A NOTE ON REPRESENTATION OF BSDE-BASED DYNAMIC RISK MEASURES AND DYNAMIC CAPITAL ALLOCATIONS.

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Abstract. We derive a representation for dynamic capital allocation when the underlying asset price process includes extreme random price movements. Moreover, we consider the representation of dynamic risk measures defined under Backward Stochastic Differential Equations (BSDE) with generators that grow quadratic-exponentially in the control variables. Dynamic capital allocation is derived from the differentiability of BSDEs with jumps. The results are illustrated by deriving a capital allocation representation for dynamic entropic risk measure and static coherent risk measure.

1. Introduction

In investment management, risk describes the changes in future value of a position, due to uncertain events. Moreover, risk measurement is the process of quantifying uncertainty in the future value of a financial position. Different types of risk measures are proposed and used in the literature and the finance industry, with Value-at-Risk (VaR) being the most popular. VaR quantifies the maximum possible financial losses over a given time horizon and confidence level. However, Artzner et al. (1999) identified the shortcomings of VaR. It fails to recognise diversification and it is not time consistent. To counter the weaknesses of VaR, Artzner et al. (1999), proposed coherent risk measure and described it as a function that satisfies four properties: translation invariance, monotonicity, subadditivity and positive homogeneity. (See also Delbaen (2002) for coherent risk measures in the general probability space). Corehent risk measure promotes diversification because the risk of holding a portfolio of assets is less than the risk of holding individual assets. Later, Föllmer and Schied (2002) and independently Frittelli and Rosazza (2002) showed that the risk of a portfolio increases nonlinearly with the size of the position because of additional liquidity. As a result, they extended the work of Artzner et al. (1999) by relaxing the properties of positive homogeneity and subadditivity to introduce the concept of convex risk measures. A convex risk measure takes into consideration that the risk of a position may increase in a nonlinear way as a position multiplies by a large factor.

In the abovementioned papers, the authors consider risk measure in a single-period setting. The ideal situation is to measure the risk of a financial position continuously throughout the investment period. Consequently, there is a need for the concept of dynamic

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risk measures. In a dynamic setting, the risk measure is updated over time according to available information. An important property of dynamic risk measure is time consistency, which describes how risk quantifications at different times are interrelated. Various authors have extended the concept of static risk measure to dynamic risk measure. Peng (1997) introduced g-expectations as nonlinear expectations based on a BSDE

$$\begin{aligned} dY(t) &= -g(t, Y(t), Z(t))dt + Z(t)dW(t), \\ Y(T) &= \xi, \end{aligned} \tag{1.1}$$

where the solution is a pair of $\mathcal{F}_t$-adapted processes $(Y(t), Z(t))$. Rosazza (2006) showed that conditional g-expectations represents a dynamic risk measures under the diffusion BSDE (see also Barrieu and Karoui (2007), Frittelli and Rosazza (2002), Peng (2004)). Jiang (2008) proved that g-expectation satisfies the translation invariance property if and only if the generator $g(t, y, z)$ is independent of y and is convex with respect to z for all t. Quenez and Sulem (2013) studied properties of dynamic risk measures based on BSDEs with jumps (see also Øksendal and Sulem (2015) for applications). An extension of quadratic BSDE to jumps was studied by Karoui, Matoussi, and Armand (2016) and they includes an application to entropic risk measures. In this paper we will work with dynamic risk measures that arise as the solution of quadratic BSDE with jumps.

Risk measures are used to determine the amount required to hold as a buffer against unexpected losses for a portfolio. Risk measures can be further used to measure the risk contribution of a subportfolio in a overall portfolio (see for example Cherny (2009), Buch and Dorfleitner (2008), Denault (2001), Kalkbrener (2005) and Tasche (2004)). Capital allocation is the problem of measuring the risk contribution of sub-portfolio in the overall portfolio risk. The methods that are mostly used and studied are the full allocation property of the Aumann-Shapley and Gradient allocation method. Denault (2001) provided the properties of coherent capital allocation. These are the "no undercut", symmetry and riskless allocation, which together justify the gradient allocation principal. The gradient allocation is the Gâteaux derivative of the risk measure of a portfolio in the direction of the subportfolio. Tasche (2004) showed that if the risk measure is smooth, then the partial derivative of the risk measure with respect to the underlying asset is the unique gradient allocation principle. As a result, the risk measure needs to be Gâteaux-differentiable for the gradient allocation to exist. Denault (2001) showed that the Aumman Shapley value is coherent and a practical approach to capital allocation. Kalkbrener (2005) further provided the properties for gradient allocation principle, and shows that the properties are satisfied if and only if the risk measure is positive homogeneous and sub-additive. The gradient allocation properties provided by Denault (2001) are shown to be equivalent to the risk measure axioms of positive homogeneity, sub-additivity and translation invariance respectively (see Buch and Dorfleitner (2008)). For more analysis on the gradient allocation method, see e.g., Tasche (2007).

This paper is motivated by the representation of BSDE-based dynamic risk measures and dynamic capital allocation. Kromer and Overbeck (2014) derived and analysed the dynamic capital allocation under the diffusion case. They use the results of Ankirchner, Imkeller, and Reis (2007) for the differentiability of the BSDE to determine the capital allocation representation under the diffusion case. Our contribution is to extend the dynamic capital allocation to include jumps. The jump-diffusion model is essential to capture the extreme movements in a risky asset, for example, caused by the announcement of an important decision made by a company or change in economic policy to the financial market (Rong (2006)). Dynamic risk measures for BSDE with jumps are studied and analysed by Quenez and Sulem (2013) and Øksendal and Sulem (2015). However, the authors did not consider the capital allocation of the risk measure under the jumps framework.

The remainder of the paper is organised as follows. In Section 2, we present the notations and define concepts that will be used throughout the paper. Section 3, we derive the representation of dynamic risk capital allocation based on the BSDE with jumps. From dynamic risk capital allocation results we derive the representation of the BSDE based dynamic convex and coherent risk measures. We conclude in Section 4 with applications of our results to the entropic risk measures.

2. Preliminaries

In this section, we introduce the main concepts and notations to be used throughout the paper. Let our source of randomness be modelled by two independent processes: the one-dimensional standard Brownian motion, $W = \{W(t), \mathcal{F}(t); 0 \leq t \leq T\}$, defined on a probability space $(\Omega^W, \mathcal{F}^W, \mathbb{P}^W)$, and the independent compensated Poisson random measure, $\tilde{N}(dt, d\zeta) := N(dt, d\zeta) - \nu(d\zeta)dt$ defined on the probability space $(\Omega^{\tilde{N}}, \mathcal{F}^{\tilde{N}}, \mathbb{P}^{\tilde{N}})$, with ν on $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$ as the Lévy measure of $N(\cdot, \cdot)$. If we let $\mathcal{B}(\mathbb{R}_0)$ denote the family of Borel sets $A \subset \mathbb{R}$. Then the Poisson random measure $N(A, t)$, counts the number of jumps of size $\Delta X \in A$ that occur on or before time t and its derivative is given by $N(d\zeta, dt)$ (Øksendal and Sulem (2005)).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the product of the canonical filtered probability spaces $(\Omega^W \times \Omega^{\tilde{N}}, \mathcal{F}^W \otimes \mathcal{F}^{\tilde{N}}, \mathbb{P}^W \otimes \mathbb{P}^{\tilde{N}})$ and the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ is the canonical filtration. We introduce the notation of the following spaces: Let $p \geq 2$.

- $L^2(\mathcal{F}_T)$ is the space of $\mathcal{F}_T$-measurable, square integrable random variable ξ.
- Let $L^p(\mathcal{F}_t)$ be the space of all real-valued $\mathcal{F}_t$-measurable, p-integrable random variables.
- $\mathbb{S}^p(\mathbb{R})$ is the space of $\mathbb{R}$-valued $Y : \Omega \times [0, T]$ càdlàg processes such that

$$\mathbb{E}[\sup_{t\in[0,T]} |Y(t)|^p] < \infty.$$

- $\mathbb{S}^\infty(\mathbb{R})$ is the space of $\mathbb{R}$-valued essentially bounded càdlàg processes Y such that

$$||Y||_{S^\infty} := ||\sup_{t\in[0,T]} |Y(t)|\,||_\infty < \infty.$$

- $\mathbb{H}^2_W(\mathbb{R})$ is the space of predictable processes $Z : \Omega \times [0,T] \to \mathbb{R}$ such that
$$\mathbb{E}[\int_0^T |Z(s)|^2 ds] < \infty.$$
- $\mathbb{H}^2_N(\mathbb{R})$ denotes the space of predictable processes $\Upsilon : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$, satisfying
$$\mathbb{E}\big[\int_0^T \int_{\mathbb{R}_0} |\Upsilon(t,\zeta)|^2 \nu(d\zeta)dt\big] < \infty.$$
- $L^\infty(\nu)$ is the space of measurable $\mathbb{R}$-valued measurable functions $v(d\zeta)$-almost everywhere (a.e.), which is essentially bounded.

We define by $\mathcal{X} \subset L^2(\mathcal{F}_T)$ the space of financial positions X and let $\mathcal{P}$ denote the $\mathbb{F}$-predictable σ-algebra on $\Omega \times [0,T]$.

We recall the martingale representation property as in Delong (2013). Suppose $M(t)$ is a $\mathcal{F}_t$-local martingale, then $M(t)$ has the following representation
$$M(t) = M(0) + \int_0^t Z(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \Upsilon(s,z)\tilde{N}(ds,d\zeta) \quad 0 \le t \le T,$$
where Z and Υ are predictable processes, integrable with respect to W and $\tilde{N}$.

Let $X(t)$ be a Lévy process with a semi-martingale decomposition $X(t) = X(0)+M(t)-V(t)$, where V is the continuous finite variation drift defined by
$$V(t) = \int_0^t \left[\mu + \frac{\sigma^2}{2} + \int_{|\zeta|<1} (e^{\Upsilon(s,\zeta)} - 1 - \Upsilon(s,\zeta))\nu(d\zeta)\right] ds$$
and M is the local martingale given by
$$M(t) = M(0) + \int_0^t Z(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} (e^{\Upsilon(s,\zeta)} - 1)\tilde{N}(ds,d\zeta).$$

Given a local martingale $M(t)$, $M(0) = 0$, then an adapted process $\Gamma(t)$ that has a stochastic differential equation of the form $d\Gamma(t) = \Gamma(t)dM(t)$, $\Gamma(0) = 1$ is the stochastic exponential of $M(t)$, denoted by $\Gamma(t) = \mathcal{E}(M)(t)$ and defined as
$$\mathcal{E}(M(t)) = \exp\{M(t) - \frac{1}{2}\langle M^c(t)\rangle\} \times \prod_{0\le s\le t} (1 + \Delta M^J(s))e^{-\Delta M^J(s)},$$
where $\langle M\rangle$ denotes a quadratic variation of a process M and M^c, M^J are continuous and discontinuous part of M, respectively. Moreover, we introduce the notion of martingales of bounded mean oscillation (*BMO*-martingales) for jump-diffusion processes as in Morlais (2009). A local Martingale M is in the class of *BMO*-martingales if there exists a constant K, $K > 0$, such that, for all $\mathcal{F}$-stopping times $\mathcal{T}$,
$$ess \sup_{\Omega} \mathbb{E}[\langle M(T)\rangle - \langle M(\mathcal{T})\rangle \mid \mathcal{F}_{\mathcal{T}}] \le K^2 \qquad \text{and} \qquad |\Delta M(\mathcal{T})|^2 \le K^2 .$$

For the diffusion case, the BMO-martingale property follows from the first condition, whilst in a jump-diffusion case, we need to ensure the boundedness of the jumps of the local martingale M. Now we recall the Kazamaki's criterion from Morlais (2009) (also see Kazamaki (2006) Theorem 2.3 for the continuous case) in the next lemma.

Lemma 2.1. *Let δ be such that: $0 < \delta < \infty$ and M a BMO martingale satisfying $\Delta M(t) \geq -1 + \delta$, $\mathbb{P}$-a.s. and for all t, then $\mathcal{E}(M)$ is a true martingale.*

Proof. See Kazamaki (1979). □

2.1. **Risk Measures and Capital Allocation notation.**

Definition 2.1. (see Artzner et al. (1999), Rosazza (2006)) A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is a static risk measure if, for any X and Y in $\mathcal{X}$, it satisfies the following axioms:

1) Monotonicity: $\rho(X) \leq \rho(Y)$, $\forall\ Y \leq X$;
2) Translation invariance: $\rho(X + m) = \rho(X) - m$, $m \in \mathbb{R}$;
3) Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$;
4) Positive homogeneity $\rho(kX) = k\rho(X)$, $k \geq 0$;
5) Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$, $\lambda \in (0, 1)$.

The functional $\rho(X)$ quantifies the risk of a financial position $X \in \mathcal{X}$. The position X is acceptable when $\rho(X) \leq 0$, and unacceptable otherwise Artzner et al. (1999). The functional $\rho(X)$ represents the capital amount that an investor can withdraw without changing the acceptability of X. Monotonicity implies that ρ is nonincreasing with respect to $X \in \mathcal{X}$. The financial meaning is that if a financial position X is always higher than Y, then the capital required to support X should be less than capital required for Y. Subadditivity allows for risk to be reduced by diversification since the risk of a portfolio $X + Y$ is bounded by the sum of individual risk of position X and Y. Translation invariance states that if you add a certain amount m to the initial investment position, then the risk of that investment will decrease by that amount m. Note that, if a position X is not acceptable, then adding an amount $\rho(X)$ to it will make the position acceptable, i.e. $\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0$. Positive homogeneity tells us that the capital required to support k identical positions is equal to k times the capital required for one position. The convexity property illustrates how the risk of a position might increase in a nonlinear way as the position is multiplied by a factor, due to liquidity risk of a large financial position.

A convex risk measure ρ whose domain includes $\mathcal{X}$ such that $\rho(X) < \infty$ where $X \in \mathcal{X}$, satisfies property 5) see Föllmer and Schied (2002) and Frittelli and Rosazza (2002), while a coherent risk measure satisfies properties 1) to 4) see Artzner et al. (1999) and Delbaen (2002). We state from (Rosazza (2006)) the following definition of a dynamic risk measure:

Definition 2.2. A mapping $(\rho_t)_{t\in[0,T]}$ is a dynamic risk measure for all $X, Y \in \mathcal{X}$ and $t \in [0, T]$, if the following properties are satisfied:

(a) $\rho_t : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t)$.
(b) ρ_0 is a static risk measure.
(c) $\rho_T(X) = -X$ for all $X \in \mathcal{X}$.

A dynamic risk measure is called coherent if it satisfies, positive homogeneity, monotonicity, translation invariance and subadditivity. A dynamic convex risk measure satisfies the convexity property and assume $\rho_t(0) = 0$ for any $t \in [0, T]$ (Rosazza (2006)).

Let $X_1, X_2, \ldots, X_n \in \mathcal{X}$ be the financial positions, with the corresponding risk contribution to the overall portfolio denoted by $\rho(X_i|X)$, $i = 1, 2, \ldots, n$. Consider a portfolio $X \in \mathcal{X}$, consisting of X_i, subportfolios, that is

$$X = \sum_{i=1}^{n} X_i.$$

The portfolio risk is given by $\rho(X)$. The capital allocation problem is allocating the overall risk $\rho(X)$ of the portfolio X to the individual subportfolios in the portfolio. That is, we require a mapping such that

$$\rho(X) = \sum_{i=1}^{n} \rho(X_i|X). \tag{2.1}$$

Such a relation is called the *full allocation property*, since the overall portfolio risk is fully allocated to the individual subportfolios in the portfolio (Tasche (2007)).

Let ρ be the risk measure that is Gâteaux differentiable at X in its domain, then gradient allocation $\rho(X_i|X)$ is determined by

$$\begin{aligned} \rho(X_i|X) &= \nabla_{X_i}\rho(X) \\ &= \lim_{\epsilon \to 0} \frac{\rho(X + \epsilon X_i) - \rho(X)}{\epsilon} \\ &= \frac{d}{d\epsilon}\rho(X + \epsilon X_i)\Big|_{\epsilon=0}. \end{aligned} \tag{2.2}$$

Note that the gradient of a continuous differentiable risk measure $\rho(X_i|X)$ is the unique allocation principle (see proposition 2.1 in Tasche (2007)). Equation (2.2) defines the static gradient allocation principle, which is the Gâteaux-derivative of X in the direction of X_i, for $i = 1, 2, \ldots, n$ (see Kromer and Overbeck (2014) and Kromer and Overbeck (2017)). The static Aumann-Shapley allocation is represented by:

$$\overline{\nabla_{X_i}\rho(X)} = \int_0^1 \nabla_{X_i}\rho(\beta X)d\beta, \qquad i = 1, 2, \ldots, n, \tag{2.3}$$

where $\beta \in [0, 1]$ is taken to be portfolio weights. If the risk measure ρ is positive homogeneous, then the Aumann-Shapley allocation reduces to the gradient allocation principle (2.2) (Denault (2001)). For the Aumann-Shapley, we do not require the risk measure to be positively homogeneous to satisfy full allocation property. However, the gradient allocation does need the risk measure to be positive homogeneous to satisfy the full allocation property. According to Kromer and Overbeck (2014), the Aumann-Shapley and Gâteaux-derivative can be jointly used to risk measures that do not satisfy the positive homogeneity property. Hence, the combination can be used for convex risk measures that do not satisfy the positive homogeneity property.

2.2. **BSDE and BSDE differentiability.** In this work, the dynamic risk measures are constructed using BSDEs. We consider a quadratic exponential BSDE, (defined as in Karoui, Matoussi, and Armand (2016)) for $t \in [0, T]$ of the form

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s), \Upsilon(s,\zeta))ds - \int_t^T Z(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon(s,\zeta)\tilde{N}(ds, d\zeta), \tag{2.4}$$

where $\xi : \Omega \to \mathbb{R}$ and $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The solution to Equation (2.4) is given by the triple $(Y(t), Z(t), \Upsilon(t)) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2_W(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$, where the adapted process $Y(t)$ is controlled by the control processes $Z(t)$ and $\Upsilon(t)$, such that $Y(T) = \xi$. The following definition from Delong (2013) defines risk measures constructed as a solution of BSDEs.

Definition 2.3. Let $\rho^g_t(\xi) := Y^\xi(t)$, $t \in [0, T]$. Then ρ is monotone, time-consistent dynamic risk measure. In addition,

(a) if g is sublinear in (z, Υ) and independent of y, then ρ is a coherent dynamic risk measure.
(b) If g is convex in (y, z, Υ), then ρ is a convex dynamic risk measure

The component Y^ξ is the solution of the BSDE (2.4). The driver g plays an essential role in the construction of risk measures by BSDE. For the existence and uniqueness of such BSDEs, the driver and terminal condition are subject to the following assumptions. We adapt the assumptions from Fujii and Takahashi (2018) (see also Briand and Hu (2006), Karoui, Matoussi, and Armand (2016), Royer (2006), Delong (2013)).

Assumption 1.

(i) The map $(\omega, t) \mapsto g(\omega, t, \cdot)$ is $\mathbb{F}$-progressively measurable. For every $(y, z, \Upsilon) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, there exist two constants $\vartheta \geq 0$ and $\gamma > 0$ and a positive $\mathbb{F}$-progressively measurable process $(\ell_t, t \in [0, T])$ such that

$$\begin{aligned} -\ell_t - \vartheta|y| - \frac{\gamma}{2}|z|^2 - \int_{\mathbb{R}_0} j_\gamma(-\Upsilon(\zeta))\nu(d\zeta) &\leq g(t, y, z, \zeta) \\ &\leq \ell_t + \vartheta|y| + \frac{\gamma}{2}|z|^2 + \int_{\mathbb{R}_0} j_\gamma(\Upsilon(\zeta))\nu(d\zeta), \end{aligned} \tag{2.5}$$

$dt \otimes d\mathbb{P}$-a.e. $(\omega, t) \in \Omega \times [0, T]$, where $j_\gamma(v) := \frac{1}{\gamma}(e^{\gamma v} - 1 - \gamma v)$.
(ii) $|\xi|$, $(\ell_t, t \in [0, T])$ are essentially bounded i.e., $||\xi||_\infty$, $||\ell||_{S^\infty} < \infty$.

Assumption 2. For $m > 0$ and $(y, z, \Upsilon), (y', z', \Upsilon') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ satisfying

$$|y|, |y'|, ||\Upsilon||_{L^\infty(\nu)}, ||\Upsilon'||_{L^\infty(\nu)} \leq m,$$

there exists some positive constant K_m depending on m such that

$$\begin{aligned}|g(t,y,z,\Upsilon)-g(t,y',z',\Upsilon')| &\le K_m(|y-y'|+||\Upsilon-\Upsilon'||_{L^2})\\ &+K_m\bigg(1+|z|+|z'|+||\Upsilon||_{L^2(\nu)}+||\Upsilon'||_{L^2(\nu)}\bigg)|z-z'| \end{aligned} \tag{2.6}$$

$dt\otimes d\mathbb{P}$ a.e. $(\omega,t)\in\Omega\times[0,T]$.

Assumption 3. For all $t\in[0,T]$, $m>0$ and $y\in\mathbb{R}, z\in\mathbb{R}, \Upsilon,\Upsilon'\in\mathbb{R}$ with $|y|, ||\Upsilon||_{L^\infty(\nu)}$, $||\Upsilon'||_{L^\infty(\nu)}\le m$, there exists a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$-measurable process $\Gamma^{y,z,\Upsilon,\Upsilon'}$ satisfying $dt\otimes d\mathbb{P}$-a.e.

$$g(t,y,z,\Upsilon)-g(t,y',z',\Upsilon')\le\int_{\mathbb{R}_0}\Gamma^{y,z,\Upsilon,\Upsilon'}(t,\zeta)[\Upsilon(t,\zeta)-\Upsilon'(t,\zeta)]\nu(d\zeta) \tag{2.7}$$

and $C_m^1(1\wedge|\zeta|)\le\Gamma_t^{y,z,\Upsilon,\Upsilon'}(\zeta)\le C_m^2(1\wedge|\zeta|)$. Here C_m^1 and C_m^2 are two constants satisfying the following conditions $C_m^1>-1$ and $C_m^2>0$ and are dependent on m.

Fujii and Takahashi (2018) (in Theorem 3.1) proved the existence of a unique bounded solution $(Y,Z,\Upsilon)\in\mathbb{S}^2\times\mathbb{H}_W^2\times\mathbb{H}_N^2$ of the BSDE (2.4). Moreover, Z belongs to the set of progressively measurable real valued functions denoted by $\mathbb{H}^2_{BMO(W)}$ satisfying

$$\bigg|\bigg|\int_0^\cdot Z(s)\bigg|\bigg|^2_{BMO(W)}=ess\sup\mathbb{E}\bigg[\int_\tau^T|Z(s)|^2ds\big|\mathcal{F}_\tau\bigg]\le K^2,\quad \mathbb{P}-a.s.$$

and Υ belongs to the set of predictable processes, denoted by $\mathbb{H}^2_{BMO(N)}$ satisfying the following

$$\bigg|\bigg|\int_0^\cdot\int_{\mathbb{R}_0}\Upsilon(\zeta)\tilde{N}(ds,d\zeta)\bigg|\bigg|^2_{BMO(N)}=ess\sup\mathbb{E}\big[\int_\tau^T\int_{\mathbb{R}_0}|\Upsilon(s,\zeta)|^2\nu(d\zeta)ds\big|\mathcal{F}_t\big]+|\Delta M(\mathcal{T})|\le K^2.$$

To define the gradient allocation, we need the differentiability for BSDE with jumps. In the Brownian case, Kromer and Overbeck (2014) used classical differentiability results for BSDEs adopted from Ankirchner, Imkeller, and Reis (2007). In our case, we use Malliavin's differentiability of the quadratic-exponential BSDE with jumps (see Ankirchner, Imkeller, and Reis (2007), Fujii and Takahashi (2018)).

As in Fujii and Takahashi (2018), we consider the following quadratic-exponential BSDE:

$$\begin{aligned}Y(t) &= \xi-\int_t^T Z(s)dW(s)-\int_t^T\int_{\mathbb{R}_0}\Upsilon(s,\zeta)\tilde{N}(ds,d\zeta)\\ &+\int_t^T g\bigg(s,Y(s),Z(s),\int_{\mathbb{R}_0}p(\zeta)G(s,\Upsilon(s,\zeta))\nu(d\zeta)\bigg)ds\,. \end{aligned} \tag{2.8}$$

for $t\in[0,T]$ where $\xi:\Omega\to\mathbb{R}$, $g:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $p^i:\mathbb{R}\to\mathbb{R}$, $G^i:[0,T]\times\mathbb{R}\to\mathbb{R}$ for each $i=1,\dots,k$. The driver $g\bigg(t,Y(t),Z(t),\int_{\mathbb{R}_0}p(\zeta)G(t,\Upsilon(t,\zeta))\nu(d\zeta)\bigg)$, satisfies Assumptions (1) and (3), where the last arguments denotes a k-dimensional vector whose i-th element is given by $\int_{\mathbb{R}_0} p^i(\zeta)G^i(s,\Upsilon^i(s,\zeta))\nu^i(d\zeta)$. Fujii and Takahashi (2018) assume that for every $i \in \{1,\dots,k\}$, the functions p^i and $G^i(t,\Upsilon)$ are continuous, with p^i satisfying $\int_{\mathbb{R}_0} |p^i(\zeta)|^2\nu^i d(\zeta) < \infty$. The function $G^i(t,v)$ is continuous in both arguments and one-time continuously differentiable with respect to v.

Assumption 4. (Fujii and Takahashi (2018)) Let $v_t = \int_{\mathbb{R}_0} p(\zeta)G(t,\Upsilon(t,\zeta))\nu(d\zeta)$ and $v'_t = \int_{\mathbb{R}_0} p(\zeta)G(t,\Upsilon'(t,\zeta))\nu(d\zeta)$.

(i) The terminal value is Malliavin differentiable; $\xi \in \mathbb{D}^{1,2}$.

(ii) For each $m > 0$ and for every $(y,z,\Upsilon) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ satisfying $|y|, ||\Upsilon||_{L^\infty(\nu)} \le m$, the driver $g(t,y,z,\Upsilon_t)$, $t\in[0,T]$ belongs to $\mathbb{D}^{1,2}$ and its Malliavin derivatives are denoted by $D_s g(t,y,z,v_t)$ and $D_{s,\zeta}g(t,y,z,v_t)$. Furthermore, the driver g is continuously differentiable with respect to its state variables.

(iii) For every $m>0$ and $(y,z,\Upsilon),(y',z',\Upsilon') \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, satisfying $|y|,|y'|,||\Upsilon||_{L^\infty(\nu)}, ||\Upsilon'||_{L^\infty(\nu)} \le m$, the Malliavin derivative of the driver satisfies the following local Lipschitz conditions;

$$|D^i_s g(t,y,z,v_t) - D^i_s g(t,y',z',v'_t)| \le K^{m,i}_s(|y-y'|+|v_t-v'_t|+(1+|z|+|z'|+|v_t|+|v'_t|)|z-z'|)$$

for ds-a.e. $s\in[0,T]$ with $i \in 1,\dots,d$, and

$$|D^i_{s,\zeta} g(t,y,z,v_t) - D^i_{s,\zeta} g(t,y',z',v'_t)| \le K^{m,i}_{s,\zeta}(|y-y'|+|v_t-v'_t|+(1+|z|+|z'|+|v_t|+|v'_t|)|z-z'|)$$

for ds-a.e. $s\in[0,T]$ with $i\in 1,\dots,k$. For ever $m>0$ and (s,ζ), $(K^{m,i}_s(t), t\in[0,T])_{i\in 1,\dots,d}$ and $(K^{m,i}_{s,\zeta}(t), t\in[0,T])_{i\in 1,\dots,d}$ are $\mathbb{R}_+$-valued $\mathbb{F}_t$-progressively measurable processes.

(iv) There exists some positive constant $r\ge 2$ such that

$$\int_{[0,T]\times\mathbb{R}^k}\left(\mathbb{E}\big[|D_{s,\zeta}\xi|^{rq} + \big(\int_0^T |D_{s,\zeta}g(t,0)|dt\big)^{rq} + ||K^m||^{2rq}\big]\right)^{\frac{1}{q}}\tilde{N}(dt,d\zeta) < \infty$$

hold for $\forall q \ge 1$ and $\forall m > 0$.

Fujii and Takahashi (2018) (in Theorem 5.1), proved that under the above assumptions the solution $(Y,Z,\Upsilon)\in\mathbb{S}^2\times\mathbb{H}^2_{BMO(W)}\times\mathbb{H}^2_{BMO(N)}$ of the BSDE (2.8) is Malliavin differentiable with respect to W and $\tilde{N}$. i.e.

(i) There exists a unique solution $(D_sY, D_sZ, D_s\Upsilon)\in\mathbb{S}^2\times\mathbb{H}^2\times\mathbb{H}^2$ to the BSDE

$$\begin{aligned} D_sY(t) &= D_s\xi - \int_t^T D_sZ(u)dW(u) - \int_t^T\int_{\mathbb{R}_0} D_s\Upsilon(u,\zeta)\tilde{N}(du,d\zeta) \\ &\quad + \int_t^T\Big[D_sg(u,\Theta) + \partial_y g(u,\Theta)D_sY(u) + \partial_z g(u,\Theta)D_sZ(u) \\ &\quad + \partial_v g(u,\Theta)\int_{\mathbb{R}_0} p(\zeta)\partial_\Upsilon G(u,\Upsilon(u,\zeta))D_s(\Upsilon(u,\zeta))\nu(d\zeta)\Big]du, \end{aligned} \tag{2.9}$$

for $0\le s\le t\le T$ where $\Theta := (Y(t), Z(t), \int_{\mathbb{R}_0} p(\zeta)G(u,\Upsilon(u,\zeta))\nu(d\zeta))$. The solution satisfies $\int_0^T ||D_sY, D_sZ, D_{s,\zeta}\Upsilon||^2 ds < \infty$.

(ii) There exists a unique solution $(D_{s,\zeta}Y, D_{s,\zeta}Z, D_{s,\zeta}\Upsilon) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$ to the BSDE

$$\begin{aligned} D_{s,\zeta}Y(t) &= D_{s,\zeta}\xi - \int_t^T D_{s,\zeta}Z(u)dW(u) - \int_t^T \int_{\mathbb{R}_0} D_{s,\zeta}\Upsilon(u,\zeta)\tilde{N}(du,d\zeta) \\ &\quad + \frac{1}{\zeta}\Bigg[\int_t^T g\Bigg(u, Y(u) + \zeta D_{s,\zeta}Y(u), Z(u) + \zeta D_{s,\zeta}Z(u), \\ &\quad \int_{\mathbb{R}_0} p(\zeta)G\big(u, \Upsilon(u,\zeta) + \zeta D_{s,\zeta}\Upsilon(u,\zeta)\big)\nu(d\zeta)\Bigg) - g(u,\Theta)\Bigg]du, \end{aligned} \tag{2.10}$$

where $0 \le s \le t \le T$, $\zeta \neq 0$ and for $\zeta^2\nu(\zeta)ds - a.e.$ $(s,\zeta) \in [0.T] \times \mathbb{R}_0$. The solution satisfies $\int_0^T \int_{\mathbb{R}_0} ||D_sY, D_sZ, D_{s,\zeta}\Upsilon||^2\zeta^2\nu(\zeta)ds < \infty$.

2.3. **Capital allocation.** For this paper, we consider the terminal condition ξ of the form $\xi(\epsilon) = \xi + \epsilon\eta$, where $\xi, \eta \in L^\infty(\mathcal{F}_T)$. We will also focus on Malliavin derivative with respect to the Brownian motion W, given by Equation (2.9). Hence, there exists a constant $c \in \mathbb{R}$ such that

$$\sup_{\epsilon \in U} ||\xi(\epsilon)||_\infty \le ||\xi||_\infty + ||\eta||_\infty \sup_{\epsilon \in U} |\epsilon| < c,$$

for every compact set $U \subset \mathbb{R}$. In addition, the functional $\epsilon \mapsto \xi(\epsilon)$ is differentiable and the derivative is given by $D_s\xi(\epsilon) = \eta$. The generator g in Equation (2.4) is defined as follows

$$g(t, z, \Upsilon(t,\zeta)) = \ell(t, z, \Upsilon) + \frac{1}{2}\gamma|z|^2 + \frac{1}{\gamma}\int_{\mathbb{R}_0} (e^{\gamma\Upsilon} - 1 - \gamma\Upsilon)\nu(dz) \tag{2.11}$$

and is a special case of the generator in Assumption 1 (i), because it is independent of the process $Y(\cdot)$. For the risk measure to satisfy the translation invariance property, the BSDE generator should be independent of $Y(\cdot)$ (Quenez and Sulem (2013)). The BSDE version of the dynamic gradient allocation is defined as the directional derivative of the risk measure ρ_t at the point ξ in the direction of η_i, that is:

$$\lim_{\epsilon \to 0} \frac{\rho_t(\xi + \epsilon\eta_i) - \rho_t(\xi)}{\epsilon} := D_{\eta_i}\rho_t(\xi) \quad i = 1, 2, \ldots, n\,. \tag{2.12}$$

and from Definition 2.3 we have that

$$D_{\eta_i}\rho_t(\xi) = D_{\eta_i}Y^\xi(t) \quad i = 1, 2, \ldots, n\,.$$

The Malliavin derivative given in Di Nunno, Øksendal, and Proske (2009) (in Definition A.10, A.16 and Lemma A.18) is as follows

$$D_{\eta_i}Y^\xi(t) = \langle D_sY(t), h\rangle = \int_0^T D_sY h_i(s)ds$$

for all

$$\eta_i = \int_0^t h_i(s)ds,$$

is a Malliavin directional derivative in the direction of η_i with respect to the Brownian motion, with $h_i \in \mathbb{D}^{1,2} \subseteq L^2([0,T])$, $i = 1,\dots,n$. See the Appendix Section for the meaning of $\mathbb{D}^{1,2}$. We observe that the Malliavin directional derivative generalizes the classical Gâteaux-derivative. The inner product $\langle D_s Y, h\rangle_H$[1] is given by

$$\begin{aligned}\langle D_s Y(t), h\rangle &= \langle D_s\xi, h\rangle - \int_t^T \langle D_s Z(u), h\rangle dW(u) - \int_t^T \int_{\mathbb{R}_0} \langle D_s \Upsilon(u,\zeta), h\rangle \tilde{N}(du, d\zeta) \\ &\quad + \int_t^T \Big[\partial_z g(u,\hat{\Theta})\langle D_s Z(u), h\rangle \\ &\quad + \partial_v g(u,\hat{\Theta}) \int_{\mathbb{R}_0} p(\zeta)\partial_\Upsilon G(u, \Upsilon(u,\zeta))\langle D_s\Upsilon(u,\zeta), h\rangle \nu(d\zeta)\Big] du \quad 0 \le t \le T. \end{aligned} \tag{2.13}$$

where $\hat{\Theta} := (Z(t), \int_{\mathbb{R}_0} p(\zeta) G(u, \Upsilon(u,\zeta))\nu(d\zeta))$. Equation (2.12) is the directional derivative of the risk measure ρ_t at the point ξ (the portfolio) in the direction of η_i (subportfolio i). It generalizes the concept given in (2.2). We also suppose that $\xi = \sum_{i=1}^n \eta_i$, that is the total sum of the subportfolio should equal to the overall portfolio. Now we are in a position to provide the main result on the representation of the dynamic risk capital allocations as a dynamic gradient allocation.

3. Representation of dynamic risk capital allocations

In this section, we derive the dynamic risk capital allocation induced from BSDEs with jumps. We also obtain the representation of BSDE based dynamic convex and dynamic coherent risk measures. We follow the approach of Kromer and Overbeck (2014) in deriving the representation of capital allocation, BSDE based dynamic convex and coherent risk measures.

Theorem 3.1. *Let $\xi, \eta_i \in L^\infty(\mathcal{F}_T)$, such that $\xi = \sum_{i=1}^n \eta_i$ for each $i = 1, 2, \dots, n$ and $D_{\eta_i} Y(t)$ exists. Suppose that $\partial_z g(t,\hat{\Theta})$ and $\partial_v g(t,\hat{\Theta}) p(\zeta)\partial_\Upsilon G(t, \Upsilon(t,\zeta))$ belong to BMO($\mathbb{P}$). Then the dynamic gradient allocations can be represented by:*

$$D_{\eta_i} Y(t) = D_{\eta_i}\rho_t(\xi) = \mathbb{E}^{\mathbb{Q}^\xi}[-\eta_i \mid \mathcal{F}_t], \quad n = 1, 2, \dots, n,$$

where $\mathbb{Q}^\xi$ is given by

$$\frac{d\mathbb{Q}^\xi}{d\mathbb{P}} := \mathcal{E}\left(\int_0^t \partial_z g(u,\hat{\Theta}) dW + \int_0^t \int_{\mathbb{R}_0} p(\zeta)\partial_v g(u,\hat{\Theta})\partial_\Upsilon G(u, \Upsilon(u,\zeta))\tilde{N}(du, d\zeta)\right)(t). \tag{3.1}$$

Proof. Since belong to BMO($\mathbb{P}$), then the stochastic integrals in (3.1) are said to be BMO($\mathbb{P}$)-martingales and the stochastic exponential is a true martingale (Morlais (2009)). From (Di Nunno, Øksendal, and Proske (2009) Theorem 12.21,) a new equivalent probability measure $\mathbb{Q}^\xi$ is defined by equation (3.1). Furthermore, the processes

$$dW^{\mathbb{Q}^\xi}(t) = dW(t) - \partial_z g(t,\hat{\Theta})dt$$

[1]Note that H is the Cameron-Martin space defined in Appendix.

and

$$\tilde{N}^{\mathbb{Q}^\xi}(dt,d\zeta) = \tilde{N}(dt,d\zeta) - p(\zeta)\partial_v g(t,\hat{\Theta})\partial_\Upsilon G(t,\Upsilon(t,\zeta))\nu(d\zeta)dt$$

are the $\mathbb{Q}^\xi$-Brownian motion and $\mathbb{Q}^\xi$-compensated random measure respectively. We define a function $\Phi_i(t)$ by $\Phi_i(t) := \mathbb{E}^{\mathbb{Q}^\xi}[-\eta_i \mid \mathcal{F}_t]$ for each $i = 1,\ldots,n$ and $t \in [0,T]$. Then from the martingale representation property there exists predictable processes $Z^{\eta_i}(t)$ and $\Upsilon^{\eta_i}(t,\zeta)$ integrable with respect to $W^{\mathbb{Q}^\xi}$ and $\tilde{N}^{\mathbb{Q}^\xi}$ respectively such that

$$\begin{aligned}
\Phi_i(t) &= \Phi_i(T) - \int_t^T Z^{\eta_i}(u)dW(u)^{\mathbb{Q}^\xi} - \int_t^T\int_{\mathbb{R}_0} \Upsilon^{\eta_i}(u,\zeta)\tilde{N}^{\mathbb{Q}^\xi}(du,d\zeta) \quad 0\le t\le T \\
&= -\eta_i - \int_t^T Z^{\eta_i}(u)dW(u) + \int_t^T Z^{\eta_i}(u)\partial_z g(u,\hat{\Theta})du - \int_t^T\int_{\mathbb{R}_0}\Upsilon^{\eta_i}(u,\zeta)\tilde{N}(du,d\zeta) \\
&\quad + \int_t^T\int_{\mathbb{R}_0} p(\zeta)\partial_v g(u,\hat{\Theta})\partial_\Upsilon G(u,\Upsilon(u,\zeta))\Upsilon^{\eta_i}(u,\zeta)\nu(d\zeta)du \quad 0\le t\le T\,. \qquad (3.2)
\end{aligned}$$

Comparing the above equation with the BSDE representing the gradient allocation (2.13) and we know that under the Assumptions 1 to 4 that (2.13) has a unique solution, we can conclude that the dynamic gradient allocation has the representation

$$D_{\eta_i}Y(t) = D_{\eta_i}\rho_t(\xi) = \mathbb{E}^{\mathbb{Q}^\xi}[-\eta_i \mid \mathcal{F}_t]\,, \quad i = 1,2,\ldots,n\,. \qquad (3.3)$$

□

Remark: This result generalizes Theorem 3.1 in Kromer and Overbeck (2014).

From the above theorem, we can immediately obtain the representation result for BSDE based dynamic convex and dynamic coherent risk measures. The results of the representation of BSDE based dynamic convex and coherent risk measures are established from the full allocation property of the Aumann-Shapley allocation (the static case given in Equation (2.3)) (Kromer and Overbeck (2014)).

Corollary 3.2. *Let $\xi \in L^\infty(\mathcal{F}_T)$. Suppose that ℓ is convex in z and Υ and $\partial_{Z^{\beta\xi}}g(s,\hat{\Theta})$, $\partial_v g(t,\hat{\Theta})p(\zeta)\partial_{\Upsilon^{\beta\xi}}G(t,\Upsilon^{\beta\xi}(t,\zeta))$ belong to the class of $BMO(\mathbb{P})$, for any $\beta\in[0,1]$, where $Z^{\beta\xi}(t)$, $\Upsilon^{\beta\xi}(t,\cdot)$ are the controls to the quadratic-exponential BSDE (2.8), with terminal condition $\rho_{t,\beta}(\xi) = -\beta\xi$. Then, the corresponding quadratic-exponential BSDE-based dynamic convex risk measure can be represented by*

$$\rho_t(\xi) = \mathbb{E}[-\Lambda^\xi(T,t)\xi \mid \mathcal{F}_t]\,,$$

where

$$\Lambda^\xi(T,t) = \int_0^1 \frac{\mathcal{E}(M^{\beta\xi}(T))}{\mathcal{E}(M^{\beta\xi}(t))}d\beta\,, \quad \forall t\in[0,T]\,, \qquad (3.4)$$

for $M^{\beta\xi}$ defined by

$$M^{\beta\xi}(t) = \int_0^t \partial_{Z^{\beta\xi}}g(s,\hat{\Theta})dW(s) + \int_0^t\int_{\mathbb{R}_0}\partial_v g(s,\hat{\Theta})p(\zeta)\partial_{\Upsilon^{\beta\xi}}G(s,\Upsilon^{\beta\xi}(s,\zeta))\tilde{N}(ds,d\zeta)\,.$$

Proof. Following Kromer and Overbeck (2014), we consider the following

$$\begin{aligned}\rho_t(\xi) &= \rho_t(1\xi) - \rho_t(0\xi) = \int_0^1 \frac{d}{d\beta}\rho_t(\beta\xi)d\beta \\ &= \int_0^1 \lim_{\epsilon\to 0} \frac{\rho_t((\beta+\epsilon)\xi) - \rho_t(\beta\xi)}{\epsilon} d\beta \\ &= \int_0^1 D_\xi \rho_t(\beta\xi) d\beta \,.\end{aligned}$$

From the previous theorem, we have

$$\rho_t(\xi) = \int_0^1 \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\xi \mid \mathcal{F}_t] d\beta \,. \tag{3.5}$$

Then since $\xi \in L^\infty(\mathcal{F}_T)$, $\mathbb{Q}^{\beta\xi}$ is an equivalent probability measure, $\forall\beta \in [0,1]$. Hence ξ is $\mathbb{Q}^{\beta\xi}$-a.s. bounded. This implies that $\int_0^1 \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\xi \mid \mathcal{F}_t]d\beta < \infty$. Define $\Lambda^\xi(t) = \mathcal{E}(M^{\beta\xi}(t))$. Then, (3.5) can be written by

$$\begin{aligned}\rho_t(\xi) &= \int_0^1 \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\xi \mid \mathcal{F}_t]d\beta = \int_0^1 \frac{1}{\Lambda^{\beta\xi}(t)} \mathbb{E}^{\mathbb{P}^{\beta\xi}}[-\Lambda^{\beta\xi}(T)\xi \mid \mathcal{F}_t]d\beta \\ &= \mathbb{E}\Big[-\Big(\int_0^1 \frac{\Lambda^{\beta\xi}(T)}{\Lambda^{\beta\xi}(t)} d\beta\Big)\xi \mid \mathcal{F}_t\Big] \\ &= \mathbb{E}[-\Lambda^\xi(T,t)\xi \mid \mathcal{F}_t],\end{aligned}$$

which completes the proof. □

Corollary 3.3. *Let $\xi \in L^\infty(\mathcal{F}_T)$. Suppose that g is of the form $g(t,z,\Upsilon) = \ell(t,z,\Upsilon)$ is convex and positively homogeneous in both z and Υ. Moreover, suppose that $\partial_z \ell(t, Z^{\beta\xi}(t), \Upsilon^{\beta\xi}(t,\cdot))$, $\partial_\Upsilon \ell(t, Z^{\beta\xi}(t), \Upsilon^{\beta\xi}(t,\cdot))$ belong to the class of $BMO(\mathbb{P})$, for any $\beta \in [0,1]$, which represent the portfolio weights. Then, the corresponding BSDE-based dynamic coherent risk measure can be represented by*

$$\rho_t(\xi) = \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\xi \mid \mathcal{F}_t] \,,$$

where the $\mathbb{Q}^{\beta\xi}$-measure is given by

$$\begin{aligned}\frac{d\mathbb{Q}^{\beta\xi}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} &= \exp\Big\{ -\int_0^t \partial_z \ell(t, Z^\xi(s), \Upsilon^\xi(s,\zeta))dW - \frac{1}{2}\int_0^t \partial_z \ell(s, Z^\xi(t), \Upsilon^\xi(s,\zeta))^2 ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \Big(\ln\big(1 - \partial_\Upsilon \ell(s, Z^\xi(s), \Upsilon^\xi(s,\zeta))\big) + \partial_\Upsilon \ell(s, Z^\xi(s), \Upsilon^\xi(s,\zeta)) \Big) \nu(d\zeta)ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \ln\big(1 - \partial_\Upsilon \ell(s, Z^\xi(t), \Upsilon^\xi(s,\zeta))\big) \tilde{N}(ds,d\zeta) \Big\}.\end{aligned} \tag{3.6}$$

Proof. From Corollary 3.2, we have the following representation

$$\rho_t(\xi) = \mathbb{E}[-\Lambda^\xi(T,t)\xi \mid \mathcal{F}_t],$$

with Λ defined in (3.4). Given that $g(t,z,\Upsilon) = \ell(t,z,\Upsilon)$ and ℓ is convex and positively homogeneuos, this implies that the corresponding BSDE-based dynamic risk measure $\rho(\cdot)$ satisfies

$$Y^{\beta\xi}(t) = \rho_t(\beta\xi) = \beta\rho_t(\xi) = \beta Y^{\xi}(t) \quad dt \otimes d\mathbb{P} - a.s.$$

for $c > 0$ and $0 \leq t \leq T$. We show this by considering two BSDEs given by

$$\begin{aligned} Y^{\beta\xi}(t) &= -\beta\xi - \int_t^T Z^{\beta\xi}(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\beta\xi}(s,\zeta)\tilde{N}(ds,d\zeta) \\ &\quad + \int_t^T g\big(s, Z^{\beta\xi}(s), \Upsilon^{\beta\xi}(s,\zeta)\big)\big)ds\,, \end{aligned}$$

and

$$\begin{aligned} Y^{\xi}(t) &= -\xi - \int_t^T Z^{\xi}(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\xi}(s,\zeta)\tilde{N}(ds,d\zeta) \\ &\quad + \int_t^T g\big(s, Z^{\xi}(s), \Upsilon^{\xi}(s,\zeta)\big)\big)ds\,. \end{aligned}$$

Then, from the proof of Proposition 6.2.3(b) in Delong Delong (2013) we conclude that $Y^{\beta\xi}(t) = \beta Y^{\xi}(t)$, $Z^{\beta\xi}(t) = \beta Z^{\xi}(t)$ and $\Upsilon^{\beta\xi}(t,\zeta) = \beta\Upsilon^{\xi}(t,\zeta)$.

The above results imply that for the representation of the BSDE coherent risk measure, the process $\mathcal{E}(M^{\beta\xi}(t))(\cdot)$ appearing in (3.4) becomes

$$\begin{aligned} &\mathcal{E}\bigg(\int_0^t \partial_z g(s, Z^{\beta\xi}(s), \Upsilon^{\beta\xi}(s,\zeta))dW(s) + \int_0^t \int_{\mathbb{R}_0} \partial_\Upsilon g(s, Z^{\beta\xi}(s), \Upsilon^{\beta\xi}(s,\zeta))\tilde{N}(ds,d\zeta)\bigg)(t) \\ &= \exp\bigg\{ - \int_0^t \partial_z g(s, \beta Z^{\xi}(t), \beta\Upsilon^{\xi}(t,\zeta))dW - \frac{1}{2}\int_0^t \partial_z g(s, \beta Z^{\xi}(s), \beta\Upsilon^{\xi}(s,\zeta))^2 ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \Big(\ln\big(1 - \partial_\Upsilon g(s, \beta Z^{\xi}(s), \beta\Upsilon^{\xi}(s,\zeta))\big) + \partial_\Upsilon g(s, \beta Z^{\xi}(s), \beta\Upsilon^{\xi}(s,\zeta))\Big)\nu(d\zeta)ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \ln\big(1 - \partial_\Upsilon g(s, \beta Z^{\xi}(s), \beta\Upsilon^{\xi}(s,\zeta))\big)\tilde{N}(ds,d\zeta)\bigg\}, \\ &= \exp\bigg\{ - \int_0^t \partial_z g(s, Z^{\xi}(s), \Upsilon^{\xi}(s,\zeta))dW - \frac{1}{2}\int_0^t \partial_z g(s, Z^{\xi}(s), \Upsilon^{\xi}(s,\zeta))^2 ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \Big(\ln\big(1 - \partial_\Upsilon g(s, Z^{\xi}(s), \Upsilon^{\xi}(s,\zeta))\big) + \partial_\Upsilon g(s, Z^{\xi}(t), \Upsilon^{\xi}(s,\zeta))\Big)\nu(d\zeta)ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \ln\big(1 - \partial_\Upsilon g(s, Z^{\xi}(t), \Upsilon^{\xi}(s,\zeta))\big)\tilde{N}(ds,d\zeta)\bigg\} \\ &= \mathcal{E}\bigg(\int_0^t \partial_z g(s, Z^{\xi}(s), \Upsilon^{\xi}(s,\zeta))dW(s) + \int_0^t \int_{\mathbb{R}_0} \partial_\Upsilon g(s, Z^{\xi}(s), \Upsilon^{\xi}(s,\zeta))\tilde{N}(ds,d\zeta)\bigg)(t) \\ &= \mathcal{E}(M^{\xi}(t))\,, \end{aligned}$$

because of the positive homogeneity of g in z and Υ. In the case of dynamic coherent risk measure, $\Lambda^{\xi}(T,t)$ is given by

$$\Lambda^{\xi}(T,t) = \int_0^1 \frac{\mathcal{E}(M^{\beta\xi}(T))}{\mathcal{E}(M^{\beta\xi}(t))} d\beta = \frac{\mathcal{E}(M^{\xi}(T))}{\mathcal{E}(M^{\xi}(t))}. \tag{3.7}$$

Hence, the BSDE-based coherent risk measure is given by

$$\rho_t(\xi) = \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\xi \mid \mathcal{F}_t],$$

where the $\mathbb{Q}$-measure is defined in (3.6).

□

We obtain similar results as Kromer and Overbeck (2014) were the exponential martingale of the BSDE based convex risk measure is dependent on all portfolio weights $\beta \in [0,1]$. The representation of the coherent risk measure is dependent only on $\beta = 1$. The difference between these two risk representations is emphasized in Equation (3.7).

4. Example

In this section we apply the results presented early to dynamic entropic risk measure and static coherent entropic risk measures to obtain the gradient capital allocation for each risk measure under the jump framework.

Example 4.1. We consider the well known dynamic entropic risk measure given by

$$\rho_t(\xi) = \frac{1}{\gamma} \ln \mathbb{E}\Big[e^{-\gamma\xi} \mid \mathcal{F}_t\Big], \quad \gamma > 0,\, t \in [0,T].$$

This example was also considered in Kromer and Overbeck (2014). It has been proved that the above entropic measure is a unique solution of the so called canonical quadratic-exponential BSDE (g,ξ) of the form (See Karoui, Matoussi, and Armand (2016))

$$\begin{aligned} \rho_t(\xi) &= -\xi + \int_t^T \Big(\frac{\gamma}{2}|Z^{\xi}(s)|^2 + \frac{1}{\gamma}\int_{\mathbb{R}_0}\Big(\exp(\gamma\Upsilon^{\xi}(s,\zeta)) - \gamma\Upsilon^{\xi}(s,\zeta) - 1\Big)\nu(d\zeta)\Big)ds \\ &\quad - \int_t^T Z^{\xi}(s)dW(s) - \int_t^T\int_{\mathbb{R}_0} \Upsilon^{\xi}(s,\zeta)\tilde{N}(ds,d\zeta). \end{aligned} \tag{4.1}$$

Note that the generator is given by

$$g(t,Z,\Upsilon(\zeta)) = \frac{\gamma}{2}|Z|^2 + \frac{1}{\gamma}\int_{\mathbb{R}_0}\Big(\exp(\gamma\Upsilon(t,\zeta)) - \gamma\Upsilon(t,\zeta) - 1\Big)\nu(d\zeta).$$

From the partial derivatives

$$\partial_z g(t,Z,\Upsilon(\zeta)) = \gamma Z$$

and

$$\partial_{\Upsilon} g(t,Z,\Upsilon(t,\zeta)) = \int_{\mathbb{R}_0}\Big(\exp(\gamma\Upsilon(t,\zeta)) - 1\Big)\nu(d\zeta)$$

.

Suppose that ξ is from a class of smooth functions such that $D_s^i(\xi)$, $D_{s,\zeta}^i(\xi)$, for $0 \le s \le t \le T$, belong to $BMO(\mathbb{P})$ and $\|\xi\|_{1,2}$ exists and is finite for $i = 1, \dots, d$ and $t \in [0, T]$. We define a function $\varphi(\xi) = e^{-\gamma\xi}$. Then from the boundedness of ξ and of any $\beta \in [0, 1]$, we have that $\varphi(\xi)$ is Malliavin differentiable and the generalized Clark-Ocone formula (Di Nunno, Øksendal, and Proske (2009), Theorem 12.20)

$$e^{-\gamma\beta\xi} = \mathbb{E}[e^{-\gamma\beta\xi}] + \int_0^T \mathbb{E}[D_s(e^{-\gamma\beta\xi}) \mid \mathcal{F}_t] dW(t) + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D_{s,\zeta}(e^{-\gamma\beta\xi}) \mid \mathcal{F}_t] \tilde{N}(dt, d\zeta) .$$

Define $\Gamma^{\beta\xi}(t) := \mathbb{E}[e^{-\gamma\beta\xi} \mid \mathcal{F}_t]$ is a positive bounded martingale for $\xi \in \mathcal{F}_T$. Then

$$\begin{aligned}
\Gamma^{\beta\xi}(t) &= \Gamma^{\beta\xi}(0) + \int_0^t \mathbb{E}[-\gamma\beta e^{-\gamma\beta\xi} D_s(\xi) \mid \mathcal{F}_s] dW(s) \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \mathbb{E}[-\gamma\beta e^{-\gamma\beta\xi} D_{s,\zeta}(\xi) \mid \mathcal{F}_s] \tilde{N}(ds, d\zeta) \\
&= \Gamma^{\beta\xi}(0) + \gamma \int_0^t \Gamma^{\beta\xi}(s) Z^{\beta\xi}(s) dW(s) + \gamma \int_0^t \int_{\mathbb{R}_0} \Gamma^{\beta\xi}(s) \Upsilon^{\beta\xi}(s, \zeta) \tilde{N}(ds, d\zeta) ,
\end{aligned}$$

where

$$Z^{\beta\xi}(t) = \frac{-\beta\mathbb{E}[e^{-\gamma\beta\xi} D_s(\xi) \mid \mathcal{F}_t]}{\mathbb{E}[e^{-\gamma\beta\xi} \mid \mathcal{F}_t]} \quad \text{and} \quad \Upsilon^{\beta\xi}(\mathrm{s}, \zeta) = \frac{-\beta\mathbb{E}[\mathrm{e}^{-\gamma\beta\xi} \mathrm{D}_{\mathrm{s},\zeta}(\xi) \mid \mathcal{F}_{\mathrm{t}}]}{\mathbb{E}[\mathrm{e}^{-\gamma\beta\xi} \mid \mathcal{F}_{\mathrm{t}}]}, \tag{4.2}$$

are the predictable control processes for the entropic risk measure defined by the BSDE (4.1). Furthermore, $Z^{\beta\xi}(\cdot)$ and $\Upsilon^{\beta\xi}(\cdot, \zeta)$ belong to the class of $BMO(\mathbb{P})$, hence $\Gamma^{\beta\xi}(t)$ satisfies the following

$$\begin{aligned}
\Gamma^{\beta\xi}(t) &= \Gamma^{\beta\xi}(0) \exp\Big\{\gamma \int_0^t Z^{\beta\xi}(s) dW(s) - \frac{\gamma^2}{2} \int_0^t |Z^{\beta\xi}(s)|^2 ds \\
&\quad + \int_0^t \int_{\mathbb{R}_0} [\ln(1 + \gamma\Upsilon^{\beta\xi}(s, \zeta)) - \gamma\Upsilon^{\beta\xi}(s, \zeta)] \nu(d\zeta) ds \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \ln(1 + \gamma\Upsilon^{\beta\xi}(s, \zeta)) \tilde{N}(ds, d\zeta) \Big\} .
\end{aligned}$$

As a result, the process $\mathcal{N}(t) := \Gamma^{\beta\xi}(t)/\Gamma^{\beta\xi}(0)$ corresponds to the stochastic exponential $\mathcal{E}$ to the process $M^{\beta\xi}(t)$ in (3.1) defined by

$$M^{\beta\xi}(t) = \int_0^t \partial_z g(s, Z^{\beta\xi}(s), \Upsilon^{\beta\xi}(s, \zeta)) dW(s) + \int_0^t \int_{\mathbb{R}_0} \partial_\Upsilon g(s, Z^{\beta\xi}(s), \Upsilon^{\beta\xi}(s, \zeta)) \tilde{N}(ds, dz) ,$$

for $t \in [0, T]$.

Now we define the equivalent probability measure under $\mathbb{Q}^{\beta\xi}$ as

$$\frac{d\mathbb{Q}^{\beta\xi}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{N}(t).$$

Under the new probability measure $\mathbb{Q}^{\beta\xi}$, the processes

$$dW^{\mathbb{Q}^{\beta\xi}}(t) = dW(t) - \gamma Z^{\beta\xi}(t)dW(t)$$

and

$$\tilde{N}^{\mathbb{Q}^{\beta\xi}}(dt, d\zeta) = \tilde{N}(dt, d\zeta) - \gamma \int_{\mathbb{R}_0} \Upsilon^{\beta\xi}(t, \zeta)\nu(d\zeta)dt$$

are the $\mathbb{Q}$- Brownian motion and $\mathbb{Q}$-compensated random measure respectively. Define a function $\Phi_i(t)$ by $\Phi_i(t) := \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\eta_i \mid \mathcal{F}_t]$ for each $i = 1, \dots, n$. Let $Z^{\eta_i}(t)$ and $\Upsilon^{\eta_i}(t, \zeta)$ be predictable processes, then from the martingale representation theorem we have,

$$\begin{aligned}
\Phi_i(t) &= -\eta_i - \int_t^T Z^{\eta_i}(s)dW^{\mathbb{Q}^{\beta\xi}}(s) - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\eta_i}(s, \zeta)\tilde{N}^{\mathbb{Q}^{\beta\xi}}(ds, d\zeta) \\
&= -\eta_i - \int_t^T Z^{\eta_i}(s)dW(s) + \int_t^T Z^{\eta_i}(s)\gamma Z^{\beta\xi}(s)ds - \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\eta_i}(s, \zeta)\tilde{N}(ds, d\zeta) \\
&\quad + \int_t^T \int_{\mathbb{R}_0} \Upsilon^{\eta_i}(s, \zeta)\gamma\Upsilon^{\beta\xi}(s, \zeta)\nu(d\zeta)ds\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Phi_i(t) = \mathbb{E}^{\mathbb{Q}^{\beta\xi}}[-\eta_i \mid \mathcal{F}_t] &= \frac{1}{\mathcal{N}(t)}\mathbb{E}^{\mathbb{P}}[-\eta_i\mathcal{N}(T) \mid \mathcal{F}_t] \\
&= \frac{\Gamma(0)}{\Gamma(t)}\mathbb{E}^{\mathbb{P}}\Big[-\eta_i\frac{\Gamma(T)}{\Gamma(0)}\Big|\mathcal{F}_t\Big] \\
&= \frac{1}{\Gamma(t)}\mathbb{E}^{\mathbb{P}}[-\eta_i\Gamma(T)|\mathcal{F}_t] \\
&= \frac{\mathbb{E}^{\mathbb{P}}[-\eta_i e^{-\gamma\beta\xi}|\mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[e^{-\gamma\beta\xi}|\mathcal{F}_t]}\,. \qquad (4.3)
\end{aligned}$$

Therefore, the gradient capital allocation of the entropic risk measure under the jump framework is given by (4.3). If ξ is the portfolio and η_i is the subportfolio. Then Equation (4.3) describes the dynamic capital risk contribution of the subportfolio η_i to the risk of portfolio ξ at time t. In addition, we can represent the BSDE-based dynamic entropic risk measure by

$$\rho_t(\xi) = \mathbb{E}^{\mathbb{P}}\Big[-\int_0^1 \Big(\frac{e^{-\gamma\beta\xi}}{\mathbb{E}^{\mathbb{P}}[e^{-\gamma\beta\xi}]}\Big)d\beta\ \eta_i\Big|\mathcal{F}_t\Big].$$

Example 4.2. In the second example we consider the static entropic coherent risk measure at level c defined by (Föllmer and Knispel (2011) in Definition 3.1) as follow

$$\rho(\xi) = \inf_{\gamma>0}\Big(\frac{c}{\gamma} + \frac{1}{\gamma}\ln\mathbb{E}\Big[e^{-\gamma\xi}\Big]\Big) \qquad (4.4)$$

for $c > 0$. From Proposition 3.1 by Föllmer and Knispel (2011) there exists a unique $\gamma_c > 0$ such that $c = \mathbb{E}^{\mathbb{Q}}[\int \frac{d\mathbb{Q}}{d\mathbb{P}} \ln(\frac{d\mathbb{Q}}{d\mathbb{P}})]$ and the infimum of Equation (4.4) is attained, i.e.

$$\rho(\xi) = \frac{c}{\gamma_c} + \frac{1}{\gamma_c} \ln \mathbb{E}\Big[e^{-\gamma_c \xi}\Big].$$

The Gâteaux-differentiable of ρ is given by

$$\nabla \rho(\beta\xi) = -\frac{e^{-\gamma_c \beta\xi}}{\mathbb{E}[e^{-\gamma_c \beta\xi}]}\,. \tag{4.5}$$

Since the entropic coherent risk measure satisfies the property of positively homogeneity (Föllmer and Knispel (2011)). Then $\nabla\rho(\beta\xi) = \nabla\rho(\xi)$ and $\Lambda^{\beta\xi}$ for this case will be given by

$$\Lambda^{\beta\xi} = \int_0^1 \nabla\rho(\xi) d\beta = -\int_0^1 \frac{e^{-\gamma_c \xi}}{\mathbb{E}[e^{-\gamma_c \xi}]} d\beta,$$

and therefore ρ form Equation (4.4) can be represented by

$$\rho(\xi) = \mathbb{E}\Big[-\Big(\int_0^1 \frac{e^{-\gamma_c \xi}}{\mathbb{E}[e^{-\gamma_c \xi}]} d\beta\Big)\xi\Big]\,.$$

5. Conclusion

We studied the capital allocation of risk measures constructed from solutions of BSDE with jumps. From the differentiability results of BSDE with jumps and the martingale representation property we were able to provide the capital allocation representation of the risk measures. We applied the representation obtained in Theorem 3.1 to entropic risk measure to achieve the allocation in terms of conditional expectation under the equivalent $\mathcal{Q}$ measure. The current results are based on a fixed time horizon, future work can study capital allocation representation of maturity independent risk measures.

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Appendix A

In this appendix, we recall from Di Nunno, Øksendal, and Proske (2009) the Clark-Ocone formula and the chain rule in the Brownian and Poisson probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let D_s^i and $D_{s,\zeta}^i$ be the Malliavin derivatives with respect to W and $\tilde{N}(dt, d\zeta)$ respectively for $i = 1, \ldots, d$ and $0 \leq s \leq t \leq T$. We denote by $\mathbb{D}^{1,2}$ the Banach space which is the closure of smooth random variables under $\|\cdot\|_{1,2}^2$ (Ocone and Karatzas (1991)). A smooth random variable F is Malliavin differentiable if and only if $F \in \mathbb{D}^{1,2} \subset L^2(\mathbb{P})$. The respective norm is defined as follows

$$\|F\|_{1,2}^2 := \mathbb{E}\Big[|F|^2 + \sum_{i=1}^{d} \int_0^T |D_s^i F|^2 ds + \sum_{i=1}^{k} \int_0^T \int_{\mathbb{R}_0} |D_{s,\zeta}^i F|^2 \zeta^2 \nu_i(d\zeta) ds\Big].$$

Theorem A.1. *Let $F \in \mathbb{D}^{1,2}$. Then*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_s F|\mathcal{F}_t] dW(t) + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D_{s,\zeta} F|\mathcal{F}_t] \tilde{N}(ds, d\zeta).$$

Theorem A.2 (Chain Rule)**.** *Let $F = F_1, \ldots, F_m \in \mathbb{D}^{1,2}$, and let $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a bounded continuously differentiable. Then*

$$D_{s,\zeta}\varphi(F) = \varphi(F + D_{s,\zeta}F) - \varphi(F).$$

$$D_s\phi(F) = \sum_{j=1}^{m} \frac{\partial}{\partial x} \phi(F_1, \ldots, F_m) D_s F_j \qquad dt \times d\mathbb{P} - a.e.$$

Definition A.1. Let $F : \Omega \to \mathbb{R}$ be a random, choose $h \in L^2([0,T])$, and consider

$$\eta(t) = \int_0^t h(s) ds \quad \in \Omega. \tag{A.1}$$

Then we define the directional derivative of F at the point $\omega \in \Omega$ in direction $\eta \in \Omega$ by

$$D_\eta F(\omega) = \frac{d}{d\epsilon}[F(\omega + \epsilon\eta)]|_{\epsilon=0},$$

if the derivative exists.

Note that the set of $\eta \in \Omega$ formulated in the form (A.1) for some $h \in L^2([0,T])$, is called the *Cameron-Martin* space and is denoted by H.

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