Nonparametric (Distribution-free) control charts:
An Updated Overview and Some Results

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Abstract

Control charts that are based on assumption(s) of a specific form for the underlying process distribution are referred to as parametric control charts. There are many applications where there is insufficient information to justify such assumption(s) and, consequently, control charting techniques with a minimal set of distributional assumption requirements are in high demand. To this end, nonparametric or distribution-free control charts have been proposed in recent years and the amount of literature on nonparametric statistical process/quality control/monitoring has grown exponentially. Chakraborti and some of his colleagues provided three detailed review papers on nonparametric control charts in 2001, 2007 and 2011, respectively. These papers have been received with considerable attention by the research community. In this paper we bring these reviews forward to 2017, discussing some of the latest developments in the area. Moreover, unlike the past reviews, which did not include the multivariate charts, here we review both univariate and multivariate charts. We end with some concluding remarks.

Keywords: CUSUM chart; EWMA chart; Median; Phase I; Phase II; Precedence; Rank; Robust; Run-length, Shewhart chart; Sign; Univariate; Multivariate

Introduction

Modern statistical process control and monitoring methods include nonparametric (or distribution-free) control (NSPC) charts which provide a more robust alternative to standard

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parametric (e.g., normal theory) charts when the form of the underlying process distribution is unknown. A key advantage of nonparametric control charts is that one doesn't need to make very specific model (shape) assumptions such as normality and their in-control (IC) run-length distribution remains the same for all continuous process distributions (of all shapes) under minimal assumptions (such as continuity and symmetry). This property ensures the full knowledge and the stability of the IC properties of the chart, which are crucial for chart implementation and application in practice. This is not the case for parametric charts whose IC run-length properties are exact only under the assumed distribution and departures from the assumed distribution can affect the chart performance, often in a significantly negative way (such as too many false alarms). Recognizing the potential, Chakraborti, Van der Laan and Bakir (2001) provided a thorough account of univariate nonparametric control charting literature up to the end of the year 2000. Chakraborti and Graham (2007) updated that review covering much of the literature up until the end of 2007. Following that, Chakraborti, Human and Graham (2011) gave an updated overview of the nonparametric control charting literature from 2007 up until the end of the year 2010. The interested reader can also see the short book by Bakir (2011) for some newer proposals. More recently, Qiu (2014) devoted two chapters to NSPC and reviews some of the available methods. However, the field of research in NSPC continues to grow at a rapid pace and many new contributions have been added in the last few years. Our list of references will show that since 2010, at least 50 papers have been published in the area, which is truly remarkable. Thus, it seems to be a good point in time to take account of what has happened in the recent past. Motivated by this, our goal in in this review is to bring the reviews of Chakraborti et al. (2001, 2007 and 2011) forward to the main body of the published literature on nonparametric control charts to date. Note that for better coverage and more completeness, we also review a few of the articles that were not covered in Chakraborti et al. (2011). In addition, we cover the multivariate nonparametric control charting literature that was not covered in any of the earlier reviews. This should be of interest to researchers as well as practitioners.

A Review of the Literature

For a review of the literature we follow the same format of presentation as in Chakraborti et al. (2001), so the charts are classified according to the three main categories, namely, Shewhart, cumulative sum (CUSUM) and exponentially weighted moving average (EWMA).

Within each category, we discuss some of the key contributions in each of the two important phases (stages) namely Phase I and Phase II as far as monitoring applications are concerned. Note that as mentioned earlier, nonparametric charts are also known as distribution-free charts as they can be applied without a specific (shape) parametric model assumption (e.g. normal) about the underlying distribution.

Univariate Nonparametric Process Monitoring

We begin with the univariate nonparametric charts. Consider the case of variables data where a continuous random variable, the quality characteristic of interest, is monitored with respect to its distribution or distributional parameters, such as, the mean, median or spread. There are generally two phases in this monitoring process. One, where the value(s) of the parameter(s) of interest is (are) specified in the IC case, known as the known parameter case (denoted Case K) and two, where values are unknown or unspecified, and thus need to be estimated before process monitoring can start. The latter is known as the unknown parameter case (denoted Case U). In Case K, process monitoring can start as soon as data become available, but in Case U, since one needs to estimate the unknown parameter(s), a preliminary or retrospective or Phase I analysis is performed so that the process can be brought under control and from the resulting IC process data, called the reference data, process parameters can be estimated, control chart limits can be calculated and then prospective monitoring of incoming, new or test data can start. This phase of the monitoring process is called Phase II monitoring. Note that there are some control charts, called self-starting charts that do not require such a strict demarcation, where process monitoring can begin as soon as some amount of "trial" (training) data become available, but we mainly focus on Phase I and Phase II charts in this review.

Nonparametric charts are useful in both of these phases with perhaps more utility in Phase I where no knowledge about the distribution seems to be typically available. However, note that although one might suggest that in Phase II we have more knowledge about the process from the Phase I analysis which we leverage from, such as fitting (estimating) a parametric distribution and using it as the model, we argue that in Phase II one often does not have the assurance of "knowing" a distribution even after a detailed Phase I analysis, since conceivably processes can change monitoring. Moreover, if one considers a fitted model in Phase II, the uncertainty associated with the estimated model (estimated parameters and the form) from Phase

I will need to be accounted for. Thus NSPC is perhaps a safer bet and a reasonable way to approach process monitoring in practice.

The keys to nonparametric control charts are the nonparametric statistical methods such as the tests and confidence intervals. Nonparametric methods are typically based on order statistics, ranks and various functions of them. The charting statistic is often selected adapting a distribution-free test statistic, meaning that its IC distribution does not depend on the underlying distribution. Once can then construct control charts based on these statistics and their statistical properties, using a classical control charting paradigm such as the Shewhart, the CUSUM and the EWMA (and their enhancements). In this sense, control charts can be viewed as graphics which are functions of suitable charting statistics, where the charting statistics depend are model dependent in the parametric case and are distribution-free in the nonparametric case. For example, in the normal theory case, we typically use the sample means and the standard deviations as the charting statistics. In the nonparametric case, we use distribution-free statistics, based on, for example, the signs and the ranks, as will be seen next.

Univariate Nonparametric Process Monitoring: Phase I Charts

The importance of a proper and effective Phase I analysis has been recognized in the literature (see Chakraborti, Human and Graham (2009), Capizzi and Masarotto (2013) and Jones-Farmer, Woodall, Steiner and Champ (2014)). Typically Shewhart-type charts have been recommended in Phase I since these charts are simple and can detect larger shifts quickly. A Phase I control charting problem bears similarities with what is known as the test of homogeneity or the k-sample problem in the statistical hypothesis testing literature. Among the available Phase I charts, a nonparametric Phase I Shewhart-type chart called the mean-rank chart was proposed by Jones-Farmer, Jordan and Champ (2009). This chart is based on the well-known Kruskal-Wallis nonparametric test (see Gibbons and Chakraborti (2010)). We start off by combining the observations from the m Phase I samples in a single pooled sample of size $N = m \times n$, ordering the observations from the lowest to the highest. Then ranks (R_{ij} where $i = 1,2,\dots,m$ and $j = 1,2,\dots,n$) are assigned to each observation of this pooled sample. The average rank for the i^{th} sample is given by $\bar{R}_i = n^{-1}\sum_{j=1}^{n} R_{ij}$. Noting that the expected value and the variance of the ranks, when the process is IC, are given by $\frac{N+1}{2}$ and $\frac{(N-n)(N+1)}{12n}$, respectively (see Gibbons and Chakraborti (2010)), the charting statistics are given by $Z_i = \left(\bar{R}_i - \frac{N+1}{2}\right)/\sqrt{\frac{(N-n)(N+1)}{12n}}, i = 1,2,3, \ldots$ Jones-Farmer et al. (2009) gave two choices for the control limits, namely, the simulated and the approximate normal theory based control limits. For the latter choice, by the central limit theorem (for large n), marginally, the standardized mean rank Z_i approximately follows a $N(0,1)$ distribution when the process is IC. However, the charting statistics $Z_1, Z_2, \ldots, Z_m$ are dependent random variables and this dependence needs to be properly accounted for. It can be shown that asymptotically, the joint distribution of $Z_1, Z_2, \ldots, Z_m$ can be approximated by a (singular) multivariate normal distribution with means equal to zero, standard deviations equal to one and a common pairwise correlation $\rho_{ij} = \sqrt{1/(m-1)}$. As m increases this correlation tends to zero, so the lower control limit (LCL) and the upper control limit (UCL) may be approximated using the quantiles of the univariate $N(0,1)$ distribution. In Phase I, the typical metric of IC performance is the false alarm probability, FAP, which is the probability of at least one false alarm. The control limits are chosen such that the attained *FAP* does not exceed a desired nominal *FAP*; typically taken to be 0.01, 0.05 or 0.10.

Note that the Jones-Farmer et al. (2009) paper was covered in the review of Chakraborti et al. (2011) but we provide the background details here since in a later section on multivariate control charts, we discuss the paper by Bell, Jones-Farmer and Billor (2014) that proposed a Phase I multivariate mean-rank chart which is the multivariate generalization of the univariate Phase I mean-rank chart. Graham, Human and Chakraborti (2010) proposed a nonparametric Phase I median chart that was also covered in the review of Chakraborti et al. (2011) and is thus not repeated here. While these charts are useful, one potential limitation is that they are not directly usable with individual data and there are constraints on the subgroup size and the number of subgroups. Along these lines, Capizzi and Masarotto (2013) proposed a distribution-free strategy for detecting shifts in process location and/or scale in Phase 1, called the recursive segmentation and permutation (RS/P) procedure. This methodology is based on a time-ordered segmentation of the data, with significance determined using a permutation approach. The authors provide an R package to implement the RS/P procedure in practice. Based on the simulation results, the authors claim that the RS/P methodology leads to an effective Phase I distribution-free procedure, both in terms of maintaining a *FAP* and OOC performance. There is a need for more work on univariate nonparametric Phase I charts as Phase I is a very important part of the overall monitoring regime. Next we consider Phase II charts.

Univariate Nonparametric Process Monitoring: Phase II Charts

Shewhart-type control charts

Shewhart-type control charts for monitoring location in Case K

Control charts based on the sign and signed-rank statistics

As in the case with normally distributed data, different types of distribution-free charts have been proposed and studied. To this end, note that the sign (SN) and the signed-rank (SR) tests are among the simplest and yet versatile one-sample distribution-free tests (see, for example, Gibbons and Chakraborti, 2010). Adapting these to the SPC setting, Amin , Reynolds and Bakir (1995) and Bakir (2004) proposed the Shewhart-type charts based on the SN and the SR statistics, called the Shewhart-SN and Shewhart-SR charts, respectively, however, these papers and several of their extensions and generalizations were reviewed in the overview paper of Chakraborti et al. (2011) and will not be discussed here. The idea is to use the distribution-free test statistics, adapting them suitably in the process monitoring framework, for example constructing Shewhart, CUSUM and EWMA charts and their generalizations based on these statistics. Note that in Case K we work with what are known as one-sample nonparametric test statistics such as the SN and the SR. In Case U the monitoring regime consists of a Phase I and a Phase II, and we work with what are known as two-sample nonparametric tests while constructing control charts. We discuss a number of these charts below.

Shewhart-type control charts with runs-type signaling rules in Case U

While the Shewhart-type chart is simple to use and is powerful in detecting larger, abrupt shifts in the process parameter, it lacks sensitivity in detecting smaller shifts. Thus there have been proposals to improve the effectiveness of the Shewhart-type charts for detecting smaller by adding runs-type signaling rules. The original set of such rules goes back to Western Electric (1956) and in the more recent literature there are both standard and improved runs-rules schemes. The former are typically of the form *w-of-*($w+v$) with $w > 1$ and $v \geq 0$ and the latter is a combination of the classical *1-of-1* runs-rule and the *w-of-*($w+v$) runs-rules. The improved scheme has the advantage of improving the performance of the charts in detecting larger shifts while maintaining its performance in detecting small to moderate shifts. Malela-Majika, Chakraborti and Graham (2016) have implemented both schemes into the Shewhart chart based on the Mann-Whitney statistic proposed by Chakraborti and Van de Wiel (2008). The authors compared their chart to the Shewhart-type precedence chart with and without signaling rules, see Chakraborti, Van der Laan and Van de Wiel (2004) and Chakraborti, Eryilmaz and Human (2009) respectively, and with its parametric counterpart, i.e. the parametric Shewhart-$\overline{X}$ chart with and without signaling rules and found that their proposed chart outperformed its competitors in detecting shifts under distributions of various shapes. Given that the runs-rules can enhance the performance of Shewhart charts in detecting small shifts, while maintaining the basic simplicity, we envision more work in this area, particularly with regard to the computation and the effect of the size of the reference sample on chart performance. Generally speaking, more data are required for nonparametric control charts, since no specific model assumptions are made for them.

Shewhart-type control charts for monitoring spread in Case U

While a lot of work has been done on monitoring the location, only a few papers are available on monitoring the spread or the dispersion or the variability of a process. Interested reader can consult Gibbons and Chakraborti (2010) for a discussion on these three concepts which are not always well understood. Das (2008) proposed two charts for monitoring variability, when the location parameter is under control, based on two nonparametric test for equality of variances by Mood (1954) and Siegel and Tukey (1960). Since the Mood test, say M, and the Tukey test, say R, are well-known, the detail will not be given here. Das (2008) used the standardized M and R statistics as the charting statistics for each chart, respectively, with control limits LCL/UCL = $\pm$ 3 and CL = 0 for each chart. Das (2008) concluded that, for all shifts under consideration, Mood's test performed best. Again, more work is necessary in this area.

There are several other nonparametric scale tests in the literature which can be adapted to control charting. One key question here is the assumption about the equality of locations. Typically, while monitoring the mean in Case U and in the normal theory setting the variances are assumed to be equal and IC, but in the distribution-free case, one requires the opposite. The full impact of this paradigm shift (see Jones-Farmer and Champ, 2010) needs to be investigated and better understood. The effect of the reference sample size on the estimation of the common median and its impact on the performance of the chart would be particularly interesting.

While a number of control charts for monitoring the location and some for monitoring the scale have been considered, there are situations where one monitors both the location and the scale parameters simultaneously. In fact, this is what is typically recommended while monitoring the mean of a normally distributed process, since the standard deviation appears in the control limits and therefore needs to be controlled (monitored) first. This has been called the joint monitoring problem. The reader is referred to McCracken and Chakraborti (2013) for a detailed review of the recent advances in joint monitoring. Joint process monitoring schemes in the normal distribution setting use two separate charts, one for the mean, and one for the standard deviation (or the variance) or a single chart using a function of these statistics. For example, a Shewhart $\overline{X}$ chart is often used, along with a Shewhart R (or an S) chart, as a scheme. Diko, Chakraborti, and Graham (2016) showed that there are issues with this monitoring scheme with regard to the false alarm rate and interpretation, and thus some researchers have recommended using a single chart for monitoring both location and scale simultaneously. Although the majority of the overview of McCracken and Chakraborti (2013) is on parametric (normal theory) charts, two nonparametric papers were cited, namely, Zou and Tsung (2010) and Mukherjee and Chakraborti (2012). We review the Mukherjee and Chakraborti's (2012) paper next.

Shewhart-type control charts for joint monitoring of location and spread in Case U

Control chart based on the Lepage statistic

In Case U, as noted earlier, process monitoring consists of both a Phase I and a Phase II and the key idea again is to consider two-sample distribution-free test statistics and adapt them for use in process monitoring. In order to monitor both the location and the scale simultaneously, Mukherjee and Chakraborti (2012) proposed a Shewhart-type chart based on the Lepage (1971) statistic (denoted Shewhart-LP). The corresponding Lepage (1971) test is a distribution-free test for testing the equality of the location and the scales of two continuous distributions based on two independent random samples. Consider two continuous distributions (the U and the V distributions) with continuous cdf's $F(x)$ and $G(y) = F(\delta y + \theta)$, respectively, with $\delta > 0$ representing the unknown scale parameter, $-\infty < \theta < \infty$ the unknown location parameter and F an unknown and continuous cdf. A random sample of size m. $U_1, U_2, \dots, U_m$, is drawn from F

and a second independent random sample of size n, $V_1, V_2, \dots, V_n$, is drawn from the cdf G. Combine the m U-observations and the n V-observations, and arrange $N = m + n$ observations from small to large, leading to N order statistics in the combined sample. Define $Z_k = 1$ or 0, according as the k^{th} order statistic is a V or a U, respectively, for $k = 1,2,\dots,N$. The Lepage statistic is a combination of the Wilcoxon rank-sum (WRS) test for location and the Ansari-Bradley (AB) test for scale. The WRS test statistic, say T_1, is defined as

$$T_1 = \sum_{k=1}^{N} kZ_k \tag{1}$$

and the AB test statistic, say T_2, is defined as

$$T_2 = \sum_{k=1}^{N} k - \frac{1}{2}(N+1)\,Z_k. \tag{2}$$

For more details on the WRS and the AB tests the interested reader is referred to Gibbons and Chakraborti (2010). Here we simply give the IC expected value and standard deviation for each statistic which are necessary for setting up the control chart. When the process is IC, $F = G$, which means that $\theta = 0$ and $\delta = 1$. In this case

$$E(T_1|\text{IC}) = \mu_1 = \frac{1}{2}n(N+1) \tag{3}$$

and

$$STDEV(T_1|\text{IC}) = \sigma_1 = \sqrt{\frac{1}{12}mn(N+1)} \tag{4}$$

respectively. Similarly for the AB statistic

$$E(T_2|\text{IC}) = \mu_2 = \begin{cases} \frac{nN}{4} & \text{if} \quad N \text{ is even} \\ \frac{n(N^2-1)}{4N} & \text{if} \quad N \text{ is odd} \end{cases} \tag{5}$$

and

$$STDEV(T_2|\text{IC}) = \sigma_2 = \begin{cases} \sqrt{\frac{1}{48}mn\frac{N^2-4}{N-1}} & \text{if} \quad N \text{ is even} \\ \sqrt{\frac{1}{48}mn\frac{(N+1)(N^2+3)}{N^2}} & \text{if} \quad N \text{ is odd} \end{cases} \tag{6}$$

respectively. Again, one is referred to Gibbons and Chakraborti (2010) for details. For j^{th} test sample, the WRS and AB statistics T_{1j} and T_{2j} and the corresponding standardized statistics, S_{1j} and S_{2j}, where $S_{1j} = \frac{T_{1j}-\mu_1}{\sigma_1}$ and $S_{2j} = \frac{T_{2j}-\mu_2}{\sigma_2}$, respectively. The charting statistic for the Shewhart-LP chart for monitoring the j^{th} Phase II (test) sample, is then given by

$$LP_j = S_{1j}^2 + S_{2j}^2. \tag{7}$$

where an out-of-control (OOC) signal is given when $LP_j \geq a$.

Note that since the LP_j statistic can only be positive, there is only an UCL for this chart. If the chart signals, the question is whether the process location or the process spread has gone OOC. This is called post signal diagnostics. Mukherjee and Chakraborti (2012) proposed a follow-up procedure for this purpose where two more design parameters are needed, say a_1 and a_2, respectively. If the chart signals at the j^{th} sample and only S_{1j}^2 exceeds a_1, a shift in location is indicated; whereas if only S_{2j}^2 exceeds a_2, a shift in scale is indicated. Finally, if both S_{1j}^2 and S_{2j}^2 exceed a_1 and a_2, respectively, a shift in both location and scale is indicated. Performance of the Shewhart-LP chart has been studied and the reader is referred to their paper for more details. Clearly, there is a need to develop more nonparametric charts for monitoring process location and process scale simultaneously. A systematic and thorough examination of the post signal diagnostic scheme, for joint monitoring schemes, would also be worthwhile. There is also the need for easier access to computational aides and better understanding the effect of the size of the reference sample on chart performance. The Shewhart-type charts are simple and effective for detecting larger shifts. Next we consider CUSUM-type univariate nonparametric charts, those, like their parametric counterparts, that are effective in detecting smaller shifts.

CUSUM-type control charts

CUSUM charts are useful and sometimes more naturally fitting in the process control environment in view of the sequential nature of data collection. We review some nonparametric CUSUM-type charts next.

CUSUM-type control charts for monitoring location in Case K

Control charts based on the sign and signed-rank statistics

Using the SN and SR statistics, Amin et al. (1995) and Bakir and Reynolds (1979) proposed CUSUM-type charts based on the SN and the SR statistics, respectively. These charts are denoted CUSUM-SN and CUSUM-SR, respectively, however, these papers were covered in the overview paper of Chakraborti et al. (2011) and will not be discussed here.

CUSUM-type control charts for monitoring location in Case U

As noted earlier, in Case U, the key idea is to consider two-sample distribution-free test statistics and adapt them for use in process monitoring. Chakraborti et al. (2004) considered a class of nonparametric Phase II Shewhart-type charts based on the so-called precedence statistics, called the precedence charts, however, this important paper was covered in the overview paper of Chakraborti et al. (2011). Thus it is logical to consider CUSUM-type charts based on precedence statistics and this is considered in Mukherjee, Graham and Chakraborti (2013). It turns out that it is more convenient to work with the so-called exceedance statistics, which are closely related to the precedence statistics. Let $U_{j,r}$ denote the number of Y observations in the j^{th} Phase II sample that exceeds $X_{(r)}$, the r^{th} ordered observation in the reference sample. The statistic $U_{j,r}$ is called an exceedance statistic and the probability $p_r = P\big(Y > X_{(r)} | \, X_{(r)}\big)$ is called an exceedance probability.

Control chart based on the exceedance statistic

Mukherjee et al. (2013) proposed a CUSUM chart based on the exceedance statistic (denoted CUSUM-EX). An upper **one-sided** CUSUM control chart based on the EX statistic is defined as

$$C_j^+ = \max\big[0, C_{j-1} + \big(U_{j,r} - \mu_U\big) - k\big] \text{ for } j = 1,2,3... \tag{8}$$

with the starting value $C_0^+ = 0$, $\mu_U = E\big(U_{j,r} \big| X_{(r)}\big) = np_r$ and which signals at the first j for which $C_j^+ \geq H$. The values of the design parameters, k and H, are found such that a nominal IC average run-length (ICARL) value is obtained. Typically, k is specified first and then, following this, H is found using a search algorithm in order to obtain some nominal ICARL. The steps for choosing the two design parameters, k and H, follow the same for other CUSUM-type charts and will not be discussed each time.

Next we review a nonparametric Phase II CUSUM-type chart useful for joint monitoring adapting the Lepage (1971) statistic. Note that we are not aware of nonparametric Phase II charts for monitoring spread. This is a useful area of research.

CUSUM-type control chart for joint monitoring of location and scale Case U

Chowdhury, Mukherjee and Chakraborti (2015) considered the distribution-free CUSUM-type chart based on the LP statistic which is expected to be more sensitive than the Shewhart-LP charts for small, sustained (upward or downward) shifts in the location. Distribution-free CUSUM charts based on the LP statistic are called CUSUM-LP charts from this point forward.

An upper **one-sided** CUSUM chart based on the LP statistic is defined as

$$C_j^+ = \max[0, C_{j-1}^+ + (LP_j - 2) - k] \text{ for } j = 1,2,3... \tag{9}$$

with the reference value $k \geq 0$, the starting value $C_0^+ = 0$ and which signals at the first j for which $C_j^+ \geq H$. It should be noted that we subtract 2 in the CUSUM since $E(LP|\text{IC}) = 2$. The interested reader is referred to Chowdhury et al. (2015) for more information on why the IC expected value of the Lepage statistic equals 2. The values of the design parameters, k and H, are found such that a nominal ICARL value is obtained. Typically, k is specified first and then, following this, H is found using a search algorithm in order to obtain some nominal ICARL. Chowdhury et al. (2015) considered values of k = 0, 3 and 6, respectively. For thepost signal diagnostics, Chowdhury et al. (2015) proposed a follow-up that makes use of the p-values of the corresponding WSR and the AB statistics, denoted p_1 and p_2, respectively. These p-values are calculated on the basis of the two samples; one with m Phase I observations and the other with the n observations from the j^{th} test sample (if a signal was given at sample number j). If p_1 is very low but not p_2, this indicates a shift in location only. Alternatively if p_1 is very high but not p_2, this indicates a shift in scale only. Finally, if both p_1 and p_2 are very low, this indicates a shift in location and scale.

The question about which Phase I sample order statistic to use in forming either the precedence (or the exceedance) statistics is an interesting one. Following the work of Mukherjee et al. (2013) which was based on the median, Graham, Mukherjee and Chakraborti (2017) investigated this choice in the design of the CUSUM-EX control chart and gave some recommendations from a practical point of view.

EWMA-type control charts

Like the CUSUM charts, the EWMA charts also take advantage of the sequentially accumulating nature of the data arising in a usual SPC environment and are known to be efficient in detecting smaller shifts, however they are easier to implement. Following the Shewhart and CUSUM-type charts described earlier, some nonparametric EWMA-type charts have been considered. They are described next.

EWMA-type control charts for monitoring location in Case K

Control charts based on the sign and signed-rank statistics

As we have noted earlier, in Case K, one can simply use the SN and the SR statistics in the monitoring framework. Following the Shewhart and CUSUM-type charts, Graham, Chakraborti and Human (2011a) considered an EWMA chart based on the SN statistic (denoted EWMA-SN). For this chart let X_i denote the i^{th} individual measurement from an unknown continuous distribution with cdf F with median θ, that is to be monitored. Define the SN statistic

$$SN_i = \sum_{j=1}^{n} sign\left(X_{ij} - \theta_0\right) \quad \text{for} \quad i = 1,2,3,\ldots \tag{10}$$

where $sign(A) = -1, 0, 1$ if $A < 0, = 0, > 0$ and θ_0 is the known or the specified or the target value of the median, θ, that is monitored. The charting statistic of the EWMA-SN chart is defined as

$$Z_i = \lambda SN_i + (1-\lambda)Z_{i-1} \qquad Z_0 = 0 \tag{11}$$

where $0 < \lambda \leq 1$ is the smoothing constant to be specified later. The exact control limits and the centerline (CL) of the EWMA-SN chart are given by

$$UCL/LCL = \pm L\sqrt{\frac{\lambda n}{2-\lambda}\left(1-(1-\lambda)^{2i}\right)} \text{ and } CL = 0 \tag{12}$$

where $L > 0$ is the distance of the control limits from the centerline.

The steady-state control limits (typically used when the EWMA chart has been running for several time periods) are based on the asymptotic standard deviation of the control statistic (Lucas and Saccucci, 1990) and are given by

$$UCL/LCL = \pm L\sqrt{\frac{\lambda n}{2-\lambda}} \text{ and } CL = 0. \tag{13}$$

If any of the charting statistics Z_i plots on or outside either of the two control limits, the process is declared OOC and a search for assignable causes is started. Otherwise, the process is considered IC and charting continues. The two design parameters, λ and L, are selected so that a nominal ICARL value is attained. Typically, λ is chosen first with small values of λ being recommended for small shifts (λ = 0.05), larger values of λ being recommended for moderate shifts (λ = 0.10) and a large value of λ being recommended for large shifts (λ = 0.20). Once λ is selected, then L is found using a search algorithm so that a nominal ICARL value is attained. The steps for choosing the two design parameters, λ and L, follow the same for other EWMA-type charts and will not be discussed each time Graham, Chakraborti and Human (2011b) proposed an nonparametric EWMA-type chart, based on the SR statistics, denoted EWMA-SR chart. The SR charts require symmetry of the underlying distribution but have performance comparable with normal theory EWMA charts based on the mean. The reader is referred to the paper for more details.

Next we discuss some EWMA-type charts in Case U.

EWMA-type control charts for monitoring location in Case U

Control chart based on the exceedance statistic

As in the case with the CUSUM-EX chart, the charting statistic of the EWMA-EX chart is obtained by sequentially accumulating the exceedance statistics $U_{1,r}$, $U_{2,r}$, $U_{3,r}$, … and is defined as

$$Z_j = \lambda\, U_{j,r} + (1-\lambda)Z_{j-1}, \qquad Z_0 = E\left(U_{j-k,r} | X_{(r)}\right) = np_r \tag{14}$$

where $0 < \lambda \leq 1$ is the smoothing constant to be specified later. Graham, Mukherjee and Chakraborti (2012) showed that the unconditional IC mean and standard deviation of Z_j are given by

$$E\left(Z_j | IC\right) = n(1-a)\left(1-(1-\lambda)^j\right)$$

and (15)

$$STDEV\left(Z_j | IC\right) = \sqrt{\left(\frac{na(1-a)}{m+2}\right)\left\{n(1-(1-\lambda)^j)^2 + \frac{\lambda(m+1)}{2-\lambda}(1-(1-\lambda)^{2j})\right\}},$$

respectively, where $a = r/(m+1)$. The exact control limits and the CL of the two-sided EWMA-EX chart are given by

$$UCL/LCL = E\left(Z_j | IC\right) \pm L \times STDEV\left(Z_j | IC\right) \text{ and } CL = E\left(Z_j | IC\right) \tag{16}$$

where L > 0 is the distance of the control limits from the centerline. The steady-state control limits and CL are given by

$$UCL/LCL = n(1-a) \pm L\sqrt{\left(\frac{na(1-a)}{m+2}\right)\left\{n + \frac{\lambda(m+1)}{2-\lambda}\right\}} \text{ and } CL = n(1-a). \tag{17}$$

If any Z_i plots on or outside either of the two control limits, the process is declared to be OOC and a search for assignable causes is started. Otherwise, the process is considered IC and charting continues.

Following the work of Graham et al. (2012) which was based on the median of the Phase I (reference) sample, Graham et al. (2017) investigated the choice of the Phase I (reference) sample order statistic used in the design of the EWMA-EX chart and gave some recommendations for selecting this order statistic from a practical point of view.

Next we review an EWMA-type chart for joint monitoring in the unknown parameter case based on the Lepage (1971) statistic.

EWMA-type control chart for joint monitoring of location and scale in Case U

Control chart based on the Lepage statistic

Chowdhury et al. (2015) stated that it's challenging to design the EWMA chart based on the Lepage statistic (denoted EWMA-LP chart), since "construction of exponentially-weighted moving average chart based on the Lepage statistic will be challenging as there is no simple explicit form of conditional variance of Lepage statistic that can be obtained". In order to circumvent this problem, we use a constant value a for the *UCL*. The value of a is obtained by making use of simulation in order to find a desirable *ARL* value. The EWMA-LP accumulates the statistics $LP_1, LP_2, LP_3, \ldots$ with the EWMA charting statistic defined as

$$Z_j = \lambda\, LP_j + (1-\lambda)Z_{j-1} \text{ for } \quad j = 1,2,3,\ldots \tag{18}$$

where the weighting constant $0 < \lambda \leq 1$ with starting value $Z_0 = 2$.

The process is considered to be IC while all the charting statistics Z_j, $j = 1,2,3,\ldots$ fall below the *UCL*, however as soon as a charting statistic falls on or above the *UCL* the process is declared OOC and typically a search for assignable causes would be started. This work is currently under review.

Another contribution to the nonparametric EWMA literature is by Zou and Tsung (2010) who integrated the powerful goodness-of-fit test with the EWMA scheme, however, details are omitted. As noted earlier, in a global sense, more work on the precedence and exceedance charts is needed, particularly to understand their performance in Phase II and how much Phase I data are needed to guarantee nominal IC and decent OOC performance. This is true for most of the

Phase II (Case U) nonparametric charts currently available in the literature. We review a few other univariate nonparametric charts in the next section.

Other Univariate Nonparametric Charts

GWMA-type control charts in Case K

A generalization of the EWMA charting scheme to monitor the mean, under normality, was considered by Sheu and Lin (2003). This is referred to as the generally weighted moving average (GWMA) charting scheme. They showed that the GWMA chart performs better in detecting small shifts in the process mean. In the nonparametric setting, Lu (2015) considered a GWMA chart for the process median based on the SN statistic, called the GWMA-SN chart. Let X be some quality characteristic of a process with target value T. Define the deviation $Y = X - T$, then $p = P(Y > 0)$ is the process proportion with $p = 0.5$ indicating the process is IC and $p \neq 0.5$ indicating the process is OOC. For a sample of size n at time t define $Y_{it} = X_{it} - T$ and $I_{it} = 1,0$ if $Y_{it} > 0$, otherwise, for $i = 1,2,\ldots,n$. Thus, the SN statistic used by Lu (2015), $N_t = \sum_{i=1}^{n} I_{it}$. It follows that N_t follows a $BIN(n,0.5)$ distribution when the process is IC. The charting statistic for the GWMA-SN chart is defined as

$$G_t = \sum_{j=1}^{t} \left(q^{(j-1)^\alpha} - q^{j^\alpha}\right) N_{t-j+1} + q^{t^\alpha} T \quad \text{for} \quad t = 1,2,3\ldots \tag{19}$$

where $0 \leq q < 1$ and $0 < \alpha \leq 1$ are two chart parameters. Note that the $\left(q^{(j-1)^\alpha} - q^{j^\alpha}\right)$ are the weights from the most recent to the oldest observation. It is seen that t the GWMA-SN chart reduces to the EWMA-SN chart for $\alpha = 1$ and $q = 1 - \lambda$, where $0 < \lambda \leq 1$, is the smoothing parameter of EWMA chart. Also, as is the case for the EWMA chart, the GWMA reduces to a Shewhart-type chart for $q = 0$ and $\alpha = 1$. Moreover, it can be shown that the IC expected value and the variance of the charting statistic are $\frac{n}{2}$ and $Q_t \frac{n}{4}$, respectively, where $Q_t = \sum_{j=1}^{t} \left(P(M = j)\right)^2$. Accordingly, the exact Shewhart-type control limits for the GWMA-SN control chart are given by

$$LCL/UCL = E(G_t|IC) \pm L \times STDEV(G_t|IC) = \frac{n}{2} \pm L\sqrt{\frac{n}{4} Q_t} \quad \text{with} \quad CL = E(G_t|IC) = \frac{n}{2} \tag{20}$$

For $t \to \infty$, the asymptotic variance of the charting statistic G_t is given by $\lim_{t\to\infty} VAR(G_t) = Q\sigma^2$, where $Q = \lim_{t\to\infty} Q_t$. Hence the steady-state control limits and the *CL* are given by

$$UCL/LCL = \frac{n}{2} \pm L\sqrt{\frac{n}{4}Q} \text{ and } CL = \frac{n}{2}. \tag{21}$$

Lu (2015) found that the GWMA-SN chart is more efficient than the EWMA-SN chart in detecting smaller shifts where larger values of q and α are recommended. For larger shifts, the GWMA-SN chart is not optimal, but can compete with the EWMA-SN chart. Clearly, a GWMA-SR chart can be considered based on the SR statistic and assuming symmetry, and in fact, this has been done by Chakraborty, Chakraborti, Human and Balakrishnan (2016). Their GWMA-SR chart performs better, as expected, for symmetric distributions and smaller shifts.

As we noted earlier, attempts have been made to enhance the smaller shift detection ability of the Shewhart charts, under normality, for example, by including supplementary runs-rules. Another attempt in this direction includes a new class charts, called the synthetic charts (Wu and Spedding (2000)). A lot of work has been done on these charts, we briefly review the nonparametric analogs.

Synthetic control charts in Case K

The synthetic control chart integrates a Shewhart chart and what is called a conforming run-length (CRL) chart. Unlike the Shewhart chart, a synthetic chart does not signal when a single charting statistic plots on or beyond the control limits. Instead, when that happens, the practitioner determines how many samples have been taken since the last time a charting statistic plotted on or beyond the control limits until the current time point, which is called the CRL. If the CRL value is sufficiently small, an OOC signal is generated.

To the best of our knowledge, only three nonparametric synthetic control charts have been proposed in the literature so far. Khilare and Shirke (2010) and Pawar and Shirke (2010) proposed nonparametric synthetic control charts based on the SN and the SR statistic, respectively, for monitoring location. Khilare and Shirke (2012) proposed a nonparametric synthetic chart to monitor the process variation using the SN statistic. First we give some definitions needed for our discussions. When referring to the zero-state mode, it is assumed that there is a nonconforming sample at time zero, whereas when referring to the steady-state mode, one assumes that the process starts and stays IC for a long time before a process shift occurs at some ‘random time’.

The synthetic SN chart integrates the operation of the nonparametric Shewhart SN chart and the CRL chart. Khilare and Shirke (2010) compared the (zero-state) *ARL* performance of the synthetic SN chart against the ordinary Shewhart SN and $\bar{X}$ charts and observed that the synthetic SN chart performs best overall and that the improvement in the *ARL* is more significant for small to moderate shifts. Note that the authors acknowledged that this good performance might be due to the fact that they assumed a zero-state mode and that under a steady-state this might not have been the case. Following their 2010 paper, in 2012 the same authors proposed a nonparametric synthetic chart to monitor the process variation using the SN-based test assuming both the zero- and steady-state modes.

The *ARL* performance of the synthetic SN chart is compared with the Shewhart S^2 and Shewhart SN charts and it is observed that the synthetic SN chart performs best overall. However, superiority of the synthetic SN chart is limited to the zero-state mode, since, when in the steady-state mode, the *ARL* performance of the synthetic chart is poor compared to the zero-state *ARL*. This occurs because the effect of the head start feature has disappeared. Next, we discuss the synthetic SR chart proposed by Pawar and Shirke (2010). This chart integrates the operation of the nonparametric Shewhart SR chart and the CRL chart. As perhaps expected, Pawar and Shirke (2010) compared the (zero-state) *ARL* performance of the synthetic SR chart against the Shewhart SR chart (i.e. the *1-of-1* chart), the *2-of-2* runs-type signaling rules SR chart and the Shewhart $\bar{X}$ chart. The authors observed that the synthetic SR chart outperforms all the other charts for all upwards shifts in the process medians. Note that although the synthetic charts have attracted a lot of interest in the literature, and more work can be done, some researchers have advised against their use. See for example, Knoth (2016) for a discussion. Again, to reiterate, more work on these charts is needed, particularly to understand their performance in Phase II and how much Phase I data are needed to guarantee nominal IC and decent OOC performance.

Adaptive control charts

Another class of control charts studied in the literature is called adaptive charts, which allows the user to vary the charting conditions depending on the recent outcomes observed in the data from the process. For example, if a control chart shows that the process is IC and stable for a long time, it might seem reasonable to extend the sampling interval. On the other hand, if the chart indicates that the process might be heading towards being out of control, one might want to collect data more frequently, thus shortening the sampling time interval. Psarakis (2015) provided an overview of the adaptive control charting literature. In that paper the definition of an adaptive control chart is given as "A control chart is considered adaptive or dynamic if at least one of the parameters *h*, *n*, or *k* are allowed to change in real time depending on the actual values from the previous sample statistics", where h denotes the sampling interval, n is the sample size and k denotes the constant which determines the width of the control limit. One type of adaptive chart has been proposed for the variable sampling interval (VSI) setting. Reynolds, Amin, Arnold and Nachlas (1988) were the first to suggest an $\bar{X}$-chart with a VSI scheme. Advantages of using VSI charts, or adaptive charts in general, include minimizing the time for shift detection, as well as the number of samples that are needed to detect a shift. In some instances this can greatly reduce the cost.

Liu, Zi, Zhang and Wang (2013) considered an adaptive nonparametric EWMA chart, called the ANE chart, for monitoring the location of a process. The chart is based on the sequential ranks (see their paper for details). Liu, Chen, Zhang and Zi (2015) considered a generalization of the ANE chart under the VSI setting. They found that adding the VSI modification improves the performance of the chart. Coelho, Graham and Chakraborti (2017) proposed a VSI SR chart and showed that it performs similar or better than the VSI SN chart that was proposed by Amin and Widmaier (1999).

Change-point model based methods in Case U

As eluded in the above discussion, another popular formulation of the monitoring problem in the literature is the change-point formulation. Hawkins, Qiu and Kang (2003) studied the change-point model under the assumption of normality. Hawkins and Deng (2010) developed a nonparametric analog, based on the Mann-Whitney statistic, however, this paper was covered in the overview paper of Chakraborti et al. (2011) and will not be discussed here. Liu, Zhang and Zi (2015) proposed a dual nonparametric CUSUM (DNC) chart based on a sequential rank for monitoring the location of a process. Let x_i denote the i^{th} independent observation from an unknown continuous distribution, F. Let τ denote the unknown change-point and let μ_0 and μ_1 ($\neq \mu_0$) denote the IC and OOC location parameters, respectively. Let R_n denote the n^{th} sequential rank, $R_n = \sum_{j=1}^{n} I\{x_n \geq x_j\}$ of an observation among n observations. As described earlier, the standardized statistic is $R_n^* = \left(R_n - E(R_n|IC)\right)/SD(R_n|IC)$ where $E(R_n|IC) = (n+1)/2$ and $SD(R_n|IC) = \sqrt{(n+1)(n-1)/12}$, respectively. Then a CUSUM charting statistic can be based on the R_n^* given by $S_n^+ = max\{0, S_{n-1}^+ + R_n^* - k\}$ and $S_n^- = min\{0, S_{n-1}^- + R_n^* + k\}$, with the starting values $S_n^- = S_n^+ = 0$. Let $S_n = max\{|S_n^-|, S_n^+\}$ and which signals at the first j for which $S_n \geq H$. The authors compare their chart to the well-known nonparametric change-point chart by Hawkins and Deng (2010) and found that it has almost the same performance for a wide range of shifts. The latter paper was covered in the review by Chakraborti et al. (2011) and will not be repeated here.

Monitoring distributions

In many industries today there is an overwhelmingly large amount of data collected almost continuously. Such data sets provide a rich environment with a large number of variables that could be monitored to the advantage of the manufacturer. Such process streams are referred to as being 'high dimensional' and SPC methods, for working with multiple data streams, are useful. Ross, Tasoulis and Adams (2011) provided a framework for detecting changes in data streams when the distributional form is unknown. This is basically a nonparametric change-point model for detecting change in the distribution and their methods make use of rank-based nonparametric tests in a streaming/monitoring context. Ross et al. (2011) started by extending the change-point model framework so that changes in the scale parameter can be found. Following this, they considered the simultaneous monitoring of location and scale parameters. They continue by introducing the idea of stream discretization which saves time by computing the ranks in a computationally effective way. Ross and Adams (2012) considered two Phase II nonparametric control charts for detecting arbitrary distribution changes. These charts are based on some well-known nonparametric goodness of fit statistics, such as the Cramer-von-Mises (CvM) and the Kolmogorov-Smirnov (KS) statistic (see Gibbons and Chakraborti (2010)). Suppose there is a change-point τ in a set of observations $\{X_1, \ldots, X_t\}$ then H_0: $X_i \sim F_0, i = 1, \ldots, t$ and H_a: $X_1, \ldots, X_\tau \sim F_0, X_{\tau+1}, \ldots, X_t \sim F_1$ where F_0 and F_1 are two continuous cdf's. Immediately following any observation X_k at time k, a change point can be tested from by partitioning the observations into two samples $S_1 = \{X_1, \ldots, X_k\}$ and $S_2 = \{X_{k+1}, \ldots, X_t\}$ with sample sizes $n_1 = k$ and $n_2 = t - k$, respectively. Then an appropriate two-sample hypothesis test can be performed. Both the CvM and the KS tests rely on the comparisons of the empirical distribution functions of the two samples defined as $\hat{F}_{S_1}(x) = \frac{1}{k}\sum_{i=1}^{k} I(X_i \le x)$ and $\hat{F}_{S_2}(x) = \frac{1}{t-k}\sum_{i=k+1}^{t} I(X_i \le x)$, respectively. The KS test is defined as $D_{k,t} = \sup_x\left|\hat{F}_{S_1}(x) - \hat{F}_{S_2}(x)\right|$ and the CvM test is defined as $W_{k,t} = \int_{-\infty}^{\infty}\left|\hat{F}_{S_1}(x) - \hat{F}_{S_2}(x)\right|^2 d\hat{F}_t(x)$ where $\hat{F}_t(x)$ is the empirical cdf of the pooled sample. For the KS and the CvM test, the null hypothesis that no change occurs is rejected at time point k, if $D_{k,t} > h_{k,t}^{(KS)}$ or if $W_{k,t} > h_{k,t}^{(CvM)}$, respectively, where $h_{k,t}^{(\cdot)}$ represents some threshold. Now, for the CvM change-point model Ross and Adams (2012) used the charting statistic $W_t = max_k\left(\frac{W_{k,t}-\mu_{k,t}}{\sigma_{k,t}}\right), 1 < k < t$ where $\mu_{k,t}$ and $\sigma_{k,t}$ denotes the mean and the standard deviation of $W_{k,t}$. A signal is given when W_t exceeds $h_t^{(CvM)}$. For the KS change-point model, the authors proposed monitoring $p_{k,t}$, the p-value associated with $D_{k,t}q_t = max_k(q_{k,t})$. Letting $q_{k,t} = 1 - p_{k,t}$, a signal is given when $q_t = max_k(q_{k,t})$ exceeds some threshold. Based on the performance results, they conclude that the CvM chart outperforms the KS chart. Monitoring the underlying distribution is an interesting problem which should include a post signal diagnostic analysis, in order to indicate the type of change that may have taken place. More work needs to be done in this area. Given these different classes of charts, using different charting statistics, one obvious question in practice is which chart should be used and when. A discussion of this important question and the recommendations follow the guidelines in the parametric case, namely, if smaller shifts are of interest a CUSUM or an EWMA-type chart should be used, whereas for larger shifts a Shewhart-type chart is recommended. The choice between a parametric and a nonparametric chart depends on what is known about the process and what assumptions can be made and justified.

Multivariate Nonparametric Process Monitoring

In recent years the interest in multivariate process monitoring has increased significantly because in today's data rich environment more quality features and variables are available and monitored that are possibly interdependent. Consequently, the interest in multivariate nonparametric control charts has gone up tremendously because it is far more difficult, if not impossible, to justify a parametric multivariate distribution assumption (such as the multivariate normal) for the underlying distribution. Suppose that p quality characteristics are of interest and let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be random vectors, each of dimension $p \times 1$, which denotes the observations over time. It is assumed that these vectors are independent with mean vectors $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots$ respectively and a common variance-covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{X}}$. Taking a cue from hypothesis testing, Hotelling's χ^2 chart is probably the most popular multivariate control chart for the mean, however, it requires the assumption of multivariate normality to maintain its nominal performance. Here we consider some nonparametric multivariate charts.

Note that although Chakraborti et al. (2001, 2007 and 2011) presented extensive overviews on the nonparametric control charting literature, they did not consider multivariate charts. That was primarily because the literature wasn't quite mature at that point in time. Some of the earlier papers in this literature include Hayter and Tusi (1994) who proposed a Shewhart-type multivariate nonparametric control charting scheme for the mean vector based on the M statistic; the maximum deviation of the observations from their sample means. Kapatou and Reynolds (1994, 1998) proposed EWMA-type multivariate nonparametric control charting schemes for the SN and SR statistics, respectively. Though, it should be noted that they are not exactly nonparametric since some elements of the covariance matrix were estimated. Liu (1995) proposed a Shewhart-type multivariate nonparametric control charting scheme based on a concept of data depth. Qiu and Hawkins (2003) proposed a CUSUM-type multivariate nonparametric control charting scheme based on the so-called antiranks of the measurements. Hamurkarouglu, Mert and Sayken (2004) proposed a nonparametric control charting scheme based on the Mahalanobis depth. Qiu (2008) proposed a CUSUM-type multivariate nonparametric control charting scheme based on log-linear modeling. Zou and Tsung (2011) proposed an EWMA-type multivariate nonparametric control charting scheme combined with the multivariate SN test; they integrated the spatial-SN test of Randles (2000) with an EWMA control charting scheme. Motivated by the use of the spatial-SN test by Zou and Tsung (2011), Zi, Zou, Zhou and Wang (2013) proposed an EWMA-type multivariate nonparametric control charting scheme for monitoring location parameters. They adapted the directional spatial-SN test to on-line sequential monitoring by incorporating the EWMA scheme. Boone and Chakraborti (2012) considered two simple Shewhart-type multivariate nonparametric control charting schemes based on the SN and the SR statistics, respectively.

In this article, not all these papers will be covered or discussed in detail due to lack of space. The reader is referred to Chapter 9 of Qiu (2014) where a good discussion about multivariate nonparametric control charts can be found. As noted earlier in the section of univariate control charts, the Phase I analysis is a very important part of the process monitoring regime and control charts are used in this phase regularly. In the multivariate setting, since more variables are involved, a Phase I analysis is just as, if not more, important. We discuss some recent papers, but more work needs to be done in this area.

Multivariate Nonparametric Process Monitoring: Phase I chart

Bell et al. (2014) proposed a Phase I multivariate mean-rank (MMR) chart which is the multivariate analog of Jones-Farmer et al. (2009)'s Phase I univariate mean-rank chart. The chart is distribution-free over the class of continuous multivariate elliptical distributions, such as the multivariate normal and the multivariate t distribution. It should be noted that most multivariate charts only have an UCL and the process is declared OOC when the charting statistic, which is usually a quadratic form, plots above the UCL and, typically, a search for assignable causes is started for the post signal diagnostics. In Phase I the UCL is chosen such that a desired FAP is attained. The authors conclude that the MMR chart outperforms the Hotelling T^2 chart when the underlying process distribution is non-normal.

Li, Dai and Wang (2014) proposed a Phase I multivariate nonparametric change-point control charting scheme based on the concept of data depth (CPDP), for individual data, which can be used to monitor the mean and/or the variance, i.e. it monitors the mean vector, the covariance matrix or both. Suppose we have n independent observations from a p dimensional multivariate distribution, $x_i \sim F_{(p)}(u_i, \Sigma_i)$, $i = 1, \dots n$. When the process is IC $u_i = \mu$ and $\Sigma_i = \Sigma$, $\forall i$, where the common μ and Σ are unknown. Assume that a step shift has occurred in the mean and/or the variance after τ observations, then the remaining $n - \tau$ observations have mean and variance $\mu^*(\neq \mu)$ and Σ^* $(\neq \Sigma)$, respectively. The authors proposed using a generalized Mann-Whitney statistic $Q(n_1) = \sum_{j=n_1+1}^{n} R_{n_1}(j)$ where

$$R_{n_1}(j) = \#\left\{x_i | D_{F_{n_1}}(x_i) < D_{F_{n_1}}\left(x_j\right), i = 1, \dots, n_1\right\} + 0.5\#\left\{x_i | D_{F_{n_1}}(x_i) = D_{F_{n_1}}\left(x_j\right), i = 1, \dots, n_1\right\}$$

where $D_{F_{n_1}}(x_i)$ denotes the data depth of x_i according to the empirical distribution of $x_1, \dots, x_{n_1}$. Although the probability of ties is zero theoretically for continuous variables, ties do occur in practice and the authors argued that, the problem of ties can be addressed by allocating a probability of a 0.5 for observations that have the same data depth. The standardized likelihood ratio statistic is defined as $SQ(n_1) = \left(E(Q(n_1)|\text{IC}) - Q(n_1)\right)/SD(Q(n_1)|\text{IC})$ where $E(Q(n_1)|\text{IC}) = n_1(n - n_1)/2$ and $SD(Q(n_1)|\text{IC}) = \sqrt{n_1(n - n_1)(n + 1)/12}$, respectively. Note that the numerator is $E(Q(n_1)|\text{IC}) - Q(n_1)$ and not the typical $Q(n_1) - E(Q(n_1)|\text{IC})$, since the former is positive with high probability. The charting statistics, $SQ(i)$, are plotted against i $(1 \le i < n)$ and a signal is given when $max_{1 \le i < n} SQ(i)$ plots above some *UCL*. The *UCL* is chosen such that a desired nominal *FAP* is attained. The authors compared their chart to the LRT chart (Sullivan and Woodall (2000)) and found that it has similar performance under the normal distribution and performs better when the underlying process distribution is non-normal.

Cheng and Shiau (2015) proposed a Phase I chart based on the spatial SN statistic for monitoring the location parameter vector of a multivariate process. They compared their chart to four competing charts, the well-known Hotelling T^2 chart, the T_{SD}^2 chart proposed by Sullivan and Woodall (1996) and the T_{MVE}^2 and T_{MCD}^2 charts proposed by Jensen, Birch and Woodall (2007), and found that their chart is the only chart that is robust to the normality assumption and that it is more powerful for the majority of OOC conditions with the exception being that their chart performs worse than the Hotelling T^2 chart for large shifts in the location vector.

One issue with both Bell et al. (2014)'s and Cheng and Shiau (2015)'s charts are that they require subgrouped data. To overcome this, Capizzi and Masarotto (2017) proposed a distribution-free multivariate Phase I chart that works both for subgrouped data and individual measurements, as the latter is fairly typical in today's data rich environments. Their chart is based on multivariate signed-ranks that integrate spatial signs and ranks of the Mahalanobis depths which is also an advantage over Bell et al. (2014)'s chart who limited their focus to Mahalanobis depths and Cheng and Shiau (2015)'s chart who limited their focus to spatial signs. Capizzi and Masarotto (2017) conclude that their proposed chart, which uses a permutation approach to calculate charting constant and thus avoids a distributional assumption, shows wider applicability and has satisfactory performance for many OOC conditions. They also provide an R package for their proposed chart.

Next we consider some Phase II nonparametric control charts for multivariate data.

Multivariate Nonparametric Process Monitoring: Phase II charts

Shewhart-type control charts for Case K

Das (2009) proposed a multivariate nonparametric control charting scheme based on the bivariate SN test (the reader is referred to Puri and Sen (1976) for more details on the bivariate SN test). Let $X_j = \left(X_{1j}, X_{2j}\right)'$, $j = 1,2,\dots,n$ be n independent stochastic vectors with cdf's $F_1(x)$,…, $F_n(x)$. We want to test whether F_1,…, F_n have n specified pairs of marginal medians. Das (2009) chose the origins so that the assumption can be made that the pair of hypothetical medians for each X_j, $j = 1,2,\dots,n$, is $0 = (0 \quad 0)'$. The null hypothesis is given by

$$H_0: F_j(0,\infty) = F_j(-\infty,0) = 0.5, \quad j = 1,2,\dots,n.$$

For each X_j define the events $\left(X_{1j} \le 0, X_{2j} \le 0\right)$ and $\left(X_{1j} \ge 0, X_{2j} \ge 0\right)$ as concordance events of the first and second kind, respectively, and define the events $\left(X_{1j} \le 0, X_{2j} \ge 0\right)$ and $\left(X_{1j} \ge 0, X_{2j} \le 0\right)$ as discordance events of the first and second kind, respectively. Let $0 < \gamma_j < 1$ denote the probability of concordance of $\left(X_{1j}, X_{2j}\right)$. Finally, denote the conditional probability of concordance (discordance) of the first kind given concordance (discordance) by θ_j $\left(\tau_j\right)$ so that the null hypothesis can be expressed as

$$H_0: \theta_j = \tau_j = 0.5, \quad j = 1,2,\dots,n.$$

The charting statistic is given by $T = (4/C)(C_1 - C/2)^2 + (4/(n-C))(D_1 - (n-C)/2)^2$ where C_i (D_i) denotes the number of concordances (discordances) of the i^{th} kind, $i = 1,2$, and $C = C_1 + C_2$ and $D = D_1 + D_2$ with $C + D = n$. The process is declared OOC (H_0 is rejected) when $T > UCL$ with $UCL = \chi^2_{2,\alpha}$ and typically a search for assignable causes would be started. Das (2009) concludes that the chart outperforms the Hotelling T^2 chart when the underlying process distribution is non-normal.

Similarly, Ghute and Shirke (2012) proposed a multivariate nonparametric control charting scheme based on the bivariate SR test of Bennett (1964). Let $X_i = (X_{1i}, X_{2i})$, $i = 1,2,\dots,n$ be a subgroup sample from some symmetric continuous bivariate distribution with $\boldsymbol{\mu}$ and Σ being the location vector and covariance matrix, respectively. The chart is for the parameter known case, so without any loss of generality assume $\boldsymbol{\mu_0} = (0 \quad 0)'$ and $\boldsymbol{\Sigma_0} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, with $-1 < \rho < 1$ and $\boldsymbol{\mu_0}$ and $\boldsymbol{\Sigma_0}$ are the known IC values of the process mean and correlation matrix, which means that the correlation coefficient ρ must also be assumed known.

The interest is in monitoring the location, so we are interested in detecting shifts in $\boldsymbol{\mu_0}$. At each inspection point in time, a SR statistic is calculated for each variate in $X_i = (X_{1i}, X_{2i})$ making use of n measurements in a sample. Recall that for the j^{th} variable, the SR statistic is

$$T_j = \sum_{i=1}^{n} C(X_{ji}) R(X_{ji}) \quad \text{for} \quad j = 1,2 \tag{22}$$

where $C(A) = 1, 0$ if $A > 0, < 0$, $R(X_{ji})$ is the rank of $|X_{ji}|$ among $|X_{j1}|, |X_{j2}|, \dots, |X_{jn}|$ and T_1 and T_2 are the two SR statistics corresponding to two variables in a sample of size n. Let $E(T_j | \boldsymbol{\mu} = \boldsymbol{\mu_0}, \mathbf{IC}) = v_j$ for $j = 1,2$. Then the two SR statistics (T_1 and T_2) are combined into a quadratic form, which is the charting statistic

$$W = (\boldsymbol{T} - \boldsymbol{v})' \widehat{\boldsymbol{\beta}}^{-\mathbf{1}} (\boldsymbol{T} - \boldsymbol{v}) \tag{23}$$

where $\boldsymbol{T} = (T_1, T_2)'$ is a 2×1 column vector of coordinate-wise univariate SR statistics, $\boldsymbol{v} = (v_1, v_2)'$ is a 2×1 column vector and $\widehat{\boldsymbol{\beta}} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$ is the variance-covariance matrix of $\boldsymbol{T}$.

As in the univariate case, for an IC process, the means are $v_1 = v_2 = \frac{n(n+1)}{4}$, the variances are $\beta_{11} = \beta_{22} = \frac{n(n+1)(2n+1)}{24}$ and the covariances are $\beta_{12} = \beta_{21} = \frac{\sum_{i=1}^{n} sgn(X_{1i}) sgn(X_{2i}) R(X_{1i}) R(X_{2i})}{4}$ (see Dietz (1982)). The process is declared OOC when $W > UCL$ with $UCL = \chi^2_{2,\alpha}$ where $\chi^2_{2,\alpha}$ is the upper 100α percentage point of the Chi-square distribution with 2 degrees of freedom and α is the Type I error probability.

Clearly, both CUSUM and EWMA type charts based on the bivariate SN and SR test statistics can be considered. Li (2015) noted that most nonparametric multivariate charts need a pre-specified tuning parameter (k in the case of a CUSUM chart and λ in the case of the EWMA chart), which requires a knowledge about the OOC distribution for implementation but, typically, the OOC distribution is unknown. This is in fact also true for their univariate counterparts, both parametric and nonparametric. Li (2015) proposed a nonparametric multivariate chart, that does not depend on a tuning parameter, using a hypothesis testing approach, based on Randles (2000)'s multivariate SN test. Dovoedo and Chakraborti (2017) noted that Randle's test is distribution-free over the class of elliptically symmetric distributions and asymptotically distribution-free over the (larger) class of directionally symmetric class of distributions. The reader is referred to Randles (2000) for details about these classes of distributions. The directionally symmetric family contains distributions such as the multivariate normal and the multivariate t distributions, and certain skewed distributions. First we give a brief overview of Randles (2000)'s multivariate SN test. Assume that $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_n\}$ is a random sample from a p-dimensional continuous population with location ϕ. In order to test H_0: $\phi = \phi_0$ against H_a: $\phi \neq \phi_0$ let $\boldsymbol{X}_i^* = \hat{A}_x(\boldsymbol{X}_i - \phi_0)$, $i = 1,2,\dots,n$ where $\hat{A}_x$ is an upper triangular matrix of dimension $p \times p$ with positive diagonal elements and a 1 in the upper-left element and satisfies

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\hat{A}_x(\boldsymbol{X}_i - \phi_0)(\boldsymbol{X}_i - \phi_0)'\hat{A}_x'}{\left\|\hat{A}_x(\boldsymbol{X}_i - \phi_0)\right\|^2}\right) = \frac{1}{p}I_p$$

where $\|\cdot\|$ and I_p denote the Euclidean norm and the identity matrix of dimension $p \times p$, respectively. Note that $\hat{A}_x(\boldsymbol{X}_i - \phi_0)/\|\hat{A}_x(\boldsymbol{X}_i - \phi_0)\|$ gives the direction vector of $\boldsymbol{X}_i^*$. Under H_0 the above transformation centers $\boldsymbol{X}_i^*$ at $\boldsymbol{0}$ and makes the direction vectors, V_i's, approximately uniformly distributed on the p-dimensional unit sphere. Under H_0 the average $\overline{V} = \sum_{i=1}^{n} V_i/n$ should be close to zero and under H_a the V_i's should cluster along the direction of $\hat{A}_x(\boldsymbol{X}_i - \phi_0)$ (and $\overline{\boldsymbol{V}}$ will point approximately towards that direction as well). The test statistic (Randles (2000)) is given by $Q = np\overline{\boldsymbol{V}}'\overline{\boldsymbol{V}}$ and H_0 is rejected for large values of Q. Li (2015) uses a similar reasoning to test whether $\{X_\tau, \dots, X_t\}$ has a similar location as $\{Y_1, \dots, Y_m\}$. Let $Y_i^* = \hat{A}_y(Y_i - \hat{\phi}_y)$, $i = 1, \dots, m$, and let $\boldsymbol{X}_i^* = \hat{A}_y(\boldsymbol{X}_i - \hat{\phi}_y)$, $i = \tau, \dots, t$ where $\left(\hat{\phi}_y, \hat{A}_y\right)$ is the solution to

$$\frac{1}{m}\sum_{i=1}^{m}\left(\frac{\hat{A}_y(\boldsymbol{Y}_i - \hat{\phi}_y)}{\left\|\hat{A}_y(\boldsymbol{Y}_i - \hat{\phi}_y)\right\|^2}\right) = 0$$

$$\frac{1}{m}\sum_{i=1}^{m}\left(\frac{\hat{A}_y\left(\boldsymbol{Y}_i - \hat{\phi}_y\right)\left(\boldsymbol{Y}_i - \hat{\phi}_y\right)'\hat{A}_y'}{\left\|\hat{A}_y\left(\boldsymbol{Y}_i - \hat{\phi}_y\right)\right\|^2}\right) = \frac{1}{p}I_p$$

where $\hat{A}_y$ is an upper triangular matrix of dimension $p \times p$ with positive diagonal elements and a 1 in the upper-left element. Similar to Randles (2000), the above transformation makes the direction vectors of $\boldsymbol{Y}_i^*$ centred at $\boldsymbol{0}$ and approximately uniformly distributed on the p-dimensional unit sphere. If $\{\boldsymbol{X}_\tau, \dots, \boldsymbol{X}_t\}$ and $\{\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_m\}$ have the same distribution then the above transformation will also center the direction vectors of $\{\boldsymbol{X}_\tau^*, \dots, \boldsymbol{X}_t^*\}$ at 0 and make them approximately uniformly distributed on the p-dimensional unit sphere. If $\{\boldsymbol{X}_\tau, \dots, \boldsymbol{X}_t\}$ has a different location, say ϕ_x, the direction vectors of $\{\boldsymbol{X}_\tau^*, \dots, \boldsymbol{X}_t^*\}$ will cluster along the direction of

$\hat{A}_y(\phi_x - \hat{\phi}_y)$. Define $V_i = X_i^*/\|X_i^*\|$ then the test statistic is given by $p(t-\tau+1)\overline{V}'_{\tau,t}\overline{V}_{\tau,t}$ where $\overline{V}_{\tau,t} = \sum_{i=\tau}^{t} V_i/(t-\tau+1)$. The charting statistic is given by

$$S_t = \max_{1\le\tau\le t} p(t-\tau+1)\overline{\boldsymbol{V}}'_{\tau,t}\overline{\boldsymbol{V}}_{\tau,t}$$

and the process is declared OOC when $S_t > UCL$ and typically post signal diagnostics, i.e., a search for assignable causes would be started. Again, the UCL is found for a specified nominal ICARL. Li (2015) concludes that the chart outperforms its nonparametric competitors. Tables are provided for easy implementation. Note again that the charting statistic is based on a suitable multivariate two-sample distribution-free test statistic and there is room for further research along this line.

Dovoedo and Chakraborti (2017) conducted a study on the performance of multivariate nonparametric Phase II control charts in terms of robustness, under estimated parameters. Such studies are important due to the inherent complexity of multivariate tests, particularly multivariate nonparametric tests. The authors stated that "While a number of recent studies have examined the IC robustness question related to the size of the reference sample for both univariate and multivariate normal theory (parametric) charts, in this paper we study the effect of parameter estimation on the performance of the multivariate nonparametric SN EWMA (MSEWMA) chart" (of Zou and Tsung (2011)). The MSEWMA chart is appealing for several reasons: (i) It is nonparametric, thus much more IC robust than the MEWMA chart; (ii) It is affine-invariant and has a strictly distribution-free property over a large class of population distributions or models (distributions with elliptical directions). That is, the IC run length distribution can attain or is always close to the nominal one, when the same control limits designed for the multivariate normal distributions are used; (iii) when the process distribution is one from the elliptical direction class, the IC average run length can be computed quickly via a one-dimensional Markov chain model; (iv) it is fast to implement with a similar computational effort to the MEWMA chart because only a multivariate median and the associated transformation matrix need to be specified (estimated) from the historical data before monitoring; and (v) it is also very efficient in detecting process shifts, especially small or moderate shifts when the process distribution is heavy tailed or skewed. The ICARL robustness and the OOC shift detection performance are both examined. Based on the results, it is seen that the required amount of the Phase I data can be very (perhaps impractically) high if one wants to use the control limits given by Zou and Tsung[1] for the known parameter case and maintain a nominal ICARL, which can limit the implementation of these useful charts in practice. To remedy this situation, using simulations, the authors obtain the "corrected for estimation" control limits that achieve a desired nominal ICARL value when parameters are estimated for a given set of Phase I data. This should be very interesting and useful to practitioners. The OOC performance of the MSEWMA chart with the correct control limits is also studied. The use of the corrected control limits with specific amounts of available reference sample is recommended. Otherwise, the performance the MSEWMA chart may be seriously affected under parameter estimation.

EWMA-type control charts in Case K

Li, Zou, Wang and Huwang (2013) proposed a nonparametric EWMA-type multivariate control charting scheme for monitoring shape parameters. Their methodology adapts a spatial-SN covariance matrix to online sequential monitoring by incorporating the EWMA procedure. The authors go into great detail about tests for shape parameters based on a spatial-SN covariance matrix. For brevity we omit the details here and simply give the charting statistic of their chart. The EWMA-type charting statistic is given by

$$\boldsymbol{\omega}_{\boldsymbol{i}} = (1-\lambda)\boldsymbol{\omega}_{\boldsymbol{i-1}} + \lambda \boldsymbol{v}_{\boldsymbol{i}}\boldsymbol{v}_{\boldsymbol{i}}' \qquad \boldsymbol{\omega}_{\boldsymbol{i}} = E(\boldsymbol{v}_{\boldsymbol{i}}\boldsymbol{v}_{\boldsymbol{i}}') \tag{24}$$

where $0 < \lambda \leq 1$ is the smoothing constant and $\boldsymbol{v}_{\boldsymbol{i}}$ is the unit vector, i.e. the standardized $\mathbf{x}_{\boldsymbol{i}}$ observations. Thus, $\boldsymbol{v}_{\boldsymbol{i}} = U\big(\mathbf{A}_0(\mathbf{x}_{\boldsymbol{i}} - \boldsymbol{\theta}_0)\big)$ where $\boldsymbol{\theta}_0$ is the specified multivariate center vector (affine-equivariant median) and $\mathbf{A}_0$ is the associated transformation matrix. An OOC signal is given when

$$Q_i = \sqrt{\tfrac{2-\lambda}{\lambda} \cdot \mathrm{tr}\left(\left(p \cdot \boldsymbol{\omega}_{\boldsymbol{i}} - \mathbf{I}_p\right)^2\right)} > L\,. \tag{25}$$

The two design parameters, λ and L, are selected so that a nominal ICARL value is attained and this procedure is discussed in the EWMA-SN section in more detail. The authors conclude that their proposed chart is very efficient, especially for small to moderate shifts. Further work should consider extending this chart to the parameter unknown case, which is a common practical situation.

Other Multivariate Nonparametric Charts

Charts based on multiple testing

Another interesting idea is to use the multiple testing (comparisons) approach. Park and Jun (2015) proposed a multivariate nonparametric EWMA chart for monitoring the location based on a series of the most recent T^2 statistics. The typical multivariate EWMA chart is given by $\boldsymbol{Z_i} = \boldsymbol{L}\boldsymbol{X_i} + (\boldsymbol{I}-\boldsymbol{L})\boldsymbol{Z_{i-1}}$ where recall that $\boldsymbol{X_i}$ is the data vector with $\boldsymbol{Z_0} = \boldsymbol{0}$ (a zero vector of order p) and $\boldsymbol{L} = diag(l_1, \dots, l_p)$ representing a diagonal matrix of p smoothing constants. Define $T_i^2 = \boldsymbol{Z}_i' \Sigma_{\boldsymbol{Z}_i}^{-1} \boldsymbol{Z_i}$ where $\Sigma_{\boldsymbol{Z}_i}$ denotes the variance-covariance matrix of $\boldsymbol{Z_i}$**.** If there is no a-prior information about the smoothing constants, then one idea is to set them all equal to l so that the variance-covariance matrix simplifies to $\Sigma_{\boldsymbol{Z}_i} = \left\{\left(l\left(1-(1-l)^{2i}\right)\right)/2-l\right\}\Sigma$, which, as i increases, reduces to $\Sigma_{\boldsymbol{Z}_i} = \{l/2-l\}\Sigma$. The authors proposed multiple testing of r hypotheses at time i, namely:

$$H_{0,i}^{(i)}\colon \boldsymbol{\mu}_i = \boldsymbol{\mu}_0 \ \text{ vs } \ H_{1,i}^{(i)}\colon \boldsymbol{\mu}_i \neq \boldsymbol{\mu}_0$$

$$H_{0,i-1}^{(i)}\colon \boldsymbol{\mu}_{i-1} = \boldsymbol{\mu}_0 \ \text{ vs } \ H_{1,i-1}^{(i)}\colon \boldsymbol{\mu}_{i-1} \neq \boldsymbol{\mu}_0$$

$$\vdots$$

$$H_{0,i-r+1}^{(i)}\colon \boldsymbol{\mu}_{i-r+1} = \boldsymbol{\mu}_0 \ \text{ vs } \ H_{1,i-r+1}^{(i)}\colon \boldsymbol{\mu}_{i-r+1} \neq \boldsymbol{\mu}_0$$

The process is declared OOC if any of the hypotheses are rejected. The BH-procedure (see Benjamini and Hochberg (1995)) is used to control the FDR of the abovementioned multiple testing as close to target level q as possible. Then steps for the BH-EWMA scheme, at time i, using an r-span and a target level (q) are as follows. First one needs to calculate the statistics $T_i^2, \dots, T_{i-r+1}^2$ corresponding to $H_{0,i}^{(i)}, \dots, H_{0,i-r+1}^{(i)}$ and the corresponding p-values (p_k). The authors state "A nonparametric density estimation based on the Parzen windows is adopted to approximate the distribution of the T-square statistics, from which the p-values are calculated." The readers are referred to the paper for more details. Sort the p-values in ascending order and let $H_{(k)}$ be the null hypothesis corresponding to $p_{(k)}$; which is the k^{th} ordered p-value. At the outset take $\hat{k} = 0$ and then find $\hat{k}$, the maximum value of k such that $p_{(k)} \leq q^k/r$. For $\hat{k}$ non-zero the hypotheses are rejected and the process is declared to be OOC. The authors perform performance comparison via simulation in terms of the OOC and IC average run length according to various non-centrality parameters. They conclude by stating that the proposed BH-EWMA chart outperforms the existing MEWMA chart. Again, one needs to perform post signal diagnostic and this would need further research. Note also that in the above formulation, $\boldsymbol{\mu}_0$ is assumed known or given as in Case K. Clearly, a question to be addressed is Case U, that is when $\boldsymbol{\mu}_0$ is to be estimated. The application of p-values (monitoring based on p-values) may be an interesting area within the SPC literature.

Change-point model based methods

Holland and Hawkins (2014) proposed a Phase II nonparametric multivariate control chart for monitoring location. Their chart is based on an approximately distribution-free multivariate generalization of the Wilcoxon-Mann-Whitney test. They extended the work of Hawkins and Deng (2010), who developed a nonparametric univariate change-point analog based on the Mann-Whitney statistic, to the multivariate setting. The authors caution that although the underlying test statistic is approximately distribution-free, their procedure may not be suitable for some multivariate distributions with unusual dependence structures between vector components.

A change-point model for the univariate case was given earlier. Now, for the multivariate setting, given a sample of $p \times 1$ random vectors we have

$$\mathbf{x}_i \sim \begin{cases} F(\boldsymbol{\mu}) & \text{for} \quad i = 1,2,\dots,\tau \\ F(\boldsymbol{\mu}+\boldsymbol{\delta}) & \text{for} \quad i = \tau+1, \tau+2 \dots \end{cases}$$

where $\boldsymbol{\delta}$ represents an arbitrary sustained shift in location. For $p = 1$ let $r^{(i)}$ denote the rank of x_i amongst the observations $\{x_1, \dots, x_n\}$ and set $R(x_i) = 2r^{(i)} - n - 1$ in order to center the ranks. Then $R(x_i) = \sum_{j=1}^{n} sign(x_i - x_j)$ and the well-known Wilcoxon-Mann-Whitney test statistic, for the difference in location between two samples $\{x_1, \dots, x_k\}$ and $\{x_{k+1}, \dots, x_n\}$, is given by $u_k = \sum_{i=1}^{k} R(x_i)$. Next, in order to outline a multivariate generalization of this test statistic, the $sign$ function is exchanged by a kernel function, $\mathbf{h}(\mathbf{x},\mathbf{y}) = -\mathbf{h}(\mathbf{y},\mathbf{x})$, so that $\mathbf{h}(\mathbf{x},\mathbf{x}) = 0$ and $\mathbf{h}(\mathbf{x},\mathbf{y})$ represents a measure of difference between $\mathbf{x}$ and $\mathbf{y}$. Choi and Marden (1997) suggested making use of a directional rank test by means of the kernel function $\mathbf{h}(\mathbf{x},\mathbf{y}) = \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}$. Then define $\mathbf{R}_n(\mathbf{x}_i) = \sum_{j=1}^{n} \mathbf{h}(\mathbf{x}_i, \mathbf{x}_j)$ so that $\mathbf{R}_n(\mathbf{x}_i)$ denotes the directional rank of $\mathbf{x}_i$ for

$i = 1,2, \dots, n$. Next, the authors introduced notation for defining within group rank vectors as follows. For each possible change point, $k \in \{1,2, \dots, n-1\}$, let $\mathbf{R}_{n,k}^{*}(\mathbf{x}_i) = \sum_{j=k+1}^{n} \mathbf{h}(\mathbf{x}_i, \mathbf{x}_j)$ and $\overline{\mathbf{r}}_n^{(k)} = \frac{1}{k}\sum_{i=1}^{k} \mathbf{R}_n(\mathbf{x}_i)$. The authors then continue with a detailed discussion on the pooled and unpooled sample covariance matrices; here we simply mention that directional rank statistic for testing for differences in location vector between $\{x_1, \dots, x_k\}$ and $\{x_{k+1}, \dots, x_n\}$ is given by $r_{k,n} = \overline{\mathbf{r}}_n^{(k)'} \widehat{\Sigma}_{k,n}^{-1} \overline{\mathbf{r}}_n^{(k)}$ with $\widehat{\Sigma}_{k,n}^{-1} = ((n-k)/nk)\widehat{\Sigma}_n$ and where $\widehat{\Sigma}_n$ denotes the unpooled covariance matrix. The scheme proposed by Holland and Hawkins (2014) entails maximizing $r_{k,n}$ across possible change-point values, $r_{\max,c,n} = \max\limits_{c<k<n-c} (r_{k,n})$. Note that c denotes the number of observations at the beginning and at the end of sequence that is not considered; see Holland and Hawkins (2014) for a detailed motivation of why the so-called quarantine constant, c, was instigated. The process is declared OOC when $r_{\max,c,n} > h_{\alpha,p,c,n}$ and typically a search for assignable causes would be started. The authors concluded that their proposed scheme outperformed its parametric counterpart for small to moderate magnitudes of shift.

Control charts using bootstrap

Determining the control limits of multivariate nonparametric charts is time consuming and complicated. Phaladiganon, Kim, Chen, Baek and Park (2011) used a bootstrap approach, as opposed to kernel density estimation (KDE) in order to find the control limits for the multivariate T^2 charts when the assumption of normality doesn't hold. The basis of bootstrapping is discussed in the univariate section of this paper. Here we simply give an outline of the bootstrapping technique in this case. Let $T_1^{2(i)}, T_2^{2(i)}, \dots, T_n^{2(i)}$ be a set of n T^2 values from the i^{th} bootstrap sample, $i = 1, \dots, B$ (with B typically being a large number), which are drawn randomly from the initial T^2 statistics (with replacement). For each bootstrap sample the $100^{(1-\alpha)th}$ percentile value is calculated with α being a value specified by the practitioner beforehand. The control limit is determined by taking an average of the B percentile values, say $\overline{T^2}_{100(1-\alpha)}$. The process is declared to be OOC if a charting statistic plots beyond $\overline{T^2}_{100(1-\alpha)}$. The authors found that the bootstrap technique outperforms the traditional T^2 chart in all cases and performed similarly to the KDE-based T^2 chart.

Miscellaneous Applications

As the realm of monitoring applications have grown and have become more pervasive in many spheres of life, data collection has become easier and quicker and computational resources have grown faster, several researchers have developed distribution-free control charts for monitoring big data and for monitoring higher dimensional process data. We briefly mention a few of these here.

Zhang, Tsung and Zou (2015) considered a method where making use of both past IC and OOC data, where the charting statistic and the control limits were computed using of support vector machines (SVMs) methodology. One issue with this approach is that the real-time (current) Phase II data is not included in the calculation of their control limits and the classifier is only trained once, based on a single data set, at the beginning of the monitoring process. Note that earlier, Deng, Runger and Tuv (2012) introduced a supervised learning approach which is based on constantly updating the classifier with real-time data. They referred to this approach as the RTC since it is based on real-time contrasts between the IC reference data and the real-time (current) data. RTC assigns one class label to the IC reference data and another class label to some available real-time data which converts the monitoring problem into a dynamic classification problem. The advantage in doing this is that, in the previous literature, a classifier is trained once based on a single data set at the beginning of monitoring process but in the RTC approach a classifier is retrained with each new observation and the statistic (such as the classification error rate) that is monitored is generated using the generalized likelihood-ratio principle. Although the RTC approach is useful and can be applied to a variety of monitoring problems, the typical problems with making use of a discrete charting statistic arises, i.e. that it might be less efficient in fault detection. He, Jiang and Deng (2016) and Wei, Huang, Jiang and Zhao (2016) proposed distance-based control charts based on support vector machines (SVMs) and kernel linear discriminant analysis (KLDA), respectively, in order to make the charting statistics continuous. Although these proposed charts outperformed the traditional RTC chart, they do not have the advantage of being able to be applied to various data types such as categorical data and missing data. In order to overcome this problem, Jang, Park and Baek (2017) proposed improved RTC charts making use of random forests with weighted voting. These improved charts outperformed the traditional RTC chart and was shown to be more effective than the RTC charts based on SVMS and KLDA.

D´avila, Runger and Tuv (2014) also make use of a supervised learning approach and showed a practical application in public health surveillance which doesn't involve a low dimensional data (which is usually the case application in public health surveillance) where 25 decision trees were used to monitor counts of a disease. Chen, Zi and Zou (2016) also showed a practical application of a semi-conductor manufacturing process which involved 591 variables and cautioned against the assumption that all 591 variables are normally distributed.

Kang and Kim (2013) proposed a control chart that makes use of a k-means clustering analysis which systematically divides the data into clusters by minimizing within-group variation and by maximizing between-group variation. The charting statistic for this chart, $\tau(x) = min_k(x-\mu_k)' \sum_k^{-1}(x-\mu_k)$ where x is the new observation that has not been classified into a cluster yet and μ_k and $\sum_k$ are the mean vector and the covariance matrix of the k^{th} cluster, respectively. The number of clusters, k, should be pre-selected by the practitioner, although Kang and Kim (2013) showed that making use of different values for k does not significantly change the result. A bootstrap method was used in order to find the control limit.

When the joint distribution of multiple quality characteristics is unknown, Liang, Xiang and Pu (2016) have extended on the idea of incorporating the least absolute shrinkage and selection operator (LASSO) into the EWMA monitoring scheme (referred to as the LEWMA chart), which was proposed by Zou and Qui (2009), by developing a robust counterpart of it that has an affine-invariance property and is nonparametric under the elliptical distribution family in that the control limit can be obtained by making use of a standard multivariate normal distribution. Since a sign statistic is used, the chart is referred to as the SLEWMA chart. This chart, used for Phase II monitoring, is compared to the LEWMA chart and the multivariate direction sign EWMA chart (denoted MDSE), which was proposed by Z et al. (2013), and not only is the computation of the control limits for the SLEWMA much simpler than the other charts, but there are many cases where the proposed SLEWMA chart outperforms its competitors. Future research could include the modification of Liang et al. (2016)'s chart so that it can be used for Phase I data.

In general, multivariate control charting, particularly multivariate nonparametric control charting, remains an area open for more contributions. There is need for more theoretical insights and practical recommendations. For example, more studies are needed for many proposed charts, along the lines of Dovoedo and Chakraborti (2017), to understand the effects of parameter estimation, amount of required data for nominal performance and perhaps considering how to adapt or modify the charts for a given amount of data. Post signal diagnostics is also an important area of further research. There is also the need for case studies and detailed explanations of what seems like a complicated methodology. To all of these ends, above all, software development is urgently needed.

Summary and Recommendations

NSPC charts provide a robust alternative for statistical process monitoring in practice when the form of the underlying distribution is unknown. The goal of this paper is to bring the review of nonparametric control charting techniques, of Chakraborti et al. (2001, 2007 and 2011), forward to 2017, covering some of the major advances and contributions. In doing so, we found that, while a lot of progress has been made in the field of NSPC, it appears that NSPC has not been fully embraced in SPC. To this end, Woodall and Montgomery (2014) stated "Despite their advantages in reducing the distributional assumptions required to design control charts with specified in-control performance, it does not seem that nonparametric methods are gaining a foothold with practitioners. This could partially be due to a lack of available statistical software for implementing the methods, a lack of familiarity, and a lack of textbook coverage. Nevertheless, this research area remains active." To address these concerns, Chakraborti, Qiu and Mukherjee (2015), in the introduction to a special issue on Nonparametric Statistical Process Control Charts, that appeared in a recent issue of *Quality and Engineering Reliability Engineering International*, wrote, "Motivated by such observations, our aim has been to bring NSPC charts to the mainstream SPC arena, since we strongly feel that it has much to offer to both the SPC practitioner and the researcher. To reiterate, as stated in the "Call for Papers" for this special issue, "Traditional control charts require the assumption that the process distribution follows a parametric form (e.g. normal). In practice, however, this assumption may not hold, in other words, the process may not follow the pre-specified parametric distribution. In the literature, it has been well demonstrated that results from the traditional control charts using the pre-specified distribution in their design may not be reliable because their actual false alarm rates could be substantially larger or smaller than the nominal false alarm rate. A direct consequence of this could be that much labor and many resources are wasted, or that many defective products are manufactured without notice. Therefore, in cases when no parametric form of the process distribution is available or when no parametric form is validated properly beforehand, control charts without requiring the specification of a parametric form for the process response distribution, or simply nonparametric (distribution-free) statistical process control (NSPC) charts, should be considered."

We feel that nonparametric statistics in general and nonparametric control charts in particular have much to offer to the quality practitioner. Thus the updated overview of the available literature will be of interest and value to a broad spectrum of readers. These charts can be beneficial when parametric model assumptions cannot be objectively justified. They have stable IC properties, are robust against outliers and their efficiency can be quite high. Further work in NSPC research and practice are encouraged and will be welcome. As we have noted in various places in the paper, there are a lot of open problems. Two major concerns are the effects of parameter estimation and examination of the reference sample size requirements. There has been very few work in this direction. It is generally agreed that nonparametric charts require more data but a systematic study will be valuable. From the practical side, nonparametric charts do not have a heavy computation burden and require a lot of extra computation compared to their parametric counterparts, but there is not much software available at the moment to apply all the control charts that are available. This would be an important area for future contributions.

Available software

Some software are now available in the area of nonparametric SPC. A vast majority of them is written in R. Most of these are focused on researchers and their own work, without much standardization, including notation and terminology. This limits their availability and accessibility to some degree. As a consequence, it is not easy for the interested users to apply nonparametric charts in routine SPC. In any case, we list some of these R packages below which can be downloaded from the CRAN archives. More work is necessary in this arena to make the full power of NSPC available to the users.

Name	**Brief description**
dfphase1	Phase I Control Charts (with Emphasis on Distribution-Free Methods) Statistical methods for retrospectively detecting changes in location and/or dispersion of univariate and multivariate variables. Data values are assumed to be independent, can be individual (one observation at each instant of time) or subgrouped (more than one observation at each instant of time). Control limits are computed, often using a permutation approach, so that a prescribed false alarm probability is guaranteed without making any parametric assumptions on the stable (in-control) distribution.
changepoint.np	Methods for Nonparametric Changepoint Detection another alternative for Phase I analysis - only univariate, only individual observations
cpm	change-point methods - parametric and nonparametric - both Phase I and Phase II
NPMVCP	nonparametric multivariate change point detection in Phase II
Spcadjust	implements a bootstrap based method to adjust monitoring schemes to take estimation error into account. Although the spcadjust is not for nonparametric charts specifically, they say that the package covers the most common setups of Shewhart, CUSUM and EWMA charts and that it is easy to add further charts
SPCModelNonpar CenterScale	updates $(X_t - \mu - \Delta/2)/\sigma$, no distributional assumptions
SPCModelNonpar	user defined updates, no distributional assumptions
qcr	Allows to generate Shewhart-type charts and to obtain numerical results of interest for a process quality control (involving continuous, attribute or count data). This package provides basic functionality for univariable and multivariable quality control analysis, including: xbar, xbar-one, S, R, ewna, cusum, mewna, mcusum and T2 charts. Additionally have nonparametric control charts multivariate
mnspc	Statistical process control for multivariate data that is not necessarily Gaussian distributed. A function is provided to categorize components of multivariate response vectors. Tools for setting up a CUSUM procedure for the transformed data are included. The CUSUM scheme can also be applied to the case when some (or all) of the multivariate response components are binary-categorical.

Acknowledgements

The authors thank two anonymous reviewers for their detailed comments and suggestions which have improved presentation.

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