Homogenization of stochastic partial differential equations in perforated porous media

by

Chigoziem Ann Emereuwa

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Declaration

I, Chigoziem Ann Emereuwa declare that the thesis, which I hereby submit for the degree *Philosophiae Doctor in Mathematics* at the University of Pretoria, is my own work and has not previously been submitted by me or any other person for a degree at this or any other tertiary institution.

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Chigoziem Emereuwa

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Name	Chigoziem Ann Emereuwa
Supervisor	Prof. Mamadou Sango
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Summary

In this thesis, we study the homogenization of a stochastic model of groundwater pollution in periodic porous media and the homogenization of a stochastic model of a single-phase fluid flow in partially fissured media.

In the first study, we investigated the flow of a fluid carrying reacting substances through a porous medium. We modeled this flow using a coupled system of equations; the velocity of the fluid is modeled using steady Stokes equations, the concentration of the solute while being moved by the fluid under the action of random forces is modeled by a stochastic convection-diffusion equation driven by a Wiener type random force and the concentration of the solute on the surface of the pore skeleton is modeled using reaction-diffusion equations. The homogenization process was carried out using the multiple scale expansion, Tartar's method of oscillating test functions and stochastic calculus together with deep probability compactness results due to Prokhorov and Skorokhod. This part of the thesis is the first in the scientific literature dealing with the important problem of groundwater pollution using stochastic partial differential equations. Our results in this regard are original. Also as a by-product of our work, we establish the first homogenization result for stochastic convection-diffusion equation

The second study is devoted to a single-phase flow under the influence of external random forces through partially fissured media arising in reservoir engineering (oil

and gas industries). We undertake to model this flow using a system of nonlinear stochastic diffusion equations with monotone operators in the pore system and the fissure system; on the interface of the pores and fissures, we prescribe transmission boundary conditions. We carried out the homogenization process using the two-scale convergence method, Prokhorov- Skorokhod compactness process and Minty's monotonicity method. While some works have been undertaken in the deterministic case and in the case of nonlinear diffusion equations with randomly oscillating coefficients, our work is novel in the sense that it uses the more advanced tool of stochastic partial differential equations driven by random forces to investigate the influence of random fluctuations on the flow. To the best of our knowledge, our work also initiates the study of stochastic evolution transmission problems by means of homogenization.

Abstract

This thesis is split into three mains parts focused on the homogenization of stochastic partial differential equations.
The first part (chapter 1) contains the introduction of the research work and important preliminary results used in the research. The main body of the work is contained in chapters two and three.

In the second part (chapter 2), we study the homogenization of a stochastic model of a flow carrying reacting particles through a periodic porous medium. The model is a coupled system of stochastic diffusion-convection, steady Stokes and reaction-diffusion equations in a perforated domain. We use different homogenization techniques namely: Tartar's method of oscillating test functions together with some results in probability theory including Prokhorov and Skorokhod compactness results and the method of asymptotic expansion to derive the homogenized system of equations.

In the third part (chapter 3), we study the homogenization of a stochastic model for flow of a single-phase fluid through a partially fissured porous medium. The model is a double-porosity model with two flow fields, one associated with the system of fissures and the other associated with the porous system. We use Nguetseng's two-scale convergence, Prokhorov and Skorokhod compactness process and Minty's monotonicity method to derive the homogenized stochastic model.

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To my late parents and siblings.

Notations

For the reader's convenience, listed here are some symbols, sets and function spaces used throughout this dissertation. Let U be an open bounded set in $\mathbb{R}^n$ with $n \in \mathbb{N}$;

- U : An open bounded subset of $\mathbb{R}^n$.
- $|U|$: The Lebesgue measure of U.
- ∂U : The boundary of U.
- $C(U)$: The space of continuous functions $u : U \to \mathbb{R}$.
- $C_0(U)$: The space of continuous functions $u : U \to \mathbb{R}$ with compact support contained in U.
- $C^\infty(U)$: The space of all infinitely differentiable functions $u : U \to \mathbb{R}$.
- $C_0^\infty(U)$ or $\mathcal{D}(U)$: The space of all infinitely differentiable functions with compact support contained in U.
- $C_0(\mathbb{R}^n)$: The space of continuous functions converging to zero at infinity.
- $Y, Y_1, Y_2, ..., Y_n$: Unit cells in $\mathbb{R}^n$.
- $C_{per}^\infty(Y)$: The restriction to Y of functions in $C^\infty(\mathbb{R}^n)$ that are Y-periodic.
- $L^p(U)-$ the space of measurable functions $\{v | v : U \to \mathbb{R}\}$ such that $\int_U |v|^p dx < \infty$.
- For $p = 2$, $(u, v)_{L^2(U)} = \int_U u(x)v(x)dx$, $||u||_{L^2(U)} = (u, u)^{\frac{1}{2}}_{L^2(U)}$.

- $L^{\infty}(U)-$ the space of essentially bounded functions in U.
- $L^p(0,T;X)-$ the space of measurable function $\phi : t \in [0,T] \to \phi(t) \in X$ such that $||\phi||_X \in L^p(0,T)$, where X is any Banach space.
- $\langle \cdot, \cdot \rangle_{X',X}-$ the duality pairing between a Banach space X and its dual X'.
- $W^{1,p}(U) - \{\phi | \phi \in L^p(U), \frac{\partial \phi}{\partial x_i} \in L^p(U), i = 1, ..., n\}$.
- $H^1(U) = W^{1,2}(U) - \{\phi | \phi \in L^2(U), \frac{\partial \phi}{\partial x_i} \in L^2(U), i = 1, ..., n\}$.
  $(u,v)_{H^1(U)} = (u,v)_{L^2(U)} + (\nabla u, \nabla v)_{L^2(U)}$.
- Let $\mathcal{V} = \{u \in C_0^{\infty}(U); u = 0 \text{ on } \mathit{\Gamma}_D \text{ a.e on } (0,T)\}$ and $V =$ closure of $\mathcal{V}$ in $H^1(U)$. where $C_0^{\infty}(U)$ denotes the space of infinitely differential functions in U.
- a.s.– almost surely.

Contents

Chapter 1

Introduction

1.1 Introduction

Homogenization in general, is a mathematical theory in the field of partial differential equations used to study differential operators with rapidly oscillating coefficients, which can be deterministic as well as random, boundary value problems with rapidly varying boundary conditions, equations in perforated domains and many other types of equations with theoretical and practical importance which arise in connection with processes taking place in heterogeneous materials/media. Heterogeneous materials can be described as having two length scales, the macroscopic scale and the microscopic scale. The theory of homogenization enables one to determine the macroscopic behaviour of processes occurring in a heterogeneous material while taking into account the behaviour of the material at the microscopic level.

The study of heterogeneous media is of great importance since they are often encountered in fields such as physics, chemistry, material science and engineering disciplines. Some examples of heterogeneous materials include composite materials and porous materials. Composite materials are materials made by combining two or more different materials with individual physical or chemical properties, resulting in a material with unique desired properties, like concrete, plastic and Carbon-Fiber-Reinforced Polymer (CFRP), while porous materials are materials consisting structurally of pores, voids or holes such as rocks, aquifers and reservoirs. The focus of this research is on the latter so we elaborate a little more on it.

The pores in porous materials are filled with fluid (gas or liquid) or they allow external substances such as particles or fluid to pass through them. The skeletal part of a porous material is called a frame or matrix, this part is usually solid. However, materials like foam can be analyzed using the idea of porous media. Different physical or chemical phenomena occur in porous media and understanding these processes require an appreciation of the interaction among them. Examples of these processes include chemical reaction, mass transport and adsorption/desorption processes. Fluid flow through porous media is a subject of great interest in theoretical and applied science and it has become a separate field of study. We refer to the monograph [14], by Bear for an authoritative source on the foundations of flow through porous media and to [54] by Ganji and Kachapi for nanofluid flow in porous media. An example of flow through porous media is groundwater flow, a fundamental problem which goes far beyond academic interest and is related to the very existence of humanity. Research in this direction has therefore attracted the attention of leading scholars in applied sciences; this will be the focus of the next chapter.

Among the porous systems encountered in nature, many are found to be fractured/fissured. These fissured porous media are made up of permeable and porous blocks interlaced by a system of fissures, the porous blocks make up the matrix of the media. Their major characteristic is that bulk of the transport happens in the system of fissures while the pores are responsible for significant fluid storage. They were first studied by reservoir engineers in petroleum engineering since most petroleum reservoirs are found in rock formations with fractures and pores. In addition to petroleum reservoirs, some groundwater resources are also fractured.

Hence, flow of fluid in fractured rock has attracted the attention of scientists and engineers due to the growing concerns of water quality and groundwater pollution. Fissured porous media are differentiated by the extent to which the system of fissures are developed within the medium. In a case where the system of fissures are well developed to the extent that they separate the matrix into individual porous rocks, the medium is called a totally fissured medium. In a totally fissured medium, it is assumed that fluid cannot flow from a porous block into another without passing through the fissures, hence there is no flow within the porous matrix, only through the fissures. A fissured medium where the system of fissures are less developed and the porous rocks may be connected, leading to some amount of flow within the matrix and through the fissures, is called a partially fissured medium, see Figure 1.1 for an illustration. We refer to [16], [2], [47], [58] (Chapter 9) and [113] for more on flow through fissured media. This type of heterogeneous medium will be the focus of chapter 3.

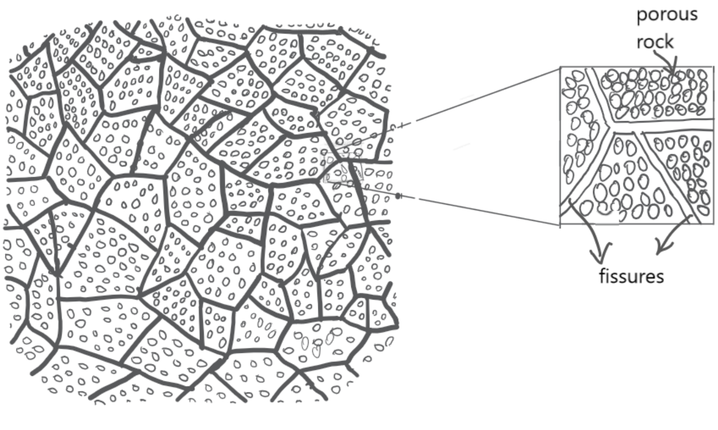

Figure 1.1: An illustration of a partially fissured material

To mathematically describe a periodic porous medium, we can say that at the microscopic scale, the porous medium consists of periodically repeating solid particles surrounded by the pores. The pore space forms a domain where a fluid (liquid or gas) flows through, while the obstacles are the perforations or holes in the domain. We assume that its properties on the microscopic scale depend on a small parameter ϵ which is the length scale of the micro structure. For simplicity,

the micro structure of a porous medium can be described by a unit cell which is repeated periodically in one or more directions. However, there are periodic micro structures whose periodicity cells are not represented by unit quadrilaterals.
In the case of fissured porous media, even though they may behave like porous media with regard to fluid flow and transport, they are treated and modeled differently since the contribution from the fissures and the fluid storage in the pores have to be taken into consideration, (see Chapter 9 by Showalter in [58] for more).

Processes and phenomena such as temperature, elasticity, molecular transport or chemical reaction taking place in heterogeneous media can be modeled using partial differential equations with the heterogeneities captured by rapid oscillations in the coefficients of the equations or by boundary values problems in perforated domains as in the case of porous media; these equations are modeled on the micro structure of the media. Solutions to these micro models are almost impossible to compute, hence the need to derive macroscopic models without oscillations that will capture the properties in the micro structure. The process by which these macro models are derived is known as homogenization.

Various differential equations arise in the theory of homogenization depending on the phenomena and type of material being modeled with periodic and non-periodic structures. To some extent, the theory can be considered a matured field as a far as deterministic problems are concerned, thanks to the development of different methods of homogenization, e.g. the method of asymptotic expansion [73], [114], [71], G-convergence by Spagnolo [123], H-convergence by L. Tartar and F. Murat [128], [87], [89], Γ-convergence by De Giorgi [45], [124], [43], Tartar's method of oscillating test functions introduced by L. Tartar and F. Murat [128], div-curl lemma by L. Tartar and F. Murat [131], [132], [62], Compensated compactness [131], [88], [90], H-measures [133], [130] and two-scale convergence by Nguetseng [91] which was further developed by Allaire [3], to mention a few. The monographs [11], [19], [32], [62], [78], [99] are great sources of wealth for the methods elaborated and results obtained over several decades.

The first rigorous investigations of homogenization of partial differential equations were undertaken by Soviet mathematicians Marchenko and Khruslov [79] in the early 1960s. They considered heterogeneous media with fined grained boundaries without any assumption of periodicity. Their tools of investigation were potential analysis and later variational methods. Afterwards, the theory of homogenization gained prominence in the 1970s and is still today, a fundamental component of applied mathematics.

There are a variety of works on deterministic models for different phenomena in porous media. However, some of the assumptions made in these type of models are different from what is encountered in practice. In man-made systems, physical/chemical processes and phenomena may be controlled, but in nature there are many unknown factors that may affect phenomena and processes. J. Bear in [13] observes that many of the uncertainties linked to modelling may be as a consequence of numerous heterogeneity of subsurface domains. In geological formations for instance, the heterogeneities can be captured in permeability and porosity values. These values are usually observed at a few locations even though they show a high degree of spatial variability at all length scales. A combination of the large spatial heterogeneity with a relatively small amount of observation lead to uncertainties about the values of the formation properties which then results in uncertainties in predicting or estimating the flow in these type of formations. The theory of stochastic processes provides a natural method for evaluating uncertainties. A random process or stochastic process is used to quantify uncertainty associated with a phenomenon or physical/chemical process. For more on extensive discussions on modelling uncertainties in porous medium, we refer to the monographs of Bear and Cheng [13] and Zhang [142].

As mentioned above, numerous uncertainties are encountered when modelling physical/chemical processes or phenomena in natural systems. Some of the data obtained for modelling these systems are contaminated with systemic noise, the type of models that capture these noise are called stochastic models, see for instance [134]. Motivated by these considerations the homogenization of partial differential equations with random coefficients was pioneered in the works of Kozlov [67] and Papanicolaou and Varadhan [100] and the methods elaborated formed the basis for subsequent research in random homogenization by many authors, for instance [17], [18], [27], [28], [42], [48], [53], [102], [66], [68]; just to cite a few. However for an even more accurate modelling of phenomena subjected to random fluctuations, the framework of stochastic partial differential equations (SPDEs) driven by noises generated by stochastic processes such as Wiener processes had to be considered. This led to the emergence of a new direction in homogenization; namely the homogenization of SPDEs which was pioneered in the work of Bensoussan in [20] and followed among others by [115], [60], [61], [117], [107], [108], [137] and some recent works on homogenization of stochastic Stokes equation in [23] and [24]; these works deal with parabolic-like SPDEs. The homogenization of hyperbolic SPDEs is far more recent and the premises can be found in [85], [86], for instance. Motivated by these considerations, modelling processes taking place in porous media using stochastic partial differential equations is of both practical and theoretic importance. This will be the central topic of the research undertaken in this thesis, through a blending of advanced tools from probability (stochastic calculus, probabilistic compactness results) and the theory of homogenization.

1.2 Preliminaries

1.2.1 Function Spaces

This section contains some definitions of function spaces needed throughout the thesis. These spaces are classical and have been treated in several books. We refer for instance to the book of Evans [50] (Chapter 5). In this section, we take Q to be an open bounded set in $\mathbb{R}^n$.

Definition 1.1. Let $p \in \mathbb{R}$ with $1 \leq p \leq +\infty$. $L^p(Q)$ is defined as the class of measurable functions f on Q such that for $1 \leq p < +\infty$,

$$\int_Q |f(x)|^p dx < +\infty,$$

with the norm,

$$||f||_{L^p(Q)} = \left[\int_Q |f(x)|^p dx\right]^{\frac{1}{p}}.$$

For $p = \infty$,

$$L^\infty(Q) = \big\{f | f : \Omega \longmapsto \mathbb{R}, f \text{ is measurable and } \exists\, C \in (0,\infty) \text{ with } |f(x)| < C, \text{ for a.e. } x \in Q\big\}$$

with the norm

$$||f||_{L^\infty(Q)} = \inf\{C > 0, |f(x)| \leq C \text{ for a.e. } x \in Q\}.$$

Definition 1.2. Let $1 \leq p \leq +\infty$. The *Sobolev space* $W^{1,p}(Q)$ is defined as the set

$$\{u | u \in L^p(Q), \frac{\partial u}{\partial x_i} \in L^p(Q), \quad i = 1, ..., n\},$$

with derivatives taken in the sense of distributions, i.e.,

$$\forall v \in \mathcal{D}(Q), \quad \left\langle \frac{\partial u}{\partial x_i}, v \right\rangle_{\mathcal{D}'(Q),\mathcal{D}(Q)} = -\left\langle u, \frac{\partial v}{\partial x_i} \right\rangle_{\mathcal{D}'(Q),\mathcal{D}(Q)}.$$

If $p = 2$, $W^{1,2}(Q)$ is written as $H^1(Q)$ i.e.,

$$H^1(Q) = \{u | u \in L^2(Q), \frac{\partial u}{\partial x_i} \in L^2(Q), \quad i = i, ..., n\}.$$

Proposition 1.3. *1. We have the Sobolev space $W^{1,p}(Q)$ endowed with*

$$||u||_{W^{1,p}(Q)} = ||u||_{L^p(Q)} + \sum_{i=1}^{n} \left|\left|\frac{\partial u}{\partial x_i}\right|\right|_{L^p(Q)},$$

is a Banach space.

2. *The space $H^1(Q)$ is equipped with the following scalar product,*

$$(u,v)_{H^1(Q)} = (u,v)_{L^2(Q)} + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(Q)}, \quad \forall u, v \in H^1(Q), \tag{1.1}$$

   and its norm is given by

$$||u||_{H^1(Q)} = \sqrt{(u,u)_{H^1(Q)}}.$$

   The space $H^1(Q)$ is a Hilbert space.

Theorem 1.4 (Extension Theorem)**.** *Suppose Q is bounded and ∂Q is C^1. Let V be an open bounded set such that $Q \subset\subset V$. Then there exists a bounded linear operator*

$$T : W^{1,p}(Q) \to W^{1,p}(\mathbb{R}^n)$$

such that $\forall u \in W^{1,P}(Q)$,

- $Tu = u$ *a.e. in Q,*
- *Tu has support within V,*
- $||Tu||_{W^{1,p}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(Q)}$, *where C is a constant that depends on p, Q, and V.*

The proof can be found in [50] (Chapter 5).

Theorem 1.5 (Sobolev Embedding Theorem)**.** *Assume that ∂Q is Lipschitz continuous. Then*

1. *if $1 \leq p < n$, $W^{1,p}(Q) \subset L^q(Q)$ with*
   (a) *compact injection for $q \in [1, s)$ where $\frac{1}{s} = \frac{1}{p} - \frac{1}{n}$,*
   (b) *continuous injection for $q = s$,*
2. *if $p = n, W^{1,p}(Q) \subset L^q(Q)$ with compact injection if $q \in [1, +\infty)$,*
3. *if $p > n$, $W^{1,p}(Q) \subset C^0(\bar{Q})$ with compact injection.*

Theorem 1.6. *Suppose ∂Q is Lipschitz continuous. Then the linear map*

$$\gamma : W^{1,p}(Q) \longmapsto L^p(\partial Q)$$

such that

$$\forall u \in W^{1,p}(Q) \cap C(Q), \quad \gamma(u) = u|_{\partial Q},$$

and

$$\forall u \in W^{1,p}(Q), \quad ||\gamma u||_{L^p(\partial Q)} \leq C||u||_{W^{1,p}(Q)},$$

where C is a constant that depends only on p, and Q, is called the trace of u on ∂Q.

The proof is contained in [50] (Chapter 5, Section 5.5).
Let us Y be defined by

$$Y = (0, l_1) \times ... \times (0, l_n), \tag{1.2}$$

where $l_1, ..., l_n$ are given positive numbers.

Definition 1.7. Let Y be defined by the relation (1.2) and f, a function defined a.e. on $\mathbb{R}^n$. The function f is called *Y-periodic* if

$$f(x + kl_i e_i) = f(x) \text{ a.e on } \mathbb{R}^n, \qquad \forall k \in \mathbb{Z}, \quad \forall i \in \{1, ..., n\}$$

where $\{e_1, ..., e_n\}$ is the canonical basis of $\mathbb{R}^n$. If $n = 1$, then f is said to be l_1-periodic.

We have the following fundamental result on the convergence of rapidly oscillating periodic functions.

Theorem 1.8. *Let $1 \leq p \leq +\infty$, and f be a Y-periodic function in $L^p(Y)$. Set*

$$f_\epsilon(x) = f\left(\frac{x}{\epsilon}\right) \quad a.e \quad on \quad \mathbb{R}^n.$$

If $p < +\infty$, then as $\epsilon \to 0$,

$$f_\epsilon \rightharpoonup \mathcal{M}_Y(f) = \frac{1}{|Y|}\int_Y f(y)\, dy \textit{ weakly in } L^p(\omega),$$

for any bounded open subset ω of $\mathbb{R}^n$.
If $p = +\infty$, then as $\epsilon \to 0$,

$$f_\epsilon \overset{*}{\rightharpoonup} \mathcal{M}_Y(f) \text{ weakly}^* \text{ in } L^\infty(\mathbb{R}^n).$$

The proof of theorem 1.8 is contained in [32].
The space $L^2(\Omega; C_{per}(Y))$ is a separable space dense in $L^2(Q; Y)$ with norm given by

$$||u||^2_{L^2(Q;C_{per}(Y))} = \int_Q (\sup_{y\in Y} |u(x, y)|)^2 \, dx.$$

Theorem 1.9. *Let $u_0 \in L^2(Q; C_{per}(Y))$, and define $u^\epsilon(x)$ by $u(x, \frac{x}{\epsilon})$ with $\epsilon > 0$. Then*

1. *$u^\epsilon \in L^2(Q)$ and $||u^\epsilon||_{L^2(Q)} \leq ||u_0||_{L^2(Q;C_{per}(Y))}$.*
2. *$u^\epsilon(x) \rightharpoonup \int_Y u_0(x, y) \, dy$ weakly in $L^2(Q)$ as $\epsilon \to 0$.*
3. *$||u^\epsilon||_{L^2(Q)} \to ||u_0||_{L^2(Q\times Y)}$ as $\epsilon \to 0$.*

The proof of this theorem can be found in [103] (Chapter 2, pg 24).

1.2.2 Some probabilistic preliminaries

This section contains some definition and classical results from stochastic analysis, analysis of partial differential equations and probability theory. For details and proofs of the results, we refer to [40], [52], [119], [122], [64].

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$ be a filtered probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of the set Ω, $\mathbb{P}$ is the probability measure and $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration of the σ-algebra $\mathcal{F}$ with $\mathcal{F}_t \subset \mathcal{F}$ $\forall t \in [0, T]$ and $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$ for $t_1 \leq t_2$.
Th pair $(D, \mathcal{G})$ consisting of a space D and the σ-algebra $\mathcal{G}$ is called a measurable space.

Definition 1.10. For a measurable space $(D, \mathcal{G})$, a map $M : (\Omega, \mathcal{F}) \mapsto (D, \mathcal{G})$ such that

$$\{\omega \in \Omega : M(\omega) \in A\} \in \mathcal{F}, \text{ for any } A \in \mathcal{G},$$

is a *random variable* with values in D.

Definition 1.11. Let $T > 0$ and $I = [0, T]$. A *stochastic process* is a collection $(M_t = M(\omega, t), t \in I)$ of random variables on $(\Omega, \mathcal{F})$ which takes values in a measurable space $(D, \mathcal{G})$, called a state space.

Definition 1.12. A stochastic process is said to be *adapted* to a filtration $\mathcal{F}_t$ if for each $t > 0$, M_t is an $\mathcal{F}_t$-measurable random variable.

Definition 1.13. A stochastic process $\{M_t\}_{t\in[0,T]}$ adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is called a *martingale* if

$$\int_\Omega |M_t| d\mathbb{P} < \infty,$$

and

$$\mathbb{E}(M_t/\mathcal{F}_s) = M_s, \quad \mathbb{P}\text{-a.s., for any } t \geq s.$$

Definition 1.14. Let $\{M_t\}_{t\in[0,T]}$ be a stochastic process. A finite valued random variable τ is known as a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ if $0 \leq \tau < \infty$ and if for any $t \in [0, T]$, the event $\{\tau \leq t\} = \{\omega : \tau(\omega) \leq t\}$ belongs to the $\mathcal{F}$.

Let X be a Banach space, and $1 \leq p \leq \infty$. We denote by $L^p(0, T; X)$ the space of measurable functions $\phi : t \in [0, T] \mapsto \phi(t) \in X$, with the norm defined by

$$||\phi||_{L^p(0,T;X)} = \left(\int_0^T ||\phi||_X^p dt \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty.$$

For $p = \infty$, $L^\infty(0, T; X)$ is the space of all essentially bounded functions on the interval $[0, T]$ with values in X equipped with the norm defined by

$$||\phi||_{L^\infty(0,T;X)} = ess \sup_{[0,T]} ||\phi||_X < \infty.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $1 \leq p, q \leq \infty$, the space $L^q(\Omega, \mathcal{F}, \mathbb{P}, L^p(0, T; X))$ is a probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ consisting of all stochastic processes

$\phi : (\omega, t) \in \Omega \times [0, T] \mapsto \phi(\omega, t, \cdot) \in X$ such that $\phi(\omega, t, \cdot)$ is progressively measurable with respect to (ω, t). Let $\mathbb{E}$ be the corresponding mathematical expectation. For $1 \leq p < \infty$, we endow this space with the norm

$$||\phi||_{L^q(\Omega,\mathcal{F},\mathbb{P},L^p(0,T;X))} = \left(\mathbb{E}||\phi||^q_{L^p(0,T;X)}\right)^{\frac{1}{q}}.$$

For $p = \infty$, the norm in the space $L^q(\Omega, \mathcal{F}, \mathbb{P}, L^\infty(0, T; X))$ is given by

$$||\phi||_{L^q(\Omega,\mathcal{F},\mathbb{P},L^p(0,T;X))} = \left(\mathbb{E}||\phi||^q_{L^\infty(0,T;X)}\right)^{\frac{1}{q}}.$$

Endowed with the above norm, $L^q(\Omega, \mathcal{F}, \mathbb{P}, L^\infty(0, T; X))$ is a Banach space.

Definition 1.15. A stochastic process $\{B_t\}_{t\in[0,T]}$ is called a one dimensional *standard Brownian process* if it satisfies the following properties;

- $B_0 = 0, \ \ \mathbb{P}$-a.s.,
- $\{B_t\}_{t\in[0,T]}$ has stationary, independent increments,
- the map $t \to B_t$ is continuous in t with probability 1,
- the increment $B_{t+s} - B_s$ has a normal distribution with variance t and mean 0.

The following result deals with stochastic integrals.

Theorem 1.16. *Let $M_t = M(\omega, t) \in L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; X))$ be a random process with values in X, then the stochastic integral*

$$I(T) = \int_0^T M(t, \cdot)dB_t,$$

exists $\mathbb{P}$-a.s., where B_t is a 1-dimensional Brownian motion.

The proof follows from the well known construction of stochastic integrals in both finite and infinite dimension, we refer to [52] and [40].

Theorem 1.17. *Let $\{M_t\}_{t\in[0,T]} \in L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; X))$, then the stochastic process $\int_0^t M_s dB_s$ is a continuous martingale with values in X. Furthermore,*

$$\mathbb{E}\int_0^t M(\cdot, s)dB_s = 0, \ \ \forall t \in [0, T].$$

The following result is of crucial important

Theorem 1.18 (**Burkhölder-Davis-Gundy inequality**). *For any $p > 0$, there exists a positive constant c_p, C_p such that for all local martingales $\{M_t\}_{t\in[0,T]}$,*

$$c_p\mathbb{E}\Big(\int_0^T ||M_t||^2 dt\Big)^{\frac{p}{2}} \leq \mathbb{E}\sup_{0\leq t\leq T}\Big|\Big|\int_0^t M_s\, dB_s\Big|\Big|_X^p \leq C_p\mathbb{E}\Big(\int_0^T ||M_t||^2 dt\Big)^{\frac{p}{2}}. \quad (1.3)$$

We recall the following inequality, known as Markov's inequality;

$$\mathbb{P}(\omega : \xi(\omega) \geq \alpha) \leq \frac{\mathbb{E}|\xi(\omega)|^k}{\alpha^k}, \quad (1.4)$$

where ξ is a nonnegative random variable and k is a positive real number. We now formulate the key result in stochastic analysis, namely ito's formula.

Definition 1.19. Let $\{M_t\}_{t\in[0,T]}$ be a process such that for any $0 \leq t_1 < t_2 \leq T$,

$$M(t_2) - M(t_1) = \int_{t_1}^{t_2} A(t)dt + \int_{t_1}^{t_2} G(t)dB(t),$$

where $A(s), G(s) \in L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; X))$, X a Hilbert space. Then M_t is said to have a stochastic differential dM_t given by

$$dM_t = A(t)dt + G(t)dB(t).$$

Theorem 1.20. *Let $dM_t = A(t)dt+G(t)dB(t)$, with $A(s), G(s) \in L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; X))$, X a Hilbert space and let $\psi(t, x)$ be a continuous function in $[0, T] \times X$ with continuous Frećhet derivatives $\psi_t, \psi_x, \psi_{xx}$, which are bounded in bounded subsets of $[0, T] \times X$. Then*

$$\psi(t, M(t)) = \psi(t, M(0)) + \int_0^t \psi_s(t, M(t))ds + \int_0^t \langle\psi_x(s, M(s)), A(s)\rangle_X ds$$
$$+ \int_0^t \langle\psi_x(s, M(s)), G(s)dB(s)\rangle_X + \frac{1}{2}\int_0^t \langle\psi_{xx}(s, M(s))G(s), G(s)\rangle_X ds, \quad \mathbb{P}\text{-a.s.,}$$
$$\forall t \in [0, T].$$

For the proof, we refer to [52], [40].

Next we introduce the fundamental probabilistic compactness results due to Prokhorov and Skorokhod. They play a key role as a bridge between homogenization of evolution deterministic problems and homogenization of SPDEs, we refer to Billingsley [25], Da Prato [40] and the original papers of Prokhorov [105] and Skorokhod [122] for details. We start with the definition of tightness of probability measures.

Definition 1.21. Let $\mathcal{S}$ be a Banach space with $\mathcal{B}(\mathcal{S})$ its Borel σ-algebra. A family of probability measures $(\mu_m)_{m\in\mathbb{N}}$ on $(\mathcal{S},\mathcal{B}(\mathcal{S}))$ is said to be tight if $\forall\varepsilon>0$, there exists a compact set $K\subset\mathcal{S}$ such that

$$\mu_m(K)>1-\varepsilon,\ \ \forall m\in\mathbb{N}.$$

Definition 1.22. A family of probability measures $(\mu_m)_{m\in\mathbb{N}}$ on $(\mathcal{S},\mathcal{B}(\mathcal{S}))$ is relatively compact if there exists a weakly convergent subsequence $(\mu_{m_k})_{k\in\mathbb{N}}$, i.e. $\exists$ a probability measure μ (not necessarily in $(\mu_m)_{m\in\mathbb{N}}$) such that

$$\lim_{k\to\infty}\int_{\mathcal{S}}\varphi(x)d\mu_{m_k}(x)=\int_{\mathcal{S}}\varphi(x)d\mu(x),$$

for any bounded and continuous function φ on $\mathcal{S}$. .

Lemma 1.23 (Prokhorov)**.** *A sequence of probability measures $(\pi_m)_{m\in\mathbb{N}}$ on $(\mathcal{S},\mathcal{B}(\mathcal{S}))$ is tight if and only if it is relatively compact.*

Lemma 1.24 (Skorokhod)**.** *Suppose $\mathcal{S}$ is a separable Banach space with $\mathcal{B}(\mathcal{S})$ as its σ-algebra. Assume that the probability measures $(\mu_m)_{m\in\mathbb{N}}$ on $(\mathcal{S},\mathcal{B}(\mathcal{S}))$ weakly converges to a probability measure μ. Then there exists random variables $\xi,\xi_1,...,\xi_m,...$ defined on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that $\mathcal{L}\{\xi_m\}=\mu_m$ and $\mathcal{L}(\xi)=\mu$ and*

$$\lim_{m\to\infty}\xi_m=\xi,\ \ \mathbb{P}\text{-}a.s.,$$

where $\mathcal{L}(\cdot)$ stands for the law of $\cdot$.

1.2.3 Monotone Operators

In this subsection we recall the definitions of monotone operators and the classical result of Minty (Minty's trick) [84]. This result will be used in Chapter 3 in the construction of the homogenized problem for a sequence of SPDEs involving monotone operators. For proofs of the results, we refer to [51], [111], [143]. For more on monotonicity and numerous applications to nonlinear PDEs, we refer to [75], [109].

Let X be a real, reflexive Banach space and X^* its dual. Let us denote the inner product (g,p) by $g(p)$ for $g \in X^*$, $p \in X$.

Definition 1.25. A mapping $\mathbf{E} : X \to X^*$ is said to be bounded if it maps bounded subsets of X to bounded subsets of X^*.

$\mathbf{E} : X \to X^*$ is continuous if $\forall p \in X$,

$$||\mathbf{E}(p) - \mathbf{E}(q)||_{X^*} \to 0 \;\text{ whenever }\; ||p-q||_X \to 0.$$

$\mathbf{E}$ is hemicontinuous if $\forall p, q, r \in X$ the map

$$t \mapsto (\mathbf{E}(p+tq), r)$$

is continuous.

Definition 1.26. A mapping $\mathbf{E} : X \to X^*$ is called coercive if

$$\frac{(\mathbf{E}(p), p)}{||p||} \to 0 \quad \text{as} \quad ||p|| \to \infty.$$

Definition 1.27. A mapping $\mathbf{E} : X \to X^*$ is said to be monotone if

$$(\mathbf{E}(p) - \mathbf{E}(q)) \cdot (p-q) \geq 0 \;\; \forall p, q \in X,$$

$\mathbf{E}$ strictly monotone if the inequality is strict whenever $p \neq q$, i.e.

$$(\mathbf{E}(p) - \mathbf{E}(q)) \cdot (p-q) > 0 \;\; \forall p, q \in X,$$

Lemma 1.28. *[Minty's trick] Let $\boldsymbol{E} : X \to X^*$ be monotone and hemicontinuous on a real Banach space X and let*

$$\langle g - \boldsymbol{E}(q), p-q \rangle \geq 0, \;\; \forall q \in X.$$

Then

$$g = \boldsymbol{E}(p).$$

1.3 Homogenization techniques

Some techniques of homogenization were briefly mentioned in the introduction; here we shall give a description of some of them, how they evolved over time and expand more on those that are directly relevant to the thesis.

The origin of homogenization theory can be traced back to the nineteenth century. They can be found in the work done by Maxwell [80], where he investigated the effective conductivity of heterogeneous media. In 1892, Rayleigh [106] studied the same problem but with periodic inclusions. From then up till the fifties, homogenization techniques and methods developed and were widely studied by physicists like Voight [136], Reuss [110] and Lifshits and Rozentsveig [74]. Mathematicians' interest in homogenization resulted in the introduction of more advanced methods and ideas.

The method of asymptotic expansions was extensively developed by Bogolyubov and Mitropolskii in the area of ordinary differential equations (ODEs) [26]. It was later formalized to be used for problems with periodic rapidly oscillating coefficients in [11], [19]; see also [72], [71] and [114]. The main goal in applications include deriving the effective properties of composite materials and the macroscopic modelling of microscopic systems, see [82] and [104].

Marchenko and Khruslov's study in [79] of partial differential equations in domains with fine grained boundaries can be considered as the first mathematically rigorous work in homogenization theory, making them pioneers in the field. They studied boundary value problems in non-periodically structured domains using potential analysis. In 1967, the G-convergence -an operator-like convergence that deals with the convergence of solutions to symmetric problems with periodic or non-periodic coefficients was introduced by Spagnolo in [123]. In the 70s, more methods emerged including Γ-convergence by De Giorgi [44], for the study of homogenization of functionals. H-convergence by L. Tartar and F. Murat [128], [87] was introduced as an extension of the G-convergence to non-symmetric problems. For problems containing the product of two weakly converging sequences, Tartar in [128] introduced Tartar's method of oscillating test functions. Tartar and Murat introduced the div-curl lemma for problems involving the product of weakly converging vector fields in systems of nonlinear PDEs, [131], [132]. The lemma is applicable to problems in physics; see e.g. [31]. The lemma was further extended to compensated compactness method also by Tartar and Murat in [90], [131]; however it is only applicable to problems with constant coefficients.

In the 1980s, a new approach was introduced independently under different names, L.Tartar named it H- measures while P. Gérard introduced it under the name microlocal defect measures [55]. In 1989, the two-scale convergence was introduced by G. Nguetseng [91] for the study of boundary value problems with periodic rapidly oscillating coefficients. This method was further developed by Allaire in [3] and in Mikelić, Bourgeat and Wright [27] introduced the stochastic two-scale convergence. Recently, the periodic unfolding method for the homogenization of periodic composites was introduced by Cioranescu, Grisco and Damlamian [35], see also [34]. In 2003, Nguetseng extended his two-scale convergence to include problems beyond the periodic setting in [94] and [95] under the name Σ-convergence. Wellander [139] in 2009 introduced the two-scale Fourier transform which is like a combination of the periodic unfolding method, two-scale convergence, and the Floquet-Bloch expansion approach to homogenization.

Next we give a brief illustration of two of the homogenization methods mentioned earlier: namely the method of asymptotic expansion and Tartar's method. For simplicity, we limit ourselves to elliptic problems. Some definitions and results on two-scale convergence is included as well.
Let Q be an open subset of $\mathbb{R}^n$ with Lipschitz boundary ∂Q. We consider the following linear second order partial differential equation with Dirichlet boundary

conditions;

$$\begin{cases} \mathcal{A}^\epsilon u^\epsilon & = f \quad \text{in } Q, \\ u^\epsilon & = 0 \quad \text{on } \partial Q, \end{cases} \tag{1.5}$$

where $f = f(x)$ is a smooth function in Q independent of ϵ, $\mathcal{A}^\epsilon = -\mathrm{div}(A^\epsilon \nabla)$, $A^\epsilon(x) = A(\frac{x}{\epsilon}) = (a_{i,j}(\frac{x}{\epsilon}))_{1\le i,j\le n}$ is such that $a_{i,j}$ are Y-periodic $\forall i,j = 1,...,n$ and $A = (a_{i,j})_{1\le i,j\le n}$ is such that there exists $\alpha.\beta \in \mathbb{R}$, $0 < \alpha < \beta$,

(i) $(A(x)\lambda, \lambda) \geq \alpha|\lambda|^2$,

(ii) $|A(x)\lambda| \leq \beta|\lambda|$,

for any $\lambda \in \mathbb{R}^n$. If conditions (i) and (ii) are satisfied by a matrix A, we say that $A \in \mathcal{M}(\alpha, \beta, Q)$.

1.3.1 Method of Asymptotic Expansion

The main idea of the method is to assume that the solution u^ϵ to (1.5) is of the form

$$u^\epsilon = u_0\big(x, \frac{x}{\epsilon}\big) + \epsilon u_1\big(x, \frac{x}{\epsilon}\big) + \epsilon^2 u_2\big(x, \frac{x}{\epsilon}\big) + ..., \tag{1.6}$$

with the terms in the expansion depending on the macroscopic variable x and the microscopic variable $\frac{x}{\epsilon}$ and Y-periodic in the second variable. Using this method, the homogenized problem and its solution are both obtained.

Suppose $u^\epsilon = u\big(x, \frac{x}{\epsilon}\big)$,

$$\frac{\partial u^\epsilon}{\partial x_i} = \frac{\partial u}{\partial x_i}\big(x, \frac{x}{\epsilon}\big) + \frac{1}{\epsilon}\frac{\partial u}{\partial y_i}\big(x, \frac{x}{\epsilon}\big) \quad i = 1, ..., n,$$

then $\mathcal{A}^\epsilon$ assumes the expansion

$$\mathcal{A}^\epsilon u^\epsilon = \frac{1}{\epsilon^2}\mathcal{A}_0 u\big(x, \frac{x}{\epsilon}\big) + \frac{1}{\epsilon}\mathcal{A}_1 u\big(x, \frac{x}{\epsilon}\big) + \mathcal{A}_2 u\big(x, \frac{x}{\epsilon}\big),$$

where $\mathcal{A}_0 = -\mathrm{div}(A(y)\nabla_y)$, $\mathcal{A}_1 = -\mathrm{div}_x(A(y)\nabla_y) - \mathrm{div}_y(A(y)\nabla_x)$, $\mathcal{A}_2 = -\mathrm{div}_x(A(y)\nabla_x)$.

Substituting (1.6) into problem (1.5), we have

$$\big(\frac{1}{\epsilon^2}\mathcal{A}_0 + \frac{1}{\epsilon}\mathcal{A}_1 + \mathcal{A}_2\big)\big(u_0 + \epsilon u_1 + \epsilon^2 u_2\big)\big(x, \frac{x}{\epsilon}\big) = f \quad \text{in } Q,$$

$$u^\epsilon = 0 \quad \text{on } \partial Q.$$

Sorting and equating equal power terms of ϵ give a sequence of problems, the first three are

$$\begin{cases} \mathcal{A}_0 u_0 & = 0 \quad \text{in } Y, \\ u_0 & Y\text{-periodic.} \end{cases} \tag{1.7}$$

$$\begin{cases} \mathcal{A}_0 u_1 & = -\mathcal{A}_1 u_0 \quad \text{in } Y, \\ u_1 & Y\text{-periodic.} \end{cases} \tag{1.8}$$

$$\begin{cases} \mathcal{A}_0 u_2 & = f - \mathcal{A}_1 u_1 \mathcal{A}_2 u_0 \quad \text{in } Y, \\ u_2 & Y\text{-periodic.} \end{cases} \tag{1.9}$$

The next step would be to solve the system of equations successively so as to determine the functions $u_i(x, \frac{x}{\epsilon})$. Starting with (1.7), the unknown u_0 is determined, then used to obtain u_1 in (1.8) and then u_0 and u_1 are to used determine u_2 in (1.9). The existence and uniqueness of problems (1.7)-(1.9) is obtained using Lax-Milgarm theorem. The calculations involved in deriving the homogenized problem are long and cumbersome making it prone to error, hence an error estimate is often required to justify the results. This method can also be applied to equations with periodic oscillations on more than one microscopic scale. We refer to [19], [32], [98] for more on the multiple-scale expansions method.

1.3.2 Tartar's method of oscillating test functions

A more rigorous method for deriving the homogenization problem is due to Tartar and involve an ingenious construction of suitable test functions.
From equation (1.5), we have the weak form

$$\int_Q A^\epsilon \nabla u^\epsilon \nabla v \, dx = \int_Q f \nabla v \, dx, \quad v \in H_0^1(Q),$$

where $v \in H_0^1(Q)$. Passing to the limit in the above equation will be a problem since we have the product of two weakly converging sequences A^ϵ and ∇u^ϵ, Tartar's method of oscillating test functions provides a solution to that. By using

test functions obtained by periodizing the solutions to a cell problem, one is able to pass to the limit in the equation above. The homogenized problem is obtained independently unlike the method of asymptotic expansions where both the homogenized problem and the homogenized solution are derived.
The main homogenization result stemming from Tartar's method goes as follows

Theorem 1.29. *Let u^ϵ be the weak solution of problem (1.5), with $f \in L^2(Q)$ and $A^\epsilon \in M(\alpha, \beta, Q)$ is Y-periodic.*
Then

- $u^\epsilon \rightharpoonup u^0$ *weakly in* $H_0^1(Q)$,
- $A^\epsilon \nabla u^\epsilon \rightharpoonup A^0 \nabla u^0$ *weakly in* $(L^2(Q))^n$,

where $u^0 \in H_0^1(Q)$ is the weak solution to the homogenized problem:

$$\begin{aligned} -div(A^0 \nabla u^0) &= f \qquad in \;\; Q, \\ u^0 &= 0 \qquad on \;\; \partial Q, \end{aligned} \tag{1.10}$$

and

$$A^0 = (a_{ij}^0)_{1 \le i,j \le n} = \frac{1}{|Y|} \int_Y a_{ij}(y)\, dy - \frac{1}{|Y|} \sum_{k=1}^{n} \int_Y a_{ik}(y) \frac{\partial \chi_j}{\partial y_k}\, dy, \tag{1.11}$$

where χ_j is the weak solution to the cell problem:

$$\begin{aligned} -div\,(A(y)\nabla\chi_j) &= -div\,(A(y)e_j) \quad in \; Y, \\ \chi_j \quad &is \quad Y\text{-}periodic. \end{aligned} \tag{1.12}$$

Proof. From Lax -Milgram theorem, (1.5) has a unique solution $u^\epsilon \in H_0^1(Q)$ for a fixed ϵ, with $f \in H^{-1}(Q)$ such that

$$\int_Q A^\epsilon \nabla u^\epsilon \nabla v\, dx = \langle f, v \rangle_{H^{-1}(Q), H_0^1(Q)}, \quad \forall v \in H_0^1(Q), \tag{1.13}$$

and we have the following estimates

$$||u^\epsilon||_{H_0^1(Q)} \le C||f||_{H^{-1}(Q)}.$$

Hence, there exists a weakly converging subsequence of $\{u^\epsilon\}$ which we still denote by $\{u^\epsilon\}$ and an element $u^0 \in H_0^1(Q)$ such that

$$u^\epsilon \rightharpoonup u^0 \quad \text{weakly in} \;\; H_0^1(Q). \tag{1.14}$$

By Sobolev embedding theorem,

$$u^\epsilon \to u^0 \quad \text{strongly in} \;\; L^2(Q). \tag{1.15}$$

Let us introduce the vector function

$$\xi^\epsilon = (\xi_1^\epsilon, ..., \xi_n^\epsilon) = A^\epsilon \nabla u^\epsilon.$$

Then (1.13) implies that

$$\int_Q \xi^\epsilon \nabla v \, dx = \langle f, v \rangle_{H^{-1}(Q), H_0^1(Q)}. \tag{1.16}$$

Since $A \in M(\alpha, \beta, Q)$, we have

$$\int_Q \xi^\epsilon \nabla v \, dx = \int_Q A^\epsilon \nabla u^\epsilon \nabla v \, dx \leq \beta ||u^\epsilon||_{H_0^1(Q)} ||v||_{H_0^1(Q)}.$$

But

$$||u^\epsilon||_{H_0^1(Q)} \leq \frac{1}{\alpha} ||f||_{H^{-1}(Q)},$$

so

$$||\xi^\epsilon||_{L^2(Q)} \leq \frac{\beta}{\alpha} ||f||_{H^{-1}(Q)}.$$

Thus (ξ^ϵ) is a uniformly bounded sequence in $(L^2(Q))^n$.
Consequently, there exists a subsequence of $\{\xi^\epsilon\}$ which we still denote by $\{\xi^\epsilon\}$ and $\xi^0 \in L^2(Q)$ such that

$$\xi^\epsilon \rightharpoonup \xi^0 \quad \text{weakly in} \;\; (L^2(Q))^n. \tag{1.17}$$

Hence passing to the limit in (1.16) gives

$$\int_Q \xi^0 \nabla v \, dx = \langle f, v \rangle_{H^{-1}(Q), H_0^1(Q)}, \qquad v \in H_0^1(Q). \tag{1.18}$$

And this is a weak formulation of the equation

$$-\mathrm{div}\, \xi^0 = f \qquad \text{in} \;\; Q. \tag{1.19}$$

Tartar's method of oscillating test functions shall be used to identify ξ^0.

Let χ_j be the solution to the following cell problem

$$
\begin{aligned}
-\text{div } (A(y)\nabla\chi_j) = -\text{div } (A(y)e_j) \quad \text{ in } Y, \\
\chi_j \text{ is } Y\text{-Periodic.}
\end{aligned}
\tag{1.20}
$$

The extension by periodicity of the solution χ_j of (1.20) still denoted by χ_j is the unique solution to the following problem:

$$
\begin{aligned}
-\text{div } (A\nabla\chi_j) &= -\text{div } (A(y)\, e_j) \quad \text{ in } \mathcal{D}'(\mathbb{R}^n), \\
\chi_j & \quad Y\text{-periodic}, \\
\mathcal{M}_Y(\chi_j) &= 0.
\end{aligned}
$$

Let

$$
w_j(x) = x_j - \chi_j\left(\frac{x}{\epsilon}\right), \tag{1.21}
$$

and define

$$
w_j^\epsilon(x) = \epsilon w_j\left(\frac{x}{\epsilon}\right) = x_j - \epsilon\chi_j\left(\frac{x}{\epsilon}\right), \qquad \text{for } j = 1, ..., n. \tag{1.22}
$$

Since $H_0^1(Q) = \overline{D(Q)}$, with respect to the H^1-norm, we have

$$
\int_Q A^\epsilon(x)\nabla w_j^\epsilon(x)\nabla v(x)\, dx = 0, \quad \forall v \in H_0^1(Q), \tag{1.23}
$$

and we arrive at the following convergence

$$
w_j^\epsilon \rightharpoonup x_j \quad \text{weakly in } \ (H^1(Q))^n, \tag{1.24}
$$

and by Sobolev embedding theorem,

$$
w_j^\epsilon \to x_j \text{ strongly in } (L^2(Q))^n. \tag{1.25}
$$

For $\varphi \in D(\mathbb{R}^n)$, let us choose $v = \varphi w_j^\epsilon$ in equation (1.13) and $v = \varphi u^\epsilon$ in equation (1.23) to get

$$
\begin{aligned}
\int_Q A^\epsilon\nabla u^\epsilon\nabla(\varphi w_j^\epsilon)\, dx &= \int_Q A^\epsilon\nabla u^\epsilon\nabla\varphi\, w_j^\epsilon\, dx + \int_Q A^\epsilon\nabla u^\epsilon\nabla w_j^\epsilon\, \varphi\, dx \\
&= \langle f, \varphi w_j^\epsilon\rangle_{H^{-1}(Q), H_0^1(Q)},
\end{aligned}
\tag{1.26}
$$

and

$$\begin{aligned}\int_Q A^\epsilon \nabla w_j^\epsilon \nabla(\varphi u^\epsilon)\,dx &= \int_Q A^\epsilon \nabla w_j^\epsilon \nabla\varphi\, u^\epsilon\,dx + \int_Q A^\epsilon \nabla w_j^\epsilon \nabla u^\epsilon \varphi\,dx \\ &= 0,\end{aligned} \tag{1.27}$$

respectively.

Using the symmetry of A, we get

$$\int_Q A^\epsilon \nabla u^\epsilon \nabla w_j^\epsilon\,\varphi\,dx = \int_Q A^\epsilon \nabla w_j^\epsilon \nabla u^\epsilon \varphi\,dx.$$

Subtracting equation (1.27) from equation (1.26) gives

$$\int_Q A^\epsilon \nabla u^\epsilon \nabla\varphi\, w_j^\epsilon\,dx - \int_Q A^\epsilon \nabla w_j^\epsilon \nabla\varphi\, u^\epsilon\,dx = \langle f, \varphi w_j^\epsilon\rangle_{H^{-1}(Q),H_0^1(Q)}. \tag{1.28}$$

Now we pass to the limit on each term as $\epsilon \to 0$.

For the first term on the left hand side of equation (1.28), equations (1.17) and (1.25) give

$$\lim_{\epsilon\to 0}\int_Q A^\epsilon \nabla u^\epsilon \nabla\varphi\, w_j^\epsilon\,dx = \int_Q \xi^0 \nabla\varphi\, x_j\,dx. \tag{1.29}$$

For the second term on the left hand side of (1.28),

$$\begin{aligned}(A^\epsilon(x)\nabla w_j^\epsilon(x))_k &= \sum_{i=1}^n a_{ik}\big(\frac{x}{\epsilon}\big)\frac{\partial w_j^\epsilon}{\partial x_i}(x) \\ &= \sum_{i=1}^n a_{ik}\big(\frac{x}{\epsilon}\big)\frac{\partial}{\partial x_i}\big(x_j - \epsilon\chi_j\big(\frac{x}{\epsilon}\big)\big) \\ &= \sum_{i=1}^n a_{ik}\big(\frac{x}{\epsilon}\big)\big(\delta_{ij} - \frac{\partial}{\partial y_i}\chi_j(y)\big), \quad y = \frac{x}{\epsilon} \\ &= a_{jk} - \sum_{i=1}^n a_{ik}\frac{\partial \chi_j}{\partial y_i}.\end{aligned}$$

Thus we have the following convergence in $(L^2(Q))^n$;

$$\begin{aligned}(A^\epsilon(x)\nabla w_j^\epsilon(x))_k &\rightharpoonup \mathcal{M}_Y(a_{jk}) - \mathcal{M}_Y\Bigg(\sum_{i=1}^n a_{ik}\frac{\partial\chi_j}{\partial y_i}\Bigg) \\ &= \frac{1}{|Y|}\int_Y a_{jk}(y)\,dy - \frac{1}{|Y|}\sum_{i=1}^n\int_Y a_{ik}(y)\frac{\partial\chi_j}{\partial y_i}(y)\,dy \\ &= A^0_{jk}.\end{aligned}$$

The strong convergence (1.15) yields,

$$\lim_{\epsilon\to 0}\int_Q A^\epsilon \nabla w_j^\epsilon \nabla\varphi\, u^\epsilon\,dx = \int_Q A^0 \nabla\varphi\, u^0\,dx. \tag{1.30}$$

Lastly, for the term on the right hand side of (1.28), equation (1.24) gives,

$$\lim_{\epsilon\to 0} \langle f, \varphi w_j^\epsilon\rangle_{H^{-1}(Q),H_0^1(Q)} = \langle f, \varphi(x_j)\rangle_{H^{-1}(Q),H_0^1(Q)}, \qquad \forall\varphi \in \mathcal{D}(Q).$$

Combining all the limits of (1.28), we get

$$\int_Q \xi^0 \nabla\varphi\, x_j\, dx - \int_Q A^0 \nabla\varphi\, u^0\, dx = \langle f, \varphi(x_j)\rangle_{H^{-1}(Q),H_0^1(Q)}, \;\; \forall\varphi \in \mathcal{D}(Q),$$

which can be rewritten as

$$\begin{aligned} \int_Q \xi^0 \nabla(\varphi x_j)\, dx - \int_Q \xi^0 e_j\, \varphi\, dx - \int_Q A^0 \nabla\varphi\, u^0\, dx& \\ = \langle f, \varphi(x_j)\rangle_{H^{-1}(Q),H_0^1(Q)}, \qquad \forall\varphi \in \mathcal{D}(Q).& \end{aligned} \tag{1.31}$$

But

$$\int_Q \xi^0 \nabla v\, dx = \langle f, v\rangle_{H^{-1}(Q),H_0^1(Q)},$$

implies that

$$\int_Q \xi^0 \nabla(\varphi x_j)\, dx = \langle f, \varphi x_j\rangle_{H^{-1}(Q),H_0^1(Q)}, \qquad \varphi \in \mathcal{D}(Q).$$

Hence, it follows from (1.31) that

$$\int_Q \xi^0 e_j\, \varphi\, dx = -\int_Q A^0 \nabla\varphi\, u^0\, dx.$$

But

$$-\int_Q A^0 \nabla\varphi\, u^0\, dx = \int_Q A^0 \nabla u^0 \varphi\, dx.$$

So

$$\int_Q \left(\xi^0 e_j - A^0 \nabla u^0\right)\varphi\, dx = 0.$$

Hence we conclude that

$$\xi^0 = A^0 \nabla u^0.$$

□

The symmetry of the operator $\mathcal{A}$ was essential in ensuring the cancellation of troubling terms, this method is also applicable in the homogenization of parabolic problems. In the case of non-symmetric operators, the adjoint operator is used.

1.3.3 Two-scale convergence

Definition 1.30. let $\{\varphi^\epsilon\}$ be a sequence of functions in $L^p(0,T;L^p(Q))$ $(1 < p < \infty)$. $\{\varphi^\epsilon\}$ is said to be two-scale convergent to $\varphi_0 = \varphi_0(t,x,y)$ with $\varphi_0 \in L^p(0,T;L^p(Q\times Y))$ if for any function $v = v(t,x,y) \in L^p((0,T)\times Q; C^\infty_{per}(Y))$, one has

$$\lim_{\epsilon\to 0}\int_0^T\int_Q \varphi^\epsilon(t,x)v(t,x,\frac{x}{\epsilon})dxdt = \int_0^T\int_Q\int_Y \varphi_0(t,x,y)v(t,x,y)dydxdt,$$

we denote this by $\varphi^\epsilon \xrightarrow{2-s} \varphi_0$ in $L^p(0,T;L^p(Q))$.

Theorem 1.31. *Let $\{\varphi^\epsilon\}$ be a bounded sequence of functions in $L^p(0,T;L^p(Q))$ with $1 < p \le \infty$. Then there exists subsequence $\{\varphi^{\epsilon'}\}$ and a function $\varphi \in L^p(0,T;L^p(Q\times Y))$ such that $\{\varphi^{\epsilon'}\}$ is two-scale convergent to φ.*

Theorem 1.32. *Let $\{\varphi^\epsilon\}$ be a sequence satisfying the assumptions of Theorem 3.9. Furthermore, let $\{\varphi^\epsilon\}$ be bounded in $L^p(0,T;W^{1,p}_0(Q))$. Then*

1. *there exists a subsequence $\{\varphi^{\epsilon'}\}$ and a couple of functions (φ,φ_1) with $\varphi \in L^p(0,T;W^{1,p}_0(Q))$ and $\varphi_1 \in L^p((0,T)\times Q;W^{1,p}_{per}(Y))$ such that up to a subsequence, $\nabla\varphi^\epsilon \xrightarrow{2-s} \nabla_x\varphi(x) + \nabla_y\varphi_1(x,y)$.*
2. *there exists a function $\varphi_0 \in L^2((0,T)\times Q;W^{1,p}_{per}(Y))$ such that up to a subsequence, $\varphi^\epsilon \xrightarrow{2-s} \varphi_0(x,y)$ and $\epsilon\nabla\varphi^\epsilon \xrightarrow{2-s} \nabla_y\varphi_0(x,y)$.*

Proposition 1.33. *Let φ^ϵ be a sequence of functions in $L^p(Q)$ such that φ^ϵ two-scale converges to $\varphi_0(x,y)$ in $L^p(Q\times Y)$. Then φ^ϵ converges weakly to $\varphi(x)$ in $L^p(Q)$, where*

$$\varphi(x) = \int_Y \varphi_0(x,y)dy, \qquad \text{in } L^p(Q).$$

Furthermore, we have

$$\lim_{\epsilon\to 0}||\varphi^\epsilon||_{L^p(Q)} \ge ||\varphi_0||_{L^p(Q\times Y)} \ge ||\varphi||_{L^p(Q)}.$$

1.4 Overview of the thesis

In this thesis, we study the homogenization of heterogeneous media under the influence of random forcing modeled using stochastic partial differential equations. We consider two problems; the first is the homogenization of a stochastic model of groundwater pollution in a periodic porous medium, studied in chapter 2 and the second is the homogenization of stochastic model of a single phase flow in a partially fissured medium, this is studied in chapter 3.

Groundwater pollution occurs when contaminants and pollutants seep through the ground surface and find their way to an underlying aquifer, these contaminants are then transported with moving groundwater to streams, rivers, e.t.c.. Our work in chapter 2, models the flow of fluid carrying reacting substances through porous medium using a coupled system of equations; the velocity of the fluid through the porous medium is modeled using steady Stokes equations, the concentration of the substance on the boundary of the pores using reaction-diffusion equation and the concentration of the substance which is being transported under the influence of an external random force by the fluid is modeled using a stochastic convection-diffusion equation. This study is the stochastic counterpart of the work investigated by Hornung and Jäger [57] in the deterministic case.
Motivated by the importance of stochastic models of groundwater pollution from various sources of contaminants extensively discussed by Bear and Verruijt in [15], this work initiates the study of groundwater pollution using SPDEs.

The porous medium is modeled as a perforated domain U with the pores (fluid phase) denoted by U^ϵ, the pore skeleton (perforations) denoted by U_0^ϵ and the surface of the pore skeleton (boundary of U^ϵ) denoted by Γ^ϵ.
We denote the velocity of the flow of the fluid by $\vec{u}^\epsilon$ and model it on U^ϵ using the following steady Stokes problem with no-slip boundary condition at the boundary

of the perforations;

$$(SS^\epsilon)\begin{cases} \epsilon^2\Delta\vec{u}^\epsilon = \nabla p^\epsilon & x\in U^\epsilon,\\ \nabla\vec{u}^\epsilon = 0 & x\in U^\epsilon,\\ \vec{\nu}\vec{u}^\epsilon = 0 & x\in \varGamma_N,\\ \vec{u}^\epsilon = \vec{u}_D & x\in \varGamma_D,\\ \vec{u}^\epsilon = 0 & x\in \varGamma^\epsilon, \end{cases}$$

where p^ϵ is the pressure within the fluid, $\vec{u}_D$ is the prescribed boundary value on $\varGamma_D$ and

$$\int_{\varGamma_D} \vec{\nu}\cdot\vec{u_D} = 0.$$

where the boundary ∂U of U consists of two parts $\varGamma_D$ and $\varGamma_N$.

In the fluid, the solute is being transported under the influence of external random force and is diffusing in the absence of any reaction. The corresponding model for the concentration of the solute v^ϵ is a stochastic convection-diffusion equation in the fluid part U^ϵ, given by;

$$(C^\epsilon)\begin{cases} dv^\epsilon(t,x) & = D\Delta v^\epsilon(t,x)dt - \vec{u}^\epsilon(x)\nabla v^\epsilon(t,x)dt + G^\epsilon(t,x)dB(t),\ t>0, x\in U^\epsilon\\ v^\epsilon(t,x) & = v_D(t,x), \qquad t>0, x\in\varGamma_D\\ \vec{\nu}\nabla v^\epsilon(t,x) & = 0, \qquad t>0, x\in\varGamma_N\\ v^\epsilon(0,x) & = v_1(x), \qquad t=0, x\in U^\epsilon\\ -D\vec{\nu}^\epsilon\nabla v^\epsilon(t,x) & = \epsilon f^\epsilon(t,x), \qquad t>0, x\in\varGamma^\epsilon \end{cases}$$

with

$$f^\epsilon(t,x) = c^\epsilon(x)v^\epsilon(t,x) - b^\epsilon(x)w^\epsilon(t,x), \tag{1.32}$$

where $t\in[0,T]$, $T\in(0,\infty)$, $\vec{u}^\epsilon$ is the velocity field, $D>0$ - the diffusion coefficient, v^ϵ represents the concentration of a solute in the fluid, w^ϵ represents the concentration of the solute on the surface of the skeleton, $(B(t))_{0\le t\le T}$ is a 1-dimensional Wiener process defined on a given filtered probability space $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{0\le t\le T})$, $G^\epsilon(t,x)$ is the intensity of the noise, v_D is the prescribed values on the boundary $\varGamma_D$ and $v_1(x)$ is the initial condition. $\varGamma^*$ is defined as $\varGamma^* = \cup\{\varGamma^k, k\in\mathbb{Z}^3\}$, and $c:\varGamma^*\to\mathbb{R}$ represents the adsorption factor and

$b : \mathit{\Gamma}^* \to \mathbb{R}$ represent the desorption factor.

On the surface of the skeleton (boundary of perforations) where the solute is diffusing and reacting with substances bound to the surface, the concentration of the solute on the surface of the skeleton w^ϵ is modeled using a diffusion-reaction equation on the surface of the skeleton $\mathit{\Gamma}^\epsilon$.

$$(R^\epsilon)\begin{cases}\partial_t w^\epsilon - \epsilon^2 E\Delta^\epsilon w^\epsilon + a^\epsilon(x)w^\epsilon = f^\epsilon & t\in(0,T], x\in\mathit{\Gamma}^\epsilon,\\ w^\epsilon = w_1(x) & t=0, x\in\mathit{\Gamma}^\epsilon,\end{cases}$$

where w^ϵ represents the concentration of the solute on the surface of the skeleton, $E > 0$ is the diffusion coefficient on the surface of the skeleton, Δ^ϵ is the Laplace-Beltrami operator on $\mathit{\Gamma}^\epsilon$, $a : \mathit{\Gamma}^* \to \mathbb{R}$ represents the reaction factor and f^ϵ is defined by (1.32).

The assumptions made on the prescribed values are described in the body of the work in Chapter two.

Our aim is to show that the sequence of solutions $(u^\epsilon, v^\epsilon, w^\epsilon)$ to the problems $(SS^\epsilon), (C^\epsilon)$ and (R^ϵ) converge in a suitable sense to the solution (u, v, w) to the following corresponding homogenized problems;

$$(S)\begin{cases}\vec{u}(x) = K\nabla P(x), & x\in U\\ \nabla\vec{U} = 0, & x\in U\\ \vec{\nu}\vec{u}(x) = \vec{u}_D(x,) & x\in\mathit{\Gamma}_D\\ \vec{\nu}\vec{u}(x) = 0, & x\in\mathit{\Gamma}_N\end{cases}$$

where K is a tensor to be defined and v is the solution to stochastic convection-diffusion equation:

$$(C)\begin{cases}dv(t,x) + F(t,x)dt = D\nabla(S\nabla v(t,x))dt - \frac{1}{|Y|}\vec{u}(x)\nabla v(t,x)dt & \\ +\frac{1}{|Y|}G(t,x)dB(t), & t>0, x\in U\\ v(t,x) = v_D(t,x), & t>0, x\in\mathit{\Gamma}_D\\ \vec{\nu}\nabla v(t,x), = 0 & t>0, x\in\mathit{\Gamma}_N\\ v(0,x) = v_1(x), & t=0, x\in U\end{cases}$$

where

$$F(t,x) = |\Gamma|(\gamma v(t,x) - \beta(t)w_1(x) - \rho(\cdot) * v(\cdot,x)(t)),$$

S is a tensor to be defined and w satisfies the homogenized reaction-diffusion equation.

$$(R)\begin{cases} \partial_t w(t,x,y) - E\nabla_y^{\Gamma} w(t,x,y) + a(y)w(t,x,y) & \\ = f(t,x,y), & t > 0, x \in U, y \in \Gamma \\ w(0,x,y) = w_1(x), & t = 0, x \in U, y \in \Gamma \end{cases}$$

with

$$f(t,x,y) = c(y)v(t,x) - b(y)w(t,x,y);$$

The proof of the convergence of (C^{ϵ}) to the homogenized problem (C) is done using Tartar's method of oscillating test functions combined with compactness results of analytic and probabilistic nature (Prokhorov-Skorokhod procedure). We derive (R) thanks to the formal asymptotic expansion method, (SS) was essentially derived by Tartar in [114].
Our work is the first dealing with the modelling of the important question of pollution by using homogenization of SPDEs. Our results are novel in this regard. Furthermore, as a by-product of our work, we establish the first homogenization results for stochastic-convection diffusion equations driven by a random force. This extends the deterministic results of Amaziane, Goncharenko and Pankratov [7] and that of Berlyand and Goncharenko [22].

Our second main investigation relates to flow through fissured porous media in Chapter 3 of this thesis. They are encountered in the study of fluid flows through natural systems, some examples include oil and water reservoirs. We consider a fissured porous medium with less developed system of fissures; this means that within the porous matrix which is responsible for fluid storage, fluid flows from one porous block to another without necessary passing through the fissures first. There are some uncertainties due to random fluctuations associated to flows within

these natural systems. Influenced by these considerations, our work models flow of a single-phase fluid affected by an external random force, through a partially fissured medium using stochastic partial differential equations.
The problem considered is a stochastic nonlinear diffusion equation driven by a Wiener type random force.
The micro-model is given by

$$
(P^\epsilon)\qquad
\begin{aligned}
c_1^\epsilon du_1^\epsilon &= \nabla \cdot \mu_1^\epsilon(x, \nabla u_1^\epsilon(t,x))dt + f_1^\epsilon(t,x)dB_1(t) \text{ in } Q_1^\epsilon,\\
c_2^\epsilon du_2^\epsilon &= \nabla \cdot \mu_2^\epsilon(x, \nabla u_2^\epsilon(t,x))dt + f_2^\epsilon(t,x)dB_2(t) \text{ in } Q_2^\epsilon,\\
c_3^\epsilon du_3^\epsilon &= \epsilon\nabla \cdot \mu_3^\epsilon(x, \epsilon\nabla u_3^\epsilon(t,x))dt + f_3^\epsilon(t,x)dB_3(t) \text{ in } Q_2^\epsilon,\\
u_1^\epsilon &= \alpha u_2^\epsilon + \beta u_3^\epsilon \text{ on } \Gamma_{1,2}^\epsilon,\\
\alpha\mu_1^\epsilon(x, \nabla u_1^\epsilon(t,x)) \cdot \vec{\nu}_1 &= \mu_2^\epsilon(x, \nabla u_2^\epsilon(t,x)) \cdot \vec{\nu}_1 \text{ on } \Gamma_{1,2}^\epsilon,\\
\beta\mu_1^\epsilon(x, \nabla u_1^\epsilon(t,x)) \cdot \vec{\nu}_1 &= \epsilon\mu_2^\epsilon(x, \epsilon\nabla u_2^\epsilon(t,x)) \cdot \vec{\nu}_1 \text{ on } \Gamma_{1,2}^\epsilon,
\end{aligned}
$$

where $t \in [0,T]$, $T \in (0,\infty)$, Q_1^ϵ represents the fissures, Q_2^ϵ represents the porous matrix and $Q_i^\epsilon,\ i = 1,2$ are periodically structured. The first equation is the conservation of mass defined in the fissures, with $u_1^\epsilon(t,x)$ representing the flow potential in the fissures. We have two components of flow potential in the matrix; $u_2^\epsilon(t,x)$ represents the usual flow through the matrix and $u_3^\epsilon(t,x)$ scaled by ϵ^p represents the very high frequency variation in the flow resulting from the relatively low permeability of the matrix (p is a positive number measuring the growth of the gradient in $\mu_i^\epsilon,\ i = 1,2,3$), $u_i^\epsilon\ i = 1,2,3$ satisfy homogeneous Dirichlet boundary conditions on $\partial Q \cap \partial Q_i^\epsilon$. These flows are assumed to satisfy corresponding conservation equations. $(B_i(t))_{0\leq t\leq T}(i = 1,2,3)$ are mutually independent standard 1-dimensional Wiener processes defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F})_{0\leq t\leq T})$.
Assumptions made on the prescribed values are stated in Chapter 3.
The aim is to show that the sequence of solutions $\vec{u}^\epsilon = [u_1^\epsilon, u_2^\epsilon, u_3^\epsilon]$ of (P^ϵ) converges in suitable topologies to the stochastic process $\vec{u} = [u_1, u_2, u_3]$ which is a solution

to the following system of SPDEs:

$$
\begin{aligned}
& d\int_{Y_1} c_1(y)u_1(t,x)dy + \frac{1}{\beta} d\int_{Y_2} c_3(y)U_3(t,x,y)dydt \\
&\quad = \nabla\cdot\int_{Y_1} \mu_1\big(y, \nabla u_1(t,x) + \nabla_y U_1(t,x,y)\big)dydt \\
&\quad + \int_{Y_1} f_1(t,x,y)dyd\tilde{B}_1(t) + \frac{1}{\beta}\int_{Y_2} f_3(t,x,y)dyd\tilde{B}_3(t), \\
&\quad t\in(0,T), x\in Q, y\in Y_i, i=1,2.
\end{aligned}
\tag{1.33}
$$

$$
\begin{aligned}
& d\int_Q\int_{Y_2} c_2(y)u_2(t,x)dydx - \frac{\alpha}{\beta} d\int_Q\int_{Y_2} c_3(y)U_3(t,x,y)dydx \\
&\quad = \nabla\cdot\left(\int_Q\int_{Y_2} \mu_2\big(y, \nabla u_2(t,x) + \nabla_y U_2(t,x,y)\big)dydx\right)dt \\
&\quad + \int_Q\int_{Y_2} f_2(t,x,y)dydxd\tilde{B}_2(t) - \frac{\alpha}{\beta}\int_Q\int_{Y_2} f_3(t,x,y)dydxd\tilde{B}_3(t), \\
&\quad t\in(0,T), x\in Q, \quad y\in Y_2.
\end{aligned}
\tag{1.34}
$$

$$
\begin{aligned}
& d\int_Q\int_{Y_2} c_3(y)U_3(t,x,y)dydx = \nabla_y\cdot\int_Q\int_{Y_2} \mu_3\big(y, \nabla_y U_3(t,x,y)\big)dydxdt \\
&\quad + \int_Q\int_{Y_2} f_3(t,x,y)dydxd\tilde{B}_3(t), \quad y\in Y_2 \\
&\quad U_3(t,x,y) \text{ and } \nabla_y\cdot\mu_3\big(y,\nabla_y U_3(t,x,y)\big)\cdot\nu \text{ are } Y\text{-periodic on } \Gamma_{2,2}, \\
&\quad \beta U_3 = u_1 - \alpha u_2 \text{ on } \Gamma_{1,2}.
\end{aligned}
\tag{1.35}
$$

and initial conditions

$$u_i(0,x) = u_i^0(x) \text{ for } i=1,2,$$

$$U_3(0,x,y) = u_3^0(x),$$

where

$$U_i \in \mathcal{D}((0,T)\times Q; W_0^{1,p}(Y)),\ i=1,2,3,$$

u_1, u_2, u_3 satisfy homogeneous Dirichlet boundary conditions and $\tilde{B} = (\tilde{B}_1, \tilde{B}_2, \tilde{B}_3)$ is an appropriate Wiener process to be determined later.

Our investigation makes systematic use of the two-scale convergence combined with probability methods. In the deterministic setting, this model was formulated and studied by Clark and Showalter [38]. The problem (P^ϵ) is the first being studied as a model to stochastic fluctuations of flow of fluids in fissured regions. Wright considered in [140] the case of randomly oscillating coefficients using stochastic two-scale convergence introduced by himself, Bourgeat and Mikelić in [27]. This case deals with stochastic differential equations and use different methods.

Plan of the Thesis

The rest of the thesis is organized as follows;
Chapter 2 is dedicated to the homogenization of a stochastic model of groundwater pollution in periodic porous media. A brief description of the flow being modeled is given, we derive necessary a priori estimates for the solution of the stochastic convection-diffusion equation. Since we are working on a probabilistic setting, we use the tightness of probability measures generated by the sequence of solutions of our micro model, and consequently the Prokhorov and Skorokhod compactness results as additional tools. Tartar's method of oscillating test functions and multiple scale expansion are used for the homogenization process.

In chapter 3, we investigate the homogenization of a stochastic diffusion model of flow of a single-phase fluid in a periodic partially fissured medium. We also use the Prokhorov and Skorokhod compactness results since we are working on SPDEs and then the two-scale convergence for the homogenization process.

In the last chapter, we give a brief conclusion and a brief highlight of our future work.

Chapter 2

Homogenization of a Stochastic Model of Groundwater Pollution in a periodic Porous Medium

2.1 Introduction

In this chapter, we investigate the homogenization of a stochastic model of groundwater pollution governed by a coupled system of stochastic convection-diffusion, reaction-diffusion and steady Stokes equations in porous medium using different homogenization techniques.

In recent times, growing interest is being devoted to the ecological challenge of groundwater contamination by hazardous industrial wastes, spills of oil and toxic liquid or agricultural activities by the use of pesticides, fertilizers etc. Even though these contaminants originate at ground surface, they soon penetrate the ground surface and seep through the unsaturated zone to the groundwater in an underlying aquifer. When it reaches the aquifers, the contaminants are transported with the moving groundwater making its way to lakes, streams and pumping wells. At times, toxic chemicals e.g. oil may make up a separate liquid phase that fill the pore space. Components of such toxic liquids may melt in percolating water becoming a source of contamination for groundwater. Chemical species transported by the water may react with each other and/or with the soil, resulting in phenomena such as adsorption, dissolution, chemical reaction and ion exchange which continually affect the concentration of the chemical constituents present in the percolating water. See Figure 2.1.

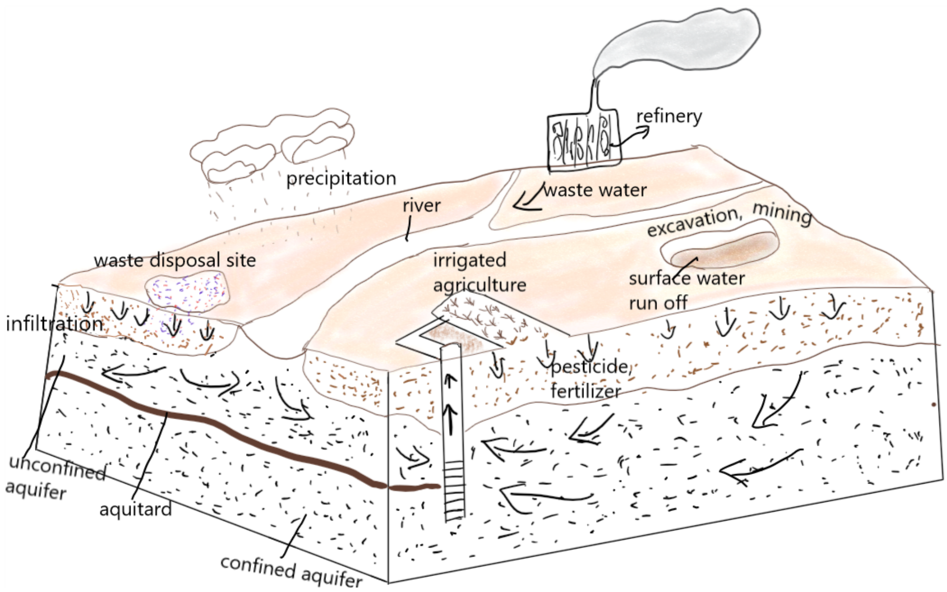

Figure 2.1: Sources of groundwater contamination

The issues have been expounded authoritatively in the monographs [13], [15], [16] and [142].

In view of the effect of groundwater pollution to the society, groundwater and contaminant flow transport modelling are being used to help with planning to remedy groundwater pollution at various hazardous waste sites, since it provides useful predictions of the rate and directions of groundwater flow and contaminant transport.

The concept of porous media is used to describe the aquifer which is a system of voids and solids filled with fluid. To model and solve problems related to groundwater pollution, detailed data on the void space is needed but this is impossible to get as measurements cannot be taken at the microscopic level. Physical or chemical processes present on the microscopic scale of porous media can be modeled using differential equations with initial and boundary conditions. The microscopic system of these materials are complex, which can make numerical simulation very cumbersome, giving rise to the need to derive a homogenized macroscopic model through the process of homogenization. The homogenized (macroscopic) model of the phenomenon under investigation is obtained by an asymptotic analysis as $\epsilon \to 0$ of the problem modeled on the microscopic scale. As it is, the limit of the solution to the microscopic problem satisfies a new differential equation with better regularity in a simpler domain, this new differential equation is the macroscopic model which is then used for applications.

Here, we undertake the investigation of the flow of a fluid transporting reacting solutes under the influence of a random external force, in a porous medium. We assume that the porous medium is made up of periodically distributed cells scaled by a small factor ϵ. Each cell consists of the fluid part and a solid part. We further assume that the liquid is incompressible and the flow of the liquid is controlled by the steady Stokes equation. This fluid contains a solute which reacts with substances bound to the surface of the solid part. The concentration of the solute in the fluid phase is described by a stochastic convection-diffusion system of equations and the concentration of the solute on the boundary of the solid phase by a diffusion-reaction equation. The homogenization of these systems of equations are investigated in the sections that follow. The homogenized stochastic convection-diffusion equations contains an extra term coming from the boundary terms of the microscopic problem. Our work is the stochastic counterpart of the work by Hornung and Jäger [57] in the deterministic case. Several auxiliary results from their work will be used here. For related works done in the deterministic case on the homogenization of Stokes and convection-diffusion equations, in addition to [57], we note the fundamental work [129] of Tartar and [4], [6], [7], [8], [22], [33],

[56], [59], [66], [70], [77], [81].

Few words are in order regarding the methodology used in the following sections. We implement Tartar's method of oscillating test functions in the construction of the homogenized problem for the stochastic convection-diffusion equation (see equation $(C^\epsilon)_{\epsilon>0}$ in Section 2.2.2). This requires, among others, some appropriate probabilistic tools such as the crucial Ito's stochastic calculus for the derivation of uniform a priori estimates and more importantly the fundamental compactness results due to Prokhorov [105] and Skorokhod [122] which are needed for the path-wise strong convergence of the sequence of solutions $(v^\epsilon)_{\epsilon>0}$ of problem (C^ϵ); as a stepping stone towards that strong convergence, we establish the tightness of a family of probability measures generated by v^ϵ and the driving Wiener process. We use the well known method of asymptotic expansions ([19], [11]) to derive the homogenized problem for the reaction-diffusion equations prescribed on the boundary of the holes (see problem (R^ϵ) in Section 2.2.2). For the steady Stokes equations (SS^ϵ), we include Tartar's proof using his method of oscillating test functions.

The plan of the chapter is as follows, In section 2.2, we state the main assumptions on the geometry of the porous medium under consideration, we introduce the microscopic models of the processes taking place in the perforated porous medium and their corresponding macroscopic homogenized problems, the existence and uniqueness of the governing stochastic convection-diffusion equation, the reaction-diffusion equation modelling the concentration of the solute and the Stokes equation for the velocity of the fluid is also included in the section. In section 2.3, we derive relevant uniform a priori estimates for the solutions of these equations. In Section 2.4 we implement Prokhorov and Skorokhod compactness procedure thanks to a relevant tightness result for a family of probability measures linked to the sequence (v^ϵ) and the Wiener process driving problem (C^ϵ), we also derive the homogenization results.

2.2 Existence and Uniqueness of Probabilistic Solution

Although we are mainly interested in the homogenization of a stochastic convection-diffusion equation, we shall include all the equations governing the fluid phase and the solid phase i.e the steady Stokes equation for the fluid's velocity and a reaction-diffusion equation for the surface of the porous medium.

2.2.1 Setting of the problem and assumptions

Let ϵ be a positive parameter taking its values in a sequence which tends to zero and let $[0, T]$ denote a time interval with $T \in (0, \infty)$.
Let U be a bounded domain in $\mathbb{R}^3$ consisting of two sub domains: the fluid phase (the pore space) filled with fluid where the transport, flow and diffusion take place and the solid phase (the perforations) where diffusion and reaction take place, see Figure 2.2.
At the microscopic level, the domain of interest is denoted by U^ϵ (the fluid phase)

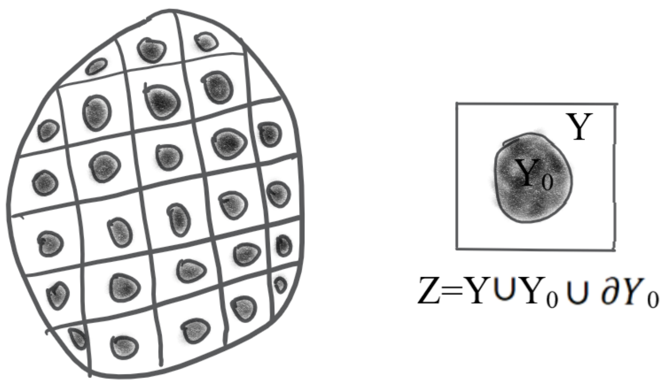

Figure 2.2: An illustration of a porous medium U consisting of ϵ-scaled periodically distributed perforations and a representative cell Z.

and the boundary of the perforations by Γ^ϵ. The piecewise boundary of U denoted by ∂U is made up of two parts

$$\partial U = \Gamma_D \cup \Gamma_N \quad \text{and} \quad \Gamma_D \cap \Gamma_D = \emptyset.$$

Let Z be a unit cell in $\mathbb{R}^3$ and denote by $Y_0 \subset Z$ - the representative obstacle (perforation), $Y = Z \backslash Y_0$ - the representative pore (fluid part), $\Gamma = \partial Y_0$ - the piecewise smooth boundary of Y_0 and $\vec{\nu}$ - the outer unit normal on ∂U with respect to U. The microscopic structure of U^ϵ and Γ^ϵ is assumed to be periodic and is obtained by the repetition of the cell Z scaled by a small parameter ϵ.

For a given scale factor $\epsilon > 0$, let us define the pore skeleton (total perforations) as follows

$$U_0^\epsilon = \bigcup_{k \in \mathbb{Z}^3} \{\epsilon Y_0^k ; Y_0^k \subset U\}.$$

Then the fluid part (pore volume) of the medium is defined by

$$U^\epsilon = U \backslash \bar{U}_0^\epsilon,$$

the surface of the skeleton (total boundary of the perforations) Γ^ϵ is defined as

$$\Gamma^\epsilon = \partial U_0^\epsilon = \{\epsilon \Gamma^k ; \epsilon \Gamma^k \subset U ; k \in \mathbb{Z}^3\},$$

$$\Gamma^* = \cup \{\Gamma^k, k \in \mathbb{Z}^3\},$$

we denote by $\vec{\nu}^\epsilon$ the inner normal on Γ^ϵ with respect to U_0^ϵ, and the perforations Γ^ϵ do not intersect with ∂U . By this construction, U^ϵ is a perforated domain. Let us define the following characteristic function

$$\chi^\epsilon = \begin{cases} 1, & x \in U^\epsilon \\ 0 & x \in U_0^\epsilon. \end{cases} \tag{2.1}$$

2.2.2 The micro model

Now we formulate the equations that model the processes at the microscopic level. The micro model consists of three components: the system of equations describing the flow of the liquid, the system of equations modelling the concentration of the solute in the fluid and the system of equations describing the concentration of the solute on the surface of the skeleton.

In the fluid part U^ϵ of the porous medium, we model the velocity of the flow $\vec{u}^\epsilon$ using the steady Stokes problem with a no-slip boundary condition at the boundary of the perforations;

$$(SS^\epsilon)\begin{cases} \epsilon^2\Delta\vec{u}^\epsilon = \nabla p^\epsilon & x \in U^\epsilon, \\ \nabla\vec{u}^\epsilon = 0 & x \in U^\epsilon, \\ \vec{\nu}\vec{u}^\epsilon = 0 & x \in \mathit{\Gamma}_N, \\ \vec{u}^\epsilon = \vec{u}_D & x \in \mathit{\Gamma}_D, \\ \vec{u}^\epsilon = 0 & x \in \mathit{\Gamma}^\epsilon, \end{cases}$$

where u^ϵ is the velocity of the fluid, p^ϵ is the pressure inside the fluid, $\vec{u}_D$ is the prescribed boundary value on $\mathit{\Gamma}_D$ and

$$\int_{\mathit{\Gamma}_D} \vec{\nu}\cdot\vec{u_D} = 0.$$

The main focus of the first part of this thesis is the system of equations modelling the concentration of the solute in the porous medium. This takes into account the solute being transported by the fluid under the influence of external random force. This external random factor affecting the concentration of the solute is captured using a stochastic process and it is represented in the model by a stochastic term $G^\epsilon(t,x)dB(t)$ where G^ϵ is the intensity of the noise. In the fluid, the solute is diffusing in the absence of any reaction. The corresponding model for the concentration of the solute v^ϵ is the following stochastic convection-diffusion equation in the fluid part U^ϵ;

$$(C^\epsilon)\begin{cases} dv^\epsilon(t,x) & = D\Delta v^\epsilon(t,x)dt - \vec{u}^\epsilon(x)\nabla v^\epsilon(t,x)dt + G^\epsilon(t,x)dB(t), \; t>0, x \in U^\epsilon \\ v^\epsilon(t,x) & = v_D(t,x), \qquad t>0, x\in \mathit{\Gamma}_D \\ \vec{\nu}\nabla v^\epsilon(t,x) & = 0, \qquad t>0, x \in \mathit{\Gamma}_N \\ v^\epsilon(0,x) & = v_1(x), \qquad t=0, x\in U^\epsilon \\ -D\vec{\nu}^\epsilon\nabla v^\epsilon(t,x) & = \epsilon f^\epsilon(t,x), \qquad t>0, x\in\mathit{\Gamma}^\epsilon \end{cases}$$

with

$$f^\epsilon(t,x) = c^\epsilon(x)v^\epsilon(t,x) - b^\epsilon(x)w^\epsilon(t,x), \tag{2.2}$$

where $t \in [0,T],\ T \in (0,\infty),\ \vec{u}^\epsilon$ is the velocity field, $D > 0$ - the diffusion coefficient, v^ϵ represents the concentration of a solute in the fluid, G^ϵ is the intensity of the noise, $(B(t))_{0\leq t\leq T}$ is a 1-dimensional Wiener process defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\leq t\leq T})$. v_D is the prescribed values on the boundary $\mathit{\Gamma}_D$ and $v_1(x)$ is the initial condition. $c : \mathit{\Gamma}^* \to\ \mathbb{R}$ represents the adsorption factor, $b : \mathit{\Gamma}^* \to \mathbb{R}$ represent the desorption factor and w^ϵ represents the concentration of the solute on the surface of the skeleton,.
The definition of f^ϵ in (2.2) describes the adsorption and desorption processes on the surface of the skeleton and its contribution to the concentration of the solute in the fluid. In particular, $c^\epsilon(x)v^\epsilon(t,x)$ captures adsorption from the fluid to the surface and $b^\epsilon(x)w^\epsilon(t,x)$ captures desorption from the surface into the fluid.
We note that all the arguments used in the work readily extend to the case when $B(t)$ is infinite-dimensional.

On the surface of the skeleton (boundary of perforations) where the solute is diffusing and reacting with substances bound to the surface, the concentration of the solute on the surface of the skeleton w^ϵ is modeled using a diffusion-reaction equation on the surface of the skeleton $\mathit{\Gamma}^\epsilon$.

$$(R^\epsilon)\begin{cases} \partial_t w^\epsilon - \epsilon^2 E\Delta^\epsilon w^\epsilon + a^\epsilon(x)w^\epsilon = f^\epsilon & t > 0, x \in \mathit{\Gamma}^\epsilon, \\ w^\epsilon = w_1(x) & t = 0, x \in \mathit{\Gamma}^\epsilon, \end{cases}$$

where w^ϵ represents the concentration of the solute on the surface of the skeleton, $E > 0$ is the diffusion coefficient on the surface of the skeleton, Δ^ϵ is the Laplace-Beltrami operator on $\mathit{\Gamma}^\epsilon$, $a : \mathit{\Gamma}^* \to \mathbb{R}$ represents the reaction factor and f^ϵ is defined by (2.2).

We make the following assumptions;
(A1) $G^\epsilon(t,x) \in L^2((0,T)\times U^\epsilon)$, with $\tilde{G}^\epsilon(t,x) \rightharpoonup G(t,x)$ weakly in $L^2((0,T)\times U)$, where $\tilde{G}^\epsilon$ denotes the extension by zero outside of U^ϵ.

(A2) a, b, c are Z-periodic, $a, b, c \geq 0 : a, b, c$ are bounded and $a^\epsilon(x) = a\big(\frac{x}{\epsilon}\big), b^\epsilon(x) = b\big(\frac{x}{\epsilon}\big), c^\epsilon(x) = c\big(\frac{x}{\epsilon}\big)$.

(A3) $v_1 \in L^2(U^\epsilon),\ w_1 \in L^2(U^\epsilon)$.

For the construction of the homogenized problems satisfied by the limits v and w of v^ϵ and w^ϵ, respectively, we need appropriate cell problems which we now introduce following [57].

Let $\vec{\mu}_j : Y \to \mathbb{R}^n$ and $\pi : Y \to \mathbb{R}$ be a pair of Z-periodic functions satisfying the following cell problem;

$$\begin{cases} \Delta_y \vec{\mu}_j(y) = \nabla_y \pi_j - \vec{e}_j, & y \in Y \\ \nabla_y \vec{\mu}_j(y) = 0 & y \in Y \\ \vec{\mu}_y(y) = 0 & y \in \mathit{\Gamma}, \end{cases} \tag{2.3}$$

where $\vec{e}_j$ is the j-th canonical vector of the basis of $\mathbb{R}^3$. Letting the mean value $\tilde{\mu}_j$ of $\vec{\mu}_j$ be defined by

$$\tilde{\mu}_j = \frac{1}{|Y|} \int_Y \vec{\mu}_j(y)\, dy, \tag{2.4}$$

we define the tensor K with elements k_{ij} by

$$k_{ij} = \tilde{\mu}_{ij} \text{ (ith component of } \tilde{\mu}_j). \tag{2.5}$$

Let $\sigma_j : \bar{Y} \to \mathbb{R},\ (j = 1, 2, 3)$ be a Z -periodic solution of the cell problem

$$\begin{cases} \Delta_y \sigma_j(y) = 0, & u \in Y \\ \vec{\nu} \nabla_y \sigma_j(y) = -\vec{\nu}\vec{e}_j, & y \in \mathit{\Gamma}. \end{cases} \tag{2.6}$$

We extend σ_j to Y_0 such that

$$\nabla_y \sigma_j(y) = 0, \quad \forall y \in Y_0.$$

Let S be the tensor whose components s_{ij} are given by

$$s_{ij} = \delta_{ij} + \int_Y \partial_i \sigma_j(y) : dy, \tag{2.7}$$

we note that K and S are positive definite and symmetric tensors.

In addition, let $l : [0, \infty) \times \mathit{\Gamma}^* \to \mathbb{R}$ be the Z-periodic solution of the cell problem

$$\begin{cases} \partial_t l(t, y) - E \Delta^{\mathit{\Gamma}} l(t, y) + (a(y) + b(y)) l(t, y) = 0 & t > 0, y \in \mathit{\Gamma}^* \\ l(0, y) = 1 & t = 0, y \in \mathit{\Gamma}^*. \end{cases} \tag{2.8}$$

We introduce the following functions related to (2.8) following verbatim the work of Hornung and Jäger [57] (pg 204).

The functions $\rho, \beta : [0.\infty) \times \mathit{\Gamma} \to \mathbb{R}$ are defined as

$$\rho(t) = \int_{\mathit{\Gamma}} l(t,y) b(y) d\mathit{\Gamma}(y),$$

$$\beta(t) = \int_{\mathit{\Gamma}} e(t,y) b(y) d\mathit{\Gamma},$$

where $e : [0,\infty) \times \mathit{\Gamma} \to \mathbb{R}$ is

$$e(t,y) = 1 - \int_0^t l(t-s,y)(a(y)+b(y))ds$$

and the constant γ is defined as

$$\gamma = \frac{1}{|\mathit{\Gamma}|} \int_{\mathit{\Gamma}} c(y) d\mathit{\Gamma}(y).$$

Now we are in the position to state the homogenized problems corresponding to (C^ϵ) and (R^ϵ).

The stochastic process v is the solution of the following SPDE

$$(C)\begin{cases} dv(t,x) + F(t,x)dt = D\nabla(S\nabla v(t,x))dt - \frac{1}{|Y|}\vec{u}(x)\nabla v(t,x)dt & \\ +\frac{1}{|Y|}G(t,x)dB(t), & t>0, x\in U \\ v(t,x) = v_D(t,x), & t>0, x\in \mathit{\Gamma}_D \\ \vec{\nu}\nabla v(t,x), = 0 & t>0, x\in \mathit{\Gamma}_N \\ v(0,x) = v_1(x), & t=0, x\in U \end{cases}$$

where

$$F(t,x) = |\mathit{\Gamma}|(\gamma v(t,x) - \beta(t)w_1(x) - \rho(\cdot) * v(\cdot,x)(t)),$$

S is defined in (2.7), $\vec{u}$ satisfies the homogenized steady Stokes equation

$$(SS)\begin{cases} \vec{u}(x) = K\nabla P(x), & x\in U \\ \nabla\vec{U} = 0, & x\in U \\ \vec{\nu}\vec{u}(x) = \vec{u}_D(x,) & x\in \mathit{\Gamma}_D \\ \vec{\nu}\vec{u}(x) = 0, & x\in \mathit{\Gamma}_N \end{cases}$$

where K is defined in (2.5) and w is the solution to the reaction-diffusion equation

$$(R)\begin{cases}\partial_t w(t,x,y) - E\nabla_y^{\Gamma} w(t,x,y) + a(y)w(t,x,y) & \\ = f(t,x,y), & t>0, x\in U, y\in \Gamma \\ w(0,x,y) = w_1(x), & t=0, x\in U, y\in \Gamma\end{cases} \tag{2.9}$$

with

$$f(t,x,y) = c(y)v(t,x) - b(y)w(t,x,y);$$

the problem (R) is the homogenized problem for (R^ϵ).

For the proof of these homogenized problems, we use Tartar's method of oscillating test functions combined with compactness results of analytic and probabilistic nature (Prokhorov-Skorokhod procedure). We derived (R) thanks to the formal asymptotic expansion method which is popular in the engineering community, thereby making our work accessible to a wider range of researchers with different backgrounds (applied mathematicians and engineers). We hereby note that a more involved mathematical derivation of (R) is possible thanks to Nguetseng-Allaire's two scale convergence (see [3], [91]).

2.2.3 Existence, uniqueness and a priori estimates for (SS^ϵ), (C^ϵ) and (R^ϵ)

Our first step is to discuss the issue of existence and uniqueness of probabilistic solution for problem (C^ϵ). Let us introduce the Hilbert space

$$V_\epsilon = \{\phi | \phi \in H^1(U^\epsilon) : \phi = 0 \text{ on } \Gamma_D \text{ a. e on } (0,T)\}$$

Definition 2.1. We define the strong probabilistic solution of problem (C^ϵ) as a stochastic process v^ϵ such that

1. $v^\epsilon \in v_D + L^2(\Omega, \mathcal{F}, \mathbb{P}, L^\infty(0,T; L^2(U^\epsilon))) \cap L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0,T; V_\epsilon)$,

2. $\forall t \in [0,T], v^\epsilon(t,.)$ satisfies

$$\begin{aligned}(v^\epsilon(t,\cdot),\phi) &+ \int_0^t (D\nabla v^\epsilon(s,\cdot),\nabla\phi)ds + \int_0^t (\epsilon f^\epsilon(s,\cdot),\phi)_{L^2(\Gamma^\epsilon)}ds \\ &+ \int_0^t (u^\epsilon(x)\nabla v^\epsilon,\phi)ds = (v^\epsilon(0,\cdot),\phi)\int_0^t (G^\epsilon(s,\cdot),\phi)dB(s), \end{aligned} \tag{2.10}$$

$$v^\epsilon(0,\cdot) = v_1(\cdot), \quad \forall\phi \in V_\epsilon.$$

Theorem 2.2. *For each $\epsilon > 0$, under the assumptions $(A1)-(A3)$, problem (C^ϵ) has a unique strong probabilistic solution*

$$v^\epsilon \in L^2(\Omega,\mathcal{F},\mathbb{P},L^\infty(0,T;L^2(U^\epsilon))) \cap L^2(\Omega,\mathcal{F},\mathbb{P},L^2(0,T;H^1(U^\epsilon))),$$

in the sense of Definition (2.1).

This result is closely related to the works of Pardoux [101] and Rozovskiĭ [112] (Theorem 4, page 90), but due to the presence of the transport term in (C^ϵ), their arguments need careful adaptation.

First, we discuss briefly the existence and uniqueness of problems (SS^ϵ) and (R^ϵ). Let us introduce the following spaces

$$\mathcal{W}^\epsilon = \{\vec{\psi}|\vec{\psi} \in \left(W^{2,2}(U^\epsilon)\right)^3 : \vec{\psi} = 0 \text{ on } \Gamma_D, \nabla\cdot\vec{\psi} = 0 \text{ in } U^\epsilon\},$$

$$\mathcal{V}^\epsilon = \{\phi|\phi \in L^2(0,T;H^1(\Gamma^\epsilon)), \partial_t\phi \in L^2(0,T;(H^1(\Gamma^\epsilon))')\}.$$

The variational formulation of problem (SS^ϵ) is then the following

$$\begin{cases}\text{Find } \vec{u}^\epsilon \in (\vec{u}_D + \mathcal{W}^\epsilon), p^\epsilon \in L^2(U^\epsilon) \text{ such that} \\ \epsilon^2 \int_{U^\epsilon} \nabla\vec{u}^\epsilon\nabla\vec{\psi}\,dx = -\int_{U^\epsilon} \nabla p^\epsilon\vec{\psi}\,dx \quad \forall\vec{\psi} \in \mathcal{W}^\epsilon.\end{cases}$$

The problem of existence and uniqueness of the solution $(\vec{u}^\epsilon, p^\epsilon)$ can be found in [129], [135], [33]. The membership of $\vec{u}^\epsilon$ to $\mathcal{W}^\epsilon$ enables us to get L^∞ regularity of $\vec{u}^\epsilon$ thanks to Sobolev embedding theorem. This will be crucial in controlling the transport term in problem (C^ϵ). It should be noted that thanks to differentiation with respect to time in the deterministic version of (C^ϵ) considered by Hornung and Jäger [57], they obtained similar results under the regularity $W^{1,2}$ for $\vec{u}^\epsilon$. However such a differentiation with respect to time is prohibited in the stochastic case due to the presence of the noise. Our approach therefore overcomes this difficulty by requiring more regularity from $\vec{u}^\epsilon$ while demanding a priori less regularity in time from v^ϵ unlike [57].

The variational formulation of problem (R^ϵ) is

$$\begin{cases} \text{Find } w^\epsilon \in \mathcal{V}^\epsilon \text{ such that} \\ \langle \partial_t w^\epsilon, \phi \rangle_{(H^1(\Gamma^\epsilon))', H^1\Gamma^\epsilon)} + \epsilon^2 E \int_{\Gamma^\epsilon} \nabla^\epsilon w^\epsilon \nabla^\epsilon \phi \, d\Gamma + \int_{\Gamma^\epsilon} a^\epsilon w^\epsilon \phi \, d\Gamma \\ = \int_{\Gamma^\epsilon} f^\epsilon \phi \, d\Gamma \quad \text{in } \mathcal{D}'(0,T), \quad \forall \phi \in \mathcal{V}^\epsilon \\ w^\epsilon(0,x) = w_1(x) \end{cases}$$

2.3 A priori Estimates

In this section, we derive and state some needed a priori estimates for solutions of problems (SS^ϵ), (C^ϵ) and (R^ϵ) uniform with respect to ϵ. Here and throughout the thesis, we shall denote by C a constant independent of ϵ.
The following technical result (see for instance [57]) will be needed.

Lemma 2.3. *For a function $\psi \in H^1(Y)$, one has the estimate*

$$||\psi||_\Gamma^2 \leq C\big(||\psi||_Y^2 + ||\nabla\psi||_Y^2\big).$$

For a function $\psi^\epsilon \in H^1(U^\epsilon)$, one has the estimate

$$\epsilon||\psi^\epsilon||_{\Gamma^\epsilon}^2 \leq C\big(||\psi^\epsilon||_{U^\epsilon}^2 + \epsilon^2||\nabla\psi^\epsilon||_{U^\epsilon}^2\big).$$

The following result is due to Tartar [129].

Lemma 2.4. *Let $\tilde{\vec{u}}^\epsilon(x)$ be the extension of $\vec{u}^\epsilon$ by zero to all of U. Then there exists an extension of p^ϵ from U^ϵ to all of U which we still denote by p^ϵ such that for a constant C independent of ϵ,*

$$||\nabla\tilde{\vec{u}}^\epsilon||_{L^2(U)} \leq \frac{C}{\epsilon}; \qquad ||\tilde{\vec{u}}^\epsilon||_{L^2(U)} \leq C, \tag{2.11}$$

$$||p^\epsilon||_{L^2(U)/\mathbb{R}} \leq C; \qquad ||\nabla p^\epsilon||_{H^{-1}(U)} \leq C, \tag{2.12}$$

and

$$\tilde{\vec{u}}^{\epsilon} \rightharpoonup u^* \text{ weakly in } L^2(U) \tag{2.13}$$

$$p^{\epsilon} \to p^* \text{ strongly in } L^2(U)/\mathbb{R}. \tag{2.14}$$

The proof of this result can be found in [76], [129]

Lemma 2.5. *For all $\alpha > 0$ and $t \geq 0$, we have the following estimate*

$$\sup_{0\le t\le T} ||w^{\epsilon}|^2_{L^2(\Gamma^{\epsilon})} + \epsilon^2 E(\nabla^{\epsilon} w^{\epsilon}, \nabla^{\epsilon} w^{\epsilon})_{L^2(0,T;L^2(\Gamma^{\epsilon}))}$$
$$\le e^{\alpha t}\big(||w^{\epsilon}(0)||^2_{L^2(\Gamma^{\epsilon})} + \frac{1}{\alpha}||c^{\epsilon}||^2_{L^{\infty}(\Gamma^{\epsilon})}\int_0^t ||v^{\epsilon}||^2_{L^2(\Gamma^{\epsilon})} ds\big) \quad \mathbb{P}\text{-}a.s.$$

The proof of this result is contained in [57](pg 210-211).

Lemma 2.6. *Under the assumptions $(A1)-(A3)$, the solution v^{ϵ} of (C^{ϵ}) satisfies the following estimate*

$$\mathbb{E}\sup_{0\le t\le T} ||v^{\epsilon}||^2_{L^2(U^{\epsilon})} + \mathbb{E}\int_0^T ||\nabla v^{\epsilon}||^2_{L^2(U^{\epsilon})} \le C.$$

Proof. Ito's Lemma gives

$$||v^{\epsilon}(t)||^2_{L^2(U^{\epsilon})} = ||v_1(x)||^2_{L^2(U^{\epsilon})} + 2\int_0^t (D\Delta v^{\epsilon}, v^{\epsilon}) ds - 2\int_0^t (\vec{u}^{\epsilon}(x)\nabla v^{\epsilon}, v^{\epsilon}) ds$$
$$+ 2\int_0^t (G^{\epsilon}, v^{\epsilon}) dB(s) + \int_0^t ||G^{\epsilon}||^2_{L^2(U^{\epsilon})} ds.$$

Integrating by parts on the second and third term on the right hand side gives

$$||v^{\epsilon}(t)||^2_{L^2(U^{\epsilon})} + 2D\int_0^t ||\nabla v^{\epsilon}||^2_{L^2(U^{\epsilon})} ds = ||v_1(x)||^2_{L^2(U^{\epsilon})} + 2\epsilon\int_0^t (b^{\epsilon} w^{\epsilon}, v^{\epsilon})_{\Gamma^{\epsilon}} ds$$
$$- 2\epsilon\int_0^t (c^{\epsilon} v^{\epsilon}, v^{\epsilon})_{\Gamma^{\epsilon}} ds + 2\int_0^t (G^{\epsilon}, v^{\epsilon}) dB(s) + \int_0^t ||G^{\epsilon}||^2_{L^2(U^{\epsilon})} ds.$$

Taking the supremum over $t \in [0, T]$ followed by the expectation on both sides, we have

$$\mathbb{E}\sup_{0\le t\le T} ||v^{\epsilon}(t)||^2_{L^2(U^{\epsilon})} + C\mathbb{E}\int_0^T ||\nabla v^{\epsilon}||^2_{L^2(U^{\epsilon})} dt \le C\mathbb{E}\Big[||v_1(x)||^2_{L^2(U^{\epsilon})}$$
$$+ \epsilon\int_0^T (b^{\epsilon} w^{\epsilon}, v^{\epsilon})_{\Gamma^{\epsilon}} dt + \epsilon\int_0^T (c^{\epsilon} v^{\epsilon}, v^{\epsilon})_{\Gamma^{\epsilon}} dt \tag{2.15}$$
$$+ \int_0^T ||G^{\epsilon}||^2_{L^2(U^{\epsilon})} dt + \sup_{0\le t\le T}\int_0^t (G^{\epsilon}, v^{\epsilon}) dB(s)\Big].$$

Using Cauchy-Schwarz's and Young's inequalities together with the assumptions on c^ϵ and b^ϵ yields

$$\begin{aligned}
&C\mathbb{E}\left[\epsilon\int_0^T (b^\epsilon w^\epsilon, v^\epsilon)_{\Gamma^\epsilon}dt + \epsilon\int_0^T (c^\epsilon v^\epsilon, v^\epsilon)_{\Gamma^\epsilon}dt\right]\\
\leq\ & C(\epsilon)\mathbb{E}\int_0^T \epsilon||w^\epsilon||^2_{L^2(\Gamma^\epsilon)}dt + C_1\mathbb{E}\int_0^T \epsilon||v^\epsilon||^2_{L^2(\Gamma^\epsilon)}dt
\end{aligned}$$

where $C_1 = C(\epsilon) + C$.

Using Lemma 2.5 and Lemma 2.3, we get

$$\begin{aligned}
&C\mathbb{E}\int_0^T \epsilon||w^\epsilon||^2_{L^2(\Gamma^\epsilon)}dt + C_1\mathbb{E}\int_0^T \epsilon||v^\epsilon||^2_{L^2(\Gamma^\epsilon)}dt\\
&\leq C\mathbb{E}\int_0^T \left(\epsilon||w^\epsilon(0)||^2_{L^2(\Gamma^\epsilon)} + \epsilon||v^\epsilon||^2_{L^\epsilon(\Gamma^\epsilon)}\right)dt\\
&\leq C\mathbb{E}\int_0^T \epsilon||w^\epsilon(0)||^2_{L^2(\Gamma^\epsilon)}dt + C\mathbb{E}\int_0^T ||v^\epsilon||^2_{L^2(U^\epsilon)}dt\\
&+ C\mathbb{E}\epsilon^2\int_0^T ||\nabla v^\epsilon||^2_{L^2(U^\epsilon)}dt.
\end{aligned} \tag{2.16}$$

Next, thanks to Burkholder-Gundy-Davis inequality followed by Cauchy-Schwarz's and Young's inequality, we infer that

$$\begin{aligned}
\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t (G^\epsilon, v^\epsilon)dB(s)\right| &\leq C\mathbb{E}\left(\int_0^T (G^\epsilon, v^\epsilon)^2dt\right)^{\frac{1}{2}}\\
&\leq C\mathbb{E}\left(\int_0^T ||G^\epsilon||^2_{L^2(U^\epsilon)}||v^\epsilon||^2_{L^2(U^\epsilon)}dt\right)^{\frac{1}{2}}\\
&\leq C\mathbb{E}\sup_{0\leq t\leq T}||v^\epsilon||_{L^2(U^\epsilon)}\left(\int_0^T ||G^\epsilon||^2_{L^2(U^\epsilon)}\right)^{\frac{1}{2}}\\
&\leq C(\varpi)\mathbb{E}\sup_{0\leq t\leq T}||v^\epsilon||^2_{L^2(U^\epsilon)}\\
&+ C(\varpi)\int_0^T ||G^\epsilon||^2_{L^2(U^\epsilon)}dt
\end{aligned} \tag{2.17}$$

for any $\varpi > 0$. Substituting (2.16) and (2.17) into (2.15), we obtain for sufficiently small ϖ that

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}||v^\epsilon(t)||^2_{L^2(U^\epsilon)} + \mathbb{E}\int_0^T ||\nabla v^\epsilon||^2_{L^2(U^\epsilon)}dt\\
&\leq C\mathbb{E}\left[||v_1(x)||^2_{L^2(U^\epsilon)} + \int_0^T \epsilon||w^\epsilon(0)||^2_{L^2(U^\epsilon)}dt + \int_0^T ||v^\epsilon||^2_{L^2(U^\epsilon)}dt + \int_0^T ||G^\epsilon||^2_{L^2(U^\epsilon)}dt\right].
\end{aligned}$$

Owing to assumptions $(A1), (A3)$ and Gronwall's lemma, the required estimate follows. □

Next we establish crucial results on the estimate of the finite difference of v^ϵ.

Lemma 2.7. *Under the assumptions* $(A1)-(A3)$ *with the replacement of the assumptions on* G^ϵ *by* $G^\epsilon \in L^4(0,T;L^2(U^\epsilon))$, v^ϵ *satisfies the following*

$$\mathbb{E}\int_0^{T-h} ||v^\epsilon(t+h)-v^\epsilon(t)||^2_{(H^1(U^\epsilon))'}dt < Ch,$$

for any $\epsilon>0$ *and small enough* $h>0$.

Proof.

$$v^\epsilon(t+h)-v^\epsilon(t) = \int_t^{t+h} D\Delta v^\epsilon(s)ds - \int_t^{t+h} \vec{u}^\epsilon(x)\nabla v^\epsilon(s)ds + \int_t^{t+h} G^\epsilon(s)dB(s).$$

Then

$$\begin{aligned}
&\Big|\Big|v^\epsilon(t+h)-v^\epsilon(t)\Big|\Big|_{(H^1(U^\epsilon))'} \\
&\leq \Big|\Big|\int_t^{t+h} D\Delta v^\epsilon(s)ds\Big|\Big|_{(H^1(U^\epsilon))'} \\
&+\Big|\Big|\int_t^{t+h} \vec{u}^\epsilon(x)\nabla v^\epsilon(s)ds\Big|\Big|_{(H^1(U^\epsilon))'} + \Big|\Big|\int_t^{t+h} G^\epsilon(s)dB(s)\Big|\Big|_{(H^1(U^\epsilon))'}.
\end{aligned}$$

Using Fubini's theorem and integrating by parts, we have

$$\begin{aligned}
&\Big|\Big|\int_t^{t+h} D\Delta v^\epsilon(s)ds\Big|\Big|_{(H^1(U^\epsilon))'} \\
&= \sup_{\phi\in H^1(U^\epsilon),||\phi||=1} \left|\left\langle \int_t^{t+h} D\Delta v^\epsilon(s)ds, \phi\right\rangle_{(H^\epsilon(U^\epsilon))',H^1(U^\epsilon)}\right| \\
&\leq \sup_{\phi\in H^1(U^\epsilon),||\phi||=1} \int_{U^\epsilon}\int_t^{t+h} D\Delta v^\epsilon(s)\phi(x)dsdx \\
&\leq \sup_{\phi\in H^1(U^\epsilon),||\phi||=1} \int_t^{t+h} \left(\int_{U^\epsilon} D\nabla v^\epsilon(s)\nabla\phi(x)dx + \int_{\Gamma^\epsilon} D\vec{\nu}^\epsilon\nabla v^\epsilon\phi(x)d\Gamma\right)ds \\
&\leq \sup_{\phi\in H^1(U^\epsilon),||\phi||=1} C\int_t^{t+h} \big(||\nabla v^\epsilon||_{L^2(U^\epsilon)}||\nabla\phi||_{L^2(U^\epsilon)} + \epsilon(f^\epsilon,\phi)_{\Gamma^\epsilon}\big)ds \\
&\leq C\int_t^{t+h}\Big(||\nabla v^\epsilon||_{L^2(U^\epsilon)} + \epsilon\big(||w^\epsilon||_{L^2(\Gamma^\epsilon)} + ||v^\epsilon||_{L^2(\Gamma^\epsilon)}\big)\Big)ds
\end{aligned}$$

Again with Fubini's theorem, Sobolev embedding theorem on $\vec{u}^\epsilon$ and Cauchy-

Schwarz's inequality, we get

$$\begin{aligned}
&\Big|\Big|\int_t^{t+h}\vec{u}^\epsilon(x)\nabla v^\epsilon(s)ds\Big|\Big|_{(H^1(U^\epsilon))'}\\
&=\sup_{\phi\in H^1(U^\epsilon),||\phi||=1}\left|\left\langle\int_t^{t+h}\vec{u}^\epsilon(x)\nabla v^\epsilon(s)ds,\phi\right\rangle_{(H^\epsilon(U^\epsilon))',H^1(U^\epsilon)}\right|\\
&\le\sup_{\phi\in H^1(U^\epsilon),||\phi||=1}\int_{U^\epsilon}\int_t^{t+h}\vec{u}^\epsilon(x)\nabla v^\epsilon(s)\phi(x)dsdx\\
&\le\sup_{\phi\in H^1(U^\epsilon),||\phi||=1}\int_t^{t+h}||\vec{u}^\epsilon(x)\nabla v^\epsilon||_{(H^1(U^\epsilon))'}||\phi||_{H^1(U^\epsilon)}ds\\
&\le C\int_t^{t+h}||\nabla v^\epsilon||_{L^2(U^\epsilon)}ds
\end{aligned}$$

Lastly, using the continuous embedding of $L^2(U^\epsilon)$ into $(H^1(U^\epsilon))'$ together with Fubini's theorem and Ito's isometry, we get

$$\begin{aligned}
&\mathbb{E}\int_0^{T-h}\Big|\Big|\int_t^{t+h}G^\epsilon(s)dB(s)\Big|\Big|^2_{(H^1U^\epsilon)'}dt\\
&\le\mathbb{E}\int_0^{T-h}\int_t^{t+h}||G^\epsilon||^2_{L^2(U^\epsilon)}dsdt.
\end{aligned}$$

Collecting the above estimates and integrating the resulting inequality over $[0,T-h]$, we infer that

$$\begin{aligned}
&\mathbb{E}\int_0^{T-h}||v^\epsilon(t+h)-v^\epsilon(t)||^2_{(H^1(U^\epsilon))'}dt\\
&\quad\le C\mathbb{E}\int_0^{T-h}\left(\int_t^{t+h}||\nabla v^\epsilon||_{L^2(U^\epsilon)}ds+\int_t^{t+h}\epsilon\Big(||w^\epsilon||_{L^2(\Gamma^\epsilon)}+||v^\epsilon||_{L^2(\Gamma^\epsilon)}\Big)ds\right)^2dt\\
&\quad+\mathbb{E}\int_0^{T-h}\left(\int_t^{t+h}||G^\epsilon||^2_{L^2(U^\epsilon)}ds\right)dt.
\end{aligned}$$

Using Cauchy-Schwarz's inequality yields

$$\begin{aligned}
&\mathbb{E}\int_0^{T-h}\left(\int_t^{t+h}||\nabla v^\epsilon||_{L^2(U^\epsilon)}ds\right)^2dt\\
&\le\mathbb{E}\int_0^{T-h}\left(\int_t^{t+h}||\nabla v^\epsilon||^2_{L^2(U^\epsilon)}ds\right)\left(\int_t^{t+h}ds\right)dt\\
&\le C\mathbb{E}h\int_0^T||\nabla v^\epsilon||^2dt.
\end{aligned}\tag{2.18}$$

Using Cauchy-Schwarz's inequality, Lemma 2.5 and Lemma 2.3 we get

$$\begin{aligned}
&\mathbb{E}\int_0^{T-h}\left(\int_t^{t+h}\epsilon||w^\epsilon||_{L^2(\Gamma^\epsilon)}+\epsilon||v^\epsilon||_{L^2(\Gamma^\epsilon)}ds\right)^2dt\\
&\le C\mathbb{E}h\int_0^T||v^\epsilon||^2_{L^2(U^\epsilon)}dt+C\epsilon^2h\int_0^T||\nabla v^\epsilon||^2_{L^2(U^\epsilon)}dt
\end{aligned}\tag{2.19}$$

Again by Cauchy-Schwarz's inequality, we have

$$\mathbb{E}\int_0^{T-h}\int_t^{t+h}||G^\epsilon||^2_{L^2(U^\epsilon)}dsdt \le \mathbb{E}\int_0^{T-h}\left(\int_t^{t+h}ds\right)^{\frac{1}{2}}\left(\int_t^{t+h}||G^\epsilon||^4_{L^2(U^\epsilon)}ds\right)^{\frac{1}{2}}dt$$
$$\le \mathbb{E}h^{\frac{1}{2}}\left(\int_0^T ||G^\epsilon||^4_{L^2(U^\epsilon)}\right)^{\frac{1}{2}}. \tag{2.20}$$

From (2.18), (2.19), (2.20) and the assumption on G^ϵ, we have

$$\mathbb{E}\int_0^{T-h}||v^\epsilon(t+h)-v^\epsilon(t)||^2_{(H^1(U^\epsilon))'}dt \le Ch.$$

□

We have obtained the following a priori estimates for v^ϵ on the porous medium U^ϵ.

$$\mathbb{E}\sup_{0\le t\le T}||v^\epsilon||^2_{L^2(U^\epsilon)} + \mathbb{E}\int_0^T ||\nabla v^\epsilon||^2_{L^2(U^\epsilon)} \le C,$$

under the assumptions $(A1)-(A3)$, and

$$\mathbb{E}\int_0^{T-h}||v^\epsilon(t+h)-v^\epsilon(t)||^2_{(H^1(U^\epsilon))'}dt \le Ch,$$

with $G^\epsilon \in L^4(0,T;L^2(U^\epsilon))$.

2.4 Homogenization results for problems (SS^ϵ), (C^ϵ), (R^ϵ)

In this section we establish the homogenization results for problems (SS^ϵ), (C^ϵ) and (R^ϵ). This chapter is organised in the following way; In section 4.1, we present the tightness property of the probability measure generated by the sequence (B, v^ϵ) which enables us to use Prokhorov's and Skorokhod's processes in constructing the sequence of random variables $(B_{\epsilon_j}, v^{\epsilon_j})$ defined on a new probability space. In section 4.2, we construct our main result which is the homogenized problem for problem (C^ϵ) using Tartar's method of oscillating test functions. Lastly in section 4.3, we include the homogenized results of problems (SS^ϵ) and (R^ϵ).

2.4.1 Tightness property of probability measures

Before we pass to the limit in the sequence of solutions to the micro model, we extend the solution v^ϵ of the micro model to the entire domain U.

Lemma 2.8.

- *Let $\phi \in H^1(Y)$ be a given function on Y, there is an extension $\tilde{\phi}$ into Y_0 and onto all of Z, such that*

$$||\tilde{\phi}||_{H^1(Z)} \leq ||\phi||_{H^1(Y)}.$$

- *There exists an extension $\tilde{\phi}^\epsilon$ of $\phi^\epsilon \in L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0,T; H^1(U^\epsilon)))$ into all U, such that*

$$||\tilde{\phi}^\epsilon||^2_{H^1(U)} \leq ||\phi^\epsilon||^2_{H^1(U^\epsilon)}$$

uniformly for $\epsilon > 0$.

Now we state some results from [121], [25] that plays a crucial role in the proof of the tightness property.

Lemma 2.9. *Let B_0, B, B_1 be Banach spaces such that $B_0 \subset B \subset B_1$ and the injection $B_0 \subset B$ is compact. For any $1 \leq p, q \leq \infty$ and $0 < s \leq 1$. Let E be a set bounded in $L^2(0,T; B_0) \cap N^{s,p}(0,T; B_1)$, where*

$$N^{s,p}(0,T; B_1) = \{v \in L^p(0,T; B_1) : \sup_{h>0} h^{-s} ||v(t+\theta) - v(t)||_{L^2(0,T-\theta,B_1)} < \infty\},$$

for small enough $h > 0$ and $|\theta| \leq 1$. Then E is relatively compact in $L^p(0,T; B)$.

Let us introduce the space $Z = Z_1$ where

$$Z_1 = \{\phi : \sup_{0\leq t\leq T} ||\phi||^2_{L^2(U)} \leq C; \int_0^T ||\nabla\phi||^2_{L^2(U)} dt \leq C \text{ and } \sup_m \frac{1}{\nu_m} \sup_{|h|\leq\mu_m} \left(\int_0^T ||\phi(t+h) - \phi(t)||^2_{(H^1(U))'} dt\right)^{\frac{1}{2}} < \infty\},$$

where ν_m, μ_m are sequences of positive real numbers such that $\nu_m, \mu_m \to 0$ as $m \to \infty$. We endow Z with the norm

$$||\phi||_Z = \sup_{0\leq t\leq T} ||\phi||_{L^2(U)} + \left(\int_0^T ||\nabla\phi||^2_{L^2(U)} dt\right)^{\frac{1}{2}} + \sup_m \frac{1}{\nu_m} \sup_{|h|\leq\mu_m} \left(\int_0^T ||\phi(t+h)-\phi(t)||^2_{(H^1(U))'} dt\right)^{\frac{1}{2}}$$

Lemma 2.10. *The space Z constructed above is a compact subset of $L^2(0,T;L^2(U))$.*

Next we consider the space $\mathcal{X} = C(0,T;\mathbb{R}) \times L^2(0,T;L^2(U))$ equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{X})$. Let Φ_ϵ be the $\mathcal{X}, \mathcal{B}(\mathcal{X})$-valued measurable map defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\Phi_\epsilon : \omega \mapsto (B(\omega), v^\epsilon).$$

We introduce the probability measures Π^ϵ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ defined by

$$\Pi^\epsilon(S) = \mathbb{P}(\Phi_\epsilon^{-1}(S)), \quad \text{for all } S \in \mathcal{B}(\mathcal{X}).$$

Lemma 2.11. *The family of probability measures $\{\Pi^\epsilon : \epsilon > 0\}$ is tight in $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$.*

The proof is carried out following [116] and [117].

Using Prokhorov's result lemma (1.23), there exists a subsequence $\{\Pi_{\epsilon_j}\}$ of $\{\Pi_\epsilon\}$ and a probability measure Π such that

$$\Pi_{\epsilon_j} \rightharpoonup \Pi \ \text{ weakly in } \mathcal{X}.$$

Using Skorokhod's result lemma (1.24), we get the existence of a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and $\mathcal{X}$ -valued random variables $(B_{\epsilon_j}, v^{\epsilon_j})$ and $(\hat{B}, v)$ defined on $(\hat{\Omega}, \hat{\mathcal{F}}, \mathbb{P})$ such that the probability law of $(B_{\epsilon_j}, v^{\epsilon_j})$ is Π_{ϵ_j} and that of $(\hat{B}, v)$ is Π. Furthermore,

$$(B_{\epsilon_j}, v^{\epsilon_j}) \longrightarrow (\hat{B}, v) \quad \text{in } \mathcal{X} \ \ \mathbb{P}\text{-a.s.} \tag{2.21}$$

Since w^ϵ is a random variable through its dependence on v^ϵ, then the Prokhorov-Skorokhod process induces the existence of a corresponding sequence of random variable w^{ϵ_j} which has the same distribution as w^ϵ.

Let $\hat{\mathcal{F}}_t$ be the σ-algebra generated by $\{\hat{B}(s), v(s), \ 0 \le s \le t\}$. We show that $\hat{B}$ is an $\hat{\mathcal{F}}_t$-adapted standard Wiener process.

Theorem 2.12. *For any $\phi \in C^\infty(U^\epsilon)$, and $t \in [0,T]$. The sequence $(B_{\epsilon_j}, v^{\epsilon_j})$ satisfies $\mathbb{P}-a.s.$*

$$\begin{aligned}(v^{\epsilon_j}(t,\cdot),\phi) &+ \int_0^t (D\nabla v^{\epsilon_j}(s,\cdot),\nabla\phi)ds + \int_0^t (\vec{u}^{\epsilon_j}(\cdot)\nabla v^{\epsilon_j}(s,\cdot),\phi)ds + \int_0^t (\epsilon f^{\epsilon_j}(s,\cdot),\phi)_{\Gamma^\epsilon} ds \\ &= (v^{\epsilon_j}(0,\cdot),\phi) + \int_0^t (G^{\epsilon j}(s,\cdot),\phi)dB_{\epsilon j}(s) + \int_0^t (f^{\epsilon_j}(s,\cdot),\phi)_{\Gamma^\epsilon} ds,\end{aligned}$$

with $(v^{\epsilon_j}(0,\cdot),\phi) = (v_1(x),\phi)$.

w^ϵ *satisfies the corresponding equation* (R^{ϵ_j}) *with* w^ϵ *and* v^ϵ *replaced by* w^{ϵ_j} *and* v^{ϵ_j} *respectively and* $\vec{u}^{\epsilon_j}$ *satisfies the corresponding equation* (SS^{ϵ_j}).

2.4.2 Construction of the homogenized problems

In the previous subsection, we obtained the limit $(\hat{B}, v)$ for the sequence $(B_{\epsilon_j}, v^{\epsilon_j})$. Now we give the homogenization results for the problem (C^ϵ) which is done using the standard homogenization process but we have used Tartar's energy method to identify the limit of $\chi^{\epsilon_j}\nabla v^{\epsilon_j}$ which is a product of two weakly converging sequences. Tartar's energy method was also used in the homogenization of the steady Stokes problem (SS^ϵ). We use the method of asymptotic expansion to obtain the homogenized for (R^ϵ).

Since the stochastic convection-diffusion equation is the main focus of this part of the thesis, we shall study its asymptotic behaviour as $\epsilon \to 0$ first and then move on to the other equations. The homogenized model (C) of (C^ϵ) is a stochastic convection-diffusion equation with an extra term coming from the adsorption and desorption boundary conditions.

Our main result is

Theorem 2.13. *Suppose the assumptions* $(A1)-(A3)$ *are satisfied, with the replacement of assumption of* G^ϵ *in* $(A1)$ *by* $G^\epsilon \in L^4(0,T;L^2(U))$. *Then there exist a probability space* $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, (\hat{\mathcal{F}})_{0\le t\le T})$ *and random variables* $(B^{\epsilon_j}, v^{\epsilon_j}, w^{\epsilon_j})$ *and* $(\hat{B}, v, w)$ *such that*

$$(B^{\epsilon_j}, v^{\epsilon_j}) \to (\hat{B}, v) \text{ in } \mathcal{X} \ \ \hat{\mathbb{P}}\text{-a.s}$$

and

$$\chi^{\epsilon_j}\nabla v^{\epsilon_j} \rightharpoonup |Y|S\nabla v \text{ weakly in } (L^2(0,T;L^2(U)))^n \ \ \hat{\mathbb{P}}\text{-a.s},$$

where $(\hat{B}, v)$ *satisfies the homogenized problem (C) with the process B replaced by* $\hat{B}$.

Furthermore w^{ϵ_j} converges to the function w which is a solution to the problem (R) obtained by formal asymptotic expansions.

For the proof of Theorem 2.13, we need the following auxiliary results which are borrowed from [57].

Lemma 2.14.

- *Let $f \in L^2(Z)$ be periodically extended to all $\mathbb{R}^n$ and $f^\epsilon(x) = f(x/\epsilon)$. Then $f^\epsilon \rightharpoonup \hat{f}$ weakly in $L^2(U)$, where*

$$\hat{f} = \int_Z f(y)dy.$$

- *If $v^\epsilon \to v$ strongly in $L^2(0,T;L^2(U))$, then $\chi^\epsilon v^\epsilon \rightharpoonup |Y|v$ weakly in $L^2(0,T;L^2(U))$.*

Lemma 2.15. *Let a, b, c be $\mathbb{Z}$-periodic and $a^\epsilon(x), b^\epsilon(x), c^\epsilon(x) \in L^\infty(\varGamma^\epsilon)$, where*

$$a^\epsilon(x) = a(\frac{x}{\epsilon}), b^\epsilon(x) = b(\frac{x}{\epsilon}), c^\epsilon(x) = c(\frac{x}{\epsilon}), \quad x \in \varGamma^\epsilon.$$

then

$$\begin{aligned} a^\epsilon &\rightharpoonup \frac{1}{|\varGamma|}\int_{|\varGamma|} a(y)dy, \text{ weakly in } L^\infty(\varGamma), \\ b^\epsilon &\rightharpoonup \frac{1}{|\varGamma|}\int_{|\varGamma|} b(y)dy, \text{ weakly in } L^\infty(\varGamma), \\ c^\epsilon &\rightharpoonup \frac{1}{|\varGamma|}\int_{|\varGamma|} c(y)dy, \text{ weakly in } L^\infty(\varGamma). \end{aligned}$$

Lemma 2.16.

- *Let v^ϵ be uniformly bounded in $L^2(\Omega, \mathbb{P}, \mathcal{F}, L^2(0,T;H^1(U)))$. Let*

$$\bar{v}^\epsilon(t,x) = \frac{1}{|\varGamma_k^\epsilon|}\int_{\varGamma^\epsilon} v^\epsilon(t,\tilde{x})d\varGamma(\tilde{x}), \qquad \text{if} \quad x \in \varGamma_k^\epsilon,$$

where $\varGamma_k^\epsilon$ is of the form

$$\varGamma_k^\epsilon = U \cap (\epsilon\varGamma^k), \quad k \in \mathbb{Z}^n.$$

Then

$$||v^\epsilon - \bar{v}^\epsilon||_{L^2(0,T;L^2(\varGamma^\epsilon))} \to 0.$$

- *Let $\bar{w}_1^\epsilon \in L^2(\varGamma^\epsilon)$ be defined as follows*

$$\bar{w}_1^\epsilon = \frac{1}{|\varGamma_k^\epsilon|}\int_{\varGamma^\epsilon} \bar{w}_1^\epsilon(\tilde{x})d\varGamma(\tilde{x}), \qquad \text{if} \quad x \in \varGamma_k^\epsilon,$$

then

$$||w_1 - \bar{w}_1^\epsilon||_{L^2(\varGamma^\epsilon)} \to 0.$$

Proposition 2.17. *For σ_k defined in (2.6), let $\sigma_k^{\epsilon_j} = \epsilon_j \sigma_k\big(\frac{x}{\epsilon_j}\big)$. Then*

- *the functions $\sigma_k^{\epsilon_j}$ satisfy the equations*

$$\begin{aligned} \Delta\sigma_k^{\epsilon_j}(x) &= 0 \quad x, \in U^\epsilon \\ \vec{\nu}^{\epsilon_j} \nabla\sigma_k^{\epsilon_j}(x) &= -\vec{\nu}^\epsilon \vec{e}_k, \quad x \in \Gamma^\epsilon, \end{aligned} \tag{2.22}$$

  i.e $(\chi^{\epsilon_j}(\nabla\sigma_k^{\epsilon_j} + \vec{e}_k), \nabla\phi)_{L^2(U)} = 0 \qquad \forall\phi \in H^1(U)$.

- $\sigma_k^{\epsilon_j} \to 0$ *strongly in* $L^2(U)$,

- $\chi^{\epsilon_j}(\delta_{ik} + \partial_i \sigma_k^{\epsilon_j}) \rightharpoonup |Y| s_{ik}$ *weakly in* $L^2(U)$.

The proof of this proposition can be found in [57] (pg 217).

Proposition 2.18. *For σ_k^ϵ defined in (2.22)*

1. $\epsilon(f^\epsilon, \sigma_k^\epsilon\phi)_{L^2((0,T);L^2(\Gamma^\epsilon))} \to 0$ *for all* $\phi \in \mathcal{D}(0, T \times U)$,

2. $\epsilon(f^\epsilon, \phi)_{L^2((0,T);L^2(\Gamma^\epsilon))} \to |Y|(F, \phi)_{L^2((0,T);L^2(U))}$ *for all* $\phi \in \mathcal{D}(0, T \times U)$

The proof is contained in [57] (pg 219 - 222).

The proof of our main result which is Theorem 2.13 will follow from the results of the next two subsections.

The convergence of the stochastic convection-diffusion problem (C^ϵ)

We now study the asymptotic behaviour of the problem (C^{ϵ_j}) when $\epsilon_j \to 0$.
Let us introduce the vector function $\vec{\xi}^{\epsilon_j}$ defined by

$$\vec{\xi}^{\epsilon_j}(t, x) = \chi^{\epsilon_j} \nabla v^{\epsilon_j}.$$

From Lemma 2.6 and the definition of χ^ϵ in (2.1), we see that $\vec{\xi}^{\epsilon_j}$ is uniformly bounded in $(L^2(0, T; L^2(U)))^n$ and we can extract a subsequence from $\vec{\xi}^{\epsilon_j}$ still denoted by $\vec{\xi}^{\epsilon_j}$ such that

$$\vec{\xi}^{\epsilon_j} \rightharpoonup \vec{\xi} \text{ weakly in } (L^2(0, T; L^2(U)))^n \ \ \hat{\mathbb{P}}\text{-a.s.} \tag{2.23}$$

For any $\phi \in \mathcal{D}(0,T)$ and $\psi \in \mathcal{D}(U)$, the weak formulation of problem (C^{ϵ_j}) is given by

$$\int_0^T \int_{U^\epsilon} dv^{\epsilon_j}\phi(t)\psi(x)dx = \int_0^T \int_{U^\epsilon} D\Delta v^{\epsilon_j}\phi(t)\psi(x)dxdt - \int_0^T \int_{U^\epsilon} \vec{u}^{\epsilon_j}\nabla v^{\epsilon_j}\phi(t)\psi(x)dxdt + \int_0^T \int_{U^\epsilon} G^{\epsilon_j}\phi(t)\psi(x)dxdB_{\epsilon_j}$$

Integrating by parts w.r.t. t on the left hand side and w.r.t x on the first term on the right hand side, we get

$$\begin{aligned} &-\int_0^T \int_{U^\epsilon} v^{\epsilon_j}\phi^{'}(t)\psi(x)dxdt + \int_0^T \int_{U^\epsilon} D\nabla v^{\epsilon_j}\phi(t)\nabla\psi(x)dxdt \\ &\quad + \int_0^T \int_{\Gamma^\epsilon} \epsilon_j f^{\epsilon_j}\phi(t)\psi(x)d\Gamma dt + \int_0^T \int_{U^\epsilon} \vec{u}^{\epsilon_j}\nabla v^{\epsilon_j}\phi(t)\psi(x)dxdt \\ &\quad = \int_0^T \int_{U^\epsilon} G^{\epsilon_j}\phi(t)\psi(x)dxdB_{\epsilon_j}, \end{aligned}$$

where $\epsilon_j f^{\epsilon_j} = -D\vec{\nu}^\epsilon\nabla v^{\epsilon_j}$. Using the definition of χ^ϵ in (2.1), the extension $\tilde{\vec{u}}^\epsilon$ of $\vec{u}^{\epsilon_j}$ by zero from U^ϵ to U and the extension $\tilde{G}^\epsilon$ of G^ϵ from U^ϵ to U we have

$$\begin{aligned} &-\int_0^T \int_U \chi^{\epsilon_j}v^{\epsilon_j}\phi^{'}(t)\psi(x)dxdt + \int_0^T \int_U D\chi^{\epsilon_j}\nabla v^{\epsilon_j}\phi(t)\nabla\psi(x)dxdt \\ &\quad + \int_0^T \int_{\Gamma^\epsilon} \epsilon_j f^{\epsilon_j}\phi(t)\psi(x)d\Gamma dt + \int_0^T \int_U \tilde{\vec{u}}^{\epsilon_j}\nabla v^{\epsilon_j}\phi(t)\psi(x)dxdt \\ &\quad = \int_0^T \int_U \tilde{G}^{\epsilon_j}\phi(t)\psi(x)dxdB_{\epsilon_j}. \end{aligned} \tag{2.24}$$

Now we take one term at a time and pass to the limits as $\epsilon_j \to 0$. The first term on the left hand side of (2.24) gives

$$-\lim_{\epsilon_j\to 0}\int_0^T \int_U \chi^{\epsilon_j}v^{\epsilon_j}\phi^{'}(t)\psi(x)dxdt = -|Y|\int_0^T \int_U v\phi^{'}(t)\psi(x)dxdt \ \ \hat{\mathbb{P}}\text{-a.s.} \tag{2.25}$$

The second term gives

$$\lim_{\epsilon_j\to 0}\int_0^T \int_U D\vec{\xi}^{\epsilon_j}\phi(t)\nabla\psi(x)dxdt = \int_0^T \int_U D\vec{\xi}\phi(t)\nabla\psi(x)dxdt \ \ \hat{\mathbb{P}}\text{-a.s.} \tag{2.26}$$

Since $\nabla\vec{u}^{\epsilon_j} = 0$ in U^ϵ, we have that $\nabla(\vec{u}^{\epsilon_j}v^{\epsilon_j}) = \vec{u}^{\epsilon_j}\nabla v^{\epsilon_j}$. Hence

$$\begin{aligned} \lim_{\epsilon_j\to 0}\int_0^T \int_U \tilde{\vec{u}}^{\epsilon_j}\nabla v^{\epsilon_j}\phi(t)\psi(x)dxdt &= \lim_{\epsilon_j\to 0}\int_0^T \int_U \nabla(\tilde{\vec{u}}^{\epsilon_j}v^{\epsilon_j})\phi(t)\psi(x)dxdt \\ &= -\lim_{\epsilon_j\to 0}\int_0^T \int_U \tilde{\vec{u}}^{\epsilon_j}v^{\epsilon_j}\phi(t)\nabla\psi(x)dxdt \\ &= -\int_0^T \int_U \vec{u}v\phi(t)\nabla\psi(x)dxdt \ \ \hat{\mathbb{P}}\text{-a.s} \\ &= \int_0^T \int_U \vec{u}\nabla v\phi(t)\psi(x)dxdt \end{aligned} \tag{2.27}$$

Lastly we show that

$$\lim_{\epsilon_j\to 0}\int_U \tilde{G}^{\epsilon_j}\phi(t)\psi(x)dxdB_{\epsilon_j} = \int_U G\phi(t)\psi(x)dxdB.$$

Recall that the extension $\tilde{G}^{\epsilon}(t,x)$ of $G^{\epsilon}(t,x)$ is such that

$$\tilde{G}^{\epsilon}(t,x) \rightharpoonup G(t,x) \quad \text{weakly in } L^2((0,T)\times U). \tag{2.28}$$

Due to the bounded variations of B_{ϵ_j}, we have the following split

$$\begin{aligned}\int_0^T\int_U \tilde{G}^{\epsilon_j}\big(t,x\big)\phi(t)\psi(x)dxdB_{\epsilon_j} &= \int_0^T\int_U \tilde{G}^{\epsilon_j}\big(t,x\big)\phi(t)\psi(x)dxd(B_{\epsilon_j}-\hat{B})\\ &+\int_0^T\int_U \tilde{G}^{\epsilon_j}\big(t,x\big)\phi(t)\psi(x)dxd\hat{B}\end{aligned} \tag{2.29}$$

For the first term on the right hand side, we adopt the concept of regularization for $\tilde{G}^{\epsilon}(t,x)$ with respect to t in the form of the following sequence

$$\tilde{G}^{\epsilon}_{\lambda}(t) = \frac{1}{\lambda}\int_0^T \rho\Big(-\frac{t-s}{\lambda}\Big)\tilde{G}^{\epsilon_j}(t,x)ds \quad \text{for } \lambda>0,$$

where ρ is a standard mollifier.

We have that $\tilde{G}^{\epsilon}_{\lambda}(t)$ is differentiable with respect to t and satisfies

$$\int_0^T ||\tilde{G}^{\epsilon}_{\lambda}(t)||^2_{L^2(U)}dt \le \int_0^T \big|\big|\tilde{G}^{\epsilon}(t,x)\big|\big|^2_{L^2(U)}dt, \text{ for any } \lambda>0,$$

and for any $\epsilon>0$,

$$\tilde{G}^{\epsilon}_{\lambda}(t) \to \tilde{G}^{\epsilon}(t,x) \text{ strongly in } L^2((0,T)\times U) \text{ as } \lambda\to 0.$$

We rewrite the first term on the right hand side of (2.29) as

$$\begin{aligned}\int_0^T\int_U \tilde{G}^{\epsilon_j}(t,x)\phi(t)\psi(x)dxd(B_{\epsilon_j}-\hat{B}) &= \int_0^T\int_U \tilde{G}^{\epsilon_j}_{\lambda}(t)\phi(t)\psi(x)dxd(B_{\epsilon_j}-\hat{B})\\ &+\int_0^T\int_U [\tilde{G}^{\epsilon_j}(t,x)-\tilde{G}^{\epsilon_j}_{\lambda}]\phi(t)\psi(x)dxd(B_{\epsilon_j}-\hat{B}).\end{aligned} \tag{2.30}$$

since $\tilde{G}^{\epsilon_j}_{\lambda}$ is differentiable, we integrate by part on the first term on the right hand side of (2.30) to get

$$\begin{aligned}\int_0^T\int_U \tilde{G}^{\epsilon_j}_{\lambda}(t)\phi(t)\psi(x)dxd(B_{\epsilon_j}-\hat{B}) &= \int_U (B_{\epsilon_j}-\hat{B})\tilde{G}^{\epsilon_j}_{\lambda}\phi(t)\psi(x)dx\Big|_0^T\\ &-\int_0^T\int_U (B_{\epsilon_j}-\hat{B})\partial_t\big(\tilde{G}^{\epsilon_j}_{\lambda}\phi(t)\big)\psi(x)dxdt.\end{aligned} \tag{2.31}$$

The conditions on G^{ϵ_j}, ϕ and ψ together with the convergence of B_{ϵ_j} to $\hat{B}$ in $C([0,T])$ $\hat{\mathbb{P}}$-a.s., give that the right hand side of (2.31) are bounded by $\kappa_1(\lambda)\zeta_1(\epsilon_j)$ where $\zeta_1(\epsilon_j)$ tends to zero as ϵ_j goes to zero.

The second term on the right hand side of (2.30) gives

$$\hat{\mathbb{E}}\left|\int_0^T [\tilde{G}^{\epsilon_j}(t,x) - \tilde{G}_\lambda^{\epsilon_j}(t)]\phi(t)\psi(x)dxd(B_{\epsilon_j} - \hat{B})\right| \leq \kappa_2(\lambda),$$

where $\kappa_2(\lambda)$ converge to zero as λ tends to zero. Then from (2.30), we have

$$\left|\hat{\mathbb{E}}\int_0^T \int_U \tilde{G}^{\epsilon_j}(t,x)\phi(t)\psi(x)dxd(B_{\epsilon_j} - \hat{B})\right| \leq \kappa_1(\lambda)\zeta_1(\epsilon_j) + \kappa_2(\lambda).$$

Hence from (2.29), we conclude that

$$\begin{aligned}&\left|\hat{\mathbb{E}}\int_0^T \int_U \tilde{G}^{\epsilon_j}(t,x)\phi(t)\psi(x)dxdB_{\epsilon_j} - \hat{\mathbb{E}}\int_0^T \int_U \tilde{G}^{\epsilon_j}(t,x)\phi(t)\psi(x)dxd\hat{B}\right|\\ &\leq \kappa_1(\lambda)\zeta_1(\epsilon_j) + \kappa_2(\lambda).\end{aligned}$$

Taking the limits as $\epsilon_j \to 0$, we get

$$\lim_{\epsilon_j\to 0}\left|\hat{\mathbb{E}}\int_0^T \int_U \tilde{G}^{\epsilon_j}(t,x)\phi(t)\psi(x)dxdB_{\epsilon_j} - \hat{\mathbb{E}}\int_0^T \int_U \tilde{G}^{\epsilon_j}(t,x)\phi(t)\psi(x)dxd\hat{B}\right| \leq \kappa_2(\lambda).$$

Since $\kappa_2(\lambda) \to 0$ as $\lambda \to 0$, we pass to the limit as $\lambda \to 0$, to get

$$\lim_{\epsilon_j\to 0}\hat{\mathbb{E}}\int_0^T \int_U \tilde{G}^{\epsilon_j}(t,x)\phi(t)\psi(x)dxdB_{\epsilon_j} = \lim_{\epsilon_j\to 0}\int_0^T \int_U \tilde{G}^{\epsilon_j}(t,x)\phi(t)\psi(x)dxd\hat{B} \tag{2.32}$$

Now taking the limit as $\epsilon_j \to 0$ on the right hand side of (2.32), using (2.28) and the convergence theorem on stochastic integral due to Rozovskii [112] (Theorem 4, pg 63), we get

$$\lim_{\epsilon_j\to 0}\int_0^T \int_U \tilde{G}^{\epsilon_j}(t,x)\phi(t)\psi(x)dxd\hat{B} = \int_0^T \int_U G(t,x)\phi(t)\psi(x)dxd\hat{B}.\ \hat{\mathbb{P}}\text{-a.s}$$

Hence we conclude from (2.32) that

$$\lim_{\epsilon_j\to 0}\int_0^T \int_U \tilde{G}^{\epsilon_j}(t,x)\phi(t)\psi(x)dxdB_{\epsilon_j} = \int_0^T \int_U G(t,x)\phi(t)\psi(x)dxd\hat{B}\ \ \hat{\mathbb{P}}\text{-a.s.} \tag{2.33}$$

Combining all the convergences i.e (2.25), (2.26), (2.27), Proposition 2.18 and (2.33), we get

$$\begin{aligned}&-|Y|\int_0^T \int_U v\phi'(t)\psi(x)dxdt + \int_0^T \int_U D\vec{\xi}\phi(t)\nabla\psi(x)dxdt + |Y|\int_0^T \int_U F\phi(t)\psi(x)dxdt\\ &\quad + \int_0^T \int_U \vec{u}\nabla v\phi(t)\psi(x)dxdt = \int_0^T \int_U G\phi(t)\psi(x)dxd\hat{B}.\end{aligned} \tag{2.34}$$

An additional integration by parts gives that $\vec{\xi}$ satisfies

$|Y|dv(t,x)-D\nabla\vec{\xi}(t,x)+|Y|F(t,x)+\vec{u}(x)\nabla v(t,x)=G(t,x)d\hat{B}$ in $U\times(0,T)$ $\mathbb{P}$-a.s.

Next we show that

$$\vec{\xi}=|Y|S\nabla v.$$

We do this using Tartar's energy method. Taking $\sigma_k^{\epsilon_j}(x)$ from Proposition 2.17 and using $\phi(t)\psi(x)\sigma_k^{\epsilon_j}(x)$ in place of $\phi(t)\psi(x)$ in equation (2.24), we obtain

$$\begin{aligned}
&-\int_0^T\int_U \chi^{\epsilon_j}v^{\epsilon_j}\phi^{'}(t)\psi(x)\sigma_k^{\epsilon_j}(x)dxdt+\int_0^T\int_U D\chi^{\epsilon_j}\nabla v^{\epsilon_j}\nabla(\psi(x)\sigma_k^{\epsilon_j}(x))\phi(t)dxdt\\
&+\int_0^T\int_{\Gamma^\epsilon}\epsilon_j f^{\epsilon_j}\phi(t)\psi(x)\sigma_k^{\epsilon_j}(x)d\Gamma dt+\int_0^T\int_U \tilde{\vec{u}}^{\epsilon_j}(x)\nabla v^{\epsilon_j}\phi(t)\psi(x)\sigma_k^{\epsilon_j}(x)dxdt\\
&=\int_0^T\int_U \tilde{G}^{\epsilon_j}\phi(t)\psi(x)\sigma_k^{\epsilon_j}(x)dxdB_{\epsilon_j}.
\end{aligned}\tag{2.35}$$

On the other hand, using $\phi(t)\psi(x)v^{\epsilon_j}$ as a test function in (2.22) for a subsequence ϵ_j, the definition of χ^ϵ in (2.1), multiplying by D and integrate over U and $(0,T)$, one obtains

$$0=\int_0^T\int_U D\chi^{\epsilon_j}\nabla\sigma_k^{\epsilon_j}\nabla(\phi(t)\psi(x)v^{\epsilon_j})dxdt+\int_0^T\int_U D\chi^{\epsilon_j}\vec{e}_k\nabla(\phi(t)\psi(x)v^{\epsilon_j})dxdt,$$

i.e

$$\begin{aligned}
0=&\int_0^T\int_U D\chi^{\epsilon_j}\nabla\sigma_k^{\epsilon_j}\phi(t)\nabla\psi(x)v^{\epsilon_j}dxdt+\int_0^T\int_U D\chi^{\epsilon_j}\nabla\sigma_k^{\epsilon_j}\phi(t)\psi(x)\nabla v^{\epsilon_j}dxdt\\
&+\int_0^T\int_U D\chi^{\epsilon_j}\vec{e}_k\phi(t)\nabla\psi(x)v^{\epsilon_j}dxdt+\int_0^T\int_U D\chi^{\epsilon_j}\vec{e}_k\phi(t)\psi(x)\nabla v^{\epsilon_j}dxdt.
\end{aligned}\tag{2.36}$$

Adding (2.35) to (2.36) gives

$$\begin{aligned}
&-\int_0^T\int_U \chi^{\epsilon_j}v^{\epsilon_j}\phi^{'}(t)\psi(x)\sigma_k^{\epsilon_j}(x)dxdt+\int_0^T\int_U D\chi^{\epsilon_j}\nabla v^{\epsilon_j}\phi(t)\nabla\psi(x)\sigma_k^{\epsilon_j}dxdt\\
&+\int_0^T\int_U D\chi^{\epsilon_j}\nabla v^{\epsilon_j}\phi(t)\psi(x)\nabla\sigma_k^{\epsilon_j}(x)dxdt+\int_0^T\int_{\Gamma^\epsilon}\epsilon_j f^{\epsilon_j}\phi(t)\psi(x)\sigma_k^{\epsilon_j}(x)d\Gamma dt\\
&+\int_0^T\int_U \vec{u}^{\epsilon_j}\nabla v^{\epsilon_j}\phi(t)\psi(x)\sigma_k^{\epsilon_j}(x)dxdt=\int_0^T\int_U D\chi^{\epsilon_j}\nabla\sigma_k^{\epsilon_j}(x)\phi(t)\nabla\psi(x)v^{\epsilon_j}dxdt\\
&+\int_0^T\int_U D\chi^{\epsilon_j}\nabla\sigma_k^{\epsilon_j}\phi(t)\psi(x)\nabla v^{\epsilon_j}dxdt+\int_0^T\int_U D\chi^{\epsilon_j}\vec{e}_k\phi(t)\nabla\psi(x)v^{\epsilon_j}dxdt\\
&+\int_0^T\int_U D\chi^{\epsilon}\vec{e}_k\phi(t)\psi(x)\nabla v^{\epsilon_j}dxdt+\int_0^T\int_U G^{\epsilon_j}\phi(t)\psi(x)\sigma_k^{\epsilon_j}(x)dxdB_{\epsilon_j}.
\end{aligned}$$

Now we take the limits as $\epsilon_j \to 0$.

Using proposition 2.17, we see that the first, second, fourth and fifth terms onthe left hand side converges to zero $\hat{\mathbb{P}}$-a.s.. The third term on the left hand side is equal to the second term on the right hand side. The first and third terms on the right hand side give

$$\begin{aligned}
&\lim_{\epsilon_j\to 0}\int_0^T\int_U D\chi^{\epsilon_j}\nabla\sigma_k^{\epsilon_j}\nabla\psi(x)\phi(t)v^{\epsilon_j}dxdt + \lim_{\epsilon_j\to 0}\int_0^T\int_U D\chi^{\epsilon_j}\vec{e}_k\nabla\psi(x)\phi(t)v^{\epsilon_j}dxdt \\
&\qquad = \lim_{\epsilon_j\to 0}\sum_i\int_0^T\int_U D\big(\chi^{\epsilon}\big(\delta_{ik}+\frac{\partial\sigma_k^{\epsilon_j}}{\partial x_i}\big)\big)\frac{\partial\psi(x)}{\partial x_i}\phi(t)v^{\epsilon_j}dxdt \\
&\qquad = |Y|D\int_0^T\int_U S\vec{e}_k\nabla\psi(x)\phi(t)vdxdt,
\end{aligned}$$

where S is defined in (2.7).

Taking the limit as $\epsilon_j \to 0$ on the fourth term on the right hand side gives

$$\begin{aligned}
\lim_{\epsilon_j\to 0}\int_0^T\int_U D\chi^{\epsilon}\vec{e}_k\nabla v^{\epsilon_j}\phi(t)\psi(x)dxdt &= \lim_{\epsilon_j\to 0}\int_0^T\int_U D\vec{\xi}^{\epsilon_j}\vec{e}_k\phi(t)\psi(x)dxdt \\
&= \int_0^T\int_U D\vec{\xi}\vec{e}_k\phi(t)\psi(x)dxdt
\end{aligned}$$

Lastly, using Bukhölder-Gundy-Davis inequality, we have

$$\begin{aligned}
&\lim_{\epsilon_j\to 0}\hat{\mathbb{E}}\sup_{t\in[0,T]}\left|\int_0^T\int_U G^{\epsilon_j}\phi(t)\psi(x)\sigma_k^{\epsilon_j}(x)dxdB_{\epsilon_j}\right| \\
&\leq C\lim_{\epsilon_j\to 0}\hat{\mathbb{E}}\Big(\int_0^T\Big(\int_U G^{\epsilon_j}\phi(t)\psi(x)\sigma_k^{\epsilon_j}(x)dx\Big)^2dt\Big)^{\frac{1}{2}} \\
&\leq C_1\lim_{\epsilon_j\to 0}\hat{\mathbb{E}}\Big(\int_0^T||G^{\epsilon_j}||^2_{L^2(U)}||\sigma_k^{\epsilon_j}||^2_{L^2(U)}dt\Big)^{\frac{1}{2}} \\
&= 0, \quad \hat{\mathbb{P}}\text{-a.s.}
\end{aligned}$$

Since $\sigma_k^{\epsilon_j} \to 0$ strongly in $L^2(U)$. Putting together all the convergences, one obtains

$$-|Y|D\int_0^T\int_U S\vec{e}_k\nabla v\phi(t)\psi(x)dxdt + D\int_0^T\int_U\vec{\xi}\vec{e}_k\phi(t)\psi(x)dxdt = 0,$$

this implies that

$$|Y|S\nabla v = \vec{\xi}, \quad \hat{\mathbb{P}}\text{-a.s.} \tag{2.37}$$

It remains to show that $v(0,x) = v_1(x)$. If we take $\zeta \in C^\infty([0,T])$ such that $\zeta(0) = 1$ and $\zeta(T) = 0$, equation (2.24) still remains valid. Recall that $\vec{\xi}^{\epsilon_j} =$

$\chi^{\epsilon_j}\nabla v^{\epsilon_j}$, taking ζ in place of ϕ in equation (2.24) gives

$$-\int_0^T\int_U \chi^{\epsilon_j}v^{\epsilon_j}\zeta'(t)\psi(x)dxdt+\int_0^T\int_U D\vec{\xi}^{\epsilon_j}\zeta(t)\nabla\psi(x)dxdt+\int_0^T\int_{\Gamma^\epsilon}\epsilon_j f^{\epsilon_j}\zeta(t)\psi(x)d\Gamma dt$$

$$+\int_0^T\int_U \tilde{\vec{u}}^{\epsilon_j}\nabla v^{\epsilon_j}\zeta(t)\psi(x)dxdt=\int_0^T\int_U \tilde{G}^{\epsilon_j}\zeta(t)\psi(x)dxdB_{\epsilon_j}+\int_U \chi^{\epsilon_j}v^{\epsilon_j}(0,x)\psi(x)dx.$$

Since $\chi^{\epsilon_j}v^{\epsilon_j}(0,x)=\chi^{\epsilon_j}v_1(x)$. Then passing to the limit as $\epsilon_j\to 0$ yields

$$-|Y|\int_0^T\int_U v\zeta'(t)\psi(x)dxdt+\int_0^T\int_U D\vec{\xi}\zeta(t)\nabla\psi(x)dxdt+|Y|\int_0^T\int_U F\zeta(t)\psi(x)dxdt$$

$$+\int_0^T\int_U \vec{u}\nabla v\zeta(t)\psi(x)dxdt=\int_0^T\int_U G\zeta(t)\psi(x)dxd\hat{B}+|Y|\int_U v_1(x)\psi(x)dx.$$

Integrating by parts with respect to time on the first term on the left hand side of the above equation gives

$$\begin{aligned}|Y|\int_0^T\int_U dv\zeta(t)\psi(x)dx&+\int_0^T\int_U D\vec{\xi}\zeta(t)\nabla\psi(x)dxdt+|Y|\int_0^T\int_U F\zeta(t)\psi(x)dxdt\\&+\int_0^T\int_U \vec{u}\nabla v\zeta(t)\psi(x)dxdt+|Y|\int_U v(0,x)\psi(x)dx\\&=\int_0^T\int_U G\zeta(t)\psi(x)dxd\hat{B}+|Y|\int_U v_1(x)\psi(x)dx.\end{aligned}$$

Since equation (2.34) is valid for $\zeta(t)\in C^\infty([0,T])$, we conclude that

$$\int_U v(0,x)\psi(x)dx=\int_U v_1(x)\psi(x)dx,\quad \forall\psi(x)\in\mathcal{D}(U).$$

Hence

$$v(0,x)=v_1(x).$$

With this result, (2.37) and $\hat{B}$ replaced with B, (2.34) becomes

$$\begin{aligned}|Y|\int_0^T\int_U dv(t,x)\phi(t)\psi(x)dx&+|Y|\int_0^T\int_U F(t,x)\phi(t)\psi(x)dxdt\\&=|Y|\int_0^T\int_U D\nabla(S\nabla v(t,x))\phi(t)\psi(x)dxdt-\int_0^T\int_U \vec{u}\nabla v(t,x)\phi(t)\psi(x)dxdt\\&+\int_0^T\int_U G(t,x)\phi(t)\psi(x)dxdB(t).\end{aligned}$$

with $v(0,x)=v_1$.

We have that (B,v) is a unique probabilistic weak solution of problem (C) . Hence by the infinite dimensional version of Yamada-Watanabe's theorem [97], we conclude that (B,v) is a unique strong solution of (C). Consequently, up to distribution (probability law) the whole sequence of solutions of problem (C^ϵ) converges to the solution of problem (C).

The convergence of the reaction diffusion equation (R^ϵ)

We study the asymptotic behaviour of the reaction-diffusion equation (R^ϵ) using the method of asymptotic expansion. The functions w^ϵ, v^ϵ are expressed in terms of a time variable t and two spatial variables x the 'slow' variable and $y = \frac{x}{\epsilon}$, the variable on the micro scale $\epsilon\Gamma$.
Let us first recall the reaction-diffusion problem (R^ϵ);

$$(R^\epsilon) \begin{cases} \partial_t w^\epsilon - \epsilon^2 E \Delta^\epsilon w^\epsilon + a^\epsilon(x) w^\epsilon = f^\epsilon & t > 0, x \in \Gamma^\epsilon, \\ w^\epsilon = w_1(x) & t = 0, x \in \Gamma^\epsilon, \end{cases}$$

where $f^\epsilon(t,x) = c^\epsilon(x)v^\epsilon(t,x) - b^\epsilon(x)w^\epsilon(t,x)$. and a, b, c are Z-periodic, $a, b, c \geq 0$: a, b, c are bounded and $a^\epsilon(x) = a\big(\frac{x}{\epsilon}\big), b^\epsilon(x) = b\big(\frac{x}{\epsilon}\big), c^\epsilon(x) = c\big(\frac{x}{\epsilon}\big)$.
We assume the following expansions

$$w^\epsilon(t,x) = w_0(t,x,\frac{x}{\epsilon}) + \epsilon w_1(t,x,\frac{x}{\epsilon}) + \epsilon^2 w_2(t,x,\frac{x}{\epsilon}) + ... \tag{2.38}$$

$$v^\epsilon(t,x) = v_0(t,x,\frac{x}{\epsilon}) + \epsilon v_1(t,x,\frac{x}{\epsilon}) + \epsilon^2 v_2(t,x,\frac{x}{\epsilon}) + ... \tag{2.39}$$

$$\Delta^\epsilon = \frac{1}{\epsilon^2}\Delta_y^\Gamma + \frac{1}{\epsilon}\Delta_{yx}^* + \Delta_x^\epsilon \tag{2.40}$$

where $\Delta_{yx}^* = \Delta_{yx}^{\Gamma,\epsilon} + \Delta_{xy}^{\epsilon,\Gamma}$ and Δ^ϵ and Δ^Γ are Laplace-Beltarmi operators on Γ^ϵ and Γ respectively.
Substituting 2.38, 2.39 and 2.40 into (R^ϵ) yields

$$\begin{cases} \partial_t[w_0 + \epsilon w_1 + \epsilon w_2 + ...] - \epsilon^2 E(\frac{1}{\epsilon^2}\Delta_y^\Gamma + \frac{1}{\epsilon}\Delta_{yx}^* + \Delta_x^\epsilon)[w_0 + \epsilon w_1 + \epsilon w_2 + ...] \\ +a\big(\frac{x}{\epsilon}\big)[w_0 + \epsilon w_1 + \epsilon w_2 + ...] - b\big(\frac{x}{\epsilon}\big)[w_0 + \epsilon w_1 + \epsilon w_2 + ...] \\ = c\big(\frac{x}{\epsilon}\big)[v_0 + \epsilon v_1 + \epsilon v_2 + ...] & t > 0, x \in U, y \in \Gamma, \\ w^\epsilon(0,x) = 0, & t = 0, x \in U \end{cases}$$

Equating terms with like powers of ϵ we get the following system of equations

$$\begin{cases} \partial_t w_0(t,x,y) - E\nabla_y^\Gamma w_0(t,x,y) + a(y)w_0(t,x,y) \\ = c(y)v_0(t,x,y) - b(y)w_0(t,x,y), & t > 0, x \in U, y \in \Gamma \\ w_0 \text{ is Z-periodic} \end{cases} \tag{2.41}$$

$$\begin{cases} \partial_t w_1(t,x,y) - E\nabla_y^{\Gamma} w_1 - E\Delta_{yx}^* w_0 + a(y)w_1 \\ = c(y)v_1 - b(y)w_1, & t > 0, x \in U, y \in \Gamma \\ w_1 \text{ is Z-periodic,} \end{cases} \tag{2.42}$$

$$\begin{cases} \partial_t w_2 - E\nabla_y^{\Gamma} w_2 - E\Delta_{yx}^* w_1 - E\Delta_x^{\epsilon} w_0 + a(y)w_2 \\ = c(y)v_2 - b(y)w_2, & t > 0, x \in U, y \in \Gamma \\ w_1 \text{ is Z-periodic,} \end{cases} \tag{2.43}$$

and

$$\begin{cases} \partial_t w_{k+2} - E\nabla_y^{\Gamma} w_{k+2} - E\Delta_{yx}^* w_{k+1} - E\Delta_x^{\epsilon} w_k + a(y)w_{k+2} \\ = c(y)v_{k+2} - b(y)w_{k+2}, & t > 0, x \in U, y \in \Gamma \\ w_{k+2} \text{ is Z-periodic,} \end{cases} \tag{2.44}$$

for $k \geq 1$.

These system of equations can be solved in succession to determine the value of each w_k. The analysis of (C^{ϵ}) using the asymptotic expansion for v^{ϵ} gives the independence of v_0 on Y, hence we have that

$$v_0(t,x,y) = v_0(t,x). \tag{2.45}$$

We then see that (2.41) with $w_0 = w_1(x)$ as initial condition is the homogenized problem (R).

Summarizing the results of the previous sections, we obtain the complete proof of Theorem 2.13.

The method of asymptotic expansion used here is a heuristic method, a rigorous convergence with appropriate topology can be given using the two scale convergence, see for instance [59].

The convergence of the steady Stokes problem (SS^ϵ)

For the convenience of the reader, we recall Tartar's proof of the convergence of the solution $\vec{u}^\epsilon$ to the steady Stokes problem (SS^ϵ) to the solution $\vec{u}$ of the homogenized problem (SS).

Let us write the cell problem (2.3) in terms of $x = \epsilon y$:

$$\begin{cases} \epsilon^2 \Delta_x \vec{\mu}_j^\epsilon = \vec{e}_j + \epsilon \nabla_x \pi_j^\epsilon, & x \in U^\epsilon \\ \nabla_x \vec{\mu}_j^\epsilon = 0, & x \in U^\epsilon \\ \vec{\mu}_j^\epsilon = 0, & x \in \Gamma^\epsilon \end{cases} \tag{2.46}$$

Since $\vec{\mu}_j(y)$ and $\pi_j(y)$ are independent of ϵ, we have

$$||\pi_j^\epsilon||_{L^2(U)} \leq C; \quad ||\vec{\mu}_j^\epsilon||_{L^2(U)} \leq C; \quad ||\nabla_x \vec{\mu}_j^\epsilon||_{L^2(U)} \leq \frac{C}{\epsilon}, \tag{2.47}$$

and a classical lemma on periodic functions yields

$$\vec{\mu}_j^\epsilon \rightharpoonup \tilde{\mu}_j \text{ weakly in } L^2(U),$$

where $\tilde{\mu}_j$ is defined in (2.4). Now we take $\phi \in \mathcal{D}(U)$ and we multiply (2.46) by $\phi \tilde{\vec{u}}^\epsilon$ and integrate over U:

$$\epsilon^2 \int_U \Delta \vec{\mu}_j^\epsilon \, \phi \tilde{\vec{u}}^\epsilon dx = \int_U \vec{e}_j \, \phi \tilde{\vec{u}}^\epsilon dx + \epsilon \int_U \nabla \pi_j^\epsilon \phi \tilde{\vec{u}}^\epsilon dx.$$

Passing to the limit on the right hand side gives

$$\int_U \vec{e}_j \phi \tilde{\vec{u}}^\epsilon dx \longrightarrow \int_U \phi \vec{u}_j^* dx,$$

$$\epsilon \int_U \nabla \pi_j^\epsilon \phi \tilde{\vec{u}}^\epsilon dx = -\epsilon \int_U \pi_j^\epsilon \nabla \phi \tilde{\vec{u}}^\epsilon dx \leq C\epsilon \to 0 \quad \text{ as } \epsilon \to 0,$$

where we have used (2.47) and (2.11). Hence

$$\epsilon^2 \int_U \Delta \vec{\mu}_j^\epsilon \phi \tilde{\vec{u}}^\epsilon dx \longrightarrow \int_U \phi \vec{u}_j^* dx. \tag{2.48}$$

On the other hand, we multiply (SS^ϵ) with $\phi \vec{\mu}_j^\epsilon$ to get

$$\epsilon^2 \int_{U^\epsilon} \Delta \tilde{\vec{u}}^\epsilon \phi \vec{\mu}_j^\epsilon dx = \int_{U^\epsilon} \nabla p^\epsilon \phi \vec{\mu}_j^\epsilon dx$$

$$\epsilon^2 \int_U \Delta \tilde{\vec{u}}^\epsilon \phi \vec{\mu}_j^\epsilon dx = \int_U \nabla p^\epsilon \phi \vec{\mu}_j^\epsilon dx.$$

(Recall that $\vec{u}^\epsilon$ is zero outside U^ϵ). Passing to the limit on the right hand side gives

$$\int_U \nabla p^\epsilon \phi \vec{\mu}_j^\epsilon dx = -\int_U p^\epsilon \frac{\partial \phi}{\partial x_i} \vec{\mu}_j^\epsilon dx \longrightarrow -\int_U p^* \frac{\partial \phi}{\partial x_i} \tilde{\mu}_{ij} dx,$$

where we have used (2.14) and (2.13). Hence

$$\epsilon^2 \int_U \Delta \tilde{\vec{u}}^\epsilon \phi \vec{\mu}_j^\epsilon dx \longrightarrow -\int_U p^* \frac{\partial \phi}{\partial x_i} \tilde{\mu}_{ij} dx. \tag{2.49}$$

Next, we compare the left hand sides of (2.48) and (2.49).Their difference yields

$$\begin{aligned}
\epsilon^2 \int_U \Delta \vec{\mu}_j^\epsilon \phi \tilde{\vec{u}}^\epsilon dx - \epsilon^2 \int_U \Delta \tilde{\vec{u}}^\epsilon \phi \vec{\mu}_j^\epsilon dx &\le \epsilon^2 \int_U \nabla \tilde{\vec{u}}^\epsilon \nabla (\phi \vec{\mu}_j^\epsilon) dx - \epsilon^2 \int_U \nabla \vec{\mu}_j^\epsilon \nabla (\phi \tilde{\vec{u}}^\epsilon) dx \\
&\le \left| \epsilon^2 \int_U \nabla \tilde{\vec{u}}^\epsilon \nabla \phi \vec{\mu}_j^\epsilon dx \right| + \left| \epsilon^2 \int_U \nabla \vec{\mu}_j^\epsilon \nabla \phi \tilde{\vec{u}}^\epsilon dx \right| \\
&\le C\epsilon \quad \to 0 \text{ as } \epsilon \to 0,
\end{aligned}$$

where we have used (2.47) and (2.11).

This means that the right hand sides of (2.48) and(2.49) are equal. Then writing them in terms of distributions, we have

$$\langle \vec{u}_j^*, \phi \rangle_{\mathcal{D}'(U),\ \mathcal{D}(U)} = \tilde{\mu}_{ij} \langle -p^*, \frac{\partial \phi}{\partial x_i} \rangle_{\mathcal{D}'(U), \mathcal{D}(U)} = K \langle \nabla p^*, \phi \rangle_{\mathcal{D}'(U), \mathcal{D}(U)},$$

which is the same as (SS^ϵ) for u^* and p^*.

Chapter 3

Homogenization of a stochastic model of a single phase flow in partially fissured media

3.1 Introduction

A fissured or fractured medium is a material made up of permeable and porous blocks interwoven by a system of fissures, the porous blocks make up the matrix of the media. Fissured media are differentiated by the extent to which the system of fissures are developed within the medium. Bulk of the fluid transport takes place through the system of fissures while significant fluid storage occurs in the porous blocks.

Flow in fissured porous domains was first investigated by reservoir engineers in the petroleum industry because many petroleum reservoirs are in fractured rock formations made up of porous blocks of rocks surrounded by fractures. The blocks have low permeability but the porosity and consequently the storage capacity of fluids is high, which led to an overestimation in well production and capacity.

Scientists and engineers have been studying this subject. Hence there are many articles and professional literature in multiple fields including hydrology, geology and environmental engineering.

There are certain characteristics of fissured media, namely, that transport occurs through the fissures while fluid storage takes place in the pore system. There are two cases of fissured media, the totally fissured media and the partially fissured media. In the case of a totally fissured medium, the porous blocks are separated by well developed system of fissures, as a result, no flow takes place within the porous matrix; the fluid only flows through the system of fissures. In the case of a partially fissured medium, the system of fissures are less developed and the porous blocks may be connected, hence there's some amount of flow within the porous matrix.

A mathematical model that describes the flow of a fluid through a fissured porous medium domain can be stated for every point in the phase considered and on the matrix-fissure interface. This description is said to be at the microscopic scale. Due to the difficulty in measuring values of variables within a phase and determining the parameters of the model, a complete description of a model at the microscopic level is difficult and a solution to a said model is almost impossible. To bypass these difficulties, a macroscopic model is derived as the limit of the microscopic model, this process can be done using various homogenization methods, for example, multiple-scale expansions, two-scale convergence e.t.c.; see for instance [37], [114] and [58].

Fluid flows through fissured media as if it has two pore systems, one for the porous matrix and the other for the system of fissures, giving rise to the concept of double porosity. The flow of a fluid through totally fissured medium can be modeled using two flow fields, one representing the porous matrix and the other representing the system of fissures. These systems are coupled to form a system of equations over the flow domain, this type of model was introduced by Barenblatt, Zheltov and Kochina in [12]; see also [9], [120].

Coeffield and Spagnuolo in [39] considered a model for single-phase flow through a totally fractured layered medium, where the fractures are horizontal and the matrix blocks are stacked vertically. The structure considered in [39] is assumed to be periodic only in one direction (vertically). In [49], Douglas, Peszyńska and Showalter extended the model for single phase flow in totally fissured media to that of a single phase fluid through periodic partially fissured media in the deterministic case; the model was constructed following [12], [138] and the macroscopic model was derived using the method of asymptotic expansion. In the microscopic model for partially fissured medium in [49], there are two flows in the matrix; a global flow within the matrix and a flow that leads to local storage. The model for partially fissured media in [49] was extended by Clark and Showalter [38] to a quasi-linear version still in the deterministic case and the corresponding macroscopic model was derived using Nguetseng's two-scale convergence. Nguetseng, Showalter and Woukeng in [93] considered a general deterministic version of the problem in [49] beyond the periodic setting using Sigma convergence.

In geological formations, such as oil reservoirs, there are many factors that affect the flows within the domain, leading to uncertainties in estimating or predicting the flow in this type of formations which could lead to overestimates in well capacity. A stochastic process or random process is used to quantify uncertainties associated with physical or chemical processes since it provides a natural method for evaluating uncertainties. In [140], Wright reformulated the model in [49] for a randomly fissured media and used stochastic two-scale convergence in the mean for the homogenization process. The homogenized problem obtained is a stochastic analog of the homogenized problem obtained in [49]. Here, we model the influence of random fluctuations on a single-phase flow through a random force driven by the Wiener process. This leads to the flow in the partially fissured media being governed by a system of stochastic partial differential equations of nonlinear diffusion type involving oscillating coefficients. Since SPDEs are more advanced and more efficient tools in modelling random fluctuations on evolution systems arising in applied sciences, our model is naturally more elaborate that Wright's [140] which captures random influence through the random pertubation of the coefficient of a partial differential equation which does not involve random forces; the PDE has essentially a deterministic form.

This chapter is devoted to the study of this nonlinear stochastic evolution problem. Our main approach is homogenization and methodologically, we make use of the two scale convergence in combination with Ito's stochastic calculus and the probabilistic compactness results of Prokhorov and Skorokhod. The crucial difference with chapter two is that we are now dealing with SPDEs with nonlinear monotone operators. the plan of the chapter is as follows; in section 3.2, we state the assumptions on the geometry of the fissured porous medium under consideration, and the function spaces relevant to our study. In section 3.3, we introduce the microscopic model, assumptions on the model and the main result. Section 3.4 contains the existence result of the governing stochastic diffusion equation and the a priori estimates for the solution of the equation. Section 3.5 is devoted to the tightness property for a family of probability measures generated by the sequence of the solution to the stochastic diffusion equation and the driving Wiener process, it also contains the Prokhorov and Skorokhod compactness procedure, see [25]. In section 3.6, we prove the convergence of the microscopic problem to the macroscopic problem using Nguetseng's two-scale convergence [91], [3] and use Minty's trick (monotonicity method) [84], [111] to identify the weak limit. For more on Minty's monotonicity method, we refer to [75], [51].

3.2 Setting of the problem and preliminaries

3.2.1 The geometry of the partially fissured domain

Let us consider ϵ to be a positive parameter taking its values in a sequence which tends to zero and $[0, T]$, a time interval with $T \in (0, \infty)$.
Let Q be a bounded domain in $\mathbb{R}^n$ consisting of two sub-domains; one representing the fissures and the other representing the matrix.
Let $Y = [0, 1]^n$ denote the unit cell of measure $|Y| = 1$ consisting of two disjoint parts, Y_1 and Y_2 representing the local structure of the fissure and the matrix respectively. We let $\chi_i(y)$ denote the characteristic function of Y_i for $i = 1, 2$ such that $\chi_1(y) + \chi_2(y) = 1$. We assume that the sets $\{y \in \mathbb{R}^n; \chi_i(y) = 1\}$ for $i = 1, 2$ are smooth, Y_i-periodic and extended to all of $\mathbb{R}^n$ periodically.

For a given scale factor $\epsilon > 0$, the sub-domains Q_1^ϵ and Q_2^ϵ of Q represent the fissures and the matrix respectively with

$$Q_i^\epsilon = \{x \in Q; \ \ \chi_i\big(\frac{x}{\epsilon}\big) = 1\}, \quad i = 1, 2.$$

Let us denote the interface of Q_1^ϵ and Q_2^ϵ lying within Q as $\Gamma_{1,2}^\epsilon = \partial Q_1^\epsilon \cap \partial Q_2^\epsilon \cap Q$ and let $\Gamma_{1,2} = \partial Y_1 \cap \partial Y_2 \cap Y$ be the interface in the representative cell Y, and $\Gamma_{2,2} = \bar{Y_2} \cap \partial Y$ (see Figure) we denote by $\vec{\nu}_i$ the outer normal on ∂Q_i^ϵ for $i = 1, 2$.

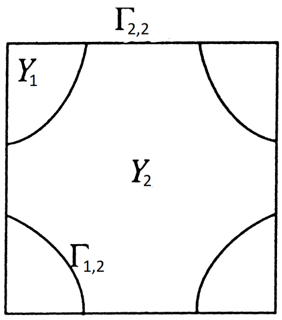

Figure 3.1: A representative cell Y of a partially fissured medium showing the local structure of the fissure Y_1, the matrix Y_2 and the interface $\Gamma_{1,2}$.

3.2.2 Function spaces

We recall some function spaces defined earlier and introduce some new ones needed throughout the chapter. Let Q be an open bounded set in $\mathbb{R}^n$. $C(Q)$ denotes the space of continuous functions $u : Q \to \mathbb{R}$, $C^\infty(Q)$ denotes the space of all infinitely differentiable functions $u : Q \to \mathbb{R}$ and $C^\infty_{per}(Y)$ denotes the restriction to Y of functions in $C^\infty(\mathbb{R}^n)$ that are Y-periodic.

For $2 \leq p \leq \infty$, we define the Sobolev space,

$$W^{1,p}(Q) = \{\phi : \phi \in L^p(Q), \frac{\partial \phi}{\partial x_j} \in L^p(Q), j = 1, ..., n\},$$

where the derivatives exists in the weak sense and $L^p(Q)$ is the usual Lebesgue space.
$W^{1,p}_0(Q)$ is the space of functions $\phi \in W^{1,p}(Q)$ with $\phi = 0$ on ∂Q, equipped with the $W^{1,p}$-norm, and $W^{1,p}_{per}$ is the closure of $C^\infty(Y)$ for the $W^{1,p}$-norm.
Let us introduce the weight vector $c(x) = [c_1, c_2, c_3]$ consisting of bounded positive functions. For $\epsilon > 0$, we define the following weights space

$$H^\epsilon = L^2(Q^\epsilon_1) \times L^2(Q^\epsilon_2) \times L^2(Q^\epsilon_2),$$

equipped with the inner product

$$\begin{aligned}([v_1, v_2, v_3], [\psi_1, \psi_2, \psi_3])_{H^\epsilon} = &\int_{Q^\epsilon_1} c^\epsilon_1(x) v_1(x) \psi_1(x) dx + \int_{Q^\epsilon_2} c^\epsilon_2(x) v_2(x) \psi_2(x) dx \\ &+ \int_{Q^\epsilon_2} c^\epsilon_3(x) v_3(x) \psi_3(x) dx,\end{aligned}$$

where $c^\epsilon_i(x) = c_i\left(\frac{x}{\epsilon}\right)$, for $i = 1, 2, 3$.
We write

$$(v_i, \psi_i)_{H^\epsilon_i} = \int_{Q^\epsilon_i} c^\epsilon_i(x) v_i(x) \psi_i(x) dx, \quad i = 1, 2, \quad (v_3, \psi_3)_{H^\epsilon_2} = \int_{Q^\epsilon_2} c^\epsilon_3(x) v_3(x) \psi_3(x) dx,$$

and

$$||\psi_i||^2_{H^\epsilon_i} = \int_{Q^\epsilon_i} c^\epsilon_i(x) |\psi_i(x)|^2 dx, \quad i = 1, 2, \quad ||\psi_3||^2_{H^\epsilon_2} = \int_{Q^\epsilon_2} c^\epsilon_3(x) |\psi_i(x)|^2 dx.$$

Let $\gamma_i^\epsilon : W^{1,p}(Q_i^\epsilon) \to L^p(\partial Q_i^\epsilon)$ for $i = 1, 2$ be the usual trace map. We define the space

$$G^\epsilon = \{[v_1, v_2, v_3] \in W^{1,p}(Q_1^\epsilon)\times W^{1,p}(Q_2^\epsilon)\times W^{1,p}(Q_2^\epsilon) : \gamma_1^\epsilon v_1 = \alpha\gamma_2^\epsilon v_2+\beta\gamma_2^\epsilon v_3 \text{ on } \Gamma_{1,2}^\epsilon\},$$

and

$$\mathcal{H}^\epsilon = H^\epsilon \cap G^\epsilon.$$

$\mathcal{H}^\epsilon$ is a Banach space equipped with the norm

$$\begin{aligned}||[v_1, v_2, v_3]||_{\mathcal{H}^\epsilon} = ||v_1||_{L^2(Q_1^\epsilon)} + ||v_2||_{L^2(Q_2^\epsilon)} + ||v_3||_{L^2(Q_2^\epsilon)} + ||\nabla v_1||_{L^p(Q_1^\epsilon)} \\ + ||\nabla v_2||_{L^p(Q_2^\epsilon)} + ||\nabla v_3||_{L^p(Q_2^\epsilon)}.\end{aligned}$$

3.3 The Micro-model

Now we develop a microscopic model for a single phase flow in a partially fissured medium.

In the fissures Q_1^ϵ, we shall denote the flow potential of the fluid by $u_1^\epsilon(t, x)$ and $-\mu_1\big(\frac{x}{\epsilon}, \nabla u_1^\epsilon\big)$ its corresponding flux. On the matrix Q_2^ϵ, we account for the global diffusion through the pore system in the matrix denoted by $u_2^\epsilon(t, x)$ with flux $-\mu_1\big(\frac{x}{\epsilon}, \nabla u_2^\epsilon\big)$ and the very high frequency spatial variation which leads to local storage in the matrix which we shall denote by $u_3^\epsilon(t, x)$ with flux $-\mu_3\big(\frac{x}{\epsilon}, \epsilon\nabla u_3^\epsilon\big)$. We specify two coefficients β and α which correspond to the proportion of the local and global phases of the total flow potential in the matrix Q_2^ϵ as measured on the interface $\Gamma_{1,2}^\epsilon$. Here, we take $\beta + \alpha = 1$ with $\beta > 0$ and $\alpha \geq 0$.

Let $\mu_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(i = 1, 2, 3)$ be some given vector fields. We make the following assumptions;

$A(1)$ $\mu_i(\cdot, \vec{\xi})$ is measurable and Y-periodic for every $\vec{\xi} \in \mathbb{R}^n$,

$A(2)$ $\mu_i(y, \cdot)$ is continuous for a.e. $y \in Y$,

$A(3)$ there are positive constants k, C, C', C_0 and $2 \leq p < \infty$ such that for every $\vec{\xi}, \vec{\zeta} \in \mathbb{R}^n$ and a.e. $y \in Y$;

$A(3.1)$ $|\mu_i(y, \vec{\xi})| \leq C|\vec{\xi}|^{p-1} + k$,

$A(3.2)\quad (\mu_i(y,\vec{\xi})-\mu_i(y,\vec{\zeta}))\cdot(\vec{\xi}-\vec{\zeta})\geq C'|\vec{\xi}-\vec{\zeta}|^p,$

$A(3.3)\quad \mu_i(y,\vec{\xi})\cdot\vec{\xi}\geq C_0|\vec{\xi}|^p-k.$

the prototype of the operators $\nabla\cdot\mu_i(y,\nabla u)$ is the p-Laplacian $\nabla\cdot(|\nabla u|^{p-2}\nabla u)$.

Let $c_i\in C_{per}(Y)$ for $i=1,2,3$ be given such that

$$0<c_0\leq c_i(y)\leq C,\quad c_i,c_i^{-1}\in L^\infty(Q). \tag{3.1}$$

For $i=1,2,3$, we can define the corresponding scaled coefficient at $x\in Q_i^\epsilon,\vec{\xi}\in\mathbb{R}^n$ by

$$c_i^\epsilon(x)=c_i\big(\frac{x}{\epsilon}\big),\quad \mu_i^\epsilon(x,\vec{\xi})=\mu_i\big(\frac{x}{\epsilon},\vec{\xi}\big).$$

The micro-model for diffusion in a partially fissured medium driven by random forces is given by the system of stochastic nonlinear diffusion equations

$$(P^\epsilon)\qquad \begin{aligned} c_1^\epsilon du_1^\epsilon &= \nabla\cdot\mu_1^\epsilon(x,\nabla u_1^\epsilon(t,x))dt+f_1^\epsilon(t,x)dB_1(t) \text{ in } Q_1^\epsilon,\\ c_2^\epsilon du_2^\epsilon &= \nabla\cdot\mu_2^\epsilon(x,\nabla u_2^\epsilon(t,x))dt+f_2^\epsilon(t,x)dB_2(t) \text{ in } Q_2^\epsilon,\\ c_3^\epsilon du_3^\epsilon &= \epsilon\nabla\cdot\mu_3^\epsilon(x,\epsilon\nabla u_3^\epsilon(t,x))dt+f_3^\epsilon(t,x)dB_3(t) \text{ in } Q_2^\epsilon,\\ u_1^\epsilon &= \alpha u_2^\epsilon+\beta u_3^\epsilon \text{ on } \Gamma_{1,2}^\epsilon,\\ \alpha\mu_1^\epsilon(x,\nabla u_1^\epsilon(t,x))\cdot\vec{\nu}_1 &= \mu_2^\epsilon(x,\nabla u_2^\epsilon(t,x))\cdot\vec{\nu}_1 \text{ on } \Gamma_{1,2}^\epsilon,\\ \beta\mu_1^\epsilon(x,\nabla u_1^\epsilon(t,x))\cdot\vec{\nu}_1 &= \epsilon\mu_2^\epsilon(x,\epsilon\nabla u_2^\epsilon(t,x))\cdot\vec{\nu}_1 \text{ on } \Gamma_{1,2}^\epsilon, \end{aligned}$$

where $t\in[0,T]$, $T\in(0,\infty)$. The first equation is the conservation of mass defined in the fissures, with $u_1^\epsilon(t,x)$ representing the flow potential in the fissures. We have two components of flow potential in the matrix; $u_2^\epsilon(t,x)$ represents the usual flow through the matrix and $u_3^\epsilon(t,x)$ scaled by ϵ^p represents the very high frequency variation in the flow resulting from the relatively low permeability of the matrix, f_i^ϵ $(i=1,2,3)$ is the intensity of the noise. These flows are assumed to satisfy corresponding conservation equations. $(B_i(t))_{0\leq t\leq T}(i=1,2,3)$ are mutually independent standard 1-dimensional Wiener processes defined on a given filtered probability space $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F})_{0\leq t\leq T})$.

We assume that

$A(4)\quad f_i^\epsilon(t,x)=f_i(t,\frac{x}{\epsilon})\in L^4((0,T)\times Q_i^\epsilon)$ for $(i=1,2)$ and $f_3^\epsilon(t,x)=f_3(t,\frac{x}{\epsilon})\in L^4((0,T)\times Q_2^\epsilon)$ are such that f_i^ϵ is uniformly bounded.

(P^ϵ) is a transmission problem of stochastic partial differential equations due to the prescribed transmission boundary conditions on the interface $\Gamma^\epsilon_{1,2}$.
Recall that on the matrix Q_2^ϵ, α and β denote the corresponding partitions for the flow potentials u_2^ϵ and u_3^ϵ respectively. The coupling on the interface is a vital element in the system. The continuity of the flow potential is represented in the first interface condition (the fourth relation in (P^ϵ)), with prescribed partitions corresponding to the global and local phases in the matrix. The fifth and sixth relations describe the flux across the interface $\Gamma^\epsilon_{1,2}$ between the flow potential in the fissures and the total flow potential in the matrix. The external boundary conditions (on ∂Q) will play no role, so we assume the following homogeneous Dirichlet boundary conditions;

$$\begin{aligned} u_1^\epsilon(t,x) &= 0, \quad x \in \partial Q_1^\epsilon \cap \partial Q, \\ u_2^\epsilon(t,x) &= 0, \quad x \in \partial Q_2^\epsilon \cap \partial Q \text{ and} \\ u_3^\epsilon(t,x) &= 0, \quad x \in \partial Q_2^\epsilon \cap \partial Q, \end{aligned} \tag{3.2}$$

and initial conditions

$$u_1^\epsilon(0,\cdot) = u_1^0(\cdot), \;\; u_2^\epsilon(0,\cdot) = u_2^0(\cdot), \;\; u_3^\epsilon(0,\cdot) = u_3^0(\cdot) \quad \text{in } H^\epsilon. \tag{3.3}$$

The aim of the chapter is to show that the sequence $\vec{u}^\epsilon = [u_1^\epsilon, u_2^\epsilon, u_3^\epsilon]$ converges in suitable topologies to the stochastic process $\vec{u} = [u_1, u_2, U_3]$ which is a solution to the following SPDEs:

$$\begin{aligned} & d\int_{Y_1} c_1(y)u_1(t,x)dy + \frac{1}{\beta} d\int_{Y_2} c_3(y)U_3(t,x,y)dydt \\ &\quad = \nabla \cdot \int_{Y_1} \mu_1\big(y, \nabla u_1(t,x) + \nabla_y U_1(t,x,y)\big)dydt \\ &\quad + \int_{Y_1} f_1(t,x,y)dyd\tilde{B}_1(t) + \int_{Y_2} \frac{1}{\beta} f_3(t,x,y)dyd\tilde{B}_3(t), \\ &\quad t \in (0,T), x \in Q, y \in Y_i, i = 1,2. \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
d\int_Q\int_{Y_2} c_2(y)u_2(t,x)dydx - \frac{\alpha}{\beta}d\int_Q\int_{Y_2} c_3(y)U_3(t,x,y)dydx \\
&= \nabla\cdot\Big(\int_Q\int_{Y_2}\mu_2\big(y,\nabla u_2(t,x)+\nabla_y U_2(t,x,y)\big)dydx\Big)dt \\
&+ \int_Q\int_{Y_2} f_2(t,x,y)dydxd\tilde{B}_2(t) - \frac{\alpha}{\beta}\int_Q\int_{Y_2} f_3(t,x,y)dydxd\tilde{B}_3(t), \\
&t\in(0,T), x\in Q, \quad y\in Y_2.
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
d\int_Q\int_{Y_2} c_3(y)U_3(t,x,y)dydx &= \nabla_y\cdot\int_Q\int_{Y_2}\mu_3\big(y,\nabla_y U_3(t,x,y)\big)dydxdt \\
&+ \int_Q\int_{Y_2} f_3(t,x,y)dydxd\tilde{B}_3(t), \quad y\in Y_2 \\
&U_3(t,x,y) \text{ and } \nabla_y\cdot\mu_3\big(y,\nabla_y U_3(t,x,y)\big)\cdot\nu \text{ are } Y\text{-periodic on } \Gamma_{2,2}, \\
&\beta U_3 = u_1-\alpha u_2 \text{ on } \Gamma_{1,2}.
\end{aligned} \tag{3.6}
$$

with initial conditions

$$u_i(0,x) = u_i^0(x) \text{ for } i=1,2,$$

$$U_3(0,x,y) = u_3^0(x),$$

where

$$U_i \in \mathcal{D}((0,T)\times Q; W_0^{1,p}(Y)),\ i=1,2,3$$

Furthermore,

$$u_i(t,x) - 0 \ \text{ on } \partial Q,\ \forall t\in[0,T],$$

and $\tilde{B} = (\tilde{B}_1,\tilde{B}_2,\tilde{B}_3)$ is an appropriate Wiener process which is the result of the Prokhorov-Skorokhod compactness process.

3.4 Existence and Uniqueness

We introduce the notion of solution of problem (P^ϵ) which is of interest to us.

Definition 3.1. For a fixed $\epsilon > 0$, we define a strong probabilistic solution of problem (P^ϵ) as a stochastic process $\vec{u} = [u_1^\epsilon, u_2^\epsilon, u_3^\epsilon]$ such that

1. $\vec{u}^\epsilon \in L^2(\Omega,\mathcal{F},\mathbb{P},L^\infty(0,T;H^\epsilon)) \cap L^2(\Omega,\mathcal{F},\mathbb{P},L^2(0,T;G^\epsilon))$.

2. For all $t \in [0,T]$, $\vec{u}^\epsilon$ satisfies

$$\sum_{i=1}^{2}(c_i^\epsilon(\cdot)u_i^\epsilon(t,\cdot),\phi_i) + (c_3^\epsilon(\cdot)u_3^\epsilon(t,\cdot),\phi_3) = \sum_{i=1}^{2}(c_i^\epsilon(\cdot)u_i^\epsilon(0,\cdot),\phi_i) + (c_3^\epsilon(\cdot)u_3^\epsilon(0,\cdot),\phi_3)$$
$$+\sum_{i=1}^{2}\int_0^t(\mu_i^\epsilon(\cdot,\nabla u_i^\epsilon(s,\cdot)),\nabla\phi_i)ds + \int_0^t(\mu_3^\epsilon(\cdot,\epsilon\nabla u_3^\epsilon(s,\cdot)),\epsilon\nabla\phi_3)ds$$
$$=\sum_{i=1}^{2}\int_0^t(f_i^\epsilon(s,\cdot),\phi_i)dB_i(s) + \int_0^t(f_3^\epsilon(s,\cdot),\phi_3)dB_3(s) \quad \mathbb{P}\text{-a.s}$$

$u_i^\epsilon(0,\cdot) = u_i^0(\cdot)$ $(i=1,2)$ and $u_3^\epsilon(0,\cdot) = u_3^0(\cdot)$, $\forall\phi_1,\phi_2,\phi_3 \in \mathcal{H}^\epsilon$.

Theorem 3.2. *For each $\epsilon > 0$, under assumptions $A(1) - A(4)$ there exists a unique solution of problem (P^ϵ) in the sense of definition 3.1.*

Theorem 3.2 has essentially been proven by Pardoux in [101] and Krylov and Rozovskii in [69] using monotonicity method.
If we weaken the condition $A(3.2)$ to the usual monotonicity condition i.e

$$(\mu_i(\vec{\xi}) - \mu_i(\vec{\zeta}))\cdot(\vec{\xi} - \vec{\zeta}) \geq 0,$$

for all $\vec{\xi},\vec{\zeta} \in \mathbb{R}^n$, we lose uniqueness of the solution but the existence still hold in a weaker sense. Indeed Bensoussan established in [21] the existence of a weak probabilistic solution in the case of one equation involving monotone operators.

3.4.1 A priori estimates

We now establish crucial a priori estimates for problem (P^ϵ).

Lemma 3.3. *We assume that ϵ is a fixed positive number, under the assumptions $A(1) - A(4)$, the solution $[u_1^\epsilon, u_2^\epsilon, u_3^\epsilon]$ of (P^ϵ) satisfies the following estimate;*

$$\begin{aligned}&\mathbb{E}\sup_{0\leq t\leq T}||u_1^\epsilon||^2_{H_1^\epsilon} + \mathbb{E}\sup_{0\leq t\leq T}||u_2^\epsilon||^2_{H_2^\epsilon} + \mathbb{E}\sup_{0\leq t\leq T}||u_3^\epsilon||^2_{H_2^\epsilon}\\ &+\mathbb{E}\sum_{i=1}^{2}\int_0^T||\nabla u_i^\epsilon||^p_{L^p(Q_i^\epsilon)}dt + \int_0^T||\epsilon\nabla u_3^\epsilon||^p_{L^p(Q_2^\epsilon)}dt \\ &\leq C.\end{aligned} \tag{3.7}$$

Proof. Using Ito's formula on the first equation of (P^ϵ) gives

$$\begin{aligned}||u_1^\epsilon(t)||^2_{H_1^\epsilon} = ||u_1^\epsilon(0)||^2_{H_1^\epsilon} + 2\int_0^t \big(\nabla\cdot\mu_1^\epsilon(x,\nabla u_1^\epsilon(s)), u_1^\epsilon(s)\big)_{L^2(Q_1^\epsilon)} ds \\ + 2\int_0^t (f_1^\epsilon(s), u_1^\epsilon(s))_{L^2(Q_1^\epsilon)} dB_1(s) + \int_0^t ||f_1^\epsilon||^2_{L^2(Q_1^\epsilon)} ds.\end{aligned} \tag{3.8}$$

Integrating by parts on the second term on the right hand side of (3.8) yields

$$\begin{aligned}2\int_0^t \big(\nabla\cdot\mu_1^\epsilon(x,\nabla u_1^\epsilon(s)), u_1^\epsilon(s)\big)_{L^2(Q_1^\epsilon)} ds = -2\int_0^t \big(\mu_1^\epsilon(x,\nabla u_1^\epsilon(s)), \nabla u_1^\epsilon(s)\big)_{L^2(Q_1^\epsilon)} ds \\ + 2\int_0^t \big(\mu_1^\epsilon(x,\nabla u_1^\epsilon(s))\cdot\vec{\nu}_1, u_1^\epsilon(s)\big)_{L^2(\Gamma_{1,2}^\epsilon)} ds,\end{aligned}$$

using the identity $u_1^\epsilon = \alpha u_2^\epsilon + \beta u_3^\epsilon$ on $\Gamma_{1,2}^\epsilon$, we get

$$\begin{aligned}||u_1^\epsilon(t)||^2_{H_1^\epsilon} + 2\int_0^t \big(\mu_1^\epsilon(x,\nabla u_1^\epsilon(s)), \nabla u_1^\epsilon(s)\big)_{L^2(Q_1^\epsilon)} ds = ||u_1^\epsilon(0)||^2_{H_1^\epsilon} \\ + 2\int_0^t \big(\alpha\mu_1^\epsilon(x,\nabla u_1^\epsilon(s))\cdot\vec{\nu}_1, u_2^\epsilon(s)\big)_{L^2(\Gamma_{1,2}^\epsilon)} ds \\ + 2\int_0^t \big(\beta\mu_1^\epsilon(x,\nabla u_1^\epsilon(s))\cdot\vec{\nu}_1, u_3^\epsilon(s)\big)_{L^2(\Gamma_{1,2}^\epsilon)} ds \\ + 2\int_0^t (f_1^\epsilon(s), u_1^\epsilon(s))_{L^2(Q_1^\epsilon)} dB_1(s) + \int_0^t ||f_1^\epsilon||^2_{L^2(Q_1^\epsilon)} ds.\end{aligned} \tag{3.9}$$

Ito's formula on the second equation in (P^ϵ) gives

$$\begin{aligned}||u_2^\epsilon(t)||^2_{H_2^\epsilon} + 2\int_0^t \big(\mu_2^\epsilon(x,\nabla u_2^\epsilon(s)), \nabla u_2^\epsilon(s)\big)_{L^2(Q_2^\epsilon)} ds = ||u_2^\epsilon(0)||^2_{H_2^\epsilon} \\ - 2\int_0^t \big(\mu_2^\epsilon(x,\nabla u_2^\epsilon(s))\cdot\vec{\nu}_1, u_2^\epsilon(s)\big)_{L^2(\Gamma_{1,2}^\epsilon)} ds \\ + 2\int_0^t (f_2^\epsilon(s), u_2^\epsilon(s))_{L^2(Q_2^\epsilon)} dB_2(s) \\ + \int_0^t ||f_2^\epsilon||^2_{L^2(Q_2^\epsilon)} ds,\end{aligned} \tag{3.10}$$

where we have used the relation $\vec{\nu}_1 = -\vec{\nu}_2$ on $\Gamma_{1,2}^\epsilon$.

Lastly, Ito's formula on the third equation on (P^ϵ) gives

$$\begin{aligned}||u_3^\epsilon(t)||^2_{H_2^\epsilon} + 2\int_0^t \big(\mu_3^\epsilon(x,\epsilon\nabla u_3^\epsilon(s)), \epsilon\nabla u_3^\epsilon(s)\big)_{L^2(Q_2^\epsilon)} ds = ||u_3^\epsilon(0)||^2_{H_2^\epsilon} \\ - 2\int_0^t \big(\epsilon\mu_3^\epsilon(x,\epsilon\nabla u_3^\epsilon(s))\cdot\vec{\nu}_1, u_3^\epsilon(s)\big)_{L^2(\Gamma_{1,2}^\epsilon)} ds \\ + 2\int_0^t (f_3^\epsilon(s), u_3^\epsilon(s))_{L^2(Q_2^\epsilon)} dB_3(s) \\ + \int_0^t ||f_2^\epsilon||^2_{L^2(Q_2^\epsilon)} ds.\end{aligned} \tag{3.11}$$

Summing (3.9), (3.10) and (3.11), we get

$$\sum_{i=1}^{2} ||u_i^\epsilon(t)||^2_{H_i^\epsilon} + ||u_3^\epsilon(t)||^2_{H_2^\epsilon} + \sum_{i=1}^{2} 2\int_0^t \big(\mu_i^\epsilon(x, \nabla u_i^\epsilon(s)), \nabla u_i^\epsilon(s)\big)_{L^2(Q_i^\epsilon)} ds$$
$$+ 2\int_0^t \big(\mu_3^\epsilon(x, \epsilon\nabla u_3^\epsilon(s)), \epsilon\nabla u_3^\epsilon(s)\big)_{L^2(Q_2^\epsilon)} ds = \sum_{i=1}^{2} ||u_i^\epsilon(0)||^2_{H_i^\epsilon} + ||u_3^\epsilon(0)||^2_{H_2^\epsilon}$$
$$+ 2\int_0^t \big(\alpha\mu_1^\epsilon(x, \nabla u_1^\epsilon(s)) \cdot \vec{\nu}_1, u_2^\epsilon(s)\big)_{L^2(\Gamma_{1,2}^\epsilon)} ds$$
$$+ 2\int_0^t \big(\beta\mu_1^\epsilon(x, \nabla u_1^\epsilon(s)) \cdot \vec{\nu}_1, u_3^\epsilon(s)\big)_{L^2(\Gamma_{1,2}^\epsilon)} ds$$
$$- 2\int_0^t \big(\mu_2^\epsilon(x, \nabla u_2^\epsilon(s)) \cdot \vec{\nu}_1, u_2^\epsilon(s)\big)_{L^2(\Gamma_{1,2}^\epsilon)} ds$$
$$- 2\int_0^t \big(\epsilon\mu_3^\epsilon(x, \epsilon\nabla u_3^\epsilon(s)) \cdot \vec{\nu}_1, u_3^\epsilon(s)\big)_{L^2(\Gamma_{1,2}^\epsilon)} ds$$
$$+ 2\int_0^t (f_1^\epsilon, u_1^\epsilon)_{L^2(Q_1^\epsilon)} ds + 2\int_0^t (f_2^\epsilon, u_2^\epsilon)_{L^2(Q_2^\epsilon)} ds + 2\int_0^t (f_3^\epsilon, u_3^\epsilon)_{L^2(Q_2^\epsilon)} ds$$
$$+ \sum_{i=1}^{2} \int_0^t ||f_i^\epsilon||^2_{L^2(Q_i^\epsilon)} ds + \int_0^t ||f_3^\epsilon||^2_{L^2(Q_2^\epsilon)} ds.$$

The boundary terms mutually cancel out thanks to the fifth and sixth relations in (P^ϵ). Using $A(3.3)$ in the resulting relation, we have

$$\sum_{i=1}^{2} ||u_i^\epsilon(t)||^2_{H_i^\epsilon} + ||u_3^\epsilon(t)||^2_{H_2^\epsilon} + 2C_0 \sum_{i=1}^{2} \int_0^t ||\nabla u_i^\epsilon||^p_{L^p(Q_i^\epsilon)} ds$$
$$+ 2C_0 \int_0^t ||\epsilon\nabla u_3^\epsilon||^p_{L^p(Q_2^\epsilon)} \leq \sum_{i=1}^{2} ||u_i^\epsilon(0)||^2_{H_i^\epsilon} + ||u_3^\epsilon(0)||^2_{H_2^\epsilon}$$
$$+ 2\sum_{i=1}^{2} \int_0^t (f_i^\epsilon, u_i^\epsilon)_{L^2(Q_i^\epsilon)} dB_i(s) + 2\int_0^t (f_3^\epsilon, u_3^\epsilon)_{L^2(Q_2^\epsilon)} dB_3(s)$$
$$+ \sum_{i=1}^{2} \int_0^t ||f_i^\epsilon||^2_{L^2(Q_i^\epsilon)} ds + \int_0^t ||f_3^\epsilon||^2_{L^2(Q_2^\epsilon)} ds + t|k|.$$

Taking the supremum over $[0,T]$ followed by the expectation in both sides, we get

$$\mathbb{E}\sup_{0\le t\le T}\sum_{i=1}^{2}||u_i^\epsilon(t)||^2_{H_i^\epsilon}+\mathbb{E}\sup_{0\le t\le T}||u_3^\epsilon(t)||^2_{H_2^\epsilon}+2C_0\mathbb{E}\sum_{i=1}^{2}\int_0^T||\nabla u_i^\epsilon||^p_{L^p(Q_i^\epsilon)}ds$$
$$+2C_0\mathbb{E}\int_0^T||\epsilon\nabla u_3^\epsilon||^p_{L^p(Q_2^\epsilon)}\le\sum_{i=1}^{2}||u_i^\epsilon(0)||^2_{H_i^\epsilon}+||u_3^\epsilon(0)||^2_{H_2^\epsilon}$$
$$+\mathbb{E}\sup_{0\le t\le T}\left[2\sum_{i=1}^{2}\int_0^t(f_i^\epsilon,u_i^\epsilon)_{L^2(Q_i^\epsilon)}dB_i(s)+\int_0^t(f_3^\epsilon,u_3^\epsilon)_{L^2(Q_2^\epsilon)}dB_3(s)\right]$$
$$+\mathbb{E}\sum_{i=1}^{2}\int_0^T||f_i^\epsilon||^2_{L^2(Q_i^\epsilon)}ds+\int_0^T||f_3^\epsilon||^2_{L^2(Q_2^\epsilon)}ds+T|k|.$$

Thanks to Burkhölder-Davis-Gundy's inequality, Cauchy-Schwarz's and Young's inequalities, we get

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}2\left|\int_0^t(f_1^\epsilon,u_1^\epsilon)_{L^2(Q_1^\epsilon)}dB_1(s)\right|&\le\mathbb{E}C\left(\int_0^T(f_1^\epsilon,u_1^\epsilon)^2_{L^2(Q_1^\epsilon)}dt\right)^{\frac{1}{2}}\\&\le\mathbb{E}C\left(\int_0^T||f_1^\epsilon||^2_{L^2(Q_1^\epsilon)}||u_1^\epsilon||^2_{L^2(Q_1^\epsilon)}dt\right)^{\frac{1}{2}}\\&\le\mathbb{E}\sup_{0\le t\le T}||u_1^\epsilon||_{L^2(Q_1^\epsilon)}\left(\int_0^T||f_1^\epsilon||^2_{L^2(Q_1^\epsilon)}dt\right)^{\frac{1}{2}}\\&\le\varpi\mathbb{E}\sup_{0\le t\le T}||u_1^\epsilon||^2_{L^2(Q_1^\epsilon)}+c(\varpi)\mathbb{E}\int_0^T||f_1^\epsilon||^2_{L^2(Q_1^\epsilon)}dt,\end{aligned}$$

where ϖ is an arbitrary positive number. Similarly, for any $\varpi>0$,

$$\mathbb{E}\sup_{0\le t\le T}2\left|\int_0^t(f_2^\epsilon,u_2^\epsilon)_{L^2(Q_2^\epsilon)}dB_2(s)\right|\le\varpi\mathbb{E}\sup_{0\le t\le T}||u_2^\epsilon||^2_{L^2(Q_2^\epsilon)}+C(\varpi)\mathbb{E}\int_0^T||f_2^\epsilon||^2_{L^2(Q_2^\epsilon)}dt,$$

and

$$\mathbb{E}\sup_{0\le t\le T}2\left|\int_0^t(f_3^\epsilon,u_3^\epsilon)_{L^2(Q_2^\epsilon)}dB_3(s)\right|\le\varpi\mathbb{E}\sup_{0\le t\le T}||u_3^\epsilon||^2_{L^2(Q_2^\epsilon)}+C(\varpi)\mathbb{E}\int_0^T||f_3^\epsilon||^2_{L^2(Q_2^\epsilon)}dt.$$

For ϖ sufficiently small, we have

$$\begin{aligned}&\mathbb{E}\sup_{0\le t\le T}||u_1^\epsilon||^2_{H_1^\epsilon}+\mathbb{E}\sup_{0\le t\le T}||u_2^\epsilon||^2_{H_2^\epsilon}+\mathbb{E}\sup_{0\le t\le T}||u_3^\epsilon||^2_{H_2^\epsilon}\\&\quad+\mathbb{E}C\left[\int_0^T||\nabla u_1^\epsilon||^p_{L^p(Q_1^\epsilon)}+||\nabla u_2^\epsilon||^p_{L^p(Q_2^\epsilon)}+||\epsilon\nabla u_3^\epsilon||^p_{L^p(Q_2^\epsilon)}dt\right]\\&\quad\le\mathbb{E}\left[||u_1^0(x)||^2_{H_1^\epsilon}+||u_2^0(x)||^2_{H_2^\epsilon}+||u_3^0(x)||^2_{H_2^\epsilon}\right]\\&\quad+C\mathbb{E}\int_0^T||f_1^\epsilon||^2_{L^2(Q_1^\epsilon)}+||f_2^\epsilon||^2_{L^2(Q_2^\epsilon)}+||f_3^\epsilon||^2_{L^2(Q_2^\epsilon)}dt+Tk.\end{aligned}$$

Based on the assumptions on u_1^0, u_2^0, u_3^0 and on $f_1^\epsilon, f_2^\epsilon, f_3^\epsilon$, we get

$$\mathbb{E}\sup_{0\le t\le T}\sum_{i=1}^{2}||u_i^\epsilon(t)||^2_{H_i^\epsilon}+\mathbb{E}\sup_{0\le t\le T}||u_3^\epsilon(t)||^2_{H_2^\epsilon}+\sum_{i=1}^{2}\mathbb{E}\int_0^T||\nabla u_i^\epsilon||^p_{L^p(Q_i^\epsilon)}dt$$
$$+\mathbb{E}\int_0^T||\epsilon\nabla u_3^\epsilon||^p_{L^p(Q_2^\epsilon)}dt\le C.$$

□

Next we establish a key estimate of the finite difference of $\vec{u}^\epsilon$ in the dual of $\mathcal{H}^\epsilon$. It plays an important role in the implementation of the compactness results. It should be noted that such an estimate was not required in the deterministic case considered in [38].

Lemma 3.4. *Under the assumptions of Lemma 3.3, the solution $\vec{u}^\epsilon=[u_1^\epsilon,u_2^\epsilon,u_3^\epsilon]$ of problem (P^ϵ) satisfies the estimate*

$$\mathbb{E}\sup_{|h|\le\delta}\int_0^{T-h}||\vec{u}^\epsilon(t+h)-\vec{u}^\epsilon(t)||^{p'}_{(\mathcal{H}^\epsilon)'}dt\le C\,max\{\delta^{\frac{1}{p-1}},\delta^{\frac{p'}{4}}\},$$

for any h such that $t+h\in[0,T]$ and $\forall\delta\le 1$.

Proof. Let $\vec{\psi}=[\psi,\psi_2,\psi_3]\in\mathcal{H}^\epsilon$, such that $||\vec{\psi}||_{\mathcal{H}^\epsilon}\le 1$, we have

$$||\vec{u}^\epsilon(t+h)-\vec{u}^\epsilon(t)||_{(\mathcal{H}^\epsilon)'}=\sup_{\vec{\psi}\in\mathcal{H}^\epsilon,||\vec{\psi}||_{\mathcal{H}^\epsilon}\le1}\left|\langle\vec{u}^\epsilon(t+h)-\vec{u}^\epsilon(t),\vec{\psi}\rangle_{(\mathcal{H}^\epsilon)',\mathcal{H}^\epsilon}\right|$$
$$=\sup_{\vec{\psi}\in\mathcal{H}^\epsilon,||\vec{\psi}||_{\mathcal{H}^\epsilon}\le1}\left|\left\langle\left[u_1^\epsilon(t+h)-u_1^\epsilon(t),u_2^\epsilon(t+h)-u_2^\epsilon(t),u_3^\epsilon(t+h)-u_3^\epsilon(t)\right],[\psi_1,\psi_2,\psi_3]\right\rangle_{(\mathcal{H}^\epsilon)',\mathcal{H}^\epsilon}\right|$$
$$\le\sup_{\vec{\psi}\in\mathcal{H}^\epsilon,||\vec{\psi}||_{\mathcal{H}^\epsilon}\le1}\left|\left[\sum_{i=1}^{2}\int_t^{t+h}\int_{Q_i^\epsilon}\nabla\cdot\mu_i^\epsilon(x,\nabla u_i^\epsilon)\psi_i dxds+\int_t^{t+h}\int_{Q_2^\epsilon}\epsilon\nabla\cdot\mu_3^\epsilon(x,\epsilon\nabla u_3^\epsilon)\psi_3dxds\right]\right|$$
$$+\left|\left|\left[\int_t^{t+h}f_1^\epsilon dB_1(s),\int_t^{t+h}f_2^\epsilon dB_2(s),\int_t^{t+h}f_3^\epsilon dB_3(s)\right]\right|\right|_{(\mathcal{H}^\epsilon)'}$$
$$\le\sup_{\vec{\psi}\in\mathcal{H}^\epsilon,||\vec{\psi}||_{\mathcal{H}^\epsilon}\le1}\left|\left[-\sum_{i=1}^{2}\int_t^{t+h}\int_{Q_i^\epsilon}\mu_i^\epsilon(x,\nabla u_i^\epsilon)\nabla\psi_i dxds-\int_t^{t+h}\int_{Q_2^\epsilon}\mu_3^\epsilon(x,\epsilon\nabla u_3^\epsilon)\epsilon\nabla\psi_3dxds\right.\right.$$
$$+\int_t^{t+h}\int_{\Gamma_{1,2}^\epsilon}\mu_1^\epsilon(x,\nabla u_1^\epsilon)\cdot\vec{\nu}_1\psi_1dxds+\int_t^{t+h}\int_{\Gamma_{1,2}^\epsilon}\mu_2^\epsilon(x,\nabla u_2^\epsilon)\cdot\vec{\nu}_2\psi_2dxds$$
$$\left.\left.+\int_t^{t+h}\int_{\Gamma_{1,2}^\epsilon}\mu_3^\epsilon(x,\nabla u_3^\epsilon)\cdot\vec{\nu}_2\psi_3dxds\right]\right|$$
$$+\left|\left|\left[\int_t^{t+h}f_1^\epsilon dB_1(s),\int_t^{t+h}f_2^\epsilon dB_2(s),\int_t^{t+h}f_3^\epsilon dB_3(s)\right]\right|\right|_{(\mathcal{H}^\epsilon)'}.$$

Since $\psi_1, \psi_2, \psi_3 \in \mathcal{H}^\epsilon$, we have $\psi_1 = \alpha\psi_2 + \beta\psi_3$ on $\Gamma^\epsilon_{1,2}$ and the terms on $\Gamma^\epsilon_{1,2}$ cancel out due to the fifth and sixth relations in (P^ϵ), so we get

$$\begin{aligned}
&||\vec{u}^\epsilon(t+h) - \vec{u}^\epsilon(t)||_{(\mathcal{H}^\epsilon)'} \leq \\
&\sup_{\vec{\psi}\in\mathcal{H}^\epsilon, ||\vec{\psi}||_{\mathcal{H}^\epsilon}\leq 1} \left|\left|\left[-\sum_{i=1}^{2}\int_t^{t+h}\int_{Q_i^\epsilon}\mu_i^\epsilon(x,\nabla u_i^\epsilon)\nabla\psi_i dxds - \int_t^{t+h}\int_{Q_2^\epsilon}\mu_3^\epsilon(x,\epsilon\nabla u_3^\epsilon)\epsilon\nabla\psi_3 dxds\right]\right|\right| \\
&+ \left|\left| \int_t^{t+h} f_1^\epsilon dB_1(s), \int_t^{t+h} f_2^\epsilon dB_2(s), \int_t^{t+h} f_3^\epsilon dB_3(s)\right|\right|_{(\mathcal{H}^\epsilon)'}.
\end{aligned} \tag{3.12}$$

Using assumption $A(3.1)$ on the first two terms on the right hand side of (3.12) gives

$$\begin{aligned}
&\sup_{\vec{\psi}\in\mathcal{H}^\epsilon, ||\vec{\psi}||_{\mathcal{H}^\epsilon}\leq 1}\left[\sum_{i=1}^{2}\int_t^{t+h}\int_{Q_i^\epsilon}|\mu_i^\epsilon(x,\nabla u_i^\epsilon)||\nabla\psi_i|dxds + \int_t^{t+h}\int_{Q_2^\epsilon}|\mu_3^\epsilon(x,\epsilon\nabla u_3^\epsilon)||\epsilon\nabla\psi_3|dxds\right] \\
&\leq \sup_{\vec{\psi}\in\mathcal{H}^\epsilon, ||\vec{\psi}||_{\mathcal{H}^\epsilon}\leq 1}\left[\sum_{i=1}^{2}\int_t^{t+h}\int_{Q_i^\epsilon}|\nabla u_i^\epsilon|^{p-1}|\nabla\psi_i|dxds + \int_t^{t+h}\int_{Q_2^\epsilon}|\epsilon\nabla u_3^\epsilon|^{p-1}|\epsilon\nabla\psi_3|dxds\right. \\
&\left. + \sum_{i=1}^{2}\int_t^{t+h}\int_{Q_i^\epsilon}k|\nabla\psi_i|dxds + \int_t^{t+h}\int_{Q_2^\epsilon}k|\epsilon\nabla\psi_3|dxds\right] \\
&\leq \sum_{i=1}^{2}\int_t^{t+h}||\nabla u_i^\epsilon||^{\frac{p}{p'}}_{L^p(Q_i^\epsilon)}ds + \int_t^{t+h}||\epsilon\nabla u_3^\epsilon||^{\frac{p}{p'}}_{L^p(Q_2^\epsilon)}ds + 3\int_t^{t+h}kds \\
&\leq h^{\frac{1}{p}}\sum_{i=1}^{2}\left(\int_t^{t+h}||\nabla u_i^\epsilon||^p_{L^p(Q_i^\epsilon)}ds\right)^{\frac{1}{p'}} + h^{\frac{1}{p}}\left(\int_t^{t+h}||\epsilon\nabla u_3^\epsilon||^p_{L^p(Q_2^\epsilon)}ds\right)^{\frac{1}{p'}} + 3\int_t^{t+h}kds,
\end{aligned}$$

where we have used Hölder's inequality.

Now we estimate the terms involving the stochastic term using the following embedding

$$\mathcal{H}^\epsilon \hookrightarrow H^\epsilon,$$

and since both $\mathcal{H}^\epsilon$ and H^ϵ are reflexive spaces, we have

$$(H^\epsilon)' \hookrightarrow (\mathcal{H}^\epsilon).'$$

Consequently,

$$
\begin{aligned}
&\Big|\Big|\Big[\int_t^{t+h} f_1^\epsilon dB_1(s), \int_t^{t+h} f_2^\epsilon dB_2(s), \int_t^{t+h} f_3^\epsilon dB_3(s)\Big]\Big|\Big|_{(\mathcal{H}^\epsilon)'}\\
&\leq \Big|\Big|\Big[\int_t^{t+h} f_1^\epsilon dB_1(s), \int_t^{t+h} f_2^\epsilon dB_2(s), \int_t^{t+h} f_3^\epsilon dB_3(s)\Big]\Big|\Big|_{(H^\epsilon)'}\\
&\leq \sup_{\vec{\psi}\in H^\epsilon, ||\vec{\psi}||_{H^\epsilon}\leq 1} \left|\Big\langle\Big[\int_t^{t+h} f_1^\epsilon dB_1(s), \int_t^{t+h} f_2^\epsilon dB_2(s), \int_t^{t+h} f_3^\epsilon dB_3(s)\Big], [\psi_1,\psi_2,\psi_3]_{(H^\epsilon)',H^\epsilon}\right|\\
&\leq \sup_{\vec{\psi}\in H^\epsilon, ||\vec{\psi}||_{H^\epsilon}\leq 1} \left|\Big\langle \int_t^{t+h} f_1^\epsilon dB_1(s), \psi_1\Big\rangle_{(H^\epsilon)',H^\epsilon} + \Big\langle \int_t^{t+h} f_2^\epsilon dB_2(s), \psi_2\Big\rangle_{(H^\epsilon)',H^\epsilon}\right.\\
&\left.+\Big\langle \int_t^{t+h} f_3^\epsilon dB_3(s), \psi_3\Big\rangle_{(H^\epsilon)',H^\epsilon}\right|\\
&\leq C\Big[\Big|\Big|\int_t^{t+h} f_1^\epsilon dB_1(s)\Big|\Big|_{H_1^\epsilon} + \Big|\Big|\int_t^{t+h} f_2^\epsilon dB_2(s)\Big|\Big|_{H_2^\epsilon} + \Big|\Big|\int_t^{t+h} f_3^\epsilon dB_3(s)\Big|\Big|_{H_2^\epsilon}\Big],
\end{aligned}
$$

where we have used the conditions (3.1) on the functions c_i.

Estimating each term at a time, we have

$$
\Big|\Big|\int_t^{t+h} f_1^\epsilon dB_1(s)\Big|\Big|_{H_1^\epsilon}^{p'} \leq \Big|\Big|\int_t^{t+h} f_1^\epsilon dB_1(s)\Big|\Big|_{L^2(Q_1^\epsilon)}^{p'}.
$$

For $h>0$, using Hölder's inequality and Fubini's theorem we get

$$
\begin{aligned}
&\mathbb{E}\sup_{h\leq\delta}\int_0^{T-h}\Big|\Big|\int_t^{t+h} f_1^\epsilon dB_1(s)\Big|\Big|_{L^2(Q_1^\epsilon)}^{p'} dt \leq \mathbb{E}\sup_{h\leq\delta}\int_0^{T-h}\left(\int_{Q_1^\epsilon}\left(\int_t^{t+h} f_1^\epsilon dB_1(s)\right)^2 dx\right)^{\frac{p'}{2}} dt\\
&\leq C\left(\mathbb{E}\sup_{h\leq\delta}\int_0^{T-h}\int_{Q_1^\epsilon}\left(\int_t^{t+h} f_1^\epsilon dB_1(s)\right)^2 dxdt\right)^{\frac{p'}{2}}\\
&\leq \left(\mathbb{E}\sup_{h\leq\delta}\int_0^{T}\int_{Q_1^\epsilon}\left(\int_t^{t+h} f_1^\epsilon dB_1(s)\right)^2 dxdt\right)^{\frac{p'}{2}}\\
&\leq \left(\int_0^{T}\int_{Q_1^\epsilon}\mathbb{E}\sup_{h\leq\delta}\left(\int_t^{t+h} f_1^\epsilon dB_1(s)\right)^2 dxdt\right)^{\frac{p'}{2}}.
\end{aligned}
$$

Using Burkhölder-Davis-Gundy's inequality, we deduce that

$$
\begin{aligned}
&\mathbb{E}\sup_{h\leq\delta}\int_0^{T-h}\Big|\Big|\int_t^{t+h} f_1^\epsilon dB_1(s)\Big|\Big|_{L^2(Q_1^\epsilon)}^{p'} dt \leq \left(\int_0^T\int_{Q_1^\epsilon}\mathbb{E}\int_t^{t+\delta}|f_1^\epsilon|^2 dsdxdt\right)^{\frac{p'}{2}}\\
&= \left(\mathbb{E}\int_0^T\int_t^{t+\delta}||f_1^\epsilon||^2_{L^2(Q_1^\epsilon)} dsdt\right)^{\frac{p'}{2}}
\end{aligned}
$$

By Hölder's inequality and the assumption on f_1^ϵ i.e. $f_1^\epsilon \in L^4(0,T;L^2(Q_1^\epsilon))$, we get

$$\mathbb{E}\sup_{h\leq\delta}\int_0^{T-h}\bigg|\bigg|\int_t^{t+h} f_1^\epsilon dB_1(s)\bigg|\bigg|^2_{L^2(Q_1^\epsilon)} dt \leq C(T)\bigg(\delta\mathbb{E}\int_0^T\int_t^{t+\delta}||f_1^\epsilon(s)||^4_{L^2(Q_1^\epsilon)}dsdt\bigg)^{\frac{p'}{4}}$$
$$\leq C\delta^{\frac{p'}{4}}.$$

Similarly,

$$\mathbb{E}\sup_{|h|\leq\delta}\int_0^{T-h}\bigg|\bigg|\int_t^{t+h} f_2^\epsilon dB_2(s)\bigg|\bigg|^{p'}_{(\mathcal{H}^\epsilon)'} dt \leq C(T)\delta^{\frac{p'}{4}},$$

and

$$\mathbb{E}\sup_{|h|\leq\delta}\int_0^{T-h}\bigg|\bigg|\int_t^{t+h} f_3^\epsilon dB_3(s)\bigg|\bigg|^{p'}_{(\mathcal{H}^\epsilon)'} dt \leq C(T)\delta^{\frac{p'}{4}}.$$

Lemma 3.3 on (3.4.1) gives

$$\begin{aligned}
&\mathbb{E}\sup_{|h|\leq\delta} h^{\frac{p'}{p}}\int_0^{T-h}\bigg[\bigg(\int_t^{t+h}||\nabla u_1^\epsilon||^p_{L^p(Q_1^\epsilon)}ds\bigg)^{\frac{1}{p'}}\bigg]^{p'}dt\\
&\leq \mathbb{E}\sup_{|h|\leq\delta} h^{\frac{p'}{p}}\int_0^{T-h}\int_t^{t+h}||\nabla u_1^\epsilon||^p_{L^p(Q_1^\epsilon)}dsdt\\
&\leq \delta^{\frac{p'}{p}}C.
\end{aligned}$$

Similarly,

$$\mathbb{E}\sup_{|h|\leq\delta} h^{\frac{p'}{p}}\int_0^{T-h}\bigg[\bigg(\int_t^{t+h}||\nabla u_2^\epsilon||^p_{L^p(Q_2^\epsilon)}ds\bigg)^{\frac{1}{p'}}\bigg]^{p'}dt \leq C\delta^{\frac{p'}{p}},$$

$$\mathbb{E}\sup_{|h|\leq\delta} h^{\frac{p'}{p}}\int_0^{T-h}\bigg[\bigg(\int_t^{t+h}||\epsilon\nabla u_3^\epsilon||^p_{L^p(Q_2^\epsilon)}ds\bigg)^{\frac{1}{p'}}\bigg]^{p'}dt \leq C\delta^{\frac{p'}{p}},$$

and

$$\mathbb{E}\sup_{|h|\leq\delta}\int_0^{T-h}\bigg(3\int_t^{t+h}kds\bigg)^{p'}dt \leq C\delta^{p'}.$$

Hence collecting all the above inequalities, we assert that

$$\mathbb{E}\sup_{|h|\leq\delta}\int_0^{T-h}||\vec{u}^\epsilon(t+h)-\vec{u}^\epsilon(t)||^{p'}_{(\mathcal{H}^\epsilon)'}dt \leq C\,max\{\delta^{\frac{1}{p-1}},\delta^{\frac{p'}{4}}\}.$$

□

One of the difficulties encountered in the homogenization of problems in perforated domain is to establish that the sequence of solutions admits a limit in the whole domain, in our case Q. From the estimates in lemmas 3.3 and 3.4, we cannot extract a convergent subsequence by weak compactness, since each u_i^ϵ $(i = 1, 2, 3)$ is defined on a space which varies in ϵ.

A similar case was studied by Cioranescu and Saint Jean Paulin in [36] and Acerbi et al in [1] where an extension of the solution to the whole domain was constructed, this extension was also proven to converge weakly to the homogenized limit. However, in [5], Allaire and Murat didn't construct an extension, a version of Rellich theorem was used. Tartar's method of oscillating test function was used in the homogenization process in [36] and [5], while Γ-convergence was used in [1]. In [3], Allaire also didn't construct an extension, the solution was extended by zero in the holes and two-scale convergence was used in the homogenization process.

As in [3], we will extend the functions $\vec{u}^\epsilon = [u_1^\epsilon, u_2^\epsilon, u_3^\epsilon]$ by zero to the whole domain Q. The domain Q has two sub-domains Q_1^ϵ representing the fissures and Q_2^ϵ representing the matrix with $Q = Q_1^\epsilon \cup Q_2^\epsilon$, hence we can assert that the flow potential u_1^ϵ defined on Q_1^ϵ equals zero on Q_2^ϵ and the flow potentials $u_2^\epsilon, u_3^\epsilon$ defined on Q_2^ϵ are zero on Q_1^ϵ.

We recall the characteristic function

$$\chi_i^\epsilon = \chi_i\big(\frac{x}{\epsilon}\big), \quad i = 1, 2.$$

We use this function to denote the extension by zero of various functions from Q_i^ϵ to Q and χ_i to denote the extension of functions from Y_i to Y.

Now we state the estimates for the extension of $\vec{u}^\epsilon = [u_1^\epsilon, u_2^\epsilon, u_3^\epsilon]$ to all of Q from Q_i^ϵ, $i = 1, 2$.

$$\mathbb{E}\sup_{0\le t\le T} ||\chi_1^\epsilon u_1^\epsilon||^2_{L^2(Q)} + \mathbb{E}\sup_{0\le t\le T} ||\chi_2^\epsilon u_2^\epsilon||^2_{L^2(Q)} + \mathbb{E}\sup_{0\le t\le T} ||\chi_2^\epsilon u_3^\epsilon||^2_{L^2(Q)} + \mathbb{E}\int_0^T ||\chi_1^\epsilon \nabla u_1^\epsilon||^p_{L^p(Q)} dt$$
$$+ \mathbb{E}\int_0^T ||\chi_2^\epsilon \nabla u_2^\epsilon||^p_{L^p(Q)} dt + \int_0^T ||\epsilon\chi_2^\epsilon \nabla u_3^\epsilon||^p_{L^p(Q)} dt \le C. \tag{3.13}$$

Let us still denote the zero extension of $\vec{u}^\epsilon = [u_1^\epsilon, u_2^\epsilon, u_3^\epsilon]$ by $\vec{u}^\epsilon = [\chi_1^\epsilon u_1^\epsilon, \chi_2^\epsilon u_2^\epsilon, \chi_2^\epsilon u_3^\epsilon]$, we state the estimate of the finite difference in the space $\mathcal{V}$

$$\mathbb{E}\sup_{|h|\le\delta}\int_0^{T-h} ||\vec{u}^\epsilon(t+h) - \vec{u}^\epsilon(t)||^{p'}_{\mathcal{V}} dt \le C\, max\{\delta^{\frac{1}{p-1}}, \delta^{\frac{p'}{4}}\}.$$

where $\mathcal{V}$ is defined as

$$\mathcal{V} =\{\vec{\psi} = [\psi_1, \psi_2, \phi_3] \in L^2(Q) \times L^2(Q) \times L^2(Q) \ \cap \ (W_0^{1,p}(Q) \times W_0^{1,p}(Q) \times W_0^{1,p}(Q)) \\ : \beta\phi_3 = \psi_1 - \alpha\psi_2, \forall y \in \Gamma_{1,2}\}.$$

3.5 Compactness result and tightness property

This section contains some results that are essential in the proof of the tightness property of the probability measures generated by the sequence $(B, \vec{u}^\epsilon)$, where $B = [B_1, B_2, B_3]$ and $\vec{u}^\epsilon = [u_1^\epsilon, u_2^\epsilon, u_3^\epsilon]$.

Let the space H be defined as

$$H = L^2(Q) \times L^2(Q) \times L^2(Q).$$

Let us consider the set Z depending on the sequences $\rho_n, \tau_n \geq 0$ of numbers such that $\rho_n, \tau_n \to 0$ as $n \to \infty$ and on the constants J, K, L, M, N, R. We define the set Z by

$$\begin{aligned} Z = \{\vec{\psi} = [\psi_1, \psi_2, \psi_3] : \sup_{0\leq t\leq T} ||\psi_1||_{L^2(Q)} \leq J; ||\psi_2||_{L^2(Q)} \leq K; ||\psi_3||_{L^2(Q)} \leq L; \\ \sum_{i=1}^{2} \int_0^T ||\nabla\psi_i||^p_{L^p(Q)} dt \leq M; \int_0^T ||\nabla\psi_3||^p_{L^p(Q)} dt \leq N \\ \text{and} \sup_{|h|\leq\rho_n} \int_0^T ||\vec{\psi}(t+h) - \vec{\psi}(t)||^{p'}_{(\mathcal{H})'} dt \leq \tau_n R, \forall n\}. \end{aligned}$$

Lemma 3.5. *The set Z is a compact subset of $L^2(0, T; L^2(Q)) \times L^2(0, T; L^2(Q)) \times L^2(0, T; L^2(Q))$.*

The proof of this lemma is similar to the proof found in [21] (Proposition 3.1).

Let $\vec{u}^\epsilon = [\chi_1^\epsilon u_1^\epsilon, \chi_2^\epsilon u_2^\epsilon, \chi_2^\epsilon u_3^\epsilon]$ and $\mathcal{S} = C(0, T; \mathbb{R}^n) \times L^2(0, T; L^2(Q)) \times L^2(0, T; L^2(Q)) \times L^2(0, T; L^2(Q))$ be equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{S})$. Let ϕ_ϵ be the $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$-valued measurable map defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\phi_\epsilon : \omega \mapsto (B(\omega), \vec{u}^\epsilon), \text{ with } B = (B_1, B_2, B_3).$$

We introduce the probability measures π^ϵ on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ defined by

$$\pi^\epsilon(S) = \mathbb{P}(\phi_\epsilon^{-1}(S)), \quad \text{for all } S \in \mathcal{B}(\mathcal{S}).$$

Lemma 3.6. *The family of probability measures $\{\pi^\epsilon : \epsilon > 0\}$ is tight in $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$.*

The proof is carried out following [21], see also [85], [116] and [117].

By Prokhorov's result (Lemma 1.23), there exists a subsequence $\{\pi_{\epsilon_j}\}$ of $\{\pi_\epsilon\}$ and a probability measure π such that

$$\pi_{\epsilon_j} \rightharpoonup \pi \ \text{ weakly in } \mathcal{S}.$$

Using Lemma (1.24) due to Skorokhod, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\mathcal{S}$-valued random variables $(B^{\epsilon_j}, \vec{u}^{\epsilon_j})$ and $(\tilde{B}, \vec{u})$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that the probability law of $(B^{\epsilon_j}, \vec{u}^{\epsilon_j})$ is π_{ϵ_j} and that of $(\tilde{B}, \vec{u})$ is π. Furthermore,

$$(B^{\epsilon_j}, \vec{u}^{\epsilon_j}) \longrightarrow (\tilde{B}, \vec{u}) \quad \text{in } \mathcal{S} \ \ \tilde{\mathbb{P}}\text{-a.s.}, \tag{3.14}$$

where $B^{\epsilon_j} = (B_1^{\epsilon_j}, B_2^{\epsilon_j}, B_3^{\epsilon_j})$, $\vec{u}^{\epsilon_j} = (u_1^{\epsilon_j}, u_2^{\epsilon_j}, u_3^{\epsilon_j})$, $\vec{u} = [u_1, u_2, u_3]$, and $\tilde{B} = (\tilde{B}_1, \tilde{B}_2, \tilde{B}_3)$. B^{ϵ_j} and $\tilde{B}$ are Wiener processes on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and the pair $(B^{\epsilon_j}, \vec{u}^{\epsilon_j})$ satisfies problem (P^{ϵ_j}) as stipulated in the following;

Theorem 3.7. *For any $\vec{\psi} = [\psi_1, \psi_2, \psi_3] \in C^\infty(Q) \times C^\infty(Q) \times C^\infty(Q)$ and $t \in [0, T]$. The sequence $(B^{\epsilon_j}, \vec{u}^{\epsilon_j})$ satisfies $\tilde{\mathbb{P}}$-a.s., the relations*

$$\begin{aligned}
(\vec{u}^{\epsilon_j}(t, \cdot), \vec{\psi})_H = (\vec{u}^{\epsilon_j}(0, \cdot), \vec{\psi})_H &+ \sum_{i=1}^{2} \int_0^t (\chi_i^{\epsilon_j} \mu_i^{\epsilon_j}(x, \nabla u_i^{\epsilon_j}), \nabla \psi_i)_{L^2(Q)} ds \\
&+ \int_0^t (\chi_2^{\epsilon_j} \mu_3^{\epsilon_j}(x, \epsilon_j \nabla u_3^{\epsilon_j}), \epsilon_j \nabla \psi_3)_{L^2(Q)} ds \\
&+ \sum_{i=1}^{2} \int_0^t (\chi_i^{\epsilon_j} f_i^{\epsilon_j}, \psi_i)_{L^2(Q)} dB^{\epsilon_j}(s) \\
&+ \int_0^t (\chi_2^{\epsilon_j} f_3^{\epsilon_j}, \psi_3)_{L^2(Q)} dB_3^{\epsilon_j}(s),
\end{aligned} \tag{3.15}$$

with $(\vec{u}^{\epsilon_j}(0, \cdot), \vec{\psi}) = (\vec{u}^0(\cdot), \vec{\psi})$, in the sense of distributions.

Since $(\vec{u}^{\epsilon_j})$ satisfies the same type of problem as (P^ϵ), we have the following corresponding estimates for $(\vec{u}^{\epsilon_j})$;

$$\begin{aligned}&\tilde{\mathbb{E}}\sup_{0\le t\le T}||\chi_1^{\epsilon_j}u_1^{\epsilon_j}||^2_{L^2(Q)}+\tilde{\mathbb{E}}\sup_{0\le t\le T}||\chi_2^{\epsilon_j}u_2^{\epsilon_j}||^2_{L^2(Q)}+\tilde{\mathbb{E}}\sup_{0\le t\le T}||\chi_2^{\epsilon_j}u_3^{\epsilon_j}||^2_{L^2(Q)}\\&+\tilde{\mathbb{E}}\int_0^T||\chi_1^{\epsilon_j}\nabla u_1^{\epsilon_j}||^p_{L^p(Q)}dt+\tilde{\mathbb{E}}\int_0^T||\chi_2^{\epsilon_j}\nabla u_2^{\epsilon_j}||^p_{L^p(Q)}dt\\&+\tilde{\mathbb{E}}\int_0^T||\epsilon_j\chi_2^{\epsilon_j}\nabla u_3^{\epsilon_j}||^p_{L^p(Q)}dt\le C,\end{aligned}\tag{3.16}$$

and on the dual $\mathcal{V}'$ we have,

$$\tilde{\mathbb{E}}\sup_{|h|\le\delta}\int_0^{T-h}||\vec{u}^{\epsilon_j}(t+h)-\vec{u}^{\epsilon_j}(t)||^{p'}_{\mathcal{V}'}dt\le C\,max\{\delta^{\frac{1}{p-1}},\delta^{\frac{p'}{4}}\},\quad \delta\le 1,$$

where $\vec{u}^{\epsilon_j}=(u_1^{\epsilon_j},u_2^{\epsilon_j},u_3^{\epsilon_j})$.

3.6 Homogenization process

In this section, we derive the homogenized problem using two-scale convergence. We start by introducing the definition of two-scale convergence and some theorems that will be useful in the convergence process. For their proofs, we refer to [3], [32].

3.6.1 Definition and some results on two-scale convergence

Definition 3.8. let $\{\varphi^\epsilon\}$ be a sequence of functions in $L^p(0,T;L^p(Q))$ $(1<p<\infty)$. $\{\varphi^\epsilon\}$ is said to be two-scale convergent to $\varphi_0=\varphi_0(t,x,y)$ with $\varphi_0\in L^p(0,T;L^p(Q\times Y))$ if for any function $v=v(t,x,y)\in L^p((0,T)\times Q;C^\infty_{per}(Y))$, one has

$$\lim_{\epsilon\to 0}\int_0^T\int_Q\varphi^\epsilon(t,x)v(t,x,\frac{x}{\epsilon})dxdt=\int_0^T\int_Q\int_Y\varphi_0(t,x,y)v(t,x,y)dydxdt,$$

we denote this by $\varphi^\epsilon\xrightarrow{2-s}\varphi_0$ in $L^p(0,T;L^p(Q))$.

Theorem 3.9. *Let $\{\varphi^\epsilon\}$ be a bounded sequence of functions in $L^p(0,T;L^p(Q))$ with $1<p\le\infty$. Then there exists subsequence $\{\varphi^{\epsilon'}\}$ and a function $\varphi\in L^p(0,T;L^p(Q\times Y))$ such that $\{\varphi^{\epsilon'}\}$ is two-scale convergent to φ.*

Theorem 3.10. *Let $\{\varphi^\epsilon\}$ be a sequence satisfying the assumptions of Theorem 3.9. Furthermore, let $\{\varphi^\epsilon\}$ be bounded in $L^p(0,T;W_0^{1,p}(Q))$. Then*

1. *there exists a subsequence $\{\varphi^{\epsilon'}\}$ and a couple of functions (φ, φ_1) with $\varphi \in L^p(0,T;W_0^{1,p}(Q))$ and $\varphi_1 \in L^p((0,T)\times Q; W^{1,p}_{per}(Y))$ such that up to a subsequence, $\nabla\varphi^\epsilon \xrightarrow{2-s} \nabla_x\varphi(x) + \nabla_y\varphi_1(x,y)$.*

2. *there exists a function $\varphi_0 \in L^2((0,T)\times Q; W^{1,p}_{per}(Y))$ such that up to a subsequence, $\varphi^\epsilon \xrightarrow{2-s} \varphi_0(x,y)$ and $\epsilon\nabla\varphi^\epsilon \xrightarrow{2-s} \nabla_y\varphi_0(x,y)$.*

Proposition 3.11. *Let φ^ϵ be a sequence of functions in $L^p(Q)$ such that φ^ϵ two-scale converges to $\varphi_0(x,y)$ in $L^p(Q\times Y)$. Then φ^ϵ converges weakly to $\varphi(x)$ in $L^p(Q)$, where*

$$\varphi(x) = \int_Y \varphi_0(x,y)dy, \qquad \text{in } L^p(Q).$$

Furthermore, we have

$$\lim_{\epsilon\to 0} ||\varphi^\epsilon||_{L^p(Q)} \le ||\varphi_0||_{L^p(Q\times Y)} \le ||\varphi||_{L^p(Q)}.$$

3.6.2 Passage to the limit

Now we study the asymptotic behaviour of (P^{ϵ_j}) as $\epsilon_j \to 0$ using the two-scale convergence method.

Lemma 3.12. *For the sequences (u_i^ϵ), there exists subsequences $(u_i^{\epsilon_j})$ such that, we have the following two-scale convergences, $\tilde{\mathbb{P}}$-almost surely:*

$$\chi_i^{\epsilon_j} u_i^{\epsilon_j} \xrightarrow{2-s} \chi_i(y)u_i(t,x),\ (i=1,2), \quad \chi_2^{\epsilon_j} u_3^{\epsilon_j} \xrightarrow{2-s} \chi_2(y)U_3(t,x,y),$$

$$\chi_i^{\epsilon_j}\nabla u_i^{\epsilon_j} \xrightarrow{2-s} \chi_i(y)[\nabla u_i(t,x) + \nabla_y U_i(t,x,y)],\ (i=1,2), \quad \chi_2^{\epsilon_j}\epsilon\nabla u_3^{\epsilon_j} \xrightarrow{2-s} \chi_2(y)\nabla_y U_3(t,x,y),$$

where $U_i \in L^p((0,T)\times Q; W^{1,p}_{per}(Y))$, $i=1,2,3$.

Furthermore, there exists some functions $\vec{g}_i \in L^{p'}((0,T)\times Q\times Y^n), u^* \in L^2(Q\times Y)$, for $i=1,2,3$ such that the following convergences hold $\tilde{\mathbb{P}}$ almost surely;

$$\chi_i^{\epsilon_j}\mu_i^{\epsilon_j}(x,\nabla u_i^{\epsilon_j}) \xrightarrow{2-s} \chi_i(y)\vec{g}_i(t,x,y),\ (i=1,2), \quad \chi_2^{\epsilon_j}\mu_3^{\epsilon_j}(x,\epsilon_j\nabla u_3^{\epsilon_j}) \xrightarrow{2-s} \chi_2(y)\vec{g}_3(t,x,y),$$

$$\chi_i^{\epsilon_j}u_i^{\epsilon_j}(T,\cdot) \xrightarrow{2-s} \chi_i(y)u_i^*(x),\ (i=1,2), \quad \chi_2^{\epsilon_j}u_3^{\epsilon_j}(T,\cdot) \xrightarrow{2-s} \chi_2(y)u_3^*(x),$$

and

$$\chi_i^{\epsilon_j} f_i^{\epsilon_j}(t,x) \xrightarrow{2-s} f_i(t,x,y) \in L^2((0,T)\times Q\times Y), \quad i=1,2,3.$$

Proof. According to lemma 3.3, the sequences $\vec{u}^\epsilon = [u_1^\epsilon, u_2^\epsilon, u_3^\epsilon]$ and $\nabla\vec{u}^\epsilon = [\nabla u_1^\epsilon, \nabla u_2^\epsilon, \nabla u_3^\epsilon]$ are bounded in $L^2(Q_i^\epsilon)$ and $L^p(Q_i^\epsilon)$ respectively. Since by definition, $\chi_1^\epsilon u_1^\epsilon$ is zero in $Q\backslash Q_1^\epsilon$, and $\chi_2^\epsilon u_2^\epsilon, \chi_2 u_3^\epsilon$ are both zero in $Q\backslash Q_2^\epsilon$, we have the estimates (3.13) and (3.16) for the subsequences $\chi_1^{\epsilon_j} u_1^{\epsilon_j}, \chi_2^{\epsilon_j} u_2^{\epsilon_j}, \chi_2^{\epsilon_j} u_3^{\epsilon_j}$.
By Theorem 3.10(1) and Theorem 2.9 in [3], we have the following two-scale convergences, $\tilde{\mathbb{P}}$-a.s.,

$$\chi_i^{\epsilon_j} u_i^{\epsilon_j} \xrightarrow{2-s} \chi_i(y) u_i(t,x), \quad i=1,2,$$

$$\chi_i^{\epsilon_j} \nabla u_i^{\epsilon_j} \xrightarrow{2-s} \chi_i(y)[\nabla u_i(t,x) + \nabla_y U_i(t,x,y)], \quad i=1,2,$$

where $u_i \in L^p(0,T;W_0^{1,p}(Q))$ and $U_i \in L^p((0,T)\times Q; W_{per}^{1,p}(Y)\backslash\mathbb{R})$, for $i=1,2$, $\tilde{\mathbb{P}}$-a.s.. Using the same argument and Theorem 3.9, $\chi_i^{\epsilon_j} f_i^{\epsilon_j} (i=1,2)$ and $\chi_2^{\epsilon_j} f_3^{\epsilon_j}$ are bounded in $L^2((0,T)\times Q)$. Hence, the subsequences two-scale converge to $\chi_i(y) f_i(t,x,y)(i=1,2)$ and $\chi_2(y) f_3(t,x,y)$ respectively in $L^2((0,T)\times Q\times Y)$, $\tilde{\mathbb{P}}$-a.s..
Theorem 3.10(2) gives

$$\chi_2^{\epsilon_j} u_3^{\epsilon_j} \xrightarrow{2-s} \chi_2(y) U_3(t,x,y)\ \tilde{\mathbb{P}}\text{-a.s. and } \epsilon_j \chi_2^{\epsilon_j} \nabla u_3^{\epsilon_j} \xrightarrow{2-s} \nabla_y U_3(t,x,y)\ \tilde{\mathbb{P}}\text{-a.s..}$$

Lastly, from $A(3.1)$, we have

$$\begin{aligned} \int_0^T \int_Q \left|\chi_i^{\epsilon_j} \mu_i(\frac{x}{\epsilon_j}, \nabla u_i^{\epsilon_j})\right|^{p'} dxdt &\le \int_0^T \int_Q \chi_i^{\epsilon_j} \left|\mu_i(\frac{x}{\epsilon_j}, \nabla u_i^{\epsilon_j})\right|^{p'} dxdt \\ &\le C \int_0^T \int_Q \chi_i^{\epsilon_j} |\nabla u_i^{\epsilon_j}|^{p-1(p')} dxdt + \int_0^T \int_Q \chi_i^{\epsilon_j} k^{p'} dxdt \\ &\le C \int_0^T \int_Q \chi_i^{\epsilon_j} |\nabla u_i^{\epsilon_j}|^p dxdt \ (i=1,2) \end{aligned}$$

Hence, by Theorem 3.9, $\chi_i^{\epsilon_j} \mu_i^{\epsilon_j}\big(x, \nabla u_i^{\epsilon_j}\big)$ $(i=1,2)$ is bounded in $L^{p'}((0,T);L^{p'}(Q))$, and similarly, $\chi_2^{\epsilon_j} \mu_3^{\epsilon_j}\big(x, \nabla u_3^{\epsilon_j}\big)$ is bounded in $L^{p'}((0,T);L^{p'}(Q))$. Consequently, the sequences two-scale converge to $\vec{g}_i(t,x,y)$ $(i=1,2)$ and $\vec{g}_3(t,x,y)$, $\tilde{\mathbb{P}}$-a.s., respectively with $\vec{g}_i(t,x,y) \in L^p((0,T)\times Q\times Y^n)$. □

Before we proceed with the homogenization process, we establish the conditions on the interface $\Gamma_{1,2}$.

Let $u^{\epsilon_j} = \chi_1^{\epsilon_j} u_1^{\epsilon_j} + \chi_2^{\epsilon_j}(\alpha u_2^{\epsilon_j} + \beta u_3^{\epsilon_j}) \in L^p(0,T;W^{1,p}(Q))$. Owing to the transmission conditions on $\Gamma_{1,2}^{\epsilon}$ in (P^ϵ), we have

$$\gamma_1^{\epsilon_j} u^{\epsilon_j} = \gamma_1^{\epsilon_j} u_1^{\epsilon_j} = \alpha\gamma_2^{\epsilon_j} u_2^{\epsilon_j} + \beta\gamma_2^{\epsilon_j} u_3^{\epsilon_j} = \gamma_2^{\epsilon_j} u^{\epsilon_j}, \text{ on } \Gamma_{1,2}^{\epsilon_j}.$$

Thus

$$\epsilon_j \nabla u^{\epsilon_j} = \epsilon_j \chi_1^{\epsilon_j} \nabla u_1^{\epsilon_j} + \chi_2^{\epsilon_j}(\alpha\epsilon_j \nabla u_2^{\epsilon_j} + \beta\epsilon_j \nabla u_3^{\epsilon_j}) \in L^p((0,T)\times Q), \ \ \mathbb{P}\text{-a.s.}.$$

Hence, according to Lemma 3.12,

$$u^{\epsilon_j} \xrightarrow{2-s} \chi_1(y)u_1(t,x) + \chi_2(y)\big(\alpha u_2(t,x) + \beta U_3(t,x,y)\big),$$

and

$$\epsilon_j \nabla u^{\epsilon_j} \xrightarrow{2-s} \chi_2(y)\beta\nabla_y U_3(t,x,y).$$

Let $\vec{\phi} \in \mathcal{D}(Q, C_{per}^{\infty}(Y^3))$, we have

$$\begin{aligned}\int_Q \epsilon_j \nabla u^{\epsilon_j}(t,x)\vec{\phi}\big(x,\frac{x}{\epsilon_j}\big)dx &= -\int_Q u^{\epsilon_j}(t,x)\epsilon_j\nabla\big(\vec{\phi}(x,\frac{x}{\epsilon_j})\big)dx \\ &= -\int_Q u^{\epsilon_j}(t,x)\big[\epsilon_j\nabla\vec{\phi}(x,\frac{x}{\epsilon_j}) + \nabla_y\vec{\phi}(x,\frac{x}{\epsilon_j})\big]dx.\end{aligned}$$

Taking the two-scale limits on both sides give

$$\begin{aligned}&\int_Q\int_Y \beta\chi_2(y)\nabla_y U_3(t,x,y)\vec{\phi}(x,y)dydx \\ &= -\int_Q\int_Y \big[\chi_1(y)u_1(t,x) + \chi_2(y)\big(\alpha u_2(t,x) + \beta U_3(t,x,y)\big)\big]\nabla_y\vec{\phi}(x,y)dydx.\end{aligned} \tag{3.17}$$

The left hand side of (3.17) can be written as

$$\begin{aligned}\int_Q\int_{Y_2} \beta\nabla_y U_3(t,x,y)\vec{\phi}(x,y)dydx = &-\int_Q\int_{Y_2} \beta U_3\nabla_y\vec{\phi}(x,y)dydx \\ &+\int_Q\int_{\partial Y_2} \beta U_3(t,x,y)\vec{\phi}(x,y)\cdot\vec{\nu}_2 dS_y dx\end{aligned} \tag{3.18}$$

while the right hand side of (3.17) can be written as

$$-\int_Q\int_{Y_1} u_1(t,x)\nabla_y\vec{\phi}(x,y)dydx - \int_Q\int_{Y_2}\big(\alpha u_2(t,x) + \beta U_3(t,x,y)\big)\nabla_y\vec{\phi}dydx. \tag{3.19}$$

From (3.17), (3.18), (3.19) and $u_1(t,x)$, $u_2(t,x)$ being independent of y we see that

$$\begin{aligned}
&\int_Q \int_{\partial Y_2} \beta U_3(t,x,y)\phi(x,y)\cdot \vec{\nu}_2 dS_y dx \\
&\quad = -\int_Q \int_{Y_1} u_1(t,x)\nabla_y \vec{\phi}(x,y) dy dx - \int_Q \int_{Y_2} \alpha u_2(t,x) \nabla_y \vec{\phi} dy dx \\
&\quad = -\int_Q \int_{\partial Y_1} u_1(t,x)\vec{\phi}(x,s)\cdot \nu_1 dS_y dx - \int_Q \int_{\partial Y_2} \alpha u_2(t,x)\vec{\phi}(x,s)\cdot \nu_2 dS_y dx.
\end{aligned}$$

Since U_3 and $\vec{\psi}$ are periodic on $\Gamma_{2,2}$ and $\nu_1 = -\nu_2$ this implies that

$$\beta U_3 + \alpha u_2 = u_1 \text{ on } \partial Y_1 \cap \partial Y_2 \equiv \Gamma_{1,2}.$$

Now we state our main result.

Theorem 3.13. *Suppose the assumptions $A(1) - A(4)$ are satisfied. Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\tilde{\mathcal{F}})_{0\le t\le T})$ and random variables $(B^{\epsilon_j}, \vec{u}^{\epsilon_j})$ and $(\tilde{B}, \vec{u})$ such that*

$$(B^{\epsilon_j}, \vec{u}^{\epsilon_j}) \to (\tilde{B}, \vec{u}) \text{ in } \mathcal{S} \quad \tilde{\mathbb{P}}\text{-a.s.},$$

where $\vec{u}^{\epsilon_j} = [u_1^{\epsilon_j}, u_2^{\epsilon_j}, u_3^{\epsilon_j}]$ and $\vec{u} = [u_1, u_2, U_3]$, and $(\tilde{B}, \vec{u})$ satisfy the homogenized problems (3.4), (3.5) and (3.6); recall that $\mathcal{S} = C(0,T;\mathbb{R}^n) \times L^2(0,T;L^2(Q)) \times L^2(0,T;L^2(Q)) \times L^2(0,T;L^2(Q))$. Furthermore, $U_3 = U_3(t,x,y) \in L^p((0,T)\times Q; W^{1,p}_{per}(Y))$.

Remark. The convergence of $[u_1^{\epsilon_j}, u_2^{\epsilon_j}, u_3^{\epsilon_j}]$ to $[u1, u_2, U_3]$ was proved in Lemma 3.12.

Proof. (of Theorem 3.13) Let $\psi_i \in \mathcal{D}(0,T;C_0^\infty(Q))$, $\ i=1,2$, and $\phi_i \in \mathcal{D}((0,T)\times Q; C^\infty_{per}(Y))$, $\ i = 1,2,3$, with $\beta\phi_3^{\epsilon_j}(t,x,y) = \psi_1(t,x) + \alpha\psi_2(t,x)$ for $y \in \Gamma_{1,2}$.
We take the triple

$$[\psi_1(t,x) + \epsilon_j \phi_1(t,x,\frac{x}{\epsilon_j}), \psi_2(t,x) + \epsilon_j\phi_2(t,x,\frac{x}{\epsilon_j}), \phi_3^{\epsilon_j}(t,x,\frac{x}{\epsilon_j})]$$

in $L^2(0,T;\mathcal{H}^\epsilon)$ as a test function, where we define

$$\phi_3^{\epsilon_j}(t,x,y) = \phi_3(t,x,y) + \frac{\epsilon_j}{\beta}\phi_1(t,x,y) - \frac{\epsilon_j \alpha}{\beta}\phi_2(t,x,y).$$

Substituting these test functions in the weak formulation (3.15) we get of problem (P^ϵ).

$$
\begin{aligned}
&-\sum_{i=1}^{2}\int_0^T\int_{Q_i^\epsilon} c_i^{\epsilon_j}u_i^{\epsilon_j}(t)\big[\psi_{it}(t,x)+\epsilon_j\phi_{it}(t,x,\frac{x}{\epsilon_j})\big]dxdt\\
&-\int_0^T\int_{Q_2^\epsilon} c_3^{\epsilon_j}u_3^{\epsilon_j}(t)\phi_{3t}^{\epsilon_j}(t,x,\frac{x}{\epsilon_j})dxdt\\
&+\int_{Q_i^\epsilon} c_i^{\epsilon_j}u_i^{\epsilon_j}(T,x)\big[\psi_i(T,x)+\epsilon_j\phi_i(T,x,\frac{x}{\epsilon_j})\big]dx\\
&+\int_{Q_2^\epsilon} c_3^{\epsilon_j}u_3^{\epsilon_j}(T,x)\phi_3^{\epsilon_j}(T,x,\frac{x}{\epsilon_j})dx\\
&-\sum_{i=1}^{2}\int_{Q_i^\epsilon} c_i^{\epsilon_j}u_i^{\epsilon_j}(0,x)\big[\psi_i(0,x)+\epsilon_j\phi_i(0,x,\frac{x}{\epsilon_j})\big]dx-\int_{Q_2^\epsilon} c_3^{\epsilon_j}u_3^{\epsilon_j}(0,x)\phi_3^{\epsilon_j}(0,x,\frac{x}{\epsilon_j})dx\\
&=-\sum_{i=1}^{2}\int_0^T\int_{Q_i^\epsilon} \mu_i^{\epsilon_j}(x,\nabla u_i^{\epsilon_j})\nabla\big[\psi_i(t,x)+\epsilon_j\phi_i(t,x,\frac{x}{\epsilon_j})\big]dxdt\\
&-\int_0^T\int_{Q_2^\epsilon} \mu_3^{\epsilon_j}(x,\epsilon_j\nabla u_3^{\epsilon_j})\epsilon_j\big[\nabla\phi_3^{\epsilon_j}(t,x,\frac{x}{\epsilon_j})+\frac{1}{\epsilon_j}\nabla_y\phi_3^{\epsilon_j}(t,x,\frac{x}{\epsilon_j})\big]dxdt\\
&+\sum_{i=1}^{2}\int_0^T\int_{Q_i^\epsilon} f_i^{\epsilon_j}\big[\psi_i(t,x)+\epsilon_j\phi_i(t,x,\frac{x}{\epsilon_j})\big]dxdB^{\epsilon_j}(t)\\
&+\int_0^T\int_{Q_2^\epsilon} f_3^{\epsilon_j}\phi_3^{\epsilon_j}(t,x,\frac{x}{\epsilon_j})dxdB_3^{\epsilon_j}(t).
\end{aligned}
\tag{3.20}
$$

Let us determine the limit of each term in this relation using the two-scale result in Lemma 3.12.

For the first term on the left hand side, we have

$$
\begin{aligned}
&\lim_{\epsilon_j\to 0}\sum_{i=1}^{2}\int_0^T\int_{Q_i^\epsilon} c_i^{\epsilon_j}u_i^{\epsilon_j}(t)\big[\psi_{it}(t,x)+\epsilon_j\phi_{it}(t,x,\frac{x}{\epsilon_j})\big]dxdt\\
&=\lim_{\epsilon_j\to 0}\sum_{i=1}^{2}\int_0^T\int_Q c_i^{\epsilon_j}\chi_i^{\epsilon_j}u_i^{\epsilon_j}(t,x)\psi_{it}(t,x)dxdt\\
&+\lim_{\epsilon_j\to 0}\epsilon_j\sum_{i=1}^{2}\int_0^T\int_Q c_i^{\epsilon_j}\chi_i^{\epsilon_j}u_i^{\epsilon_j}(t,x)\phi_{it}(t,x,\frac{x}{\epsilon_j})dxdt\\
&=\sum_{i=1}^{2}\int_0^T\int_Q\int_{Y_i} c_i(y)u_i(t,x)\psi_{it}(t,x)dxdt,\quad \tilde{\mathbb{P}}\text{-a.s..}
\end{aligned}
$$

The second term yields

$$\lim_{\epsilon_j\to 0}\int_0^T\int_{Q_2^\epsilon} c_3^{\epsilon_j}u_3^{\epsilon_j}(t,x)\phi_{3t}^{\epsilon_j}(t,x,\frac{x}{\epsilon_j})dxdt = \lim_{\epsilon_j\to 0}\int_0^T\int_Q c_3^{\epsilon_j}\chi_2^{\epsilon_j}u_3^{\epsilon_j}(t,x)\phi_{3t}^{\epsilon_j}(t,x,\frac{x}{\epsilon_j})dxdt$$
$$= \int_0^T\int_Q\int_{Y_2} c_3(y)U_3(t,x,y)\phi_{3t}(t,x,y)dxdt, \quad \tilde{\mathbb{P}}\text{-a.s..}$$

Similarly, using the definitions of $\chi_i^{\epsilon_j}$ and taking the limit at $\epsilon_j \to 0$ on the remaining terms on the left hand side of (3.20) give, $\tilde{\mathbb{P}}$-a.s.,

$$\sum_{i=1}^2\int_Q\int_{Y_i} c(y)u_i^*(x)\psi_i(T,x)dydx + \int_Q\int_{Y_2} c_3(y)u_3^*(x)\phi_3(T,x,y)dydx$$
$$-\sum_{i=1}^2\int_Q\int_{Y_i} c_i(y)u_i^0(x)\psi_i(0,x)dydx - \int_Q\int_{Y_2} c_3(y)u_3^0(x)\phi_3(0,x,y)dydx.$$

Taking the limit as $\epsilon_j \to 0$ in the first and second terms on the right hand side of (3.20), we obtain

$$-\sum_{i=1}^2\int_0^T\int_Q\int_{Y_i}\vec{g}_i(t,x,y)\big[\nabla\psi_i(t,x)+\nabla_y\phi_i(t,x,y)\big]dydxdt$$
$$-\int_0^T\int_Q\int_{Y_2}\vec{g}_3(t,x,y)\nabla_y\phi_3(t,x,y)dydxdt, \quad \tilde{\mathbb{P}}\text{-a.s..}$$

Lastly, we deal with the limits of the last two terms on the right hand side of (3.20); which are stochastic integrals.

For the integral involving f_1^ϵ, we have

$$\begin{aligned}&\int_0^T\int_Q \chi_1^{\epsilon_j}f_1^{\epsilon_j}(t,x)\big[\psi_1(t,x)+\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})\big]dxdB_1^{\epsilon_j}(t)\\ &=\int_0^T\int_Q \chi_1^{\epsilon_j}f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxdB_1^{\epsilon_j}(t)\\ &+\int_0^T\int_Q \chi_1^{\epsilon_j}f_1^{\epsilon_j}(t,x)\epsilon_j\phi_1(t,x,\frac{x}{\epsilon_j})dxdB_1^{\epsilon_j}(t).\end{aligned} \tag{3.21}$$

We start with the first term on the right hand side of (3.21). Since $B_1^{\epsilon_j}(t)$ has unbounded variations, some care is needed. We first split the integral as

$$\begin{aligned}&\int_0^T\int_Q \chi_1^{\epsilon_j}f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxdB_1^{\epsilon_j}(t)\\ &=\int_0^T\int_Q \chi_1^{\epsilon_j}f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxd\big(B_1^{\epsilon_j}(t)-\tilde{B}_1(t)\big)\\ &+\int_0^T\int_Q \chi_1^{\epsilon_j}f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxd\tilde{B}_1(t).\end{aligned} \tag{3.22}$$

For the first term on the right hand side of (3.22), we adopt the process of regularization for $\chi_1^{\epsilon_j} f^{\epsilon_j}(t,x)$ with respect to t in the following form

$$\chi_1^{\epsilon_j} f_{1,\lambda}^{\epsilon_j}(t) = \frac{1}{\lambda}\int_0^T \rho\big(-\frac{t-s}{\lambda}\big)\chi_1^{\epsilon_j} f_1^{\epsilon_j}(s,x)ds, \quad \text{for } \lambda > 0,$$

where ρ is a standard mollifier.

Now we have that $\chi_1^{\epsilon_j} f_{1,\lambda}^{\epsilon_j}$ is differentiable with respect to t and satisfies the following relation

$$\int_0^T ||\chi_1^{\epsilon_j} f_{1,\lambda}^{\epsilon_j}(t)||_{L^2(Q)}^2 dt \leq \int_0^T ||\chi_1^{\epsilon_j} f_1^{\epsilon_j}||_{L^2(Q)}^2 dt, \quad \forall\lambda > 0 \text{ and } \forall\epsilon_j > 0,$$

and

$$\chi_1^{\epsilon_j} f_{1,\lambda}^{\epsilon_j}(t,x) \to \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x) \;\text{ strongly in } L^2((0,T)\times Q) \quad \text{as } \lambda \to 0.$$

We write the first term on the right hand side of (3.22) as

$$\begin{aligned}
&\int_0^T \int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxd\big(B_1^{\epsilon_j}(t) - \tilde{B}_1(t)\big)\\
&= \int_0^T \int_Q \chi_1^{\epsilon} \epsilon_j f_{1,\lambda}^{\epsilon_j}(t,x)\psi_1(t,x)dxd\big(B_1^{\epsilon_j}(t) - \tilde{B}_1(t)\big)\\
&+ \int_0^T \int_Q [\chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x) - \chi_1^{\epsilon_j} f_{1,\lambda}^{\epsilon_j}]\psi_1(t,x)dxd\big(B_1^{\epsilon_j}(t) - \tilde{B}_1(t)\big).
\end{aligned} \tag{3.23}$$

Since $\chi_1^{\epsilon_j} f_{1,\lambda}^{\epsilon_j}$ is differentiable, we integrate by parts on the first term in the right hand side of (3.23) to get

$$\begin{aligned}
&\int_0^T \int_Q \chi_1^{\epsilon_j} f_{1,\lambda}^{\epsilon_j}(t,x)\psi_1(t,x)dxd\big(B_1^{\epsilon_j}(t) - \tilde{B}_1(t)\big)\\
&= \int_Q \big(B^{\epsilon_j}(t) - \tilde{B}_1(t)\big)\chi_1^{\epsilon_j} f_{1,\lambda}^{\epsilon_j}\psi_1(t,x)ds\Big|_0^T\\
&- \int_0^T \int_Q \big(B_1^{\epsilon_j}(t) - \tilde{B}_1(t)\big)\partial_t\big(\chi_1^{\epsilon_j} f_{1,\lambda}^{\epsilon_j}(t,x)\psi_1(t,x)\big)dxdt.
\end{aligned} \tag{3.24}$$

The condition on $\chi_1^{\epsilon_j} f_1^{\epsilon_j}$ and ψ_1 together with the convergence of $B_1^{\epsilon_j}$ to $\tilde{B}_1(t)$ in $C([0,T])$ $\tilde{\mathbb{P}}$-a.s., give that the right hand side of (3.24) is bounded by a positive number $\kappa_1(\lambda)\eta_1(\epsilon_j)$, where $\eta_1(\epsilon_j)$ vanishes as ϵ tends to zero, while $\kappa_1(\lambda)$ is finite. Thanks to Burhölder-Davis Gundy inequality and the convergence of $\chi_1^{\epsilon_j} f_{1,\lambda}^{\epsilon_j}$ to $\chi_1^{\epsilon_j} f_1^{\epsilon_j}$, the second term on the right hand side of (3.23) is estimated as

$$\tilde{\mathbb{E}}\Big|\int_0^T \int_Q [\chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x) - \chi_1^{\epsilon_j} f_{1,\lambda}^{\epsilon_j}(t)]\psi_1(t,x)dxd\big(B_1^{\epsilon_j}(t) - \tilde{B}_1(t)\big)\Big| \leq \kappa_2(\lambda),$$

where $\kappa_2(\lambda)$ converge to zero as $\lambda \to 0$.

Hence from (3.23), we have

$$\tilde{\mathbb{E}}\bigg|\int_0^T\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxd\big(B_1^{\epsilon_j}(t)-\tilde{B}_1(t)\big)\bigg| \leq \kappa_1(\lambda)\eta_1(\epsilon_j)+\kappa_2(\lambda).$$

From (3.22), we conclude that

$$\bigg|\tilde{\mathbb{E}}\int_0^T\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxdB_1^{\epsilon_j}(t)-\tilde{\mathbb{E}}\int_0^T\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxd\tilde{B}_1(t)\bigg|$$
$$\leq \kappa_1(\lambda)\eta_1(\epsilon_j)+\kappa_2(\lambda).$$

Passing to the limit as $\epsilon_j \to 0$, we get

$$\lim_{\epsilon_j\to 0}\bigg|\tilde{\mathbb{E}}\int_0^T\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxdB_1^{\epsilon_j}(t)-\tilde{\mathbb{E}}\int_0^T\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxd\tilde{B}_1(t)\bigg|$$
$$\leq \kappa_2(\lambda).$$

But since the left hand side of this relation is independent of λ, and $\kappa_2(\lambda) \to 0$ as $\lambda \to 0$, we can pass to the limit on both sides as $\lambda \to 0$ to get

$$\lim_{\epsilon_j\to 0}\int_0^T\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxdB_1^{\epsilon_j}(t)=\lim_{\epsilon_j\to 0}\int_0^T\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxd\tilde{B}_1(t).$$

By Lemma 3.12, we have the two-scale convergence of $\chi_1^{\epsilon_j} f_1^{\epsilon_j}$ to $\chi_1(y)f_1(t,x,y)$ $\tilde{\mathbb{P}}$-a.s., which implies weak convergence

Hence by using the convergence result for stochastic integrals in [112] (Theorem 4, pg 63), we get

$$\lim_{\epsilon_j\to 0}\int_0^T\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\psi_1(t,x)dxdB_1^{\epsilon_j}(t)=\int_0^T\int_Q \chi_1(y)f_1(t,x,y)\psi_1(t,x)dxd\tilde{B}_1(t),$$

$\tilde{\mathbb{P}}$-a.s..

Now we show that

$$\lim_{\epsilon_j\to 0}\epsilon_j\int_0^T\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\phi_1\big(t,x,\frac{x}{\epsilon_j}\big)dxdB_1^{\epsilon_j}(t)=0,\ \ \tilde{\mathbb{P}}\text{-a.s..}$$

With the assumptions of $f_1^{\epsilon_j}$ and Burkhölder-Davis-Gundy's inequality, we have

$$\lim_{\epsilon_j\to 0}\epsilon_j\tilde{\mathbb{E}}\sup_{t\in[0,T]}\bigg|\int_0^T\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\phi_1\big(t,x,\frac{x}{\epsilon_j}\big)dxdB_1^{\epsilon_j}(t)\bigg|$$
$$\leq C\lim_{\epsilon_j\to 0}\epsilon_j\tilde{\mathbb{E}}\bigg(\int_0^T\bigg(\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}(t,x)\phi_1\big(t,x,\frac{x}{\epsilon_j}\big)dx\bigg)^2dt\bigg)^{\frac{1}{2}}$$
$$\leq C\lim_{\epsilon_j\to 0}\epsilon_j\tilde{\mathbb{E}}\bigg(\int_0^T ||\chi_1^{\epsilon_j} f_1^{\epsilon_j}||_{L^2(Q)}||\phi_1\big(t,x,\frac{x}{\epsilon_j}\big)||_{L^2(Q)}dt\bigg)^{\frac{1}{2}}$$
$$\leq C\lim_{\epsilon_j\to 0}\epsilon_j\bigg(\int_0^T ||\chi_1^{\epsilon_j} f_1^{\epsilon_j}||_{L^2(Q)}dt\bigg)^{\frac{1}{2}} \to 0\ \ \tilde{\mathbb{P}}\text{-a.s..}$$

Combining the above convergences, we assert that

$$\begin{aligned}&\lim_{\epsilon_j\to 0}\int_0^T\int_Q \chi_1^{\epsilon_j} f_1^{\epsilon_j}\big[\psi_1(t,x)+\epsilon_j\phi_1\big(t,x,\frac{x}{\epsilon_j}\big)\big]dxdB_1^{\epsilon_j}(t)\\ &=\int_0^T\int_Q\int_{Y_1} f_1(t,x,y)\psi_1(t,x)dxd\tilde{B}_1(t).\ \ \mathbb{P}\text{-a.s.}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\lim_{\epsilon_j\to 0}\int_0^T\int_Q \chi_2^{\epsilon_j} f_2^{\epsilon_j}\big[\psi_2(t,x)+\epsilon_j\phi_2\big(t,x,\frac{x}{\epsilon_j}\big)\big]dxdB_2^{\epsilon_j}(t)\\ &=\int_0^T\int_Q\int_{Y_2} f_2(t,x,y)\psi_2(t,x)dxd\tilde{B}_2(t),\ \ \tilde{\mathbb{P}}\text{-a.s.}.\end{aligned}$$

and

$$\begin{aligned}&\lim_{\epsilon_j\to 0}\int_0^T\int_Q \chi_2^{\epsilon_j} f_3^{\epsilon_j}\phi_3^{\epsilon_j}\big(t,x,\frac{x}{\epsilon_j}\big)dxdB_3^{\epsilon_j}(t)\\ &=\int_0^T\int_Q\int_{Y_2} f_3(t,x,y)\phi_3(t,x,y)dxd\tilde{B}_3(t),\ \ \mathbb{P}\text{-a.s.}.\end{aligned}$$

Combining all the above convergences yield

$$\begin{aligned}&-\sum_{i=1}^{2}\int_0^T\int_Q\int_{Y_i} c_i(y)u_i(t,x)\psi_{it}(t,x)dydxdt\\ &-\int_0^T\int_Q\int_{Y_2} c_3(y)U_3(t,x,y)\phi_{3t}(t,x,y)dydxdt\\ &+\sum_{i=1}^{2}\int_Q\int_{Y_i} c_i(y)u_i^*(x)\psi_i(T,x)dydx+\int_Q\int_{Y_2} c_3(y)u_3^*(x)\phi_3(T,x,y)dydx\\ &-\sum_{i=1}^{2}\int_Q\int_{Y_i} c_i(y)u_i^0(x)\psi_i(0,x)dydx-\int_Q\int_{Y_2} c_3(y)u_3^0(x)\phi_3(0,x,y)dydx\\ &+\sum_{i=1}^{2}\int_0^T\int_Q\int_{Y_i}\vec{g}_i(t,x,y)\big[\nabla\psi_i(t,x)+\nabla_y\phi_i(t,x,y)\big]dydxdt\\ &+\int_0^T\int_Q\int_{Y_2}\vec{g}_3(t,x,y)\nabla_y\phi_3(t,x,y)dydxdt\\ &-\sum_{i=1}^{2}\int_0^T\int_Q\int_{Y_i} f_i(t,x,y)\psi_i(t,x)dydxd\tilde{B}_i(t)\\ &-\int_0^T\int_Q\int_{Y_2} f_3(t,x,y)\phi_3(t,x,y)dydxd\tilde{B}_3(t)=0,\quad \mathbb{P}\text{-a.s.}.\end{aligned}\tag{3.25}$$

Let us decouple equation (3.25) by making specific choices of the test functions ψ_1, ψ_2, ϕ_1, ϕ_2, ϕ_3.

Let ψ_1 be such that $\psi_1(t) = 0$ at $t = 0$ and $t = T$ and choose ϕ_3 such that $\beta\phi_3(t,x,y) = \psi_1(t,x)$ for $y \in Y_2$ and $\psi_2, \phi_1, \phi_2 = 0$. Then we get the following equation;

$$
\begin{aligned}
&-\int_0^T \int_Q \int_{Y_1} c_1(y) u_1(t,x) \psi_{1t}(t,x) dy dx dt - \frac{1}{\beta} \int_0^T \int_Q \int_{Y_2} c_3(y) U_3(t,x,y) \psi_{1t}(t,x) dy dx dt \\
&+ \int_Q \int_{Y_1} c_1(y) u_1^*(x) \psi_1(T,x) dy dx + \frac{1}{\beta} \int_Q \int_{Y_2} c_3(y) u_3^*(x) \psi_1(T,x) dy dx \\
&- \int_Q \int_{Y_1} c_1(y) u_1^0(x) \psi_1(0,x) dy dx - \frac{1}{\beta} \int_Q \int_{Y_2} c_3(y) u_3^0(x) \psi_1(0,x) dy dx \\
&+ \int_0^T \int_Q \int_{Y_1} \vec{g}_1(t,x,y) \nabla \psi_1(t,x) dy dx dt \\
&+ \frac{1}{\beta} \int_0^T \int_Q \int_{Y_2} \vec{g}_3(t,x,y) \nabla_y \psi_1(t,x) dy dx dt \\
&- \int_0^T \int_Q \int_{Y_1} f_1(t,x,y) \psi_1(t,x) dy dx d\tilde{B}_1(t) \\
&- \frac{1}{\beta} \int_0^T \int_Q \int_{Y_2} f_3(t,x,y) \psi_1(t,x) dy dx d\tilde{B}_3(t) = 0.
\end{aligned}
$$

Integrating by parts with respect to t in the first and second terms on the left hand side and with respect to x in the seventh term gives

$$
\begin{aligned}
&\int_0^T \int_Q \int_{Y_1} c_1(y) du_1(t,x) \psi_1(t,x) dy dx \\
&+ \frac{1}{\beta} \int_0^T \int_Q \int_{Y_2} c_3(y) dU_3(t,x,y) \psi_1(t,x) dy dx \\
&- \int_0^T \int_Q \int_{Y_1} \nabla \cdot \vec{g}_1(t,x,y) \psi_1(t,x) dy dx dt \\
&+ \int_0^T \int_{\partial Q} \int_{Y_1} \vec{g}_1(t,x,y) \cdot \vec{\nu_1} \psi_1(t,x) dy dS_x dt \\
&+ \frac{1}{\beta} \int_0^T \int_Q \int_{Y_2} \vec{g}_3(t,x,y) \nabla_y \psi_1(t,x) dy dx dt \\
&- \int_0^T \int_Q \int_{Y_1} f_1(t,x,y) \psi_1(t,x) dy dx d\tilde{B}_1(t) \\
&- \frac{1}{\beta} \int_0^T \int_Q \int_{Y_2} f_3(t,x,y) \psi_1(t,x) dy dx d\tilde{B}_3(t) = 0.
\end{aligned}
\tag{3.26}
$$

This is the weak formulation of the following macro-fissure equation

$$\begin{aligned}
&\Big(\int_{Y_1} c_1(y)dy\Big)du_1(t,x) + \frac{1}{\beta}\Big(\int_{Y_2} c_3(y)dU_3(t,x,y)dy\Big) \\
&\qquad = \nabla\cdot\Big(\int_{Y_1} \vec{g}_1(t,x,y)dy\Big)dt + \Big(\int_{Y_1} f_1(t,x,y)dy\Big)d\tilde{B}_1(t) \qquad (3.27)\\
&\qquad + \frac{1}{\beta}\Big(\int_{Y_2} f_3(t,x,y)dy\Big)d\tilde{B}_3(t).
\end{aligned}$$

Similarly, let $\psi_1, \phi_1, \phi_2 = 0$ and ψ_2 be such that $\psi_2(t,x) = 0$ at $t = 0$ and $t = T$ and ϕ_3 be such that $\beta\phi_3 = -\alpha\psi_2$, for $y \in Y_1$. Then we obtain the following

$$\begin{aligned}
&-\int_0^T \int_Q \int_{Y_2} c_2(y)u_2(t,x)\psi_{2t}(t,x)dydxdt \\
&+\frac{\alpha}{\beta}\int_0^T \int_Q \int_{Y_2} c_3(y)U_3(t,x,y)\psi_{2t}(t,x)dydxdt \\
&+\int_Q \int_{Y_2} c_2(y)u_2^*(x)\psi_2(T,x)dydx - \frac{\alpha}{\beta}\int_Q \int_{Y_2} c_3(y)u_3^*(x)\psi_2(T,x)dydx \\
&-\int_Q \int_{Y_2} c_2(y)u_2^0(x)\psi_2(0,x)dydx + \frac{\alpha}{\beta}\int_Q \int_{Y_2} c_3(y)u_3^0(x)\psi_2(0,x)dydx \\
&+\int_0^T \int_Q \int_{Y_2} \vec{g}_2(t,x,y)\nabla\psi_2(t,x)dydxdt \\
&-\frac{\alpha}{\beta}\int_0^T \int_Q \int_{Y_2} \vec{g}_3(t,x,y)\nabla_y\psi_2(t,x)dydxdt \\
&-\int_0^T \int_Q \int_{Y_2} f_2(t,x,y)\psi_2(t,x)dydxd\tilde{B}_2(t) \\
&+\frac{\alpha}{\beta}\int_0^T \int_Q \int_{Y_2} f_3(t,x,y)\psi_2(t,x)dydxd\tilde{B}_3(t) = 0.
\end{aligned}$$

Integrating by parts with respect to t in the first and second terms on the left hand side and with respect to x in the seventh term gives

$$\begin{aligned}
&\int_0^T \int_Q \int_{Y_2} c_2(y) du_2(t,x)\psi_2(t,x) dydx \\
&- \frac{\alpha}{\beta}\int_0^T \int_Q \int_{Y_2} c_3(y) dU_3(t,x,y)\psi_2(t,x) dydx \\
&- \int_0^T \int_Q \int_{Y_2} \nabla \cdot \vec{g}_2(t,x,y)\psi_2(t,x) dydxdt \\
&+ \int_0^T \int_{\partial Q} \int_{Y_2} \vec{g}_2(t,x,y) \cdot \vec{\nu_2}\psi_2(t,x) dydS_x dt \qquad (3.28) \\
&+ \frac{\alpha}{\beta}\int_0^T \int_Q \int_{Y_2} \vec{g}_3(t,x,y)\nabla_y \psi_2(t,x) dydxdt \\
&- \int_0^T \int_Q \int_{Y_2} f_2(t,x,y)\psi_2(t,x) dydxd\tilde{B}_2(t) \\
&+ \frac{\alpha}{\beta}\int_0^T \int_Q \int_{Y_2} f_3(t,x,y)\psi_2(t,x) dydxd\tilde{B}_3(t) = 0.
\end{aligned}$$

Since $\psi_2 \in \mathcal{D}((0,T); C_0^\infty(Q))$ is arbitrary, we have that (3.28) is the weak formulation of the following macro-matrix equation

$$\begin{aligned}
&\left(\int_{Y_2} c_2(y) dy\right) du_2(t,x) - \frac{\alpha}{\beta}\left(\int_{Y_2} c_3(y) dU_3(t,x,y) dy\right) \\
&\quad = \nabla \cdot \left(\int_{Y_2} \vec{g}_2(t,x,y) dy\right) dt + \left(\int_{Y_2} f_2(t,x,y) dy\right) d\tilde{B}_2(t) \qquad (3.29) \\
&\quad - \frac{\alpha}{\beta}\left(\int_{Y_2} f_3(t,x,y) dy\right) d\tilde{B}_3(t).
\end{aligned}$$

Next, let $\psi_1, \psi_2, \phi_1, \phi_2 = 0$ and ϕ_3 be such that $\phi_3(t) = 0$ at $t = 0$ and $t = T$ on $\Gamma_{1,2}$, together with $\beta U_3 + \alpha u_2 = u_1,$ on $\partial Y_1 \cap \partial Y_2 = \Gamma_{1,2}$, we get the cell equation

$$\begin{aligned}
&- \int_0^T \int_Q \int_{Y_2} c_3(y) U_3(t,x,y)\phi_{3t}(t,x,y) dydxdt + \int_Q \int_{Y_2} c_3(y) u_3^*(x)\phi_3(T,x,y) dydx \\
&- \int_Q \int_{Y_2} c_3(y) u_3^0(x)\phi_3(0,x,y) dydx + \int_0^T \int_Q \int_{Y_2} \vec{g}_3(t,x,y)\nabla_y \phi_3(t,x,y) dydxdt \\
&- \int_0^T \int_Q \int_{Y_2} f_3(t,x,y)\phi_3(t,x,y) dydxd\tilde{B}_3(t) = 0.
\end{aligned}$$

Integrating by parts with respect to t in the first and third terms in the left hand side gives

$$
\begin{aligned}
&-\int_0^T\int_Q\int_{Y_2} c_3(y)dU_3(t,x,y)\phi_3(t,x,y)dydx\\
&+\int_0^T\int_Q\int_{Y_2} \nabla_y\cdot\vec{g}_3(t,x,y)\phi_3(t,x,y)dydxdt\\
&-\int_0^T\int_Q\int_{\partial Y_2} \vec{g}_3(t,x,y)\cdot\vec{\nu_2}\phi_3(t,x,y)dS_ydxdt\\
&-\int_0^T\int_Q\int_{Y_2} f_3(t,x,y)\phi_3(t,x,y)dydxd\tilde{B}_3(t)=0.
\end{aligned}
\tag{3.30}
$$

This is a weak formulation of the following the cell equation;

$$
\begin{aligned}
c_3(y)dU_3(t,x,y) = \nabla_y\cdot\vec{g}_3(t,x,y) + f_3(t,x,y)d\tilde{B}_3(t),\quad y\in Y_2\\
U_3 \text{ and } \vec{g}_3\cdot\nu \text{ are } Y\text{-periodic on } \Gamma_{2,2},\\
\beta U_3 = u_1-\alpha u_2 \text{ on } \Gamma_{1,2}.
\end{aligned}
\tag{3.31}
$$

Lastly, letting $\psi_1,\psi_2,\phi_3=0$ and $\phi_1,\phi_2\in\mathcal{D}((0,T)\times Q;C^\infty_{per}(Y))$, we obtain

$$
\int_0^T\int_Q\int_{Y_1}\vec{g}_1(t,x,y)\nabla_y\phi_1(t,x,y)dydxdt+\int_0^T\int_Q\int_{Y_2}\vec{g}_2(t,x,y)\nabla_y\phi_2(t,x,y)dydxdt=0,
$$

which is weak formulation of the following system of equations;

$$
\begin{aligned}
\nabla_y\cdot\vec{g}_i(t,x,y)=0,\ \ y\in Y_i,\\
\vec{g}_i\cdot\vec{\nu}=0 \text{ on } \Gamma_{1,2} \text{ and } \vec{g}\cdot\vec{\nu} \text{ is } Y\text{-periodic on } \partial Y_i\cap\partial Y,\ \text{ for } i=1,2.
\end{aligned}
\tag{3.32}
$$

We will have our homogenized problem when we have identified the terms $\vec{g}_1,\vec{g}_2$ and $\vec{g}_3$.

Next we split (3.25) using special choices of test functions $\psi_1,\psi_2,\psi_3,\phi_1,\phi_2,\phi_3$ in order to be able to use Ito's formula.

In the first stage, choosing $\psi_2,\phi_1,\phi_2,\phi_3=0$ and $\psi_1\in\mathcal{D}(0,T;W_0^{1,p}(Q))$, we have

$$
\begin{aligned}
&\left(\int_{Y_1}c_1(y)dy\right)du_1(t,x)=\nabla\cdot\left(\int_{Y_1}\vec{g}_1(t,x,y)dy\right)\\
&+\left(\int_{Y_1}f_1(t,x,y)dy\right)d\tilde{B}_1(t).
\end{aligned}
\tag{3.33}
$$

Next choosing $\psi_1,\phi_1,\phi_2,\phi_3=0$ and $\psi_2\in\mathcal{D}(0,T;W_0^{1,p}(Q))$, we have

$$
\begin{aligned}
&\left(\int_{Y_2}c_2(y)dy\right)du_2(t,x)=\nabla\cdot\left(\int_{Y_2}\vec{g}_2(t,x,y)dy\right)\\
&+\left(\int_{Y_2}f_2(t,x,y)dy\right)d\tilde{B}_2(t).
\end{aligned}
\tag{3.34}
$$

Now we choose $\psi_1, \psi_2, \phi_1, \phi_2 = 0$ and $\phi_3 \in \mathcal{D}([0,T] \times Q; W^{1,p}_{per}(Y))$, to get

$$c_3(y)dU_3(t,x,y) = \nabla_y \cdot \vec{g}_3(t,x,y) + \left(\int_{Y_2} f_3(t,x,y)dy \right) d\tilde{B}_3(t). \tag{3.35}$$

Next we choose $\psi_1, \psi_2, \phi_2, \phi_3 = 0$ and $\phi_1 \in \mathcal{D}([0,T] \times Q; W^{1,p}_{per}(Y))$, to get

$$\int_0^T \int_Q \int_{Y_1} \vec{g}_1(t,x,y) \nabla_y \phi_1(t,x,y) dydxdt = 0. \tag{3.36}$$

Lastly, choosing $\psi_1, \psi_2, \phi_1, \phi_3 = 0$ and $\phi_2 \in \mathcal{D}([0,T] \times Q; W^{1,p}_{per}(Y))$, we have

$$\int_0^T \int_Q \int_{Y_2} \vec{g}_2(t,x,y) \nabla_y \phi_2(t,x,y) dydxdt = 0. \tag{3.37}$$

Ito's formula on (3.33) - (3.35) at $t = T$ and adding (3.36) and (3.37) we get

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{2} \int_Q \int_{Y_i} c_1(y)|u_i(T)|^2 dydx + \frac{1}{2}\int_Q \int_{Y_2} c_3(y)|U_3(T)|^2 dydx \\
&- \frac{1}{2}\int_Q \int_{Y_2} c_3(y)|U_3(0)|^2 dydx - \frac{1}{2}\sum_{i=1}^{2} \int_Q \int_{Y_i} c_1(y)|u_i(0)|^2 dydx \\
&+ \sum_{i=1}^{2} \int_0^T \int_Q \int_{Y_i} \vec{g}_i(t,x,y) \nabla u_i(t,x) dydxdt \\
&+ \int_0^T \int_Q \int_{Y_2} \vec{g}_3(t,x,y) \nabla_y U_3(t,x,y) dydxdt \\
&- \sum_{i=1}^{2} \int_0^T \int_Q \int_{Y_i} f_i(t,x,y) u_i(t,x) dydxdB_i(t) \\
&- \int_0^T \int_Q \int_{Y_2} f_3(t,x,y) U_3(t,x,y) dydxdB_3(t) \\
&- \frac{1}{2}\sum_{i=1}^{2} \int_0^T \int_Q \int_{Y_i} |f_i(t,x,y)|^2 dydxdt - \frac{1}{2}\int_0^T \int_Q \int_{Y_2} |f_3(t,x,y)|^2 dydxdt \\
&+ \sum_{i=1}^{2} \int_0^T \int_Q \int_{Y_i} \vec{g}_i(t,x,y) \nabla_y U_i(t,x,y) dydxdt = 0, \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{3.38}$$

Now we identify $\vec{g}_1, \vec{g}_2, \vec{g}_3$. For this, we use Minty's trick (Lemma 1.28) [84]; see also [21] and [30]. Let $\vec{\varphi}, \vec{\xi} \in C_0^\infty([0,T] \times Q; C^\infty_{per}(Y))^3$ and $\eta_1, \eta_2, \eta_3 \in C_0^\infty([0,T] \times Q; C^\infty_{per}(Y))$ and for $\epsilon > 0$, we define the functions

$$\lambda_i^\epsilon(t,x) = \chi_i\big(\frac{x}{\epsilon}\big)\nabla u_i(t,x) + \epsilon \chi_i\big(\frac{x}{\epsilon}\big)\nabla \eta_i\big(t,x,\frac{x}{\epsilon}\big) + \sigma \vec{\varphi}\big(t,x,\frac{x}{\epsilon}\big) \;\; i = 1,2,$$

$$\lambda_3^\epsilon(t,x) = \chi_2\big(\frac{x}{\epsilon}\big)(\epsilon \nabla \eta_3\big(t,x,\frac{x}{\epsilon}\big)) + \sigma \vec{\xi}\big(t,x,\frac{x}{\epsilon}\big).$$

Since $\mu_i\big(\frac{x}{\epsilon}, \lambda_i^\epsilon(t,x)\big)$ and $\lambda_i^\epsilon(t,x)$ $(i=1,2,3)$ arise from an admissible test function, we have the following two-scale convergence

$$\lambda_i^{\epsilon_j}(t,x) \xrightarrow{2-s} \lambda_i(t,x,y) = \chi_i(y)\nabla u_i(t,x) + \chi_i(y)\nabla_y \eta_i(t,x,y) + \sigma\varphi(t,x,y) \quad (i=1,2),$$

$$\lambda_3^{\epsilon_j}(t,x) \xrightarrow{2-s} \lambda_3(t,x,y) = \chi_2(y)\nabla_y \eta_3(t,x,y) + \sigma\vec{\xi}(t,x,y).$$

By $A(3.2)$, we get

$$\sum_{i=1}^{2}\int_0^T\int_Q \big(\chi_i^{\epsilon_j}\mu_i^{\epsilon_j}(x,\nabla u_i^{\epsilon_j}) - \chi_i^{\epsilon_j}\mu_i^{\epsilon_j}(x,\lambda_i^{\epsilon_j})\big)(\nabla u_i^{\epsilon_j} - \lambda_i^{\epsilon_j})dxdt$$
$$+\int_0^T\int_Q \big(\chi_2^{\epsilon_j}\mu_3^{\epsilon_j}(x,\epsilon_j\nabla u_3^{\epsilon_j}) - \chi_2^{\epsilon_j}\mu_3^{\epsilon_j}(x,\lambda_3^{\epsilon_j})\big)(\epsilon_j\nabla u_3^{\epsilon_j} - \lambda_3^{\epsilon_j})dxdt \geq 0.$$

Expanding the above inequality yields

$$\begin{aligned}
&\sum_{i=1}^{2}\int_0^T\int_Q \chi_i^{\epsilon_j}\mu_i^{\epsilon_j}(x,\nabla u_i^{\epsilon_j})\nabla u_i^{\epsilon_j}dxdt - \sum_{i=1}^{2}\int_0^T\int_Q \chi_i^{\epsilon_j}\mu_i^{\epsilon_j}(x,\nabla u_i^{\epsilon_j})\lambda_i^{\epsilon_j}dxdt\\
&-\sum_{i=1}^{2}\int_0^T\int_Q \chi_i^{\epsilon_j}\mu_i^{\epsilon_j}(x,\lambda_i^{\epsilon_j})(\nabla u_i^{\epsilon_j} - \lambda_i^{\epsilon_j})dxdt\\
&+\int_0^T\int_Q \chi_i^{\epsilon_j}\mu_3^{\epsilon_j}(x,\epsilon\nabla u_3^{\epsilon_j})\epsilon_j\nabla u_3^{\epsilon_j}dxdt\\
&-\int_0^T\int_Q \chi_2^{\epsilon_j}\mu_3^{\epsilon_j}(x,\epsilon\nabla u_3^{\epsilon_j})\lambda_3^{\epsilon_j}dxdt\\
&-\int_0^T\int_Q \chi_2^{\epsilon_j}\mu_3^{\epsilon_j}(x,\lambda_3^{\epsilon_j})(\epsilon_j\nabla u_3^{\epsilon_j} - \lambda_3^{\epsilon_j})dxdt \geq 0.
\end{aligned} \tag{3.39}$$

Recall that Ito's formula on (P^{ϵ_j}) yields

$$\begin{aligned}
&\sum_{i=1}^{2} c_i^{\epsilon_j}(x)|u_i^{\epsilon_j}(t)|^2 + c_3^{\epsilon_j}(x)|u_3^{\epsilon_j}(t)|^2 + 2\sum_{i=1}^{2}\int_0^t (\mu_i^{\epsilon_j}(x,\nabla u_i^{\epsilon_j}), \nabla u_i^{\epsilon_j}(s))ds\\
&+2\int_0^t (\mu_3^{\epsilon_j}(x,\epsilon_j\nabla u_3^{\epsilon_j}), \epsilon_j\nabla u_3^{\epsilon_j}(s))ds = \sum_{i=1}^{2} c_i^{\epsilon_j}(x)|u_i^{\epsilon_j}(0)|^2 + c_3^{\epsilon_j}(x)|u_3^{\epsilon_j}(0)|^2\\
&+2\sum_{i=1}^{2}\int_0^t (f_i^{\epsilon_j}(s), u_i^{\epsilon_j}(s))dB^{\epsilon_j}(s) + 2\int_0^t (f_3^{\epsilon_j}(s), u_3^{\epsilon_j}(s))dB_3^{\epsilon_j}(s)\\
&+\sum_{i=1}^{2}\int_0^t ||f_i^{\epsilon_j}||^2_{L^2(Q_i^\epsilon)}ds + \int_0^t ||f_3^{\epsilon_j}||^2_{L^2(Q_2^\epsilon)}ds, \quad \mathbb{P}\text{-a.s.}.
\end{aligned} \tag{3.40}$$

Adding a suitable zero to (3.39) and using (3.40) gives

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{2}\int_Q c_i^{\epsilon_j}(x)\chi_i^{\epsilon_j}|u_i^{\epsilon_j}(0)|^2dx - \frac{1}{2}\sum_{i=1}^{2}\int_Q c_i^{\epsilon_j}(x)\chi_i^{\epsilon_j}|u_i^{\epsilon_j}(T)|^2dx \\
&+\frac{1}{2}\int_Q c_3^{\epsilon_j}(x)\chi_2^{\epsilon_j}|u_3^{\epsilon_j}(0)|^2dx - \frac{1}{2}\int_Q c_3^{\epsilon_j}(x)\chi_2^{\epsilon_j}|u_3^{\epsilon_j}(T)|^2dx \\
&+\sum_{i=1}^{2}\int_0^T\int_Q \chi_i^{\epsilon_j}f_i^{\epsilon_j}u_i^{\epsilon_j}dxdB_i(t) + \sum_{i=1}^{2}\frac{1}{2}\int_0^T\int_Q|\chi_i^{\epsilon_j}f_i^{\epsilon_j}|^2dxdt \\
&+\int_0^T\int_Q \chi_2^{\epsilon_j}f_3^{\epsilon_j}u_3^{\epsilon_j}dxdB_3(t) + \frac{1}{2}\int_0^T\int_Q|\chi_2^{\epsilon_j}f_3^{\epsilon_j}|^2dxdt \\
&-\sum_{i=1}^{2}\int_0^T\int_Q \chi_i^{\epsilon_j}\mu_i^{\epsilon_j}(x,\nabla u_i^{\epsilon_j})\lambda_i^{\epsilon_j}dxdt \\
&-\sum_{i=1}^{2}\int_0^T\int_Q \chi_i^{\epsilon_j}\mu_i^{\epsilon_j}(x,\lambda_i^{\epsilon_j})(\nabla u_i^{\epsilon_j}-\lambda_i^{\epsilon_j})dxdt \\
&-\int_0^T\int_Q \chi_2^{\epsilon_j}\mu_3^{\epsilon_j}(x,\epsilon\nabla u_3^{\epsilon_j})\lambda_3^{\epsilon_j}dxdt \\
&-\int_0^T\int_Q \chi_2^{\epsilon_j}\mu_3^{\epsilon_j}(x,\lambda_3^{\epsilon_j})(\epsilon\nabla u_3^{\epsilon_j}-\lambda_3^{\epsilon_j})dxdt \geq 0.
\end{aligned}
\tag{3.41}
$$

Recall that $\nabla u_i^{\epsilon_j} \xrightarrow{2-s} \nabla u_i(t,x) + \nabla_y U_i(t,x,y)$, $\quad i = 1,2,\quad$ and $\epsilon\nabla u_3^{\epsilon_j} \xrightarrow{2-s} \nabla_y U_3(t,x,y)$.

We now take the limit as $\epsilon_j \to 0$ to get

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{2}\int_Q\int_{Y_i} c_i(y)|u_i(0)|^2 dydx + \frac{1}{2}\int_Q\int_{Y_2} c_3(y)|U_3(0)|^2 dydx \\
&+\sum_{i=1}^{2}\int_0^T\int_Q\int_{Y_i} f_i u_i dydxd\tilde{B}_i(t) + \frac{1}{2}\sum_{i=1}^{2}\int_0^T\int_Q\int_{Y_i} |f_i|^2 dydxdt \\
&+\int_0^T\int_Q\int_{Y_2} f_3 U_3 dydxd\tilde{B}_3(t) + \frac{1}{2}\int_0^T\int_Q\int_{Y_2} |f_3|^2 dydxdt \\
&-\int_0^T\int_Q\int_{Y_1} \vec{g}_1(\nabla u_1 + \nabla_y \eta_1 + \sigma\varphi) dydxdt \\
&-\int_0^T\int_Q\int_{Y_2} \mu_1(y,\lambda_1)(\nabla_y U_1 - \nabla_y \eta_1 - \sigma\varphi) dydxdt \\
&-\int_0^T\int_Q\int_{Y_1} \vec{g}_2(\nabla u_2 + \nabla_y \eta_2 + \sigma\varphi) dydxdt \\
&-\int_0^T\int_Q\int_{Y_2} \mu_2(y,\lambda_2)(\nabla_y U_2 - \nabla_y \eta_2 - \sigma\varphi) dydxdt \\
&-\int_0^T\int_Q\int_{Y_2} \vec{g}_3(\nabla_y \eta_3 + \sigma\xi) dydxdt \\
&-\int_0^T\int_Q\int_{Y_2} \mu_3(y,\lambda_3)(\nabla_y U_3 - \nabla_y \eta_3 - \sigma\xi) dydxdt \\
&\geq \frac{1}{2}\lim_{\epsilon_j\to 0}\int_{Q_1^\epsilon} c_1^{\epsilon_j}(x)|u_1^{\epsilon_j}(T)|^2 dx + \frac{1}{2}\lim_{\epsilon_j\to 0}\int_{Q_2^\epsilon} c_2^{\epsilon_j}(x)|u_2^{\epsilon_j}(T)|^2 dx \\
&+\frac{1}{2}\lim_{\epsilon_j\to 0}\int_{Q_2^\epsilon} c_3^{\epsilon_j}(x)|u_3^{\epsilon_j}(T)|^2 dx,
\end{aligned}
\tag{3.42}
$$

where we omit the variable (t,x,y) in order to avoid cumbersome writing.

We use (3.38) in (3.42) and replace $\eta_i(t,x,y)$ by $U_i(t,x,y)$ $(i = 1,2,3)$ to get

$$
\begin{aligned}
&-\int_0^T\int_Q\int_{Y_1} \vec{g}_1\sigma\vec{\varphi} dydxdt + \int_0^T\int_Q\int_{Y_1} \mu_1(y, \nabla u_1 + \nabla_y U_1 + \sigma\vec{\varphi})\sigma\vec{\varphi} dydxdt \\
&-\int_0^T\int_Q\int_{Y_2} \vec{g}_2\sigma\vec{\varphi} dydxdt + \int_0^T\int_Q\int_{Y_2} \mu_2(y, \nabla u_2 + \nabla_y U_2 + \sigma\vec{\varphi})\sigma\vec{\varphi} dydxdt \\
&-\int_0^T\int_Q\int_{Y_2} \vec{g}_3\sigma\vec{\xi} dydxdt + \int_0^T\int_Q\int_{Y_2} \mu_3(y, \nabla_y U_3 + \sigma\vec{\xi})\sigma\vec{\xi} dydxdt \\
&\geq \frac{1}{2}\lim_{\epsilon_j\to 0}\int_{Q_1^\epsilon} c_1^{\epsilon_j}(x)|u_1^{\epsilon_j}(T)|^2 dx + \frac{1}{2}\lim_{\epsilon_j\to 0}\int_{Q_2^\epsilon} c_2^{\epsilon_j}(x)|u_2^{\epsilon_j}(T)|^2 dx \\
&+\frac{1}{2}\lim_{\epsilon_j\to 0}\int_{Q_2^\epsilon} c_3^{\epsilon_j}(x)|u_3^{\epsilon}(T)|^2 dx - \frac{1}{2}\int_Q\int_{Y_1} c_1(y)|u_1(T)|^2 dydx \\
&-\frac{1}{2}\int_Q\int_{Y_2} c_2(y)|u_2(T)|^2 dydx - \frac{1}{2}\int_Q\int_{Y_2} c_3(y)|U_3(T)|^2 dydx.
\end{aligned}
\tag{3.43}
$$

Now let us set $\vec{\varphi} = \chi_1\vec{\theta_1} + \chi_2\vec{\theta_2}$, where we take $\theta_i \in C_0^\infty([0,T] \times Q; C^\infty(Y_i))$, $(i = 1, 2)$.

Using Proposition 3.11, the right hand side of (3.43) is nonnegative. Thus (3.43) becomes

$$
\begin{aligned}
&-\int_0^T \int_Q \int_{Y_1} \vec{g}_1 \sigma \vec{\theta}_1 dydxdt + \int_0^T \int_Q \int_{Y_1} \mu_1(y, \nabla u_1 + \nabla_y U_1 + \sigma\vec{\theta}_1)\sigma\vec{\theta}_1 dydxdt \\
&-\int_0^T \int_Q \int_{Y_2} \vec{g}_2 \sigma \vec{\theta}_2 dydxdt + \int_0^T \int_Q \int_{Y_2} \mu_2(y, \nabla u_2 + \nabla_y U_2 + \sigma\vec{\theta}_2)\sigma\vec{\theta}_2 dydxdt \\
&-\int_0^T \int_Q \int_{Y_2} \vec{g}_3 \sigma \vec{\xi} dydxdt + \int_0^T \int_Q \int_{Y_2} \mu_3(y, \nabla_y U_3 + \sigma\vec{\xi})\sigma\vec{\xi} dydxdt \geq 0.
\end{aligned}
$$

Following Bensoussan's argument in [21], first we divide the above equation by σ and then let $\sigma \to 0$ to obtain

$$
\begin{aligned}
&\int_0^T \int_Q \int_{Y_1} [\mu(y, \nabla u_1 + \nabla_y U_1) - \vec{g}_1]\vec{\theta}_1 dydxdt \\
&+ \int_0^T \int_Q \int_{Y_2} [\mu(y, \nabla u_2 + \nabla_y U_2) - \vec{g}_2]\vec{\theta}_2 dydxdt \\
&+ \int_0^T \int_Q \int_{Y_1} [\mu(y, \nabla_y U_3) - \vec{g}_3]\vec{\xi} dydxdt \geq 0 \quad \forall \ \vec{\theta}_1, \vec{\theta}_2, \vec{\xi}.
\end{aligned}
$$

Hence owing to Minty's trick (Lemma 1.28) [84] and as implemented by Bensoussan in [21], we conclude that

$$
\begin{aligned}
\vec{g}_1(t,x,y) &= \mu_1(y, \nabla u_1 + \nabla_y U_1) \text{ in } (0,T) \times Q \times Y_1 \ \ \mathbb{P}\text{-a.s} \\
\vec{g}_2(t,x,y) &= \mu_2(y, \nabla u_2 + \nabla_y U_2) \text{ in } (0,T) \times Q \times Y_2 \ \ \mathbb{P}\text{-a.s} \\
\vec{g}_3(t,x,y) &= \mu_3(y, \nabla_y U_3) \quad \text{ in } (0,T) \times Q \times Y_2 \ \ \mathbb{P}\text{-a.s}
\end{aligned}
$$

This completes the proof of Theorem 3.13. $\square$

Chapter 4

Conclusion

Due to the relevance of stochastic models in applied science and environmental engineering, we studied the stochastic model of groundwater flow and pollution and stochastic diffusion model of single-phase flow through partially fissured medium.

In the first part of the research, we initiated the investigation of coupled stochastic diffusion-convection, reaction-diffusion and steady Stokes equations governing processes of groundwater contamination. The porous medium is modeled as a perforated domain and we made use of the powerful method of homogenization as our main tool of investigation coupled with some crucial compactness results of both analytic and probabilistic nature; in particular we successfully implemented Prokhorov and Skorokhod compactness procedures. We constructed the corresponding macroscopic homogenized problems using both Tartar's method of oscillating test functions and the formal asymptotic expansion method.

In the second part of the research, we investigated a double-porosity model for flow of single-phase fluid through a partially fissured medium. The medium is modeled as a domain consisting of periodic perforated domain and a system of fractures with a transmission condition at the interface of the sub-domains. We used Nguetseng's two-scale convergence, Minty's monotonicity method and Prokhorov-Skorokhod compactness process for the homogenization process.

To the best of our knowledge, our work is the first to systematically investigate process of groundwater contamination governed by stochastic partial differential equations in perforated porous medium and to use the more advanced tool of stochastic partial differential equations driven by random forces to study the random fluctuations on a flow through partially fissured media.

The novelty of the research is that it opens several avenues for the development of the homogenization theory for SPDEs for theoretical and practical problems in applied science. Here are some open problems:

1. The corrector result for a flow in a partially fissured medium modeled using SPDEs.
2. Homogenization of stochastic convection-diffusion equation with levy process.

Bibliography

[1] Acerbi E., Chiado Piat V., Dal Maso G. and Percivale D., An extension theorem from connected sets and homogenization in general periodic domain. Nonlinear Analysis: Theory, Methods and Applications, **18**(5), 481–496, (1992).

[2] Adler P.M., Thovert J-F. and Mourzenko V.V., Fractured porous media. oxford, (2012).

[3] Allaire G., Homogenization and two-scale convergence. SIAM J. Math. Anal., 23, 1482-1518 (1992).

[4] Allaire G., Homogenization of the stokes flow in a connected porous medium. Asymptotic Analysis, 2, no. 3, 203–222 (1989).

[5] Allaire G and Murat F, Homogenization of the Neumann problem with nonisolated holes. Asymptotic Analysis, **7**(2), 81–95, (1993).

[6] Allaire G. and Raphael A. L., Homogenization of a convection diffusion model with reaction in a porous medium. Mathematical problems in Mechanics, 344, no. 8, 523–528 (2007).

[7] Amaziane B., Goncharenko M. and Pankratov L., Homogenization of a convection-diffusion equation in perforated domains with a weak adsorption, Zeitschrift für angewandte Mathematik und Physik, 58, no. 4, 592–611 (2007).

[8] Amirat Y., Hamdache K., Ziani A., Some results on homogenization of convection-diffusion equations. Archive for Rational Mechanics and Analysis, 114, no. 2, 155–178 (1991).

[9] Arbogast T., Douglas Jr. J., Hornung U., Modeling of naturally fractured reservoirs by formal homogenization techniques. Frontiers in pure and applied mathematics, R. Dautray (eds), Elsevier, 1–19, (1991).

[10] Babuska I., Homogenization approach in engineering. Lecture notes in economics and mathematical systems, Springer-Verlag, 137–153, (1975).

[11] Bakhvalov N. and Panasenko G., Homogenisation: averaging processes in periodic media. Mathematical problems in the mechanics of composite materials. Kluwer Academic Publishers Group, Dordrecht, 1989.

[12] Barenblatt G.I., Zheltov I.P. and Kochina I.N. , Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks[strata]. Journal of Applied Mathematics and Mechanics **24**(5), 1286–1303 (1960).

[13] Bear J., Cheng A. D., modeling Groundwater Flow and Contaminant Transport 23, of Theory and Applications of Transport in Porous Media. Springer, (2010).

[14] Bear J., Dynamics of fluids in porous media. American Elsevier, New York, 1972.

[15] Bear J., Verruijt A., Modeling Groundwater Flow and Pollution. Springer Netherlands, 1987.

[16] Bear J., Tsang C-F. and De Marsily G., Flow and Contaminant Transport in Fractured Rock. Elsevier Science, 1993.

[17] Beliaev A. Yu, The homogenization of Stokes flows in random porous domains of general type. Asymptotic Analysis, **19**(2), 81–94, (1999).

[18] Beliaev A. Yu, Kozlov S.M., Darcy equation for random porous media. Comm. Pure Appl. Math., **49**(1), 1–34, (1996).

[19] Bensoussan A., Lions J. L. and Papanicolaou G., Asymptotic Analysis for Periodic Structures. North-Holland, Amsterdam, (1978).

[20] Bensoussan A., Homogenization of a class of stochastic partial differential equations. Composite media and homogenization theory (Trieste, 1990), 47—65, Progr. Nonlinear Differential Equations Appl., 5, Birkhuser Boston, Boston, MA, (1991).

[21] Bensoussan A., Some existence results for stochastic partial differential equations in: Stochastic partial differential equations and applications., Pitman Research Notes in Mathematics, Da Prato G. and Tubaro L. (eds), **268**. 37–53 (1992).

[22] Berlyand L., Goncharenko M. , The averaging of the diffusion equation in a porous medium with weak absorption, Journal of Soviet Mathematics, 52, no. 5, 3428–3435 (1990).

[23] Bessaih H., Efendiev Y. and Maris F., Homogenization of the evolution Stokes equation in a perforated domain with a stochastic Fourier boundary condition, Netw. Heterog. Media, 10, no. 2, 343–367 (2015).

[24] Bessaih H., Efendiev Y., Maris F., Homogenization of the evolution Stokes equation in a perforated domain with a stochastic Fourier boundary condition. Netw. Heterog. Media 10 (2015), no. 2, 343–367.

[25] Billingsley P., Convergence of Probability Measures. Wiley Series in Probability and Statistics. Wiley, 2nd edition (1999).

[26] Bogolyubov N.N and Mitropolskii Y.A., Asymptotic methods in the theory of nonlinear oscillations. Gordon and Breach science, (1961).

[27] Bourgeat A. , Mikelić A. and Wright S., Stochastic two-scale convergence in the mean and applications, J. Reine Angew. Math. 456, 19–51 (1994).

[28] Bourgeat A., Piatnitski A. L., Averaging of a singular random source term in a diffusion convection equation, SIAM J. Math. Anal.,no. 6, 2626—2651 42 (2010).

[29] Browder F.E., Nonlinear operators and nonlinear equations of evolution in Banach spaces. Proceedings of Symposia in Pure Mathematics **18**, American Mathematical Society, (1976)

[30] Chiado Piat V. and Defranceschi A., Homogenization of monotone operators. Nonlinear Analysis: Theory, Methods and Applications, **14**(9), 717–732, (1990).

[31] Christiansen S.H., A Div-Curl Lemma for Edge Elements. SIAM Journal on Numerical Analysis, **43**(1), 116-126, (2005).

[32] Cioranescu D. and Donato P., An introduction to homogenization. Oxford Lecture Series in Mathematics and its Applications, 17. The Clarendon Press, Oxford University Press, New York, 1999.

[33] Cioranescu D., Donato P., Ene H., Homogenization of the stokes problem with non-homogeneous slip boundary conditions, Mathematical Methods in the Applied Sciences, 19, no.11, 857–881 (1996).

[34] Cioranescu D., Damlamian A. and Grisco G., The Periodic Unfolding Method in Homogenization, SIAM Journal on Mathematical Analysis, **40**(4), 1585-1620,(2008).

[35] Cioranescu D, Damlamian A. and Grisco G., The Periodic Unfolding Method in Homogenization, Comptes Rendus de L'Academie Sciences, **335**(1), 99-104, (2002).

[36] Cioranescu D and Saint Jean Paulin J., Homogenization in open sets with holes. Journal of Mathematical Analysis and Applications, **71**(2), 590–607, (1979).

[37] Clark G.W, Derivation of microstructure models of fluid flow by homogenization, Journal of Mathematical Analysis **226**(2), 364–376, (1998).

[38] Clark G.W. and Showalter R.E., Two-scale convergence of a model for flow in a partially fissured medium. Electronic Journal of Differential Equations, **1999**(2), 1–20, (1999).

[39] Coffeirld D.J. and Spagnuolo A.M., A model for single phase flow in layered porous media. Electronic Journal of Differential Equations, **2010**(94), 1–19, (2010).

[40] Da Prato G., Introduction to stochastic analysis and Malliavin calculus. Lecture notes **7**, Edizioni della Normale, (2008).

[41] Dal Maso G. and Defranceschi A., Correctors for the Homogenization of monotone operators, Differential Integral Equations, **3**(6), 1151–1166, (1990).

[42] Dal Maso G. and Modica L., Nonlinear stochastic homogenization and ergodic theory, J. Rein. Ang. Math. B.368, 27-42, (1986).

[43] Dal Maso G., An introduction to Γ-convergence. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhauser Boston, Inc., Boston, MA, 1993.

[44] De Giorgi, Sulla convergenca di alcune successioni di integrali del tipo dellarea, Rendi Conti di Mat., **8**, 277–294, (1975).

[45] De Giorgi E, G-operators and Γ-convergence, Proceedings of the International Congress of Mathematicians, 1175–1191, (1984).

[46] Defranceschi A., An introduction to homogenization and G-convergence, Lecture notes, School on Homogenization, (1993).

[47] Dietrich P., Helmig R., Sauter M., Hötzl H., Köngeter J., Teutsch G. (Eds.), Flow and transport in fractured porous media, Springer, (2005).

[48] Diop M. A. and Pardoux E., Averaging of a parabolic partial differential equation with random evolution. Seminar on Stochastic Analysis, Random Fields and Applications IV, 111–128, Progr. Probab., 58, Birkhauser, Basel, (2004).

[49] Douglas J., Peszyńska M. and Showalter R.E., Single phase flow in partially fissured media. Transport in porous media **28** 285–306, (1997).

[50] Evans L.C., Partial Differential Equations, AMS, Providence, (1998).

[51] Evans L.C., Weak convergence methods for nonlinear partial differential equations. CBMS Regional Conference Series in Mathematics, 74. American Mathematical Society, (1990).

[52] Friedman A., Stochastic differential equations and application. Dover publication, (2006).

[53] Gorb Y., Maris F., Vernescu B., Homogenization for rigid suspensions with random velocity-dependent interfacial forces. J. Math. Anal. Appl. 420 (2014), no. 1, 632–668.

[54] Ganji D.D. and Kachapi S.H.H, Applications of nonlinear systems in nanomechanics and nanofluids, Analytical methods and Applications, Elsevier, (2015).

[55] Gérard P., Microlocal defects measures, Communications in Partial Differential Equations, 1761-1794, (1991).

[56] Goncharenko M. and . Khil'kova L., Homogenized model of diffusion in porous media with nonlinear absorption on the boundary, Ukrainian Mathematical Journal, 67, no. 9, 1349–1366 (2016).

[57] Hornung U. and Jäger W., Diffusion, convection, adsorption, and reaction of chemicals in porous media, Journal of Differential Equations, 92(2), 2199–225 (1991).

[58] Hornung U., Homogenization and porous media, Springer-Verlag, Berlin, New York, (1997)

[59] Hornung U., Jager W., and Mikelić A., Reactive transport through an array of cells with semi-permeable membranes, ESAIM: Mathematical modeling and Numerical Analysis, 28, no. 1 59–94 (1994).

[60] Ichihara N., Homogenization problem for stochastic partial differential equations of zakai type, Stochastics and Stochastic reports, 76, no. 3, 243–266 (2004).

[61] Ichihara N., Homogenization of stochastic partial differential equations derived from nonlinear filtering with feedback, Journal of the mathematical society of Japan, **57**(2), 593–603, (2005).

[62] Jikov V.V., and Kozlov, S.M. and Oleinik, O. A., Homogenization of differential operators and integral functionals, Springer-Verlag, Berlin, (1994).

[63] Kalamkarov L.A., Andrianov I.V., and Danishevs'kyy V.V., Asymptotic Homogenization of Composite Materials and Structures, Applied mechanics review, **62**(3), (2009).

[64] Karatzas I. and Shreve S.E, Brownian Motion and Stochastic Calculus, Springer-Verlag, (1987).

[65] Khruslov E.Y., Asymptotic behaviour of solutions of the second boundary value problems under fragmentation of the boundary of the domain, Mathematics of the USSR-Sbornik, 35, no. 2 266–282 (1979).

[66] Kleptsyna M. and Pyatnitskii A., Homogenization of a random non-stationary convection-diffusion problem, Russian Mathematical Surveys, 57, no. 4, 729–751 (2002).

[67] Kozlov S., Averaging differential operators with almost periodic rapidly oscillating coefficients, Mathematics of the USSR-Sbornik, 35, no. 4, 481–498 (1979).

[68] Kozlov S.M., The averaging of random operators,Sbornik: Mathematics **37**(2), 167–180, (1980).

[69] Krylov N.V. and Rozovskii B.L., Stochastic evolution equations. Journal of Soviet Mathematics, **16**(4), 1233-1277, (1981).

[70] Kumar K., Neuss-Radu M., and Pop I., Homogenization of a pore scale model for precipitation and dissolution in porous media, IMA Journal of Applied Mathematics, 81, no. 5, 877–897 (2016).

[71] Larsen E.W and Keller J.B, Asymptotic solution of neutron transport problems for small mean free paths, Journal of mathematical physics **15**(1), 75–81, (1974).

[72] Larsen E.W, Neutron transport and diffusion in inhomogeneous media I, Journal of Mathematical Physics **16**(7), 1421–1427, (1975).

[73] Larsen E.W, Neutron transport and diffusion in inhomogeneous media II, Nuclear Science and Engineering **60**(4), 357–368, (1976).

[74] Lifshits I.M. and Rozentsveig L.N., On the theory of elastic properties of polycrystals. ZhETF, **16**, (1946).

[75] Lions J.L., Quelques methodes de re solution des problmes aux limites nonlinaires. Gauthier-Villars, Paris, 1969.

[76] Lipton R. and Avellaneda M., Darcy's law for slow viscous flow past a stationary array of bubbles, Proceedings of the Royal Society of Edinburgh, 114A, 71–79 (1990).

[77] Mauri R., Dispersion, convection and reaction in porous media, Physics of Fluids, 3, no. 5, 743–756 (1991).

[78] Marchenko V. A. and Khruslov E. Ya.: Homogenization of partial differential equations. Progress in Mathematical Physics, 46. Birkhauser, Boston, 2006.

[79] Marchenko V.A and Khruslov E.Y, Boundary value problem with fine-grained boundary, Matematicheskii Sbornik, **65**(107), 458–472, (1964).

[80] Maxwell J.C., A treatise on electricity and magnetism, Clarendon Press, (1881).

[81] Mikelić A. and Aganović I., Homogenization in a porous media under a non-homogeneous boundary condition. Bollettino Della Unione Matematica Italia, 1, no. 2, 171–180 (1987).

[82] Milton G.W., The Theory of Composites. Cambridge University Press, (2002).

[83] Minty G.J., Monotone (nonlinear) operators in Hilbert space, Duke Mathematical journal, **29**(3), 341–346,(1962).

[84] Minty G.J, On a "Monotonicity" method for the solution of nonlinear equations in Banach spaces. Proceedings of the National Academy of the United States of America, **50**(6), 1038–1041, (1963).

[85] Mohammed M., Sango M., Homogenization of linear hyperbolic stochastic partial differential equation with rapidly oscillating coefficients: The two scale convergence method, Asymptotic Analysis 91(3-4), 341-371, (2015).

[86] Mohammed M., Homogenization of nonlinear hyperbolic stochastic equation via Tartar's method, J. Hyper. Differential Equations, 14:02, 323-340 (2017).

[87] Murat F., H-Convergence, Séminaire d'analyse fonctionnelle et numérique de l'Universiteéd'Alger, (1977).

[88] Murat F., Compacité par compensation, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, **5**(3), 489–507, (1978).

[89] Murat F. and Tartar L., H-Convergence, Topics in the Mathematical modeling of composite material, **31**, 21–43, (1997).

[90] Murat F., Compacité par Compensation: Condition nécessaire et suffisante de continuité faible sous une hypothése de rang constant, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, **8**, 69–102, (1981).

[91] Nguetseng G., A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal, 20, no. 3, 608–623 (1989).

[92] Nguetseng G., Asymptotic analysis of a stiff variational problem arising in mechanics, SIAM Journal on Mathematical analysis, **21**(6), 1394-1414, (1989).

[93] Nguetseng G., Showalter R.E., Woukeng J-L, Diffusion of a single phase fluid through a general deterministic partially fissured medium. Electronic Journal of Differential Equations, **2014**(164), 1–26, (2014).

[94] Nguetseng G., Homogenization structures and applications I, Zeitschrift fuür Analysis und ihre Anwendungen- Journal for Analysis and its Applications, **22**(1), 73-107 (2003).

[95] Nguetseng G., Homogenization structures and applications II, Zeitschrift für Analysis und ihre Anwendungen- Journal for Analysis and its Applications, **23**(3), 483-508, (2004).

[96] Oleinik O.A., Shamaev A.S. and Yosifian G.A, Mathematical Problems in Elasticity and Homogenization, Elsevier, Amsterdam, (1992).

[97] Ondrejat M. , Uniqueness for stochastic evolution equation in banach spaces, Dissertationes Mathematicae, 426, 1–63 (2004).

[98] Panasenko G., Multi-scale modeling for Structures and Composites, Springer, (2005).

[99] Pankov A., G-Convergence and Homogenization of Nonlinear Partial Differential Operators, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 422, 1997.

[100] Papanicolaou G.C. and Varadhan S.R.S., Boundary value problems with rapidly oscillating random coefficients in Random fields, Colloq. Math. Soc. Janos Bolyai, 27 (Esztergom, 1979), 835–873. North-Holland, Amsterdam-New York (1981).

[101] Pardoux E., Equations aux derivees Partielles Stochastiques Non Lineaires Monotones. These, Universite Paris VI (1975).

[102] Pardoux E. and A. Piatnitski, Homogenization of a nonlinear random parabolic partial differential equation. Stochastic Processes and their Applications, 104, no. 1, 1–27 (2003).

[103] Pavliotis A.G. and Stuart A.M., Multiscale Methods: Averaging and Homogenization. Texts in Applied Mathematics, Springer, **53**, (2008).

[104] Pobedrya B.E, Principles of computational mechanics of composites. Mechanics of Composite materials, **32**(6), 504-515, (1996).

[105] Prokhorov Yu. V., Convergence of random processes and limit theorems in probability theory, Teor. Veroyatnost. i Primenen., 1, 177–238 (1956).

[106] Rayleigh J.W, On the influence of obstacles arranged in rectangular order upon the properties of a medium, Philosophical magazine, 481-491, **34**(211) series 5, (1892).

[107] Razafimandimby P.A., Sango M. and Woukeng J-L, Homogenization of a stochastic nonlinear reaction-diffusion equation with a large reaction term: the almost periodic framework, Journal of Mathematical Analysis and Applications, **394**(1), 186–212, (2012).

[108] Razafimandimby P. and Woukeng J-L, Homogenization of Nonlinear stochastic partial differential equations in a general Ergodic environment, Stochastic Analysis and Applications, **31**(5), 755–784, (2013).

[109] Renardy M. and Rogers R.C., An introduction to partial differential equations, Text in Applied Mathematics 13 (2nd ed), Springer-Verlag, (2004).

[110] Reuss A. Berechnung der fliehgrenze von mischkristallen auf grund der plastizitatsbedingung für eimkristalle. Z. Angew. Math. und Mech., **9**, (1929).

[111] Roubíček, Nonlinear partial differential equations with applications, International Series of Numerical mathematics **153**, Birkhauser, (2013).

[112] Rozovskii B., Stochastic Evolution Systems: Linear theory and applications to nonlinear filtering. Springer, Netherlands, (1990).

[113] Sahimi M., FLow and transport in porous media and fractured rock: from classical methods to modern approaches, Wiley, (2011).

[114] Sanchez-Palencia E., Non-Homogeneous Media and Vibration Theory (Lecture Notes in Physics). Springer 1980.

[115] Sango M., Asymptotic behaviour of a stochastic evolution problem in a varying domain, Stochastic Analysis and Applications, 20, no. 6, 1331–1358 (2002).

[116] Sango M., Splitting-up scheme for nonlinear stochastic hyperbolic equations, Forum Mathematicum, 25, no. 5, 931–965 (2013).

[117] Sango M., Homogenization of stochastic semilinear parabolic equations with non-Lipschitz forcings in domains with fine grained boundaries, Communications in mathematical sciences, 12(2):345–382, (2014).

[118] Shamaev A.S., Averaging of solutions and eigenvalues for boundary value problems for elliptic equations in perforated domains, Russian Mathematical survey, **37**(2), 253–254, (1982).

[119] Shiryayev A.N, Probability, Springer-Verlag, (95) Graduate texts in Mathematics, (1984).

[120] Showalter R.E. and Walkington N.J., Micro-structure models of diffusion in fissured media. Journal of Mathematical Analysis and Applications, **155**(1), 1–20, (1991).

[121] Simon J., Compact sets in the space $L^p(0,T;B)$, Annali di Mathematica Pura ed Applicata, 146, no. IV, 65–96 (1987).

[122] Skorohod A.V., Limit theorems for stochastic processes, Teor. Veroyatnost. i Primenen. 1, 289–319 (1956).

[123] Spagnolo S., Sul limite delle soluzioni di problemi di cauchy relativi all'equazione del calore, Annali della Scuola Normale Superiore di Pisa **22**(3), 657–699, (1967).

[124] Spagnolo S. and De Giorgi E., Sulla convergenza degli integrali dell"energia per operatori ellittici del secondo ordine, Bollettino della Unione Matematica Italiana 4, **8**, 391–411, (1973).

[125] Spagnolo S., Convergence in energy for elliptic operators, Proceedings of the 3rd symposium on the numerical solution of partial differential eqautions, Academic Press, 469-498, (1976).

[126] Svanstedt N., Multiscale stochastic homogenization of monotone operators, Networks and Heterogeneous Media, **2**(1), 181–192, (2007).

[127] Svanstedt N., Multiscale stochastic homogenization of convection-diffusion equations, Applications of Mathematics, 53, no. 2, 143–155 (2008).

[128] Tartar L. Cours peccot au collége de France Unpublished (1977).

[129] Tartar L., Non-homogeneous Media and Vibration Theory 127, Lecture Notes in physics, chapter Incompressible fluid flow in a porous medium-convergence of the homogenization process, 368–377, Springer-Verlag, Berlin (1980).

[130] Tartar L., H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations, Proceedings of the Royal Society of Edinburgh, 193–230, 115A (1990).

[131] Tartar L., Compensated compactness and applications to partial differential equations, Nonlinear Analysis and Mechanics Heriot-Watt Symposium IV, Pitman **39**, 136–212, (1979).

[132] Tartar L., The General Theory of Homogenization, Springer, (2009).

[133] Tartar L., Multi-scale H-measures, Discrete and Continuous Dynamical Systems - Series S, **8**(1), 77–90, (2015).

[134] Taylor H. and Karlin S., Introduction to stochastic modeling. Academic press (1998).

[135] Temam R., Navier-Stokes Equations: Theory and numerical analysis 2, Studies in mathematics and its applications. North-Holland Pub. Co, Amsterdam (1977).

[136] Voight W., Lehrbuch der kristallpfysik. Berlin, Teubner, (1928).

[137] Wang W, Cao D. and Duan J., Effective macroscopic dynamics of stochastic differential equations in perforated domains, SIAM Journal of Mathematical Analysis, **38**, 1508–1527, (2007).

[138] Warren J.E. and Root P.J., The behavior of naturally fractured reservoirs. Society of Petroleum Engineers Journal **3**(3), 245–255, (1963).

[139] Wellander N., The two-scale Fourier transform approach to homogenization; periodic homogenization in Fourier space, Asymptotic Analysis, **62**, 1-40, (2009).

[140] Wright S., On a diffusion of a single-phase slightly compressible fluid through a randomly fissured medium. Mathematical Methods in the Applied Sciences, **24**, 805–825, (2001).

[141] Wright S., On the steady-state flow of an incompressible fluid through a randomly perforated porous medium. Journal of Differential equations **146**(DE983436), 261-286, (1998).

[142] Zhang D., Stochastic Methods for Flow in Porous Media: coping with uncertainties. Academic Press (2002).

[143] Zhang, Z., Monotonicity methods in PDE, Charleton University Mathematics Journal, **1**(1), 1–10, (2010).

[144] Zhikov V.V, Kozlov S.M and Oleĭnik O.A., Averaging and G-convergence of differential operators, Russian Mathematical Survey, **34**, 65–133, (1979).