FINITE GROUPS WITH SOME RESTRICTION ON THE VANISHING SET

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Abstract. Let x be an element of a finite group G and denote the order of x by $\mathrm{ord}(x)$. We consider a finite group G such that $\gcd(\mathrm{ord}(x), \mathrm{ord}(y)) \leqslant 2$ for any two vanishing elements x and y contained in distinct conjugacy classes. We show that such a group G is solvable. When G with the property above is supersolvable, we show that G has a normal metabelian 2-complement.

1. Introduction

Let G be a finite group. An element $g \in G$ is a vanishing element if there exists an irreducible character χ of G such that $\chi(g) = 0$. The set of all vanishing elements of G is denoted by $\mathrm{Van}(G)$. A classical theorem of Burnside [15, Theorem 3.15] implies that $\mathrm{Van}(G)$ is non-empty when G is non-abelian. Note that

$$\mathrm{Van}(G) = \bigcup_{i=1}^{r} v\mathcal{C}_i,$$

where each $v\mathcal{C}_i$ is a vanishing conjugacy class. We denote the order of the elements contained in a vanishing conjugacy class $v\mathcal{C}_i$ by $\mathrm{ord}(v\mathcal{C}_i)$. Many authors have studied finite groups G with certain restrictions on the set $\mathrm{Van}(G)$ (see [9, 6, 10, 11, 17, 22]). We shall discuss some of that work here. Let p be a fixed prime. Dolfi, Pacifici, Sanus and Spiga in [9] studied finite groups G such that $\mathrm{ord}(v\mathcal{C}_i) \neq p^n$ for some n, for all $i \in \{1, 2, \ldots, r\}$. They showed that G has a normal Sylow p-subgroup [9, Theorem A]. On the other hand, in [6], the authors studied finite groups G such that $\mathrm{ord}(v\mathcal{C}_i) = p^n$ for some n, for all $i \in \{1, 2, \ldots, r\}$, and proved that G is either a p-group or G has a homomorphic image which is a Frobenius group with a complement of p-power order. Robati [22] recently proved that if $\mathrm{Van}(G)$ contains three conjugacy classes of G, then the group G is solvable.

In this article, we investigate finite groups G with the property below:

$$\gcd(\mathrm{ord}(v\mathcal{C}_i), \mathrm{ord}(v\mathcal{C}_j)) = 1 \textit{ for } i \neq j,\ i, j \in \{1, 2, \ldots, r\}. \qquad (\star)$$

We also investigate finite groups G with a more general property:

$$\gcd(\mathrm{ord}(v\mathcal{C}_i), \mathrm{ord}(v\mathcal{C}_j)) \leqslant 2 \textit{ for } i \neq j,\ i, j \in \{1, 2, \ldots, r\}. \qquad (\star\star)$$

In particular, using the classification of finite simple groups we show that if G has property $(\star\star)$, then G is solvable :

Theorem A. *Let G be a finite group. If G satisfies property $(\star\star)$, then G is solvable.*

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Remark. If $\gcd(\mathrm{ord}(v\mathcal{C}_i), \mathrm{ord}(v\mathcal{C}_j)) \leqslant 3$, then G is not necessarily solvable for S_5 satisfies this property. Let $\mathrm{Vo}(G)$ be the set of orders of vanishing elements of G. Then if for every $a, b \in \mathrm{Vo}(G)$, $\gcd(a, b) = 1$, then G is also not necessarily solvable: A_5 is a counterexample.

A theorem of Thompson [15, Corollary 12.2] states that, given a prime number p, if every character degree of a non-linear character of G is a multiple of p, then the group G has a normal p-complement. In [9, Corollary B], it was shown that if G is a finite group and if $p \mid a$ for all $a \in \mathrm{Vo}(G)$ for some fixed prime p, then G has a normal nilpotent p-complement. This does not necessarily hold when G satisfies property $(\star\star)$. An example is S_4, since $\gcd(\mathrm{ord}(v\mathcal{C}_i), \mathrm{ord}(v\mathcal{C}_j)) \leqslant 2$ for all $v\mathcal{C}_i, v\mathcal{C}_j \subseteq \mathrm{Van}(S_4)$, that is, S_4 satisfies property $(\star\star)$ but S_4 does not have a normal 2-complement or 3-complement. However, if G is supersolvable or $\mathbf{O}_2(G) = 1$, then G has a normal 2-complement with one exception: some Frobenius groups with a homomorphic image isomorphic to S_4, as the following result states:

Theorem B. *Let G be a finite non-abelian group satisfying property $(\star\star)$.*

(a) *If G is supersolvable, then G has a normal metabelian 2-complement*
(b) *If $\mathbf{O}_2(G) = 1$, then either*
  (i) *G has a normal 2-complement of Fitting height at most 3, or*
  (ii) *G is a Frobenius group which has an abelian kernel and a Frobenius complement isomorphic to S_4.*

In [7, Proposition 2.7], Chillag showed that if G is a non-abelian group, then G is a Frobenius group with an abelian odd order kernel and a complement of order 2 if and only if every irreducible character of G vanishes on at most one conjugacy class. In this article, we prove a new characterisation of these Frobenius groups:

Corollary C. *Let G be a finite non-abelian group. Then G has property $(\star)$ if and only if G is a Frobenius group with an abelian kernel and complement of order two.*

2. Preliminaries

In this section we shall list some properties of vanishing elements needed to prove our results.

Lemma 2.1. *Let G be an finite group and let N be a normal subgroup of G. Then the following statements hold:*

(a) *If G satisfies property $(\star)$, then G/N satisfies property $(\star)$.*
(b) *If G satisfies property $(\star\star)$, then G/N satisfies property $(\star\star)$.*

Proof. The result follows by the standard observation that $xN \in \mathrm{Van}(G/N)$ implies that $xN \subseteq \mathrm{Van}(G)$. □

Lemma 2.2. [21, Lemma 2] *Let G be a finite solvable group. Suppose M, N are normal subgroups of G.*

(a) *If $M \setminus N$ is a conjugacy class and $\gcd(|M{:}N|, |N|) = 1$, then M is a Frobenius group with kernel N and prime order complement.*
(b) *If $G \setminus N$ is a conjugacy class, then G is a Frobenius group with an abelian kernel and complement of order two.*

For a positive integer m, set $\pi(m) := \{p \mid p \text{ divides } m, \text{ where } p \text{ is prime}\}$.

Corollary 2.3. [10, Corollary 2.6] *Let G be a finite group and let K be a nilpotent normal subgroup of G. If $K \cap \mathrm{Van}(G) \neq \emptyset$, then there exists $g \in K \cap \mathrm{Van}(G)$ whose order is divisible by every prime in $\pi(|K|)$.*

Lemma 2.4. [16, Theorem D] *Let G be a finite solvable group. If x is a non-vanishing element of G, then $x\mathbf{F}(G)$ is a 2-element of $G/\mathbf{F}(G)$. If G is not nilpotent, then x lies in the penultimate term of the Fitting series.*

A non-linear irreducible character χ of G is said to be of p-defect zero if p does not divide $|G|/\chi(1)$. By a result of Brauer (see [15, Theorem 8.17]), if χ is an irreducible character of p-defect zero of G, then $\chi(g) = 0$ whenever p divides the order of g in G. The existence of p-defect zero characters is guaranteed in finite simple groups G for almost all primes $p \geqslant 5$ dividing $|G|$ as the following result shows:

Lemma 2.5. [14, Corollary 2.2] *Let G be a non-abelian finite simple group and p be a prime. If G is a finite group of Lie type, or if $p \geqslant 5$, then there exists $\chi \in \mathrm{Irr}(G)$ of p-defect zero.*

Lemma 2.6. [4, Lemma 2.2] *Let G be a finite group, N a normal subgroup of G and p be a prime. If N has an irreducible character of p-defect zero, then every element of N of order divisible by p is a vanishing element in G.*

Lemma 2.7. [3, Lemma 5] *Let G be a finite group, and $N = S_1 \times \cdots \times S_k$ a minimal normal subgroup of G, where every S_i is isomorphic to a non-abelian simple group S. If $\theta \in \mathrm{Irr}(S)$ extends to $\mathrm{Aut}(S)$, then $\varphi = \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$ extends to G.*

Lemma 2.8. [19, Theorem 1.1] *Suppose that N is a minimal normal non-abelian subgroup of a finite group G. Then there exists an irreducible character θ of N such that θ is extendible to G with $\theta(1) \geqslant 5$.*

The number theory result below follows easily.

Lemma 2.9. *Let p be and f be a positive integer. If $q = p^f \geqslant 32$, then $f < (q-2)/2$.*

We end this section by stating a result on groups in which every irreducible character vanishes on at most two conjugacy classes.

Theorem 2.10. [2, Theorem 1] *Let G be a non-abelian finite group in which every irreducible character vanishes on at most two conjugacy classes. Then one of the following holds:*

(a) *$G \cong \mathrm{A}_5$ or $G \cong \mathrm{PSL}_2(7)$;*
(b) *G is solvable and one of the following holds:*
 (i) *G has a subgroup Z with $|Z| \leqslant 2$ such that G/Z is Frobenius group with a Frobenius complement of order 2 and an abelian Frobenius kernel of odd order.*
 (ii) *$G/Z = FA$ is a semidirect product, where $|A| \leqslant 2$, $|Z| \leqslant 2$ and F is a Frobenius group with a Frobenius complement of order 3 and a nilpotent Frobenius kernel of class at most 2.*

3. Theorem A

Proof of Theorem A. We prove the result by induction on $|G|$. Let N be a non-trivial normal subgroup of G. Then G/N satisfies property $(\star\star)$ by Lemma 2.1(b) and hence G/N is a solvable group. If N_1 and N_2 are two minimal normal subgroups of

G, then G/N_1 and G/N_2 are solvable. Hence G is solvable. We may assume that G has a unique non-abelian minimal normal subgroup N. If $N = S_1 \times S_2 \times \cdots \times S_k$, where $S_i \cong S$, S is a simple group and $i = 1, 2, \ldots, k$, then by Lemma 2.8, there exists $\theta \in \mathrm{Irr}(N)$ which is extendible to G. Note that $\theta = \phi_1 \times \phi_2 \times \cdots \times \phi_k$ with $\phi_i \in \mathrm{Irr}(S_i)$ for each $i \in \{1, 2, \ldots, k\}$. Suppose that $k \geqslant 2$. Since ϕ_1 is non-linear, we may assume that ϕ_1 vanishes on a p-element $x_1 \in S_1$ for some prime p by [20, Theorem B]. Suppose that p is odd. Note that $2 \mid |N|$ and let $y_2 \in S_2$ be a 2-element. Then $\theta(x_1y_2) = \phi_1(x_1)\phi_2(y_2)\cdots\phi_k(1) = 0$ and $p \mid \gcd(\mathrm{ord}(x_1), \mathrm{ord}(x_1y_2))$. Hence G does not satisfy $(\star\star)$. Suppose that p is even. Then there is a prime $q \geqslant 5$ such that $q \mid |N|$ since by [15, Theorem 3.10], $\pi(|G|) \geqslant 3$. Let $y_2 \in S_2$ be a q-element. Note that x_1y_2 and y_2 are vanishing elements of G by Lemma 2.5 and Lemma 2.6. Since $q \mid \gcd(\mathrm{ord}(y_2), \mathrm{ord}(x_1y_2))$, the result follows.

We may assume that N is a simple group. Since N is the unique minimal normal subgroup of G, $\mathbf{C}_G(N) = 1$ and so G is almost simple. Let N be a sporadic simple group or ${}^2\mathrm{F}_4(2)'$. Table 3 below contains an irreducible character θ of N of p-defect zero for some odd prime p and two elements of distinct orders divisible by p. The result that G does not satisfy property $(\star\star)$ follows from Lemma 2.6. We shall use the character tables and notation in the Atlas [8].

N	$\theta(1)$	$\mathrm{ord}(v\mathcal{C}_1)$	$\mathrm{ord}(v\mathcal{C}_2)$
M_{11}	$\chi_9(1) = 45$	$3A$	$6A$
M_{12}	$\chi_8(1) = 55$	$5A$	$10A$
M_{22}	$\chi_3(1) = 45$	$3A$	$6A$
M_{23}	$\chi_3(1) = 45$	$3A$	$6A$
M_{24}	$\chi_3(1) = 45$	$5A$	$10A$
J_2	$\chi_{18}(1) = 225$	$5A$	$10A$
Suz	$\chi_{10}(1) = 10725$	$5A$	$10A$
HS	$\chi_{19}(1) = 1750$	$5A$	$10A$
M^cL	$\chi_9(1) = 1750$	$5A$	$10A$
Co_3	$\chi_6(1) = 896$	$7A$	$14A$
Co_2	$\chi_5(1) = 1771$	$7A$	$14A$
Co_1	$\chi_4(1) = 1771$	$11A$	$22A$
He	$\chi_9(1) = 1275$	$5A$	$10A$
Fi_{22}	$\chi_4(1) = 1001$	$7A$	$14A$
Fi_{23}	$\chi_8(1) = 106743$	$7A$	$14A$
Fi'_{24}	$\chi_4(1) = 249458$	$11A$	$22A$
HN	$\chi_2(1) = 133$	$7A$	$14A$
Th	$\chi_6(1) = 30628$	$13A$	$39A$
B	$\chi_{11}(1) = 3214743741$	$11A$	$22A$
M	$\chi_9(1) = 36173193327999$	$17A$	$34A$
J_1	$\chi_9(1) = 120$	$3A$	$6A$
$O'N$	$\chi_8(1) = 32395$	$5A$	$10A$
J_3	$\chi_2(1) = 85$	$5A$	$10A$
Ly	$\chi_5(1) = 48174$	$7A$	$14A$
Ru	$\chi_2(1) = 378$	$7A$	$14A$
J_4	$\chi_9(1) = 1187145$	$5A$	$10A$
${}^2\mathrm{F}_4(2)'$	$\chi_8(1) = 325$	$5A$	$10A$

Suppose that N is an alternating group A_n, $n \geqslant 5$. For $N \cong \mathrm{A}_5$ and $N \cong \mathrm{A}_7$, our result follows by consulting the Atlas [8]. Assume that $N \cong \mathrm{A}_6 \cong \mathrm{PSL}_2(9)$. Then for an almost simple G such that $|G/N| \leqslant 2$, our result follows by consulting the explicit character tables in the Atlas [8]. For the case when $|G/N| = 4$, we obtain the character table in GAP [13] and our result follows.

Let $N \cong \mathrm{A}_n$, $n \geqslant 8$. By Lemma 2.5, N has a 5-defect zero character. Note that N has two elements (12345) and (12345)(678) of orders 5 and 15, respectively. These two elements are vanishing elements of G by Lemma 2.6 and hence G does not satisfy property ($\star\star$).

Suppose that N has an element of order $2r$, r an odd prime. Then N has an irreducible character θ of r-defect zero by Lemma 2.5. Since N has elements of order r and $2r$, the result follows since these two elements are vanishing elements by Lemma 2.6. We may assume that N has no element of order $2r$, r an odd prime. Then the centralizer of each involution contained in N is a 2-group. It follows from [24, III, Theorem 5] that N is isomorphic to one of the following: $\mathrm{PSL}_2(p)$, where p is a Fermat or Mersenne prime; $\mathrm{PSL}_2(9)$; $N \cong \mathrm{PSL}_3(4)$; $N \cong {}^2\mathrm{B}_2(2^{2n+1})$, $n \geqslant 1$.

Thus far, we have dealt with the cases when $N \cong \mathrm{PSL}_2(5) \cong \mathrm{A}_5$ and $N \cong \mathrm{PSL}_2(9) \cong \mathrm{A}_6$. For $N \cong \mathrm{PSL}_2(7)$ the result follows by checking the character tables in the Atlas [8]. Suppose that $N \cong \mathrm{PSL}_2(p)$, where $p \geqslant 17$, is a Fermat or Mersenne prime. Then the centralizer of an involution in N is a dihedral group of order 2^n, $n \geqslant 4$. Hence N contains elements of order 4 and 8. Using Lemmas 2.5 and 2.6, we conclude that G does not satisfy property ($\star\star$).

Suppose $N \cong \mathrm{PSL}_3(4)$. Then for an almost simple G such that $|G/N| \leqslant 6$, our result follows by checking the explicit character tables in the Atlas [8]. For the case when $|G/N| = 12$, we obtain the character table from GAP [13] and by checking the pertinent information, our result follows.

We may assume that $N \cong {}^2\mathrm{B}_2(2^{2n+1})$ with $n \geqslant 1$. The result basically follows from [18, Proposition 3.13] but we shall prove it here for completeness. Now N has two conjugacy classes of elements of order 4 by [25, Proposition 18]. Since the outer automorphism group is cyclic of odd order $2n + 1$, the outer automorphisms cannot fuse these two conjugacy classes to one conjugacy class in G. Hence G has two conjugacy classes of order 4 and so G does not satisfy property ($\star\star$). This concludes our argument. □

4. Normal 2-complements

Given a finite set of positive integers Y, the prime graph $\Pi(Y)$ is defined as the undirected graph whose vertices are the primes p such that there exists an element of Y divisible by p, and two distinct vertices p, q are adjacent if and only if there exists an element of Y divisible by pq. The vanishing prime graph of G, denoted by $\Gamma(G)$, is the prime graph $\Pi(\mathrm{Vo}(G))$. We shall state a result on solvable groups with disconnected vanishing prime graphs. We first recall two definitions:

A group G is said to be a 2-*Frobenius group* if there exists two normal subgroups F and L of G such that G/F is a Frobenius group with kernel L/F and L is a Frobenius group with kernel F.

A group G is said to be a *nearly* 2-*Frobenius group* if there exist two normal subgroups F and L of G with the following properties: $F = F_1 \times F_2$ is nilpotent, where F_1 and F_2 are normal subgroups of G. Furthermore, G/F is a Frobenius group with kernel L/F, G/F_1 is a Frobenius group with kernel L/F_1, and G/F_2 is a 2-Frobenius group.

Theorem 4.1. [11, Theorem A] *Let G be a finite solvable group. Then $\Gamma(G)$ has at most two connected components. Moreover, if $\Gamma(G)$ is disconnected, then G is either a Frobenius group or a nearly 2-Frobenius group.*

The following is a classification of Frobenius complements.

Theorem 4.2. [5, Theorem 1.4] *Let G be a Frobenius group with Frobenius complement M. Then M has a normal subgroup N such that all Sylow subgroups of N are cyclic and one of the following holds:*

(a) $M/N \cong 1$;
(b) $M/N \cong \mathrm{V}_4$, *the Sylow 2-subgroup of the alternating group* A_4;
(c) $M/N \cong \mathrm{A}_4$;
(d) $M/N \cong \mathrm{S}_4$;
(e) $M/N \cong \mathrm{A}_5$;
(f) $M/N \cong \mathrm{S}_5$.

Proof Theorem B. We first assume that G satisfies property $(\star)$. Since G is solvable, G contains at most two vanishing conjugacy classes by Theorem 4.1. If G has one vanishing class, then every irreducible character of G vanishes on at most one conjugacy class and by [7, Proposition 2.7], G is a Frobenius group with a Frobenius complement of order 2 and an odd order kernel. Hence G has an abelian normal 2-complement. Suppose that G has exactly two conjugacy classes. Note that $\mathrm{ord}(v\mathcal{C}_1) \neq \mathrm{ord}(v\mathcal{C}_2)$. Then every irreducible character of G vanishes on at most two conjugacy classes. Using Theorem 2.10, we have two cases. Suppose that Theorem 2.10(b)(ii) holds. Then G has at least two vanishing conjugacy classes of order 3, a contradiction. Assume that Theorem 2.10(b)(i) holds. Then G has one vanishing conjugacy class with elements of order 2 contained in the Frobenius complement of G/Z. By Lemma 2.2(b), G is a Frobenius group with an abelian kernel and complement of order two, that is, $Z = 1$ and the result follows.

Assume that G satisfies property $(\star\star)$ and suppose $\gcd(\mathrm{ord}(v\mathcal{C}_i), \mathrm{ord}(v\mathcal{C}_j)) = 2$ for some $i \neq j$. Let G be nilpotent. Then $G = P_2 \times H$ with P_2, the Sylow 2-subgroup of G and H, a nilpotent group of odd order. If H is non-abelian, then H has a vanishing h of G by [16, Theorem B]. This means that $\chi(h) = \theta_1(1) \times \theta_2(h) = \theta_2(h) = 0$ for some $\chi = \theta_1 \times \theta_2 \in \mathrm{Irr}(P_2 \times H)$ with $\theta_1 \in \mathrm{Irr}(P_2)$ and $\theta_2 \in \mathrm{Irr}(H)$. It follows that for any $x \in P_2$, $\chi(xh) = \theta_1(x) \times \theta_2(h) = 0$ and so xh and h are vanishing elements of G, a contradiction. Thus H is abelian which implies that G has a normal abelian 2-complement.

Let $2 = p_1 < p_2 < \cdots < p_n$ and let P_i be a Sylow p_i-subgroup of G for $i \in \{1, 2, \ldots, n\}$. Suppose that G is a non-nilpotent supersolvable group and consider $\mathbf{F}(G)$. If $\pi(|\mathbf{F}(G)|) = 1$, then by [1, Theorems 6.2.5 and 6.2.2], $\mathbf{F}(G)$ is the p_n-subgroup P_n of G. Let M be a normal subgroup of G such that $M/\mathbf{Z}(\mathbf{F}(G))$ is a chief factor of G. Then $M \setminus \mathbf{Z}(\mathbf{F}(G))$ is a conjugacy class of G since $M \setminus \mathbf{Z}(\mathbf{F}(G)) \subseteq \mathrm{Van}(G)$ and $\gcd(\mathrm{ord}(v\mathcal{C}_i), \mathrm{ord}(v\mathcal{C}_j)) \leqslant 2$. Hence $M/\mathbf{Z}(\mathbf{F}(G))$ is cyclic and so M is abelian. Thus $M = \mathbf{F}(G) = P_n$.

Suppose that p_{n-1} is an odd prime. Then $P_{n-1}\mathbf{F}(G)/\mathbf{F}(G)$ is a normal subgroup of $G/\mathbf{F}(G)$. Thus $P_{n-1}\mathbf{F}(G) \setminus \mathbf{F}(G)$ is a conjugacy class since $\gcd(\mathrm{ord}(v\mathcal{C}_i), \mathrm{ord}(v\mathcal{C}_j)) \leqslant 2$. By Lemma 2.2(b), $\mathbf{F}(G)P_{n-1}P_{n-2}$ is a Frobenius group of kernel $\mathbf{F}(G)P_{n-1}$ and complement of order p_{n-2}. This means that the kernel $\mathbf{F}(G)P_{n-1}$ is nilpotent, that is, $\mathbf{F}(G)P_{n-1} = \mathbf{F}(G) \times P_{n-1}$. Since $P_{n-1}P_{n-2} \cdots P_1$ is supersolvable and since by [1, Theorems 6.2.5 and 6.2.2], P_{n-1} is normal in $P_{n-1}P_{n-2} \cdots P_1$, we obtain that P_{n-1} is a normal subgroup of G, a contradiction since $\mathbf{F}(G) = P_n$. Then $P_{n-2} = P_1$ is a Sylow 2-subgroup of G and so $\pi(|G|) \leqslant 3$. Hence $\mathbf{F}(G)P_{n-1}$ is a metabelian normal 2-complement of G.

Suppose that $\pi(|\mathbf{F}(G)|) \geqslant 2$. Let $\mathbf{F}(G) = Q_1 \times Q_2 \times \cdots \times Q_n$, where $2 = q_1 < q_2 < \cdots < q_n$ and let Q_i be the Sylow q_i-subgroup of $\mathbf{F}(G)$ for $i \in \{1, 2, \ldots, m\}$. Note that $Q_i \setminus \mathbf{Z}(Q_i) \subseteq \mathrm{Van}(G)$ by [16, Theorem B] since G is supersolvable and so all the non-vanishing elements of G are contained in $\mathbf{Z}(\mathbf{F}(\mathbf{G})) = \mathbf{Z}(Q_1) \times \mathbf{Z}(Q_2) \times \cdots \times \mathbf{Z}(Q_n)$. If q_i is an odd prime and there exists a q_i-element m which is a vanishing element, then by Corollary 2.3, there exists a vanishing element mn whose order is divisible by every prime in $\pi(|\mathbf{F}(G)|)$, a contradiction since $\gcd(\mathrm{ord}(m), \mathrm{ord}(mn)) > 2$. Hence Q_i is abelian for all odd q_i's. Consider Q_i, the Sylow 2-subgroup of $\mathbf{F}(G)$. If $Q_1 \setminus \mathbf{Z}(Q_1)$ has an element of order greater than 2, then by Corollary 2.3 and the above argument, we obtain a contradiction. Then $Q_1 \setminus \mathbf{Z}(Q_1)$ consists only of involutions. By [6, Theorem C], Q_1 is a direct product of an elementary abelian 2-group and a Frobenius group with Frobenius complement of order 2, a contradiction. Hence Q_1 is abelian and therefore $\mathbf{F}(G)$ is abelian. Thus G is metabelian.

Finally suppose $2 = p_1 < p_2 < \cdots < p_n$ and let P_i be a Sylow p_i-subgroup of G for $i \in \{1, 2, \ldots, n\}$. By [1, Theorems 6.2.5 and 6.2.2], $H = P_nP_{n-1} \ldots P_2$ is a normal subgroup of G and therefore H is a metabelian 2-complement as required.

We may now assume that G is not supersolvable and $\mathbf{O}_2(G) = 1$. Suppose that $\Gamma(G)$ consists of a single vertex of an odd prime. Since G satisfies property $(\star\star)$, we have that G has one vanishing conjugacy class. This means that every irreducible character of G vanishes on at most one conjugacy class and by [7, Proposition 2.7], G is a Frobenius group with a Frobenius complement of order 2, a contradiction. So if $\Gamma(G)$ is connected, then property $(\star\star)$ implies that every element of $\mathrm{Vo}(G)$ is divisible by 2. By [9, Corollary B], G has a normal nilpotent 2-complement as required. We may assume that $\Gamma(G)$ is disconnected. Then by Theorem 4.1, G is either a Frobenius group or a nearly 2-Frobenius group.

Suppose that G is a Frobenius group. Assume further that the Frobenius complement of G has odd order. Let H be a maximal subgroup of G that contains G'. Then $|G/H| = p$ for some odd prime p. Note that $\mathbf{F}(G) \leq H$. Hence $G \setminus H \subseteq \mathrm{Van}(G)$. Since p divides the order of every element in $G \setminus H$. By Lemma 2.2(a), $G \setminus H$ has at least two conjugacy classes, $v\mathcal{C}_1$ and $v\mathcal{C}_2$ say, and $p \mid \gcd(\mathrm{ord}(v\mathcal{C}_1), \mathrm{ord}(v\mathcal{C}_2))$, contradicting our hypothesis. We may assume that the Frobenius complement has even order. Denote it by M. Then the Frobenius kernel K is abelian. Using Theorem 4.2, we have that M has a unique normal subgroup N such that all the Sylow subgroups of N are cyclic and $M/N \in \{1, \mathrm{V}_4, \mathrm{A}_4, \mathrm{S}_4, \mathrm{A}_5, \mathrm{S}_5\}$. Since G is solvable we need not consider $\mathrm{A}_5, \mathrm{S}_5$. Note that N is metacyclic and supersolvable by [23, p. 290]. Suppose that $M/N \in \{1, \mathrm{V}_4\}$. Then since N is supersolvable, the Hall $2'$-subgroup R of N is normal in $N = M$. Now KR is a normal 2-complement of G. Also note that KR is of derived length at most 3 and thus of Fitting height at most 3, as required. Suppose $M/N \cong \mathrm{A}_4$. Then there exists a normal subgroup T of G such that $|G/T| = 3$. The result follows using the argument above in the case when the Frobenius complement is of odd order. We now suppose that $M/N \cong \mathrm{S}_4$. Note that $\Gamma(G)$ has two connected components and vertex 3 is isolated. A Sylow 2-subgroup T of G is a generalized quaternion. This means that $|T| = 8$ and $|N|$ is of odd order, otherwise G does not satisfy property $(\star\star)$. If there is a prime $r \neq 3$ such that $r \mid |N|$, then using [11, Proposition 3.2], there exists an vanishing element g such that $\mathrm{ord}(g)$ is either divisible by $3s$ or rs for some prime s such that

$s \mid |M|$, a contradiction. But that means the Frobenius complement has a cyclic Sylow 3-subgroup of order k greater than 3. Then G has vanishing elements of orders 3 and k, a contradiction. Hence $N = 1$ and therefore $G/K \cong \mathrm{S}_4$. The result follows.

Suppose that G is a nearly 2-Frobenius group. Then there exist two normal subgroups F and L of G with the following properties: $F = F_1 \times F_2$ is nilpotent, where F_1 and F_2 are normal subgroups of G. Furthermore, G/F is a Frobenius group with kernel L/F, G/F_1 is a Frobenius group with kernel L/F_1, and G/F_2 is a 2-Frobenius group. Since G/F_2 is a 2-Frobenius group and G/F is a Frobenius group with kernel L/F, it follows that L/F_2 is a Frobenius group with kernel F/F_2. By [11, Remark 1.2], G/L is cyclic and L/F is cyclic with $|L/F|$ odd. If $|G/L|$ is odd, then using the argument in the first part of the previous paragraph, we obtain a contradiction. Hence we may assume that $|G/L|$ is even. Since G/F_1 is a Frobenius group with kernel L/F_1, we conclude that L/F_1 is nilpotent and $\gcd(|G/L|, |L/F_1|) = 1$. Note that $|F|$ is odd since $\mathbf{O}_2(G) = 1$. We consider G/F, a Frobenius group with a cyclic kernel L/F and a cyclic Frobenius complement G/L. It follows that the Hall $2'$-subgroup J/L of G/L is cyclic. Hence J/L is cyclic, L/F_1 is nilpotent and F_1 is nilpotent, that is J is a normal 2-complement of G with Fitting height at most 3. This concludes our proof. □

Proof of Corollary C. If G satisfies property $(\star)$, then G is a Frobenius group with an abelian kernel and complement of order two by the argument in the first paragraph of the proof of Theorem B. The converse holds because if G is a Frobenius group with an abelian kernel and complement of order two, then $\mathrm{Van}(G)$ contains only one conjugacy class of elements of order two. □

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