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Fuzzy $\mathfrak{b}$-Metric Spaces: Fixed Point Results for ψ-Contraction Correspondences and Their Application

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Abstract: In this paper we introduce the concepts of ψ-contraction and monotone ψ-contraction correspondence in "fuzzy $\mathfrak{b}$-metric spaces" and obtain fixed point results for these contractive mappings. The obtained results generalize some existing ones in fuzzy metric spaces and "fuzzy $\mathfrak{b}$-metric spaces". Further we address an open problem in $\mathfrak{b}$-metric and "fuzzy $\mathfrak{b}$-metric spaces". To elaborate the results obtained herein we provide an example that shows the usability of the obtained results.

Keywords: fixed point; correspondence; fuzzy $\mathfrak{b}$-metric space; $\mathfrak{b}$-metric space

MSC: 47H09; 47H10; 54H25

1. Introduction

Several kinds of nonlinear problems arising in various branches of the sciences can be formulated as a "fixed point problem" mathematically $\mathfrak{f}\mathfrak{x} = \mathfrak{x}$ (an operator equation) where $\mathfrak{f}$ is some nonlinear operator defined on some topological structure. The Banach [1] contraction principle is a significant tool for solving fixed point problems. The simple and constructive nature of its proof has attracted the attention of several researchers around the globe to generalize this famous tool. There are several generalizations; among them, one is to modify the underlying space. In this regard, a framework of "probabilistic metric spaces" is a matter of great interest for scientists and mathematicians (for details, see [2–4]). Kramosil and Michalek [5] defined the "fuzzy metric space". In [6], George and Veeramani modified the concept of "fuzzy metric spaces" using the continuous $\mathfrak{t}$-norm. This modification is the generalization of the "probabilistic metric space" to the fuzzy situation. Afterwards the "fuzzy $\mathfrak{b}$-metric space" was defined in [7] which generalizes the "fuzzy metric space" and "$\mathfrak{b}$-metric space".

The fixed point results in "fuzzy metric space" have deep roots (for details, see [8]). This work has been appreciated by researchers (see [9,10]). This work was extended by several researchers in various ways (compare with [11–21]). Among one of them, in 1969, Nadler proposed Banach's contraction principle for correspondence in Hausdorff metric spaces (see [22]). Various extensions of this work were subsequently proposed by several authors (for details, see [23]). In 1993, Czerwik [24,25] proposed the first Banach fixed point theorem for both single and multivalued mappings in "$\mathfrak{b}$-metric spaces", introduced by Bourbaki and Bakhtin [26,27]. Afterwards, this concept was extended for some particular types of contractions in the context of "$\mathfrak{b}$-metric spaces" (see [28]). In this direction, many researchers studied and extended various well known fixed point results for several types of contractive mappings in the framework of "$\mathfrak{b}$-metric spaces" [29–31].

In general, fixed point theory remained successful in challenging and solving various problems and has contributed significantly to many real-world problems. However, various strong fixed point theorems are proven under strong assumptions. Particularly, in "fuzzy metric spaces", some of these assumptions can lead to some induced norms. Some assumptions do not hold in general or can lead to reformulations as a particular problem in normed vector spaces. The recent trend of research has been dedicated to studying the fundamentals of fixed point theorems and relaxing their conditions by replacing these strong assumptions with weaker ones.

The aim of the work presented in this paper is to provide some fixed point results in "fuzzy $\mathfrak{b}$-metric spaces" and to improve their conditions and assumptions by addressing the open questions and challenges outlined in the literature by identifying the ties between "fuzzy $\mathfrak{b}$-metric spaces" and "pseudo $\mathfrak{b}$-metric spaces".

This paper starts with a brief introduction to "$\mathfrak{b}$-metric spaces" and "fuzzy $\mathfrak{b}$-metric spaces" along with the required concepts. Afterwards we describe the relation between these two particular spaces. Along with these details, some basic techniques and ways of improving some current fixed point result are also discussed. Finally, an application of Banach's contraction to linear equations is provided.

2. Background and Relevant Literature

This section will serve as an introduction to some fundamental concepts related to "$\mathfrak{b}$-metric spaces" and "fuzzy $\mathfrak{b}$-metric spaces". Further, some basic definitions and known results are discussed which will be needed in the sequel.

Definition 1. *[32] Let $\mathcal{X}$ be a nonempty set, define a real valued function $\mathfrak{d} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that for a given real number $\mathfrak{s} \geq 1$ satisfies the conditions:*

1. $\mathfrak{d}(\mathfrak{x}, \mathfrak{y}) = 0$ *if and only if* $\mathfrak{x} = \mathfrak{y}$,
2. $\mathfrak{d}(\mathfrak{x}, \mathfrak{y}) = \mathfrak{d}(\mathfrak{y}, \mathfrak{x})$,
3. $\mathfrak{d}(\mathfrak{x}, \mathfrak{z}) \leq \mathfrak{s}[\mathfrak{d}(\mathfrak{x}, \mathfrak{y}) + \mathfrak{d}(\mathfrak{y}, \mathfrak{z})]$, *for all* $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{X}$,

the pair $(\mathcal{X}, \mathfrak{d})$ is called a "$\mathfrak{b}$-metric space".

It is important to discuss that every "$\mathfrak{b}$-metric space" is not necessarily a "metric space" [32]. With $\mathfrak{s} = 1$, every "$\mathfrak{b}$-metric space" is a "metric space". If we replace Condition 1 with the following:
• If $\mathfrak{x} = \mathfrak{y}$ implies $\mathfrak{d}(\mathfrak{x}, \mathfrak{y}) = 0$,
then $(\mathcal{X}, \mathfrak{d})$ is called a "pseudo $\mathfrak{b}$-metric space". Moreover, it has been shown that several metric fixed point theorems can be extended to "$\mathfrak{b}$-metric spaces" (see [33]). It is important to mention that the "$\mathfrak{b}$-metric" is not continuous (see [34]). The notion of the "$\mathfrak{b}$-metric space" was introduced for the generalization of the fixed point theorem for single valued mappings and correspondences (see [24,25]).

Definition 2. *Let $\mathcal{X}$ be a nonempty set and $\mathfrak{s} \geq 1$ be a real number. A fuzzy subset $\mathcal{M}$ of $\mathcal{M} : \mathcal{X} \times \mathcal{X} \times [0, +\infty) \to [0, 1]$ is called a "fuzzy $\mathfrak{b}$-metric" on $\mathcal{X}$ if the following conditions are satisfied for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{X}$ and $c \in \mathbb{R}$.*

$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = 0$, *for all non-positive real numbers* $\mathfrak{t}$,
$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = 1$, *for all* $\mathfrak{t} \in \mathbb{R}^+$ *if and only if* $\mathfrak{x} = \mathfrak{y}$,
$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \mathcal{M}(\mathfrak{y}, \mathfrak{x}, \mathfrak{t})$,
$\mathcal{M}(\mathfrak{x}, \mathfrak{z}, \mathfrak{s}(\mathfrak{t} + h)) \geq \min\{\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}), \mathcal{M}(\mathfrak{y}, \mathfrak{z}, h)\}$, *for all* $\mathfrak{s} \geq 1$,
$\mathcal{M}(\mathfrak{c}\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \frac{\mathfrak{t}}{|\mathfrak{c}|})$, *for* $\mathfrak{c} \neq 0$,
$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $\sup_{\mathfrak{t}}\{\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})\} = 1$.

The pair $(\mathcal{X}, \mathcal{M})$ is said to be a "fuzzy $\mathfrak{b}$-metric space".

It is important to discuss that for $\mathfrak{s} = 1$, every "fuzzy $\mathfrak{b}$-metric space" will reduced to a "fuzzy metric space". The following example explains the concept of the "fuzzy $\mathfrak{b}$-metric space".

Example 1. *Suppose that $(\mathcal{X}, \mathfrak{d})$ is a "$\mathfrak{b}$-metric space". Define*

$$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \begin{cases} \frac{\mathfrak{t}}{\mathfrak{t}+\mathfrak{d}^{\mathfrak{r}}(\mathfrak{x},\mathfrak{y})} & \mathfrak{t} > 0, \\ 0 & \mathfrak{t} \leq 0. \end{cases}$$

Then, $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})$ is a "fuzzy $\mathfrak{b}$-metric space" for all $\mathfrak{r} \in \mathbb{R}^+$.

Definition 3. *Let $(\mathcal{X}, \mathcal{M})$ be a "fuzzy $\mathfrak{b}$-metric space". We define the following subset of $\mathcal{X}$, as:*

$$B_{\mathfrak{r}}(\mathfrak{x}_0, \mathfrak{t}_0) = \{\mathfrak{x} \in \mathcal{X} : \mathcal{M}(\mathfrak{x}, \mathfrak{x}_0, \mathfrak{t}_0) > \mathfrak{r}\},$$

where $\mathfrak{x}_0 \in \mathcal{X}$, $\mathfrak{r} \in (0,1)$ and $\mathfrak{t}_0 > 0$.

Let $(\mathcal{X}, \mathcal{M})$ be a "fuzzy $\mathfrak{b}$-metric space" and define an open set $O \subseteq \mathcal{X}$ as follows. An element $\mathfrak{x} \in O$ if and only if there exist $\mathfrak{r} \in (0,1)$ and $\mathfrak{t}_0 > 0$ such that $B_{\mathfrak{r}}(\mathfrak{x}, \mathfrak{t}_0) \subseteq O$. Let $\tau_{\mathcal{M}}$ be a topology induced by $\mathcal{M}$ on $\mathcal{X}$ which contains all open sets (for details, see [35,36]). Therefore, with $\tau_{\mathcal{M}}$, some topological notions such that the convergent sequence, Cauchy sequence, closed set, complete set and closure of a set are meaningful. Let $\mathcal{X}$ be a "fuzzy metric space" and suppose that $\mathfrak{d}_{\mathfrak{r}} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ for each $\mathfrak{r} \in (0,1)$ is defined as:

$$\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}, \mathfrak{y}) = \sup\{\mathfrak{t} : \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) \leq \mathfrak{r}\}.$$

Then, $\mathfrak{d}_{\mathfrak{r}}$ is known as a pseudo metric. One can verify that if $\mathcal{X}$ is a fuzzy $\mathfrak{b}$-metric then $\mathfrak{d}_{\mathfrak{r}}$ is a pseudo $\mathfrak{b}$-metric (for details, see [37]). The family of the pseudo $\mathfrak{b}$-metric $\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}, \mathfrak{y})$ generates a topology on $\mathcal{X}$ which is the same as the topology generated by $\tau_{\mathcal{M}}$. Therefore, $(\mathcal{X}, \mathcal{M})$ is complete if and only if $(\mathcal{X}, \mathfrak{d}_{\mathfrak{r}})$ is complete. It is easy to show that for $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{w} \in \mathcal{X}$ and $\mathfrak{q} \in [0, \infty)$, $\mathcal{M}(\mathfrak{z}, \mathfrak{w}, \mathfrak{q}\mathfrak{t}) \geq \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})$ give that $\mathfrak{d}_{\mathfrak{r}}(\mathfrak{z}, \mathfrak{w}) \leq \mathfrak{q}\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}, \mathfrak{y})$, for each $\mathfrak{r} \in (0,1)$. If we define such pseudo metrics in a "fuzzy $\mathfrak{b}$-metric space" then it can lead to a smooth proof for many fixed point theorems in "fuzzy $\mathfrak{b}$-metric spaces".

In the following lemma, some equivalences are provided as:

Lemma 1. *Let $(\mathcal{X}, \mathcal{M})$ be a "fuzzy $\mathfrak{b}$-metric space".*

- *A sequence $\{\mathfrak{x}_n\} \in \mathcal{X}$ is convergent and converges to $\mathfrak{x} \in \mathcal{X}$ if $\lim_{n} \mathcal{M}(\mathfrak{x}_n, \mathfrak{x}, \mathfrak{t}) = 1$ for all $\mathfrak{t} > 0$ and denoted as $\mathfrak{x}_n \to \mathfrak{x}$.*
- *If $\lim_{n,m} \mathcal{M}(\mathfrak{x}_n, \mathfrak{x}_m, \mathfrak{t}) = 1$ for all sufficiently large m, n and for any $\mathfrak{t} > 0$ then $\mathfrak{x}_n$ is called a Cauchy sequence in $\mathcal{X}$.*
- *If every Cauchy sequence is convergent in $\mathcal{X}$ then $\mathcal{X}$ is called a "complete fuzzy $\mathfrak{b}$-metric space".*
- *A subset $\mathfrak{C}$ of $\mathcal{X}$ is a complete space if and only if it is complete with induced pseudo $\mathfrak{b}$-metric $\mathfrak{d}_{\mathfrak{r}}$ for every $\mathfrak{r} \in (0,1)$.*
- *A subset $\mathfrak{C} \subset \mathcal{X}$ is open if for every $\mathfrak{x} \in \mathfrak{C}$ there exist $\mathfrak{t}, \mathfrak{r} > 0$ such that $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) > \mathfrak{r}$ implies $\mathfrak{y} \in \mathfrak{C}$.*
- *A subset $\mathfrak{C} \subset \mathcal{X}$ is closed if it contains all of its limit points.*
- *The closure of $\mathfrak{C}$ denoted by $\overline{\mathfrak{C}}$ is defined as the set of all points of $\mathcal{X}$ that are the limit points of some sequence in $\mathfrak{C}$.*

The following theorem is an equivalent to the "Banach fixed point theorem" in "fuzzy metric space".

Theorem 1. *[8] Let $(\mathcal{X}, \mathcal{M})$ be a "complete fuzzy metric space" and $\mathfrak{T} : \mathcal{X} \to \mathcal{X}$. If*

$$\mathcal{M}(\mathfrak{T}\mathfrak{x}, \mathfrak{T}\mathfrak{y}, \mathfrak{t}) \geq \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \frac{\mathfrak{t}}{\mathfrak{k}}),$$

for $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$, $\mathfrak{k} \in (0,1)$ and $\mathfrak{t} \in \mathbb{R}$. Then $\mathfrak{T}$ has a fixed point.

Theorem 2. *[37] Suppose $\mathcal{X}$ is a "complete fuzzy metric space", $\mathfrak{T} : \mathcal{X} \to \mathcal{X}$ is a single valued mapping and for every $\mathfrak{d}_{\mathfrak{r}}$ there exists a constant $\mathfrak{k}_{\mathfrak{r}}$ with $0 < \mathfrak{k}_{\mathfrak{r}} < 1$ such that $\mathfrak{d}_{\mathfrak{r}}(\mathfrak{T}(\mathfrak{x}), \mathfrak{T}(\mathfrak{y})) \leq \mathfrak{k}_{\mathfrak{r}}\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}, \mathfrak{y})$ for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$. Then there exists a unique point $\mathfrak{z} \in \mathcal{X}$ such that $\mathfrak{T}(\mathfrak{z}) = \mathfrak{z}$.*

By a correspondence $\mathfrak{f}$ on a set $\mathcal{X}$ we mean a relation that assigns to each $\mathfrak{x}$ in $\mathcal{X}$ a nonempty subset of $\mathcal{X}$. For a correspondence $\mathfrak{f}$ an element $\mathfrak{x} \in \mathcal{X}$ is said to be a fixed point if $\mathfrak{x} \in \mathfrak{f}(\mathfrak{x})$. It is worthwhile mentioning that it is not necessary for the fixed point of a correspondence to be unique (see Example 3). Define

$$\mathcal{M}(\mathfrak{a}, \mathfrak{f}(\mathfrak{b}), \mathfrak{t}_0) = \sup_{\mathfrak{t} < \mathfrak{t}_0} \sup_{\mathfrak{y} \in \mathfrak{f}(\mathfrak{b})} \mathcal{M}(\mathfrak{a}, \mathfrak{y}, \mathfrak{t}).$$

In this paper we define the ψ-contractive and monotone ψ-contractive correspondence and prove some results for the existence of fixed points for these contractive conditions in "fuzzy $\mathfrak{b}$-metric spaces", where $\psi \in \Psi$ and Ψ consists of all the functions $\psi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ being continuous, nondecreasing and $\psi(1) = 1$. It is important to mention that several researchers have obtained fixed points of correspondence satisfying the contractive conditions via the Hausdorff distance [38–43]. We improve Theorem 1 in a short and comprehensive way and obtain the result without using the Hausdorff distance. Further we answer an open problem related to the "Banach fixed point theorem" in "$\mathfrak{b}$-metric space".

3. Main Results

In the sequel, it is assumed that $(\mathcal{X}, \mathcal{M})$ is a "complete fuzzy $\mathfrak{b}$-metric space" with some $\mathfrak{s} > 1$ and $\mathfrak{f}$ is a closed correspondence i.e. for every $\mathfrak{y}_n \in \mathcal{X}$ such that $\mathfrak{y}_n \in \mathfrak{f}(\mathfrak{x}_n)$, for all $\mathfrak{x}_n \in \mathcal{X}$ then the following implication holds:

$$\mathfrak{x}_n \to \mathfrak{x},\ \mathfrak{y}_n \to \mathfrak{y} \quad \text{implies} \quad \mathfrak{y} \in \mathfrak{f}(\mathfrak{x}).$$

The following lemma is a handy tool that will be used in the sequel.

Lemma 2. *[38] A sequence $\{\mathfrak{x}_n\}$ in a "$\mathfrak{b}$-metric space" $(\mathcal{X}, \mathfrak{d})$ is a $\mathfrak{b}$-Cauchy sequence if there exists $\mathfrak{k} \in [0,1)$ such that:*

$$\mathfrak{d}(\mathfrak{x}_n, \mathfrak{x}_{n+1}) \leq \mathfrak{k}\mathfrak{d}(\mathfrak{x}_{n-1}, \mathfrak{x}_n),$$

for every $n \in \mathbb{N}$.

It is also verified that Lemma 2 holds for "pseudo $\mathfrak{b}$-metric spaces" as well.

Definition 4. *Let $\mathfrak{C}$ be a nonempty subset of $(\mathcal{X}, \mathcal{M})$. A correspondence $\mathfrak{f} : \mathfrak{C} \rightsquigarrow \mathcal{X}$ is said to be an $\epsilon - \psi$-contraction ($\epsilon.\psi$. $\mathfrak{C}$) if $\psi \in \Psi$, $\epsilon \geq 1$, $\mathfrak{L} > 0$ and for every $\mathfrak{y} \in \mathfrak{C}$ there exists $\mathfrak{w} \in \mathfrak{f}(\mathfrak{y})$ such that:*

$$\psi(\mathcal{M}(\mathfrak{z}, \mathfrak{w}, \frac{\mathfrak{t}}{\mathfrak{s}^{\epsilon}})) \geq \min\{\psi(S(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})),\ \psi(I(\mathfrak{x}, \mathfrak{y}, \frac{\mathfrak{t}}{\mathfrak{L}}))\}, \tag{1}$$

for every $\mathfrak{x} \in \mathfrak{C}$, $\mathfrak{z} \in \mathfrak{f}(\mathfrak{x})$ where

$$\begin{aligned} S(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) \quad = \quad & \min\{\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}), \mathcal{M}(\mathfrak{x}, \mathfrak{f}(\mathfrak{x}), \mathfrak{t}), \frac{\mathcal{M}(\mathfrak{y}, \mathfrak{f}(\mathfrak{y}), \mathfrak{t})\mathcal{M}(\mathfrak{x}, \mathfrak{f}(\mathfrak{x}), \mathfrak{t})}{\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})}, \\ & \mathcal{M}(\mathfrak{x}, \mathfrak{f}(\mathfrak{y}), 2\mathfrak{s}\mathfrak{t}),\ \mathcal{M}(\mathfrak{y}, \mathfrak{f}(\mathfrak{x}), 2\mathfrak{s}\mathfrak{t})\}, \end{aligned}$$

and

$$I(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = \max\{\min\{\mathcal{M}(\mathfrak{x},\mathfrak{f}(\mathfrak{x}),\mathfrak{t}),\ \mathcal{M}(\mathfrak{y},\mathfrak{f}(\mathfrak{y}),\mathfrak{t})\},\mathcal{M}(\mathfrak{x},\mathfrak{f}(\mathfrak{y}),\mathfrak{t}),\mathcal{M}(\mathfrak{y},\mathfrak{f}(\mathfrak{x}),\mathfrak{t})\}.$$

For $\epsilon = 1$, $(\epsilon.\psi.\ \mathfrak{C})$ is called ψ-contractive or $(\psi.\ \mathfrak{C})$. The following theorem is a generalization of theorem 1 for $(\psi.\mathfrak{C})$ correspondences in "fuzzy $\mathfrak{b}$-metric spaces".

Theorem 3. *Every $(\psi.\mathfrak{C})$ correspondence $\mathfrak{f}$ has a fixed point.*

Proof. Let $\mathfrak{x}_0$ be any element in the domain of $\mathfrak{f}$. If $\mathfrak{x}_0 \in \mathfrak{f}(\mathfrak{x}_0)$ then $\mathfrak{x}_0$ is the fixed point of $\mathfrak{f}$ and we have obtained the required result. However, if $\mathfrak{x}_0 \notin \mathfrak{f}(\mathfrak{x}_0)$ then choose an arbitrary element $\mathfrak{x}_1 \in \mathfrak{f}(\mathfrak{x}_0)$. By the definition of the $(\psi.\mathfrak{C})$ correspondence there exists an $\mathfrak{x}_2 \in \mathfrak{f}(\mathfrak{x}_1)$ such that

$$\psi(\mathcal{M}(\mathfrak{x}_2,\mathfrak{x}_1,\frac{\mathfrak{t}}{\mathfrak{s}})) \geq \min\{\psi(S(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t})),\ \psi(I(\mathfrak{x}_1,\mathfrak{x}_0,\frac{\mathfrak{t}}{\mathfrak{L}}))\}, \tag{2}$$

Now we have to compute $S(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t})$ and $I(\mathfrak{x}_1,\mathfrak{x}_0,\frac{\mathfrak{t}}{\mathfrak{L}})$ where

$$\begin{aligned} S(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t}) &= \min\{\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_1,\mathfrak{f}(\mathfrak{x}_1),\mathfrak{t}),\tfrac{\mathcal{M}(\mathfrak{x}_0,\mathfrak{f}(\mathfrak{x}_0),\mathfrak{t})\mathcal{M}(\mathfrak{x}_1,\mathfrak{f}(\mathfrak{x}_1),\mathfrak{t})}{\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t})},\mathcal{M}(\mathfrak{x}_1,\mathfrak{f}(\mathfrak{x}_0),2\mathfrak{s}\mathfrak{t}), \\ &\quad \mathcal{M}(\mathfrak{x}_0,\mathfrak{f}(\mathfrak{x}_1),2\mathfrak{s}\mathfrak{t})\}, \\ &\geq \min\{\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t}),\tfrac{\mathcal{M}(\mathfrak{x}_0,\mathfrak{x}_1,\mathfrak{t})\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t})}{\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t})},1,\ \mathcal{M}(\mathfrak{x}_0,\mathfrak{x}_2,2\mathfrak{s}\mathfrak{t})\}, \\ &= \min\{\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t}),\ \mathcal{M}(\mathfrak{x}_0,\mathfrak{x}_2,2\mathfrak{s}\mathfrak{t})\}, \\ &\geq \min\{\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t}),1,\ \min\{\mathcal{M}(\mathfrak{x}_0,\mathfrak{x}_1,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t})\}, \end{aligned} \tag{3}$$

and

$$\begin{aligned} I(\mathfrak{x}_1,\mathfrak{x}_0,\tfrac{\mathfrak{t}}{\mathfrak{L}}) &= \max\{\min\{\mathcal{M}(\mathfrak{x}_1,\mathfrak{f}(\mathfrak{x}_1),\tfrac{\mathfrak{t}}{\mathfrak{L}}),\ \mathcal{M}(\mathfrak{x}_0,\mathfrak{f}(\mathfrak{x}_0),\tfrac{\mathfrak{t}}{\mathfrak{L}})\},\mathcal{M}(\mathfrak{x}_1,\mathfrak{f}(\mathfrak{x}_0),\tfrac{\mathfrak{t}}{\mathfrak{L}}), \\ &\quad \mathcal{M}(\mathfrak{x}_0,\mathfrak{f}(\mathfrak{x}_1),\tfrac{\mathfrak{t}}{\mathfrak{L}})\} \\ &\geq \max\{\min\{\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\tfrac{\mathfrak{t}}{\mathfrak{L}}),\ \mathcal{M}(\mathfrak{x}_0,\mathfrak{x}_1,\tfrac{\mathfrak{t}}{\mathfrak{L}})\},1,\mathcal{M}(\mathfrak{x}_0,\mathfrak{x}_2,\tfrac{\mathfrak{t}}{\mathfrak{L}})\} = 1, \end{aligned}$$

if $\min\{\mathcal{M}(\mathfrak{x}_0,\mathfrak{x}_1,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t})\} = \mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t})$ then from Inequality 3 we have $S(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t}) \geq \mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t})$. Then, Inequality 2 becomes

$$\begin{aligned} \psi(\mathcal{M}(\mathfrak{x}_2,\mathfrak{x}_1,\tfrac{\mathfrak{t}}{\mathfrak{s}})) &\geq \min\{\psi(\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t})),\ \psi(1)\} \\ &\geq \min\{\psi(\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t})),\ 1\} \\ &\geq \psi(\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t})). \end{aligned}$$

From the above inequality we have

$$\psi(\mathcal{M}(\mathfrak{x}_2,\mathfrak{x}_1,\frac{\mathfrak{t}}{\mathfrak{s}})) \geq \psi(\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t})).$$

Since ψ is an increasing function, hence we have

$$\mathcal{M}(\mathfrak{x}_2,\mathfrak{x}_1,\frac{\mathfrak{t}}{\mathfrak{s}}) \geq \mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t}).$$

As discussed in the previous section, the related correspondence implies

$$\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_1,\mathfrak{x}_2) \leq \frac{1}{\mathfrak{s}}\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_1,\mathfrak{x}_2),$$

where $\mathfrak{d}_{\mathfrak{r}}$ is a pseudo $\mathfrak{b}$-metric induced by a $\mathfrak{b}$-fuzzy metric $\mathcal{M}$. Then the above inequality is true if $\mathfrak{x}_1 = \mathfrak{x}_2$. In this case, $\mathfrak{x}_1$ is a fixed point of $\mathfrak{f}$ and the proof is complete. If not then $\mathfrak{x}_1 \neq \mathfrak{x}_2$. In this case

$\min\{\mathcal{M}(\mathfrak{x}_0,\mathfrak{x}_1,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t})\}=\mathcal{M}(\mathfrak{x}_0,\mathfrak{x}_1,\mathfrak{t})$. From Inequality 3 we have $S(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t})\geq\mathcal{M}(\mathfrak{x}_0,\mathfrak{x}_1,\mathfrak{t})$ and $I(\mathfrak{x}_1,\mathfrak{x}_0,\frac{\mathfrak{t}}{\mathfrak{L}})\geq 1$.

$$\begin{aligned}\psi(\mathcal{M}(\mathfrak{x}_2,\mathfrak{x}_1,\frac{\mathfrak{t}}{\mathfrak{s}})) &\geq \min\{\psi(\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t})),\ \psi(1)\}\\ &\geq \min\{\psi(\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t})),\ 1\}\\ &\geq \psi(\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_0,\mathfrak{t})).\end{aligned}$$

Hence, $\psi(\mathcal{M}(\mathfrak{x}_2,\mathfrak{x}_1,\frac{\mathfrak{t}}{\mathfrak{s}}))\geq\psi(\mathcal{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{t}))$. Continuing in this way we obtain a sequence $\{\mathfrak{x}_n\}$ for each $n\geq 1$ such that $\mathfrak{x}_{n+1}\in\mathfrak{f}(\mathfrak{x}_n)$ and it satisfies:

$$\psi(\mathcal{M}(\mathfrak{x}_{n+1},\mathfrak{x}_n,\frac{\mathfrak{t}}{\mathfrak{s}}))\geq\min\{\psi(S(\mathfrak{x}_n,\mathfrak{x}_{n-1},\mathfrak{t})),\ \psi(I(\mathfrak{x}_n,\mathfrak{x}_{n-1},\frac{\mathfrak{t}}{\mathfrak{L}}))\}.$$

If $\mathfrak{x}_{n+1}=\mathfrak{x}_n$ for some $n\in\mathbb{N}$ then $\mathfrak{f}$ has a fixed point. We assume that $\mathfrak{x}_{n+1}\neq\mathfrak{x}_n$. It is easy to show that $I(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t})=1$. Now we have

$$\begin{aligned}S(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}) &= \min\{\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{f}(\mathfrak{x}_{n-1}),\mathfrak{t}),\\ &\quad \tfrac{\mathcal{M}(\mathfrak{x}_n,\mathfrak{f}(\mathfrak{x}_n),\mathfrak{t})\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{f}(\mathfrak{x}_{n-1}),\mathfrak{t})}{\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t})},\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{f}(\mathfrak{x}_n),2\mathfrak{s}\mathfrak{t}),\mathcal{M}(\mathfrak{x}_n,\mathfrak{f}(\mathfrak{x}_{n-1}),2\mathfrak{s}\mathfrak{t})\},\\ &\geq \min\{\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}),\\ &\quad \mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},2\mathfrak{s}\mathfrak{t}),\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_n,2\mathfrak{s}\mathfrak{t})\},\\ &\geq \min\{\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}),\min\{\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t})\},\\ &= \min\{\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}),\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t})\}.\end{aligned}$$

If $S(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t})\geq\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t})$ then we have

$$\begin{aligned}\psi(\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\frac{\mathfrak{t}}{\mathfrak{s}})) &\geq \min\{\psi(\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t})),\ \psi(1)\}\\ &= \psi(\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t})).\end{aligned}$$

Since ψ is nondecreasing, so

$$\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\frac{\mathfrak{t}}{\mathfrak{s}})\geq\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\mathfrak{t}).$$

This implies that $\mathfrak{d}_\mathfrak{r}(\mathfrak{x}_n,\mathfrak{x}_{n+1})\leq\frac{1}{\mathfrak{s}}\mathfrak{d}_\mathfrak{r}(\mathfrak{x}_n,\mathfrak{x}_{n+1})$ where $\mathfrak{d}_\mathfrak{r}$ is a "pseudo $\mathfrak{b}$-metric" induced by a "$\mathfrak{b}$-fuzzy metric" $\mathcal{M}$. Since $\mathfrak{x}_n\neq\mathfrak{x}_{n+1}$ then above inequality generates a contradiction.
Hence we have

$$S(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t})\geq\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t}).$$

This implies

$$\psi(\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\frac{\mathfrak{t}}{\mathfrak{s}}))\geq\min\{\psi(\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t})),\ \psi(1)\}=\psi(\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t})).$$

Thus we have $\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1},\frac{\mathfrak{t}}{\mathfrak{s}})\geq\mathcal{M}(\mathfrak{x}_{n-1},\mathfrak{x}_n,\mathfrak{t})$, for all $n\in\mathbb{N}$. Hence

$$\mathfrak{d}_\mathfrak{r}(\mathfrak{x}_{n+1},\mathfrak{x}_n)\leq\frac{1}{\mathfrak{s}}\mathfrak{d}_\mathfrak{r}(\mathfrak{x}_{n-1},\mathfrak{x}_n).$$

Lemma 2 implies that $\{\mathfrak{x}_n\}$ is a Cauchy sequence by $\mathfrak{d}_\mathfrak{r}$ for each $\mathfrak{r}\in(0,1)$. Thus, $\{\mathfrak{x}_n\}$ is a Cauchy sequence in $(\mathcal{X},\mathcal{M})$. Since $(\mathcal{X},\mathcal{M})$ is a "complete fuzzy $\mathfrak{b}$-metric space" there exists an $\mathfrak{x}\in\mathcal{X}$ such that $\lim_{n\to\infty}\mathcal{M}(\mathfrak{x}_n,\mathfrak{x},\mathfrak{t})=1$. As $\mathfrak{x}_n\in\mathfrak{f}(\mathfrak{x}_{n-1})$, $\mathfrak{x}_n\to\mathfrak{x}$, $\mathfrak{x}_{n-1}\to\mathfrak{x}$ and $\mathfrak{f}$ is closed, this implies $\mathfrak{x}\in\mathfrak{f}(\mathfrak{x})$. □

Clearly, $(\epsilon.\psi.\mathfrak{C})$ is $(\psi.\mathfrak{C})$. Thus, the Theorem 3 also holds for $(\epsilon.\psi.\mathfrak{C})$. The following theorem is equivalent to Nadler's theorem in [22] in the "fuzzy $\mathfrak{b}$-metric space".

Theorem 4. *Suppose that $\mathfrak{f} : \mathcal{X} \rightsquigarrow \mathcal{X}$ is a correspondence in the "fuzzy $\mathfrak{b}$-metric space" such that for $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$ and $\mathfrak{z} \in \mathfrak{f}(\mathfrak{x})$ there is $\mathfrak{w} \in \mathfrak{f}(\mathfrak{y})$ satisfying the following condition*

$$\mathcal{M}(\mathfrak{z}, \mathfrak{w}, \mathfrak{t}) \geq \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \frac{\mathfrak{t}}{\mathfrak{k}}),$$

where $\mathfrak{k} \in (0, 1)$ and $\mathfrak{t} \in \mathbb{R}$. Then, $\mathfrak{f}$ has a fixed point.

Proof. The proof follows using similar arguments as in Theorem 3. □

The following example supports Theorems 3 and 4.

Example 2. *Let $\mathcal{X} = [0, 1]$ and $(\mathcal{X}, \mathcal{M})$ be a "fuzzy $\mathfrak{b}$-metric space" where*

$$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \begin{cases} \frac{\mathfrak{t}}{\mathfrak{t}+(\mathfrak{x}-\mathfrak{y})^2} & \mathfrak{t} > 0, \\ 0 & \mathfrak{t} \leq 0. \end{cases}$$

As in Example 1, $(\mathcal{X}, \mathcal{M})$ is a "complete fuzzy $\mathfrak{b}$-metric space" with $\mathfrak{s} = 2$. Let $\mathfrak{f} : \mathcal{X} \rightsquigarrow \mathcal{X}$ be defined as $\mathfrak{f}(\mathfrak{x}) = \{\frac{\mathfrak{x}}{2}\}$. It is straightforward to see that for each $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$,

$$\mathcal{M}(\frac{\mathfrak{x}}{2}, \frac{\mathfrak{y}}{2}, \frac{\mathfrak{t}}{4}) \geq \min\{S(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}),\ I(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})\}.$$

Thus, for $\mathfrak{L} = 1$, $\epsilon = 2$ and $\psi(\mathfrak{t}) = \mathfrak{t}$, Theorem 3 is satisfied and $\mathfrak{f}$ has a fixed point.

Example 3. *Let $\mathcal{X} = [0, \frac{1}{2}] \cup \{1\}$ and $(\mathcal{X}, \mathcal{M})$ be a "fuzzy $\mathfrak{b}$-metric space" where*

$$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \begin{cases} 0 & |\mathfrak{x} - \mathfrak{y}| \geq \mathfrak{t}, \\ 1 & |\mathfrak{x} - \mathfrak{y}| < \mathfrak{t}. \end{cases}$$

Define a correspondence $\mathfrak{f}$ on $\mathcal{X}$ as

$$\mathfrak{f}(\mathfrak{x}) = \begin{cases} \frac{1}{4} & \mathfrak{x} = 1, \\ \{\frac{1}{4}, \frac{1}{2}\} & \mathfrak{x} \neq 1. \end{cases}$$

We claim that for some $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$ and $\mathfrak{z} \in \mathfrak{f}(\mathfrak{x})$ there exists a $\mathfrak{w} \in \mathfrak{f}(\mathfrak{y})$ such that $\mathcal{M}(\mathfrak{z}, \mathfrak{w}, \mathfrak{t}) \geq \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \frac{\mathfrak{t}}{\mathfrak{k}})$ where $\mathfrak{k} \in (0, 1)$. Suppose that for $\mathfrak{k} = \frac{1}{2}$ and

$$\mathcal{M}(\mathfrak{z}, \mathfrak{w}, \mathfrak{t}) = 1 \text{ or } \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \frac{\mathfrak{t}}{\mathfrak{k}}) = 0,$$

then the claim holds. However,

$$\mathcal{M}(\mathfrak{z}, \mathfrak{w}, \mathfrak{t}) = 0 \text{ and } \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \frac{\mathfrak{t}}{\mathfrak{k}}) = 1,$$

is impossible. Without loss of generality we can suppose that $\mathfrak{y} = 1$ and $\mathfrak{x} \neq 1$. Indeed, if $\mathfrak{x} \neq 1$ and $\mathfrak{y} \neq 1$ we can choose $\mathfrak{z} = \mathfrak{w}$ and therefore, $\mathcal{M}(\mathfrak{z}, \mathfrak{w}, \mathfrak{t}) = 1$. Take $\mathfrak{z} = \frac{1}{4}$ and $\mathfrak{w} \in \{\frac{1}{4}, \frac{1}{2}\}$. If $\mathfrak{z} = \mathfrak{w}$ then $\mathcal{M}(\mathfrak{z}, \mathfrak{w}, \mathfrak{t}) = 1$ is a contradiction. If $\mathfrak{z} = \frac{1}{4}$ and $\mathfrak{w} = \frac{1}{2}$ then

$$\mathcal{M}(\frac{1}{4}, \frac{1}{2}, \mathfrak{t}) = 0 \textit{ and } \mathcal{M}(\mathfrak{x}, 1, 2\mathfrak{t}) = 1$$

implies that $\mathfrak{t} \leq \frac{1}{4}$ and $|1 - \mathfrak{x}| < 2\mathfrak{t} \leq \frac{1}{2}$, that is $\frac{1}{2} < \mathfrak{x}$ which is impossible. By Theorem 4, $\mathfrak{x} = \frac{1}{2}$ is a fixed point of the mapping $\mathfrak{f}$.

Corollary 1. *Suppose that $\mathfrak{T} : \mathcal{X} \to \mathcal{X}$ is a single valued mapping on a "fuzzy $\mathfrak{b}$-metric space" satisfying*

$$\mathcal{M}(\mathfrak{T}(\mathfrak{x}), \mathfrak{T}(\mathfrak{y}), \mathfrak{t}) \geq \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \frac{\mathfrak{t}}{\mathfrak{k}}),$$

for every $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$, $\mathfrak{k} \in (0,1)$ and $\mathfrak{t} \in \mathbb{R}$. Then, $\mathfrak{T}$ has a unique fixed point.

Corollary 1 is a "fuzzy $\mathfrak{b}$-metric" version of the "Banach fixed point theorem".

Corollary 2. *Suppose that $\mathfrak{T} : \mathcal{X} \to \mathcal{X}$ is a single valued mapping on a "complete $\mathfrak{b}$-metric space" $(\mathcal{X}, \mathfrak{d})$ and*

$$\mathfrak{d}(\mathfrak{T}(\mathfrak{x}), \mathfrak{T}(\mathfrak{y})) \leq \mathfrak{k}\mathfrak{d}(\mathfrak{x}, \mathfrak{y}),$$

holds for every $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$, $\mathfrak{k} \in (0,1)$. Then, $\mathfrak{T}$ has a fixed point.

Proof. The inequality $\mathfrak{d}(\mathfrak{T}(\mathfrak{x}), \mathfrak{T}(\mathfrak{y})) \leq \mathfrak{k}\mathfrak{d}(\mathfrak{x}, \mathfrak{y})$ implies that

$$\frac{\mathfrak{t}}{\mathfrak{t} + \mathfrak{d}(\mathfrak{T}(\mathfrak{x}), \mathfrak{T}(\mathfrak{y}))} \geq \frac{\mathfrak{t}}{\mathfrak{t} + \mathfrak{k}\mathfrak{d}(\mathfrak{x}, \mathfrak{y})}.$$

Therefore

$$\mathcal{M}(\mathfrak{T}(\mathfrak{x}), \mathfrak{T}(\mathfrak{y}), \mathfrak{t}) \geq \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \frac{\mathfrak{t}}{\mathfrak{k}})$$

Note that every "$\mathfrak{b}$-metric" is a "fuzzy $\mathfrak{b}$-metric", as shown in Example 1. The rest of the proof follows by using Corollary 1. □

Remark 1. *Corollary 2 has been proven for $\mathfrak{k} \in [\frac{1}{\mathfrak{s}}, 1)$. It is an open problem whether $\mathfrak{T}$ has a fixed point when $\frac{1}{\mathfrak{s}} \leq \mathfrak{k} < 1$. Actually we replied to this important question in the Corollary 2.*

Theorem 5. *Let $\mathfrak{x}_0 \in \mathcal{X}$, $\mathfrak{r} > 0$ and $\mathfrak{t}_0 > 0$. Suppose that $\mathfrak{f} : \overline{B_{\mathfrak{r}}(\mathfrak{x}_0, \mathfrak{t}_0)} \rightsquigarrow \mathcal{X}$ is an $(\epsilon.\psi.\mathfrak{C})$ correspondence where $\epsilon > 1$. Suppose that there exists $\mathfrak{x}_1 \in \mathfrak{f}(\mathfrak{x}_0)$ such that*

$$\mathcal{M}(\mathfrak{x}_1, \mathfrak{x}_0, \frac{\mathfrak{s}^{\epsilon-1} - 1}{\mathfrak{s}^{\epsilon}}\mathfrak{t}_0) > \mathfrak{r}.$$

Then, $\mathfrak{f}$ has a fixed point.

Proof. Since $\mathcal{M}(\mathfrak{x}_1, \mathfrak{x}_0, \frac{\mathfrak{s}^{\epsilon-1}-1}{\mathfrak{s}^{\epsilon}}\mathfrak{t}_0) > \mathfrak{r}$ we have $\mathcal{M}(\mathfrak{x}_1, \mathfrak{x}_0, \mathfrak{t}_0) \geq \mathcal{M}(\mathfrak{x}_1, \mathfrak{x}_0, \frac{\mathfrak{s}^{\epsilon-1}-1}{\mathfrak{s}^{\epsilon}}\mathfrak{t}_0) > \mathfrak{r}$. This implies that $\mathfrak{x}_1 \in B_{\mathfrak{r}}(\mathfrak{x}_0, \mathfrak{t}_0)$. By the similar arguments as in the proof of Theorem 3 there exists $\mathfrak{x}_2 \in \mathfrak{f}(\mathfrak{x}_1)$ such that

$$\mathcal{M}(\mathfrak{x}_2, \mathfrak{x}_1, \frac{\mathfrak{t}_0}{\mathfrak{s}^{\epsilon}}) \geq \mathcal{M}(\mathfrak{x}_1, \mathfrak{x}_0, \mathfrak{t}_0) > \mathfrak{r}.$$

Therefore, $\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_2, \mathfrak{x}_1) \leq \frac{1}{\mathfrak{s}^{\epsilon}}\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_0) \leq \frac{\mathfrak{s}^{\epsilon-1}-1}{\mathfrak{s}^{\epsilon}}\mathfrak{t}_0$. Following on the same lines we have

$$\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_{n+1}, \mathfrak{x}_n) \leq (\frac{1}{\mathfrak{s}^{\epsilon}})^n \mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_0),$$

for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}
\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_n, \mathfrak{x}_0) &\leq \mathfrak{s}\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_0) + \mathfrak{s}^2\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) + \mathfrak{s}^3\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_2, \mathfrak{x}_3) + \ldots + \mathfrak{s}^n\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_{n-1}, \mathfrak{x}_n), \\
&\leq \mathfrak{s}\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_0)[1 + (\frac{\mathfrak{s}}{\mathfrak{s}^\epsilon}) + (\frac{\mathfrak{s}}{\mathfrak{s}^\epsilon})^2 + \ldots + (\frac{\mathfrak{s}}{\mathfrak{s}^\epsilon})^{n-2} + \frac{\mathfrak{s}^{n-2}}{\mathfrak{s}^{\epsilon(n-1)}}], \\
&\leq \mathfrak{s}\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_0)[1 + (\frac{\mathfrak{s}}{\mathfrak{s}^\epsilon}) + (\frac{\mathfrak{s}}{\mathfrak{s}^\epsilon})^2 + \ldots + (\frac{\mathfrak{s}}{\mathfrak{s}^\epsilon})^{n-2} + (\frac{\mathfrak{s}}{\mathfrak{s}^\epsilon})^{n-1}], \\
&\leq \frac{\mathfrak{s}}{1 - \frac{1}{\mathfrak{s}^{\epsilon-1}}}\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_0) < \frac{1}{\mathfrak{s}^\epsilon}\mathfrak{d}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_0) \\
&< \mathfrak{t}_0,
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, the sequence $\mathfrak{x}_n \in B_{\mathfrak{r}}(\mathfrak{x}_0, \mathfrak{t}_0)$, for all $n \in \mathbb{N}$. Following similar arguments to those in the proof of Theorem 3, we deduce that $\{\mathfrak{x}_n\}$ is a Cauchy sequence. By the closeness of $\overline{B_{\mathfrak{r}}(\mathfrak{x}_0, \mathfrak{t}_0)}$ and the completeness of $\mathcal{X}$ there exists an $\mathfrak{x} \in \overline{B_{\mathfrak{r}}(\mathfrak{x}_0, \mathfrak{t}_0)}$ such that $\mathfrak{x}_n \to \mathfrak{x}$. $\mathfrak{f}$ is closed and we have $\mathfrak{x} \in \mathfrak{f}(\mathfrak{x})$. □

A correspondence $\mathfrak{f} : \mathcal{X} \rightsquigarrow \mathcal{X}$ is called monotone if for all $\mathfrak{x} \preceq \mathfrak{y}$, $u \in \mathfrak{f}(\mathfrak{x})$ and $v \in \mathfrak{f}(\mathfrak{y})$ we have $u \preceq v$ (for details, see [44,45]). Suppose that in Definition 4 (defined in [46]), $(\mathcal{X}, \mathcal{M})$ is equipped with a partial order relation $\preceq$ and Inequality 1 holds for $\mathfrak{x}, \mathfrak{y} \in \mathfrak{C}$ where $\mathfrak{x} \preceq \mathfrak{y}$. Then, $\mathfrak{f}$ is said to be monotone ψ-contractive (briefly, monotone $(\psi.\mathfrak{C})$). The following theorem is a generalization of theorem 1 to monotone $\psi.\ \mathfrak{C}$ correspondences in "ordered fuzzy $\mathfrak{b}$-metric space".

Theorem 6. *Let $(\mathcal{X}, \mathcal{M})$ be a complete order "fuzzy $\mathfrak{b}$-metric space" and $\mathfrak{f}$ be a monotone $(\psi.\mathfrak{C})$ such that $\mathfrak{x}_0 \preceq \mathfrak{f}(\mathfrak{x}_0)$ for some $\mathfrak{x}_0 \in \mathcal{X}$. Then, $\mathfrak{f}$ has a fixed point.*

Proof. The proof is closely modeled on Theorem 3. □

4. Application to Linear Equations

In this section we will provide an application of the "Banach fixed point theorem" on "fuzzy $\mathfrak{b}$-metric spaces" to linear equations. Now, consider the linear system

$$\begin{gathered}
\mathfrak{a}_{11}\mathfrak{x}_1 + \mathfrak{a}_{12}\mathfrak{x}_2 + \cdots + \mathfrak{a}_{1n}\mathfrak{x}_n = \mathfrak{b}_1, \\
\mathfrak{a}_{21}\mathfrak{x}_1 + \mathfrak{a}_{22}\mathfrak{x}_2 + \cdots + \mathfrak{a}_{2n}\mathfrak{x}_n = \mathfrak{b}_2, \\
\vdots \\
\mathfrak{a}_{n1}\mathfrak{x}_1 + \mathfrak{a}_{n2}\mathfrak{x}_2 + \cdots + \mathfrak{a}_{nn}\mathfrak{x}_n = \mathfrak{b}_n,
\end{gathered}$$

which has a unique solution. It is equivalent to show that the following linear system has a unique solution.

$$\begin{gathered}
\mathfrak{c}_{11}\mathfrak{x}_1 + \mathfrak{c}_{12}\mathfrak{x}_2 + \cdots + \mathfrak{c}_{1n}\mathfrak{x}_n = \mathfrak{b}'_1 \\
\mathfrak{c}_{21}\mathfrak{x}_1 + \mathfrak{c}_{22}\mathfrak{x}_2 + \cdots + \mathfrak{c}_{2n}\mathfrak{x}_n = \mathfrak{b}'_2 \\
\vdots \\
\mathfrak{c}_{n1}\mathfrak{x}_1 + \mathfrak{c}_{n2}\mathfrak{x}_2 + \cdots + \mathfrak{c}_{nn}\mathfrak{x}_n = \mathfrak{b}'_n
\end{gathered}$$

where $\mathfrak{c}_{ij} = \frac{\mathfrak{a}_{ij}}{2nM}$, $i, j \in \{1, \ldots, n\}$, $\mathcal{M} = \sqrt{\max_{i,j} \mathfrak{a}_{ij}^2}$ and $\mathfrak{b}' = [\frac{\mathfrak{b}_1}{2nM}, \ldots, \frac{\mathfrak{b}_n}{2nM}]^{\mathfrak{T}}$. For this we consider the "fuzzy $\mathfrak{b}$-metric space" generated by

$$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \frac{\mathfrak{t}}{\mathfrak{t} + \max\limits_{1 \leq j \leq n} |\mathfrak{x}_j - \mathfrak{y}_j|^2},$$

for all $\mathfrak{x}, \mathfrak{y} \in \mathbb{R}^n$. Consider the mapping $\mathfrak{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$\mathfrak{T}(\mathfrak{x}) = \mathfrak{C}\mathfrak{x} + \mathfrak{b}'$$

where $\mathfrak{x} \in \mathbb{R}^n$, $\mathfrak{b}'$ is a column matrix having entries from $\mathbb{R}$ and $\mathfrak{C}$ is an $n \times n$ matrix with $\mathfrak{c}_{ij}$ arrays. It is essay to show that $\max\limits_{i,j} \mathfrak{c}_{ij}^2 = \max\limits_{i,j} \frac{\mathfrak{a}_{ij}^2}{4n^2\mathcal{M}^2}$. Now we have to show that the self-mapping $\mathfrak{T}$ satisfies the "Banach's contraction principle" on "fuzzy $\mathfrak{b}$-metric spaces".

$$\begin{aligned} \max_{1\le i\le n} \Big(\sum_{j=1}^{n} \mathfrak{c}_{ij} \left|\mathfrak{x}_j - \mathfrak{y}_j\right|\Big)^2 &\le \max_{1\le i\le n} \sum_{j=1}^{n} \mathfrak{c}_{ij}^2 \sum_{j=1}^{n} \left|\mathfrak{x}_j - \mathfrak{y}_j\right|^2, \\ &\le n^2 \max_{i,j} \mathfrak{c}_{ij}^2 \max_{j} \left|\mathfrak{x}_j - \mathfrak{y}_j\right|^2, \\ &\le \frac{1}{4} \max_{j} \left|\mathfrak{x}_j - \mathfrak{y}_j\right|^2. \end{aligned}$$

This implies that $\mathcal{M}(\mathfrak{T}(\mathfrak{x}), \mathfrak{T}(\mathfrak{y}), \mathfrak{t}) \ge \mathcal{M}(\mathfrak{x}, \mathfrak{y}, 4\mathfrak{t})$. Corollary 1 implies that $\mathfrak{T}$ has a fixed point. Therefore, the linear system has a unique solution.

5. Conclusions

In this article we defined the ψ-contraction and monotone ψ-contraction correspondence and obtained fixed point result in the "fuzzy $\mathfrak{b}$-metric space". As a consequence of our main result we obtained the Banach contraction principle in the "fuzzy $\mathfrak{b}$-metric space". Further we addressed an open problem in which we generalized the interval of contraction and proved that our results were also valid if contractive constant $\mathfrak{k}$ lied in $[\frac{1}{\mathfrak{s}}, 1)$, where $\mathfrak{s} \ge 1$. As an application of our result we obtained a solution of the system of n linear equations in the "fuzzy $\mathfrak{b}$-metric space". Further we provided examples that further elaborated the useability of our result.

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