to serve as a toolkit for the subsequent sections. Most of the results in this section follow from [18].

Let $\theta > 0$ be a fixed positive real parameter, and define

$$\Theta := \theta\Omega = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then the map

$$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x \cdot \Theta y$$

defines a symplectic bilinear form on $\mathbb{R}^2$, i.e. a bilinear map that is alternating ($x \cdot \Theta x = 0$ for all $x \in \mathbb{R}^2$) and nondegenerate ($x \cdot \Theta y = 0$ for all $y \in \mathbb{R}^2$ implies $x = 0$). Note, in particular, that for all $x, y \in \mathbb{R}^2$ it holds that $x \cdot \Theta y = -y \cdot \Theta x$. This symplectic bilinear form will be used to define the Moyal product on S and in so doing will serve to deform the commutative algebra of Schwartz functions equipped with pointwise multiplication to a noncommutative one. For the definition below to make sense, we mention here that $S \subset L^2\left(\mathbb{R}^2\right)$. This will be proven shortly in Proposition 4.2.1.

We define the Moyal $\star$-product on S by

$$(f \star g)(x) = \frac{1}{(\pi\theta)^2} \int f(x+y)\, g(x+z)\, e^{-i2y\cdot\Theta^{-1}z} d^2y d^2z \tag{4.2.1}$$

for all $f, g \in S$. The commutator of the $\star$-product was introduced in the context of phase-space quantum mechanics by Moyal [30] using a series development in powers of θ, where the first nontrivial term gives the Poisson bracket. The form above followed later when Moyal's series development was considered an asymptotic expansion of oscillatory integrals. In [17] and references therein, it is discussed how the usual pointwise product of Schwartz functions is recovered in the classical limit $\theta \to 0$ from left $\star$-multiplication, with convergence in the Schwartz topology.

Furthermore, we define on S an involution given by complex conjugation of functions

$$f^*(x) := \overline{f(x)} \tag{4.2.2}$$

and pointwise multiplication by a coordinate as

$$(\mu_j f)(x) := x_j f(x) \tag{4.2.3}$$

for $j = 1, 2$. Also, we define a flip operator by

$$\check{f}(x) := f(-x) \tag{4.2.4}$$

The following notation will be useful when performing calculations with the partial derivatives:

$$\hat{\partial}_1 f := \partial_2 f \quad \text{and} \quad \hat{\partial}_2 f := -\partial_1 f \tag{4.2.5}$$

In a similar vein to (4.1.4) and (4.1.5), for a multi-index $\alpha \in \mathbb{N}_0^2$, we write

$$\mu^\alpha = \mu_1^{\alpha_1} \mu_2^{\alpha_2} \quad \text{and} \quad \hat{\partial}^\alpha = \hat{\partial}_1^{\alpha_1} \hat{\partial}_2^{\alpha_2} \tag{4.2.6}$$

For the remainder of this dissertation, we make the following normalisations, as per [18], in order to simplify the definition (4.2.1) and the subsequent calculations:

- For integrals over $\mathbb{R}^2$, we use the Haar measure $dx := \frac{1}{2\pi} d^2x$, where d^2x is the usual Lebesgue measure. This will relieve us of powers of 2π in the Fourier transforms we define below.
- We use the bilinear form

$$\langle f, g \rangle := \int f(x)\, g(x)\, dx \tag{4.2.7}$$

and the sesquilinear form

$$(f|g) := 1/2 \langle f^*, g \rangle = 1/2 \int \overline{f(x)} g(x)\, dx \tag{4.2.8}$$

whenever the integrals converge.

- We set $\theta = 2$ so that $\Theta = 2\Omega$.

By applying these normalisations, it is simple to show that (4.2.1) simplifies to the following two equivalent definitions of the Moyal (twisted) product: If $f, g \in S$ and $s, t, u, v, w \in \mathbb{R}^2$, then

$$(f \star g)(u) := \int \int f(v)\, g(w)\, e^{i(u\cdot\Omega v + v\cdot\Omega w + w\cdot\Omega u)} dv dw \tag{4.2.9}$$

$$= \int \int f(u+s)\, g(u+t)\, e^{is\cdot\Omega t} ds dt \tag{4.2.10}$$

As a tool for later calculations, we define another product on S, which we will see is related to the Moyal product via certain Fourier transforms. For $f, g \in S$, we define the twisted convolution $\diamond$ by

$$(f \diamond g)(u) := \int f(u-t)\, g(t)\, e^{-iu\cdot\Omega t} dt \tag{4.2.11}$$

The normalisations above also allow us to define the space of measurable functions that are square-integrable with respect to the Haar measure as follows: Let $L^2(\mathbb{R}^2)$ denote the set of complex-valued measurable functions on $\mathbb{R}^2$ which satisfy

$$\int |f(x)|^2 \, dx < \infty$$

Note that $L^2(\mathbb{R}^2)$ becomes a Hilbert space when equipped with inner product $(\cdot|\cdot)$, as defined in (4.2.8). The induced norm is given by

$$\|f\|_2 := (f|f)^{\frac{1}{2}} = \left(1/2 \int |f(x)|^2 \, dx\right)^{\frac{1}{2}} \tag{4.2.12}$$

for all $f \in L^2(\mathbb{R}^2)$. Furthermore, we define $\|\cdot\|_\infty$ on S as the seminorm $\|\cdot\|_{\alpha,\beta}$ in Definition 4.1.8 with $\alpha, \beta = 0$, i.e.

$$\|f\|_\infty := \sup_{x \in \mathbb{R}^2} |f(x)| = \|f\|_{0,0} \tag{4.2.13}$$

for all $f \in S$. Finally, we define the family of seminorms $\left\{\|\cdot\|_{N,\beta}\right\}_{N \in \mathbb{N}_0, \beta \in \mathbb{N}_0^2}$ on S, where

$$\|f\|_{N,\beta} := \sup_{x \in \mathbb{R}^2} (1+|x|)^N \left|\partial^\beta f(x)\right| < \infty \tag{4.2.14}$$

for all $f \in S$. This family of seminorms is equivalent to the family $\left\{\|\cdot\|_{\alpha,\beta}\right\}_{\alpha,\beta \in \mathbb{N}_0^2}$, in the sense that they generate the same topology on S [37, Chapter 7].

Proposition 4.2.1. *S is dense in $L^2(\mathbb{R}^2)$.*

Proof. First, we show that $S \subset L^2(\mathbb{R}^2)$. Let $f \in S$ be arbitrary. Using the seminorms defined in (4.2.14), we find that

$$\begin{aligned}
\|f\|_2 &= \left(1/2 \int |f(x)|^2 \, dx\right)^{\frac{1}{2}} \\
&= \left(1/2 \int (1+|x|)^{-6} (1+|x|)^6 |f(x)|^2 \, dx\right)^{\frac{1}{2}} \\
&\leq \left(1/2 \int (1+|x|)^{-6} \, dx\right)^{\frac{1}{2}} \sup_{x \in \mathbb{R}^2} (1+|x|)^3 |f(x)| \\
&\leq C \|f\|_{3,0}
\end{aligned}$$

for some constant $C < \infty$. Also, $\|f\|_{3,0} < \infty$ follows from the fact that $\|f\|_{N,\beta} < \infty$ for all $N \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^2$ when $f \in S$. Thus $f \in L^2(\mathbb{R}^2)$. Since $f \in S$ was chosen arbitrarily, it follows that $S \subset L^2(\mathbb{R}^2)$.

Now, let $C_0^\infty(\mathbb{R}^2)$ denote the set of smooth functions of compact support on $\mathbb{R}^2$. In other words, it is the set of infinitely differentiable functions f such that the closure of the set $\{x : f(x) \neq 0\}$ is compact. Clearly

$$C_0^\infty(\mathbb{R}^2) \subset S$$

It is well known, see for example [16, Proposition 8.17], that $C_0^\infty\left(\mathbb{R}^2\right)$ is dense in $L^2\left(\mathbb{R}^2\right)$. Therefore, S is also dense in $L^2\left(\mathbb{R}^2\right)$. □

A similar proof gives $S \subset L^1\left(\mathbb{R}^2\right)$. Moreover, if $f \in S$, then the function $\mathbb{R}^2 \to \mathbb{C} : x \mapsto x^\alpha f(x)$ is in $L^1\left(\mathbb{R}^2\right)$, where we use the notation from (4.1.4).

Now we define an ordinary Fourier transform $\mathfrak{F}$ and two symplectic Fourier transforms F and $\check{F}$ on S by:

$$(\mathfrak{F}f)(u) := \int f(t)\, e^{-it\cdot u} dt \tag{4.2.15}$$

$$(Ff)(u) := \int f(t)\, e^{-it\cdot\Omega u} dt \tag{4.2.16}$$

$$\left(\check{F}f\right)(u) := \int f(t)\, e^{it\cdot\Omega u} dt \tag{4.2.17}$$

Note that these integrals make sense because $S \subset L^1\left(\mathbb{R}^2\right)$.

Lemma 4.2.2. *The transforms $\mathfrak{F}$, F and $\check{F}$ are commuting isomorphisms (of Fréchet spaces) of S onto S, and satisfy the following properties for all $f, g \in S$:*

(1) $Ff(u) = \mathfrak{F}f(\Omega u)$
(2) $\check{F}f(u) = \mathfrak{F}f(-\Omega u) = F\left(\check{f}\right)$
(3) $(Ff)^* = \check{F}(f^*)$
(4) $F^2 = \check{F}^2 = I$
(5) $F\left(\hat{\partial}_j f\right) = -i\mu_j Ff$
(6) $F(\mu_j f) = -i\hat{\partial}_j Ff$
(7) $\langle Ff, g\rangle = \left\langle f, \check{F}g\right\rangle$
(8) $(Ff|g) = (f|Fg)$
(9) $\langle \mathfrak{F}f, g\rangle = \langle f, \mathfrak{F}g\rangle$

Proof. The fact that the transforms are commuting isomorphisms of S onto S can be seen in [35, Chapter IX]. Properties (1)-(3) follow directly from the definitions (4.2.15), (4.2.16) and (4.2.17). Let $f \in S$. Then, since $S \subset$

$L^1(\mathbb{R}^2)$, we may apply Fubini's theorem to satisfy property (4) as follows:

$$
\begin{aligned}
\left(F^2 f\right)(u) &= F\int f(t)\, e^{-it\cdot\Omega u}dt \\
&= \int\left(\int f(t)\, e^{it\cdot\Omega t'}dt\right)e^{-it'\cdot\Omega u}dt' \\
&= \int\int f(t)\, e^{it'\cdot\Omega t}e^{-it'\cdot\Omega u}dtdt' \\
&= \int\int f(t)\, e^{it'\cdot\Omega(t-u)}dt'dt \\
&= \int f(t)\,\delta(t-u)\,dt \\
&= f(u)
\end{aligned}
$$

where $\delta(\cdot)$ denotes the Dirac delta function, which was first treated with mathematical rigour as a generalised function or distribution in [38]. Thus $F^2 = I$ and similarly $\breve{F}^2 = I$.

To prove property (5), first note that if $f \in S$, then there exists a constant $C > 0$ such that

$$
|f(t)| \leq C(1+|t|)^{-1}
$$

It follows, for $j = 1, 2$, that

$$
\begin{aligned}
\lim_{t_j\to\pm\infty}\left|f(t)\, e^{-it\cdot\Omega u}\right| &\leq \lim_{t_j\to\pm\infty}|f(t)| \\
&\leq \lim_{t_j\to\pm\infty} C(1+|t|)^{-1} \\
&= 0
\end{aligned}
$$

Now we use integration by parts, where the boundary terms go to zero due to the expression above: For $j = 1$,

$$
\begin{aligned}
F\left(\hat{\partial}_1 f\right)(u) &= \int\left(\hat{\partial}_1 f\right)(t)\, e^{-it\cdot\Omega u}dt \\
&= \int\int(\partial_2 f)(t)\, e^{-it\cdot\Omega u}dt_2dt_1 \\
&= \int\int\frac{\partial}{\partial t_2}\left(f(t)\, e^{-it\cdot\Omega u}\right)dt_2dt_1 - \int\int f(t)\,\frac{\partial}{\partial t_2}\left(e^{-it\cdot\Omega u}\right)dt_2dt_1 \\
&= \int f(t)\, e^{-it\cdot\Omega u}\Big|_{t_2=-\infty}^{t_2=\infty}dt_1 - iu_1\int f(t)\, e^{-it\cdot\Omega u}dt \\
&= 0 - iu_1 Ff(u) \\
&= -i\mu_1 Ff(u)
\end{aligned}
$$

For $j = 2$, we find $F\left(\hat{\partial}_2 f\right)(u) = -i\mu_2 Ff(u)$ by following the same steps as above, while taking into account the difference in signs coming from

$\hat{\partial}_2 f = -\partial_1 f$. Hence, $F\left(\hat{\partial}_j f\right) = -i\mu_j F f$, as required.

Property (6) is found similarly to property (5). Properties (7) and (8) are found by using properties (1)-(3) in the definitions of $\langle\cdot,\cdot\rangle$ and $(\cdot|\cdot)$ in (4.2.7) and (4.2.8) respectively. Property (9) follows from definitions (4.2.7) and (4.2.15). $\square$

We now proceed to prove certain properties of the Moyal product that will allow us to characterise the Moyal algebra.

Proposition 4.2.3. *The Moyal product satisfies the Leibniz rule: If $f, g \in S$, then*

$$\partial_j (f \star g) = \partial_j f \star g + f \star \partial_j g$$

Proof. Let $f, g \in S$. Note that S is closed under pointwise multiplication and $S \subset L^1(\mathbb{R}^2)$. The partial derivative with respect to u_j of the integrand in (4.2.10) exists and is uniformly bounded by an L^1 function. Therefore, the dominated convergence theorem implies the Leibniz integral rule, which allows us to differentiate (4.2.10) under the integral:

$$\begin{aligned}
&\partial_j (f \star g)(u) \\
&= \int\int \frac{\partial}{\partial u_j}\Big(f(u+s)\, g(u+t)\Big) e^{is\cdot\Omega t} ds dt \\
&= \int\int \Big(\partial_j f(u+s)\Big) g(u+t)\, e^{is\cdot\Omega t} ds dt + \int\int f(u+s)\Big(\partial_j g(u+t)\Big) e^{is\cdot\Omega t} ds dt \\
&= (\partial_j f \star g)(u) + (f \star \partial_j g)(u)
\end{aligned}$$

$\square$

Proposition 4.2.4. *Pointwise multiplication by a coordinate obeys*

$$\mu_j (f \star g) = f \star \mu_j g + i\left(\hat{\partial}_j f\right) \star g = (\mu_j f) \star g - i f \star \hat{\partial}_j g$$

for all $f, g \in S$.

Proof. Using (4.2.16) and Lemma 4.2.2(5) in the definition (4.2.9), we find

$$
\begin{aligned}
&\mu_j\left(f\star g\right)(u)\\
&=\int\int u_j f(v)\, g(w)\, e^{i(u\cdot\Omega v+v\cdot\Omega w+w\cdot\Omega u)}dvdw\\
&=\int\int u_j f(v)\, g(w)\, e^{-iv\cdot\Omega(u-w)}e^{iw\cdot\Omega u}dvdw\\
&=\int u_j\left(Ff\right)(u-w)\, g(w)\, e^{-iu\cdot\Omega w}dw\\
&=\int \left(u_j-w_j+w_j\right)\left(Ff\right)(u-w)\, g(w)\, e^{-iu\cdot\Omega w}dw\\
&=\int w_j\left(Ff\right)(u-w)\, g(w)\, e^{-iu\cdot\Omega w}dw+\int\left(u_j-w_j\right)\left(Ff\right)(u-w)\, g(w)\, e^{-iu\cdot\Omega w}dw\\
&=\int\int f(v)\, w_j g(w)\, e^{-iv\cdot\Omega(u-w)}e^{iw\cdot\Omega u}dvdw+\int\left(\mu_j Ff\right)(u-w)\, g(w)\, e^{iw\cdot\Omega u}dw\\
&=\int\int f(v)\left(\mu_j g\right)(w)\, e^{-iv\cdot\Omega(u-w)}e^{iw\cdot\Omega u}dvdw+i\int\left(F\left(\hat{\partial}_j f\right)\right)(u-w)\, g(w)\, e^{iw\cdot\Omega u}dw\\
&=\int\int f(v)\left(\mu_j g\right)(w)\, e^{-iv\cdot\Omega(u-w)}e^{iw\cdot\Omega u}dvdw+i\int\int\left(\hat{\partial}_j f\right)(v)\, g(w)\, e^{-iv\cdot\Omega(u-w)}e^{iw\cdot\Omega u}dvdw\\
&=\left(f\star\mu_j g\right)(u)+\left(i\left(\hat{\partial}_j f\right)\star g\right)(u)
\end{aligned}
$$

for all $f, g \in S$. The second equality is found similarly using Lemma 4.2.2(6). □

Proposition 4.2.5. *If $f, g \in S$, then $(f \star g) \in S$.*

Proof. We offer a proof via induction on the formulas in Propositions 4.2.3 and 4.2.4. The number of combinations available for distributing higher order derivatives and polynomials over the Moyal product will be expressed by binomial coefficients, which are defined, for $n, \ell \in \mathbb{N}_0$ such that $n \geq \ell \geq 0$, by

$$\binom{n}{\ell}=\frac{n!}{(n-\ell)!\ell!}=\binom{n}{n-\ell}$$

It will be useful to remember Pascal's rule: For $1 \leq \ell \leq n+1$, it holds that

$$\binom{n}{\ell}+\binom{n}{\ell-1}=\binom{n+1}{\ell}$$

We extend this notation to include multi-indices: For $\alpha, \beta \in \mathbb{N}_0^2$, we write

$$\binom{\alpha}{\beta}:=\binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}=\frac{\alpha!}{\beta!\,(\alpha-\beta)!}=\binom{\alpha}{\alpha-\beta}$$

where

$$\alpha\pm\beta=(\alpha_1\pm\beta_1,\alpha_2\pm\beta_2)\qquad\text{and}\qquad\alpha!=\alpha_1!\alpha_2!$$

Let $f, g \in S$ be arbitrary. First, we use induction on the Leibniz rule. From Proposition 4.2.3 we have that

$$\partial_j (f \star g) = \partial_j f \star g + f \star \partial_j g = \sum_{\ell=0}^{1} \binom{1}{\ell} \partial_j^{1-\ell} f \star \partial_j^{\ell} g$$

This serves as the basis for the induction. Now fix $n \in \mathbb{N}$ and suppose that

$$\partial_j^n (f \star g) = \sum_{\ell=0}^{n} \binom{n}{\ell} \partial_j^{n-\ell} f \star \partial_j^{\ell} g$$

as the induction hypothesis. The induction step follows:

$$\begin{aligned}
&\partial_j^{n+1} (f \star g) \\
&= \partial_j \left(\sum_{\ell=0}^{n} \binom{n}{\ell} \partial_j^{n-\ell} f \star \partial_j^{\ell} g \right) \\
&= \sum_{\ell=0}^{n} \binom{n}{\ell} \partial_j \left(\partial_j^{n-\ell} f \star \partial_j^{\ell} g \right) \\
&= \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\partial_j^{n+1-\ell} f \star \partial_j^{\ell} g + \partial_j^{n-\ell} f \star \partial_j^{\ell+1} g \right) \\
&= \sum_{\ell=0}^{n} \binom{n}{\ell} \partial_j^{n+1-\ell} f \star \partial_j^{\ell} g + \sum_{\ell=0}^{n} \binom{n}{\ell} \partial_j^{n-\ell} f \star \partial_j^{\ell+1} g \\
&= \binom{n}{0} \partial_j^{n+1} f \star g + \sum_{\ell=1}^{n} \binom{n}{\ell} \partial_j^{n+1-\ell} f \star \partial_j^{\ell} g + \binom{n}{n} f \star \partial_j^{n+1} g + \sum_{\ell=1}^{n} \binom{n}{\ell-1} \partial_j^{n+1-\ell} f \star \partial_j^{\ell} g \\
&= \binom{n}{0} \partial_j^{n+1} f \star g + \binom{n}{n} f \star \partial_j^{n+1} g + \sum_{\ell=1}^{n} \binom{n+1}{\ell} \partial_j^{n+1-\ell} f \star \partial_j^{\ell} g \\
&= \binom{n+1}{0} \partial_j^{n+1} f \star g + \binom{n+1}{n+1} f \star \partial_j^{n+1} g + \sum_{\ell=1}^{n} \binom{n+1}{\ell} \partial_j^{n+1-\ell} f \star \partial_j^{\ell} g \\
&= \sum_{\ell=0}^{n+1} \binom{n+1}{\ell} \partial_j^{n+1-\ell} f \star \partial_j^{\ell} g
\end{aligned}$$

Since this holds for arbitrary $n \in \mathbb{N}$, it follows by induction that

$$\partial_j^n (f \star g) = \sum_{\ell=0}^{n} \binom{n}{\ell} \partial_j^{n-\ell} f \star \partial_j^{\ell} g \tag{4.2.5.1}$$

Now consider multi-indices $\gamma, \epsilon \in \mathbb{N}_0^2$ with components $k, \ell, m, n \in \mathbb{N}_0$ such that $\gamma = (m, n)$ and $\epsilon = (k, \ell)$. Repeated application of (4.2.5.1), expressed in terms of the notation (4.1.5), gives

$$\begin{aligned}
\partial^{\gamma}(f\star g) &= \partial_1^m\partial_2^n(f\star g)\\
&= \partial_1^m\sum_{\ell=0}^{n}\binom{n}{\ell}\partial_2^{n-\ell}f\star\partial_2^{\ell}g\\
&= \sum_{\ell=0}^{n}\binom{n}{\ell}\partial_1^m\left(\partial_2^{n-\ell}f\star\partial_2^{\ell}g\right)\\
&= \sum_{\ell=0}^{n}\binom{n}{\ell}\sum_{k=0}^{m}\binom{m}{k}\partial_1^{m-k}\partial_2^{n-\ell}f\star\partial_1^{k}\partial_2^{\ell}g\\
&= \sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{k}\binom{n}{\ell}\partial^{(m-k,n-\ell)}f\star\partial^{(k,\ell)}g\\
&= \sum_{\epsilon\leq\gamma}\binom{\gamma}{\epsilon}\partial^{\gamma-\epsilon}f\star\partial^{\epsilon}g \qquad (4.2.5.2)
\end{aligned}$$

Note that $f\star g\in C^{\infty}$ because $f,g\in C^{\infty}$.

Next, we use induction on the pointwise multiplication rule. From Proposition 4.2.4 we have that

$$\mu_j(f\star g)=(\mu_j f)\star g-if\star\hat{\partial}_j g=\sum_{\ell=0}^{1}(-i)^{\ell}\binom{1}{\ell}\mu_j^{1-\ell}f\star\hat{\partial}_j^{\ell}g$$

which serves as the basis for the induction. Now fix $n\in\mathbb{N}$ and suppose that

$$\mu_j^n(f\star g)=\sum_{\ell=0}^{n}(-i)^{\ell}\binom{n}{\ell}\mu_j^{n-\ell}f\star\hat{\partial}_j^{\ell}g$$

as the induction hypothesis. Then we use similar manipulations as in the case above to find

$$\mu_j^{n+1}(f\star g)=\sum_{\ell=0}^{n+1}(-i)^{\ell}\binom{n+1}{\ell}\mu_j^{n+1-\ell}f\star\hat{\partial}_j^{\ell}g$$

Since this holds for arbitrary $n\in\mathbb{N}$, it follows by induction that

$$\mu_j^n(f\star g)=\sum_{\ell=0}^{n}(-i)^{\ell}\binom{n}{\ell}\mu_j^{n-\ell}f\star\hat{\partial}_j^{\ell}g \qquad (4.2.5.3)$$

Now consider multi-indices $\alpha,\beta\in\mathbb{N}_0^2$ with components $k,\ell,m,n\in\mathbb{N}_0$ such that $\alpha=(m,n)$ and $\beta=(k,\ell)$. Note that $|\beta|=k+\ell$. Repeated application of (4.2.5.3), expressed in terms of the notation (4.2.8), gives

$$\begin{aligned}
\mu^{\alpha}(f\star g) &= \mu_1^m\mu_2^n(f\star g)\\
&= \mu_1^m\sum_{\ell=0}^{n}(-i)^{\ell}\binom{n}{\ell}\mu_2^{n-\ell}f\star\hat{\partial}_2^{\ell}g\\
&= \sum_{\ell=0}^{n}(-i)^{\ell}\binom{n}{\ell}\mu_1^m\left(\mu_2^{n-\ell}f\star\hat{\partial}_2^{\ell}g\right)\\
&= \sum_{\ell=0}^{n}(-i)^{\ell}\binom{n}{\ell}\sum_{k=0}^{m}(-i)^{k}\binom{m}{k}\mu_1^{m-k}\mu_2^{n-\ell}f\star\hat{\partial}_1^{k}\hat{\partial}_2^{\ell}g\\
&= \sum_{k=0}^{m}\sum_{\ell=0}^{n}(-i)^{k+\ell}\binom{m}{k}\binom{n}{\ell}\mu^{(m-k,n-\ell)}f\star\hat{\partial}^{(k,\ell)}g\\
&= \sum_{\beta\le\alpha}(-i)^{|\beta|}\binom{\alpha}{\beta}\mu^{\alpha-\beta}f\star\hat{\partial}^{\beta}g \qquad (4.2.5.4)
\end{aligned}$$

Combining (4.2.5.2) and (4.2.5.4) gives

$$\begin{aligned}
\mu^{\alpha}\partial^{\gamma}(f\star g) &= \mu^{\alpha}\sum_{\epsilon\le\gamma}\binom{\gamma}{\epsilon}\partial^{\gamma-\epsilon}f\star\partial^{\epsilon}g\\
&= \sum_{\epsilon\le\gamma}\binom{\gamma}{\epsilon}\mu^{\alpha}\left(\partial^{\gamma-\epsilon}f\star\partial^{\epsilon}g\right)\\
&= \sum_{\epsilon\le\gamma}\binom{\gamma}{\epsilon}\sum_{\beta\le\alpha}(-i)^{|\beta|}\binom{\alpha}{\beta}\mu^{\alpha-\beta}\partial^{\gamma-\epsilon}f\star\hat{\partial}^{\beta}\partial^{\epsilon}g\\
&= \sum_{\beta\le\alpha}\sum_{\epsilon\le\gamma}(-i)^{|\beta|}\binom{\alpha}{\beta}\binom{\gamma}{\epsilon}\mu^{\alpha-\beta}\partial^{\gamma-\epsilon}f\star\hat{\partial}^{\beta}\partial^{\epsilon}g \qquad (4.2.5.5)
\end{aligned}$$

for all $\alpha,\gamma\in\mathbb{N}_0^2$. From the definition of $\star$ in (4.2.9), we find the following norm inequality:

$$\begin{aligned}
\|f\star g\|_{\infty} &= \sup_{x\in\mathbb{R}^2}|(f\star g)(x)|\\
&= \sup_{x\in\mathbb{R}^2}\left|\int\int f(v)\,g(w)\,e^{i(x\cdot\Omega v+v\cdot\Omega w+w\cdot\Omega x)}dvdw\right|\\
&\le \sup_{x\in\mathbb{R}^2}\int\int\left|f(v)\,g(w)\,e^{i(x\cdot\Omega v+v\cdot\Omega w+w\cdot\Omega x)}\right|dvdw\\
&\le \int|f(v)|\,dv\int|g(w)|\,dw\\
&= \|f\|_1\|g\|_1 \qquad (4.2.5.6)
\end{aligned}$$

Note that $\mu^{\alpha-\beta}\partial^{\gamma-\epsilon}f\in S$ and $\hat{\partial}^{\beta}\partial^{\epsilon}g\in S$ whenever $f,g\in S$. Therefore, from (4.2.5.6) and (4.2.5.5), it holds for all $\alpha, \gamma \in \mathbb{N}_0^2$ that

$$\begin{aligned}
\|f \star g\|_{\alpha,\gamma} &= \|\mu^\alpha \partial^\gamma (f \star g)\|_\infty \\
&= \left\| \sum_{\beta \leq \alpha} \sum_{\epsilon \leq \gamma} (-i)^{|\beta|} \binom{\alpha}{\beta} \binom{\gamma}{\epsilon} \mu^{\alpha-\beta} \partial^{\gamma-\epsilon} f \star \hat{\partial}^\beta \partial^\epsilon g \right\|_\infty \\
&\leq \sum_{\beta \leq \alpha} \sum_{\epsilon \leq \gamma} \binom{\alpha}{\beta} \binom{\gamma}{\epsilon} \left\| \mu^{\alpha-\beta} \partial^{\gamma-\epsilon} f \star \hat{\partial}^\beta \partial^\epsilon g \right\|_\infty \\
&\leq \sum_{\beta \leq \alpha} \sum_{\epsilon \leq \gamma} \binom{\alpha}{\beta} \binom{\gamma}{\epsilon} \left\| \mu^{\alpha-\beta} \partial^{\gamma-\epsilon} f \right\|_1 \left\| \hat{\partial}^\beta \partial^\epsilon g \right\|_1 \qquad (4.2.5.7) \\
&< \infty
\end{aligned}$$

Hence, Definition 4.1.8 is satisfied so that $(f \star g) \in S$. □

On our way towards proving associativity of $\star$, we consider the way the Fourier transforms in (4.2.15), (4.2.16) and (4.2.17) intertwine the twisted product $\star$ and the twisted convolution $\diamond$. This will allow us to perform relatively simple calculations in $\diamond$ before transferring the results to $\star$. The next proposition regards these intertwining properties and the subsequent one proves associativity.

Proposition 4.2.6. *For all $f, g \in S$, it holds that*

$$f \star g = (Ff) \diamond g = f \diamond \left(\breve{F} g\right) \qquad (4.2.6.1)$$

$$f \diamond g = (Ff) \star g = f \star \left(\breve{F} g\right) \qquad (4.2.6.2)$$

$$\mathfrak{F}(f \star g) = (\mathfrak{F} f) \diamond (\mathfrak{F} g) \qquad (4.2.6.3)$$

$$\mathfrak{F}(f \diamond g) = (\mathfrak{F} f) \star (\mathfrak{F} g) \qquad (4.2.6.4)$$

Proof. Let $f, g \in S$ be arbitrary. We use definitions (4.2.9) and (4.2.11) for $\star$ and $\diamond$ respectively. The Fourier transforms $\mathfrak{F}$, F and $\breve{F}$ are defined in (4.2.15), (4.2.16) and (4.2.17). We find

$$\begin{aligned}
(f \star g)(u) &= \int \int f(v)\, g(w)\, e^{i(u \cdot \Omega v + v \cdot \Omega w + w \cdot \Omega u)} dv dw \\
&= \int \int f(v)\, g(w)\, e^{-iv \cdot \Omega (u-w)} e^{iw \cdot \Omega u} dv dw \\
&= \int (Ff)(u-w)\, g(w)\, e^{-iu \cdot \Omega w} dw \\
&= \Big((Ff) \diamond g \Big)(u)
\end{aligned}$$

which proves the first equality in (4.2.6.1). The remaining equality in (4.2.6.1) and those in (4.2.6.2) follow similarly. We find (4.2.6.3) by applying Lemma 4.2.2 (1) and making appropriate substitutions as follows:

$$\begin{aligned}
\mathfrak{F}(f \star g)(u) &= \int (f \star g)(s)\, e^{-is\cdot u} ds \\
&= \int\int\int f(v)\, g(w)\, e^{i(s\cdot\Omega v + v\cdot\Omega w + w\cdot\Omega s)} dv dw\, e^{-is\cdot u} ds \\
&= \int\int\int f(v)\, g(w)\, e^{-iv\cdot\Omega(s-w)} e^{iw\cdot\Omega s} e^{-is\cdot u} dv dw ds \\
&= \int\int (Ff)(s-w)\, g(w)\, e^{-is\cdot\Omega w} e^{-is\cdot u} dw ds \\
&= \int\int (\mathfrak{F}f)(\Omega s - \Omega w)\, g(w)\, e^{-is\cdot(\Omega w + u)} dw ds \\
&= \int\int (\mathfrak{F}f)(u-t)\, g(r)\, e^{-i(-\Omega u + \Omega t + r)\cdot(\Omega r + u)} dr dt \\
&= \int\int (\mathfrak{F}f)(u-t)\, g(r)\, e^{-i(t\cdot r + \Omega t\cdot u)} dr dt \\
&= \int (\mathfrak{F}f)(u-t) \left(\int g(r)\, e^{-ir\cdot t} dr \right) e^{-iu\cdot\Omega t} dt \\
&= \int (\mathfrak{F}f)(u-t)\, (\mathfrak{F}g)(t)\, e^{-iu\cdot\Omega t} dt \\
&= \Big((\mathfrak{F}f) \diamond (\mathfrak{F}g) \Big)(u)
\end{aligned}$$

A similar proof gives (4.2.6.4). Note that expressions exactly analogous to (4.2.6.3) and (4.2.6.4) hold with $\mathfrak{F}$ replaced by either F or $\breve{F}$. □

Proposition 4.2.7. *The Moyal product $\star : S \times S \to S$ is an associative product.*

Proof. Let $f, g, h \in S$. We begin by showing that $\diamond$ is associative:

$$\begin{aligned}
((f \diamond g) \diamond h)(u) &= \int (f \diamond g)(u-t)\, h(t)\, e^{-iu\cdot\Omega t} dt \\
&= \int\int f(u-t-s)\, g(s)\, e^{-i(u-t)\cdot\Omega s} ds\, h(t)\, e^{-iu\cdot\Omega t} dt \\
&= \int\int f(u-t-s)\, g(s)\, h(t)\, e^{-i(u\cdot\Omega t+(u-t)\cdot\Omega s)} dsdt \\
&= \int\int f(u-v)\, g(v-t)\, h(t)\, e^{-i(u\cdot\Omega v - t\cdot\Omega v)} dvdt \\
&= \int f(u-v) \left(\int g(v-t)\, h(t)\, e^{-iv\cdot\Omega t} dt \right) e^{-iu\cdot\Omega v} dv \\
&= \int f(u-v)\, (g \diamond h)(v)\, e^{-iu\cdot\Omega v} dv \\
&= (f \diamond (g \diamond h))(u)
\end{aligned}$$

Now, by applying (4.2.6.3) multiple times and using the associativity of $\diamond$ above, we find

$$\begin{aligned}
\mathfrak{F}((f \star g) \star h) &= \mathfrak{F}(f \star g) \diamond \mathfrak{F}h \\
&= (\mathfrak{F}f \diamond \mathfrak{F}g) \diamond \mathfrak{F}h \\
&= \mathfrak{F}f \diamond (\mathfrak{F}g \diamond \mathfrak{F}h) \\
&= \mathfrak{F}f \diamond \mathfrak{F}(g \star h) \\
&= \mathfrak{F}(f \star (g \star h))
\end{aligned}$$

Hence, $(f \star g) \star h = f \star (g \star h)$; the Moyal product is associative on S. □

Proposition 4.2.8. *Complex conjugation of functions $f \mapsto f^*$ is an involution for the Moyal product.*

Proof. Using definition (4.2.10), we find that

$$\begin{aligned}
(f \star g)^*(u) &= \left(\int\int f(u+s)\, g(u+t)\, e^{is\cdot\Omega t} dsdt \right)^* \\
&= \int\int \overline{f(u+s) g(u+t)} e^{-is\cdot\Omega t} dsdt \\
&= \int\int g^*(u+t)\, f^*(u+s)\, e^{it\cdot\Omega s} dtds \\
&= (g^* \star f^*)(u)
\end{aligned}$$

for all $f, g \in S$. Furthermore, we have that $(f^*)^* = f$ and that $(\alpha f + \beta g)^* = \overline{\alpha} f^* + \overline{\beta} g^*$ for all $\alpha, \beta \in \mathbb{C}$. Thus, complex conjugation defines an involution for the Moyal product. □

Proposition 4.2.9. *The Moyal product* $\star : S \times S \to S$ *is a jointly continuous bilinear map.*

Proof. The natural topology of S (Schwartz topology) is equivalently generated by the two families of seminorms $\{p_{\alpha\gamma}\}_{\alpha,\gamma\in\mathbb{N}_0^2}$ and $\{q_{\alpha\gamma}\}_{\alpha,\gamma\in\mathbb{N}_0^2}$, where

$$p_{\alpha\gamma}(f) := \|\mu^{\alpha}\partial^{\gamma} f\|_{\infty} = \sup_{x\in\mathbb{R}^2} |x^{\alpha}\partial^{\gamma} f(x)| \tag{4.2.9.1}$$

and

$$q_{\alpha\gamma}(f) := \|\mu^{\alpha}\partial^{\gamma} f\|_{1} = \int |x^{\alpha}\partial^{\gamma} f(x)|\, dx \tag{4.2.9.2}$$

In other words, each $p_{\alpha\gamma}(f)$ is continuous in the topology generated by $\{q_{\alpha\gamma}\}_{\alpha,\gamma\in\mathbb{N}_0^2}$ and each $q_{\alpha\gamma}(f)$ is continuous in the topology generated by $\{p_{\alpha\gamma}\}_{\alpha,\gamma\in\mathbb{N}_0^2}$. See, for example, [34, Chapter V].

Let $\eta = (\eta_1, \eta_2) \in \mathbb{N}_0^2$ such that $\eta_1 = \beta_2$ and $\eta_2 = \beta_1$. Then, using the inequality in (4.2.5.7), it follows that for all $f, g \in S$ and $\alpha, \gamma \in \mathbb{N}_0^2$

$$\begin{aligned}
p_{\alpha\gamma}(f \star g) &= \|\mu^{\alpha}\partial^{\gamma}(f\star g)\|_{\infty} \\
&\le \sum_{\beta\le\alpha}\sum_{\epsilon\le\gamma}\binom{\alpha}{\beta}\binom{\gamma}{\epsilon}\left\|\mu^{\alpha-\beta}\partial^{\gamma-\epsilon}f\right\|_1 \left\|\hat{\partial}^{\beta}\partial^{\epsilon}g\right\|_1 \\
&= \sum_{\beta\le\alpha}\sum_{\epsilon\le\gamma}\binom{\alpha}{\beta}\binom{\gamma}{\epsilon}\left\|\mu^{\alpha-\beta}\partial^{\gamma-\epsilon}f\right\|_1 \left\|\partial_1^{\beta_2+\epsilon_1}\partial_2^{\beta_1+\epsilon_2}g\right\|_1 \\
&= \sum_{\beta\le\alpha}\sum_{\epsilon\le\gamma}\binom{\alpha}{\beta}\binom{\gamma}{\epsilon}\left\|\mu^{\alpha-\beta}\partial^{\gamma-\epsilon}f\right\|_1 \left\|\partial^{\eta+\epsilon}g\right\|_1 \\
&= \sum_{\beta\le\alpha}\sum_{\epsilon\le\gamma}\binom{\alpha}{\beta}\binom{\gamma}{\epsilon} q_{\alpha-\beta,\gamma-\epsilon}(f)\, q_{0,\eta+\epsilon}(g)
\end{aligned}$$

By [34, Theorem V.2], $\star : S \times S \to S$ is a jointly continuous bilinear map. Actually, this theorem demonstrates that the maps $\star : S \to S : f \mapsto f \star g$ and $\star : S \to S : g \mapsto f \star g$ are continuous for all $f \in S$ and $g \in S$ separately. However, in the world of Fréchet spaces, separate continuity and joint continuity are equivalent [29, Theorem 1.29]. □

Note that the previous two propositions also hold for $\diamond$; that is,

$$(f \diamond g)^* = g^* \diamond f^*$$

for all $f, g \in S$ and $\diamond : S \times S \to S$ is a jointly continuous bilinear map.

Proposition 4.2.10. *The integral of the Moyal product has the tracial property: For all* $f, g \in S$

$$\int (f\star g)(u)\, du = \int (g \star f)(u)\, du = \int f(u)\, g(u)\, du = \langle f, g\rangle$$

Proof. Recall that the ordinary convolution $*$ between functions in S (without the twist induced by the symplectic form) is defined, for example in [34, Section IX.1], by

$$(f * g)(u) := \int f(u-t)\, g(t)\, dt$$

for all $f, g \in S$, where convolution in the u-domain corresponds to multiplication in the t-domain. Moreover, the convolution satisfies the following properties as per [34, Theorem IX.3]: For all $f, g \in S$

$$f * g = g * f \qquad \text{and} \qquad \mathfrak{F}(fg) = \mathfrak{F}f * \mathfrak{F}g$$

where $\mathfrak{F}$ is the Fourier transform defined in (4.2.15).

Let $f, g \in S$. Using the properties above, together with definition (4.2.11) and the relation (4.2.6.3), we find that

$$\begin{aligned}
\int (f \star g)(u)\, du &= (\mathfrak{F}(f \star g))(0) \\
&= (\mathfrak{F}f \diamond \mathfrak{F}g)(0) \\
&= \int (\mathfrak{F}f)(-t)\, (\mathfrak{F}g)(t)\, dt \\
&= (\mathfrak{F}f * \mathfrak{F}g)(0) \\
&= (\mathfrak{F}g * \mathfrak{F}f)(0) \\
&= \int (\mathfrak{F}g)(-t)\, (\mathfrak{F}f)(t)\, dt \\
&= (\mathfrak{F}g \diamond \mathfrak{F}f)(0) \\
&= (\mathfrak{F}(g \star f))(0) \\
&= \int (g \star f)(u)\, du
\end{aligned}$$

and

$$\begin{aligned}
\int (f \star g)(u)\, du &= (\mathfrak{F}f * \mathfrak{F}g)(0) \\
&= \mathfrak{F}(fg)(0) \\
&= \int f(u)\, g(u)\, du
\end{aligned}$$

□

The cyclicity in the tracial identity in Proposition 4.2.10 will allow us to extend the Moyal product via duality to larger spaces than S in the next section. Before we do so, there is an important consequence of this identity that will become useful when performing the extensions.

Proposition 4.2.11. *If $f, g, h \in S$, then*

$$\langle f \star g, h\rangle = \langle f, g \star h\rangle = \langle g, h \star f\rangle \tag{4.2.11.1}$$

$$\langle f \diamond g, h\rangle = \langle f, \breve{g} \diamond h\rangle = \left\langle g, h \diamond \breve{f}\right\rangle \tag{4.2.11.2}$$

$$(h|f \star g) = (f^* \star h|g) = (h \star g^*|f) \tag{4.2.11.3}$$

Proof. Let $f, g, h \in S$. Applying Proposition 4.2.7 and Proposition 4.2.10 to $\langle \cdot, \cdot\rangle$ gives

$$\begin{aligned}
\langle f \star g, h\rangle &= \int (f \star g)(u)\, h(u)\, du \\
&= \int \Big((f \star g) \star h\Big)(u)\, du \\
&= \int \Big(f \star (g \star h)\Big)(u)\, du \\
&= \int f(u)\,(g \star h)(u)\, du \\
&= \langle f, g \star h\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle f, g \star h\rangle &= \int \Big(f \star (g \star h)\Big)(u)\, du \\
&= \int \Big((g \star h) \star f\Big)(u)\, du \\
&= \int \Big(g \star (h \star f)\Big)(u)\, du \\
&= \int g(u)\,(h \star f)(u)\, du \\
&= \langle g, h \star f\rangle
\end{aligned}$$

This proves (4.2.11.1), which can be used, together with Proposition 4.2.6 and properties (2) and (4) from Lemma 4.2.2, to find (4.2.11.2) as follows:

$$\begin{aligned}
\langle f \diamond g, h\rangle &= \left\langle f \star \breve{F} g, h\right\rangle \\
&= \left\langle f, \left(\breve{F} g\right) \star h\right\rangle \\
&= \left\langle f, \left(F\left(\breve{F} g\right)\right) \diamond h\right\rangle \\
&= \langle f, (F^2 \breve{g}) \diamond h\rangle \\
&= \langle f, \breve{g} \diamond h\rangle
\end{aligned}$$

The second equality in (4.2.11.2) is found similarly. Finally, the first equality in (4.2.11.3) follows from (4.2.11.1), Proposition 4.2.8, and definition (4.2.8):

$$\begin{aligned}(h|f \star g) &= 1/2 \langle h^*, f \star g \rangle \\ &= 1/2 \langle h^* \star f, g \rangle \\ &= 1/2 \langle (f^* \star h)^*, g \rangle \\ &= (f^* \star h|g)\end{aligned}$$

The second equality in (4.2.11.3) follows similarly. □

Armed with sufficient properties of the Moyal $\star$-product between Schwartz functions, we are ready to define and characterise the algebra of the Moyal plane: Let

$$A := (S, \star) \tag{4.2.18}$$

be the algebra obtained by equipping the Schwartz space S with the Moyal $\star$-product and an involution defined by complex conjugation. We call A the algebra of the Moyal plane. The following corollary assembles the results from Theorem 4.1.9 and Propositions 4.2.5, 4.2.7 and 4.2.8.

Corollary 4.2.12. *A is a noncommutative, non-unital, associative, involutive Fréchet algebra.*

Chapter 5

Extension of $\star$ via duality

In the previous chapter, we established a calculus for functions in the Schwartz space S with Moyal product $\star$. Here, we extend the Moyal product to a larger space of tempered distributions via duality. The dual space of tempered distributions is defined in Section 5.1 and shown to contain S. In Section 5.2, we systematically extend the Moyal product to certain subspaces of the dual space. In particular, the extension reaches a space that contains all polynomials. This will enable us to define an orthonormal basis for $L^2\left(\mathbb{R}^2\right)$ in Chapter 6, where the basis elements will be composed of Moyal products between functions in S and certain polynomials.

5.1 The Schwartz space S is contained in its dual

Here we define the dual space of S and show that it contains S. The following definition draws from [34, Section V.1] and [34, Section V.3].

Definition 5.1.1. *The **topological dual space** of the topological vector space X is the family of continuous linear functionals on X, and is denoted X'. The topological dual space of $S\left(\mathbb{R}^2\right)$, denoted by $S'\left(\mathbb{R}^2\right)$, is called the **space of tempered distributions**.*

Henceforth, we let

$$S' := S'\left(\mathbb{R}^2\right) \tag{5.1}$$

Note that S' is topologised by the strong dual topology, that of uniform convergence on bounded subsets of S. Formally, the topology of S' is generated by the family of seminorms $\{\rho_B : B \subset S \quad \text{is bounded}\}$, where

$$\rho_B\left(T\right) := \sup_{f \in B} \left|T\left(f\right)\right| \tag{5.2}$$

for all $T \in S'$.

For a linear functional T on S to be in S', it must be continuous. By [34, Theorem V.2], this is equivalent to the existence of a seminorm $\|\cdot\|_{\alpha,\beta}$ such that

$$|T(f)| \leq C \|f\|_{\alpha,\beta}$$

for all $f \in S$ and some constant C.

The following notation emphasises the action of S' on S, i.e. the action of a continuous linear functional on a Schwartz function: For $T \in S'$ and $h \in S$, we write

$$\langle T, h \rangle := T(h) \tag{5.3}$$

If $f \in S$, then we define a linear functional $f(\cdot)$ on S by

$$f(h) = \langle f, h \rangle = \int f(x) h(x) \, dx$$

for all $h \in S$. Note that for $f, h \in S$, (5.3) gives $\langle f(\cdot), h \rangle = \langle f, h \rangle$. Hence, (5.3) is indeed an extension of $\langle \cdot, \cdot \rangle$ from $S \times S$ to $S' \times S$. The following proposition shows that S is a subspace of S':

Proposition 5.1.2. *If $f \in S$, then $f(\cdot) \in S'$. Furthermore, if $f_1, f_2 \in S$ such that $f_1(\cdot) = f_2(\cdot)$, then $f_1 = f_2$.*

Proof. Let $f \in S$. Using Hölder's inequality, we find that

$$\begin{aligned} |f(g)| &= \left| \int f(x) g(x) \, dx \right| \\ &\leq \int |f(x) g(x)| \, dx \\ &= \|fg\|_1 \\ &\leq \|f\|_1 \|g\|_\infty \\ &\leq C \|g\|_{0,0} \end{aligned}$$

for all $g \in S$ and for some constant C. Thus $f(\cdot)$ is a continuous linear functional on S, i.e. $f(\cdot) \in S'$.

Let $f_1, f_2 \in S$ such that $f_1 \neq f_2$, then $f_1 \neq f_2$ in $L^2(\mathbb{R}^2)$ since S is dense in $L^2(\mathbb{R}^2)$ by Proposition 4.2.1. This implies that $f_1(\cdot) \neq f_2(\cdot)$ in $(L^2(\mathbb{R}^2))'$, since the dual space $(L^2(\mathbb{R}^2))'$ is exactly $L^2(\mathbb{R}^2)$. Finally, since $(L^2(\mathbb{R}^2))'$ is dense in S', we conclude that $f_1(\cdot) \neq f_2(\cdot)$ in S'. This statement is equivalent to the proposed result by contraposition. □

Owing to Proposition 5.1.2, it is no longer necessary to distinguish between f and $f(\cdot)$ when $f \in S$. Henceforth, we write $f(\cdot)$ simply as f when $f \in S$.

5.2 Extending the Moyal product

Before we extend the Moyal product to spaces of tempered distributions, we extend the notions of partial differentiation, pointwise multiplication by a coordinate, and involution to S': For $T \in S'$ and $h \in S$, we define

$$\langle \partial_j T, h \rangle := -\langle T, \partial_j h \rangle \tag{5.4}$$

$$\langle \mu_j T, h \rangle := \langle T, \mu_j h \rangle \tag{5.5}$$

$$(T|h) := \frac{1}{2}\langle T^*, h \rangle := \frac{1}{2}\langle T, h^* \rangle^* \tag{5.6}$$

Furthermore, we extend the flip operator and the Fourier transforms to S': For $T \in S'$ and $h \in S$, we define

$$\left\langle \breve{T}, h \right\rangle := \left\langle T, \breve{h} \right\rangle \tag{5.7}$$

$$\langle \mathfrak{F} T, h \rangle := \langle T, \mathfrak{F} h \rangle \tag{5.8}$$

$$\langle FT, h \rangle := \left\langle T, \breve{F} h \right\rangle \tag{5.9}$$

$$\left\langle \breve{F} T, h \right\rangle := \langle T, Fh \rangle \tag{5.10}$$

Proposition 5.2.1. *The operations defined by* (5.4) *to* (5.10) *extend the corresponding operations on* S*.*

Proof. Let $T \in S'$. First consider (5.4). Note that $\partial_j T$ defines a linear functional on S, since for all $h \in S$

$$\begin{aligned}
(\partial_j T)(h) &= \langle \partial_j T, h \rangle \\
&= -\langle T, \partial_j h \rangle \\
&= -T(\partial_j h)
\end{aligned}$$

where $\partial_j h \in S$ and T is a linear functional on S. Seeing as both ∂_j and T are continuous, the composition $\partial_j T = -T \circ \partial_j$ is continuous. Hence, $\partial_j T$ is a continuous linear functional, i.e. $\partial_j T \in S'$. Now observe that

$$\begin{aligned}
\langle \partial_j f, h \rangle &= \int (\partial_j f)(x)\, h(x)\, dx \\
&= -\int f(x)\, (\partial_j h)(x)\, dx \\
&= -\langle f, \partial_j h \rangle
\end{aligned}$$

for all $f, h \in S$, which shows, on account of Proposition 5.1.2, that $\partial_j : S' \to S'$ is indeed an extension of $\partial_j : S \to S$. Similar proofs show that (5.5), (5.6) and (5.7) extend the corresponding operations on S. By following the same pattern as the proof above, and applying properties (2), (7) and (9) from Lemma 4.2.2, it follows that (5.8), (5.9) and (5.10) are extensions of the Fourier transforms on S. □

Now we start our extension of the Moyal product. First, we define the Moyal product between one element in S' and one element in S. For $T \in S'$ and $f, h \in S$, let $T \star f$ be defined by

$$\langle T \star f, h \rangle := \langle T, f \star h \rangle \tag{5.11}$$

and let $f \star T$ be defined by

$$\langle f \star T, h \rangle := \langle T, h \star f \rangle \tag{5.12}$$

The twisted convolution is extended similarly: For $T \in S'$ and $f, h \in S$, let $T \diamond f$ be defined by

$$\langle T \diamond f, h \rangle := \left\langle T, \breve{f} \diamond h \right\rangle \tag{5.13}$$

and let $f \diamond T$ be defined by

$$\langle f \diamond T, h \rangle := \left\langle T, h \diamond \breve{f} \right\rangle \tag{5.14}$$

Proposition 5.2.2. *The products $T \star f$ and $f \star T$ map into S' and extend the corresponding operations on $S \times S$ to $S' \times S$ and $S \times S'$ respectively. Furthermore, the convolutions $T \diamond f$ and $f \diamond T$ map into S' and extend the corresponding operations on $S \times S$ to $S' \times S$ and $S \times S'$ respectively.*

Proof. Let $T \in S'$ and $f, h \in S$ be arbitrary. Note that

$$\begin{aligned} (T \star f)(h) &= \langle T \star f, h \rangle \\ &= \langle T, f \star h \rangle \\ &= T(f \star h) \end{aligned}$$

Since $f \star h \in S$ and $\star$ is continuous in S by Proposition 4.2.9, it follows that $T \star f$ is a continuous linear functional on S and thus $T \star f \in S'$. Similarly, $f \star T \in S'$. From (4.2.11.1) in Proposition 4.2.11 we have that

$$\langle g \star f, h \rangle = \langle g, f \star h \rangle$$

for all $f, g, h \in S$, which shows, on account of Proposition 5.1.2, that $\star : S' \times S \to S'$ extends the corresponding operation on S, that is $\star : S \times S \to S$. Similarly,

$$\langle f \star g, h \rangle = \langle g, h \star f \rangle$$

shows that $f \star T : S \times S' \to S'$ is indeed an extension of the corresponding operation on S. Analogous proofs, with reference to (4.2.11.2) from Proposition 4.2.11, show that $T \diamond f$ and $f \diamond T$ are extensions of the corresponding operations on S. $\square$

Many of the formulas in Chapter 4 involving $f, g \in S$ extend to analogous ones for $T \in S'$ and $f \in S$. The following proposition groups a few such extended formulas for later use.

Proposition 5.2.3. *If $T \in S'$ and $f \in S$, then*

$$\partial_j (T \star f) = (\partial_j T) \star f + T \star (\partial_j f) \tag{5.2.3.1}$$

$$\partial_j (f \star T) = (\partial_j f) \star T + f \star (\partial_j T) \tag{5.2.3.2}$$

$$\mu_j (T \star f) = (\mu_j T) \star f - iT \star \left(\hat{\partial}_j f\right) \tag{5.2.3.3}$$

$$\mu_j (f \star T) = f \star (\mu_j T) + i \left(\hat{\partial}_j f\right) \star T \tag{5.2.3.4}$$

$$(T \star f)^* = f^* \star T^* \tag{5.2.3.5}$$

$$(f \star T)^* = T^* \star f^* \tag{5.2.3.6}$$

$$(T^*)^* = T \tag{5.2.3.7}$$

Proof. Let $T \in S'$ and $f \in S$ be arbitrary. Throughout this proof, we use the fact that $T \star f, f \star T \in S'$ from Proposition 5.2.2; this allows us to use the extensions that we have defined thus far. In particular, we use the extensions defined in (5.11) and (5.12) without further mention.

We derive (5.2.3.1) and (5.2.3.2) by means of definition (5.4) and Proposition 4.2.3:

$$\begin{aligned}
\langle \partial_j (T \star f), h \rangle &= - \langle T \star f, \partial_j h \rangle \\
&= - \langle T, f \star (\partial_j h) \rangle \\
&= - \langle T, \partial_j (f \star h) - (\partial_j f) \star h \rangle \\
&= \langle \partial_j T, f \star h \rangle + \langle T, (\partial_j f) \star h \rangle \\
&= \langle (\partial_j T) \star f, h \rangle + \langle T \star (\partial_j f), h \rangle \\
&= \langle (\partial_j T) \star f + T \star (\partial_j f), h \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle \partial_j (f \star T), h \rangle &= - \langle f \star T, \partial_j h \rangle \\
&= - \langle T, (\partial_j h) \star f \rangle \\
&= - \langle T, \partial_j (h \star f) - h \star (\partial_j f) \rangle \\
&= \langle \partial_j T, h \star f \rangle + \langle T, h \star (\partial_j f) \rangle \\
&= \langle (\partial_j f) \star T + f \star (\partial_j T), h \rangle
\end{aligned}$$

for all $h \in S$, which proves (5.2.3.1) and (5.2.3.2). We find (5.2.3.3) and (5.2.3.4) via definition (5.5) and Proposition 4.2.4:

$$\begin{aligned}
\langle \mu_j (T \star f), h \rangle &= \langle T \star f, \mu_j h \rangle \\
&= \langle T, f \star (\mu_j h) \rangle \\
&= \left\langle T, \mu_j (f \star h) - i \left(\hat{\partial}_j f\right) \star h \right\rangle \\
&= \langle \mu_j T, f \star h \rangle - \left\langle T, i \left(\hat{\partial}_j f\right) \star h \right\rangle \\
&= \left\langle (\mu_j T) \star f - iT \star \left(\hat{\partial}_j f\right), h \right\rangle
\end{aligned}$$

and

$$\begin{aligned}\langle \mu_j (f \star T), h\rangle &= \langle f \star T, \mu_j h\rangle \\ &= \langle T, (\mu_j h) \star f\rangle \\ &= \left\langle T, \mu_j (h \star f) + ih \star \left(\hat{\partial}_j f\right)\right\rangle \\ &= \left\langle f \star (\mu_j T) + i \left(\hat{\partial}_j f\right) \star T, h\right\rangle\end{aligned}$$

for all $h \in S$, which proves (5.2.3.3) and (5.2.3.3). Finally, using definition (5.6) and Proposition 4.2.8, we find that

$$\begin{aligned}\langle (T \star f)^*, h\rangle &= \langle T \star f, h^*\rangle^* \\ &= \langle T, f \star h^*\rangle^* \\ &= \langle T, (h \star f^*)^*\rangle^* \\ &= \langle T^*, h \star f^*\rangle \\ &= \langle f^* \star T^*, h\rangle\end{aligned}$$

and

$$\begin{aligned}\langle (f \star T)^*, h\rangle &= \langle f \star T, h^*\rangle^* \\ &= \langle T, h^* \star f\rangle^* \\ &= \langle T, (f^* \star h)^*\rangle^* \\ &= \langle T^*, f^* \star h\rangle \\ &= \langle T^* \star f^*, h\rangle\end{aligned}$$

and

$$\begin{aligned}\langle (T^*)^*, h\rangle &= \langle T^*, h^*\rangle^* \\ &= \langle T, (h^*)^*\rangle \\ &= \langle T, h\rangle\end{aligned}$$

for all $h \in S$, which proves (5.2.3.5), (5.2.3.6) and (5.2.3.7). □

In order to interpret the Moyal product between a polynomial and a function in S, we require an element in S' to act as identity for the Moyal product. Let $\mathbf{1}$ denote the constant function on $\mathbb{R}^2$ with value 1, i.e. for all $x \in \mathbb{R}^2$

$$\mathbf{1}(x) = 1 \tag{5.15}$$

Proposition 5.2.4. $\mathbf{1}$ *is the identity for the* $\star$*-product i.e. for all* $f \in S$

$$\mathbf{1} \star f = f \star \mathbf{1} = f$$

Proof. First, note that $\imath$ defines a linear functional on S, since for all $f \in S$

$$\imath(f) = \langle \imath, f \rangle = \int \imath(x) f(x)\, dx = \int f(x)\, dx$$

and $S \subset L^1(\mathbb{R}^2)$, as discussed in Section 4.2. Also,

$$\begin{aligned} |\imath(f)| &\leq \int |f(x)|\, dx \\ &= \|f\|_1 \\ &= q_{00}(f) \end{aligned}$$

for all $f \in S$, where $\{q_{\alpha\gamma}\}_{\alpha,\gamma \in \mathbb{N}_0^2}$ is the family of seminorms defined in (4.2.9.2). Hence, $\imath$ is a continuous linear functional on S, i.e. $\imath \in S'$.

By Proposition 5.2.2, $\imath \star f, f \star \imath \in S'$. Using definitions (5.11) and (5.12), together with the cyclicity in Proposition 4.2.10, we see that for all $f, h \in S$

$$\begin{aligned} \langle \imath \star f, h \rangle &= \langle \imath, f \star h \rangle \\ &= \int (f \star h)(u)\, du \\ &= \langle f, h \rangle \\ &= \int (h \star f)(u)\, du \\ &= \langle \imath, h \star f \rangle \\ &= \langle f \star \imath, h \rangle \end{aligned}$$

Thus, by (5.3) and Proposition 5.1.2, it holds for all $f \in S$ that

$$\imath \star f = f = f \star \imath \tag{5.2.4.1}$$

□

The key fact regarding this identity element is that

$$(\mu_j \imath)(x) = x_j \tag{5.16}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. When performing manipulations with x_j, one should be able to distinguish between x_j performing as a function and x_j performing as a coordinate. However, for the sake of clarity, we define the functions

$$X_j : \mathbb{R}^2 \to \mathbb{R} : (x_1, x_2) \mapsto x_j \tag{5.17}$$

for $j = 1, 2$.

Proposition 5.2.5. *$X_j \in S'$ for $j = 1, 2$. Furthermore, for all $f \in S$,*

$$X_j \star f = \mu_j f + i\hat{\partial}_j f \quad \text{and} \quad f \star X_j = \mu_j f - i\hat{\partial}_j f$$

Proof. First, note that X_j defines a linear functional on S, since for all $f \in S$

$$X_j(f) = \langle X_j, f \rangle = \int X_j(x) f(x)\, dx = \int x_j f(x)\, dx$$

and, as discussed in Section 4.2, $(x_1, x_2) \mapsto x_j f(x)$ is in $L^1(\mathbb{R}^2)$. Consider the multi-index $\eta = (\eta_1, \eta_2) = (2-j, j-1) \in \mathbb{N}_0^2$. Then

$$\begin{aligned} |X_j(f)| &\leq \int |x_j f(x)|\, dx \\ &= \|\mu^\eta f\|_1 \\ &= q_{\eta,0}(f) \end{aligned}$$

for all $f \in S$. Hence, X_j is a continuous linear functional on S, i.e. $X_j \in S'$.

By Proposition 5.2.2, $X_j \star f \in S'$ and $f \star X_j \in S'$. Using (5.16), definition (5.11), and Propositions 4.2.4, 4.2.10 and 5.2.4, we see that for all $f, h \in S$

$$\begin{aligned} \langle X_j \star f, h \rangle &= \langle X_j, f \star h \rangle \\ &= \int x_j (f \star h)(x)\, dx \\ &= \int (\mu_j \mathtt{1})(x) (f \star h)(x)\, dx \\ &= \int ((\mu_j \mathtt{1}) \star f)(x) h(x)\, dx \\ &= \langle (\mu_j \mathtt{1}) \star f, h \rangle \\ &= \left\langle \mu_j (\mathtt{1} \star f) + i \mathtt{1} \star \hat{\partial}_j f, h \right\rangle \\ &= \left\langle \mu_j f + i \hat{\partial}_j f, h \right\rangle \end{aligned}$$

Thus, by (5.3) and Proposition 5.1.2, it holds for all $f \in S$ that

$$X_j \star f = \mu_j f + i \hat{\partial}_j f$$

Similarly, using definition (5.12), we find

$$f \star X_j = \mu_j f - i \hat{\partial}_j f$$

$\square$

The following corollary states the same result as above, only in terms of explicit coordinates rather than functions.

Corollary 5.2.6. *For all $f \in S$ and $x = (x_1, x_2) \in \mathbb{R}^2$:*

(1) $x_1 \star f = (x_1 + i\partial_2) f$
(2) $x_2 \star f = (x_2 - i\partial_1) f$
(3) $f \star x_1 = (x_1 - i\partial_2) f$
(4) $f \star x_2 = (x_2 + i\partial_1) f$

In order to further extend the Moyal product, we now define, as per [18], the algebra $\mathfrak{M}$ as the intersection of two subsets of S', one comprising the left multipliers in S and the other the right multipliers in S.

$$\mathfrak{M}_L := \left\{L \in S' : L \star f \in S \quad \forall f \in S\right\} \tag{5.18}$$

$$\mathfrak{M}_R := \left\{R \in S' : f \star R \in S \quad \forall f \in S\right\} \tag{5.19}$$

$$\mathfrak{M} := \mathfrak{M}_L \cap \mathfrak{M}_R \tag{5.20}$$

It is clear from Propositions 4.2.5, 5.1.2 and 5.2.4 that $S \subset \mathfrak{M}$ and $\mathbf{1} \in \mathfrak{M}$.

Proposition 5.2.7. *$\mathfrak{M}$ is closed under partial differentiation and pointwise multiplication by a coordinate. In particular, all polynomials lie in $\mathfrak{M}$.*

Proof. Let $M \in \mathfrak{M}$ and $f \in S$ be arbitrary. By definition (5.20), $M \in \mathfrak{M}_L$ and $M \in \mathfrak{M}_R$. Moreover, $M \in S'$ because $\mathfrak{M} \subset S'$. Therefore, the formulas in Proposition 5.2.3 hold for M. In particular, from (5.2.3.1), it holds that

$$(\partial_j M) \star f = \partial_j (M \star f) - M \star (\partial_j f)$$

Since $M \in \mathfrak{M}_L$, we know from definition (5.18) that $M \star f \in S$ for all $f \in S$. Furthermore, we know that ∂_j maps S into S. Accordingly, with reference to the equation above, both terms on the right are in S; thus $(\partial_j M) \star f \in S$. This holds for arbitrary $f \in S$; therefore, we may deduce, from definition (5.18), that

$$\partial_j M \in \mathfrak{M}_L \tag{5.2.7.1}$$

Once again appealing to Proposition 5.2.3, specifically to (5.2.3.2), we have

$$f \star (\partial_j M) = \partial_j (f \star M) - (\partial_j f) \star M$$

Since $M \in \mathfrak{M}_R$, definition (5.19) states that $f \star M \in S$ for all $f \in S$. As before, both terms on the right are in S. Therefore, $f \star (\partial_j M) \in S$, which in turn implies that

$$\partial_j M \in \mathfrak{M}_R \tag{5.2.7.2}$$

as per definition (5.19). Together, (5.2.7.1) and (5.2.7.2) satisfy definition (5.20); we conclude that

$$\partial_j M \in \mathfrak{M}$$

This holds for arbitrary $M \in \mathfrak{M}$; hence, $\mathfrak{M}$ is closed under partial differentiation.

We proceed to the second part of the proof by invoking formulas (5.2.3.3) and (5.2.3.4), which give

$$(\mu_j M) \star f = \mu_j (M \star f) + iM \star \left(\hat{\partial}_j f\right)$$

and

$$f \star (\mu_j M) = \mu_j (f \star M) - i \left(\hat{\partial}_j f\right) \star M$$

respectively. Employing the same reasoning as before, we infer that $\mu_j M \in \mathfrak{M}_L$ and $\mu_j M \in \mathfrak{M}_R$. Definition (5.20) implies then that

$$\mu_j M \in \mathfrak{M}$$

Since this holds for arbitrary $M \in \mathfrak{M}$, we conclude that $\mathfrak{M}$ is closed under pointwise multiplication by a coordinate. We apply this result to the specific case $\mathbb{1} \in \mathfrak{M}$. It follows from (5.16) that $(\mu_j \mathbb{1})(x) = x_j \in \mathfrak{M}$. When considering x_j as a function via (5.17), i.e. $X_j(x) = x_j$, it becomes clear that repeated pointwise multiplication by a coordinate generates higher-dimensional polynomials that remain in $\mathfrak{M}$. Hence, we may induce that all polynomials lie in $\mathfrak{M}$. □

We now extend the Moyal product to the case where one element is in $\mathfrak{M}$ and one is in S'. Since $\mathfrak{M} \subset S'$, this extension includes products between elements in $\mathfrak{M}$. This will allow us, in particular, to handle the Moyal product between polynomials. For $R \in \mathfrak{M}_R$, $L \in \mathfrak{M}_L$, and $T \in S'$, define $T \star L$ by

$$\langle T \star L, h \rangle := \langle T, L \star h \rangle \tag{5.21}$$

and define $R \star T$ by

$$\langle R \star T, h \rangle := \langle T, h \star R \rangle \tag{5.22}$$

for all $h \in S$.

Proposition 5.2.8. *The products $T \star L$ and $R \star T$ map into S' and extend the corresponding operations on $S' \times S$ and $S \times S'$ to $S' \times \mathfrak{M}_L$ and $\mathfrak{M}_R \times S'$ respectively. In particular, they define the Moyal product between elements in $\mathfrak{M}$. Moreover, $\mathfrak{M}$ becomes an associative $*$-algebra under the Moyal product.*

Proof. Let $R \in \mathfrak{M}_R$, $L \in \mathfrak{M}_L$, and $T \in S'$ be arbitrary. Then for all $h \in S$

$$\begin{aligned}(T \star L)(h) &= \langle T \star L, h \rangle \\ &= \langle T, L \star h \rangle \\ &= T(L \star h)\end{aligned}$$

Since $L \star h \in S$ by definition (5.18), and T is a continuous linear functional on S, it follows that $T \star L$ is a continuous linear functional on S and thus $T \star L \in S'$. Similarly, $R \star T \in S'$. By comparing definitions (5.21) and (5.22) to definitions (5.11) and (5.12) respectively, we see that the former are indeed extensions of the corresponding operations on $S' \times S$ and $S \times S'$ respectively.

Considering that $\mathfrak{M} \subset S'$, $\mathfrak{M} \subset \mathfrak{M}_L$ and $\mathfrak{M} \subset \mathfrak{M}_R$, definitions (5.21) and (5.22) define, in particular, the Moyal product between two elements in $\mathfrak{M}$. Explicitly, if $M, N \in \mathfrak{M}$, then

$$\langle M \star N, h \rangle := \langle M, N \star h \rangle \tag{5.2.8.1}$$

and

$$\langle N \star M, h \rangle := \langle M, h \star N \rangle \tag{5.2.8.2}$$

for all $h \in S$.

Now we prove that $\mathfrak{M}$ is an associative $*$-algebra when equipped with the Moyal product. Let $M, N, P \in \mathfrak{M}$ and $f, g, h \in S$ be arbitrary. First, using the associativity of $\star$ on S from Proposition 4.2.7, we note that

$$\begin{aligned} \langle (M \star f) \star g, h \rangle &= \langle M \star f, g \star h \rangle \\ &= \langle M, f \star (g \star h) \rangle \\ &= \langle M, (f \star g) \star h \rangle \\ &= \langle M \star (f \star g), h \rangle \end{aligned} \tag{5.2.8.3}$$

Then, from (5.2.8.1) and (5.2.8.3), we have

$$\begin{aligned} \langle (M \star N) \star g, h \rangle &= \langle M \star N, g \star h \rangle \\ &= \langle M, N \star (g \star h) \rangle \\ &= \langle M, (N \star g) \star h \rangle \\ &= \langle M \star (N \star g), h \rangle \end{aligned} \tag{5.2.8.4}$$

Since $N \star g \in S$, it follows from definition (5.20) that $M \star (N \star g) \in S$. Then (5.2.8.4) implies that $(M \star N) \star g \in S$, which gives

$$M \star N \in \mathfrak{M}_L \tag{5.2.8.5}$$

by definition (5.18). Similarly, we find that

$$\begin{aligned} \langle g \star (M \star N), h \rangle &= \langle M \star N, h \star g \rangle \\ &= \langle M, N \star (h \star g) \rangle \\ &= \langle M, (N \star h) \star g \rangle \\ &= \langle g \star M, N \star h \rangle \\ &= \langle (g \star M) \star N, h \rangle \end{aligned} \tag{5.2.8.6}$$

Since $g \star M \in S$, it follows from definition (5.20) that $(g \star M) \star N \in S$. Then (5.2.8.6) implies that $g \star (M \star N) \in S$, which gives

$$M \star N \in \mathfrak{M}_R \tag{5.2.8.7}$$

by definition (5.19). Together, (5.2.8.5) and (5.2.8.7) satisfy definition (5.20) so that

$$M \star N \in \mathfrak{M}$$

for all $M, N \in \mathfrak{M}$. Associativity follows from (5.2.8.1) and (5.2.8.4):

$$\begin{aligned}
\langle (M \star N) \star P, h \rangle &= \langle M \star N, P \star h \rangle \\
&= \langle M, N \star (P \star h) \rangle \\
&= \langle M, (N \star P) \star h \rangle \\
&= \langle M \star (N \star P), h \rangle
\end{aligned}$$

Regarding the involution, note that (5.2.3.7) from Proposition 5.2.3 holds for $M \in \mathfrak{M}$, because $\mathfrak{M} \subset S'$. Furthermore, using (5.2.3.6), together with definitions (5.6), (5.2.8.1) and (5.2.8.2), we see that

$$\begin{aligned}
\langle (M \star N)^*, h \rangle &= \langle M \star N, h^* \rangle^* \\
&= \langle M, N \star h^* \rangle^* \\
&= \langle M, (h \star N^*)^* \rangle^* \\
&= \langle M^*, h \star N^* \rangle \\
&= \langle N^* \star M^*, h \rangle
\end{aligned}$$

Hence, $\mathfrak{M}$ is an associative $*$-algebra under the Moyal product. □

Having sufficiently extended the Moyal product, we conclude this chapter by proving a result analogous to the tracial property in Proposition 4.2.10.

Proposition 5.2.9. *For all $M \in \mathfrak{M}$ and $f \in S$,*

$$\int (M \star f)(u)\, du = \int (f \star M)(u)\, du$$

Proof. Let $M \in \mathfrak{M}$ and $f \in S$ be arbitrary. By definition (5.20), $M \star f \in S$ and $f \star M \in S$; therefore, remembering that $S \subset L^1(\mathbb{R}^2)$, the proposed integrals make sense. Since $\mathbf{1} \in \mathfrak{M}$ and $M \in \mathfrak{M}$, we may use definitions (5.2.8.1) and (5.2.8.2), together with Proposition 5.2.4, to find

$$\begin{aligned}
\int (M \star f)(u)\, du &= \langle \mathbf{1}, M \star f \rangle \\
&= \langle \mathbf{1} \star M, f \rangle \\
&= \langle M, f \star \mathbf{1} \rangle \\
&= \langle M, \mathbf{1} \star f \rangle \\
&= \langle M \star \mathbf{1}, f \rangle \\
&= \langle \mathbf{1}, f \star M \rangle \\
&= \int (f \star M)(u)\, du
\end{aligned}$$

□

Chapter 6

A family of functions in S as a basis for $L^2\left(\mathbb{R}^2\right)$

In this chapter, we construct an orthonormal basis (total orthonormal set) for $L^2\left(\mathbb{R}^2\right)$ as a family of functions in S. This will enable us to expand functions in S in terms of this basis; these expansions will allow us to represent S as a sequence space in the next chapter, where a sequence will correspond to a function in S via the coefficients in its basis expansion. We define the family of functions and all its ingredients in Section 6.1. In Sections 6.2 and 6.3, we examine in detail the constituents of the family, and their interactions, on our way towards proving orthonormality. Section 6.4 introduces the orthogonal polynomials that are used to prove completeness in Section 6.5.

6.1 Defining the family of functions $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$

Let $z, \overline{z} \in \mathfrak{M}$ be defined by

$$z := \frac{1}{\sqrt{2}}\left(x_1 + ix_2\right) \quad \text{and} \quad \overline{z} := \frac{1}{\sqrt{2}}\left(x_1 - ix_2\right) \tag{6.1}$$

where $(x_1, x_2) \in \mathbb{R}^2$. Also, define corresponding derivatives

$$\partial := \frac{1}{\sqrt{2}}\left(\partial_1 - i\partial_2\right) \quad \text{and} \quad \overline{\partial} := \frac{1}{\sqrt{2}}\left(\partial_1 + i\partial_2\right) \tag{6.2}$$

Furthermore, let $G \in \mathfrak{M}$ be given by

$$G := z\overline{z} = \frac{1}{2}\left(x_1^2 + x_2^2\right) = \frac{1}{2}x \cdot x \tag{6.3}$$

and define a Gaussian function

$$f_{00} := 2e^{-G} \tag{6.4}$$

We introduce the following notation for integer $\star$-powers of z:

$$z^{\star n} := z \star z \star \cdots \star z \qquad \text{(n times)} \tag{6.5}$$

for $n \in \mathbb{N}_0$, where $z^{\star 0} = \mathbb{1}$. Finally, define the family of functions $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ such that

$$f_{mn} := \left(2^{m+n} m! n!\right)^{-\frac{1}{2}} \overline{z}^{\star m} \star f_{00} \star z^{\star n} \tag{6.6}$$

for all $m, n \in \mathbb{N}_0$.

Proposition 6.1.1. $\{f_{mn}\}_{m,n\in\mathbb{N}_0} \subset S \subset L^2\left(\mathbb{R}^2\right)$.

Proof. Since $z, \overline{z} \in \mathfrak{M}$, $z^{\star n} \in \mathfrak{M}$ and $\overline{z}^{\star m} \in \mathfrak{M}$ for all $m, n \in \mathbb{N}_0$ by Proposition 5.2.8. Furthermore, since f_{00} in (6.4) is defined in terms of an exponential function, it is infinitely differentiable and all of its derivates are rapidly decreasing in the Schwartz space sense. Hence, $f_{00} \in S$. It follows from definition (5.20) for $\mathfrak{M}$ that $\overline{z}^{\star m} \star f_{00} \star z^{\star n} \in S$ for all $m, n \in \mathbb{N}_0$. In (6.6), we have then that $f_{mn} \in S$ for all $m, n \in \mathbb{N}_0$ and thus $\{f_{mn}\}_{m,n\in\mathbb{N}_0} \subset S$. The $L^2\left(\mathbb{R}^2\right)$ inclusion was proved in Proposition 4.2.1. □

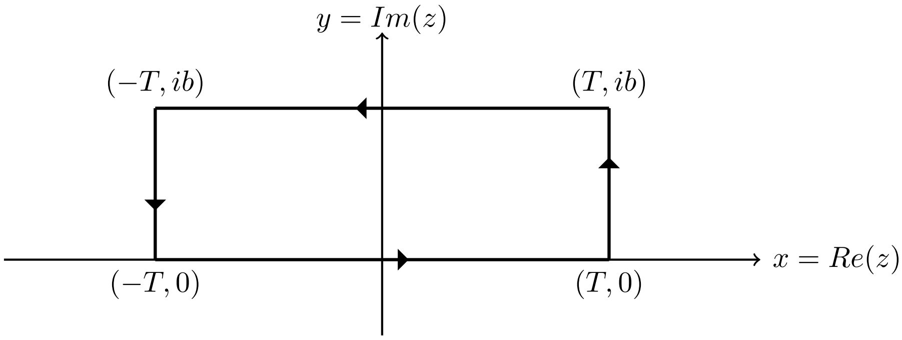
6.2 Properties of f_{00}

In order to prove orthonormality of the family $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$, we require enough control over the functions f_{mn} to handle their interactions with one another. It seems fitting to start our investigation with an analysis of their constituents. First, we consider f_{00} and work towards proving its idempotence. The upcoming theorem will help us to evaluate some recurring integrals along the way. Note that we continue to normalise as per Section 4.2.

Theorem 6.2.1. *Let t denote a variable in $\mathbb{R}$. Then for $c \in \mathbb{C}$ and $p > 0$,*

$$\int_{-\infty}^{\infty} e^{-p(t+c)^2} dt = \sqrt{\frac{1}{2p}}$$

Proof. For an arbitrary complex number $c = a + ib$, where $a, b \in \mathbb{R}$, let $T \in \mathbb{R}$ such that $T > |a|$. Let $\Gamma_c(T)$ denote the closed rectangular contour in the complex plane that starts at the point $(-T, 0)$ on the Real axis and has vertices $(T, 0)$, (T, ib), and $(-T, ib)$. Note that we associate the point (x, y) in the complex plane with the complex number $z = x + iy$.



Let

$$f(z) := e^{-pz^2}$$

where $p > 0$ is arbitrary. If we put $z = x + iy$ and split $f(z)$ into its real and imaginary parts, then it is a simple calculation to show that these two parts satisfy the Cauchy-Riemann equations. Moreover, both parts are continuously differentiable in the sense of real-valued functions; therefore, $f(z)$ is an entire function (analytic in $\mathbb{C}$). Specifically, $f(z)$ is analytic inside the region bounded by $\Gamma_c(T)$. By Cauchy's integral theorem, the contour integral of $f(z)$ along $\Gamma_c(T)$ is zero:

$$\oint\limits_{\Gamma_c(T)} f(z)\,dz = 0$$

We can brake this contour integral into four line integrals - each along one of the straight lines that form $\Gamma_c(T)$:

$$\oint\limits_{\Gamma_c(T)} f(z)\,dz = \int\limits_{-T}^{T} f(x)\,dx + \int\limits_{0}^{b} f(T+iy)\,(idy) + \int\limits_{T}^{-T} f(x+ib)\,dx + \int\limits_{b}^{0} f(-T+iy)\,(idy)$$
$$= 0$$

In the limit $T \to \infty$, since $e^{-p(t+c)^2} \to 0$ as $|t| \to \infty$, the second and fourth terms above tend to zero. Furthermore, it is well known that the first term tends to $\sqrt{\frac{1}{2p}}$ as $T \to \infty$ (it tends to $\sqrt{\frac{\pi}{p}}$ if one uses the ordinary Lebesgue measure). Since the total contour integral is zero, we are left with the third term as negative the first term. So, in the limit $T \to \infty$, after reversing the integration limits, we have

$$\int\limits_{-\infty}^{\infty} f(x+ib)\,dx = \sqrt{\frac{1}{2p}}$$

Substituting $x = t + a = t + c - ib$, we arrive at the required result

$$\int\limits_{-\infty}^{\infty} f(t+c)\,dt = \sqrt{\frac{1}{2p}}$$

□

Corollary 6.2.2. *Let u denote a variable in $\mathbb{R}^2$. Then*

$\int e^{-u \cdot u} du = \frac{1}{2}$ $\qquad$ $\int e^{-\frac{1}{2}u \cdot u} du = 1$

This result is found by evaluating the dot product $u \cdot u$, then rewriting the integral as separate integrals over the components of u, and finally applying Theorem 6.2.1 to each integral separately. Now we are armed to evaluate f_{00}.

Proposition 6.2.3. *f_{00} is a unit vector in $L^2\left(\mathbb{R}^2\right)$.*

Proof. Note that $f_{00}^* = f_{00}$, since the exponent is real-valued, as per (6.4). In the L^2-norm, we have

$$\begin{aligned}
\|f_{00}\|_2^2 &= (f_{00}|f_{00}) \\
&= \frac{1}{2}\int f_{00}^* f_{00} du \\
&= 2\int e^{-u \cdot u} du \\
&= 2\left(\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

where we have employed Corollary 6.2.2 to solve the integral. Thus, f_{00} is a unit vector in $L^2\left(\mathbb{R}^2\right)$. □

Proposition 6.2.4. *f_{00} is a fixed point for the Fourier transform $\mathfrak{F}$; that is, $\mathfrak{F}f_{00} = f_{00}$.*

Proof. Invoking definition (4.2.15) for $\mathfrak{F}$, and evaluating the integral via Theorem 6.2.1, we find that

$$\begin{aligned}
\mathfrak{F}f_{00}(u) &= \int f_{00}(t)\, e^{-it \cdot u} dt \\
&= \int 2e^{-\frac{1}{2}t \cdot t} e^{-it \cdot u} dt \\
&= \int 2e^{-\frac{1}{2}(t \cdot t + 2it \cdot u - u \cdot u)} e^{-\frac{1}{2}u \cdot u} dt \\
&= 2e^{-\frac{1}{2}u \cdot u} \int e^{-\frac{1}{2}(t+iu)^2} dt \\
&= f_{00}(u)
\end{aligned}$$

□

Proposition 6.2.5. *f_{00} is an idempotent in S for the $\star$-product; that is, $f_{00} \star f_{00} = f_{00}$.*

Proof. It will be convenient to prove that f_{00} is an idempotent for the twisted convolution $\diamond$ before using Fourier transforms to carry the result over to the $\star$-product. So, definition (4.2.11) for $\diamond$ gives

$$\begin{aligned}
(f_{00} \diamond f_{00})(u) &= \int f_{00}(u-t)\, f_{00}(t)\, e^{-iu\cdot\Omega t} dt \\
&= \int 2e^{-\frac{1}{2}(u-t)\cdot(u-t)} 2e^{-\frac{1}{2}t\cdot t} e^{-iu\cdot\Omega t} dt \\
&= 4 \int e^{-(t\cdot t - u\cdot t + iu\cdot\Omega t)} e^{-\frac{1}{2}u\cdot u} dt \\
&= 4e^{-\frac{1}{2}u\cdot u} \int e^{-\left(t_1^2+t_2^2-u_1t_1-u_2t_2+iu_1t_2-iu_2t_1\right)} dt \\
&= 2f_{00}(u) \int e^{-t_1^2+t_1(u_1+iu_2)} dt_1 \int e^{-t_2^2+t_2(u_2-iu_1)} dt_2
\end{aligned}$$

Now, let $c_1 = -\frac{1}{2}(u_1 + iu_2)$ and $c_2 = -\frac{1}{2}(u_2 - iu_1)$ so that $c_1, c_2 \in \mathbb{C}$. If we substitute these in the expression above and complete the squares, the integrands have the form necessary to implement Theorem 6.2.1. We solve, noting that $c_1^2 + c_2^2 = 0$, as follows:

$$\begin{aligned}
(f_{00} \diamond f_{00})(u) &= 2f_{00}(u) \int e^{-t_1^2-2t_1c_1} dt_1 \int e^{-t_2^2-2t_2c_2} dt_2 \\
&= 2f_{00}(u) \int e^{-(t_1+c_1)^2} e^{c_1^2} dt_1 \int e^{-(t_2+c_2)^2} e^{c_2^2} dt_2 \\
&= 2f_{00}(u)\, e^{c_1^2+c_2^2} \int e^{-(t_1+c_1)^2} dt_1 \int e^{-(t_2+c_2)^2} dt_2 \\
&= 2f_{00}(u) \sqrt{1/2}\sqrt{1/2} \\
&= f_{00}(u)
\end{aligned}$$

Hence, $f_{00} \diamond f_{00} = f_{00}$, which means f_{00} is an idempotent for the twisted convolution $\diamond$. Applying (4.2.6.4) from Proposition 4.2.6 to f_{00} and appealing to Proposition 6.2.4, the idempotence of f_{00} subject to $\diamond$ leads to

$$\begin{aligned}
f_{00} \star f_{00} &= \mathfrak{F} f_{00} \star \mathfrak{F} f_{00} \\
&= \mathfrak{F}(f_{00} \diamond f_{00}) \\
&= \mathfrak{F} f_{00} \\
&= f_{00}
\end{aligned}$$

which proves that f_{00} is an idempotent in S for the $\star$-product. □

6.3 Orthonormality of $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$

Now that we have a grip on the Gaussian function f_{00}, we want to investigate the $\star$-product between f_{00} and $\star$-powers of $z, \overline{z}$ as they appear in (6.6). This will prepare us to handle the interactions between different members of the family $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ on our way towards proving orthonormality. Having said that, we start with the following:

Proposition 6.3.1. *If $f \in S$, then*

(1) $z \star f = zf + \overline{\partial} f$
(2) $\overline{z} \star f = \overline{z} f - \partial f$
(3) $f \star z = zf - \overline{\partial} f$
(4) $f \star \overline{z} = \overline{z} f + \partial f$

Proof. Using Corollary 5.2.6, we find (1) by

$$\begin{aligned} z \star f &= \frac{1}{\sqrt{2}} x_1 \star f + \frac{i}{\sqrt{2}} x_2 \star f \\ &= \frac{1}{\sqrt{2}} (x_1 + i\partial_2) f + \frac{i}{\sqrt{2}} (x_2 - i\partial_1) f \\ &= \frac{1}{\sqrt{2}} (x_1 + ix_2) f + \frac{1}{\sqrt{2}} (\partial_1 + i\partial_2) f \\ &= zf + \overline{\partial} f \end{aligned}$$

and similarly for (2)-(4). □

Proposition 6.3.2. *The following equalities hold in $\mathfrak{M}$:*

(1) $\overline{z} \star z = G - 1$
(2) $z \star \overline{z} = G + 1$
(3) $z \star \overline{z} - \overline{z} \star z = 2$

Proof. Since $z, \overline{z} \in \mathfrak{M}$, it follows from Proposition 5.2.8 that $\overline{z} \star z \in \mathfrak{M}$. We use Propositions 5.2.9 and 6.3.1, and the associativity in (5.2.8.4), to find that

$$\begin{aligned} \langle \overline{z} \star z, f \rangle &= \int \Big((\overline{z} \star z) \star f \Big) (x)\, dx \\ &= \int \Big(\overline{z} \star (z \star f) \Big) (x)\, dx \\ &= \int \Big(\overline{z} \star \big(zf + \overline{\partial} f \big) \Big) (x)\, dx \\ &= \int \big(\overline{z} z f + \overline{z} \overline{\partial} f \big) (x)\, dx \\ &= \langle \overline{z} z, f \rangle - \int \Big(\big(\overline{\partial} \overline{z} \big) f \Big) (x)\, dx \\ &= \langle \overline{z} z - \mathbb{1}, f \rangle \\ &= \langle G - \mathbb{1}, f \rangle \end{aligned}$$

for all $f \in S$, where $\overline{\partial}\overline{z} = 1$. Thus, we have proved (1). A similar proof gives (2). Together, (1) and (2) imply (3). □

Note that (3) resembles the canonical commutation relation for the creation and annihilation operators, where we have identified Planck's constant with $\theta = 2$ in Section 4.2.

Proposition 6.3.3. *For all $m, n \in \mathbb{N}$, it holds that*

$$f_{00} \star z^{\star n} \star \overline{z}^{\star m} \star f_{00} = \delta_{mn} 2^n n! f_{00}$$

Proof. We will frequently use $\partial f_{00} = -\overline{z} f_{00}$ and $\overline{\partial} f_{00} = -z f_{00}$. The first step is to prove that

$$\overline{z}^{\star m} \star f_{00} = 2^m \overline{z}^m f_{00} \qquad \forall m \in \mathbb{N} \tag{6.3.3.1}$$

We do so via induction. Using Proposition 6.3.1, we obtain a basis for the induction as

$$\begin{aligned} \overline{z} \star f_{00} &= \overline{z} f_{00} - \partial f_{00} \\ &= \overline{z} f_{00} + \overline{z} f_{00} \\ &= 2\overline{z} f_{00} \end{aligned}$$

Suppose, as inductive hypothesis, that $\overline{z}^{\star m} \star f_{00} = 2^m \overline{z}^m f_{00}$ for some fixed $m \in \mathbb{N}$. The inductive step follows:

$$\begin{aligned} \overline{z}^{\star m+1} \star f_{00} &= \overline{z} \star \overline{z}^{\star m} \star f_{00} \\ &= \overline{z} \star \left(2^m \overline{z}^m f_{00}\right) \\ &= \overline{z} \left(2^m \overline{z}^m f_{00}\right) - \partial \left(2^m \overline{z}^m f_{00}\right) \\ &= 2^m \overline{z}^{m+1} f_{00} + 2^m \overline{z}^{m+1} f_{00} \\ &= 2^{m+1} \overline{z}^{m+1} f_{00} \end{aligned}$$

Since this holds for arbitrary $m \in \mathbb{N}$, induction proves (6.3.3.1) for all $m \in \mathbb{N}$.

Now consider the expression $f_{00} \star z^{\star n} \star \overline{z}^{\star m} \star f_{00}$ in the different cases. Case 1: Let $n > m$. Using (6.3.3.1) and Proposition 6.3.1, we find

$$\begin{aligned} z^{\star n} \star \overline{z}^{\star m} \star f_{00} &= z^{\star n} \star \left(2^m \overline{z}^m f_{00}\right) \\ &= z^n \left(2^m \overline{z}^m f_{00}\right) + \overline{\partial}^n \left(2^m \overline{z}^m f_{00}\right) \\ &= z^n \left(2^m \overline{z}^m f_{00}\right) + 2^m \left(\overline{\partial}^n \overline{z}^m\right) f_{00} + 2^m \overline{z}^m \left(\overline{\partial}^n f_{00}\right) \\ &= z^n \left(2^m \overline{z}^m f_{00}\right) + 2^m \left(\overline{\partial}^n \overline{z}^m\right) f_{00} - 2^m \overline{z}^m \left(z^n f_{00}\right) \\ &= 2^m \left(\overline{\partial}^n \overline{z}^m\right) f_{00} \\ &= 0 \end{aligned}$$

which implies

$$f_{00} \star z^{\star n} \star \overline{z}^{\star m} \star f_{00} = 0 \quad \text{for} \quad n > m \tag{6.3.3.2}$$

Case 2: Let $n < m$. Using (6.3.3.2) and Proposition 4.2.8, we find

$$f_{00} \star z^{\star n} \star \overline{z}^{\star m} \star f_{00} = \left(f_{00} \star z^{\star m} \star \overline{z}^{\star n} \star f_{00}\right)^* = 0 \quad \text{for} \quad n < m \tag{6.3.3.3}$$

Case 3: Let $n = m$. Using (6.3.3.1) and Proposition 6.3.1, together with the idempotence of f_{00} in Proposition 6.2.5, we find

$$\begin{aligned}
f_{00} \star z^{\star n} \star \overline{z}^{\star n} \star f_{00} &= f_{00} \star z^{\star n} \star \left(2^n \overline{z}^n f_{00}\right) \\
&= f_{00} \star \left(z^n \left(2^n \overline{z}^n f_{00}\right) + \overline{\partial}^n \left(2^n \overline{z}^n f_{00}\right)\right) \\
&= f_{00} \star \left(z^n 2^n \overline{z}^n f_{00} + 2^n \left(\overline{\partial}^n \overline{z}^n\right) f_{00} + 2^n \overline{z}^n \left(\overline{\partial}^n f_{00}\right)\right) \\
&= f_{00} \star \left(2^n z^n \overline{z}^n f_{00} + 2^n \left(\overline{\partial}^n \overline{z}^n\right) f_{00} - 2^n \overline{z}^n z^n f_{00}\right) \\
&= f_{00} \star 2^n \left(\overline{\partial}^n \overline{z}^n\right) f_{00} \\
&= 2^n n! f_{00} \star f_{00} \\
&= 2^n n! f_{00} \qquad \text{for} \quad n = m
\end{aligned} \tag{6.3.3.4}$$

Combining the three cases in (6.3.3.2), (6.3.3.3) and (6.3.3.4) gives the desired result, namely

$$f_{00} \star z^{\star n} \star \overline{z}^{\star m} \star f_{00} = \delta_{mn} 2^n n! f_{00} \quad \text{for} \quad m, n \in \mathbb{N}$$

□

We now have sufficient ammunition to tackle the interactions between different members of the family $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$. We arrive at the orthonormality of the family $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ in $L^2\left(\mathbb{R}^2\right)$ via two properties, interesting in and of themselves, that reveal an analogy between $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ and a certain class of matrices.

Proposition 6.3.4. *Let $k, \ell, m, n \in \mathbb{N}_0$. Then $f_{mn} \star f_{k\ell} = \delta_{nk} f_{m\ell}$.*

Proof. We use Propositions 6.3.3 and 4.2.7 to find

$$\begin{aligned}
f_{mn} \star f_{k\ell} &= \left(\left(2^{m+n} m! n!\right)^{-\frac{1}{2}} \overline{z}^{\star m} \star f_{00} \star z^{\star n}\right) \star \left(\left(2^{k+\ell} k! \ell!\right)^{-\frac{1}{2}} \overline{z}^{\star k} \star f_{00} \star z^{\star \ell}\right) \\
&= \left(2^{m+n+k+\ell} m! n! k! \ell!\right)^{-\frac{1}{2}} \overline{z}^{\star m} \star \left(f_{00} \star z^{\star n} \star \overline{z}^{\star k} \star f_{00}\right) \star z^{\star \ell} \\
&= \left(2^{m+n+k+\ell} m! n! k! \ell!\right)^{-\frac{1}{2}} \overline{z}^{\star m} \star \left(\delta_{nk} \left(2^n n!\right)^{\frac{1}{2}} \left(2^k k!\right)^{\frac{1}{2}} f_{00}\right) \star z^{\star \ell} \\
&= \delta_{nk} \left(2^{m+\ell} m! \ell!\right)^{-\frac{1}{2}} \overline{z}^{\star m} \star f_{00} \star z^{\star \ell} \\
&= \delta_{nk} f_{m\ell}
\end{aligned}$$

□

Proposition 6.3.5. *Let $m, n \in \mathbb{N}_0$. Then $f_{mn}^* = f_{nm}$.*

Proof. It follows from definition (5.6) and Propositions 5.2.9 and 6.3.1 that

$$\begin{aligned}
\langle z^*, f \rangle &= \langle z, f^* \rangle^* \\
&= \left(\int (z \star f^*)(x)\ dx \right)^* \\
&= \left(\int z\overline{f}\ dx \right)^* \\
&= \int \overline{z} f\ dx \\
&= \int (\overline{z} \star f)(x)\ dx \\
&= \langle \overline{z}, f \rangle
\end{aligned}$$

for all $f \in S$. Hence, $z^* = \overline{z}$. Likewise, we find that $\overline{z}^* = z$. Now, by Proposition 5.2.8,

$$\begin{aligned}
f_{mn}^* &= \left(2^{m+n} m! n!\right)^{-\frac{1}{2}} \left(\overline{z}^{\star m} \star f_{00} \star z^{\star n}\right)^* \\
&= \left(2^{n+m} n! m!\right)^{-\frac{1}{2}} \overline{z}^{\star n} \star f_{00} \star z^{\star m} \\
&= f_{nm}
\end{aligned}$$

□

Proposition 6.3.6. *Let $k, \ell, m, n \in \mathbb{N}_0$. Then $(f_{mn} | f_{k\ell}) = \delta_{mk}\delta_{n\ell}$.*

Proof. We use here the results from Propositions 4.2.10, 5.2.9, 6.2.5, 6.3.3, 6.3.4, 6.3.5, and definition (6.6):

$$
\begin{aligned}
2\left(f_{mn}|f_{k\ell}\right) &= \langle f_{mn}^{*}, f_{k\ell}\rangle \\
&= \langle f_{nm}, f_{k\ell}\rangle \\
&= \int f_{nm}\left(u\right) f_{k\ell}\left(u\right) du \\
&= \int \left(f_{nm} \star f_{k\ell}\right)\left(u\right) du \\
&= \delta_{mk} \int f_{n\ell}\left(u\right) du \\
&= \delta_{mk} \left(2^{n+\ell} n!\ell!\right)^{-\frac{1}{2}} \int \left(\overline{z}^{\star n} \star f_{00} \star z^{\star\ell}\right)\left(u\right) du \\
&= \delta_{mk} \left(2^{n+\ell} n!\ell!\right)^{-\frac{1}{2}} \int \left(\overline{z}^{\star n} \star f_{00} \star f_{00} \star z^{\star\ell}\right)\left(u\right) du \\
&= \delta_{mk} \left(2^{n+\ell} n!\ell!\right)^{-\frac{1}{2}} \int \left(f_{00} \star z^{\star\ell} \star \overline{z}^{\star n} \star f_{00}\right)\left(u\right) du \\
&= \delta_{mk}\delta_{n\ell} \int f_{00}\left(u\right) du \\
&= 2\delta_{mk}\delta_{n\ell}
\end{aligned}
$$

where the last step follows from definition (6.4) and Corollary 6.2.2. □

Proposition 6.3.4 shows that when one takes the $\star$-product between elements of $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$, the second index of the first element has to equal the first index of the second element in order to return a nonzero function in $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$. This immediately brings matrix multiplication to mind, where the number of columns in the first matrix has to equal the number of rows in the second matrix. Proposition 6.3.5 shows that complex conjugation of elements in $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ is similar to transposition of matrices. In fact, in the following chapter we will see that this family allows us to fashion a correspondence between S and a certain space of infinite-dimensional matrices. Finally, Proposition 6.3.6 proves orthonormality of the family $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$.

6.4 Orthogonal polynomials

Now we proceed to prove completeness of the family $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$. Our method is to show that all the Hermite functions on $\mathbb{R}^2$ can be expressed as linear combinations of elements of $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$. Since the set of all Hermite functions $\{h_{k\ell}\}_{k,\ell\in\mathbb{N}_0}$ is complete in $L^2\left(\mathbb{R}^2\right)$, it will follow that $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ is complete in $L^2\left(\mathbb{R}^2\right)$. To this end, we present here an interlude on orthogonal polynomials: We state the definitions of certain classes of such polynomials and describe their relationships with one another.

The Jacobi, Laguerre and Hermite polynomials are examples of families of classical orthogonal polynomials. These families arise from the investigation of certain linear differential equations of the Sturm-Liouville type. After multiplication by a weight function, the orthogonal functions thus obtained are the eigenfunctions of a Sturm-Liouville problem, which have discrete spectra. Since the space of quadratically integrable functions is separable, these orthogonal families consist of at most a denumerable infinity of elements. The elements of such an orthogonal family can be expressed by a generalised Rodrigues' formula. For each family of orthogonal polynomials, we give the Rodrigues' formula and the equivalent explicit expression as found in [15, Chapter X].

The Jacobi polynomials are defined, for $\alpha > -1$, $\beta > -1$, by

$$\begin{aligned} P_n^{(\alpha,\beta)}(x_1) &= \frac{(-1)^n}{2^n n!}(1-x_1)^{-\alpha}(1+x_1)^{-\beta}\frac{d^n}{dx_1^n}\left[(1-x_1)^{\alpha+n}(1+x_1)^{\beta+n}\right] \\ &= 2^{-n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x_1-1)^{n-m}(x_1+1)^m \end{aligned} \tag{6.7}$$

for all $x_1 \in \mathbb{R}$ and $n \in \mathbb{N}_0$, where $P_n^{(\alpha,\beta)}(-x_1) = (-1)^n P_n^{(\beta,\alpha)}(x_1)$. The Laguerre polynomials are defined, for $\alpha > -1$, by

$$\begin{aligned} L_n^{\alpha}(x_1) &= \frac{e^{x_1}}{n!}x_1^{-\alpha}\frac{d^n}{dx_1^n}\left(e^{-x_1}x_1^{n+\alpha}\right) \\ &= \sum_{m=0}^{n}\binom{n+\alpha}{n-m}\frac{(-x_1)^m}{m!} \end{aligned} \tag{6.8}$$

for all $x_1 \in \mathbb{R}$ and $n \in \mathbb{N}_0$, where $L_0^{\alpha}(x_1) = 1$ and $L_n^{\alpha}(0) = \binom{n+\alpha}{n}$. The Hermite polynomials are defined by

$$\begin{aligned} H_n(x_1) &= (-1)^n e^{x_1^2}\frac{d^n}{dx_1^n}e^{-x_1^2} \\ &= n!\sum_{m=0}^{[n/2]}\frac{(-1)^m(2x_1)^{n-2m}}{m!\,(n-2m)!} \end{aligned} \tag{6.9}$$

for all $x_1 \in \mathbb{R}$ and $n \in \mathbb{N}_0$, where

$$[n/2] = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

and $H_0(x_1) = 1$, $H_{2m}(0) = \frac{(-1)^m(2m!)}{m!}$, and $H_{2m+1}(0) = 0$. We also have that

$$H_n(-x_1) = (-1)^n H_n(x_1)$$

and

$$H'_n(x_1) = 2nH_{n-1}(x_1) = 2x_1H_n(x_1) - H_{n+1}(x_1) \tag{6.10}$$

The Hermite polynomials are connected to the Laguerre polynomials via

$$\sum_{k=0}^{n} \binom{n}{k} H_{2k}(x_1) H_{2n-2k}(x_2) = (-1)^n n! L_n^0 \left(x_1^2 + x_2^2\right) \tag{6.11}$$

The Hermite functions on $\mathbb{R}$ are defined in terms of the Hermite polynomials, as per [41, Chapter 4], as the functions in $L^2(\mathbb{R})$ given by

$$h_k(x_1) := \left(2^{k-1}k!\right)^{-\frac{1}{2}} H_k(x_1) e^{-\frac{x_1^2}{2}} \tag{6.12}$$

for all $x_1 \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Note that this definition differs only from that in [41, Chapter 4] inasmuch as we accommodate the normalisation choices made in Section 4.2. As shown in [41, Chapter 4], the family $\{h_k : k \in \mathbb{N}_0\}$ is an orthonormal basis for $L^2(\mathbb{R})$. Because the Hermite functions are orthogonal polynomials, the k'th one being exactly of degree k, their span contains all polynomials.

Now, note that $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ is isomorphic to $L^2\left(\mathbb{R}^2\right) = L^2(\mathbb{R} \times \mathbb{R})$ via the identification of $(f \otimes g)(x) \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ with the function $f(x_1)g(x_2) \in L^2\left(\mathbb{R}^2\right)$ for all $f, g \in L^2(\mathbb{R})$, where $x_1, x_2 \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^2$. This allows us to construct an orthonormal basis for $L^2\left(\mathbb{R}^2\right)$ from the basis for $L^2(\mathbb{R})$. The Hermite functions on $\mathbb{R}^2$ are defined as the functions in $L^2\left(\mathbb{R}^2\right)$ given by

$$\begin{aligned}(h_k \otimes h_\ell)(x) &= h_k(x_1) h_\ell(x_2) \\ &= \left(2^{k+\ell-2}k!\ell!\right)^{-1/2} H_k(x_1) H_\ell(x_2) e^{-1/2\left(x_1^2+x_2^2\right)}\end{aligned} \tag{6.13}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $k, \ell \in \mathbb{N}_0$. The family $\{h_k \otimes h_\ell : k, \ell \in \mathbb{N}_0\}$ is an orthonormal basis for $L^2\left(\mathbb{R}^2\right)$.

In [42]-[44], Wünsche introduces the Laguerre two-dimensional polynomials as

$$\begin{aligned}L_{m,n}(z_1, z_2) &= exp\left(-\frac{\partial^2}{\partial z_1 \partial z_2}\right) z_1^m z_2^n \\ &= \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} z_1^{m-j} z_2^{n-j}\end{aligned} \tag{6.14}$$

for two independent complex variables z_1 and z_2, where $L_{m,n}(0,0) = (-1)^n n! \delta_{mn}$ and $L_{0,0}(z_1, z_2) = 1$. The Laguerre two-dimensional polynomials are related to the usual Laguerre polynomials in (6.8) by

$$L_{m,n}(z_1,z_2) = (-1)^n\, n! z_1^{m-n} L_n^{m-n}(z_1 z_2)$$
$$= (-1)^m\, m! z_2^{n-m} L_m^{n-m}(z_1 z_2) \tag{6.15}$$

If the second variable is complex conjugated to the first variable, that is $z_2 = z_1^*$, (6.15) can be simplified by transferring to polar coordinates. If we write $z = x_1 + ix_2$ and $z^* = x_1 - ix_2$ in polar coordinates, we have $z = re^{i\varphi}$ and $z^* = re^{-i\varphi}$ such that $r^2 = zz^* = x_1^2 + x_2^2$. Then

$$L_{m,n}(x_1+ix_2, x_1-ix_2) = (-1)^n\, n! r^{m-n} L_n^{m-n}\left(r^2\right) e^{i(m-n)\varphi}$$
$$= (-1)^n\, n! r^{m-n} L_n^{m-n}\left(r^2\right) e^{-i(m-n)\varphi} \tag{6.16}$$

As shown in [45], the Laguerre two-dimensional polynomials are related to products of Hermite polynomials by

$$L_{m,n}(x_1+ix_2, x_1-ix_2) = (-1)^n \sum_{j=0}^{m+n} \left(\frac{i}{2}\right)^{m+n-j} P_j^{(m-j,n-j)}(0)\, H_j(x_1)\, H_{m+n-j}(x_2) \tag{6.17}$$

where the coefficients are given by Jacobi polynomials evaluated at $x_1 = 0$. Furthermore, (6.17) can be inverted so that

$$H_m(x_1)\, H_n(x_2) = i^n \sum_{j=0}^{m+n} 2^j P_j^{(m-j,n-j)}(0)\, L_{j,m+n-j}(x_1+ix_2, x_1-ix_2)$$
$$= i^n \sum_{j=0}^{m+n} (-2)^j\, P_j^{(m-j,n-j)}(0)\, j! r^{m+n-2j} L_j^{m+n-2j}\left(r^2\right) e^{i(2j-m-n)\varphi} \tag{6.18}$$

This concludes our orthogonal interlude.

6.5 Completeness of $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$

We now proceed as follows: We define an operator B on S and show that to each eigenvalue corresponds both functions in $\{h_k \otimes h_\ell\}_{k,\ell\in\mathbb{N}_0}$ and functions in $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$. In other words, the eigenspace of each eigenvalue contains functions from both these families. Using the fact that $\{h_k \otimes h_\ell\}_{k,\ell\in\mathbb{N}_0}$ is an orthonormal basis for $L^2\left(\mathbb{R}^2\right)$, we will express each eigenfunction f_{mn} in terms of the basis elements, specifically those in the corresponding eigenspace. Invertibility of the coefficients in these expansions will imply that each $h_k \otimes h_\ell$ can be expressed in terms of elements of $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$. This will be sufficient to prove completeness of $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$.

Define the Hermite operator $B : S \to L^2(\mathbb{R}^2)$ by

$$B := |x|^2 - \Delta = \left(x_1^2 + x_2^2\right) - \left(\partial_1^2 + \partial_2^2\right) \tag{6.19}$$

Using definition (6.13), it can readily be shown that the Hermite functions are contained in S; hence, both $\{h_k \otimes h_\ell\}_{k,\ell\in\mathbb{N}_0}$ and $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ are in the domain of B. First, we consider the action of B on the basis $\{h_k \otimes h_\ell\}_{k,\ell\in\mathbb{N}_0}$. This will give us the eigenvalue corresponding to each eigenfunction in the basis.

Proposition 6.5.1. *$B(h_k \otimes h_\ell) = 2(k+\ell+1)(h_k \otimes h_\ell)$ for all $k, \ell \in \mathbb{N}_0$.*

Proof. For $j = 1, 2$, we have

$$\begin{aligned}(x_j - \partial_j)(x_j + \partial_j) h_k(x_j) &= (x_j - \partial_j)(x_j h_k(x_j) + \partial_j h_k(x_j)) \\ &= x_j^2 h_k(x_j) + x_j \partial_j h_k(x_j) - \partial_j x_j h_k(x_j) - \partial_j^2 h_k(x_j) \\ &= \left(x_j^2 - \partial_j^2\right) h_k(x_j) - h_k(x_j)\end{aligned}$$

which gives

$$\left(x_j^2 - \partial_j^2\right) h_k(x_j) = (x_j - \partial_j)(x_j + \partial_j) h_k(x_j) + h_k(x_j) \tag{6.5.1.1}$$

Using (6.10), together with definition (6.12), we find that for all $k \in \mathbb{N}_0$

$$\begin{aligned}(x_j + \partial_j) h_k(x_j) &= \left(2^{k-1}k!\right)^{-\frac{1}{2}} \left[x_j H_k(x_j) e^{-\frac{x_j^2}{2}} + (\partial_j H_k(x_j)) e^{-\frac{x_j^2}{2}} + H_k(x_j) \partial_j e^{-\frac{x_j^2}{2}}\right] \\ &= \left(2^{k-1}k!\right)^{-\frac{1}{2}} 2k H_{k-1}(x_j) e^{-\frac{x_j^2}{2}} \\ &= \sqrt{2k} \left(2^{(k-1)-1}(k-1)!\right)^{-\frac{1}{2}} H_{k-1}(x_j) e^{-\frac{x_j^2}{2}} \\ &= \sqrt{2k}\, h_{k-1}(x_j) \end{aligned} \tag{6.5.1.2}$$

Similarly, we find

$$\begin{aligned}(x_j - \partial_j) h_k(x_j) &= \left(2^{k-1}k!\right)^{-\frac{1}{2}} \left[x_j H_k(x_j) e^{-\frac{x_j^2}{2}} - (\partial_j H_k(x_j)) e^{-\frac{x_j^2}{2}} - H_k(x_j) \partial_j e^{-\frac{x_j^2}{2}}\right] \\ &= \left(2^{k-1}k!\right)^{-\frac{1}{2}} H_{k+1}(x_j) e^{-\frac{x_j^2}{2}} \\ &= \sqrt{2(k+1)} \left(2^{(k+1)-1}(k+1)!\right)^{-\frac{1}{2}} H_{k+1}(x_j) e^{-\frac{x_j^2}{2}} \\ &= \sqrt{2(k+1)}\, h_{k+1}(x_j) \end{aligned} \tag{6.5.1.3}$$

By applying (6.5.1.2) and (6.5.1.3) to (6.5.1.1), we obtain

$$\begin{aligned}\left(x_j^2 - \partial_j^2\right) h_k(x_j) &= (x_j - \partial_j)(x_j + \partial_j) h_k(x_j) + h_k(x_j) \\ &= (x_j - \partial_j) \sqrt{2k}\, h_{k-1}(x_j) + h_k(x_j) \\ &= \sqrt{2k}\sqrt{2(k-1+1)}\, h_k(x_j) + h_k(x_j) \\ &= (2k+1) h_k(x_j) \end{aligned} \tag{6.5.1.4}$$

for all $k \in \mathbb{N}_0$ and $j = 1, 2$. The required result follows by using (6.5.1.4), together with definitions (6.19) and (6.13): For all $k, \ell \in \mathbb{N}_0$

$$\begin{aligned}
B\,(h_k \otimes h_\ell) &= \left(|x|^2 - \Delta\right) h_k\,(x_1)\, h_\ell\,(x_2) \\
&= \left[\left(x_1^2 + x_2^2\right) - \left(\partial_1^2 + \partial_2^2\right)\right] h_k\,(x_1)\, h_\ell\,(x_2) \\
&= \left(x_1^2 - \partial_1^2\right) h_k\,(x_1)\, h_\ell\,(x_2) + \left(x_2^2 - \partial_2^2\right) h_k\,(x_1)\, h_\ell\,(x_2) \\
&= (2k+1)\, h_k\,(x_1)\, h_\ell\,(x_2) + (2\ell+1)\, h_k\,(x_1)\, h_\ell\,(x_2) \\
&= 2\,(k+\ell+1)\,(h_k \otimes h_\ell)
\end{aligned}$$

□

Next, we consider the action of B on elements of the family $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ via a series of propositions.

Proposition 6.5.2. *$Bf = G \star f + f \star G$ for all $f \in S$.*

Proof. It follows from Propositions 6.3.1 and 6.3.2 that

$$\begin{aligned}
Bf &= |x|^2 f - \Delta f \\
&= 2z\overline{z}f - 2\overline{\partial}\partial f \\
&= \left(z\overline{z} + \overline{\partial}\overline{z} - z\partial - \overline{\partial}\partial\right) f + \left(z\overline{z} - \overline{\partial}\overline{z} + z\partial - \overline{\partial}\partial\right) f \\
&= z \star (\overline{z}f - \partial f) + (\overline{z}f + \partial f) \star z \\
&= z \star \overline{z} \star f + f \star \overline{z} \star z \\
&= (G + \mathbf{1}) \star f + f \star (G - \mathbf{1}) \\
&= G \star f + f \star G
\end{aligned}$$

for all $f \in S$. □

Proposition 6.5.3. *If $m, n \in \mathbb{N}_0$, then*
(1) $z \star f_{mn} = \sqrt{2m} f_{m-1,n}$
(2) $\overline{z} \star f_{mn} = \sqrt{2m+2} f_{m+1,n}$
(3) $f_{mn} \star z = \sqrt{2n+2} f_{m,n+1}$
(4) $f_{mn} \star \overline{z} = \sqrt{2n} f_{m,n-1}$
with $f_{mn} = 0$ if m or n is -1.

Proof. Using definition (6.6), we find (2) by

$$\begin{aligned}
\overline{z} \star f_{mn} &= \left(2^{m+n} m!n!\right)^{-\frac{1}{2}} \overline{z} \star \overline{z}^{\star m} \star f_{00} \star z^{\star n} \\
&= \sqrt{2m+2}\left(2^{m+1+n}\,(m+1)!n!\right)^{-\frac{1}{2}} \overline{z}^{\star m+1} \star f_{00} \star z^{\star n} \\
&= \sqrt{2m+2} f_{m+1,n}
\end{aligned}$$

Using Propositions 4.2.8 and 6.3.5, then applying (2), we find (3):

$$\begin{aligned} f_{mn} \star z &= (\overline{z} \star f_{nm})^* \\ &= \left(\sqrt{2n+2} f_{n+1,m}\right)^* \\ &= \sqrt{2n+2} f_{m,n+1} \end{aligned}$$

We find (1) by applying Proposition 6.3.1 and equation (6.3.3.1):

$$\begin{aligned} z \star f_{mn} &= \left(2^{m+n} m!n!\right)^{-\frac{1}{2}} z \star \overline{z}^{\star m} \star f_{00} \star z^{\star n} \\ &= \left(2^{m+n} m!n!\right)^{-\frac{1}{2}} z \star \left(2^m \overline{z}^m f_{00}\right) \star z^{\star n} \\ &= \left(2^{m+n} m!n!\right)^{-\frac{1}{2}} \left[z \left(2^m \overline{z}^m f_{00}\right) + \overline{\partial} \left(2^m \overline{z}^m f_{00}\right)\right] \star z^{\star n} \\ &= \left(2^{m+n} m!n!\right)^{-\frac{1}{2}} \left[2^m z \overline{z}^m f_{00} + 2^m \left(\overline{\partial} \overline{z}^m\right) f_{00} + 2^m \overline{z}^m \left(\overline{\partial} f_{00}\right)\right] \star z^{\star n} \\ &= \left(2^{m+n} m!n!\right)^{-\frac{1}{2}} 2^m m \overline{z}^{m-1} f_{00} \star z^{\star n} \\ &= 2m \left(2^{m+n} m!n!\right)^{-\frac{1}{2}} 2^{m-1} \overline{z}^{m-1} f_{00} \star z^{\star n} \\ &= \sqrt{2m} \left(2^{m-1+n} (m-1)!n!\right)^{-\frac{1}{2}} \overline{z}^{\star m-1} \star f_{00} \star z^{\star n} \\ &= \sqrt{2m} f_{m-1,n} \end{aligned}$$

Then (4) follows:

$$\begin{aligned} f_{mn} \star \overline{z} &= (z \star f_{nm})^* \\ &= \left(\sqrt{2n} f_{n-1,m}\right)^* \\ &= \sqrt{2n} f_{m,n-1} \end{aligned}$$

□

We include here a result showing the action of the derivatives on the elements of $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$; however, it will only be used in later chapters.

Proposition 6.5.4. *If $m, n \in \mathbb{N}_0$, then*
(1) $\partial f_{mn} = \sqrt{n/2} f_{m,n-1} - \sqrt{1/2\,(m+1)} f_{m+1,n}$
(2) $\overline{\partial} f_{mn} = \sqrt{m/2} f_{m-1,n} - \sqrt{1/2\,(n+1)} f_{m,n+1}$

Proof. We find (1) as follows:

$$\begin{aligned}
\partial f_{mn} &= \left(2^{m+n} m!n!\right)^{-\frac{1}{2}} \partial \left(\overline{z}^{\star m} \star f_{00} \star z^{\star n}\right) \\
&= \left(2^{m+n} m!n!\right)^{-\frac{1}{2}} \left[\overline{z}^{\star m} \star \partial f_{00} \star z^{\star n} + \overline{z}^{\star m} \star f_{00} \star \partial z^{\star n}\right] \\
&= \left(2^{m+n} m!n!\right)^{-\frac{1}{2}} \left[\overline{z}^{\star m} \star \left(-\overline{z} f_{00}\right) \star z^{\star n} + \overline{z}^{\star m} \star f_{00} \star n z^{\star n-1}\right] \\
&= \left(2^{m+n} m!n!\right)^{-\frac{1}{2}} \left[n\overline{z}^{\star m} \star f_{00} \star z^{\star n-1} - 2^{-1}\overline{z}^{\star m+1} \star f_{00} \star z^{\star n}\right] \\
&= \sqrt{n/2} \left(2^{m+n-1} m! \left(n-1\right)!\right)^{-\frac{1}{2}} \overline{z}^{\star m} \star f_{00} \star z^{\star n-1} \\
&\qquad - \sqrt{1/2\left(m+1\right)} \left(2^{m+1+n} \left(m+1\right)!n!\right)^{-\frac{1}{2}} \overline{z}^{\star m+1} \star f_{00} \star z^{\star n} \\
&= \sqrt{n/2} f_{m,n-1} - \sqrt{1/2\left(m+1\right)} f_{m+1,n}
\end{aligned}$$

(2) is found similarly. □

Proposition 6.5.5. *If $m, n \in \mathbb{N}_0$, then*
(1) $G \star f_{mn} = \left(2m+1\right) f_{mn}$
(2) $f_{mn} \star G = \left(2n+1\right) f_{mn}$

Proof. Using Propositions 6.3.2 and 6.5.3, we find

$$\begin{aligned}
G \star f_{mn} &= \left(\overline{z} \star z + \mathbf{1}\right) \star f_{mn} \\
&= \overline{z} \star z \star f_{mn} + \mathbf{1} \star f_{mn} \\
&= \overline{z} \star \left(\sqrt{2m} f_{m-1,n}\right) + f_{mn} \\
&= \sqrt{2\left(m-1\right)+2}\sqrt{2m} f_{mn} + f_{mn} \\
&= \left(2m+1\right) f_{mn}
\end{aligned}$$

and

$$\begin{aligned}
f_{mn} \star G &= f_{mn} \star \left(\overline{z} \star z + \mathbf{1}\right) \\
&= f_{mn} \star \overline{z} \star z + f_{mn} \star \mathbf{1} \\
&= \sqrt{2n} f_{m,n-1} \star z + f_{mn} \\
&= \sqrt{2\left(n-1\right)+2}\sqrt{2n} f_{mn} + f_{mn} \\
&= \left(2n+1\right) f_{mn}
\end{aligned}$$

□

Proposition 6.5.6. *$B f_{mn} = 2\left(m+n+1\right) f_{mn}$ for all $m, n \in \mathbb{N}_0$.*

Proof. Using Propositions 6.5.5 and 6.5.2, we find

$$\begin{aligned}
B f_{mn} &= G \star f_{mn} + f_{mn} \star G \\
&= \left(2m+1\right) f_{mn} + \left(2n+1\right) f_{mn} \\
&= 2\left(m+n+1\right) f_{mn}
\end{aligned}$$

□

Corollary 6.5.7. *There exists constants* $c_{mn}^{k\ell}$ *such that for all* $m, n \in \mathbb{N}_0$*,*

$$f_{mn} = \sum_{k+\ell=m+n} c_{mn}^{k\ell} h_k \otimes h_\ell$$

Proof. Let $m, n \in \mathbb{N}_0$ be arbitrary. From Proposition 6.5.6, we see that f_{mn} is in the eigenspace, say E_{mn}, of the eigenvalue $2(m+n+1)$ of B. Proposition 6.5.1 implies that $h_k \otimes h_\ell$ is in E_{mn} whenever $k+\ell = m+n$. Since $\{h_k \otimes h_\ell\}_{k,\ell\in\mathbb{N}_0}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$, the set of eigenvectors $\{h_k \otimes h_\ell : k+\ell = m+n\}$ spans the eigenspace E_{mn}. Hence, f_{mn} can be written as a linear combination of these eigenvectors. Since this holds for arbitrary $m, n \in \mathbb{N}_0$, we have the required result. ☐

Now we want to show that each $h_k \otimes h_\ell$ can be expressed as a linear combination of elements in $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$. We do so by finding an explicit expression for the expansion in Corollary 6.5.7, and then showing that the coefficients are invertible.

Proposition 6.5.8. *For all* $m, n \in \mathbb{N}_0$*, we have*

$$f_{mn} = \left(2^{m+n} m! n!\right)^{-\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \binom{m}{k} \binom{n}{k} k! 2^{m+n-k} \overline{z}^{m-k} z^{n-k} f_{00}$$

Proof. We will prove this by induction over n. From definition (6.6) for f_{mn} and equation (6.3.3.1), we have as inductive basis that for all $m \in \mathbb{N}_0$

$$\begin{aligned}
f_{m0} &= (2^m m!)^{-\frac{1}{2}} \overline{z}^{\star m} \star f_{00} \\
&= (2^m m!)^{-\frac{1}{2}} 2^m \overline{z}^m f_{00} \\
&= (2^m m!)^{-\frac{1}{2}} \binom{m}{0} \binom{0}{0} 2^m \overline{z}^m f_{00} \\
&= \left(2^{m+0} m! 0!\right)^{-\frac{1}{2}} \sum_{k=0}^{0} (-1)^k \binom{m}{k} \binom{0}{k} k! 2^{m+0-k} \overline{z}^{m-k} z^{0-k} f_{00}
\end{aligned}$$

Suppose, as inductive hypothesis, that for some fixed $n \in \mathbb{N}$

$$f_{mn} = \left(2^{m+n} m! n!\right)^{-\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \binom{m}{k} \binom{n}{k} k! 2^{m+n-k} \overline{z}^{m-k} z^{n-k} f_{00}$$

for all $m \in \mathbb{N}_0$. Let $M = \left(2^{m+n+1} m! (n+1)!\right)^{-\frac{1}{2}}$. Then, using Propositions

6.5.3 and 6.3.1, the inductive step follows: For all $m \in \mathbb{N}_0$

$$\begin{aligned}
&f_{m,n+1} \\
&= (2n+2)^{-\frac{1}{2}} f_{mn} \star z \\
&= \left(2^{m+n+1} m!\,(n+1)!\right)^{-\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \binom{m}{k}\binom{n}{k} k! 2^{m+n-k} \overline{z}^{m-k} z^{n-k} f_{00} \star z \\
&= M \sum_{k=0}^{n} (-1)^k \binom{m}{k}\binom{n}{k} k! 2^{m+n-k} \left(2\overline{z}^{m-k} z^{n+1-k} f_{00} - (m-k)\,\overline{z}^{m-k-1} z^{n-k} f_{00}\right) \\
&= M \sum_{k=0}^{n} (-1)^k \binom{m}{k}\binom{n}{k} k! 2^{m+n+1-k} \overline{z}^{m-k} z^{n+1-k} f_{00} \\
&\qquad + M \sum_{k=0}^{n} (-1)^{k+1} \binom{m}{k+1}\binom{n}{k} (k+1)! 2^{m+n-k} \overline{z}^{m-k-1} z^{n-k} f_{00} \\
&= M \left[\binom{m}{0}\binom{n}{0} 2^{m+n+1} \overline{z}^{m} z^{n+1} + (-1)^{n+1} \binom{m}{n+1}\binom{n}{n} (n+1)! 2^{m} \overline{z}^{m-n-1}\right] f_{00} \\
&\qquad + M \sum_{k=1}^{n} (-1)^k \left[\binom{m}{k}\binom{n}{k} + \binom{m}{k}\binom{n}{k-1}\right] k! 2^{m+n+1-k} \overline{z}^{m-k} z^{n+1-k} f_{00} \\
&= M \left[\binom{m}{0}\binom{n+1}{0} 2^{m+n+1} \overline{z}^{m} z^{n+1} + (-1)^{n+1} \binom{m}{n+1}\binom{n+1}{n+1} (n+1)! 2^{m} \overline{z}^{m-n-1}\right] f_{00} \\
&\qquad + M \sum_{k=1}^{n} (-1)^k \binom{m}{k}\binom{n+1}{k} k! 2^{m+n+1-k} \overline{z}^{m-k} z^{n+1-k} f_{00} \\
&= \left(2^{m+n+1} m!\,(n+1)!\right)^{-\frac{1}{2}} \sum_{k=0}^{n+1} (-1)^k \binom{m}{k}\binom{n+1}{k} k! 2^{m+n+1-k} \overline{z}^{m-k} z^{n+1-k} f_{00}
\end{aligned}$$

Since this holds for arbitrary $n \in \mathbb{N}$ and for $n = 0$, the required result follows by induction. □

The following proposition expresses the f_{mn}, after switching to polar coordinates, in terms of the Laguerre polynomials L_n^α, as defined in (6.8). Note that we can write $z, \overline{z}$ in polar coordinates by defining $\rho e^{i\alpha} := x_1 + ix_2$ such that $\rho^2 = x_1^2 + x_2^2$. Then we have

$$z := \frac{1}{\sqrt{2}}(x_1 + ix_2) = \frac{1}{\sqrt{2}}\rho e^{i\alpha} \qquad \overline{z} := \frac{1}{\sqrt{2}}(x_1 - ix_2) = \frac{1}{\sqrt{2}}\rho e^{-i\alpha}$$

Also, note that

$$f_{00} := 2e^{-z\overline{z}} = 2e^{-1/2\left(x_1^2+x_2^2\right)} = 2e^{-\rho^2/2}$$

Proposition 6.5.9. *For all $m, n \in \mathbb{N}_0$, we can write*

$$f_{mn} = 2\,(-1)^n \sqrt{\frac{n!}{m!}} e^{-i\alpha(m-n)} \rho^{m-n} L_n^{m-n}\left(\rho^2\right) e^{-\rho^2/2}$$

Proof. We switch to polar coordinates in the expression derived in Proposition 6.5.8. After some rearrangement, we use definition (6.8) to obtain an expression in terms of the Laguerre polynomials. For all $m, n \in \mathbb{N}_0$, we can write

$$\begin{aligned}
f_{mn} &= \left(2^{m+n} m! n!\right)^{-\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \binom{m}{k} \binom{n}{k} k! 2^{m+n-k} \bar{z}^{m-k} z^{n-k} f_{00} \\
&= \left(2^{m+n} m! n!\right)^{-\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \binom{m}{k} \binom{n}{k} k! 2^{m+n-k} \left(2^{-\frac{1}{2}} \rho e^{-i\alpha}\right)^{m-k} \left(2^{-\frac{1}{2}} \rho e^{i\alpha}\right)^{n-k} 2e^{-\rho^2/2} \\
&= 2\left(m! n!\right)^{-\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \binom{m}{k} \binom{n}{k} k! \rho^{m-n} \rho^{2(n-k)} e^{-i\alpha(m-n)} e^{-\rho^2/2} \\
&= 2\sqrt{\frac{n!}{m!}} e^{-i\alpha(m-n)} \rho^{m-n} \frac{1}{n!} \left(\sum_{k=0}^{n} (-1)^k \frac{m! n!}{(m-k)! k! \, (n-k)!} \rho^{2(n-k)} \right) e^{-\rho^2/2} \\
&= 2\sqrt{\frac{n!}{m!}} e^{-i\alpha(m-n)} \rho^{m-n} \frac{1}{n!} \left((-1)^n \sum_{k=0}^{n} \frac{m! n!}{(m-n+k)! \, (n-k)! k!} \left(-\rho^2\right)^k \right) e^{-\rho^2/2} \\
&= 2\sqrt{\frac{n!}{m!}} e^{-i\alpha(m-n)} \rho^{m-n} \frac{1}{n!} \left((-1)^n \, n! \sum_{k=0}^{n} \binom{m}{n-k} \frac{\left(-\rho^2\right)^k}{k!} \right) e^{-\rho^2/2} \\
&= 2\left(-1\right)^n \sqrt{\frac{n!}{m!}} e^{-i\alpha(m-n)} \rho^{m-n} \left(\sum_{k=0}^{n} \binom{n+m-n}{n-k} \frac{\left(-\rho^2\right)^k}{k!} \right) e^{-\rho^2/2} \\
&= 2\left(-1\right)^n \sqrt{\frac{n!}{m!}} e^{-i\alpha(m-n)} \rho^{m-n} L_n^{m-n}\left(\rho^2\right) e^{-\rho^2/2}
\end{aligned}$$

□

Proposition 6.5.10. *The family of functions* $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ *is complete in* $L^2\left(\mathbb{R}^2\right)$.

Proof. The following equalities are obtained by applying (6.16) and (6.17) to the expression found in Proposition 6.5.9. Furthermore, we use (6.13), (6.7), and Corollary 6.5.7. For all $m, n \in \mathbb{N}_0$, it holds that

$$
\begin{aligned}
& f_{mn}(x) \\
=& 2(-1)^n \sqrt{\frac{n!}{m!}} e^{-i\alpha(m-n)} \rho^{m-n} L_n^{m-n}\left(\rho^2\right) e^{-\rho^2/2} \\
=& 2(-1)^n (n!m!)^{-\frac{1}{2}} n! \rho^{m-n} L_n^{m-n}\left(\rho^2\right) e^{-i\alpha(m-n)} e^{-\rho^2/2} \\
=& 2(-1)^n (n!m!)^{-\frac{1}{2}} \sum_{k=0}^{m+n} \left(\frac{i}{2}\right)^{m+n-k} P_k^{(m-k,n-k)}(0)\, H_k(x_1)\, H_{m+n-k}(x_2)\, e^{-1/2\left(x_1^2+x_2^2\right)} \\
=& \frac{2(-1)^n}{(n!m!)^{\frac{1}{2}}} \sum_{k=0}^{m+n} \left(\frac{i}{2}\right)^{m+n-k} P_k^{(m-k,n-k)}(0) \left(2^{m+n-2} k!\,(m+n-k)!\right)^{\frac{1}{2}} \left(h_k \otimes h_{m+n-k}\right)(x) \\
=& \sum_{k+\ell=m+n} (-1)^n (n!m!)^{-\frac{1}{2}} \left(\frac{i}{2}\right)^{\ell} P_k^{(m-k,n-k)}(0)\, 2^{(k+\ell)/2} (k!\ell!)^{\frac{1}{2}} \left(h_k \otimes h_\ell\right)(x) \\
=& \sum_{k+\ell=m+n} (-1)^n i^\ell 2^{(k-\ell)/2} \left(\frac{k!\ell!}{n!m!}\right)^{\frac{1}{2}} P_k^{(m-k,n-k)}(0) \left(h_k \otimes h_\ell\right)(x) \\
=& \sum_{k+\ell=m+n} 2^{(k-\ell)/2} i^{2n+\ell} \left(\frac{(k+\ell)!}{n!\,(k+\ell-n)!}\right)^{\frac{1}{2}} \left(\frac{(k+\ell)!}{k!\ell!}\right)^{-\frac{1}{2}} P_k^{(m-k,n-k)}(0) \left(h_k \otimes h_\ell\right)(x) \\
=& \sum_{k+\ell=m+n} 2^{(k-\ell)/2} i^{2n+\ell} \binom{k+\ell}{n}^{\frac{1}{2}} \binom{k+\ell}{k}^{-\frac{1}{2}} P_k^{(m-k,n-k)}(0) \left(h_k \otimes h_\ell\right)(x)
\end{aligned}
\tag{6.5.10.1}
$$

Recall from Corollary 6.5.7 that, for any $m, n \in \mathbb{N}_0$, f_{mn} can be written as a linear combination of elements of the set $\{h_k \otimes h_\ell : k + \ell = m + n\}$. In other words, there exist constants $c_{mn}^{k\ell}$ such that for all $m, n \in \mathbb{N}_0$,

$$
f_{mn} = \sum_{k+\ell=m+n} c_{mn}^{k\ell} h_k \otimes h_\ell \tag{6.5.10.2}
$$

By comparing (6.5.10.1) and (6.5.10.2), both of which hold for all $m, n \in \mathbb{N}_0$, an explicit expression for the $c_{mn}^{k\ell}$ is given by

$$
c_{mn}^{k\ell} = 2^{(k-\ell)/2} i^{2n+\ell} \binom{k+\ell}{n}^{\frac{1}{2}} \binom{k+\ell}{k}^{-\frac{1}{2}} P_k^{(m-k,n-k)}(0) \tag{6.5.10.3}
$$

Since each $c_{mn}^{k\ell}$ in (6.5.10.3) defines a constant polynomial, it follows that (6.5.10.2) is invertible. Thus, there exist constants $b_{k\ell}^{mn}$ such that for all $k, \ell \in \mathbb{N}_0$

$$
h_k \otimes h_\ell = \sum_{m+n=k+\ell} b_{k\ell}^{mn} f_{mn} \tag{6.5.10.4}
$$

This means that every function in $\{h_k \otimes h_\ell\}_{k,\ell\in\mathbb{N}_0}$ can be expressed as a linear combination of elements of $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$. We know that the family

$\{h_k \otimes h_\ell\}_{k,\ell\in\mathbb{N}_0}$ is an orthonormal basis for $L^2\left(\mathbb{R}^2\right)$; specifically, this family is complete, i.e.

$$\overline{span\,\{h_k \otimes h_\ell\}_{k,\ell\in\mathbb{N}_0}} = L^2\left(\mathbb{R}^2\right)$$

where the overbar denotes the closure. So (6.5.10.4) implies that $L^2\left(\mathbb{R}^2\right)$ is equal to the closure of a set consisting of certain linear combinations of elements in $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$. In particular, it follows that

$$\overline{span\,\{f_{mn}\}_{m,n\in\mathbb{N}_0}} = L^2\left(\mathbb{R}^2\right)$$

Hence, the family $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ is complete in $L^2\left(\mathbb{R}^2\right)$. □

Corollary 6.5.11. $\{f_{mn}\}_{m,n\in\mathbb{N}_0} \subset S \subset L^2\left(\mathbb{R}^2\right)$ *is an orthonormal basis for* $L^2\left(\mathbb{R}^2\right)$.

This result follows from Propositions 6.1.1, 6.3.6 and 6.5.10. The first proves the inclusion of the family in S, while the other two prove orthonormality and completeness of the family respectively. The power of this result lies in the fact that we can expand any element of $L^2\left(\mathbb{R}^2\right)$ in terms of this basis. In particular, we can expand any element of S in terms of this basis. More precisely, for every $f \in S$, there exist constants c_{mn} such that

$$f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn} \tag{6.20}$$

Chapter 7

Sequence representation of S

We represent S as a sequence space of coefficients after expansion in the basis $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$. Before we define the relevant sequence space, we define a family of seminorms that generates a topology on S equivalent to the Schwartz topology in Section 7.2. This family of seminorms will be instrumental in proving the equivalence between S and its corresponding sequence space in Section 7.3. The seminorms will be defined via a certain unbounded operator; therefore, we start by recalling some relevant definitions in Section 7.1. We conclude this chapter by finding a matricial form of the Moyal product and extending it to $L^2\left(\mathbb{R}^2\right)$ in Section 7.4.

7.1 Unbounded operators

The following definitions can be found in [23, Chapter 10].

Definition 7.1.1. *Let $T : \mathfrak{D}(T) \to H$ be a (possibly unbounded) densely defined linear operator in a complex Hilbert space H. Then the* ***adjoint*** *operator $T^* : \mathfrak{D}(T^*) \to H$ of T is defined as follows: The domain $\mathfrak{D}(T^*)$ consists of all $y \in H$ such that there is a $y^* \in H$ satisfying*

$$\langle Tx, y\rangle = \langle x, y^*\rangle$$

for all $x \in \mathfrak{D}(T)$. For each such $y \in \mathfrak{D}(T^*)$, the ***adjoint*** *operator T^* is then defined in terms of that y^* by*

$$y^* = T^*y$$

In other words, an element $y \in H$ is in $\mathfrak{D}(T^*)$ if (and only if) for that y the inner product $\langle Tx, y\rangle$, considered as a function of x, can be represented in the form $\langle Tx, y\rangle = \langle x, y^*\rangle$ for all $x \in \mathfrak{D}(T)$. Also, each such y determines the corresponding y^* uniquely since $\mathfrak{D}(T)$ is dense in H, by assumption.

Definition 7.1.2. *Let $T : \mathfrak{D}(T) \to H$ be a linear operator which is densely defined in a complex Hilbert space H. Then T is called* ***symmetric*** *if*

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

for all $x, y \in \mathfrak{D}(T)$.

7.2 New seminorms

Here we define a new family of seminorms in terms of an unbounded operator W and prove that it generates a topology on S equivalent to the Schwartz topology. Let

$$Wf := G \star f \star G \tag{7.1}$$

for all $f \in S$, where G is defined in (6.3). Since $G \in \mathfrak{M}$, it follows from definition (5.20) that $Wf \in S$ for all $f \in S$. Owing to Proposition 4.2.1, we may consider W as an operator in $L^2(\mathbb{R}^2)$ with domain S, i.e.

$$W : S \to L^2(\mathbb{R}^2) : f \mapsto G \star f \star G$$

Proposition 7.2.1. *$W : S \to L^2(\mathbb{R}^2)$ is a densely defined, unbounded, symmetric operator.*

Proof. Note that W is densely defined because S is dense in $L^2(\mathbb{R}^2)$ by Proposition 4.2.1. Using Proposition 6.5.5, we see that, for all $m, n \in \mathbb{N}_0$,

$$Wf_{mn} = G \star f_{mn} \star G = (2m+1)(2n+1) f_{mn}$$

Thus, W has arbitrarily large eigenvalues and is unbounded. Now we prove that W is symmetric. Propositions 5.2.8, 6.3.2 and 6.3.5 give

$$\begin{aligned} G^* &= (\overline{z} \star z + \mathbb{1})^* \\ &= z^* \star \overline{z}^* + \mathbb{1} \\ &= \overline{z} \star z + \mathbb{1} \\ &= G \end{aligned}$$

Using this result, together with definition (4.2.8) and Propositions 4.2.10, 5.2.3, 5.2.8 and 5.2.9, we find that

$$\begin{aligned}
2\,(Wf|g) &= \langle (G\star f\star G)^*\,, g\rangle \\
&= \langle G\star f^*\star G, g\rangle \\
&= \int (G\star f^*\star G)\,(x)\,g\,(x)\,dx \\
&= \int (G\star f^*\star G\star g)\,(x)\,dx \\
&= \int (f^*\star G\star g\star G)\,(x)\,dx \\
&= \int f^*\,(x)\,(G\star g\star G)\,(x)\,dx \\
&= \langle f^*, G\star g\star G\rangle \\
&= 2\,(f|Wg)
\end{aligned}$$

for all $f, g \in S$. Hence, W is symmetric. □

Now we define a family of seminorms on S. For each $k \in \mathbb{N}_0$, let

$$|||f|||_k := \left\| W^k f\right\|_2 \tag{7.2}$$

for all $f \in S$. It is simple to show that $|||\cdot|||_k$ is a seminorm for each $k \in \mathbb{N}_0$. Our aim is to show that the family of seminorms $\{|||\cdot|||_k\}_{k\in\mathbb{N}_0}$ generates the Schwartz topology on S. We do so in the following propositions.

Proposition 7.2.2. *For all $f \in S$ and any $m > \frac{1}{2}$,*

$$\|f\|_\infty \leq K2^m\Big(\,\|f\|_2 + \|\Delta^m f\|_2\Big)$$

where $\Delta := \partial_1^2 + \partial_2^2$ and $K = \left(\int \frac{du}{(1+|u|^2)^{2m}}\right)^{\frac{1}{2}}$.

Proof. Let $f \in S$ and $u \in \mathbb{R}^2$ be arbitrary. We use the Fourier transform F as defined in (4.2.16). From Lemma 4.2.2 (5), one has

$$\begin{aligned}
|u|^2\,Ff\,(u) &= \left(u_1^2 + u_2^2\right) Ff\,(u) \\
&= u_1\,(iF\,(\partial_2 f))\,(u) + u_2\,(iF\,(\partial_1 f))\,(u) \\
&= i^2F\,(\partial_2\,(\partial_2 f))\,(u) + i^2F\,(\partial_1\,(\partial_1 f))\,(u) \\
&= -F\left(\partial_2^2 f\right)(u) - F\left(\partial_1^2 f\right)(u) \\
&= -F\,(\Delta f)\,(u)
\end{aligned}$$

Iterating gives, for all $r \in \mathbb{N}_0$,

$$|u|^{2r}\,Ff\,(u) = (-1)^r\,F\,(\Delta^r f)\,(u) \tag{7.2.2.1}$$

Also, F satisfies the Plancherel formula:

$$\begin{aligned}
\int |Ff(u)|^2\, du &= \int Ff(u)\, \overline{Ff(u)} du \\
&= \int \int f(t)\, e^{-it\cdot Su} dt \int \overline{f(t')} e^{it'\cdot Su} dt' du \\
&= \int \int \int e^{i(t'-t)\cdot Su} du\, f(t)\, \overline{f(t')} dt dt' \\
&= \int \int \delta(t'-t)\, f(t)\, \overline{f(t')} dt dt' \\
&= \int \int \delta(t'-t)\, \overline{f(t')} dt' f(t)\, dt \\
&= \int \overline{f(t)} f(t)\, dt \\
&= \int |f(t)|^2\, dt \qquad (7.2.2.2)
\end{aligned}$$

We can combine (7.2.2.1) and (7.2.2.2) to find

$$\int \left| |u|^{2r} Ff(u) \right|^2 du = \int |F(\Delta^r f)(u)|^2\, du = \int |\Delta^r f(t)|^2\, dt \qquad (7.2.2.3)$$

Next, the Cauchy-Schwartz inequality implies that for any $m > \frac{1}{2}$

$$\begin{aligned}
\|f\|_\infty &= \sup_{u\in\mathbb{R}^2} |f(u)| \\
&\leq \int |Ff(u)|\, du \\
&= \int \left(1+|u|^2\right)^m |Ff(u)| \left(1+|u|^2\right)^{-m} du \\
&\leq \left| \int \left(1+|u|^2\right)^m |Ff(u)| \left(1+|u|^2\right)^{-m} du \right| \\
&\leq \left(\int \left| \left(1+|u|^2\right)^m |Ff(u)| \right|^2 du \int \left| \left(1+|u|^2\right)^{-m} \right|^2 du \right)^{\frac{1}{2}} \\
&= K \left(\int \left(1+|u|^2\right)^{2m} |Ff(u)|^2\, du \right)^{\frac{1}{2}} \qquad (7.2.2.4)
\end{aligned}$$

where $K = \left(\int \frac{du}{(1+|u|^2)^{2m}} \right)^{\frac{1}{2}} < \infty$. As a final ingredient in our proof, consider the function

$$\frac{(1+s)^n}{(1+s^n)} \qquad \text{where} \quad s \geq 0$$

It attains a maximum value of 2^{n-1}. Therefore, it holds that

$$(1+s)^{2m} \leq 2^{2m-1}\left(1+s^{2m}\right)$$

Since $|u|^2 \geq 0$, the above inequality implies that

$$\left(1+|u|^2\right)^{2m} \leq 2^{2m-1}\left(1+|u|^{4m}\right) \tag{7.2.2.5}$$

Finally, in (7.2.2.4), we use (7.2.2.5), (7.2.2.2), and (7.2.2.3) to find

$$\begin{aligned}
\|f\|_\infty^2 &\leq K^2 \int \left(1+|u|^2\right)^{2m} |Ff(u)|^2 \, du \\
&\leq K^2 \int 2^{2m-1}\left(1+|u|^{4m}\right) |Ff(u)|^2 \, du \\
&= K^2 2^{2m-1} \left(\int |Ff(u)|^2 \, du + \int |u|^{4m} |Ff(u)|^2 \, du \right) \\
&= K^2 2^{2m-1} \left(\int |f(t)|^2 \, dt + \int \left| |u|^{2m} Ff(u) \right|^2 du \right) \\
&= K^2 2^{2m-1} \left(\int |f(t)|^2 \, dt + \int |\Delta^m f(t)|^2 \, dt \right) \\
&= K^2 2^{2m} \left(\|f\|_2^2 + \|\Delta^m f\|_2^2 \right) \\
&\leq K^2 2^{2m} \Big(\|f\|_2 + \|\Delta^m f\|_2 \Big)^2
\end{aligned}$$

The required result follows:

$$\|f\|_\infty \leq K 2^m \Big(\|f\|_2 + \|\Delta^m f\|_2 \Big)$$

□

Proposition 7.2.3. *For any multi-indices $\alpha, \beta \in \mathbb{N}_0^2$, there exists some $k \in \mathbb{N}_0$ such that*

$$\left\| \mu^\alpha \partial^\beta f \right\|_2 \leq \left\| W^k f \right\|_2$$

for all $f \in S$.

Proof. Let $f \in S$ and $\alpha, \beta \in \mathbb{N}_0^2$ be arbitrary. We can expand f in terms of the basis $\{f_{mn}\}_{m,n \in \mathbb{N}_0}$ as

$$f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn}$$

Let $g := \mu^\alpha \partial^\beta f$. Then $g \in S$, which means we can also expand g in terms of the basis as

$$g = \sum_{m,n=0}^{\infty} d_{mn} f_{mn}$$

Now, define

$$k_{mn} := 2 \left(\frac{\ln|d_{mn}| - \ln|c_{mn}|}{\ln(2m+1) + \ln(2n+1)} \right)$$

for all $m, n \in \mathbb{N}_0$. Choose

$$k = \min \{k_i \in \mathbb{N}_0 : k_i \geq k_{mn} \quad \forall m, n \in \mathbb{N}_0\}$$

It is easy to show that $|d_{mn}|^2 \leq |c_{mn}|^2 (2m+1)^k (2n+1)^k$ for all $m, n \in \mathbb{N}_0$. We obtain the required result as follows:

$$\begin{aligned}
\left\|\mu^\alpha \partial^\beta f\right\|_2^2 &= \|g\|_2^2 \\
&= \left\|\sum_{m,n=0}^{\infty} d_{mn} f_{mn}\right\|_2^2 \\
&= \sum_{m,n=0}^{\infty} |d_{mn}|^2 \\
&\leq \sum_{m,n=0}^{\infty} |c_{mn}|^2 (2m+1)^k (2n+1)^k \\
&\leq \sum_{m,n=0}^{\infty} |c_{mn}|^2 (2m+1)^{2k} (2n+1)^{2k} \\
&= \left\|W^k \sum_{m,n=0}^{\infty} c_{mn} f_{mn}\right\|_2^2 \\
&= \left\|W^k f\right\|_2^2
\end{aligned}$$

□

Proposition 7.2.4. *For any $k \in \mathbb{N}_0$, there exist complex numbers $C_{\alpha,\beta}$ with $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$ such that*

$$W^k f = \sum_{\alpha,\beta \in \mathbb{N}_0^2} C_{\alpha,\beta} x^\alpha \partial^\beta f$$

for all $f \in S$, where $C_{\alpha,\beta} \neq 0$ is possible only if $\alpha_1, \alpha_2, \beta_1, \beta_2 \leq 4k$. Note that the $C_{\alpha,\beta}$ depend on k but not on f.

Proof. We prove this by induction. Let $f \in S$ and consider the case $k = 1$. First, Propositions 6.3.1 and 6.3.2 give

$$\begin{aligned}
G \star f &= (\bar{z} \star z + \mathbf{1}) \star f \\
&= \bar{z} \star \left(zf + \bar{\partial} f\right) + f \\
&= \bar{z} z f - z \partial f + \bar{z} \bar{\partial} f - \partial \bar{\partial} f \\
&= \frac{1}{2}\left(x_1^2 + x_2^2 + 2i x_1 \partial_2 - 2i x_2 \partial_1 - \partial_1^2 - \partial_2^2\right) f
\end{aligned}$$

Similarly,

$$\begin{aligned} f \star G &= f \star (\overline{z} \star z + \mathbb{1}) \\ &= \frac{1}{2}\left(x_1^2 + x_2^2 - 2ix_1\partial_2 + 2ix_2\partial_1 - \partial_1^2 - \partial_2^2\right) f \end{aligned}$$

Then

$$\begin{aligned} Wf &= G \star f \star G \\ &= \frac{1}{4}\left(x_1^4 + x_2^4 + \partial_1^4 + \partial_2^4\right) f \\ &\quad + \frac{1}{2}\left(x_1^2 x_2^2 + x_1^2\partial_2^2 - x_1^2\partial_1^2 + x_2^2\partial_1^2 - x_2^2\partial_2^2 + \partial_1^2\partial_2^2\right) f \\ &\quad - 2\left(x_1\partial_1 + x_1x_2\partial_1\partial_2 + x_2\partial_2\right) f - f \end{aligned}$$

This proves the case $k = 1$. Now suppose the result holds for arbitrary $k \in \mathbb{N}_0$. Then replacing f in the expression above by $W^k f$ shows that the result holds for $k+1$. □

Proposition 7.2.5. *The topology on S generated by the family of seminorms $\{|||\cdot|||_k\}_{k\in\mathbb{N}_0}$ coincides with the Schwartz topology.*

Proof. Let $f \in S$ be arbitrary. Recall that the Schwartz topology is generated by the family of seminorms $\{p_{\alpha\gamma}\}_{\alpha,\gamma\in\mathbb{N}_0^2}$ as per (4.2.9.1). To prove the equivalence of the topologies, we have to show that each $p_{\alpha\gamma}$ is bounded above by a linear combination of elements of the family $\{|||\cdot|||_k\}_{k\in\mathbb{N}_0}$, and that each $|||\cdot|||_k$ is bounded above by a linear combination of elements of the family $\{p_{\alpha\gamma}\}_{\alpha,\gamma\in\mathbb{N}_0^2}$.

We start by proving the first bound. Let $\alpha, \gamma \in \mathbb{N}_0^2$ be arbitrary multi-indices. Note that, for $m \in \mathbb{N}$, we can write

$$\Delta^m f = \left(\partial_1^2 + \partial_2^2\right)^m f = \sum_{n=0}^{m} \binom{m}{n} \partial_1^{2(m-n)}\partial_2^{2n} f$$

Since $f \in S$, we know that $\mu^\alpha\partial^\gamma f \in S$, so that Proposition 7.2.2 holds for $\mu^\alpha\partial^\gamma f \in S$. We apply Proposition 7.2.2, together with the expression above, to the definition for $p_{\alpha\gamma}$:

$$\begin{aligned} p_{\alpha\gamma}(f) &= \|\mu^\alpha\partial^\gamma f\|_\infty \\ &\le K2^m\Big(\|\mu^\alpha\partial^\gamma f\|_2 + \|\Delta^m\mu^\alpha\partial^\gamma f\|_2\Big) \\ &= K2^m\left(\|\mu^\alpha\partial^\gamma f\|_2 + \left\|\sum_{n=0}^{m}\binom{m}{n}\partial_1^{2(m-n)}\partial_2^{2n}\mu^\alpha\partial^\gamma f\right\|_2\right) \\ &\le K2^m\left(\|\mu^\alpha\partial^\gamma f\|_2 + \sum_{n=0}^{m}\binom{m}{n}\left\|\partial_1^{2(m-n)}\partial_2^{2n}\mu^\alpha\partial^\gamma f\right\|_2\right) \end{aligned} \tag{7.2.5.1}$$

Let $\beta = (\beta_1, \beta_2) = (2(m-n), 2n) \in \mathbb{N}_0^2$. Also, let C_β be constants such that $C_\beta = \binom{m}{n}\binom{\alpha}{\beta}\beta!$. This makes C_β greater or equal to all the constants that come out of the second term in (7.2.5.1). Then (7.2.5.1) becomes, after using the triangle inequality,

$$\begin{aligned} p_{\alpha\gamma}(f) \leq K2^{|\beta|} \|\mu^\alpha \partial^\gamma f\|_2 \\ + K2^{|\beta|} \sum_{\beta/2=0}^{|\beta|/2} C_\beta \left(\left\| \mu^{\alpha-\beta} \partial^\gamma f \right\|_2 + \left\| \mu^{\alpha-(0,\beta_2)} \partial^{\gamma+(\beta_1,0)} f \right\|_2 \right) \\ + K2^{|\beta|} \sum_{\beta/2=0}^{|\beta|/2} C_\beta \left(\left\| \mu^{\alpha-(\beta_1,0)} \partial^{\gamma+(0,\beta_2)} f \right\|_2 + \left\| \mu^{\alpha} \partial^{\gamma+\beta} f \right\|_2 \right) \end{aligned} \tag{7.2.5.2}$$

We can apply Proposition 7.2.3 to each of these five terms. In fact, we can apply it simultaneously to the linear combination of the last four terms. We incorporate the constants C_β into a new constant by setting $C_\ell = \sup\{C_\beta\}$ for any nonzero $\ell \in \mathbb{N}_0$. There exists some $k \in \mathbb{N}_0$ (first term) and constants C_ℓ that are nonzero for finitely many $\ell \in \mathbb{N}_0$ (last four terms) such that

$$\begin{aligned} p_{\alpha\gamma}(f) &\leq K2^{|\beta|} \left\| W^k f \right\|_2 + K2^{|\beta|} \sum_{\ell \in \mathbb{N}_0} C_\ell \left\| W^\ell f \right\|_2 \\ &= K2^{|\beta|} \left(\left\| W^k f \right\|_2 + \sum_{\ell \in \mathbb{N}_0} C_\ell \left\| W^\ell f \right\|_2 \right) \\ &= K2^{|\beta|} \left(|||f|||_k + \sum_{\ell \in \mathbb{N}_0} C_\ell |||f|||_\ell \right) \end{aligned} \tag{7.2.5.3}$$

Since this holds for arbitrary $\alpha, \gamma \in \mathbb{N}_0^2$, we have shown that each $p_{\alpha\gamma}$ is bounded above by a linear combination of elements of the family $\{|||\cdot|||_k\}_{k \in \mathbb{N}_0}$.

Now we prove the reverse inequality. First note that since $f \in S$, $f(x) \to 0$ as $|x| \to \infty$. In fact, $f(x) \to 0$ faster than the inverse of any polynomial. Specifically,

$$\int \left(1 + |x|^8\right) |f(x)|^4 \, dx \leq \sup_{x \in \mathbb{R}^2} \left| \left(1 + |x|^8\right) |f(x)|^4 \right| \tag{7.2.5.4}$$

Recall that (7.2.2.5), which holds for $m = 2$ in particular, gives

$$\left(1 + |u|^2\right)^4 \leq 2^3 \left(1 + |u|^8\right) \tag{7.2.5.5}$$

Next, by applying the Cauchy-Swartz inequality, then (7.2.5.4) and (7.2.5.5), we find

$$
\begin{aligned}
2\left\|f\right\|_2^2 &= \int \left|f\left(x\right)\right|^2 dx \\
&= \int \left(1+\left|x\right|^2\right)^{-2}\left(1+\left|x\right|^2\right)^2 \left|f\left(x\right)\right|^2 dx \\
&\leq \left|\int \left(1+\left|x\right|^2\right)^{-2}\left(1+\left|x\right|^2\right)^2 \left|f\left(x\right)\right|^2 dx\right| \\
&\leq \left(\int \left|\left(1+\left|x\right|^2\right)^{-2}\right|^2 dx \int \left|\left(1+\left|x\right|^2\right)^2 \left|f\left(x\right)\right|^2\right|^2 dx\right)^{\frac{1}{2}} \\
&= \left(\int \left(1+\left|x\right|^2\right)^{-4} dx\right)^{\frac{1}{2}} \left(\int \left(1+\left|x\right|^2\right)^4 \left|f\left(x\right)\right|^4 dx\right)^{\frac{1}{2}} \\
&= \sqrt{\frac{\pi}{3}}\left(\int \left(1+\left|x\right|^2\right)^4 \left|f\left(x\right)\right|^4 dx\right)^{\frac{1}{2}} \\
&\leq \sqrt{\frac{\pi}{3}}\left(\int 2^3\left(1+\left|x\right|^8\right) \left|f\left(x\right)\right|^4 dx\right)^{\frac{1}{2}} \\
&\leq \sqrt{\frac{8\pi}{3}}\left(\sup_{x\in\mathbb{R}^2}\left|\left(1+\left|x\right|^8\right) \left|f\left(x\right)\right|^4\right|\right)^{\frac{1}{2}} \\
&= \sqrt{\frac{8\pi}{3}}\left(\left\|\left(1+\left|x\right|^8\right) \left|f\left(x\right)\right|^4\right\|_\infty\right)^{\frac{1}{2}} \\
&\leq \sqrt{\frac{8\pi}{3}}\left(\left\|f\right\|_\infty^4 + \left\|\left|x\right|^8 \left|f\left(x\right)\right|^4\right\|_\infty\right)^{\frac{1}{2}} \\
&= \sqrt{\frac{8\pi}{3}}\left(\left\|f\right\|_\infty^4 + \left\|\left|x\right|^2 f\left(x\right)\right\|_\infty^4\right)^{\frac{1}{2}} \\
&\leq \sqrt{\frac{8\pi}{3}}\left(\left\|f\right\|_\infty + \left\|\left|x\right|^2 f\left(x\right)\right\|_\infty\right)^2
\end{aligned}
$$

Thus, we have

$$
\left\|f\right\|_2 \leq \left(\frac{8\pi}{3}\right)^{\frac{1}{4}}\left(\left\|f\right\|_\infty + \left\|\left|x\right|^2 f\left(x\right)\right\|_\infty\right) \tag{7.2.5.6}
$$

This already proves the needed inequality for $k=0$, since

$$
\begin{aligned}
\left\|\left|f\right|\right\|_0 &= \left\|f\right\|_2 \\
&\leq \left(\frac{8\pi}{3}\right)^{\frac{1}{4}}\left(\left\|f\right\|_\infty + \left\|\left|x\right|^2 f\left(x\right)\right\|_\infty\right) \\
&\leq \left(\frac{8\pi}{3}\right)^{\frac{1}{4}}\left(p_{00}\left(f\right) + p_{\delta 0}\left(f\right) + p_{\epsilon 0}\left(f\right)\right)
\end{aligned} \tag{7.2.5.7}
$$

where $\delta = (\delta_1, \delta_2) = (2,0)$ and $\epsilon = (\epsilon_1, \epsilon_2) = (0,2)$ in $\mathbb{N}_0^2$. This shows that $|||\cdot|||_0$ is bounded by a linear combination of elements in $\{p_{\alpha\gamma}\}_{\alpha,\gamma\in\mathbb{N}_0^2}$.

For the remainder of the proof consider an arbitrary $k \in \mathbb{N}$. Choose $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$ such that $\alpha_1, \alpha_2, \beta_1, \beta_2 \le 4k$. Then we know from Proposition 7.2.4 that there exist constants $C_{\alpha,\beta}$ such that

$$W^k f = \sum_{\alpha,\beta\in\mathbb{N}_0^2} C_{\alpha,\beta} x^\alpha \partial^\beta f$$

Also, let $\delta = (\delta_1, \delta_2) = (2,0)$ and $\epsilon = (\epsilon_1, \epsilon_2) = (0,2)$ be multi-indices in $\mathbb{N}_0^2$. Using the expression above, together with (7.2.5.6), and noting that $|x|^2 = x_1^2 + x_2^2$, we find that

$$\begin{aligned}
|||f|||_k &= \left\| W^k f \right\|_2 \\
&= \left\| \sum_{\alpha,\beta\in\mathbb{N}_0^2} C_{\alpha,\beta} x^\alpha \partial^\beta f \right\|_2 \\
&\le \sum_{\alpha,\beta\in\mathbb{N}_0^2} |C_{\alpha,\beta}|^2 \left\| x^\alpha \partial^\beta f \right\|_2 \\
&\le \sum_{\alpha,\beta\in\mathbb{N}_0^2} |C_{\alpha,\beta}|^2 \left(\frac{8\pi}{3}\right)^{\frac{1}{4}} \left(\left\| x^\alpha \partial^\beta f \right\|_\infty + \left\| |x|^2 x^\alpha \partial^\beta f \right\|_\infty \right) \\
&= \sum_{\alpha,\beta\in\mathbb{N}_0^2} |C_{\alpha,\beta}|^2 \left(\frac{8\pi}{3}\right)^{\frac{1}{4}} \left(\left\| x^\alpha \partial^\beta f \right\|_\infty + \left\| x_1^2 x^\alpha \partial^\beta f + x_2^2 x^\alpha \partial^\beta f \right\|_\infty \right) \\
&\le \sum_{\alpha,\beta\in\mathbb{N}_0^2} |C_{\alpha,\beta}|^2 \left(\frac{8\pi}{3}\right)^{\frac{1}{4}} \left(\left\| x^\alpha \partial^\beta f \right\|_\infty + \left\| x^{\alpha+\delta} \partial^\beta f \right\|_\infty + \left\| x^{\alpha+\epsilon} \partial^\beta f \right\|_\infty \right) \\
&= \sum_{\alpha,\beta\in\mathbb{N}_0^2} |C_{\alpha,\beta}|^2 \left(\frac{8\pi}{3}\right)^{\frac{1}{4}} \left(p_{\alpha\beta}(f) + p_{\alpha+\delta,\beta}(f) + p_{\alpha+\epsilon,\beta}(f) \right)
\end{aligned} \tag{7.2.5.8}$$

Together, (7.2.5.7) and (7.2.5.8) prove that each $|||\cdot|||_k$ is bounded above by a linear combination of elements of the family $\{p_{\alpha\gamma}\}_{\alpha,\gamma\in\mathbb{N}_0^2}$.

Having proven inequalities both ways, we see that the topology generated by $\{|||\cdot|||_k\}_{k\in\mathbb{N}_0}$ is contained in the topology generated by $\{p_{\alpha\gamma}\}_{\alpha,\gamma\in\mathbb{N}_0^2}$ and vice versa. Hence, the topology on S generated by the family of seminorms $\{|||\cdot|||_k\}_{k\in\mathbb{N}_0}$ coincides with the Schwartz topology. □

7.3 The sequence space

Now we proceed to represent S as a sequence space. Define the Fréchet space $\overline{s}$ of rapidly decreasing double sequences c by

$$\overline{s} := \left\{ c = (c_{mn})_{m,n=0}^{\infty} : r_k(c) < \infty \text{ for every } k \in \mathbb{N}_0 \right\} \tag{7.3}$$

where

$$r_k(c) := \left(\sum_{m,n=0}^{\infty} (2m+1)^{2k} (2n+1)^{2k} |c_{mn}|^2 \right)^{\frac{1}{2}} \tag{7.4}$$

for each $k \in \mathbb{N}_0$. The topology for $\overline{s}$ is generated by the family of seminorms $\{r_k\}_{k\in\mathbb{N}_0}$.

We want to show that the spaces S and $\overline{s}$ are topologically isomorphic. In order to do so, we need to define a homeomorphism between the spaces, that is, a bijective correspondence that preserves the topological structure involved. We will make use of the fact, as proved in Chapter 6, that every $f \in S$ can be expanded as a linear combination of the elements of the basis $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$, i.e. there exist constants c_{mn} such that $f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn}$.

For every $f \in S$, we let $c \in \overline{s}$ be the sequence of coefficients $(c_{mn})_{m,n=0}^{\infty}$ in the basis expansion. To make this explicit, we define the map

$$I_S : S \to \overline{s} : f \mapsto c$$

such that

$$I_S(f) = I_S\left(\sum_{m,n=0}^{\infty} c_{mn} f_{mn} \right) := (c_{mn})_{m,n=0}^{\infty} = c \tag{7.5}$$

for all $f \in S$.

Proposition 7.3.1. *$I_S : S \to \overline{s} : f \mapsto c$ is a homeomorphism from S onto $\overline{s}$; in other words, S and $\overline{s}$ are topologically isomorphic.*

Proof. In Proposition 7.2.5, we proved that the seminorms $\{|||\cdot|||_k\}_{k\in\mathbb{N}_0}$ equivalently generate the Schwartz topology on S, which is generated by the seminorms $\{p_{\alpha\gamma}\}_{\alpha,\gamma\in\mathbb{N}_0^2}$. This means that

$$f \in S \quad \text{iff} \quad p_{\alpha\gamma}(f) < \infty \quad \forall \alpha, \gamma \in \mathbb{N}_0^2$$

or equivalently

$$f \in S \quad \text{iff} \quad |||f|||_k = \left\| W^k f \right\|_2 < \infty \quad \forall k \in \mathbb{N}_0 \tag{7.3.1.1}$$

Using the symmetry of W from Proposition 7.2.1, we find that for all $f \in S$ and every $k \in \mathbb{N}_0$

$$\begin{aligned}
\left\|W^k f\right\|_2^2 &= \left\|W^k \sum_{m,n=0}^{\infty} c_{mn} f_{mn}\right\|_2^2 \\
&= \sum_{m,n=0}^{\infty} |c_{mn}|^2 \left(W^k f_{mn} | W^k f_{mn}\right) \\
&= \sum_{m,n=0}^{\infty} (2m+1)^{2k} (2n+1)^{2k} |c_{mn}|^2 (f_{mn}|f_{mn}) \\
&= \sum_{m,n=0}^{\infty} (2m+1)^{2k} (2n+1)^{2k} |c_{mn}|^2 \\
&= r_k(c)^2
\end{aligned}$$

Hence,

$$|||f|||_k = \left\|W^k f\right\|_2 = r_k(c) \tag{7.3.1.2}$$

Let $f \in S$. Then $I_S(f) = c$ such that, from (7.3.1.1) and (7.3.1.2), $r_k(c) = |||f|||_k < \infty$ for all $k \in \mathbb{N}_0$. It follows from the definition (7.3) for $\overline{s}$ that $c \in \overline{s}$. Hence, I_S maps S into $\overline{s}$. Since the seminorms $|||\cdot|||_k$ are in fact norms ($|||f|||_k = 0 \implies f = 0$), the equality in (7.3.1.2) shows that I_S is injective.

To show surjectivity, consider an arbitrary $c \in \overline{s}$. So c is the double sequence $c = (c_{mn})_{m,n=0}^{\infty}$ such that $r_k(c) < \infty$ for every $k \in \mathbb{N}_0$. Now, define double sequences $c^{MN} \in \overline{s}$ for all $M, N \in \mathbb{N}_0$ by

$$c^{MN} := \left(c_{mn}^{MN}\right)_{m,n=0}^{\infty}$$

where the elements are given by

$$c_{mn}^{MN} := \begin{cases} c_{mn} & \text{if } m \le M \text{ and } n \le N \\ 0 & \text{otherwise} \end{cases}$$

Clearly, for each $k \in \mathbb{N}_0$,

$$r_k\left(c^{MN} - c\right) \to 0 \quad \text{as} \quad M, N \to \infty$$

In other words, the sequence $\left(c^{MN}\right)_{M,N\in\mathbb{N}_0}$ of partial sequences in $\overline{s}$ converges to $c \in \overline{s}$. Thus, $\left(c^{MN}\right)_{M,N\in\mathbb{N}_0}$ is Cauchy in $\overline{s}$, i.e. for $M_1, N_1, M_2, N_2 \in \mathbb{N}_0$,

$$r_k\left(c^{M_1N_1} - c^{M_2N_2}\right) \to 0 \quad \text{as} \quad M_1, N_1, M_2, N_2 \to \infty \tag{7.3.1.3}$$

Furthermore, consider the sequence $\left(f^{MN}\right)_{M,N\in\mathbb{N}_0}$ in S, where the functions $f^{MN} \in S$ are defined, for all $M, N \in \mathbb{N}_0$, by

$$f^{MN} := \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn} f_{mn}$$

Then, for all $M, N \in \mathbb{N}_0$

$$I_S\left(f^{MN}\right) = \left(c_{mn}\right)_{m\leq M, n\leq N} = \left(c_{mn}^{MN}\right)_{m,n=0}^{\infty} = c^{MN}$$

It follows from the fact that $\left(c^{MN}\right)_{M,N\in\mathbb{N}_0}$ is Cauchy in $\overline{s}$, as per (7.3.1.3), and from (7.3.1.2), that for $M_1, N_1, M_2, N_2 \in \mathbb{N}_0$, and for each $k \in \mathbb{N}_0$,

$$\begin{aligned}
\left|\!\left|\!\left| f^{M_1N_1} - f^{M_2N_2} \right|\!\right|\!\right|_k &= \left\| W^k\left(f^{M_1N_1} - f^{M_2N_2}\right)\right\|_2 \\
&= r_k\Big(I_S\left(f^{M_1N_1}\right) - I_S\left(f^{M_2N_2}\right)\Big) \\
&= r_k\left(c^{M_1N_1} - c^{M_2N_2}\right) \\
&\to 0 \quad \text{as} \quad M_1, N_1, M_2, N_2 \to \infty
\end{aligned}$$

Thus, the sequence $\left(f^{MN}\right)_{M,N\in\mathbb{N}_0}$ is Cauchy in S. Since S is complete by Theorem 4.1.9, $\left(f^{MN}\right)_{M,N\in\mathbb{N}_0}$ converges to some $f \in S$. Let us denote the limit by $f := \lim_{M,N\to\infty} f^{MN}$. Then

$$\begin{aligned}
I_S(f) &= I_S\left(\lim_{M,N\to\infty} f^{MN}\right) \\
&= \lim_{M,N\to\infty} I_S\left(f^{MN}\right) \\
&= \lim_{M,N\to\infty} c^{MN} \\
&= c
\end{aligned}$$

Since we considered an arbitrary $c \in \overline{s}$, it follows that for every $c \in \overline{s}$, there exists an $f \in S$ such that $I_S(f) = c$. Hence, I_S is surjective.

To conclude, I_S is a bijection between S and $\overline{s}$. Moreover, (7.3.1.2) shows the equivalence of the topologies; I_S preserves the topological structure. Hence, I_S is a homeomorphism or, equivalently, S and $\overline{s}$ are topologically isomorphic. □

7.4 Matricial form of $\star$ and its extension to $L^2\left(\mathbb{R}^2\right)$

The sequence representation of S in the previous section suggests that we can define a matricial form of the Moyal product. Specifically, we will show that the Moyal product between functions in S corresponds to the matrix product between the corresponding sequences (which can be viewed as infinite-dimensional matrices) in $\overline{s}$. This matricial form provides a simple way to extend the Moyal product to spaces larger than S. In particular, we find its extension to $L^2(\mathbb{R}^2)$.

Proposition 7.4.1. *If $c, b \in \overline{s}$ correspond respectively to $f, g \in S$ as coefficient sequences in the basis $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$, i.e.*

$$f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn} \qquad\qquad g = \sum_{k,\ell=0}^{\infty} b_{k\ell} f_{k\ell}$$

then the sequence corresponding to the Moyal (twisted) product $f \star g \in S$ is the matrix product $cb \in \overline{s}$, where

$$(cb)_{m\ell} = \sum_{n=0}^{\infty} c_{mn} b_{n\ell}$$

Proof. Using Proposition 6.3.4, together with the continuity of $\star$ in S, we find

$$\begin{aligned} f \star g &= \left(\sum_{m,n=0}^{\infty} c_{mn} f_{mn} \right) \star \left(\sum_{k,\ell=0}^{\infty} b_{k\ell} f_{k\ell} \right) \\ &= \sum_{m,n,k,\ell=0}^{\infty} c_{mn} b_{k\ell} f_{mn} \star f_{k\ell} \\ &= \sum_{m,n,k,\ell=0}^{\infty} c_{mn} b_{k\ell} \delta_{nk} f_{m\ell} \\ &= \sum_{m,n,\ell=0}^{\infty} c_{mn} b_{n\ell} f_{m\ell} \end{aligned}$$

Since $f \star g \in S$, it can be expressed in terms of the basis. So the expression above implies that

$$f \star g = \sum_{m,\ell=0}^{\infty} (cb)_{m\ell} f_{m\ell}$$

with the coefficients given by

$$(cb)_{m\ell} = \sum_{n=0}^{\infty} c_{mn} b_{n\ell}$$

Using the Cauchy-Schwartz inequality, we see that for all $k \in \mathbb{N}_0$

$$\begin{aligned}
r_k(cb)^2 &= \sum_{m,\ell=0}^{\infty} (2m+1)^{2k} (2\ell+1)^{2k} \left|(cb)_{m\ell}\right|^2 \\
&= \sum_{m,\ell=0}^{\infty} (2m+1)^{2k} (2\ell+1)^{2k} \left|\sum_{n=0}^{\infty} c_{mn} b_{n\ell}\right|^2 \\
&\leq \sum_{m,\ell=0}^{\infty} (2m+1)^{2k} (2\ell+1)^{2k} \sum_{r=0}^{\infty} |c_{mr}|^2 \sum_{s=0}^{\infty} |b_{\ell s}|^2 \\
&\leq \sum_{m,r=0}^{\infty} (2m+1)^{2k} (2r+1)^{2k} |c_{mr}|^2 \sum_{\ell,s=0}^{\infty} (2\ell+1)^{2k} (2s+1)^{2k} |b_{\ell s}|^2 \\
&= r_k(c)^2 \, r_k(b)^2
\end{aligned}$$

Since $c, b \in \overline{s}$ with finite seminorms, the expression above implies that

$$r_k(cb) \leq r_k(c) \, r_k(b) < \infty$$

Hence, $cb \in \overline{s}$. □

The proposition above suggests that we can extend the topological correspondence between S and $\overline{s}$ to an algebraic correspondence. Recall that $A = (S, \star)$ (see (4.2.18) and Corollary 4.2.12). Define the map η on A such that

$$\eta(f) := c \tag{7.6}$$

for all $f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn} \in A$.

Proposition 7.4.2. *A and $\eta(A)$ are algebraically $*$-isomorphic:*

$$A \simeq \eta(A)$$

where $\eta(A)$ is the algebra obtained by equipping $\overline{s}$ with matrix multiplication and an involution given by complex transposition.

Proof. Let $f, g \in A$ with basis expansions

$$f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn} \quad \text{and} \quad g = \sum_{m,n=0}^{\infty} b_{mn} f_{mn}$$

Note that η, considered as a map on the underlying topological vector space of A, is exactly $I_S : S \to \overline{s}$, which is a homeomorphism by Proposition 7.3.1. It is also clear that η is linear. Furthermore, it follows from Proposition 7.4.1 that the $\star$-product in A corresponds to the matrix product in $\overline{s}$ such that

$$\eta(f \star g) = cb = \eta(f)\, \eta(g)$$

Also, complex conjugation of a function in A corresponds to conjugate transposition in $\overline{s}$:

$$\eta\left(f^*\right)=\eta\left(\sum_{m,n=0}^{\infty}\overline{c_{mn}}f_{nm}\right)=c^*=\eta\left(f\right)^*$$

by Proposition 6.3.5. Thus, η is a bijective $*$-homomorphism from A to $\eta(A)$, where $\eta(A)$ is the algebra $\overline{s}$ equipped with matrix multiplication and an involution given by complex transposition. Hence, η is a $*$-isomorphism and $A \simeq \eta(A)$. $\square$

The matricial form of $\star$ in Proposition 7.4.1 gives a way of defining the $\star$-product between elements of $L^2\left(\mathbb{R}^2\right)$. Remember that any $g \in L^2\left(\mathbb{R}^2\right)$ can be expanded in terms of the basis $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ such that $g=\sum_{m,n=0}^{\infty} b_{mn}f_{mn}$. The norm on $L^2\left(\mathbb{R}^2\right)$, as defined in (4.2.12), can be written in terms of the coefficients of the basis expansion as

$$\|g\|_2=(g|g)^{\frac{1}{2}}=\left(\sum_{m,n=0}^{\infty}|b_{mn}|^2\right)^{\frac{1}{2}} \tag{7.7}$$

such that $\|g\|_2<\infty$ for all $g\in L^2\left(\mathbb{R}^2\right)$. Since S is a subspace of $L^2\left(\mathbb{R}^2\right)$, the inclusion map $\iota : S \to L^2\left(\mathbb{R}^2\right)$, defined by $\iota(f) = f$ for all $f \in S$, is injective and $\iota(S) = S$ is dense in $L^2\left(\mathbb{R}^2\right)$ by Proposition 4.2.1. $L^2\left(\mathbb{R}^2\right)$ is the Hilbert completion of S with respect to the norm $\|\cdot\|_2$.

If $f, g \in L^2\left(\mathbb{R}^2\right)$, such that $f=\sum_{m,n=0}^{\infty} c_{mn}f_{mn}$ and $g=\sum_{m,n=0}^{\infty} b_{mn}f_{mn}$, then we define

$$f\star g := \sum_{m,n=0}^{\infty}\left(\sum_{k=0}^{\infty}c_{mk}b_{kn}\right)f_{mn} \tag{7.8}$$

Proposition 7.4.3. *For all $f, g \in L^2\left(\mathbb{R}^2\right)$, the series (7.8) converges in $L^2\left(\mathbb{R}^2\right)$ so that $f\star g\in L^2\left(\mathbb{R}^2\right)$ with*

$$\|f\star g\|_2\leq\|f\|_2\,\|g\|_2$$

This defines a map

$$\star : L^2\left(\mathbb{R}^2\right)\times L^2\left(\mathbb{R}^2\right)\to L^2\left(\mathbb{R}^2\right)$$

that extends the $\star$-product on S.

Proof. Let $f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn} \in L^2\left(\mathbb{R}^2\right)$ and $g = \sum_{m,n=0}^{\infty} b_{mn} f_{mn} \in L^2\left(\mathbb{R}^2\right)$. Using (7.8), together with the Cauchy-Schwartz inequality, we find that

$$\begin{aligned}
\|f \star g\|_2^2 &= \left\| \sum_{m,n=0}^{\infty} \left(\sum_{k=0}^{\infty} c_{mk} b_{kn} \right) f_{mn} \right\|_2^2 \\
&= \sum_{m,n=0}^{\infty} \left| \sum_{k=0}^{\infty} c_{mk} b_{kn} \right|^2 \\
&\leq \sum_{m,n=0}^{\infty} \left(\sum_{k=0}^{\infty} |c_{mk}| \, |b_{kn}| \right)^2 \\
&\leq \sum_{m,n=0}^{\infty} \left(\sum_{k=0}^{\infty} |c_{mk}|^2 \right) \left(\sum_{k=0}^{\infty} |b_{kn}|^2 \right) \\
&= \left(\sum_{m,k=0}^{\infty} |c_{mk}|^2 \right) \left(\sum_{k,n=0}^{\infty} |b_{kn}|^2 \right) \\
&= \|f\|_2^2 \, \|g\|_2^2
\end{aligned}$$

Since $\|f\|_2$ and $\|g\|_2$ are finite for $f, g \in L^2\left(\mathbb{R}^2\right)$, it follows that $\|f \star g\|_2$ is finite so that $f \star g \in L^2\left(\mathbb{R}^2\right)$. Thus (7.8) defines a map

$$\star : L^2\left(\mathbb{R}^2\right) \times L^2\left(\mathbb{R}^2\right) \to L^2\left(\mathbb{R}^2\right)$$

which, when restricted to $S \times S$, is exactly the $\star$-product on S as per Proposition 7.4.1. Hence, the $\star$-product on $L^2\left(\mathbb{R}^2\right)$ extends the $\star$-product on S. $\square$

In fact, $L^2\left(\mathbb{R}^2\right)$ equipped with the $\star$-product and complex conjugation is a Banach $*$-algebra, since it is the completion (with respect to a submultiplicative norm) of an associative $*$-algebra.

Chapter 8

Equivalent representations of A

In this chapter, we define equivalent representations of the algebra of the Moyal plane A. This will allow us to switch between the representation that will form part of the spectral triple of the Moyal plane and a representation that makes calculation of the spectral distance tractable. We start by assigning some notations: If H is a Hilbert space, let

- $\mathfrak{L}(H)$ denote the space of all linear operators on H
- $\mathfrak{B}(H)$ denote the space of all bounded linear operators on H
- $K(H)$ denote the space of all compact operators on H
- $HS(H)$ denote the space of all Hilbert-Schmidt operators on H.

Note, from [31, Section 2.4], that $\mathfrak{B}(H)$, $K(H)$, and $HS(H)$ are C^*-algebras when equipped with the operator norm. Moreover, we have the following inclusions

$$HS(H) \subset K(H) \subset \mathfrak{B}(H)$$

where $K(H)$ is dense in $\mathfrak{B}(H)$.

8.1 Representation as Hilbert-Schmidt operators

First, we recall a basic definition from [31, Section 2.4].

Definition 8.1.1. *If $\{e_k\}$ is an orthonormal basis for a Hilbert space H, then an operator $T \in \mathfrak{B}(H)$ is called* ***Hilbert-Schmidt*** *if*

$$\|T\|_{HS} = \left(\sum_{k=0}^{\infty} \|Te_k\|_H^2 \right)^{\frac{1}{2}} < \infty$$

where $\|\cdot\|_{HS}$ is called the Hilbert-Schmidt norm and $\|\cdot\|_H$ is the norm on H.

Let $\ell^2(\mathbb{N}_0)$ denote the Hilbert space of square-summable sequences of complex numbers

$$\ell^2(\mathbb{N}_0) := \{\psi = (\psi_n)_{n=0}^{\infty} : \psi_n \in \mathbb{C} \ \forall n \in \mathbb{N}_0 \ \text{ and } \ \|\psi\|_{\ell^2} < \infty\} \tag{8.1}$$

where the norm is defined by

$$\|\psi\|_{\ell^2} := \langle \psi, \psi \rangle_{\ell^2}^{\frac{1}{2}} = \left(\sum_{n\in\mathbb{N}_0} |\psi_n|^2 \right)^{\frac{1}{2}} \tag{8.2}$$

via the inner product $\langle \psi, \phi \rangle_{\ell^2} := \psi^* \phi = \sum_{n=0}^{\infty} \overline{\psi_n} \phi_n$, which is defined for all $\psi, \phi \in \ell^2 (\mathbb{N}_0)$. Furthermore, let $\{e_n\}_{n\in\mathbb{N}_0}$ denote the canonical orthonormal basis of $\ell^2 (\mathbb{N}_0)$, where each e_i is given by a sequence with a 1 in the i'th position and zeros elsewhere, i.e.

$$e_0 = (1, 0, 0, \cdots) \qquad e_1 = (0, 1, 0, \cdots) \qquad \text{etc.}$$

Throughout this chapter, we will represent the functions $f, g \in A$ as sequences $\eta (f) = c \in \eta (A)$ and $\eta (g) = b \in \eta (A)$ respectively, via the basis expansions

$$f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn} \qquad \text{and} \qquad g = \sum_{m,n=0}^{\infty} b_{mn} f_{mn}$$

as done in Proposition 7.4.2. Let us now return to the map $\eta : A \to \eta (A) : f \mapsto c$ defined in (7.6). This time we consider it as a map into the space of linear operators on $\ell^2 (\mathbb{N}_0)$:

$$\eta : A \to \mathfrak{L} \left(\ell^2 (\mathbb{N}_0)\right) \tag{8.3}$$

where the rapid decay sequences in $\eta (A)$ act on vectors in $\ell^2 (\mathbb{N}_0)$ by row by column multiplication. Explicitly, for every $f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn} \in A$, and each basis element $e_k \in \{e_n\}_{n\in\mathbb{N}_0}$ of $\ell^2 (\mathbb{N}_0)$, we have

$$\eta (f) e_k = c e_k = \sum_{m=0}^{\infty} c_{mk} e_m \tag{8.4}$$

The sum on the right is exactly the k'th column of the matrix $c = (c_{mn})_{m,n=0}^{\infty}$ and is clearly in $\ell^2 (\mathbb{N}_0)$. This implies that $\eta (f)$ determines a linear operator on $\ell^2 (\mathbb{N}_0)$ for every $f \in A$ and justifies our consideration of η as a map into $\mathfrak{L} \left(\ell^2 (\mathbb{N}_0)\right)$.

Proposition 8.1.2. *η maps A into the C^*-algebra of Hilbert-Schmidt operators on $\ell^2 (\mathbb{N}_0)$:*

$$\eta : A \to HS \left(\ell^2 (\mathbb{N}_0)\right)$$

Proof. Let $f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn} \in A$ be arbitrary. Then $c \in \overline{s}$ so that $r_k(c) < \infty$ for all $k \in \mathbb{N}_0$ (see Section 7.3). Using Definition 8.1.1 for the Hilbert-Schmidt norm and (8.2) for the ℓ^2-norm, we find

$$\begin{aligned}
\|\eta(f)\|_{HS}^2 &= \sum_{\ell=0}^{\infty} \|\eta(f) e_\ell\|_{\ell^2}^2 \\
&= \sum_{\ell=0}^{\infty} \|c e_\ell\|_{\ell^2}^2 \\
&= \sum_{\ell=0}^{\infty} \left\| \sum_{m=0}^{\infty} c_{m\ell} e_m \right\|_{\ell^2}^2 \\
&= \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} |c_{n\ell}|^2 \\
&= r_0(c)^2 \\
&< \infty
\end{aligned}$$

By Definition 8.1.1 for a Hilbert-Schmidt operator, $\eta(f) \in HS\left(\ell^2(\mathbb{N}_0)\right)$ for all $f \in A$, as required. □

Proposition 8.1.3. *η is a faithful representation of A on $\ell^2(\mathbb{N}_0)$.*

Proof. Proposition 8.1.2 proves that $\eta(f) \in HS\left(\ell^2(\mathbb{N}_0)\right)$ for all $f \in A$. Since $HS\left(\ell^2(\mathbb{N}_0)\right) \subset \mathfrak{B}\left(\ell^2(\mathbb{N}_0)\right)$, it follows that $\eta(f) \in \mathfrak{B}\left(\ell^2(\mathbb{N}_0)\right)$ for all $f \in A$. Therefore, η maps A into the C^*-algebra $\mathfrak{B}\left(\ell^2(\mathbb{N}_0)\right)$:

$$\eta : A \to \mathfrak{B}\left(\ell^2(\mathbb{N}_0)\right)$$

Moreover, $\eta : A \to \mathfrak{B}\left(\ell^2(\mathbb{N}_0)\right)$ is a $*$-homomorphism, since for all $f, g \in A$ and $\psi, \phi \in \ell^2(\mathbb{N}_0)$ we have

$$\begin{aligned}
\eta(f \star g)\psi &= (cb)\psi \\
&= c(b\psi) \\
&= c\eta(g)\psi \\
&= \eta(f)\eta(g)\psi
\end{aligned}$$

and

$$\begin{aligned}
\langle \psi, \eta(f^*)\phi \rangle_{\ell^2} &= \langle \psi, c^*\phi \rangle_{\ell^2} \\
&= \psi^* c^* \phi \\
&= (c\psi)^* \phi \\
&= \langle c\psi, \phi \rangle_{\ell^2} \\
&= \langle \eta(f)\psi, \phi \rangle_{\ell^2} \\
&= \langle \psi, \eta(f)^* \phi \rangle_{\ell^2}
\end{aligned}$$

Since η is injective, it follows that $\eta : A \to \mathfrak{B}\left(\ell^2(\mathbb{N}_0)\right)$ is a faithful representation of A on $\ell^2(\mathbb{N}_0)$. $\square$

Since $\ell^2(\mathbb{N}_0)$ is a Hilbert space with orthonormal basis $\{e_m\}_{m\in\mathbb{N}_0}$, it is well known that $\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)$ is a Hilbert space with orthonormal basis $\{e_m \otimes e_n\}_{m,n\in\mathbb{N}_0}$ when equipped with the inner product

$$\langle \psi_1 \otimes \phi_1, \psi_2 \otimes \phi_2\rangle_{\ell^2\otimes\ell^2} := \langle \psi_1, \psi_2\rangle_{\ell^2} \langle \phi_1, \phi_2\rangle_{\ell^2} \tag{8.5}$$

which is defined for all $\psi_1, \psi_2, \phi_1, \phi_2 \in \ell^2(\mathbb{N}_0)$. The induced norm is given by

$$\|\psi_1 \otimes \phi_1\|_{\ell^2\otimes\ell^2} := \|\psi_1\|_{\ell^2} \|\phi_1\|_{\ell^2} \tag{8.6}$$

Let I denote the identity operator on $\ell^2(\mathbb{N}_0)$. Then we define the map

$$\eta \otimes I : A \to \mathfrak{B}\left(\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)\right) \tag{8.7}$$

such that

$$(\eta \otimes I)(f) := \eta(f) \otimes I$$

for all $f \in A$ and

$$(\eta(f) \otimes I)(e_m \otimes e_n) := \eta(f)\, e_m \otimes I e_n \tag{8.8}$$

for each basis element $e_m \otimes e_n \in \{e_m \otimes e_n\}_{m,n\in\mathbb{N}_0}$ of $\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)$.

Proposition 8.1.4. *$\eta \otimes I$ is a faithful representation of A on $\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)$.*

Proof. Consider the map

$$\iota : \mathfrak{B}\left(\ell^2(\mathbb{N}_0)\right) \to \mathfrak{B}\left(\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)\right) : T \mapsto T \otimes I$$

ι is clearly a $*$-homomorphism and is injective because if $\iota(T) = T \otimes I = 0$, then

$$0 = (T \otimes I)(\psi_1 \otimes \phi_1) = (T\psi_1) \otimes \phi_1$$

for all $\psi_1, \phi_1 \in \ell^2(\mathbb{N}_0)$, which implies

$$0 = \langle (T\psi_1) \otimes \phi_1, \psi_2 \otimes \phi_1\rangle_{\ell^2\otimes\ell^2} = \langle T\psi_1, \psi_2\rangle_{\ell^2} \|\phi_1\|_{\ell^2}$$

for all $\psi_1, \psi_2, \phi_1 \in \ell^2(\mathbb{N}_0)$. It follows that $0 = \langle T\psi_1, \psi_2\rangle_{\ell^2}$ for all $\psi_1, \psi_2 \in \ell^2(\mathbb{N}_0)$; therefore, $T = 0$.

Since $\eta : A \to \mathfrak{B}\left(\ell^2(\mathbb{N}_0)\right)$ is an injective $*$-homomorphism by Proposition 8.1.3, the composition $\eta \otimes I = \iota \circ \eta$ is an injective $*$-homomorphism. Hence, $\eta \otimes I$ is a faithful representation of A on $\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)$. $\square$

8.2 Representation via the Moyal product

Let $L : A \to \mathfrak{L}\left(L^2\left(\mathbb{R}^2\right)\right)$ be the left multiplication operator defined by

$$L(f)\,g := f \star g \qquad \text{for all} \quad f \in A, \quad g \in L^2\left(\mathbb{R}^2\right) \tag{8.9}$$

Since $f \star g \in L^2\left(\mathbb{R}^2\right)$ for all $f \in A$ and $g \in L^2\left(\mathbb{R}^2\right)$, in view of Proposition 7.4.3, L does in fact map A into $\mathfrak{L}\left(L^2\left(\mathbb{R}^2\right)\right)$.

Proposition 8.2.1. *L is a faithful representation of A on $L^2\left(\mathbb{R}^2\right)$*

Proof. First, note that for all $f \in A$, from Proposition 7.4.3,

$$\begin{aligned}
\|L(f)\|_{op} &= \sup_{0 \neq g \in L^2(\mathbb{R}^2)} \left\{ \frac{\|f \star g\|_2}{\|g\|_2} \right\} \\
&\leq \sup_{0 \neq g \in L^2(\mathbb{R}^2)} \left\{ \frac{\|f\|_2 \, \|g\|_2}{\|g\|_2} \right\} \\
&= \|f\|_2
\end{aligned}$$

Thus, $L(f) \in \mathfrak{B}\left(L^2\left(\mathbb{R}^2\right)\right)$ for all $f \in A$. Therefore, L maps A into the C^*-algebra $\mathfrak{B}\left(L^2\left(\mathbb{R}^2\right)\right)$:

$$L : A \to \mathfrak{B}\left(L^2\left(\mathbb{R}^2\right)\right)$$

Moreover, $L : A \to \mathfrak{B}\left(L^2\left(\mathbb{R}^2\right)\right)$ is a homomorphism because

$$L\left(f \star f'\right) g = L(f)\, L\left(f'\right) g$$

for all $f, f' \in A$ and $g \in L^2\left(\mathbb{R}^2\right)$ by the associativity of the Moyal product. The Moyal product defined in (7.8) is continuous on account of the inequality proven in Proposition 7.4.3. So, for $f \in L^2\left(\mathbb{R}^2\right)$, let (f_n) be a sequence in A that converges in $L^2\left(\mathbb{R}^2\right)$ to f. Then, using (4.2.11.3) from Proposition 4.2.11, we find that for all $g, h \in A$

$$\begin{aligned}
(h|f \star g) &= \lim_{n\to\infty} (h|f_n \star g) \\
&= \lim_{n\to\infty} (f_n^* \star h|g) \\
&= (f^* \star h|g)
\end{aligned}$$

since $f_n^* \to f^*$ in $L^2\left(\mathbb{R}^2\right)$. Hence,

$$(h|f \star g) = (f^* \star h|g) \tag{8.2.1.1}$$

for all $g, h \in A$ and $f \in L^2\left(\mathbb{R}^2\right)$. Next, for $h \in L^2\left(\mathbb{R}^2\right)$, let (h_n) be a sequence in A that converges in $L^2\left(\mathbb{R}^2\right)$ to h. Then, using (8.2.1.1), we find that for all $g \in A$ and $f \in L^2\left(\mathbb{R}^2\right)$

$$\begin{aligned}
(h|f \star g) &= \lim_{n\to\infty} (h_n|f \star g) \\
&= \lim_{n\to\infty} (f^* \star h_n|g) \\
&= (f^* \star h|g)
\end{aligned}$$

Hence,

$$(h|f \star g) = (f^* \star h|g) \tag{8.2.1.2}$$

for all $g \in A$ and $f, h \in L^2(\mathbb{R}^2)$. Next, for $g \in L^2(\mathbb{R}^2)$, let (g_n) be a sequence in A that converges in $L^2(\mathbb{R}^2)$ to g. Then, using (8.2.1.2), we find that for all $f, g, h \in L^2(\mathbb{R}^2)$

$$\begin{aligned}(h|f \star g) &= \lim_{n\to\infty} (h|f \star g_n) \\ &= \lim_{n\to\infty} (f^* \star h|g_n) \\ &= (f^* \star h|g)\end{aligned} \tag{8.2.1.3}$$

Finally, (8.2.1.3) allows us to find

$$\begin{aligned}(L(f)^* g|h) &= (g|L(f)h) \\ &= (g|f \star h) \\ &= (f^* \star g|h) \\ &= (L(f^*)g|h)\end{aligned}$$

for all $f \in A$ and $g, h \in L^2(\mathbb{R}^2)$. Hence, $L : A \to \mathfrak{B}(L^2(\mathbb{R}^2))$ is a $*$-homomorphism. Injectivity of L follows easily from (7.8). Thus $L : A \to \mathfrak{B}(L^2(\mathbb{R}^2))$ is a faithful representation of A on $L^2(\mathbb{R}^2)$. $\square$

The next step is to show that the representations $L(A)$ and $(\eta \otimes I)(A)$ are equivalent. We do so by finding a certain intertwining map between them. If H is a Hilbert space and I_{dex} is a set that indexes any orthonormal basis of H, then H is isometrically isomorphic to $\ell^2(I_{dex})$. Applied to the Hilbert space $L^2(\mathbb{R}^2)$, which has basis $\{f_{mn}\}_{m,n\in\mathbb{N}_0}$ indexed by the set $\mathbb{N}_0 \times \mathbb{N}_0$, this statement implies that $L^2(\mathbb{R}^2)$ is isometrically isomorphic to $\ell^2(\mathbb{N}_0 \times \mathbb{N}_0)$, which in turn is isomorphic to the Hilbert space $\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)$ with orthonormal basis $\{e_m \otimes e_n\}_{m,n\in\mathbb{N}_0}$. Hence, there exists a unitary operator U from $L^2(\mathbb{R}^2)$ onto $\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)$ that preserves the vector space and inner product structures, therefore, also the topological structure. Explicitly, define the map $U : L^2(\mathbb{R}^2) \to \ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)$ such that

$$U(f_{mn}) := e_m \otimes e_n \tag{8.10}$$

for all $m, n \in \mathbb{N}_0$.

Proposition 8.2.2. *$(L^2(\mathbb{R}^2), L)$, $(\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0), \eta \otimes I)$, and $(\ell^2(\mathbb{N}_0), \eta)$ are isometrically $*$-isomorphic representations of A:*

$$L(A) \simeq (\eta \otimes I)(A) \simeq \eta(A)$$

In particular, for all $f \in A$,

$$UL(f)U^* = (\eta \otimes I)(f)$$

Proof. Since $U : L^2(\mathbb{R}^2) \to \ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)$ maps orthonormal basis elements to orthonormal basis elements, it is straightforward to show that it is a bijection that preserves the inner product; in other words, U is a unitary operator. We prove that $U : L^2(\mathbb{R}^2) \to \ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)$ is an intertwining map for the representations L and $\eta \otimes I$ of A. Let $f = \sum\limits_{m,n=0}^{\infty} c_{mn} f_{mn} \in A$ and $e_i, e_j \in \ell^2(\mathbb{N}_0)$ be arbitrary. It follows from (8.4), (8.7), (8.8), and Proposition 6.3.4 that

$$\begin{aligned}
UL(f)U^*(e_i \otimes e_j) &= UL(f) f_{ij} \\
&= U\left(\sum_{m,n=0}^{\infty} c_{mn} f_{mn} \star f_{ij}\right) \\
&= U\left(\sum_{m,n=0}^{\infty} c_{mn} \delta_{ni} f_{mj}\right) \\
&= U\left(\sum_{m=0}^{\infty} c_{mi} f_{mj}\right) \\
&= \sum_{m=0}^{\infty} c_{mi} e_m \otimes e_j \\
&= \eta(f) e_i \otimes I e_j \\
&= (\eta(f) \otimes I)(e_i \otimes e_j) \\
&= (\eta \otimes I)(f)(e_i \otimes e_j)
\end{aligned}$$

Thus $UL(f)U^* = (\eta \otimes I)(f)$ for all $f \in A$; in other words, U intertwines the representations L and $\eta \otimes I$. Since U is also a unitary operator, it follows that $\left(L^2(\mathbb{R}^2), L\right)$ and $\left(\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0), \eta \otimes I\right)$ are unitarily equivalent representations, i.e. isometrically $*$-isomorphic representations of A:

$$L(A) \simeq (\eta \otimes I)(A)$$

Note that the $*$-isomorphic correspondence between $(\eta \otimes I)(A)$ and $\eta(A)$ follows from Proposition 8.1.4 by restricting the map ι to $\eta(A)$. Applying [31, Lemma 6.3.2] shows that the norms are preserved:

$$\|(\eta \otimes I)(f)\|_{op} = \|\eta(f) \otimes I\|_{op} = \|\eta(f)\|_{op} \|I\|_{op} = \|\eta(f)\|_{op}$$

for all $f \in A$. Therefore, $\left(\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0), \eta \otimes I\right)$ and $\left(\ell^2(\mathbb{N}_0), \eta\right)$ are isometrically $*$-isomorphic representations of A:

$$(\eta \otimes I)(A) \simeq \eta(A)$$

$\square$

Let

- $\overline{\eta(A)}$ denote the closure of $\eta(A)$ in $\mathfrak{B}\left(\ell^2(\mathbb{N}_0)\right)$
- $\overline{L(A)}$ denote the closure of $L(A)$ in $\mathfrak{B}\left(L^2(\mathbb{R}^2)\right)$.

Proposition 8.2.3. *$\overline{L(A)}$ is isometrically $*$-isomorphic to $\overline{\eta(A)}$:*

$$\overline{L(A)} \simeq \overline{\eta(A)}$$

Proof. We know, from Proposition 8.2.2, that there exists an isometric $*$-isomorphism

$$\varphi : L(A) \to \eta(A)$$

with inverse

$$\varphi^{-1} : \eta(A) \to L(A)$$

Since both are isometric, they have isometrically $*$-homomorphic extensions

$$\overline{\varphi} : \overline{L(A)} \to \overline{\eta(A)}$$

and

$$\overline{\varphi^{-1}} : \overline{\eta(A)} \to \overline{L(A)}$$

It is simple to show, using limit arguments, that the properties of φ and φ^{-1} extend to $\overline{\varphi}$ and $\overline{\varphi^{-1}}$ respectively, and that

$$\overline{\varphi^{-1}} \circ \overline{\varphi} = Id_{\overline{L(A)}} \quad \text{and} \quad \overline{\varphi} \circ \overline{\varphi^{-1}} = Id_{\overline{\eta(A)}}$$

i.e. $\overline{\varphi}$ and $\overline{\varphi^{-1}}$ are inverses. Hence, $\overline{\varphi}$ is an isometric $*$-isomorphism:

$$\overline{L(A)} \simeq \overline{\eta(A)}$$

□

Chapter 9

Pure states of A

In this chapter, we determine the pure states of A. We start by showing that the C^*-closure of A is isomorphic to the algebra of compact operators. The latter has been comprehensively studied and the form of its pure states is well-known. As discussed in Chapter 1, a pre-C^*-algebra has the same pure states as its closure; therefore, we arrive at the pure states of A via the pure states of the algebra of compact operators.

9.1 The C^*-algebra of compact operators

We set out to prove that $\overline{\eta(A)}$ is isomorphic to the algebra

$$\mathbb{K} := K\left(\ell^2(\mathbb{N}_0)\right)$$

of compact operators on $\ell^2(\mathbb{N}_0)$. To this end, consider the following from [3, II.8.2]:

Definition 9.1.1. *An **inductive system** of C^*-algebras is a collection*

$$\{(A_i, \phi_{ij}) : i, j \in \Omega, i \leq j\}$$

where Ω is a directed set, the A_i are C^*-algebras, and $\phi_{ij} : A_i \to A_j$ is a $*$-homomorphism such that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ for $i \leq j \leq k$. Each ϕ_{ij} is norm-decreasing, so there is a naturally induced C^*-seminorm on the algebraic direct limit defined, for $a \in A_i$, by

$$\|a\| = \lim_{j>i} \|\phi_{ij}(a)\| = \inf_{j>i} \|\phi_{ij}(a)\|$$

*The completion of the algebraic limit (with elements of seminorm zero divided out) is a C^*-algebra called the **inductive limit** of the system, denoted by*

$$\lim_{\to} (A_i, \phi_{ij})$$

There is a natural $*$-homomorphism ϕ_i from A_i to the inductive limit. If all the connecting maps are injective (and hence isometric), the algebraic direct limit may be thought of as the union of the A_i, and the inductive limit as the completion of this union.

Let

$$M_k(\mathbb{C}) \tag{9.1}$$

denote the C^*-algebra of $k \times k$ matrices with complex entries, equipped with matrix multiplication and complex transposition. Furthermore, let

$$\phi_{k,k+\ell} : M_k(\mathbb{C}) \to M_{k+\ell}(\mathbb{C}) \tag{9.2}$$

be the $*$-homomorphism that acts on a $k \times k$ matrix by adding ℓ rows of zeros and ℓ columns of zeros. Note that each $\phi_{k,k+\ell}$ is injective. In accordance with Definition 9.1.1 above,

$$M_\infty(\mathbb{C}) = \bigcup_{k \geq 1} M_k(\mathbb{C}) \tag{9.3}$$

is, by definition, the algebraic direct limit of the inductive system $\{(M_k, \phi_{k,k+\ell}) : k, \ell \in \mathbb{N}\}$. Then the inductive limit is the completion

$$\lim_{\rightarrow} (M_k, \phi_{k,k+\ell}) = \overline{M_\infty(\mathbb{C})} \tag{9.4}$$

As seen in [3, II.8.2.2], this inductive limit is isomorphic to the C^*-algebra of compact operators on $\ell^2(\mathbb{N}_0)$, i.e.

$$\overline{M_\infty(\mathbb{C})} \simeq \mathbb{K} \tag{9.5}$$

Proposition 9.1.2. *$\overline{\eta(A)}$ is isomorphic to $\mathbb{K}$:*

$$\overline{\eta(A)} \simeq \mathbb{K}$$

Proof. In Proposition 7.4.2, we have shown that $A \simeq \eta(A)$, which allows us to consider elements of A as infinite-dimensional matrices. This enables us to identify $M_k(\mathbb{C})$ with a certain subalgebra of A for each $k \geq 1$. Specifically, for each $k \geq 1$,

$$M_k(\mathbb{C}) \simeq \{f \in A : c_{mn} = 0 \text{ whenever } m \geq k \text{ or } n \geq k\}$$

We have not only $M_k(\mathbb{C}) \subset \eta(A)$ for each $k \geq 1$ but also

$$M_\infty(\mathbb{C}) = \bigcup_{k \geq 1} M_k(\mathbb{C}) \subset \eta(A)$$

Then (9.5) implies that

$$\mathbb{K} \subset \overline{\eta(A)} \tag{9.1.2.1}$$

To prove the reverse inclusion, we use Proposition 8.1.2 and note that all Hilbert-Schmidt operators are compact operators:

$$\eta(A) \subset HS\left(\ell^2(\mathbb{N}_0)\right) \subset \mathbb{K}$$

This implies

$$\overline{\eta(A)} \subset \mathbb{K} \tag{9.1.2.2}$$

Combining (9.1.2.1) and (9.1.2.2) gives the required result. □

9.2 The pure states

Now we can use a result from [31, Section 5.1] that describes the pure states of $\mathbb{K}$ to determine the pure states of A.

Proposition 9.2.1. *Any unit vector $\psi = \sum_{n=0}^{\infty} \psi_n e_n \in \ell^2(\mathbb{N}_0)$ determines a pure state ω_ψ of A as*

$$\omega_\psi(f) = \langle \psi, \eta(f)\psi \rangle_{\ell^2} = \sum_{m,n=0}^{\infty} \psi_m^* \psi_n a_{mn}$$

where $f = \sum_{m,n=0}^{\infty} a_{mn} f_{mn} \in A$. Moreover, any pure state of A comes from such a unit vector.

Proof. As shown in Theorem 5.1.7 and Example 5.1.1 in [31], if H is a Hilbert space and $K(H)$ is the C^*-algebra of compact operators on H, then the pure states of $K(H)$ are the vector states of the irreducible representation given by

$$\omega_x : K(H) \to \mathbb{C} : u \mapsto \langle x, u(x) \rangle$$

where x is a unit vector in H and $u \in K(H)$.

As discussed in Chapter 1, the pure states of A are uniquely determined by those of its C^*-completion $\overline{\eta(A)}$. From Propositions 8.2.3 and 9.1.2, we have

$$\overline{L(A)} \simeq \overline{\eta(A)} \simeq \mathbb{K}$$

so that the pure states of A are exactly those of $\mathbb{K}$. Therefore, the pure states of A are the positive linear functionals

$$\omega_\psi(f) = \langle \psi, \eta(f)\psi \rangle_{\ell^2}$$

where ψ is a unit vector in $\ell^2(\mathbb{N}_0)$ and $f \in A$. Consider an arbitrary unit vector $\psi = \sum_{n=0}^{\infty} \psi_n e_n \in \ell^2(\mathbb{N}_0)$, i.e.

$$\|\psi\|_{\ell^2}^2 = \sum_{n=0}^{\infty} |\psi_n|^2 = 1$$

Also, let $f = \sum\limits_{m,k=0}^{\infty} a_{mk} f_{mk} \in A$ be arbitrary. Then, using (8.4), we find that

$$\begin{aligned}
\omega_\psi (f) &= \langle \psi, \eta (f) \psi \rangle_{\ell^2} \\
&= \left\langle \sum_{n=0}^{\infty} \psi_n e_n, \eta (f) \sum_{n=0}^{\infty} \psi_n e_n \right\rangle_{\ell^2} \\
&= \left\langle \sum_{n=0}^{\infty} \psi_n e_n, \sum_{n=0}^{\infty} \psi_n \sum_{m=0}^{\infty} a_{mn} e_m \right\rangle_{\ell^2} \\
&= \sum_{m,n=0}^{\infty} \psi_m^* \psi_n a_{mn}
\end{aligned}$$

as required. □

Proposition 9.2.2. *The pure states of A are equivalently given by*

$$\omega_\psi (f) = (v_n | L (f) v_n)$$

where $f \in A$ and each

$$v_n := U^* (\psi \otimes e_n) \qquad (n \in \mathbb{N}_0)$$

is defined in terms of a unit vector $\psi \in \ell^2 (\mathbb{N}_0)$ and U from (8.10).

Proof. Let $\psi = \sum\limits_{m=0}^{\infty} \psi_m e_m \in \ell^2 (\mathbb{N}_0)$ be an arbitrary unit vector, i.e.

$$\|\psi\|_{\ell^2}^2 = \sum_{m=0}^{\infty} |\psi_m|^2 = 1$$

Then

$$\|v_n\|_2 = \|U^* (\psi \otimes e_n)\|_2 = \|\psi \otimes e_n\|_{\ell^2 \otimes \ell^2} = \|\psi\|_{\ell^2} = 1$$

so that v_n is a unit vector in $L^2 (\mathbb{R}^2)$. It follows from Propositions 8.2.2 and 9.2.1 that

$$\begin{aligned}
(v_n | L (f) v_n) &= \langle U v_n, U L (f) U^* U v_n \rangle_{\ell^2 \otimes \ell^2} \\
&= \langle \psi \otimes e_n, (\eta (f) \otimes I) (\psi \otimes e_n) \rangle_{\ell^2 \otimes \ell^2} \\
&= \langle \psi, \eta (f) \psi \rangle_{\ell^2} \\
&= \omega_\psi (f)
\end{aligned}$$

for all $f \in A$. □

Chapter 10

The spectral triple of the Moyal plane

At last we are in a position to assemble the spectral triple of the Moyal plane. In this chapter, we define the constituents of our spectral triple and use all the preparatory work in the prior chapters to prove that it satisfies Definition 1.1.1.

10.1 Construction

We consider the noncommutative involutive algebra A defined in (4.2.18) as

$$A := (S, \star)$$

Let

$$H := L^2\left(\mathbb{R}^2\right) \otimes \mathbb{C}^2 \tag{10.1}$$

be the complex vector space $L^2\left(\mathbb{R}^2\right) \otimes \mathbb{C}^2$ equipped with the inner product $\langle \cdot, \cdot \rangle_H : H \times H \to \mathbb{C}$ defined by

$$\langle \psi, \phi \rangle_H := \int \left(\psi_1^* \phi_1 + \psi_2^* \phi_2\right) dx \tag{10.2}$$

for all $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in H$ with $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2\left(\mathbb{R}^2\right)$.

Define a Dirac operator $D : \mathfrak{D}(D) \to H$, with domain $\mathfrak{D}(D) := S \otimes \mathbb{C}^2$, by

$$D := -i\sigma^j \partial_j = -i\sqrt{2} \begin{pmatrix} 0 & \overline{\partial} \\ \partial & 0 \end{pmatrix} \tag{10.3}$$

where we use the Einstein convention of summing over repeated indices, and where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

are the Pauli matrices. Furthermore, we define a mapping $\pi : A \to \mathfrak{L}(H)$ such that

$$\pi(f) := L(f) \otimes I_2 \tag{10.4}$$

for all $f \in A$, where L is given by (8.9) and I_2 denotes the 2×2 identity matrix. The resulting operator acts on H such that

$$\pi(f)\psi = \begin{pmatrix} L(f)\psi_1 \\ L(f)\psi_2 \end{pmatrix} = \begin{pmatrix} f \star \psi_1 \\ f \star \psi_2 \end{pmatrix} \in H \tag{10.5}$$

for all $\psi \in H$, since $f \star \psi_1, f \star \psi_2 \in L^2(\mathbb{R}^2)$ by Proposition 7.4.3.

We will show that

$$(A, H, D) \tag{10.6}$$

is a spectral triple as per Definition 1.1.1.

10.2 Verification

Proposition 10.2.1. *H is a Hilbert space.*

Proof. H is clearly an inner product space. The inner product $\langle \cdot, \cdot \rangle_H$ induces a norm on H:

$$\|\psi\|_H = \sqrt{\|\psi_1\|_2^2 + \|\psi_2\|_2^2} \tag{10.2.1.1}$$

To show that H is complete, let $\left(\psi^{(m)}\right)$ be an arbitrary Cauchy sequence in H with $\psi^{(m)} = \begin{pmatrix} \psi_1^{(m)} \\ \psi_2^{(m)} \end{pmatrix}$. It is clear, from (10.2.1.1), that $\left(\psi_1^{(m)}\right)$ and $\left(\psi_2^{(m)}\right)$ are Cauchy in $L^2(\mathbb{R}^2)$. Since $L^2(\mathbb{R}^2)$ is complete, both $\left(\psi_1^{(m)}\right)$ and $\left(\psi_2^{(m)}\right)$ converge in $L^2(\mathbb{R}^2)$. Let $\psi_1, \psi_2 \in L^2(\mathbb{R}^2)$ denote these limits, i.e.

$$\psi_1^{(m)} \to \psi_1 \qquad \text{as} \qquad m \to \infty$$

and

$$\psi_2^{(m)} \to \psi_2 \qquad \text{as} \qquad m \to \infty$$

Using these limits, we define $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, which is clearly in H. From (10.2.1.1), it follows that

$$\psi^{(m)} \to \psi \in H \qquad \text{as} \qquad m \to \infty$$

Since $\left(\psi^{(m)}\right)$ was an arbitrarily chosen Cauchy sequence in H, it follows that every Cauchy sequence in H converges. Hence, H is complete and thus a Hilbert space. □

Proposition 10.2.2. (π, H) *is a faithful representation of* A.

Proof. It follows directly from definition (10.4) and Proposition 8.2.1 that $\pi(f) \in \mathfrak{B}(H)$ for all $f \in A$ and that $\pi : A \to \mathfrak{B}(H)$ is an injective $*$-homomorphism. Hence, π is a faithful representation of A on H. $\square$

Proposition 10.2.3. $D : \mathfrak{D}(D) \to H$ *is symmetric.*

Proof. Consider the derivatives ∂ and $\overline{\partial}$ as operators in $L^2(\mathbb{R}^2)$ with domain S. That is

$$\partial : S \to L^2(\mathbb{R}^2) : f \mapsto \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2) f$$

and

$$\overline{\partial} : S \to L^2(\mathbb{R}^2) : f \mapsto \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2) f$$

Since S is dense in $L^2(\mathbb{R}^2)$ by Proposition 4.2.1, ∂ and $\overline{\partial}$ are densely defined and admit adjoint operators. For all $f, g \in S$, we have that

$$\begin{aligned}(f|\partial^* g) &= (\partial f|g)\\ &= 1/2 \int \left(\overline{\partial} f^*\right) g\, dx\\ &= 1/2 \int f^* \left(-\overline{\partial} g\right) dx\\ &= \left(f| \left(-\overline{\partial}\right) g\right) \end{aligned} \tag{10.2.3.1}$$

Since S is dense in $L^2(\mathbb{R}^2)$, it follows that $\mathfrak{D}(D) := S \otimes \mathbb{C}^2$ is dense in H. Therefore,

$$D : \mathfrak{D}(D) \to H$$

is densely defined. For all $\psi, \phi \in \mathfrak{D}(D)$, we have from (10.2.3.1) that

$$\begin{aligned}\langle D\psi, \phi\rangle_H &= \left\langle -i\sqrt{2}\begin{pmatrix} 0 & \overline{\partial} \\ \partial & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\right\rangle_H\\ &= \left\langle -i\sqrt{2}\begin{pmatrix} \overline{\partial}\psi_2 \\ \partial\psi_1 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\right\rangle_H\\ &= \left(-i\sqrt{2}\left(\overline{\partial}\psi_2\right)|\phi_1\right) + \left(-i\sqrt{2}\left(\partial\psi_1\right)|\phi_2\right)\\ &= \left(\psi_1| - i\sqrt{2}\left(\overline{\partial}\phi_2\right)\right) + \left(\psi_2| - i\sqrt{2}\left(\partial\phi_1\right)\right)\\ &= \left\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, -i\sqrt{2}\begin{pmatrix} \overline{\partial}\phi_2 \\ \partial\phi_1 \end{pmatrix}\right\rangle_H\\ &= \langle \psi, D\phi\rangle_H\end{aligned}$$

Thus D is symmetric on $\mathfrak{D}(D)$. $\square$

Proposition 10.2.4. *Let A, B be bounded linear operators on $L^2\left(\mathbb{R}^2\right)$. Then*

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{op} = \max\{\|A\|_{op}, \|B\|_{op}\}$$

Proof. Let ψ_1, ψ_2 be arbitrary unit vectors in $L^2\left(\mathbb{R}^2\right)$ and $a, b \geq 0$ such that $a^2 + b^2 = 1$. Then $\psi = \begin{pmatrix} a\psi_1 \\ b\psi_2 \end{pmatrix}$ is a unit vector in H, since

$$\begin{aligned} \|\psi\|_H^2 &= \left\| \begin{pmatrix} a\psi_1 \\ b\psi_2 \end{pmatrix} \right\|_H^2 \\ &= a^2 \|\psi_1\|_2^2 + b^2 \|\psi_2\|_2^2 \\ &= a^2 + b^2 \\ &= 1 \end{aligned}$$

Since any unit vector in H can be written in this form, we have

$$\begin{aligned} \left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \psi \right\|_H^2 &= \left\| \begin{pmatrix} aA\psi_1 \\ bB\psi_2 \end{pmatrix} \right\|_H^2 \\ &= a^2 \|A\psi_1\|_2^2 + b^2 \|B\psi_2\|_2^2 \\ &\leq \left(a^2 + b^2\right) \max\{\|A\psi_1\|_2, \|B\psi_2\|_2\}^2 \\ &= \max\{\|A\psi_1\|_2, \|B\psi_2\|_2\}^2 \end{aligned}$$

The supremum in the operator norm is reached by choosing either $a = 1, b = 0$ or $a = 0, b = 1$ to find equality in the above equation, since both cases return $\|\psi\|_H^2 = 1$:

$$\begin{aligned} \left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{op} &= \sup_{\psi \in H} \left\{ \left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \psi \right\|_H : \|\psi\|_H = 1 \right\} \\ &= \max\{\|A\|_{op}, \|B\|_{op}\} \end{aligned}$$

□

Proposition 10.2.5. *$[D, \pi(f)] : \mathfrak{D}(D) \to H$ is bounded for all $f \in A$.*

Proof. Let $f \in A$ be arbitrary. First, note that for all $\psi_1 \in A$ and $j = 1, 2$, we have from Proposition 4.2.3 that

$$\begin{aligned} [\partial_j, L(f)]\, \psi_1 &= \partial_j L(f)\, \psi_1 - L(f)\, \partial_j \psi_1 \\ &= \partial_j (f \star \psi_1) - f \star \partial_j \psi_1 \\ &= (\partial_j f) \star \psi_1 \\ &= L(\partial_j f)\, \psi_1 \end{aligned}$$

This gives

$$\begin{aligned}
[\partial, L(f)] &= \frac{1}{\sqrt{2}}[\partial_1, L(f)] - \frac{i}{\sqrt{2}}[\partial_2, L(f)] \\
&= \frac{1}{\sqrt{2}} L(\partial_1 f) - \frac{i}{\sqrt{2}} L(\partial_2 f) \\
&= L(\partial f) \qquad (10.2.5.1)
\end{aligned}$$

and similarly

$$\left[\overline{\partial}, L(f)\right] = L\left(\overline{\partial} f\right) \qquad (10.2.5.2)$$

on S. Using (10.2.5.1) and (10.2.5.2), we find that for all $\psi \in S \otimes \mathbb{C}^2$

$$\begin{aligned}
[D, \pi(f)]\psi &= D\pi(f)\psi - \pi(f) D\psi \\
&= -i\sqrt{2}\begin{pmatrix} 0 & \overline{\partial} \\ \partial & 0 \end{pmatrix}\begin{pmatrix} L(f)\psi_1 \\ L(f)\psi_2 \end{pmatrix} + i\sqrt{2}\,\pi(f)\begin{pmatrix} \overline{\partial}\psi_2 \\ \partial\psi_1 \end{pmatrix} \\
&= -i\sqrt{2}\begin{pmatrix} \overline{\partial} L(f)\psi_2 - L(f)\overline{\partial}\psi_2 \\ \partial L(f)\psi_1 - L(f)\partial\psi_1 \end{pmatrix} \\
&= -i\sqrt{2}\begin{pmatrix} 0 & \left[\overline{\partial}, L(f)\right] \\ [\partial, L(f)] & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
&= -i\sqrt{2}\begin{pmatrix} 0 & L\left(\overline{\partial} f\right) \\ L(\partial f) & 0 \end{pmatrix}\psi
\end{aligned}$$

Thus for all $f \in A$

$$[D, \pi(f)] = -i\sqrt{2}\begin{pmatrix} 0 & L\left(\overline{\partial} f\right) \\ L(\partial f) & 0 \end{pmatrix} \qquad (10.2.5.3)$$

on $S \otimes \mathbb{C}^2$.

Now we can calculate the norm. Note that $\mathfrak{B}(H)$ is a C^*-algebra with the operator norm a C^*-norm. Since S is dense in $L^2\left(\mathbb{R}^2\right)$, we can use Proposition 10.2.4, together with (10.2.5.3), to find

$$\begin{aligned}
\|[D, \pi(f)]\|_{op}^2 &= \|[D, \pi(f)]^* [D, \pi(f)]\|_{op} \\
&= 2\left\|\begin{pmatrix} L(\partial f)^* L(\partial f) & 0 \\ 0 & L\left(\overline{\partial} f\right)^* L\left(\overline{\partial} f\right) \end{pmatrix}\right\|_{op} \\
&= 2\max\left\{\left\|L(\partial f)^* L(\partial f)\right\|_{op}, \left\|L\left(\overline{\partial} f\right)^* L\left(\overline{\partial} f\right)\right\|_{op}\right\} \\
&= 2\max\left\{\|L(\partial f)\|_{op}^2, \left\|L\left(\overline{\partial} f\right)\right\|_{op}^2\right\}
\end{aligned}$$

Thus,

$$\|[D, \pi(f)]\|_{op} = \sqrt{2}\max\left\{\|L(\partial f)\|_{op}, \left\|L\left(\overline{\partial} f\right)\right\|_{op}\right\} \qquad (10.2.5.4)$$

We know, from Proposition 8.2.1, that $L(f) \in \mathfrak{B}\left(L^2\left(\mathbb{R}^2\right)\right)$ for all $f \in A$. This implies that both norms in the maximum are finite, so that $\|[D, \pi(f)]\|_{op}$ is finite. Hence, $[D, \pi(f)]$ is a bounded operator for all $f \in A$. ☐

Corollary 10.2.6. (A, H, D) *is a spectral triple.*

Proof. Corollary 4.2.12 shows that A is an involutive algebra. Propositions 10.2.1 and 10.2.2 show that π is a faithful representation of A on the Hilbert space H. Proposition 10.2.3 proves that $D : \mathfrak{D}(D) \to H$ is a symmetric operator and Proposition 10.2.5 proves that $[D, \pi(f)] : \mathfrak{D}(D) \to H$ is bounded for every $f \in A$. Hence, (A, H, D) satisfies Definition 1.1.1 for a spectral triple. ☐

Chapter 11

The spectral distance on the Moyal plane

In this chapter, we explicitly calculate the spectral distance between certain pure states of A. The first step toward calculating the spectral distance is to conveniently characterise the unit ball, which is defined as

$$B_D := \left\{ a \in A : \left\| [D, \pi(a)] \right\|_{op} \leq 1 \right\} \tag{11.1}$$

We do so by investigating the relation between the coefficients of an arbitrary $a \in A$ and those of $[D, \pi(a)]$ when expanding in the matrix basis. We start by finding a more convenient expression for the norm. Define $\|\cdot\|_L : A \to \mathbb{R}^+$ such that

$$\|a\|_L := \|L(a)\|_{op}$$

for all $a \in A$. Note that $\|\cdot\|_L$ is clearly a C^*-norm on A, because $\|\cdot\|_{op}$ is a C^*-norm and L is a faithful representation of A by Proposition 8.2.1. Then we can write (10.2.5.4) as

$$\left\| [D, \pi(a)] \right\|_{op} = \sqrt{2} \max \left\{ \|\partial a\|_L , \left\|\overline{\partial} a\right\|_L \right\} \tag{11.2}$$

for all $a \in A$.

Next, we find the relation between coefficients of an arbitrary $a \in A$ and those of ∂a and $\overline{\partial} a$. Note that $\partial a, \overline{\partial} a \in A$ whenever $a \in A$; therefore, we can expand them in terms of our basis. If

$$a = \sum_{m,n=0}^{\infty} a_{mn} f_{mn} \in A$$

then we let

$$\partial a := \sum_{m,n=0}^{\infty} \alpha_{mn} f_{mn} \quad ; \quad \overline{\partial} a := \sum_{m,n=0}^{\infty} \beta_{mn} f_{mn} \tag{11.3}$$

Proposition 11.0.1. *The coefficients of ∂a and $\overline{\partial} a$, as per* (11.3), *can be expressed as functions of the coefficients of $a \in A$ by*

(1) $\alpha_{m+1,n} = \sqrt{1/2\,(n+1)}a_{m+1,n+1} - \sqrt{1/2\,(m+1)}a_{m,n}$

(2) $\alpha_{0,n} = \sqrt{1/2\,(n+1)}a_{0,n+1}$

(3) $\beta_{m,n+1} = \sqrt{1/2\,(m+1)}a_{m+1,n+1} - \sqrt{1/2\,(n+1)}a_{m,n}$

(4) $\beta_{m,0} = \sqrt{1/2\,(m+1)}a_{m+1,0}$

for all $m, n \in \mathbb{N}_0$.

Proof. Let $a = \sum_{m,n=0}^{\infty} a_{mn} f_{mn} \in A$. Using Proposition 6.5.4, we find the derivative

$$\begin{aligned}
\partial a &= \partial \left(\sum_{m,n=0}^{\infty} a_{mn} f_{mn} \right) \\
&= \sum_{m,n=0}^{\infty} a_{mn} \left(\sqrt{n/2} f_{m,n-1} - \sqrt{1/2\,(m+1)} f_{m+1,n} \right) \\
&= \sum_{m,n=0}^{\infty} a_{m,n+1} \sqrt{1/2\,(n+1)} f_{mn} - \sum_{m,n=0}^{\infty} a_{m-1,n} \sqrt{m/2} f_{mn} \\
&= \sum_{m,n=0}^{\infty} \left(a_{m,n+1} \sqrt{1/2\,(n+1)} - a_{m-1,n} \sqrt{m/2} \right) f_{mn}
\end{aligned}$$

Comparing this expression to $\partial a := \sum_{m,n=0}^{\infty} \alpha_{mn} f_{mn}$ from (11.3), we see that

$$\alpha_{mn} = \sqrt{1/2\,(n+1)} a_{m,n+1} - \sqrt{m/2} a_{m-1,n}$$

for all $m, n \in \mathbb{N}_0$, where a negative index means the coefficient is zero. This proves both (1) and (2). Similarly,

$$\begin{aligned}
\overline{\partial} a &= \overline{\partial} \left(\sum_{m,n=0}^{\infty} a_{mn} f_{mn} \right) \\
&= \sum_{m,n=0}^{\infty} a_{mn} \left(\sqrt{m/2} f_{m-1,n} - \sqrt{1/2\,(n+1)} f_{m,n+1} \right) \\
&= \sum_{m,n=0}^{\infty} \left(a_{m+1,n} \sqrt{1/2\,(m+1)} - a_{m,n-1} \sqrt{n/2} \right) f_{mn}
\end{aligned}$$

Comparing this expression to $\overline{\partial}a := \sum_{m,n=0}^{\infty} \beta_{mn} f_{mn}$ from (11.3), we see that

$$\beta_{mn} = \sqrt{1/2\,(m+1)}a_{m+1,n} - \sqrt{n/2}a_{m,n-1}$$

for all $m, n \in \mathbb{N}_0$, which proves (3) and (4). $\square$

The following proposition exhibits some necessary constraints for an element in A to belong to the unit ball B_D. We present the constraints on the coefficients of the derivatives ∂a and $\overline{\partial}a$ but they can easily be transferred to equivalent constraints on the coefficients of a via Proposition 11.0.1.

Proposition 11.0.2. *If $a \in B_D$, then*

$$\sum_{p=0}^{\infty} |\alpha_{mp}|\,|\varphi_{pn}| \leq \frac{1}{\sqrt{2}} \quad \text{and} \quad \sum_{p=0}^{\infty} |\beta_{mp}|\,|\varphi_{pn}| \leq \frac{1}{\sqrt{2}}$$

for all $m, n \in \mathbb{N}_0$ and any unit vector $\varphi = \sum_{m,n=0}^{\infty} \varphi_{mn} f_{mn} \in L^2\left(\mathbb{R}^2\right)$.

Proof. Let $a \in B_D$ and ∂a, $\overline{\partial}a$ as in (11.3). Also, let $\varphi = \sum_{m,n=0}^{\infty} \varphi_{mn} f_{mn}$ be an arbitrary unit vector in $L^2\left(\mathbb{R}^2\right)$, i.e.

$$\|\varphi\|_2^2 = \left\| \sum_{m,n=0}^{\infty} \varphi_{mn} f_{mn} \right\|_2^2 = \sum_{m,n=0}^{\infty} |\varphi_{mn}|^2 = 1 \tag{11.0.2.1}$$

Using (7.8), where we defined the Moyal product between elements in $L^2\left(\mathbb{R}^2\right)$, we find

$$\begin{aligned} \|\partial a \star \varphi\|_2^2 &= \left\| \sum_{m,n=0}^{\infty} \alpha_{mn} f_{mn} \star \sum_{m,n=0}^{\infty} \varphi_{mn} f_{mn} \right\|_2^2 \\ &= \left\| \sum_{m,n=0}^{\infty} \left(\sum_{p=0}^{\infty} \alpha_{mp}\varphi_{pn} \right) f_{mn} \right\|_2^2 \\ &= \sum_{m,n=0}^{\infty} \left| \sum_{p=0}^{\infty} \alpha_{mp}\varphi_{pn} \right|^2 \end{aligned} \tag{11.0.2.2}$$

Now note that by definition (11.1), $a \in B_D$ implies $\|[D, \pi(a)]\|_{op} \leq 1$, which in turn implies $\|\partial a\|_L \leq \frac{1}{\sqrt{2}}$ by (11.2). From the definition of $\|\cdot\|_L$, we then have

$$\|\partial a\|_L = \sup_{0 \neq b \in L^2(\mathbb{R}^2)} \left\{ \frac{\|\partial a \star b\|_2}{\|b\|_2} \right\} \leq \frac{1}{\sqrt{2}}$$

In particular,

$$\frac{\|\partial a \star \varphi\|_2^2}{\|\varphi\|_2^2} \leq \frac{1}{2}$$

After substituting (11.0.2.1) and (11.0.2.2), the inequality reads

$$\sum_{m,n=0}^{\infty} \left| \sum_{p=0}^{\infty} \alpha_{mp} \varphi_{pn} \right|^2 \leq \frac{1}{2}$$

Since the summations with respect to m and n contain only nonnegative terms, the inequality holds, in particular, for each summand separately. Consequently, for all $m, n \in \mathbb{N}_0$,

$$\left| \sum_{p=0}^{\infty} \alpha_{mp} \varphi_{pn} \right| \leq \frac{1}{\sqrt{2}} \tag{11.0.2.3}$$

Now, fix $m \in \mathbb{N}_0$ and define $\tilde{\varphi}^m = \sum\limits_{p,n=0}^{\infty} \tilde{\varphi}_{pn}^m f_{pn} \in L^2\left(\mathbb{R}^2\right)$ such that

$$\tilde{\varphi}_{pn}^m := e^{-i \arg(\alpha_{mp})} \left|\varphi_{pn}\right| \quad \text{for all} \quad p, n \in \mathbb{N}_0$$

$\tilde{\varphi}^m$ is a unit vector in $L^2\left(\mathbb{R}^2\right)$:

$$\begin{aligned}
\|\tilde{\varphi}^m\|_2^2 &= \left\| \sum_{p,n=0}^{\infty} \tilde{\varphi}_{pn}^m f_{pn} \right\|_2^2 \\
&= \sum_{p,n=0}^{\infty} \left|\tilde{\varphi}_{pn}^m\right|^2 \\
&= \sum_{p,n=0}^{\infty} \left| e^{-i \arg(\alpha_{mp})} \left|\varphi_{pn}\right| \right|^2 \\
&= \sum_{p,n=0}^{\infty} \left|\varphi_{pn}\right|^2 \\
&= 1
\end{aligned}$$

Since (11.0.2.3) holds for any unit vector in $L^2\left(\mathbb{R}^2\right)$, it holds specifically for $\tilde{\varphi}^m$, i.e.

$$\left| \sum_{p=0}^{\infty} \alpha_{mp} \tilde{\varphi}_{pn}^m \right| \leq \frac{1}{\sqrt{2}} \tag{11.0.2.4}$$

for all $n \in \mathbb{N}_0$. Note that

$$\left| \sum_{p=0}^{\infty} \alpha_{mp} \tilde{\varphi}_{pn}^m \right| = \left| \sum_{p=0}^{\infty} \alpha_{mp} e^{-i \arg(\alpha_{mp})} \left|\varphi_{pn}\right| \right| = \sum_{p=0}^{\infty} \left|\alpha_{mp}\right| \left|\varphi_{pn}\right| \tag{11.0.2.5}$$

Substituting (11.0.2.5) in (11.0.2.4) gives

$$\sum_{p=0}^{\infty} |\alpha_{mp}| \, |\varphi_{pn}| \leq \frac{1}{\sqrt{2}} \quad \text{for all} \quad n \in \mathbb{N}_0$$

We can repeat this same procedure for any $m \in \mathbb{N}_0$ because m was chosen arbitrarily. Thus, we conclude that

$$\sum_{p=0}^{\infty} |\alpha_{mp}| \, |\varphi_{pn}| \leq \frac{1}{\sqrt{2}} \quad \text{for all} \quad m, n \in \mathbb{N}_0$$

as required. An analogous argument for $\overline{\partial} a$ proves the inequality for the coefficients β_{mp}. □

Proposition 11.0.3. *If $a \in B_D$, then*

$$|\alpha_{mn}| \leq \frac{1}{\sqrt{2}} \quad \textit{and} \quad |\beta_{mn}| \leq \frac{1}{\sqrt{2}}$$

for all $m, n \in \mathbb{N}_0$.

Proof. Let $a \in B_D$. Fix $k, \ell \in \mathbb{N}_0$ and define $\phi = \sum_{m,n=0}^{\infty} \phi_{mn} f_{mn}$ such that

$$\phi_{mn} := \begin{cases} 1 & \text{if} \quad m = k \quad \text{and} \quad n = \ell \\ 0 & \text{otherwise} \end{cases}$$

Then ϕ is a unit vector in $L^2\left(\mathbb{R}^2\right)$, since

$$\|\phi\|_2^2 = \left\| \sum_{m,n=0}^{\infty} \phi_{mn} f_{mn} \right\|_2^2 = \sum_{m,n=0}^{\infty} |\phi_{mn}|^2 = |\phi_{k\ell}|^2 = 1$$

Since $a \in B_D$, Proposition 11.0.2 holds for any unit vector and for all $m, n \in \mathbb{N}_0$. In particular, it holds for our unit vector ϕ and our fixed $\ell \in \mathbb{N}_0$. Therefore,

$$\sum_{p=0}^{\infty} |\alpha_{mp}| \, |\phi_{p\ell}| \leq \frac{1}{\sqrt{2}}$$

for all $m \in \mathbb{N}_0$. The definition of ϕ gives

$$\sum_{p=0}^{\infty} |\alpha_{mp}| \, |\phi_{p\ell}| = |\alpha_{mk}| \, |\phi_{k\ell}| = |\alpha_{mk}|$$

which simplifies the previous inequality to

$$|\alpha_{mk}| \leq \frac{1}{\sqrt{2}} \quad \text{for all} \quad m \in \mathbb{N}_0$$

Since k was chosen arbitrarily, it holds that

$$|\alpha_{mk}| \leq \frac{1}{\sqrt{2}} \quad \text{for all} \quad m, k \in \mathbb{N}_0$$

as required. A similar argument shows that

$$|\beta_{mk}| \leq \frac{1}{\sqrt{2}} \quad \text{for all} \quad m, k \in \mathbb{N}_0$$

□

Unfortunately, the spectral distance is not calculable between all the pure states of A. However, it can be calculated between those pure states that are determined by unit vectors with only one nonzero component. These pure states correspond to the diagonal elements of A. Remember that we only have to search the supremum in the spectral distance over self-adjoint elements of A as per Lemma 1.3.2. We assume $a^* = a$ and only consider elements in A^{sa}. This implies that $(\partial a)^* = (\overline{\partial} a)$, i.e. $\overline{\alpha_{nm}} = \beta_{mn}$ for all $m, n \in \mathbb{N}_0$. The pure states of A are the vector states of $\eta(A)$ on $\ell^2(\mathbb{N}_0)$, as per Proposition 9.2.1. For unit vectors $\psi = e_m$, the pure states are given by

$$\begin{aligned} \omega_m(a) &= \langle e_m, a e_m \rangle_{\ell^2} \\ &= a_{mm} \end{aligned}$$

for all $m \in \mathbb{N}_0$. Finally, we can calculate the spectral distance between these pure states.

Proposition 11.0.4. *The spectral distance between pure states ω_m and ω_n such that $n < m$ is*

$$d(\omega_m, \omega_n) = \sum_{k=n+1}^{m} \frac{1}{\sqrt{k}}$$

Proof. For any $a = \sum_{m,n=0}^{\infty} a_{mn} f_{mn} \in A^{sa}$ with $\partial a = \sum_{m,n=0}^{\infty} \alpha_{mn} f_{mn}$, Proposition 11.0.1 implies that

$$\omega_{n+1}(a) - \omega_n(a) = a_{n+1,n+1} - a_{n,n} = \sqrt{\frac{2}{n+1}} \alpha_{n+1,n} \tag{11.0.4.1}$$

for all $n \in \mathbb{N}_0$. Also, Proposition 11.0.3 shows that for any $a \in B_D$

$$|\alpha_{mn}| \leq \frac{1}{\sqrt{2}} \tag{11.0.4.2}$$

for all $m, n \in \mathbb{N}_0$. Combining (11.0.4.1) and (11.0.4.2), we find that for any $a \in B_D$

$$|\omega_{n+1}(a) - \omega_n(a)| = \sqrt{\frac{2}{n+1}} |\alpha_{n+1,n}| \leq \frac{1}{\sqrt{n+1}} \tag{11.0.4.3}$$

for all $n \in \mathbb{N}_0$. In the spectral distance formula, (11.0.4.3) implies that

$$d\left(\omega_{n+1}, \omega_n\right) \leq \frac{1}{\sqrt{n+1}} \tag{11.0.4.4}$$

for all $n \in \mathbb{N}_0$. For the spectral distance $d\left(\omega_m, \omega_n\right)$, repeated application of the triangle inequality, together with the inequality (11.0.4.4), gives

$$\begin{aligned} d\left(\omega_m, \omega_n\right) &\leq d\left(\omega_m, \omega_{n+1}\right) + d\left(\omega_{n+1}, \omega_n\right) \\ &\leq \sum_{k=n+1}^{m} d\left(\omega_k, \omega_k\right) \\ &\leq \sum_{k=n+1}^{m} \frac{1}{\sqrt{k}} \end{aligned} \tag{11.0.4.5}$$

for all $m, n \in \mathbb{N}_0$ such that $n < m$.

If we can find an $a \in B_D$ such that the upper bound in (11.0.4.5) is attained, then this upper bound is exactly the supremum in the spectral distance formula. To this end, consider the function $a^{(m)} = \sum\limits_{p,q=o}^{\infty} a_{pq}^{(m)} f_{pq}$ with coefficients defined by

$$a_{pq}^{(m)} := -\delta_{pq} \sum_{k=p}^{m} \frac{1}{\sqrt{k+1}} \tag{11.0.4.6}$$

for all $p, q \in \mathbb{N}_0$, where an empty sum is equal to zero. Note that $\left(a_{pq}^{(m)}\right)_{p,q=0}^{\infty}$ is a rapid decay sequence, since only finitely many terms are nonzero. Thus $a^{(m)} \in A$. Also, $\left(a_{pq}^{(m)}\right)_{p,q=0}^{\infty}$ is diagonal in the matrix basis.

First, we show that $a^{(m)} \in B_D$. Let $\partial a^{(m)} = \sum\limits_{p,q=0}^{\infty} \alpha_{pq}^{(m)} f_{pq}$. The coefficients of $\partial a^{(m)}$ are related to those of $a^{(m)}$ as per Proposition 11.0.1.

Applying (11.0.4.6) gives: For all $p, q \in \mathbb{N}_0$,

$$
\begin{aligned}
\alpha_{p+1,q}^{(m)} &= \sqrt{1/2\,(q+1)}a_{p+1,q+1}^{(m)} - \sqrt{1/2\,(p+1)}a_{p,q}^{(m)} \\
&= \begin{cases} 0 & \text{if } \ q \neq p \\ 0 & \text{if } \ q = p > m \\ \sqrt{1/2\,(p+1)} \left(-\sum\limits_{k=p+1}^{m} \frac{1}{\sqrt{k+1}} + \sum\limits_{k=p}^{m} \frac{1}{\sqrt{k+1}} \right) & \text{if } \ q = p \leq m \end{cases} \\
&= \begin{cases} 0 & \text{if } \ q \neq p \\ 0 & \text{if } \ q = p > m \\ \sqrt{1/2\,(p+1)} \left(\frac{1}{\sqrt{p+1}} \right) & \text{if } \ q = p \leq m \end{cases} \\
&= \begin{cases} \frac{1}{\sqrt{2}} & \text{if } \ q = p \leq m \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{11.0.4.7}
$$

Now, for any $\psi = \sum\limits_{p,q=0}^{\infty} \psi_{pq} f_{pq} \in L^2\left(\mathbb{R}^2\right)$, we use (11.0.4.7) to find

$$
\begin{aligned}
\left\| \partial a^{(m)} \star \psi \right\|_2^2 &= \left\| \sum_{p,q=0}^{\infty} \alpha_{pq}^{(m)} f_{pq} \star \sum_{p,q=0}^{\infty} \psi_{pq} f_{pq} \right\|_2^2 \\
&= \left\| \sum_{p,q=0}^{\infty} \left(\sum_{r=0}^{\infty} \alpha_{pr}^{(m)} \psi_{rq} \right) f_{pq} \right\|_2^2 \\
&= \sum_{p,q=0}^{\infty} \left| \sum_{r=0}^{\infty} \alpha_{pr}^{(m)} \psi_{rq} \right|^2 \\
&= \sum_{p,q=0}^{\infty} \left| \alpha_{p,p-1}^{(m)} \psi_{p-1,q} \right|^2 \\
&= \sum_{p,q=0}^{\infty} \left| \alpha_{p+1,p}^{(m)} \psi_{pq} \right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \right|^2 \sum_{p=0}^{m} \sum_{q=0}^{\infty} \left| \psi_{pq} \right|^2 \\
&\leq \frac{1}{2} \sum_{p,q=0}^{\infty} \left| \psi_{pq} \right|^2 \\
&= \frac{1}{2} \left\| \psi \right\|_2^2
\end{aligned}
$$

In the definition of $\|\cdot\|_L$, this inequality implies that

$$\begin{aligned}\left\|\partial a^{(m)}\right\|_L &= \sup_{0\neq\psi\in L^2(\mathbb{R}^2)}\left\{\frac{\left\|\partial a^{(m)}\star\psi\right\|_2}{\|\psi\|_2}\right\}\\ &\leq \frac{1}{\sqrt{2}}\sup_{0\neq\psi\in L^2(\mathbb{R}^2)}\left\{\frac{\|\psi\|_2}{\|\psi\|_2}\right\}\\ &= \frac{1}{\sqrt{2}} \qquad (11.0.4.8)\end{aligned}$$

Since $\left(a^{(m)}\right)^* = a^{(m)}$, we have that $\left(\partial a^{(m)}\right)^* = \overline{\partial} a^{(m)}$. Then it follows from the fact that $\|\cdot\|_L$ is a C^*-norm that

$$\left\|\partial a^{(m)}\right\|_L = \left\|\left(\partial a^{(m)}\right)^*\right\|_L = \left\|\overline{\partial} a^{(m)}\right\|_L \qquad (11.0.4.9)$$

Finally, using (11.0.4.8) and (11.0.4.9), we find that

$$\begin{aligned}\left\|\left[D,\pi\left(a^{(m)}\right)\right]\right\|_{op} &= \sqrt{2}\max\left\{\left\|\partial a^{(m)}\right\|_L, \left\|\overline{\partial} a^{(m)}\right\|_L\right\}\\ &= \sqrt{2}\max\left\{\left\|\partial a^{(m)}\right\|_L\right\}\\ &\leq \frac{\sqrt{2}}{\sqrt{2}}\\ &= 1 \qquad (11.0.4.10)\end{aligned}$$

Hence, $a^{(m)} \in B_D$.

Next, we show that the upper bound in (11.0.4.5) is reached by $a^{(m)}$. Consider that

$$\begin{aligned}\omega_{p+1}\left(a^{(m)}\right) - \omega_p\left(a^{(m)}\right) &= a^{(m)}_{p+1,p+1} - a^{(m)}_{p,p}\\ &= -\sum_{k=p+1}^{m}\frac{1}{\sqrt{k+1}} + \sum_{k=p}^{m}\frac{1}{\sqrt{k+1}}\\ &= \frac{1}{\sqrt{p+1}}\end{aligned}$$

for all $p \leq m-1$. Then, for any $m > n$,

$$\begin{aligned}\left|\omega_m\left(a^{(m)}\right) - \omega_n\left(a^{(m)}\right)\right| &= \left|\sum_{k=n+1}^{m}\left(\omega_k\left(a^{(m)}\right) - \omega_{k-1}\left(a^{(m)}\right)\right)\right|\\ &= \left|\sum_{k=n+1}^{m}\frac{1}{\sqrt{k}}\right|\\ &= \sum_{k=n+1}^{m}\frac{1}{\sqrt{k}}\end{aligned}$$

Thus, the upper bound in (11.0.4.5) is attained by $a^{(m)}$. Together with the fact that $a^{(m)} \in B_D$ by (11.0.4.10), this implies that

$$d(\omega_m, \omega_n) = \sum_{k=n+1}^{m} \frac{1}{\sqrt{k}}$$

whenever $n < m$, as required. □

Notice that the distance between nearest points ω_k and ω_{k-1} is

$$d(\omega_k, \omega_{k-1}) = \frac{1}{\sqrt{k}}$$

Furthermore, if $p < q$, then the distance between ω_q and ω_p is the sum over the path joining the two points, i.e.

$$d(\omega_q, \omega_p) = \sum_{k=p+1}^{q} d(\omega_{k-1}, \omega_k)$$

In other words, for any n such that $p < n < q$, ω_n is a middle point between ω_p and ω_q. In this case, the triangle inequality becomes an equality:

$$d(\omega_q, \omega_p) = d(\omega_q, \omega_n) + d(\omega_n, \omega_p)$$

We may interpret this geometrically: The points corresponding to the specific pure states that we considered lie in a straight line in our noncommutative geometry - the Moyal plane.

Outlook

We conclude this dissertation by discussing possibilities for future study and mentioning a few applications of noncommutative geometry in physics.

After describing the Moyal plane by a spectral triple, we computed the spectral distance only between certain pure states. Our restriction to the pure states was based on the fact that pure states correspond to points in the commutative case. However, there are instances where purity of state does not seem to be an adequate criteria for characterising a point in a noncommutative geometry. For example, in the cut-off geometries developed in [13], one is forced to approximate the pure states by non-pure states. The spectral distance between pure states is infinite and is made finite by truncating the pure states, thus yielding non-pure states. Also, as explained in [25], the product of a manifold by $\mathbb{C}^2$ allows the Pythagoras equality to hold between pure states. In the Moyal plane, the same equality holds between all translated states independent of the purity of state. It is thus worthwhile to investigate the spectral distance in the Moyal plane between arbitrary states and to consider other classes of states, such as coherent states and normal states.

As explained in [40], the principle of gravitational stability against localisation leads us to expect a quantised spacetime at the Planck scale. In many models of quantum gravity, a noncommutative spacetime is defined by replacing the spacetime coordinates by the generators of a noncommutative C^*-algebra of operators that obey certain commutation relations. The induced uncertainty relation implies the existence of a minimal length scale. In order to accommodate a quantised length, these models define a quantum length operator. In [27] it is shown that between certain classes of states, the spectral distance and the quantum length capture the same metric information.

One of the first concrete examples of physics in noncommutative spacetime was Yang-Mills theory on a noncommutative torus [10]. Since then, many attempts have been made to interpret the Standard Model in terms of noncommutative geometry. [40] provides an account of the most recent developments in noncommutative quantum field theory and its ties to string theory and gravity.

In the theory of optimal transport, Connes's spectral distance appears as a noncommutative version of the Wasserstein distance of order 1 [36]. In [25], it is suggested that this view affords the possibility of the spectral distance providing an interpretation of the Higgs field as a cost function on spacetime.

One of the most successful applications of noncommutative geometry appears in the study of the integer quantum Hall effect. Within Connes's formalism, [2] shows that the Hall conductivity is quantised and that plateaux occur when the Fermi energy varies in a region of localised states.

Spectral triples have been applied to physical models in string theory. A noncommutative σ-model is explored in [28], where the parameter space and the target space are noncommutative tori. It should be interesting to attempt similar constructions for other noncommutative spaces.

In [14], we studied noncommutative Ricci flow in a simple matrix geometry, namely a finite-dimensional representation of a noncommutative torus. Future work could include the study of Ricci flow in other noncommutative geometries.

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