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Doctoral Thesis

A quasi-Hopf structure in marginally deformed $\mathcal{N} = 4$ Super Yang-Mills Theory

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Declaration of Authorship

I, Siphesihle Hector Dlamini, declare that this thesis titled, '**A quasi-Hopf structure in marginally deformed $\mathcal{N} = 4$ Super Yang-Mills Theory**' and the work presented in it is my own. I confirm that this work submitted for a PhD degree at the University of Pretoria is my own and is expressed in my own words. Any uses made within it of the works of other authors in any form (e.g., ideas, equations, figures, text, tables, programs) are properly acknowledged at any point of their use. A list of the references employed is included.

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Date:

"I am just going outside and may be some time" [1]

– capt. L. Oates

Abstract

The $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions admits deformations and the exactly marginal deformations of its $SU(3)$ R-symmetry sub-sector are known as *Leigh-Strassler*. Leigh-Strassler deformations break the $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 1$ while preserving conformal symmetry. With exactly marginal deformations only the F-terms are deformed thus Leigh-Strassler deformations only affect the superpotential in the lagrangian. In this thesis we study the symmetry of the marginally deformed $\mathcal{N} = 4$ SYM and demonstrate that its algebraic structure can be understood in terms of quasi-Hopf algebras. Quasi-Hopf algebras have a notion of twisting due to Drinfeld which makes them a natural mathematical language with which to treat deformations. Furthermore the deformation of the $\mathcal{N} = 4$ SYM superpotential is automated by the definition of a suitable star product.

Acknowledgements

"...when what is desired comes, it is a tree of life"

- Prov. 13:12, *The Bible*

It has been a long journey which required blood, sweat and a lot of coffee. Yet it is during this time that the Lord Jesus as grace has been sweet to me. I thank the Lord for His grace that has brought me to this point. I would like to thank my family for all the sacrifices they underwent in order to support me. Thanks Ma and big bro

"For who distinguishes you? And what do you have that you did not receive?..."

- 1 Cor. 4:7, *The Bible*

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By now at least two things are evident:

1. this work belongs to more than one person

2. this page cannot contain them all.

Hector Dlamini

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Abbreviations and Symbols

Abbreviations:

HA	**H**opf **A**lgebra(s)
qHA	**q**uasi-**H**opf **A**lgebra(s)
YBE	**Y**ang **B**axter **E**quation
CFT	**C**onformal **F**ield **T**heory
SCFT	**S**uper **C**onformal **F**ield **T**heory
SCT	**S**pecial **C**onformal **T**ransformation
SUSY	**SU**per**SY**mmetry
SYM	**S**uper **Y**ang-**M**ills
LS	**L**eigh-**S**trassler

Symbols:

$\bullet$	Product map of a/an group/algebra
m or .	Product map on a module [matrix multiplication in our case]
Δ	Co-product map of a co-algebra
$\circ$	Map composition symbol
id	Identity map
$\mathbb{1}$	Identity element

To my family:
Tryphina and S'bonelo

Chapter 1

Introduction

It is a long-term dream of physics research to obtain a model that captures all known physics phenomena under one umbrella. To this end much attention has been invested in unifying the already existing descriptions of known phenomena. Such an undertaking is ambitious but not hopeless because the existing descriptions provide hints on what ingredients must be included in the unified model. Symmetry is one such hint. One naturally expects the unified model to reduce to its ingredient models in the appropriate limits. This reduction should also be consistent at the level of symmetry. If U is a symmetry group of M_U where M_U is a model that unifies models M_A and M_B whose symmetry groups are A and B then we expect that the group U reduce to A or B in the respective limits. By this we can test any proposed unified model from the view point of symmetry. The details of the test would involve a specification of the mechanism by which the symmetry of the unified model reduces to the ingredient model.

An inefficient way of approaching this search for a unified model is to first propose a model and then test whether its symmetry group reduces to that of the ingredient model(s). An alternative approach is to build the symmetry groups of the ingredient models into the unified model; this ascertains that the unified model will reduce to its ingredient models. The latter approach has proved successful and testimony to this is modern physics.

In modern physics we have, on the one hand, *Special Relativity* [SR] which is a successful model for the description of physics in inertial frames. Its brilliance is especially realized at speeds comparable to that of light where it serves to preserve causality among many things. The requirement of SR as far as symmetry is concerned is Lorentz invariance.

On the other hand there is Quantum Mechanics [QM] which also is successful in its domain, the physics of (small) particles that constitute matter. QM is probabilistic and thus the likelihood of any event must at least be 0 and at most 1. This restricts the allowed transformations in QM to those that preserve probabilities and thus we are led to the idea of unitarity. In the construct of a model which combines SR and QM, Lorentz invariance and unitarity become the guideline and out of this came forth *Quantum Field Theory* [QFT] with some of the earliest pioneering work found in [2].

The customary way of constructing a QFT is to begin with a Lagrangian formalism of a classical field theory and then impose a quantization prescription. Our method of quantization is the Feynman path integral prescription which consists of defining the generating functional [1] as

$$\mathcal{Z}[\phi, g] = \int \mathcal{D}\phi \; e^{iS[\phi,g]} \tag{1.1}$$

where g is a (set of) coupling constant(s) in which the 'strength' of the interactions is encoded. ϕ is a collection of fields, possibly with different spins, in the theory and S is the action, which in d-dimensions is given by

$$S[\phi, g] = \int d^d x \; \mathcal{L}(\phi, \partial\phi) \tag{1.2}$$

$\mathcal{L}$ is the Lagrange density and its variation with respect to a field gives the equations of motion of that specific field. By using *Noether*'s theorem one can obtain the conserved charges corresponding to the symmetries of the classical theory described by $\mathcal{L}$. At the quantum level, symmetries are transformations that leave Lagrangian density $\mathcal{L}$ and the measure $\mathcal{D}\phi$ unchanged. Some classical symmetries do not survive the quantization process and as a result quantities/observables/operators of the quantum theory will have anomalies. n-point functions are central quantities in understanding scattering processes in QFT and for a field $\phi_i \equiv \phi(x_i)$, a generic n-point function is defined by

$$G^{(n)}(x_1, x_2, \ldots, x_n) \equiv \langle\Omega| \, T\left(\phi_1\phi_2\ldots\phi_n\right) |\Omega\rangle = \frac{\int \mathcal{D}\phi e^{iS[\phi]}\phi_1\phi_2\ldots\phi_n}{\int \mathcal{D}\phi e^{iS[\phi]}} \tag{1.3}$$

where $|\Omega\rangle$ represents the vacuum state of the (interacting) theory. Completely solving a QFT is synonymous with calculating all n-point functions because n-point functions [2] are related to the S-matrix elements of the theory via the *LSZ reduction formula* [3][4]. However, the computation of n-point functions is a task which involves an infinity of integrals over momenta whose limits are $-\infty$ and $+\infty$ and often times these integrals are divergent. The divergences can appear in the large momentum limit and these are known as Ultra-Violet [UV] divergences. For theories with massless particles divergences will appear in the low momentum limit; these are known as Infra-Red [IR] divergences. In order to curb the divergences and obtain finite results one needs to define a cut-off

[1] or partition function

[2] (connected and amputated)

scale, that is, a point (momentum) beyond which we admit ignorance. More formally, in order to compute scattering amplitudes i.e. elements of the S-matrix, from n-point functions defined at an energy scale, M, we must introduce a cut-off scale, Λ. If the fields are normalized at this scale then use the LSZ formula to obtain the scattering amplitudes. Otherwise one would need to renormalize the fields as

$$\phi_b(x_i) \longrightarrow \phi(x_i) = Z^{-1/2}(M)\phi_b(x_i) \tag{1.4}$$

So then the renormalized n-point function $G^{(n)}(\bar{x}; M, g)$ is related to the non-renormalized n-point function $G_b^{(n)}(\bar{x}; \Lambda, g_b)$ via

$$G^{(n)}(\bar{x}; M, g(M)) = Z^{-n/2}(M)G_b^{(n)}(\bar{x}; \Lambda, g_b) \tag{1.5}$$

where b signifies the non-renormalized quantities, which are usually referred to as bare quantities and $\bar{x}$ is a list of the n coordinates of each field in the n-point function. The bare n-point function depends on bare fields, ϕ_b, bare coupling parameters g_b and cut-off scale, Λ. The renormalized n-point function on the other hand depends on renormalized fields, ϕ, coupling parameters, g, and the scale of renormalization M. The implications of this observation means we can write

$$\frac{d}{dM}G_b^{(n)}(\bar{x}; \Lambda, g_b) = 0 \tag{1.6}$$

The choice of the renormalization scale is arbitrary and thus the same theory could equally be defined at a different scale. What is of interest is the effect of a shift in this scale has; an infinitesimal shift $M \longrightarrow M + \delta M$ will also have corresponding shifts in the fields and coupling parameters so that the bare n-point functions $G_b^{(n)}$ are unchanged (1.6). The fields correspondingly shift as $\phi \longrightarrow \phi + \delta\phi$ which can be written as $(1+\delta\eta)\phi$, where $\delta\eta$ is a dimensionless parameter. The shift in the coupling parameter is $g \longrightarrow g + \delta g$. Yet the renormalized n-point functions will be shifted as $G^{(n)} \longrightarrow (1 + n\delta\eta)G_n$. The variation of $G^{(n)}$ as a function of the renormalization scale, M, and coupling, g, is

$$\delta G^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial g}\delta g = n\delta\eta G^{(n)} \tag{1.7}$$

which, after multiplying by $M/\delta M$, can be manipulated to obtain the *Callan-Symanzik equation* [5], [6]:

$$\left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial g} + n\gamma\right] G_n(\bar{x}; M, g(M)) = 0 \tag{1.8}$$

where β and γ are given by

$$\beta = \frac{M}{\delta M}\delta g \quad \text{and} \quad \gamma = -\frac{M}{\delta M}\delta\eta \tag{1.9}$$

The parameters γ and $\beta(g)$, respectively known as the anomalous dimension and the β function, are dimensionless and independent of spacetime $\bar{x}$ and cut-off scale, Λ. $\beta(g)$ shows the dependence of the coupling constant g on the renormalization scale M and can thus be expressed as

$$\beta(g) = M\frac{d}{dM}g(M) \tag{1.10}$$

hence $\beta(g)$ encodes the renormalization group [RG] flow of the theory. The values of g for which $\beta(g) = 0$ are known as fixed points. Fixed points are a matter of interest because they indicate scale invariance of theory and in 2 dimensional unitary QFTs scale invariance has been shown to imply conformal invariance, which is an even richer symmetry [7], [8]. Although scale invariance does not guarantee conformal symmetry in dimensions other than 2 ([9], [8],[10]), fixed points have nonetheless been associated with conformal invariance. In chapter 2 we shall discuss the meaning and usefulness of conformal symmetry in solving QFTs. Typically the beta function is positive, meaning the coupling constant grows with increase of the energy scale, strong coupling at short distances. What is surprising is that there are non-Abelian gauge theories which behave vice versa. This is seen in the fact that at one-loop the beta function of a non-Abelian gauge theory with gauge group G is

$$\beta(g) = -\frac{g^3}{16\pi^2}\Big(\frac{11}{3}C_2(G) - \frac{4}{3}N_f C(r)\Big) \tag{1.11}$$

Here $C_2(G)$ is a quadratic Casimir of the gauge group G. N_f refers to the number of fermion species present in the theory and these fermions are assumed to all belong to a representation r; r is an irreducible representation of the gauge group G and $C(r)$ is a Casimir of r. Both $C(r)$ and $C_2(G)$ are constants which can be determined via the generators of the group. Note in (1.11) that for small values of N_f the beta function is negative which suggests that the coupling constant becomes weak with increasing scale. This is useful because non-Abelian gauge theories are suited for studying Quantum Chromodynamics [QCD] and the field theories studied in this work are of the same type.

Having presented a brief overview of QFT, we now can consider the outline of the document. In chapter 2 we present the symmetries typically encountered in physics research albeit with a view to build up to $\mathcal{N} = 4$ Super Yang-Mills theory, some of whose properties are reviewed in chapter 3. In chapter 4 is a mini presentation of aspects of quasi-Hopf algebra [qHA] structures. Chapter 5 marks the beginning of novel results which involves uncovering the qHA structure that is associated to the $SU(3)$ sector of $\mathcal{N} = 4$ SYM. Another novel result is the computation of a Drinfel'd twist that deforms qHA structure of $\mathcal{N} = 4$ SYM to that of Leigh-Strassler [LS] theories. In Chapter 6 we define a suitable star product which enables the automation and realization of the deformation from $\mathcal{N} = 4$ SYM to Leigh-Strassler theories at the level of the Lagrangian. The results presented in Chapters 5 and 6 are original contributions by the author and have been published in [11]. Finally in Chapter 7 is a test and discussion of the qHA symmetry of LS theories which then concludes with an outlook.

Chapter 2

Symmetry in Physics

In this section we review symmetries often encountered in physics and also consider theories which possess such symmetries.

2.1 Lorentz and Poincaré symmetry

We begin with Lorentz invariance, a symmetry which pertains to space and time. Such a symmetry is the result of the postulates of *Special relativity* [SR], namely that

1. physics be equivalent in all inertial frames

2. and that the speed of light be the same for all inertial frames

Due to the second postulate space and time must be unified into spacetime. In SR, time is allowed to dilate and space to contract so to preserve the constancy of the speed of light. Thus time need not advance at the same rate in every inertial frame. Observers in different frames will not necessarily agree on measured lengths. The consequence of the first postulate is that time and length (space) as measured in one frame need not be the same as in another inertial frame yet the physics in one frame must be compatible/consistent with that of the other frame. Since the physics as measured in frame A with coordinates x^μ is to be consistent with physics as measured in frame A' with coordinates $x^{\mu'}$ there must be a transformation which relates data in frame A to data in frame A', such a transformation is called a *Lorentz* transformation.

Lorentz symmetry

In order to be concrete we will work in $(1+3)$-dimensional flat Minkowski spacetime $M = \mathbb{R}^{1+3}$ with metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Lorentz transformations are linear coordinate transformations

$$x^{\mu} \longrightarrow x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} \tag{2.1}$$

which preserve the metric. The consequence is that (2.1) will also preserve spacetime "lengths". The most obvious Lorentz transformation is the trivial transformation where frames A and A' are the same, i.e. $\Lambda = \mathbb{1}$. Since spacetime lengths are to be invariant under such transformations we therefore can deduce the properties of Λ by noting that [12]

$$\begin{aligned} x^{\mu} x_{\mu} &= x'^{\rho} x'_{\rho} \\ x^{\mu} \eta_{\mu\nu} x^{\nu} &= x'^{\rho} \eta_{\rho\sigma} x'^{\sigma} \\ x^{\mu} \eta_{\mu\nu} x^{\nu} &= \Lambda^{\rho}_{\ \mu} x^{\mu} \eta_{\rho\sigma} \Lambda^{\sigma}_{\ \nu} x^{\nu} \\ \Rightarrow \eta_{\mu\nu} &= \eta_{\rho\sigma} \Lambda^{\rho}_{\ \mu} \Lambda^{\sigma}_{\ \nu} \end{aligned} \tag{2.2}$$

For later convenience we write the last equality as

$$\eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^{\rho}_{\ \mu} \Lambda^{\sigma}_{\ \nu} = \Lambda_{\sigma\mu} \Lambda^{\sigma}_{\ \nu} \tag{2.3}$$

and then raise the index μ to produces the relation

$$\eta^{\tau\mu} \eta_{\mu\nu} = \eta^{\tau}_{\ \nu} = \delta^{\tau}_{\ \nu} = \eta^{\tau\mu} \Lambda_{\sigma\mu} \Lambda^{\sigma}_{\ \nu} = \Lambda_{\sigma}^{\ \tau} \Lambda^{\sigma}_{\ \nu} = (\Lambda^{\tau}_{\ \sigma})^{T} (\Lambda^{\sigma}_{\ \nu}) \tag{2.4}$$

The last equality of (2.2) is the defining relation which the Λ's must observe in order to be Lorentz transformations. As matrices, the Λ's which satisfy this relation define the non-compact Lie group, $O(1,3;\mathbb{R})$ which is a subset of $GL(4;\mathbb{R})$, the set of all 4×4 real invertible matrices. The Lorentz group has four disconnected parts which correspond to four classification of Lorentz transformations. In order to see these parts we first take the determinant of the last equality of (2.2) and learn that

$$\begin{aligned} \det(\eta)\det(\Lambda^{T})\det(\Lambda) &= \det(\eta) \\ \det(\Lambda^{T})\det(\Lambda) &= 1 \\ \det(\Lambda)^{2} &= 1 \\ \therefore \det(\Lambda) &= \pm 1 \end{aligned} \tag{2.5}$$

Λ is called *proper* when $\det(\Lambda) = +1$ and *improper* when $\det(\Lambda) = -1$; these are 2 of the four parts. Note that improper Λ's cannot contain the identity, as such they do not define a (sub-)group but proper Λ's do define a subgroup of the full Lorentz group. We shall focus on the proper transformations. In order to show the 2 other parts of the Lorentz group we excerpt the temporal component from the last equality of (2.2) to obtain

$$\eta_{00} = \eta_{\rho\sigma}\Lambda^{\rho}_{\ 0}\Lambda^{\sigma}_{\ 0} \tag{2.6}$$

$$-1 = -\left(\Lambda^{0}_{\ 0}\right)^2 + \sum_i \left(\Lambda^{i}_{\ 0}\right)^2 \tag{2.7}$$

$$\Rightarrow \left(\Lambda^{0}_{\ 0}\right)^2 = 1 + \sum_i \left(\Lambda^{i}_{\ 0}\right)^2 \tag{2.8}$$

According to (2.8) there is a lower bound on the square of the time-component of Λ, $(\Lambda^{0}_{\ 0})^2 \geq 1$. This bound means the time-component itself is also constrained to $|\Lambda^{0}_{\ 0}| \geq 1$. The Λ's for which $\Lambda^{0}_{\ 0} \geq 1$ correspond to *orthochronous* transformation while those for which $\Lambda^{0}_{\ 0} \leq -1$ to *non-orthochronous* transformations. For the present work we focus our efforts on the proper and orthochronous subgroup[1] which is denoted by $SO^{+}(1, 3; \mathbb{R})$.

From (2.4), we consider the generators of this group by expanding the Λ's in a small parameter ϵ. Doing so gives the infinitesimal form of Lorentz transformations

$$\Lambda^{\mu}_{\ \nu} = \eta^{\mu}_{\ \nu} + \epsilon\omega^{\mu}_{\ \nu} + \dots \tag{2.9}$$

and substituting this into (2.3). Keeping at most terms linear in the expansion parameter ϵ means

$$\begin{aligned}
\eta_{\mu\nu} &= \Lambda_{\sigma\mu}\Lambda^{\sigma}_{\ \nu} \\
&= \left(\eta_{\sigma\mu} + \epsilon\omega_{\sigma\mu} + \dots\right)\left(\eta^{\sigma}_{\ \nu} + \epsilon\omega^{\sigma}_{\ \nu} + \dots\right) \\
&= \eta_{\mu\nu} + \epsilon\left[\omega_{\nu\mu} + \omega_{\mu\nu}\right] + \dots \\
&\Longrightarrow \omega_{\mu\nu} = -\omega_{\nu\mu}
\end{aligned} \tag{2.10}$$

This antisymmetric tensor $\omega_{\mu\nu}$ corresponds to rotations and boosts which are generated by $M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ with $\mu, \nu = 0, \dots, 3$ and these generators obey the commutation relations

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right) \tag{2.11}$$

These relations describe the algebra $\mathfrak{so}(1, 3; \mathbb{R})$. Define $J_i = \frac{1}{2}\,\epsilon_{ijk}M_{jk}$ with $i, j, k = 1, 2, 3$ and $K_i = M_{0i}$ so that the Lorentz generators are divided into spatial rotations, J_i and boosts, K_i. The commutation relations (2.11) then become

$$\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk}J_k \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k \\
&[J_i, K_j] = i\epsilon_{ijk}K_k
\end{aligned} \tag{2.12}$$

In order to recognize Lorentz invariance we have to know the irreducible representations of $so(1, 3; \mathbb{R})$ and the linear combinations of J_i and K_j are rather useful to uncover

[1]also known as the *restricted Lorentz group*.

some of these representation. Let $A_i := \frac{1}{2}(J_i + iK_i)$ and $B_i := \frac{1}{2}(J_i - iK_i)$. These new generators satisfy

$$[A_i, B_j] = 0 \quad , \ [A_i, A_j] = i\epsilon_{ijk}A_k \quad \text{and } [B_i, B_j] = i\epsilon_{ijk}B_k \tag{2.13}$$

The first commutation relation shows that the A_i's and B_i's do not mix and the others are the $\mathfrak{su}(2)$ relations. Having begun with elements of $\mathfrak{so}(1,3;\mathbb{R})$, after a redefinition we arrived at two copies of $\mathfrak{su}(2)$. It is important to note that during the definitions of A_i and B_i we considered the generators J_i and K_j as elements of a complex vector space and effectively complexified $\mathfrak{so}(1,3;\mathbb{R})$. The honest decomposition is then given by

$$\mathfrak{so}(1,3;\mathbb{R})_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} \tag{2.14}$$

which tells us that, when complexified, the Lie algebra which underlies the proper and orthochronous Lorentz group is locally homomorphic to two copies of the complexified $\mathfrak{su}(2)$ algebra. From the representation theory of Lie algebra we know that there is a one-to-one correspondence between the representations of a complexified form of an algebra to the representations of its real form ([13],[14],[15]). This means we can use the representations of $\mathfrak{su}(2)$, which are many, to find those of $\mathfrak{so}(3,1,\mathbb{R})$. Since spin is a useful index to label $\mathfrak{su}(2)$ representations we can therefore employ it to label representations of $\mathfrak{so}(1,3;\mathbb{R})$; a scalar whose spin is 0 is labeled by the couple $(0,0)$. A 4-vector having spin 1 is labeled by

$$\left(\frac{1}{2},\frac{1}{2}\right) - \text{vector} \tag{2.15}$$

and Weyl spinors are labeled by the usual

$$\left(\frac{1}{2},0\right) - \text{chiral spinor} \quad , \quad \left(0,\frac{1}{2}\right) - \text{antichiral spinor} \tag{2.16}$$

Besides $SU(2)$, the Lorentz subgroup $SO^+(1,3;\mathbb{R})$ is homeomorphic to $SL(2,\mathbb{C})$, the Lie group of invertible complex 2×2 matrices with unit determinant. It suffices to show that for a given Λ in $SO^+(1,3;\mathbb{R})$ there is an element N in $SL(2;\mathbb{C})$. To demonstrate the connection we first define the set

$$\sigma^\mu = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{2.17}$$

made up of the identity matrix together with the Pauli matrices. By direct calculation it is easy to see that

$$\frac{1}{2}\sigma^\mu \bar{\sigma}_\mu = \mathbb{1}_{2\times 2} \tag{2.18}$$

which means any 2×2 complex matrix can be written as a linear combination of σ's.

Using the set (2.17), any 4-vector $x^\mu = (-x_0, x_1, x_2, x_3)$ can be written as a 2×2 complex matrix

$$x_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} =: X \tag{2.19}$$

Note that the matrix X is hermitian and its determinant is equal to $-x^\mu x_\mu$ [2]. Moreover the set (2.17) can be used to map a 2×2 hermitian complex matrix X to a 4-vector x_μ by simply noting that

$$\mathrm{Tr}\left(X\bar{\sigma}^\nu\right) = \mathrm{Tr}\left(x_\mu \sigma^\mu \bar{\sigma}^\nu\right) = x_\mu \mathrm{Tr}\left(\sigma^\mu \bar{\sigma}^\nu\right) = 2x^\nu \tag{2.20}$$

This is due to the fact that the σ matrices satisfy the relation $\mathrm{Tr}(\sigma^\mu\bar{\sigma}^\nu) = 2\eta^{\mu\nu}$. The adjoint action of $SL(2;\mathbb{C})$ on X is given by

$$X \longrightarrow \tilde{X} = NXN^\dagger \quad N \in SL(2;\mathbb{C}) \tag{2.21}$$

and it is clear that this action preserves the hermiticity of X since $(NXN^\dagger)^\dagger = NXN^\dagger$. Furthermore the determinants of $\tilde{X}$ and X are equal because $\det(N) = 1$ which means adjoint action of $SL(2;\mathbb{C})$ leaves the inner product of vectors unchanged:

$$\det(\tilde{X}) = \det(N)\det(X)\det(N^\dagger) = -x_\mu x^\mu \tag{2.22}$$

The lesson from (2.20) is that there is a transformed 4-vector $\tilde{x}^\mu$ which corresponds to the $SL(2,\mathbb{C})$ transformed matrix $\tilde{X}$ [13] i.e.

$$2\tilde{x}^\nu = \mathrm{Tr}(\tilde{X}\bar{\sigma}^\nu) \tag{2.23}$$

The connection between $SL(2,\mathbb{C})$ and $SO^+(1,3;\mathbb{R})$ is established using (2.21) and (2.19) as follows

$$\begin{aligned}
2\tilde{x}^\nu &= \mathrm{Tr}(\tilde{X}\bar{\sigma}^\nu) && (2.24)\\
&= \mathrm{Tr}(NXN^\dagger\bar{\sigma}^\nu) && (2.25)\\
&= \mathrm{Tr}(Nx_\mu\sigma^\mu N^\dagger\bar{\sigma}^\nu) && (2.26)\\
&= \mathrm{Tr}(\bar{\sigma}^\nu N\sigma_\mu N^\dagger)x^\mu && (2.27)\\
\tilde{x}^\nu &= \frac{1}{2}\mathrm{Tr}(\bar{\sigma}^\nu N\sigma_\mu N^\dagger)x^\mu && (2.28)\\
&= \Lambda^\nu{}_\mu x^\mu && (2.29)
\end{aligned}$$

[2] The '$-$' is due to the choice of metric $(-,+,+,+)$ and disappears for the mostly minus choice.

In (2.29), the index structure allows us to make the identification

$$\Lambda^{\nu}_{\ \mu} \equiv \frac{1}{2}[\mathrm{Tr}(\bar{\sigma}^{\nu} N \sigma_{\mu} N^{\dagger})] \tag{2.30}$$

and as advertised, for a Lorentz Λ there is a corresponding complex simple linear matrix N. It must be noted that because both N and $-N$ give rise to the same Λ, so the map is 2:1. This isomorphism is further reinforced by the fact that the Lie algebra $sl(2,\mathbb{C})$ is known to be isomorphic to $su(2)_{\mathbb{C}} \oplus su(2)_{\mathbb{C}}$ ([13],[16]) hence the spinor representation of (proper orthochronous) Lorentz transformations has a chiral and antichiral part. The transformation of fields in the fundamental and anti-fundamental spinor representations are respectively given by

$$\psi_{\alpha} \to N_{\alpha}^{\ \beta}\psi_{\beta} \quad \text{and} \quad \bar{\chi}_{\dot{\alpha}} \to (N^{\dagger})_{\dot{\alpha}}^{\ \dot{\beta}}\bar{\chi}_{\dot{\beta}} \text{ where } \alpha,\beta,\dot{\alpha},\dot{\beta} = \{1,2\} \tag{2.31}$$

where ψ_{α} is left-handed Weyl spinor and $\bar{\chi}_{\dot{\alpha}}$ a right-handed Weyl spinor. The indices of the spinors are raised/lowered using the $SU(2)$ invariant tensor

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\epsilon^{\alpha\beta} = -\epsilon^{\dot{\alpha}\dot{\beta}} \tag{2.32}$$

The value of this discussion will soon be realized when we come to supersymmetry where our representations will contain both spinor and vectors. In anticipation we define the mixed tensors $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ in following way

$$(\sigma^{\mu\nu})_{\ \beta}^{\alpha} = \frac{i}{4}\left(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu}\right)_{\ \beta}^{\alpha} \tag{2.33}$$

$$(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\ \dot{\beta}} = \frac{i}{4}\left(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu}\right)_{\dot{\alpha}}^{\ \dot{\beta}} \tag{2.34}$$

Poincaré symmetry

A generalization of Lorentz symmetry is achievable by extending (2.1) to include translations

$$x^{\mu} \longrightarrow x'^{\mu} = \Lambda^{\mu}_{\ \nu}x^{\nu} + a^{\mu} \tag{2.35}$$

where a is a constant vector and (2.35) is a Poincaré transformation. The effect of this extension is that there is a new generator, P_{μ}, which performs infinitesimal translations and thus in addition to (2.11) there are also the relations

$$[P_{\mu}, P_{\nu}] = 0 \quad , \quad [P_{\mu}, M_{\sigma\rho}] = i\left(g_{\mu\sigma}P_{\rho} - g_{\mu\rho}P_{\sigma}\right) \tag{2.36}$$

The group generated by P_{μ} and $M_{\mu\nu}$ is called *Poincaré* since it arises from (2.35) and a field theory is Poincare invariant if it remains unchanged under Poincaré transformations.

For the Poincaré algebra one can define two *Casimirs*; Casimirs are operators that commute with every element of the algebra. The first Casimir of the Poincaré algebra is defined as $C_1 := P_\mu P^\mu$ and its commutation with P_μ and $M_{\mu\nu}$ is simple to confirm using (2.36). The eigenvalues of Casimirs provide a means to label representations of a group. The eigenvalue, t, of C_1 on a state $|k^\mu\rangle$ with momentum, k^μ, is

$$t = \begin{cases} m^2 & \text{for } |k^\mu\rangle\,, \quad \text{a massive state} \\ 0 & \text{for } |k^\mu\rangle\,, \quad \text{a massless state} \end{cases} \tag{2.37}$$

The second Casimir is given by $C_2 = W_\mu W^\mu$ where W_μ is the *Pauli-Lubanski spin vector* defined as

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma} \quad , \quad \epsilon_{0123} = 1 \tag{2.38}$$

By expanding C_2 one arrives at $W^2 = -m^2\mathbf{J}^2$, with m^2 the mass eigenvalue and $\mathbf{J}^2 = j(j+1)$ is the spin eigenvalue, hence C_2 can be used to label massive representations by their spin j. The case of massless representations require a label since $W^2 = 0 = P^2$ but from its definition (2.38) it is evident that the spin vector W_μ is parallel to the P_ν [17], that is

$$W_\mu = \lambda P_\mu \tag{2.39}$$

where λ is known as *helicity* defined as

$$\lambda = \frac{\mathbf{P}\cdot\mathbf{J}}{|\mathbf{P}|} \tag{2.40}$$

Helicity is a projection of spin $\mathbf{J}$ in the direction of motion $\mathbf{P}$, thus a particle of spin s moving in the z-direction has helicity $\lambda = s_z$. In general then multiplets of the Poincaré group will be labelled by mass, spin and helicity, that is in a Poincaré multiplet the particles have the same mass and same spin. Next we discuss a symmetry even more general: the conformal symmetry.

2.2 Conformal Symmetry

A theory is said to be conformal if it is invariant under conformal transformations. Following the exposition of [18], [19] and [20], we devote this section to making the definition of conformal transformations exact. At heart, a conformal transformation is a change of coordinates whose effect amounts to a positive re-scaling by an overall factor of the metric of the manifold. Let $\tilde{M} = \mathbb{R}^{p+q}$ be a manifold with a flat Minkowskian metric $\eta_{\mu\nu}$ of signature (p,q) so that $p+q=d$; apart from the generalized dimension and metric signature, $\tilde{M}$ is the same as M defined in Section 2.1. A transformation

$x^\mu \to x'^\mu$ is conformal if it gives rise to a metric transformation of the form

$$\eta_{\mu\nu}(x) \to \eta'_{\mu\nu}(x') = \Omega(x)\eta_{\mu\nu}(x) \quad \text{with} \quad \Omega(x) > 0 \tag{2.41}$$

Using continuity and differentiability of M, the condition for a transformation to be conformal can be written as

$$\eta'_{\mu\nu}(x')\frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x'^\nu}{\partial x^\rho} = \Omega(x)\eta_{\sigma\rho}(x) \tag{2.42}$$

Expressing the relation between x and x' to first order in a small parameter $\epsilon(x) << 1$ gives

$$x'^\mu = x^\mu + \epsilon^\mu(x) + \mathcal{O}(\epsilon^2) + \dots \tag{2.43}$$

This allows for an infinitesimal formulation of the constraint (2.42) and makes it easy to isolate the generators of the conformal group to which the transformations belong. Keeping terms that are linear in the parameter ϵ means (2.42) becomes

$$\Omega(x)\eta_{\sigma\rho}(x) = \eta'_{\mu\nu}(x')\frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x'^\nu}{\partial x^\rho} \tag{2.44}$$

$$= \eta'_{\mu\nu}\left[\delta^\mu_\sigma + \partial_\sigma\epsilon^\mu + \dots\right]\left[\delta^\nu_\rho + \partial_\rho\epsilon^\nu + \dots\right] \tag{2.45}$$

$$= \eta'_{\mu\nu}\left[\delta^\mu_\sigma\ \delta^\nu_\rho + \delta^\nu_\rho\partial_\sigma\epsilon^\mu + \delta^\mu_\sigma\partial_\rho\epsilon^\nu + \dots\right] \tag{2.46}$$

$$= \eta'_{\sigma\rho} + \eta'_{\mu\rho}\partial_\sigma\epsilon^\mu + \eta'_{\sigma\nu}\partial_\rho\epsilon^\nu + \dots \tag{2.47}$$

$$= \eta'_{\sigma\rho} + \Big(\partial_\sigma\epsilon_\rho + \partial_\rho\epsilon_\sigma\Big) + \dots \tag{2.48}$$

The terms in the parentheses of the last equality in (2.44) must be proportional to $\eta'_{\mu\nu}$ in order for (2.43) to satisfy (2.42). This means

$$\Big[\partial_\sigma\epsilon_\rho + \partial_\rho\epsilon_\sigma\Big] = Q(x)\eta'_{\sigma\rho}(x) \tag{2.49}$$

and by tracing this equation we learn that $Q(x)$ is given by

$$\eta'^{\sigma\rho}\Big[\partial_\sigma\epsilon_\rho + \partial_\rho\epsilon_\sigma\Big] = Q(x)\eta'^{\sigma\rho}\eta'_{\sigma\rho} = Q(x)d \tag{2.50}$$

$$Q(x) = \frac{2}{d}\left[\partial\cdot\epsilon\right] \tag{2.51}$$

where d is the dimension of the manifold. The conformal requirement can now be restated in terms of the function ϵ^μ, that is, (2.43) will satisfy (2.48) if the differential equation

$$\Big[\partial_\sigma\epsilon_\rho + \partial_\rho\epsilon_\sigma\Big] = \frac{2}{d}\left[\partial\cdot\epsilon\right]\eta'_{\sigma\rho} \tag{2.52}$$

holds true for ϵ^μ. Comparing this to (2.42), we find that the scale factor to first order in ϵ is given by

$$\Omega(x) = 1 + \frac{2}{d}\partial \cdot \epsilon \tag{2.53}$$

In order to determine the generators of the conformal group, the dimension d of the manifold must be specified since the infinitesimal conformal condition (2.52) depends on it. Although at this point there is no reason to suspect this, it turns out that CFTs in $d = 2$ are very special in that their conformal group is generated by the infinite dimensional algebra known as the *Virasoro algebra*. This fact has been extremely useful in string theory because the worldsheet string dynamics are governed by a 2-dimensional CFT. Two dimensional CFTs have found many uses in the gauge-gravity duality where they have been shown to be holographically dual to theories of gravity in an AdS_3 background [21], [22].

The scope of this work is focused on CFTs in dimensions higher than 2 so then the rest of the discussion on CFTs is conducted with the assumption that $d > 2$. Next we apply two partial derivatives, one contravariant and one covariant, on (2.52) and repackage to obtain

$$\partial_\rho\partial_\sigma(\partial \cdot \epsilon) + \Box\partial_\rho\epsilon_\sigma = \frac{2}{d}\partial_\rho\partial_\sigma(\partial \cdot \epsilon) \tag{2.54}$$

and switching the indices $\rho \leftrightarrow \sigma$ produces an equally valid equation

$$\partial_\sigma\partial_\rho(\partial \cdot \epsilon) + \Box\partial_\sigma\epsilon_\rho = \frac{2}{d}\partial_\sigma\partial_\rho(\partial \cdot \epsilon) \tag{2.55}$$

The sum of (2.54) and (2.55) with the use of (2.52) means ϵ_ρ must satisfy

$$2\partial_\rho\partial_\sigma(\partial \cdot \epsilon) + \Box\left(\partial_\rho\epsilon_\sigma + \partial_\sigma\epsilon_\rho\right) = \frac{4}{d}\partial_\rho\partial_\sigma(\partial \cdot \epsilon) \tag{2.56}$$

$$\left[(d-2)\partial_\rho\partial_\sigma + \eta'_{\rho\sigma}\Box\right](\partial \cdot \epsilon) = 0 \tag{2.57}$$

$$\left[(d-1)\Box\right](\partial \cdot \epsilon) = 0 \tag{2.58}$$

From the last equality we deduce that ϵ is at most quadratic in x^μ. Thus ϵ is of the form

$$\epsilon_\rho = a_\rho + b_{\rho\sigma}x^\sigma + c_{\rho\sigma\tau}x^\sigma x^\tau \tag{2.59}$$

the first term of which corresponds to the familiar translations and these are generated by the momentum operator $P_\rho = -i\partial_\rho$. By substituting the second term into (2.52) one uncovers that the symmetric part of $b_{\rho\sigma}$ is proportional to the metric with a fixed constant of proportionality.

$$b_{\rho\sigma} + b_{\sigma\rho} = \frac{2}{d}(b^\lambda{}_\lambda)\eta_{\rho\sigma} := \alpha\ \eta_{\rho\sigma} \tag{2.60}$$

Thus the symmetric part of $b_{\rho\sigma}$ re-scales the metric and thus corresponds to dilations which are generated by $D = -ix^\rho\partial_\rho$. Hence we can write

$$b_{\rho\sigma} = \alpha\eta_{\rho\sigma} + \omega_{\sigma\rho} \tag{2.61}$$

where $\omega_{\sigma\rho}$ is the antisymmetric part of $b_{\rho\sigma}$. As before tensor $\omega_{\sigma\rho}$ corresponds to rotations whose generators are $M_{\mu\nu}$ as in the the Lorentz symmetry case. For the quadratic term a similar analysis can be done in order to determine the $c_{\rho\sigma\tau}$. This terms corresponds to novel transformations known as *Special Conformal Transformations* [SCT] and these are generated by $K_\rho = -i\left[(2x_\rho x^\sigma\partial_\sigma) + (x\cdot x)\partial_\rho\right]$[3]. The finite transformations are summarized in the table below:

Rotations	$x^\mu \longrightarrow M^\mu_\nu x^\nu$
Translations	$x^\mu \longrightarrow x^\mu + a^\mu$
Dilation	$x^\mu \longrightarrow \alpha x^\mu$
SCT	$x^\mu \longrightarrow \frac{x^\mu - b^\mu x^2}{1-2b^\nu x_\nu + b^2x^2}$

Table 2.1: Finite conformal group transformations

The aforementioned generators define the conformal algebra whose commutation relations are

$$\begin{aligned}
[D, P_\rho] &= iP_\rho\ , & [D, K_\rho] &= -iK_\rho \\
[K_\rho, P_\sigma] &= 2i\left(\eta_{\rho\sigma}D - M_{\rho\sigma}\right)\ , & [K_\rho, M_{\sigma\tau}] &= i\left(\eta_{\rho\sigma}K_\tau - \eta_{\rho\tau}K_\sigma\right) \\
[P_\rho, M_{\sigma\tau}] &= i\left(\eta_{\rho\sigma}P_\tau - \eta_{\rho\tau}P_\sigma\right)\ , & [M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right)
\end{aligned}$$

They generate the group $SO(p,q)$ and for our case this group will reduce to $SO(2,d)$ because we shall work in flat pseudo-Minkowski spacetime. The $SO(2,d)$ structure is more apparent when the generators are repackaged as follows[18],[19]:

$$\begin{aligned}
L_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu)\ , & L_{-1,0} &= D \\
L_{\mu,\nu} &= M_{\mu\nu}\ , & L_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu)
\end{aligned} \tag{2.62}$$

[3]The SCT generator, K_ρ, should not be confused with the generator for boosts, K_i

The new generators are antisymmetric $L_{ab} = -L_{ba}$ and their commutation relations assume the form [19]:

$$[L_{ab}, L_{cd}] = i(\eta_{ad}L_{bc} + \eta_{bc}L_{ad} - \eta_{ac}L_{bd} - \eta_{bd}L_{ac})$$
$$\text{with } a, b = \{-1, 0, 1, 2, \ldots, d\} \tag{2.63}$$

Here η_{ab} is a diagonal metric diag$(-1, 1, \ldots, 1, -1)$. The generators (2.62) with commutations (2.63) describe the isometries of a $(d+1)$-dimensional AdS space embedded onto a flat $(2, d)$-dimensional pseudo-Minkowski space. We shall describe this space later in section 3.1.1.1. One of the major consequences of conformal symmetry is that it demands that a theory remain unchanged even though the metric is re-scaled, hence a conformally symmetric theory will have the same dynamics at short distances as it does at long distances,i.e. no running of coupling constants. Based on everyday experience, one might rule out the possibility of finding such a symmetry in nature. However CFTs play an important role in efforts to describe and understand critical phenomena (i.e. magnetization, phase transition, etc.) of systems [23], [24]. This is mostly related to the fact that at critical points, systems tend to be insensitive to scale and correlation lengths approach infinity. In condensed matter physics, there have been experimental studies concerned with validating the theoretical results [25].

There is another reason why this symmetry is very attractive to physicists: computability. The presence of conformal symmetry in a QFT allows one the ability to compute 2-point and 3-point functions, up to a constant, purely from symmetry, albeit for fields of the *quasi-primary* type. Quasi-primary fields are defined by their transformation properties which are that they transform as

$$\phi_i(x) \longrightarrow \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi_i(x) \tag{2.64}$$

when $x^\mu \longrightarrow x'^\mu$ is a conformal i.e (2.41) is satisfied. Δ is known as the scaling dimension of the quasi-primary field ϕ_i.

Imposing conformal invariance of the action and integration measure of the theory under the transformation $x^\mu \longrightarrow x'^\mu$ has far-reaching consequences. For instance, a 2-point function of spinless fields ϕ_i with scaling dimension Δ_i will be of the form

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\frac{\Delta_2}{d}} \langle\phi_1(x_1')\phi_2(x_2')\rangle \tag{2.65}$$

and invariance under re-scaling $x'^{\mu} = \lambda x^{\mu}$ would imply the Jacobians are λ implying

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2\lambda(x_2)\rangle \tag{2.66}$$

However the requirement of translation and rotation invariance means the 2-point function depends only on separation of spacetime points which in general will be a function $f(|x_1 - x_2|)$ which because of (2.66) must transform as $f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)$. We conclude that in general the 2-point function will be of the form:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.67}$$

where C_{12} is a constant. Similar symmetry arguments are also possible in the case of 3-point functions [19]. Conformal symmetry has found use in condensed matter physics for the computation of critical data and in describing critical phenomena in order to understand the experimental data of condensed matter systems [26], [27]. From experimental data suggests that at critical points a large variety systems tend to behave the same, that is they belong to the same universality class. So their critical data does not depend on the details of the system. CFTs are a suitable framework to recover the critical data and describe critical phenomena.

2.3 Supersymmetry

Thus far we have reviewed external symmetries that is, those that pertain to spacetime transformations and their generators are either scalars, 4-vectors or 2-tensors. In this section we shall consider a symmetry which combines both internal[4] and external symmetries. It is a curious fact that in nature particles come with either integer or half-integer spin. Naturally one may wonder if there is a way to understand this dichotomy or to do even better and unify the divide between particle types. The latter is more in line with the pursuit in physics of obtaining a unified description of nature. Such a unification would necessitate the combination of groups that describe the internal features with those that describe external ones. Coleman and Mandula based on a few assumptions about the S-matrix, were able to argue that such extensions of the Poincaré algebra could not be done without trivializing the dynamics of the theory. The Coleman-Mandula theorem assumes that, for a a non-trivial interacting quantum field theory whose S-matrix has symmetry group G, the following hold [28]:

- the group G has a subgroup which is locally isomorphic to the Poincaré group,
- trivial scattering is forbidden, thus scattering angles are not limited to 0° and 180°,

[4] Internal symmetries are symmetries of transformations in internal space e.g. charge conjugation

- all particles transform as positive-energy representations of the Poincaré group,
- elastic scattering amplitudes are assumed to be analytic functions of the Mandelstam variables in the neighbourhood of a physical region,
- for a given mass M of a particle type, the set $T = \{m|m < M\}$ of masses m of the same particle type is always finite.

Their argument has come to be known as a no-go theorem [29]. Under these assumptions or constraint on a QFT, the algebra that generates the symmetries of the S-matrix can at best contain P_μ, $M_{\mu\nu}$ and Lorentz scalars B_l which obey the relations

$$[P_\mu, B_l] = 0 = [M_{\mu\nu}, B_l] \quad \text{and} \quad [B_l, B_m] = iC_{lmn}B_n \tag{2.68}$$

where C_{lmn} are structure constants of the Lie algebra that generates the compact group of internal symmetries. So the symmetry group G of the S-matrix would have to be a direct product of the Poincaré group with the group of internal symmetry, that is, external and internal symmetries do not affect one another. Note that the generators B_l are Lorentz scalars. By including generators with a spinor label, the authors of [30] by-passed the Coleman-Mandula theorem and found a way of extending the Poincaré algebra to include generators of internal and external symmetry algebras. This extension comes at a cost because the introduction of spinor generators would require $\mathbb{Z}_2$ grading, thus normal Lie algebras are not sufficient. The $\mathbb{Z}_2$ grading means the algebra structure must include both commutators and anticommutators and for generators T_i we have

$$T_iT_j - (-1)^{x_ix_j}T_jT_i = if^k_{ij}T_k \ , \quad \text{here} \quad x_i = \begin{cases} 0 & \text{if } T_i \text{ is a boson} \\ 1 & \text{if } T_i \text{ is a fermion} \end{cases} \tag{2.69}$$

These graded algebras are also known as *superalgebras* and their spinor generators are called supercharges Q_α. The supercharges are generators of supersymmetry since their action on bosons gives fermions and vice versa, effectively abolishing the partition between bosons and fermions. To be sure, supersymmetry is a conjectured symmetry and at the time of writing had not yet been observed in nature.

2.3.1 Super-Poincaré algebra

A symmetry algebra is generalized to a superalgebra by adding supercharges Q^A_α which are fermionic, hence anticommute. After adding supercharges, the extended Poincaré algebra becomes

$$\begin{aligned} [Q^A_{\ \alpha}, P_\mu] &= 0 & [\bar{Q}^A_{\ \dot{\alpha}}, P_\mu] &= 0 \\ \{Q^A_{\ \alpha}, \bar{Q}_{B\dot{\beta}}\} &= 2\delta^A_{\ B}\,(\sigma^\mu)_{\alpha\dot{\beta}}\,P_\mu & [P_\mu, P_\nu] &= 0 \\ [Q^A_{\ \alpha}, M^{\mu\nu}] &= (\sigma^{\mu\nu})_\alpha^{\ \beta}\,Q^A_{\ \beta} & \{Q^A_{\ \alpha}, Q^B_{\ \beta}\} &= \epsilon_{\alpha\beta}Z^{AB} \end{aligned} \tag{2.70}$$

where σ^μ and $\sigma^{\mu\nu}$ are as defined in (2.17) and (2.33) respectively. $Z^{AB} = -Z^{BA}$ is the central charge of the algebra and here 'central' refers to the fact that Z^{AB} commutes with every generator of the algebra. The uppercase indices $A, B = \{1, 2, \ldots, \mathcal{N}\}$ count the number of supercharges present while $\alpha, \dot{\alpha}, \beta, \dot{\beta} = \{1, 2\}$ specify the elements of Q or $\bar{Q}$. The object ϵ is the $SU(2)$ invariant tensor in (2.32). By complex conjugation one can obtain the relations for $\bar{Q}$ with dotted indices. Suppose there is an internal symmetry generated by R_a's which satisfy the relations $[R_a, R_b] = if_{abc}R_c$. This internal symmetry will mix with supersymmetry so that the commutation relations of generators R_a with supercharges will take the form $[Q^A_\alpha, R_a] \propto Q^B_\alpha$. This is made abundantly clear by recalling that super-Lie algebra needs to be closed and the Jacobi identities for generators of different gradings, i.e. $[odd, even] \sim odd$ [31]. Now we can write

$$[Q^A_\alpha, R_a] = S_a{}^A{}_B Q^B_\alpha \ , \quad \text{for } S_a{}^A{}_B \text{ a constant} \tag{2.71}$$

The commutation relations imply that the R_a's rotate the supercharges into one another. Such a symmetry is called an *R-symmetry* and if the supercharges Q are completely unrelated i.e. the central charge is zero, $Z^{AB} = 0$, then the R-symmetry is $U(\mathcal{N})_R$ for a model with $\mathcal{N}$ supercharges. If the central charges are not zero then the R-symmetry is a subset of $U(\mathcal{N})$.

Another interesting observation is that in a supersymmetric theory there are as many bosonic degrees of freedom as there are fermionic ones. To see this we first define the fermion number operator $(-)^{N_F}$ as

$$(-)^{N_F}\,|b\rangle = +1\,|b\rangle \quad \text{and} \quad (-)^{N_F}\,|f\rangle = -1\,|f\rangle \tag{2.72}$$

with kets $|b\rangle$ and $|f\rangle$ representing a boson and a fermion respectively. By applying the operator $(-)^{N_F}$ on the anticommutator $\{Q^A_{\ \alpha}, \bar{Q}_{B\dot{\beta}}\} = 2\delta^A_{\ B}\,(\sigma^\mu)_{\alpha\dot{\beta}}\,P_\mu$ and tracing, one is led to conclude that there the number of bosons matches that of fermions

$$2\mathrm{Tr}\left((-)^{N_F}\left[\delta^A_B\left(\sigma^\mu\right)_{\alpha\dot\beta}P_\mu\right]\right) = \mathrm{Tr}\left[(-)^{N_F}\{Q^A_\alpha, \bar{Q}_{B\dot\beta}\}\right] \tag{2.73}$$

$$= \mathrm{Tr}\left[(-)^{N_F}(Q^A_\alpha\bar{Q}_{B\dot\beta} + \bar{Q}_{B\dot\beta}Q^A_\alpha)\right] \tag{2.74}$$

$$= \mathrm{Tr}\left[(-Q^A_\alpha(-)^{N_F}\bar{Q}_{B\dot\beta} + Q^A_\alpha(-)^{N_F}\bar{Q}_{B\dot\beta})\right] \tag{2.75}$$

$$= 0 \tag{2.76}$$

where we have used the cyclicity of the trace and the fact the the number operator anticommutes with the supercharge Q. The last equality must hold for all values of the indices so that for any state of eigenmomentum p_μ expanding the LHS gives

$$2\mathrm{Tr}\left((-)^{N_F}\left[\delta^A_B\left(\sigma^\mu\right)_{\alpha\dot\beta}P_\mu\right]\right) = 2\left(\sigma^\mu\right)_{\alpha\dot\beta}p_\mu\mathrm{Tr}\left[(-)^{N_F}\right] \tag{2.77}$$

$$= 0\ ,\ \forall\ \ \alpha, \dot\beta, A \text{ and } B \tag{2.78}$$

$$\implies\ \mathrm{Tr}\left[(-)^{N_F}\right] = 0 \tag{2.79}$$

From (2.79) we conclude that the total number of bosons minus that of fermions is zero.

2.3.2 SUSY and its features

We next use the non-trivial supersymmetric part of the algebra above and consider the multiplets of supersymmetry in 4d while maintaining $\mathcal{N}$ unspecified. As in the case of Poincaré symmetry where we used the eigenvalues of $C_1 = P_\mu P^\mu$ because it was a Casimir, so shall we do in case of the super-Poincaré algebra. This is because C_1 is still a Casimir. Thus multiplets will be labeled by their mass as before. The square of the Pauli-Lubanski vector does not commute with all the generators of the super-Poincaré algebra thus there is need of a new Casimir. Things are rather more involved because the Casimirs will be affected by the amount of supersymmetry present. In addition, the usefulness of a Casimir is related to whether the multiplet is massive or massless. Since $\mathcal{N}$ is kept unspecified we shall define a Casimir for the massless case because of its relevance to the present work.

2.3.2.1 Massless multiplets

The massless case has a Casimir defined as $C_2 = C_{\mu\nu}C^{\mu\nu}$ where

$$C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu, \quad \text{and} \quad B_\mu = W_\mu - \frac{1}{4}\delta^A_B\bar{Q}^A_{\dot\alpha}(\bar\sigma_\mu)^{\dot\alpha\beta}Q_{B\beta} \tag{2.80}$$

Consider a massless particle, $|p, \lambda\rangle$, in a frame where its 4-momentum is $p^\mu = (E, 0, 0, E)$ then anticommutator $\{Q^A_\alpha, \bar{Q}_{B\dot{\beta}}\} = 2\delta^A_B\,(\sigma^\mu)_{\alpha\dot{\beta}}\,P_\mu$ means

$$\{Q^A_\alpha, \bar{Q}_{B\dot{\beta}}\}\,|p,\lambda\rangle = 2\delta^A_B\,(\sigma^\mu)_{\alpha\dot{\beta}}\,P_\mu\,|p,\lambda\rangle \tag{2.81}$$

$$= 2\delta^A_B\left(\sigma^0 p_0 + \sigma^3 p_3\right)_{\alpha\dot{\beta}}|p,\lambda\rangle \tag{2.82}$$

$$= 4E\delta^A_B \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} |p,\lambda\rangle \tag{2.83}$$

$$\therefore\ \{Q^A_\alpha, \bar{Q}_{B\dot{\beta}}\} = 4E\delta^A_B \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \Longrightarrow Q^A_2 = 0 \tag{2.84}$$

the anticommutation relation is now reduce to

$$\{Q^A_1, \bar{Q}_{B\dot{1}}\} = 4E\delta^A_B \tag{2.85}$$

The other non-trivial supersymmetry algebra relation involving the central charges becomes trivial, $Z^{AB} = 0$, thanks to the fact that $Q^A_2 = 0$. This effect is transparent from the relation $\{Q^A_1, Q^B_2\} = 0 = \epsilon_{12} Z^{AB} = -Z^{AB} = Z^{BA}$. Thanks to the re-definitions:

$$a^A := \frac{Q^A_1}{2\sqrt{E}} \quad , \quad a^{A\dagger} := \frac{Q^{A\dagger}_1}{2\sqrt{E}} \tag{2.86}$$

the anticommutator (2.85) takes a form that is useful in building multiplets

$$\{a^A, a^\dagger_B\} = \delta^A_B\ ,\ \{a^A, a^B\} = 0 = \{(a^A)^\dagger, (a^B)^\dagger\} \tag{2.87}$$

In this form the a's and $a^\dagger$'s are reminiscent of ladder operators and in fact they are. To show this fact consider a specific case of the SUSY algebra relation $[Q^A_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^{\ \beta}\,Q^A_\beta$, namely: $[Q^A_\alpha, M_{12}] = [Q^A_\alpha, J_3]$. Computing this commutation relation gives

$$[Q^A_\alpha, J_3] = (\sigma^{12})_\alpha^{\ \beta} Q^A_\beta = \begin{cases} \frac{1}{2} Q^A_1 & \text{for } \ \alpha = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.88}$$

a direct consequence of $Q^A_2 = 0$. If $|p, \lambda\rangle$ is a state vector representing a particle with momentum p and helicity λ then the action of a and $a^\dagger$ on $|p, \lambda\rangle$ is given by

$$\begin{aligned} J_3 a^A\,|p,\lambda\rangle &= (a^A J_3 - [a^A, J_3])\,|p,\lambda\rangle = (a^A J_3 - \frac{1}{2}a^A)\,|p,\lambda\rangle = (\lambda - \frac{1}{2})a^A\,|p,\lambda\rangle \\ J_3 a^{B\dagger}\,|p,\lambda\rangle &= (a^B J_3 + [a^{B\dagger}, J_3])\,|p,\lambda\rangle = (a^{B\dagger} J_3 + \frac{1}{2}a^{B\dagger})\,|p,\lambda\rangle = (\lambda + \frac{1}{2})a^{B\dagger}\,|p,\lambda\rangle \end{aligned} \tag{2.89}$$

The state $a^{B\dagger}|p,\lambda\rangle$ has helicity larger than $|p,\lambda\rangle$ by a half, thus $a^\dagger$ raises the helicity by a half while a lowers it the same amount. We can now build multiplets.

Let $|\Omega\rangle$ be a state with momentum p and helicity λ such that $a^A|\Omega\rangle = 0$ then

- $\mathcal{N} = 1$ chiral multiplet: $\lambda = 0$

State	Helicity	Field
$\lvert\Omega\rangle$	$\lambda = 0$	1 complex scalar
$a^\dagger\lvert\Omega\rangle$	$\lambda = \pm\frac{1}{2}$	1 Weyl spinor,

- $\mathcal{N} = 1$ vector multiplet: $\lambda = \frac{1}{2}$

State	Helicity	Field
$\lvert\Omega\rangle$	$\lambda = \pm\frac{1}{2}$	1 Weyl spinor
$a^\dagger\lvert\Omega\rangle$	$\lambda = \pm 1$	1 Gauge field

For $\mathcal{N} = 1$ we only have one raising operator, $a^\dagger$, which anticommutes thus it cannot be used to raise the helicity twice and the tables above exhaust the $\mathcal{N} = 1$ SUSY states. The states with negative helicities are obtained via CPT[5] conjugation. Invariance under CPT conjugation is a physical requirement we impose on our theory. The case of interest to us is the vector multiplet for $\mathcal{N} = 4$ with a restriction to helicities no larger than one. Restricting helicity to one means our theory will not take gravity into account. In this setting the different states are tabulated below

- $\mathcal{N} = 4$ vector multiplet: $\lambda = 1$

State	Helicity	Field
$\lvert\Omega\rangle$, $a^{1\dagger}a^{2\dagger}a^{3\dagger}a^{4\dagger}\lvert\Omega\rangle$	$\lambda = \pm 1$	1 Gauge field
$a^{1\dagger}a^{2\dagger}\lvert\Omega\rangle$, $a^{3\dagger}a^{4\dagger}\lvert\Omega\rangle$, etc.	$\lambda = 0$	3 complex scalars
$a^{1\dagger}\lvert\Omega\rangle$, $a^{1\dagger}a^{2\dagger}a^{3\dagger}\lvert\Omega\rangle$, etc.	$\lambda = \pm\frac{1}{2}$	4 Weyl spinors

Note that the $\mathcal{N} = 4$ vector multiplet has the same field content as an $\mathcal{N} = 1$ vector multiplet combined with 3 $\mathcal{N} = 1$ chiral multiplets, that is:

$$\mathcal{N} = 4 \text{ vector multiplet} = 1 \times (\mathcal{N} = 1 \text{ vector multiplet}) \oplus\ 3 \times (\mathcal{N} = 1 \text{ chiral multiplets}) \tag{2.90}$$

This reducibility will be useful later in this work.

[5] Charge Parity and Time.

2.3.3 Superspace and superfields

2.3.3.1 $\mathcal{N}=1$ Superspace

We have seen that the notation of 4-vectors in field theories in $(3+1)$-dimensions makes Lorentz invariance readily apparent. Here we shall briefly present the notions of *superspace* and *superfields* as a formalism which helps manifest supersymmetry in field theory. A superfield is a function G of variables x^μ,θ_α and $\bar{\theta}^{\dot{\alpha}}$ where x^μ are the familiar spacetime coordinates while θ_α and $\bar{\theta}^{\dot{\alpha}}$ are Grassmann coordinates which are hermitian conjugate to one another. x^μ is said to be Grassmann even and $\theta, \bar{\theta}$ Grassmann odd because they commute according to the relations:

$$\begin{aligned} \theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}(\theta^\gamma\theta_\gamma) \equiv -\frac{1}{2}\epsilon^{\alpha\beta}(\theta\theta) \\ \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \ \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}(\bar{\theta}_{\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}) \equiv \ \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}(\bar{\theta}\bar{\theta}) \\ \text{and}\quad \theta_\alpha\bar{\theta}_{\dot{\alpha}} &= \ \frac{1}{2}\theta(\sigma_\mu)\bar{\theta}(\sigma^\mu)_{\alpha\dot{\alpha}} \end{aligned} \tag{2.91}$$

The introduction of superspace may seem arbitrary at first but a look at the commutation relation

$$\{Q_{\ \alpha}, \bar{Q}_{\dot{\beta}}\} = 2\,(\sigma^\mu)_{\alpha\dot{\beta}}\,P_\mu \tag{2.92}$$

gives insight and calls for a geometric view. From the anticommutator we note that the effect of the supercharges (LHS) is equivalent to spacetime translation (RHS) P_μ. First we convert the anticommutator to a commutator with the help of Grassmann coordinates to get

$$[\xi^\alpha Q_\alpha, \bar{\xi}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}] = 2(\xi\sigma^\mu\bar{\xi})P_\mu \tag{2.93}$$

A super-Poincaré group element that performs finite superspace translations would thus be given by

$$T(x,\theta,\bar{\theta}) = \exp\left[i(x^\mu P_\mu + \theta Q + \bar{\theta}\bar{Q})\right] \tag{2.94}$$

Here it worth point out that superspace can also be viewed as the super-Poincaré group *modulo* the homogeneous generators $M_{\mu\nu}$ and a typical element of a such a group will definitely take the form in (2.94). In order to isolate the effect of the supercharges we can compute the product $\tilde{T} := T(0,\xi,\bar{\xi})\cdot T(x,\theta,\bar{\theta})$ by expanding, simplifying using the SUSY algebra and re-summing[6] the exponentials to obtain another group element which depends on the combination of the coordinates $\tilde{T}(x^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}, \theta+\xi, \bar{\theta}+\bar{\xi})$. The action of supercharges has induced a shift in both the bosonic and fermionic coordinates,

[6] using the Baker-Campbell-Hausdorff formula

thus allowing us to represent the supercharges in this way

$$Q_\alpha := i\left(-\frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right) \tag{2.95}$$

$$\bar{Q}^{\dot{\alpha}} := i\left(\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu\right) \tag{2.96}$$

The fermionic directions are represented by the Grassmann coordinates, θ and $\bar{\theta}$, are 2-component Weyl spinors whose differential and integral properties are

$$\begin{aligned} &\frac{\partial}{\partial\theta^\alpha}(\theta^\beta) = \delta^\beta_{\ \alpha}\ , \quad \frac{\partial}{\partial\theta^\alpha}(\bar{\theta}^{\dot{\beta}}) = 0 \ \text{ and h.c} \\ &\int \mathrm{d}\theta\ \theta = 1\ , \qquad \int \mathrm{d}\theta\ 1 = 0 \ \text{ and h.c} \end{aligned} \tag{2.97}$$

Superspace is effectively an extension of spacetime to include fermionic directions. The advantage of the superspace formalism is that a power series expansion of superfields $\Phi(x,\theta,\bar{\theta})$ eventually truncates because of the Grassmann coordinates.

2.3.4 Superfields

A scalar superfield, with spinor indices suppressed, will generally be of the form

$$\begin{aligned} S(x,\theta,\bar{\theta}) = &\ \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ &+ \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\eta(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \end{aligned} \tag{2.98}$$

where $\{\phi(x), m(x), n(x), d(x)\}$, are complex scalar functions of spacetime while contained in the set $\{\psi(x), \chi(x), \lambda(x), \eta(x)\}$ are 2-component spinors and $v_\mu(x)$ is a complex vector. There are 4×2 components from the scalars, 8 from the vector, 4×2 from the left-handed Weyl spinors and another 4×2 from the right-handed Weyl spinors. This all adds up to a total of 32 components = 16 bosonic + 16 fermionic. The form in (2.98) is most general and any other possible terms either vanish or can be shown to be related to the ones contained in (2.98). The superfield has more degrees of freedom than what we need to describe the supermultiplets presented above, thus we must impose appropriate restrictions on it. By appropriate we shall mean that the restrictions must first reduce the degrees of freedom to those permissible for SUSY multiplets and secondly the restrictions must be SUSY covariant. To this end we first note that while the Lorentz derivative ∂_μ is SUSY covariant, since $[P_\mu, Q_\alpha] = 0 = [P_\mu, \bar{Q}_{\dot{\alpha}}]$, the Grassmann derivative ∂_α is not. The effect of a superspace translation by parameter ϵ

$(x, \theta, \bar{\theta}) \longrightarrow (\tilde{x}, \tilde{\theta}, \tilde{\bar{\theta}}) = (x + i\theta\sigma\bar{\epsilon} - i\epsilon\sigma\bar{\theta}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon})$ on the Grassmann derivative is

$$\partial_\alpha = \frac{\partial\tilde{\theta}}{\partial\theta^\alpha}\frac{\partial}{\partial\tilde{\theta}} + \frac{\partial\tilde{x}}{\partial\theta^\alpha}\frac{\partial}{\partial\tilde{x}} \tag{2.99}$$

$$= \frac{\partial}{\partial\tilde{\theta}^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\frac{\partial}{\partial\tilde{x}^\mu} \tag{2.100}$$

By direct calculation one can show that the RHS of (2.100) anticommutes with Q_α and $\bar{Q}^{\dot{\alpha}}$. This observation is an inspiration for a definition of a SUSY covariant derivative, so then we define the super derivatives in superspace by:

$$D_\alpha := \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu} \tag{2.101}$$

$$\bar{D}_{\dot{\alpha}} := -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu} \tag{2.102}$$

Using these definitions one can show that the super derivatives satisfy the anti-commutation relations

$$\left.\begin{array}{l} \{D_\alpha, D_\beta\} = 0 = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} \\ \{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \end{array}\right\} \Longrightarrow D^3 = 0 = \bar{D}^3 \tag{2.103}$$

These super derivatives together with their properties will prove useful in the construction of SUSY invariant Lagrangians and reducing the extra component fields in order to match the SUSY multiplets. Now we shall consider two important types of superfields together with their defining constraints

2.3.4.1 Chiral superfields

A chiral superfield Φ is defined by the constraint

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{2.104}$$

The Φ that solves this constraint will have a chiral field as its highest spin component field and on-shell its component field content will match that of the chiral multiplet. In order to manifest the meaning of constraint on the superfield Φ we shift the coordinates as follows

$$(x^\mu, \theta, \bar{\theta}) \longrightarrow (y^\mu, \theta^{'}, \bar{\theta}^{'}) = (x^\mu + i\theta\sigma^\mu\bar{\theta}, \theta, \bar{\theta}) \tag{2.105}$$

and the effect of such a shift on the covariant derivatives is

$$\begin{aligned} D_\alpha &= \frac{\partial\theta'^\beta}{\partial\theta^\alpha}\frac{\partial}{\partial\theta'^\beta} + \frac{\partial y^\mu}{\partial\theta^\alpha}\frac{\partial}{\partial y^\mu} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}'^{\dot{\alpha}}\frac{\partial}{\partial y^\mu} \\ &= \frac{\partial}{\partial\theta'^\alpha} + 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}'^{\dot{\alpha}}\frac{\partial}{\partial y^\mu} \end{aligned} \tag{2.106}$$

and by the exact same treatment for the other derivative we arrive at

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \tag{2.107}$$

The chiral superfield constraint is now $\bar{D}_{\dot{\alpha}}\Phi(y^\mu,\theta,\bar{\theta}) = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\Phi(y^\mu,\theta,\bar{\theta})$. The power series expansion of a chiral superfield in the y-coordinate is

$$\Phi(y^\mu,\theta,\bar{\theta}) = \phi(y) + \sqrt{2}\theta\psi + \theta\theta F(y) \tag{2.108}$$

where ϕ and F are scalar component fields and ψ is a fermionic 2-component field. This expression, however, is in terms of the variable y^μ which is a combination of spacetime and Grassmann coordinates. Returning to the $(x^\mu,\theta,\bar{\theta})$ will produce extra terms so that general the chiral superfield solution of (2.104) is [17]

$$\Phi(x,\theta,\bar{\theta}) = \phi + \sqrt{2}\theta\psi + \theta\theta F + i\theta\sigma\bar{\theta}\partial_\mu\phi - \frac{i}{\sqrt{2}}\theta\theta\sigma^\mu\bar{\theta}\partial_\mu\psi + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu\phi \tag{2.109}$$

The presence of the scalar field $F(x)$ exceeds the number of allowed bosonic component fields in an $\mathcal{N}=1$ chiral multiplet on-shell. This extra degree of freedom can be removed by requiring that $F(x)$ not be dynamic. We can integrate $F(x)$ out using the equations of motion to express it in terms of ϕ and ψ so that on-shell we have the correct bosonic and fermionic degrees of freedom allowed in the chiral multiplet.

The two useful properties of chiral superfields Φ_i are that sums and products of chiral superfields are themselves also chiral superfields

$$\bar{D}_{\dot{\alpha}}(\Phi_i + \Phi_j) = \bar{D}_{\dot{\alpha}}\Phi_i + \bar{D}_{\dot{\alpha}}\Phi_j = 0 \tag{2.110}$$

since Φ_i's are chiral to begin with. For the case of a product of chiral superfields we have [32]

$$\bar{D}_{\dot{\alpha}}(\Phi_i\Phi_j) = \left(\bar{D}_{\dot{\alpha}}\Phi_i\right)\Phi_j + (-1)^{[\Phi_i]}\Phi_i\left(\bar{D}_{\dot{\alpha}}\Phi_j\right) = 0 \tag{2.111}$$

where $[\Phi_i]$ is the grading of the chiral superfield Φ_i. Based on these two facts it is clear that any holomorphic function, $\mathcal{W}$, of chiral superfields Φ will also be chiral:

$$\bar{D}_{\dot{\alpha}}\mathcal{W}(\Phi) = -\frac{\partial\mathcal{W}}{\partial\Phi}\frac{\partial\Phi}{\partial\theta^{\dot{\alpha}}} = \frac{\partial\mathcal{W}}{\partial\Phi}\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{2.112}$$

Holomorphic functions $\mathcal{W}(\Phi)$ are referred to as the superpotential and they introduce interactions in supersymmetric field theories, so do their antiholomorphic counterparts $\bar{\mathcal{W}}(\bar{\Phi})$. The defining constraint of an antichiral superfield $\bar{\Phi} \equiv \Phi^\dagger$ is $D_\alpha\Phi^\dagger = 0$. The sums and products of antichiral superfields are also antichiral superfields, thus any antiholomorphic function of antichiral superfields is a chiral superfield. This is not true a product of a chiral with antichiral superfields; i.e. $\Phi\bar{\Phi}$ is neither chiral nor antichiral. From the definition of the supercharges (2.95) the infinitesimal transformations $\Phi \longrightarrow \Phi + \delta\Phi = \Phi + i(\xi Q + \bar{\xi}\bar{Q})\Phi$ means the supersymmetric variation of the component fields is

$$\delta_\xi \phi = \sqrt{2}\xi\psi \tag{2.113}$$

$$\delta_\xi \psi = \sqrt{2}(\xi F - i\sigma^\mu \bar{\xi}\partial_\mu \phi) \tag{2.114}$$

$$\delta_\xi F = \sqrt{2}i\psi\sigma^\mu\bar{\xi}\partial_\mu \tag{2.115}$$

the field F transforms as a total derivative and can be used to construct Lagrangians that are invariant under supersymmetric transformations. In fact the terms with highest allowed degree in the Grassmann coordinates θ will transform as a total derivative. In the chiral field case this term has auxiliary field F as its coefficient and auxiliary field D for the vector superfield case. These auxiliary fields are eventually integrated out in order to obtain the correct on-shell degrees of freedom. Note that the terms

$$\int d\theta^2\ \mathcal{W}(\Phi) \quad \text{and} \quad \int d\bar{\theta}^2\ \bar{\mathcal{W}}(\bar{\Phi}) \tag{2.116}$$

are $\mathcal{N} = 1$ SUSY invariant where $\mathcal{W}(\Phi)$ and $\bar{\mathcal{W}}(\bar{\Phi})$ are chiral and anti-chiral respectively. The superfield $\mathcal{W}(\Phi)$ is known as a superpotential of the theory; it is (Grassmann) integrated over half of superspace since the highest non-vanishing degree in the coordinate θ is 2. If one is interested in constructing a renormalizable theory in 4 spacetime dimensions then each term of the Lagrangian density must at most have mass dimension of 4; the implication is that the most general allowed form of the superpotential as a polynomial of chiral superfields is given by

$$\mathcal{W}(\Phi_i) = f_i\Phi_i + \frac{g_{ij}}{2}\Phi_i\Phi_j + \frac{f_{ijk}}{3!}\Phi_i\Phi_j\Phi_k \tag{2.117}$$

The form of the superpotential follows from dimensional analysis arguments. From the fact that

$$\int \mathrm{d}\theta\ \theta = 1$$

and $[\theta] = -1/2$ it is clear that $[\mathrm{d}\theta] = 1/2$. Thus $[\mathrm{d}\theta^2] = 1$ and the contribution of $[\mathcal{W}(\Phi)] = 3$. From the chiral superfield term $\sqrt{2}\theta\psi$ one can deduce that $[\Phi] = 1$ because $[\theta] = -1/2$ and $[\psi] = 3/2$ which means the superpotential will be a holomorphic polynomial of Φ with degree 3 at most [13].

2.3.4.2 Vector superfields

Another constraint useful in removing the extra component fields from a general superfield V is the reality condition

$$V^\dagger = V \tag{2.118}$$

A superfield that satisfies this constraint is known as a vector superfield because its component field content corresponds to the vector multiplet. Imposing (2.118) on the superfield (2.98) means

$$\psi = \bar{\xi}^\dagger = \xi \quad , \quad \eta = \bar{\lambda}^\dagger = \lambda \tag{2.119}$$

$$m = n^* \quad , \quad d \text{ and } \phi \text{ are real functions} \tag{2.120}$$

and v_μ is a real vector field which hence lends its name to the full superfield that obeys the constraint (2.118). Vector superfields have become important as in addition to removing unnecessary component fields they naturally allow for a superspace construction of supersymmetric gauge field theories. Making the substitutions (2.119) on superfield (2.98) means a vector superfield V takes the general form [33]

$$\begin{aligned} V(x,\theta,\bar{\theta}) = \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\psi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}m^*(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ + \theta\theta\bar{\theta}\bar{\eta}(x) + \bar{\theta}\bar{\theta}\theta\eta(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \end{aligned} \tag{2.121}$$

It is easy to see that the sums of vector superfields are themselves also vector superfields. Moreover the sums of chiral with antichiral superfields are also vector superfields, as are the products: $\Phi + \bar{\Phi}$ and $\Phi\bar{\Phi}$ are vector superfields. And for renormalization purposes we shall be interested in vector superfields that are products of chiral and antichiral superfields i.e. $\Phi\bar{\Phi}$. When it comes to the construction of SUSY invariant Lagrangians we shall turn to the highest Grassmann order term $\theta^2\bar{\theta}^2$, known as the D-term. This term transforms into a spacetime derivative of d under SUSY transformation $\xi, \bar{\xi}$:

$$\delta_{\xi,\bar{\xi}} d(x) = \frac{i}{2}\left[\partial_\mu \eta\sigma^\mu\bar{\xi} - \partial_\mu\bar{\eta}\bar{\sigma}^\mu\xi\right] \tag{2.122}$$

As in the case of the chiral superfield, the vector superfield also has more degrees of freedom than allowed in an $\mathcal{N} = 1$ vector multiplet: 8 bosonic + 8 fermionic. We shall introduce the notion of invariance under gauge transformation for superfields in order to reduce the extra (redundant) degrees of freedom. At heart, gauge invariance amounts to appropriate replacements such that certain terms remain unchanged under a given transformation. In Quantum Electrodynamics, the derivative ∂_μ is replaced with D_μ which contains a gauge field A_μ with appropriate transformation rules so that, under the transformation $\psi \longrightarrow e^{i\alpha(x)}\psi$, the term $D_\mu\psi$ remains unchanged up to a phase, $e^{i\alpha(x)}$.

The strategy here is the same in that we define a supersymmetric gauge transformation of a vector superfield V to be

$$\begin{aligned} V(x,\theta,\bar{\theta}) \longrightarrow \tilde{V}(x,\theta,\bar{\theta}) &= V(x,\theta,\bar{\theta}) + \Phi(x,\theta,\bar{\theta}) + \bar{\Phi}(x,\theta,\bar{\theta}) \\ &= V(x,\theta,\bar{\theta}) + i\Lambda(x,\theta,\bar{\theta}) - i\bar{\Lambda}(x,\theta,\bar{\theta}) \end{aligned} \tag{2.123}$$

where $\Phi = i\Lambda$ is a chiral superfield. Based on the definition of gauge transformation we replace terms in the vector superfield in order to guarantee that (2.122) is not sacrificed in the process. The replacement of terms in (2.121) is as follows

$$\eta \longrightarrow \eta - \frac{i}{2}\sigma^\mu \partial_\mu \bar{\psi}\ , \qquad d \longrightarrow d - \frac{1}{4}\Box\phi \tag{2.124}$$

with the result that the vector superfield is not given by

$$\begin{aligned} V = \phi + \theta\psi + \bar{\theta}\bar{\psi} + \theta^2 m + \bar{\theta}^2 m^* + \theta\sigma^\mu\bar{\theta}v_\mu + \theta^2\bar{\theta}(\bar{\eta} + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\psi) \\ + \bar{\theta}^2\theta(\eta - \frac{i}{2}\sigma^\mu\partial_\mu\bar{\psi}) + \theta^2\bar{\theta}^2(d - \frac{1}{4}\Box\phi) \end{aligned} \tag{2.125}$$

Given that $\Phi = a + \sqrt{2}\theta\xi + \theta^2 F$, under gauge transformation (2.123), the component fields of vector superfield (2.125) transform as

$$\phi \to \phi + a + a^* \tag{2.126}$$

$$\psi \to \psi + \sqrt{2}\xi \tag{2.127}$$

$$m \to m + F \tag{2.128}$$

$$m^* \to m^* + F^* \tag{2.129}$$

$$d \to d \tag{2.130}$$

$$\eta \to \eta \tag{2.131}$$

As promised, d and η are invariant under gauge transformation and the F-term is still useful to construct supersymmetric gauge field Lagrangians. The vector field v_μ transforms to $v_\mu + i\partial_\mu[a - a^*]$. The introduction of gauge transformation provides more conditions to use in removing the 'redundant' fields i.e. the extra component fields of V can be removed by an appropriate choice of the component fields of Φ in (2.123), this choice is *gauge fixing*. A well-known gauge is the *Wess-Zumino* gauge where the component fields of Φ are chosen such that ([34],[13]) :

$$a + a^* = -\phi \tag{2.132}$$

$$F = -m \tag{2.133}$$

$$\xi = -\frac{1}{\sqrt{2}}\psi \tag{2.134}$$

The Wess-Zumino gauge simply removes fields ϕ, ψ and m so that the vector superfield now becomes

$$V_{WZ} = \theta^2\bar{\theta}\bar{\eta} + \bar{\theta}^2\theta\eta + \theta^2\bar{\theta}^2 d + \theta\sigma^\mu\bar{\theta}\left[v_\mu + i\partial_\mu(a - a^*)\right] \tag{2.135}$$

d is an auxiliary field which can be integrated out using the equations of motion. The convenience of the Wess-Zumino gauge is in the fact that

$$V_{WZ}^2 = \frac{1}{2}\theta^2\bar{\theta}v^2 \tag{2.136}$$

$$V_{WZ}^3 = 0 \tag{2.137}$$

so then the exponential of the vector superfield in the Wess-Zumino gauge is exactly

$$e^{V_{WZ}} = 1 + V_{WZ} + \frac{1}{2}V_{WZ}^2 \tag{2.138}$$

Such an exponential makes appearance when one extends $U(1)$ gauge transformation to supersymmetric field theories where the transformation of chiral superfields is given by

$$\Phi \longrightarrow e^{-ig\Lambda}\Phi \quad \text{and} \quad \bar{\Phi} \longrightarrow e^{ig\Lambda}\bar{\Phi} \tag{2.139}$$

Vector superfields like $\bar{\Phi}\Phi$ are invariant for as long as $g\Lambda$ is a constant, which is a case analogous to abelian gauge transformations in QFT [35]. For the non-abelian case $g\Lambda$ is allowed to depend on superspace coordinates and vector superfield $\bar{\Phi}\Phi$ is modified to $\bar{\Phi}e^{gV}\Phi$ where V is a vector superfield which, under (2.139), transforms as

$$V \longrightarrow V + ig(\Lambda - \bar{\Lambda}) \tag{2.140}$$

where Λ is a chiral superfield in the sense of Section 2.3.4.1. Note that in QFT the phase angle α being a constant means $\bar{D}_{\dot{\beta}}\alpha = 0$, hence in the language of superspace α is a chiral superfield [7]. The requirement that Λ be a chiral superfield is natural.

The term $\bar{\Phi}e^{gV}\Phi$ is function of chiral and antichiral superfields. It is known as the *Kähler* potential, denoted with $\mathcal{K}(\Phi, \bar{\Phi})$, and serves as the kinetic part of the Lagrangian density. $\mathcal{K}$ is integrated over all of superspace

$$\int d^4x d\theta^2 d\bar{\theta}^2 \mathcal{K}(\Phi, \bar{\Phi}) \tag{2.141}$$

[7] and a antichiral superfield since $D_\beta\alpha = 0$

From this we observe that 4-dimensional SUSY invariant actions in $\mathcal{N} = 1$ superspace will schematically be of the form

$$S = \int d^4x \Big[d^2\theta d^2\bar{\theta}\ \mathcal{K}(\Phi, \bar{\Phi}) + d^2\theta\ \mathcal{W}(\Phi) + d^2\bar{\theta}\ \bar{\mathcal{W}}(\bar{\Phi}) \Big] \tag{2.142}$$

Here the volume elements can be written as $d^8z \equiv d^4xd^2\theta d^2\bar{\theta}$, $d^6z \equiv d^4xd^2\theta$ and $d^6\bar{z} \equiv d^4xd^2\bar{\theta}$ and the action becomes

$$S = \int\ d^8z\ \mathcal{K}(\Phi, \bar{\Phi}) + \int\ d^6z\ \mathcal{W}(\Phi) + \int\ d^6\bar{z}\bar{\mathcal{W}}(\bar{\Phi}) \tag{2.143}$$

The Kähler potential is given by $\mathcal{K}(\Phi e^{gV}, \bar{\Phi})$ and in the abelian case the chiral superfields are vectors and e^{gV} a matrix. In the non-abelian case the chiral superfields themselves become matrices so that the Lagrangian is traced over i.e.

$$S = \int\ d^8z\ \ \mathrm{Tr}\, \mathcal{K}(\Phi, \bar{\Phi}) + \int\ d^6z\ \ \mathrm{Tr}\, \mathcal{W}(\Phi) + \int\ d^6\bar{z}\, \mathrm{Tr}\, \bar{\mathcal{W}}(\bar{\Phi}) \tag{2.144}$$

2.3.4.3 Spinor superfields

Thus far we have considered chiral and vector superfields and these are without a free index, making them Lorentz scalar superfields i.e. spin 0. In supersymmetric field theories it is also possible to construct spinor superfields using the super-derivatives of a vector superfield:

$$\begin{aligned} W_\alpha &:= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V(x, \theta, \bar{\theta}) \\ \bar{W}_{\dot{\alpha}} &:= -\frac{1}{4}DD\bar{D}_{\dot{\alpha}} V(x, \theta, \bar{\theta}) \end{aligned} \tag{2.145}$$

These spinors superfields are the supersymmetry analogue of a gauge field strength in QFT, $F_{\mu\nu} = \partial_{(\mu}A_{\nu)}$ as such they are thought of as fermionic field strengths. Recalling the definitions (2.101) and anticommutation relations (2.103) of super-covariant derivatives we conclude that W_α and $\bar{W}_{\dot{\alpha}}$ are chiral and anti-chiral respectively [36]

$$\bar{D}_{\dot{\beta}} W_\alpha = -\frac{1}{4}\bar{D}_{\dot{\beta}}\bar{D}\bar{D}D_\alpha V(x, \theta, \bar{\theta}) = 0 \tag{2.146}$$

$$D_\beta \bar{W}_{\dot{\alpha}} = -\frac{1}{4}D_\beta DD\bar{D}_{\dot{\alpha}} V(x, \theta, \bar{\theta}) = 0 \tag{2.147}$$

Under the supersymmetric gauge transformation $V \longrightarrow V + i(\Lambda - \bar{\Lambda})$ where Λ and $\bar{\Lambda}$ are chiral and antichiral superfields, the spinor superfield strength, W_α, is invariant since

$$W_\alpha \longrightarrow W_\alpha^{'} = -\frac{1}{4}\bar{D}\bar{D}D_\alpha(V + i\Lambda - i\bar{\Lambda}) \tag{2.148}$$
$$= -\frac{1}{4}\left(\bar{D}\bar{D}D_\alpha V + i\bar{D}\bar{D}D_\alpha\Lambda - i\bar{D}\bar{D}D_\alpha\bar{\Lambda}\right) \tag{2.149}$$
$$= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \tag{2.150}$$

This is a by-product of

$$\bar{D}\bar{D}D_\alpha\Phi = \bar{D}(\{\bar{D}, D_\alpha\} + D_\alpha\bar{D})\Lambda \tag{2.151}$$
$$= \bar{D}(\{\bar{D}, D_\alpha\})\Lambda \tag{2.152}$$
$$= \bar{D}(2i\sigma^\mu\partial_\mu)\Lambda \tag{2.153}$$
$$= 2i\sigma\partial_\mu\bar{D}\Lambda \tag{2.154}$$
$$= 0 \tag{2.155}$$

By the same argument one can show that $\bar{D}\bar{D}D_\alpha\bar{\Lambda} = 0$, hence W_α is super-gauge invariant. The solution of the spinor superfield strength in the Wess-Zumino gauge after changing to the y-coordinate is given by

$$W_\alpha = -i\lambda_\alpha(y) + \theta\theta\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}}(y) + \delta^\beta_\alpha\theta_\beta D(y) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha^{\ \beta}\theta_\beta F_{\mu\nu}(y) \tag{2.156}$$

with $F_{\mu\nu} = \partial_{[\mu}v_{\nu]}$, the abelian gauge field strength. The spinor superfield term in the action will be

$$\int d^6 z W^\alpha W_\alpha \tag{2.157}$$

This term is SUSY invariant since W_α is chiral. By adding (2.157) and its hermitian conjugate to (2.143) we are guaranteed that the kinetic term of the gauge fields is complete. For the non-abelian case a super-gauge invariant spinor superfield is defined as [28]

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V + \frac{1}{8}\bar{D}\bar{D}[V, \bar{D}_\alpha V] \tag{2.158}$$

which in terms of the y-coordinate in the Wess-Zumino gauge means

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \theta\theta\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}} - \frac{i}{2}\theta\theta\sigma^\mu_{\alpha\dot{\alpha}}[v_\mu, \lambda^{\dot{\alpha}}] \tag{2.159}$$

and the gauge field strength is now given by

$$F_{\mu\nu} = \partial_{[\mu}v_{\nu]} - \frac{i}{2}[v_\mu, v_\nu] \tag{2.160}$$

Admittedly our discussion of supersymmetric fields has focus on the chiral relations. The antichiral relations can be obtained by (hermitian) conjugation. Thus we can conclude our discussion of the section of symmetries commonly found in physics.

Chapter 3

$\mathcal{N} = 4$ SYM and Friends

3.1 $\mathcal{N} = 4$ SYM in 4d

The $\mathcal{N} = 4$ SYM theory in 4-dimensions is an example of a model which possesses all the symmetries discussed in Chapter 2 thus making it rather special. If one is interested in fields with spin no larger than 1 then the maximum amount of supersymmetry allowed is 4. $\mathcal{N} = 4$ means the theory has total of 16 real supercharges. So $\mathcal{N} = 4$ SYM is maximally supersymmetric. Its field content is made of the gauge field, A_μ, 4 Weyl spinors,$\psi^\alpha_a, \bar{\psi}^b_{\dot{\alpha}}$ and 6 real scalar fields, ϕ^i all of which are in the adjoint representation of the gauge group which we take to be $SU(N)$. So under the action of $U \in SU(N)$, the Weyl spinor fields and scalar fields will transform as $\psi \longrightarrow U\psi U^{-1}$ and $\phi \longrightarrow U\phi U^{-1}$. The gauge field on the other hand will transform as

$$A_\mu \longrightarrow UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \tag{3.1}$$

because it appears through the covariant derivative, D_μ which is defined as

$$D_\mu := \partial_\mu - igA_\mu \tag{3.2}$$

Since we are in 4-dimensional Minkowski spacetime the index, μ, runs from 1 to 4. So then $SO(1,3)$ is the spacetime symmetry enjoyed by the gauge and scalar fields. The Weyl spinors, whose indices α and $\dot{\alpha}$ run from 1 to 2, also enjoy the $\mathfrak{so}^+(1,3)$ symmetry which is realized as two copies of $\mathfrak{su}(2)$, one for the chiral Weyl spinors and the other for the anti-chiral Weyl spinors [See (2.14) and (2.15)]. The Lagrangian of this 4-dimensional

$\mathcal{N}=4$ super-Poincaré invariant theory is [37][38]:

$$\mathcal{L}=\mathrm{Tr}\Bigg[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu}+\frac{\theta_I}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}-i(\bar{\psi}_{\dot{\alpha}})^a(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}D_\mu(\psi_\beta)_a-D_\mu\phi^iD^\mu\phi_i$$
$$+gC_i^{ab}(\psi^\alpha)_a[\phi_i,(\psi_\alpha)_b]+g\bar{C}_{iab}(\bar{\psi}_{\dot{\alpha}})^a[\phi^i,(\bar{\psi}^{\dot{\alpha}})^b]+\frac{g^2}{2}[\phi^i,\phi^j][\phi_i,\phi_j]\Bigg] \tag{3.3}$$

and it is invariant under $\mathcal{N}=4$ super-Poincaré transformations. The non-abelian gauge field strength $F_{\mu\nu}$ is given by $F_{\mu\nu}=\partial_{[\mu}A_{\nu]}+i[A_\mu,A_\nu]$. $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol while the constants C_i^{ab} and $\bar{C}_{iab}$ are the *Clebsch-Gordan* coefficients of the $SO(6)$ sector. To understand how they eventually enter the scene we first note that in the Lagrangian above the fields are matrix-valued and thus carry symmetry group indices. The group indices $\{a,b\}$ which run from 1 to 4 are for the spinors and they are indicative of the $SU(4)$ internal symmetry that rotates the spinors into another and it is an R-symmetry. The 6 scalar fields also carry indices for the $SO(6)$ group which pertains to their internal symmetry; these indices $\{i,j\}$ run from 1 to 6. At the algebra level $\mathfrak{so}(6)$ is isomorphic to $\mathfrak{su}(4)$ which means the spinors can be written in the same representation as the scalar fields with help of the Clebsch-Gordan coefficients. So these coefficients allow for translation between the $SO(6)$ and $SU(4)$ representations.

A dimensional analysis[1] of each term of (3.3) shows that the Lagrangian has dimensionless couplings, i.e. $[\theta_I]=0=[g]$. Thus it follows immediately that this theory is scale invariant. For the bosonic sub-sector of the theory, scale invariance together with Poincaré symmetry in (3+1)-dimensions give rise to conformal symmetry described by $\mathfrak{so}(2,4)\simeq\mathfrak{su}(2,2)$ [37]. This symmetry is enhanced by the presence of supersymmetry because the Special Conformal Transformation [SCT] generators K_μ do not commute with SUSY generators Q_α, meaning special conformal symmetry mixes with supersymmetry to produce supersymmetric special conformal transformations [SUSY SCT] that are generated by $S^\alpha,\bar{S}_{\dot{\alpha}}$. The SUSY SCT generators are fermionic in nature because of the grading of the supercharges i.e. $[\mathrm{Even},\mathrm{Odd}]=\mathrm{Odd}$. The commutation relations of the SUSY SCT generators with the Lorentz generators are [39]:

$$\begin{aligned}
&[S_\alpha,K_\mu]=0=[\bar{S}^{\dot{\alpha}},K_\mu], \qquad [S_\alpha,D]=-\frac{i}{2}S_\alpha,\ \text{and}\ [\bar{S}^{\dot{\alpha}},K_\mu]=-\frac{i}{2}\bar{S}^{\dot{\alpha}}\\
&[S_\alpha,P_\mu]=-i(\sigma_\mu)_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}}, \qquad [\bar{S}_{\dot{\alpha}},P_\mu]=-i(\bar{\sigma}_\mu)^{\beta}_{\dot{\alpha}}\,Q_\beta\\
&[S_\alpha,M_{\mu\nu}]=-\frac{i}{2}(\sigma_\mu\bar{\sigma}_\nu)_\alpha^{\ \beta}S_\beta, \quad [\bar{S}^{\dot{\alpha}},M_{\mu\nu}]=-\frac{i}{2}(\bar{\sigma}_\mu\sigma_\nu)^{\dot{\alpha}}_{\ \dot{\beta}}\bar{S}^{\dot{\beta}}
\end{aligned} \tag{3.4}$$

The total number of real supercharges thus increased from 16 to 32. The superalgebra

[1] $[A_\mu]=1$, $[\psi]=\frac{3}{2}$,$[\phi^i]=1$

corresponding to $\mathcal{N}=4$ SYM is thus $\mathfrak{psu}(2,2|4)$ due to the enhancement by superconformal symmetry. $\mathcal{N}=4$ SYM is the unique maximally SUSY field theory in 4d whether constructed from ground up or from a higher-dimensional theory. When an $\mathcal{N}=1$ SUSY field theory in 10-dimensions is, by Kaluza-Klein compactification, reduced to 4-dimensions, one obtains $\mathcal{N}=4$ SYM. Moreover $\mathcal{N}=4$ SYM, being independent of a renormalization energy scale, has a vanishing β function and under perturbative treatment its n-point functions show no signs of UV divergences. These observations persist even to the quantum level. This makes it a good toy model on which to develop understanding and tools for solutions.

3.1.1 The role of $\mathcal{N}$= 4 SYM:

3.1.1.1 In AdS/CFT duality

Much time has been spent studying this theory and this effort has proven useful because $\mathcal{N}=4$ SYM is linked to many other models. Our knowledge of it has been employed to understanding these other models. In the AdS/CFT correspondence, $\mathcal{N}=4$ SYM is conjectured to be dual to IIB String theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ [21]. Inasmuch as the correspondence is conjectural, there are features that fuel our reasons to believe in it. One such feature is the global symmetry of either one of the theories. It is useful to digress a little and first discuss AdS space.

A (d+1)-dimensional AdS space of radius R is defined by the constraint

$$-X_0^2 - X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -R^2 \tag{3.5}$$

and this can be embedded in a flat (d+2)-dimensional space with a pseudo-Minkwoski metric given by

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2 \tag{3.6}$$

The following definition of the embedding coordinates X_t, $t=0,1,\ldots,d+1$ gives a solution to (3.5) [40]

$$\begin{aligned} X_0 &= R\sec\rho\cos\tau\ , \\ X_{d+1} &= R\sec\rho\sin\tau\ , \\ X_i &= R\tan\rho\,\xi_i\ ,\ i=1,\ldots,d \end{aligned} \tag{3.7}$$

where ξ's satisfy $\xi^i\xi_i = \xi^i\delta_{ij}\xi^j = 1$ where $i,j=1,\ldots,d$. ρ is an AdS radial coordinate which takes values in $[0,\pi/2$ while τ and the ξ's are angular coordinates which are restricted to $-\pi<\tau\leq+\pi$ and $-1\leq\xi_i\leq+1$. In these coordinates, (ρ,τ,ξ) – which are usually called *global coordinates*, the AdS metric is given by

$$ds^2 = \frac{R^2}{\cos^2\rho}\left(-d\tau^2 + d\rho^2 + \sin^2\rho\sum_{i=1}^{d} d{\xi_i}^2\right) \tag{3.8}$$

There is another choice of coordinates in which the metric (3.8) assumes a compact form, they are called the *Poincaré* cooridnates $(z, \bar{x}, t)$. We first combine one 'time-like' coordinate X_0 with one space-like coordinate X_d to define light-cone coordinates u and v

$$\begin{aligned} u &:= \frac{X_0 - X_d}{R^2} \equiv \frac{1}{z} \\ v &:= \frac{X_0 + X_d}{R^2} \end{aligned} \tag{3.9}$$

Then we change the other remaining coordinates, (d-1) space-like and 1 time-like coordinates, to

$$\begin{aligned} x_i &:= \frac{X_i}{Ru} \\ t &:= \frac{X_{d+1}}{Ru} \end{aligned} \tag{3.10}$$

The constraint (3.5) allows us to resolve the light-cone coordinate v in terms of u, the x_i's and t. At this point, the embedding coordinates can be written as

$$\begin{aligned} X_0 &= \frac{1}{2z}\left(z^2 + R^2 + \bar{x}^2 - t^2\right) \\ X_d &= \frac{1}{2z}\left(z^2 - R^2 + \bar{x}^2 - t^2\right) \\ X_{d+1} &= \frac{Rt}{z} \\ X_i &= \frac{Rx^i}{z} \quad , \ i = 1, \ldots, d-1 \end{aligned} \tag{3.11}$$

where we have defined $\bar{x}^2 = \delta_{ij}x^i x^j$ with $i, j = 1, \ldots, d-1$ and used an equality defined earlier, $u = z^{-1}$. In these coordinates, the AdS metric is

$$ds^2 = \frac{R^2}{z^2}\left[dz^2 + d\bar{x}^2 - dt^2\right] \tag{3.12}$$

The full isometry group of this (d+1)-dimensional AdS space is $SO(2, d)$ even though only part of it is apparent in the metric[2].

We are now in a position to compare the global symmetries of $\mathcal{N} = 4$ SYM and type IIB string theory in $AdS_5 \times S^5$. The AdS_5 part of the product manifold has an isometry group $SO(2, 4)$ while the S^5 part has isometry group $SO(6)$. Recall that the global symmetry of SYM is given by the superconformal group $SU(2, 2|4)$.

The isometry group of a 5-sphere is $SO(6)$ which is isomorphic to $SU(4)$.

[2] $SO(1, d-1) \times SO(1, 1)$ is Poincaré coordinates, $SO(d) \times SO(2)$ in global coordinates.

The isometry group of AdS_5 is $SO(2,4) \equiv SU(2,2)$ and that of S^5 is $SU(4)$ then it follows that the product space has $SU(2,2|4)$ as a global symmetry group, matching $\mathcal{N} = 4$ SYM. The AdS/CFT duality has a strong-weak coupling relation and thus can be employed to study the strongly coupled regime of one theory in terms of the weakly coupled regime of its dual theory. Thus our knowledge of $\mathcal{N} = 4$ SYM can be used to study IIB string theory [41]. In some cases the flow of information is reversed, that is, the theories to which $\mathcal{N} = 4$ SYM is dual either unveil its properties or supply a novel platform with tools which can be used to calculate quantities previously impossible to calculate.

3.1.1.2 In Integrability

Early work [42] and especially the AdS/CFT correspondence [21] have led to the suspicion that gauge theories admit a string description in the planar limit. In the case of $\mathcal{N} = 4$ SYM being dual to IIB string theory evidence for this suspicion is contained in their spectra. The spectrum of non-interacting strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ matches the spectrum of single-trace operators in $\mathcal{N} = 4$ SYM. By considering operators of the form $\mathrm{Tr}(ZZ \ldots Z) = \mathrm{Tr}(Z^J)$, which have come to be known as BMN operators, the authors of [41] were able to recover the string spectrum from planar $\mathcal{N} = 4$ SYM for large J. The insertion of a field ϕ in $\mathrm{Tr}(Z^J)$ is considered as an impurity on the field theory side. This impurity is understood on the string description as an excitation. The anomalous dimensions of BMN operators can be predicted from string theory. This is because string theory suggests that the dimension of a BMN operator is equal to the mass of the corresponding string state [41], [43], [44]. Thus we have an operator/string correspondence. By playing the 'same game' backwards, it is clear that computing the anomalous dimensions of operators allows the reconstruction of the string spectrum. From renormalization arguments, the computation of anomalous dimensions of a field ϕ is done via field strength renormalization factor Z_ϕ. For composite operators, the renormalization factor $\mathcal{Z}$ is generally a matrix[3]. So anomalous dimensions can also be represented/collected into a matrix Γ which is given by

$$\Gamma = \frac{d\mathcal{Z}}{d \ln \Lambda} \frac{1}{\mathcal{Z}} \tag{3.13}$$

In an attempt to understand operators beyond the BMN type at one-loop, this line of reasoning was employed in [45]. There it was found that the one-loop matrix of anomalous dimensions, Γ, is hermitian. Moreover a single-trace operator of length J made of the scalar fields belonging to the $SO(6)$ sector of $\mathcal{N} = 4$ SYM has J sites and can be considered an element of a Hilbert space $V^{\otimes J}$ where $V = \mathbb{R}^6$. A useful picture to have in mind is that of a spin chain. If the Hilbert space $V^{\otimes J}$ is thought of as spin

[3] a mixing matrix

chain of length J then the single trace operators can be regarded as states of the spin chain.

As an example, consider a gauge field theory with a sub-sector made of two scalar fields X and Y. The Hilbert space $V^{\otimes 2}$ associated to this sector will then be a spin chain with two sites and the operators $\mathrm{Tr}(XX), \mathrm{Tr}(YY), \mathrm{Tr}(XY)$ and $\mathrm{Tr}(XY)$ would be states of the spin chain. The following identification can be made

$$\mathrm{Tr}(XX) \Longleftrightarrow |\uparrow\uparrow\rangle \tag{3.14}$$

$$\mathrm{Tr}(XY) \Longleftrightarrow |\uparrow\downarrow\rangle \tag{3.15}$$

$$\mathrm{Tr}(YX) \Longleftrightarrow |\downarrow\uparrow\rangle \tag{3.16}$$

$$\mathrm{Tr}(YY) \Longleftrightarrow |\downarrow\downarrow\rangle \tag{3.17}$$

Either (3.15) or (3.17) can be considered a ground state so that the others are excited states. The mixing matrix corresponding to this sub-sector of the gauge field theory will serve as the Hamiltonian of/in the spin chain picture. This mini example serves to clarify the underlying logic without caring for the details since these vary by sector and model. For a discussion of in-depth detials can be found in [46][47].

Returning to our specific model, the operator $\mathrm{Tr}\big(Z^J\big)$ is was mapped to the ground state of a spin chain with J sites and Γ, served as the Hamiltonian operator of the spin system. For the one-loop calculation Γ was found to be [45]

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{J} \left[K_{l,l+1} + 2(\mathrm{I}_{l,l+1} - P_{l,l+1})\right] \tag{3.18}$$

where K is a trace operator, I an identity operator and P a permutation operator. As a Hamiltonian, Γ at one-loop corresponds to a spin chain with nearest-neighbour interaction. This Hamiltonian is the same as that of a Heisenberg XXX-spin chain model with J-sites and thus is integrable. This mapping avails the computational resources used in integrability. This method is made efficient and simple by focusing only on the dilatation operator of $\mathcal{N} = 4$ SYM because the dilatation computes the scaling dimensions of operators [38],[48].

3.2 The Friends: Marginal deformations

The $\mathcal{N} = 4$ SYM possesses a large amount of symmetry and it seems expected that it enjoys all these benefits. The next task would be then to find other theories, with less symmetry, for which similar analyses are possible. A good starting point in this quest is to deform $\mathcal{N} = 4$ SYM by introducing or modifying terms in its Lagrangian. We shall be interested in marginal deformations, deformations that reduce supersymmetry but keep conformal symmetry intact. To appreciate this we first re-write the action corresponding to (3.3) in the language of $\mathcal{N} = 1$ superspace introduced earlier

$$S = \int \Big[\mathrm{d}^8 z \ \mathrm{Tr}(e^{-gV}\bar{\Phi}^i e^{gV}\Phi^i) + \frac{1}{2g^2}\Big(\mathrm{d}^6 z \ \mathrm{Tr}(W^\alpha W_\alpha) + \mathrm{d}^6 \bar{z} \ \mathrm{Tr}(\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}})\Big) + \\ ig\Big(\mathrm{d}^6 z \ \mathrm{Tr}(\Phi^1\Phi^2\Phi^3 - \Phi^1\Phi^3\Phi^2) + \mathrm{d}^6\bar{z} \ \mathrm{Tr}(\bar{\Phi}^1\bar{\Phi}^2\bar{\Phi}^3 - \bar{\Phi}^1\bar{\Phi}^3\bar{\Phi}^2)\Big)\Big] \tag{3.19}$$

where Φ^i's are chiral superfields, W_α is the spinor superfield strength and V is a vector superfield as presented in Chapter 2. The term

$$\mathcal{W}_{\mathcal{N}=4} := g\mathrm{Tr}(\Phi^1\Phi^2\Phi^3 - \Phi^1\Phi^3\Phi^2) = g\mathrm{Tr}\Big(\Phi^1[\Phi^2, \Phi^3]\Big) \tag{3.20}$$

is the superpotential. In this work we shall focus on Leigh-Strassler deformations which only deform the superpotential to

$$\mathcal{W}_{LS} = \kappa\mathrm{Tr}\Big[\Phi^1\Phi^2\Phi^3 - q\Phi^1\Phi^3\Phi^2 + \frac{h}{3}\Big((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\Big)\Big] \tag{3.21}$$

where q and h are the deformation parameters. In principle there are more general deformations that can be done on $\mathcal{N} = 4$ SYM but if we demand that the deformations be exactly marginal then (3.21) is the most general form that the superpotential assumes [49]. This class of theories is called Leigh-Strassler [LS] credit to the authors who, by using the NSVZ beta function [50], demonstrated finiteness provided that there is a function which γ which relates parameters of the theory [51]. Such a function must satisfy the condition

$$\gamma(g, \kappa, q, h) = 0 \tag{3.22}$$

This condition arises because of a distinguishing feature of Leigh-Strassler theories which we point out. The scale independence of the Leigh-Strassler theory requires that the scaling coefficients of each of the chiral fields that appear in the superpotential must vanish and so also for that of the gauge coupling. However the chiral fields in Leigh-Strassler theories possess a $\mathbb{Z}_3$ symmetry which mandates that their anomalous dimensions be the same. This anomalous dimension is proportional to the function $\gamma(g, \kappa, q, h)$, thus the condition (3.22) guarantees a vanishing anomalous dimension. The existence of the function γ implies that fixed points of Leigh-Strassler theories constitute a manifold. At one-loop order the condition (3.22) can be solved to obtain [51]:

$$2g^2 = \kappa\bar{\kappa}\Big[\frac{2}{N^2}(q+1)(\bar{q}+1) + \Big(1 - \frac{4}{N^2}\Big)\left(q\bar{q} + h\bar{h} + 1\right)\Big] \tag{3.23}$$

The Leigh-Strassler deformations break the supersymmetry of SYM from $\mathcal{N}=4$ SYM to $\mathcal{N}=1$, while preserving conformal symmetry. This in principle means that Leigh-Strassler theories can be studied using the same tools used for $\mathcal{N}=4$ SYM , i.e. AdS/CFT duality. An example of this is the real β-deformed theory which is obtained by setting $q=e^{i\beta}, \bar{q}=e^{-i\beta}$ and $h=\bar{h}=0$ with $\beta\in\mathbb{R}$. In [52] the gravity dual of the real β deformed theory was obtained. The key ingredient here was that real β deformation breaks the $SU(3)$ symmetry of the superpotential down to $U(1)\times U(1)$ symmetry. This remnant symmetry which enabled the authors of [52] to obtain the gravity dual to the β-deformed gauge theory by a procedure now known as a TsT transformation. This method of obtaining a gravity dual of a deformed field theory is useful ([53], [54])but it depends on the existence of $U(1)\times U(1)$ symmetry which is realized geometrically. In [52] $U(1)\times U(1)$ issued from the $SU(3)$ part of the $SU(4)$ of the R-symmetry. A general Leigh-Strassler deformation however will break the $SU(3)$ subgroup down to the discrete Δ_{27} group [55].

Next we can consider integrability as in [56] where an $SU(2)$ sector of a q-deformed $\mathcal{N}=4$ SYM was shown to be integrable at one-loop. The spin chain Hamiltonian of the sector corresponds to that of a parity violating XXZ Heisenberg spin chain [57] [58]. Attempts to match the full q-deformed $SO(6)$ sector of $\mathcal{N}=4$ SYM to the SO(6) XXZ spin chain were unsuccessful [59]. From (3.19) it is clear that Leigh-Strassler deformations affect the superpotential, a function of 3 chiral superfields Φ^i. The Φ's constitute an $SU(3)$ sector whose corresponding one-loop spin chain Hamiltonian is [60]

$$H_{l,l+1}=\frac{1}{(q\bar{q}+h\bar{h}+1)}\begin{bmatrix} h\bar{h} & 0 & 0 & 0 & 0 & \bar{h} & 0 & -\bar{h}q & 0 \\ 0 & 1 & 0 & -q & 0 & 0 & 0 & 0 & h \\ 0 & 0 & q\bar{q} & 0 & -h\bar{q} & 0 & -\bar{q} & 0 & 0 \\ 0 & -\bar{q} & 0 & q\bar{q} & 0 & 0 & 0 & 0 & -h\bar{q} \\ 0 & 0 & -\bar{h}q & 0 & h\bar{h} & 0 & \bar{h} & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 1 & 0 & -q & 0 \\ 0 & 0 & -q & 0 & h & 0 & 1 & 0 & 0 \\ -h\bar{q} & 0 & 0 & 0 & 0 & -\bar{q} & 0 & q\bar{q} & 0 \\ 0 & \bar{h} & 0 & -\bar{h}q & 0 & 0 & 0 & 0 & h\bar{h} \end{bmatrix} \tag{3.24}$$

This Hamiltonian is related to the R-matrix via [57],[60]

$$H=-iP\frac{d}{du}R(u)\bigg|_{u=0} \tag{3.25}$$

here P is the permutation operator and u is the spectral parameter. The R-matrix of Leigh-Strassler deformed $\mathcal{N} = 4$ SYM in the quantum limit ($u \longrightarrow \infty$) is given by

$$R_{qh} = \frac{1}{2d^2}\begin{bmatrix} t_1 & 0 & 0 & 0 & 0 & -2\bar{h} & 0 & 2\bar{h}q & 0 \\ 0 & 2\bar{q} & 0 & t_3 & 0 & 0 & 0 & 0 & 2h\bar{q} \\ 0 & 0 & 2q & 0 & -2h & 0 & t_2 & 0 & 0 \\ 0 & t_2 & 0 & 2q & 0 & 0 & 0 & 0 & -2h \\ 0 & 0 & 2\bar{h}q & 0 & t_1 & 0 & -2\bar{h} & 0 & 0 \\ 2h\bar{q} & 0 & 0 & 0 & 0 & 2\bar{q} & 0 & t_3 & 0 \\ 0 & 0 & t_3 & 0 & 2h\bar{q} & 0 & 2\bar{q} & 0 & 0 \\ -2h & 0 & 0 & 0 & 0 & t_2 & 0 & 2q & 0 \\ 0 & -2\bar{h} & 0 & 2\bar{h}q & 0 & 0 & 0 & 0 & t_1 \end{bmatrix} \tag{3.26}$$

with $d^2 = (1 + q\bar{q} + h\bar{h})/2$, $t_1 = 1 - h\bar{h} + q\bar{q}$, $t_2 = -1 + h\bar{h} + q\bar{q}$ and $t_3 = 1 + h\bar{h} - q\bar{q}$. [4]Note that the R-matrix, consequently also the Hamiltonian, acts on a basis in which rows and columns are labeled by

$$\{|1\ 1\rangle, |1\ 2\rangle, |1\ 3\rangle, |2\ 1\rangle, |2\ 2\rangle, |2\ 3\rangle, |3\ 1\rangle, |3\ 2\rangle, |3\ 3\rangle\}$$

The R-matrix is a key component of the puzzle because of its usefulness in characterizing Hamiltonians of integrable models. To be sure, quantum integrable models are characterized by their RTT relations, much like classical integrable models could be characterized by their Poisson-involution relations [61] [62]. These relations have the form [See Appendix C for further details]

$$R(u,v)\hat{T}_a(u)\hat{T}_b(v) = \hat{T}_b(v)\hat{T}_a(u)R(u,v) \tag{3.27}$$

where $\hat{T}_a(u), \hat{T}_b(v)$ are monodromy matrices and $R(u,v)$ is spectral parameter dependent R-matrix with spectral parameters u and v. Note that the commutation relations of $\hat{T}$ are defined via R and thus R characterizes the model. The model is said to be integrable if R satisfies the spectral-parameter-dependent Yang-Baxter equation [YBE]:

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \tag{3.28}$$

and its corresponding Hamiltonian, obtained via (3.25), will also be integrable. Taking the spectral parameter to infinity means (3.28) becomes $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. We shall encounter these expressions in the following chapters.

The process of actually obtaining the actual conserved charges is carried out by the algebraic Bethe Ansatz methods, whose details are discussed in [63][64] and references

[4]The functions t_i have been introduced for brevity of the notation

therein. We now have seen the role of the R-matrix from the physics view point. In the next chapter we present the mathematics in which it appears and that appearance will be sufficient ground for us to marry the mathematics to the physics.

Chapter 4

The theory of quasi-Hopf Algebras

In this section we review the definitions and properties of quasi-Hopf algebras [qHA]. Our approach is to first present (regular) Hopf algebras and thereafter quasi-Hopf algebras as generalizations of Hopf algebras [HA]. We first review the structures that are necessary for the definition of a Hopf algebra. A more detailed and complete exposition of quasi-Hopf algebras can be found in refs [65][66][67].

4.1 Hopf Algebras

4.1.1 Algebras

To begin let V be an abelian group with an additive composition rule $+$ and k a field with zero characteristic[1] . Then the triple $(V, +; k)$ together with the multiplicative action of the group $(k-\{0\})$ on V define a vector space if they are compatible. Compatibility here means the multiplicative action (which from here on will be called scalar multiplication[2]) of $(k - \{0\})$ and the additive composition '+' of V satisfy

$$\alpha\,(v_i + v_j) = \alpha v_i + \alpha v_j, \tag{4.1}$$

$$(\alpha + \beta)\, v_i = \alpha v_i + \beta v_i \tag{4.2}$$

for all scalars $\alpha, \beta \in k$ and 'vectors' $v_i,\ v_j \in V$. A vector space can be endowed with a multiplicative composition: $\bullet$ in addition to the additive composition $+$. This enriches it into an algebra defined over a field k. That is to say the quadruple $(V, \bullet, +; k)$ is called

[1]For the multiplicative identity 1 of the field there exists no n for which the sum $\sum_{i=1}^{n} 1$ vanishes.
[2]In this notation scalar multiplication is implied hence no symbol is used.

an *algebra* if the action of k on V is also compatible with: $\bullet$. Hence

$$\alpha(v_i \bullet v_j) = v_i \bullet (\alpha v_j), \; \forall\; \alpha \in \; k \text{ and } v_i, v_j \in V \tag{4.3}$$

It is important to note that $\bullet$ is a linear map from $V \otimes V$ to V and we shall require that $\bullet$ be associative which means

$$(\bullet \otimes \mathrm{id})(v_i \otimes v_j \otimes v_k) = (v_i \bullet v_j) \otimes v_k = v_i \otimes (v_j \bullet v_k) = (\mathrm{id} \otimes \bullet)(v_i \otimes v_j \otimes v_k) \;\in\; V \otimes V \tag{4.4}$$

By $\mathbb{1}_V$ we denote the multiplicative identity of V so that $(\mathbb{1}_V \bullet v_i) = (v_i \bullet \mathbb{1}_V) = v_i$. For reasons soon to be clear, for any element $v_i \in V$ we define a linear map $\eta_{v_i} : k \longrightarrow V$ as follows $\eta_{v_i}(1) := v_i$. These maps are nothing more than a re-scaling[3] of the elements of V and $\eta_{v_i}(\alpha) = \alpha v_i$ returns the element itself. We employ this fact to represent $\mathbb{1}_V$ as a map with a special designation, namely $\eta \equiv \eta_{\mathbb{1}_V}(1) = \mathbb{1}_V$ and since η is map representation of $\mathbb{1}_V$ we call it a *unit map*. Then we note that

$$\left.\begin{aligned}(\eta \otimes \mathrm{id})(k \otimes V)\\ (\mathrm{id} \otimes \eta)(V \otimes k)\end{aligned}\right\} \in V \otimes V \tag{4.5}$$

It is customary that the above definition of an algebra be summarized with the help of commutative diagrams which in turn serve as good mnemonic devices. Figure 4.1 below shows the commutative diagrams.

$$\begin{tikzcd}
 & V \otimes V \otimes V \arrow[dl, "(id\otimes\bullet)"'] \arrow[dr, "(\bullet\otimes id)"] & \\
V \otimes V \arrow[dr, "\bullet"'] & & V \otimes V \arrow[dl, "\bullet"] \\
 & V &
\end{tikzcd}$$

(A) Associativity of product map $\bullet$

$$\begin{tikzcd}
V \otimes V \arrow[dr, "\bullet"] & & V \otimes V \arrow[dl, "\bullet"'] \\
k \otimes V \arrow[u, "\eta\otimes\mathrm{id}"] \arrow[r, "\cong"'] & V \arrow[r, "\cong"'] & V \otimes k \arrow[u, "\mathrm{id}\otimes\eta"']
\end{tikzcd}$$

(B) Unit map

FIGURE 4.1: Algebra properties

4.1.2 Co-algebras

When considering the diagrammatic summary of the maps of an algebra, it is natural to wonder whether the reversal of arrows defines a meaningful mathematical structure.

[3] stretching, shrinking and/or reflecting

Such a definition is possible and the resulting structure is called a *co-algebra*[4] and its corresponding commutative diagram is:

$$\begin{tikzcd}
& W \otimes W \otimes W & \\
W \otimes W \arrow[ur, "\mathrm{id}\otimes\Delta"] & & W \otimes W \arrow[ul, "\Delta\otimes\mathrm{id}"'] \\
& W \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

(A) Co-associativity of co-product map Δ

$$\begin{tikzcd}
W \otimes W \arrow[d, "\epsilon\otimes\mathrm{id}"'] & & W \otimes W \arrow[d, "\mathrm{id}\otimes\epsilon"] \\
k \otimes W \arrow[r, "\cong"'] & W \arrow[ul, "\Delta"'] \arrow[ur, "\Delta"] & W \otimes k \arrow[l, "\cong"]
\end{tikzcd}$$

(B) Co-unit map

FIGURE 4.2: Co-algebra properties

Formally, a co-algebra over a field k is a quintuple $(W, +, \Delta, \epsilon; k)$ where the triple $(W, +; k)$ is a vector space. Here Δ and ϵ are linear maps which complement '$\bullet$' and η respectively, hence they are fitly named *co-multiplication or co-product* and *co-unit*, ditto. And Figure 4.2 implies that $\Delta : W \longrightarrow W \otimes W$ and $\epsilon : W \longrightarrow k$. In general the action of Δ on any $w \in W$ can be written as

$$\Delta(w_i) = \sum_{jk} {\alpha_i}^{jk} w_j \otimes w_k =: \sum w_{(1)} \otimes w_{(2)} \tag{4.6}$$

hence Δ shares out w to $W \otimes W$, a tensor product of two copies of the vector space W. The rightmost expression, written in what is known as *Sweedler notation* [68], helps to keep the expressions clean by suppressing the coefficients with the indices and then labelling the vector space copies. Any element $w_i \in W$ on which co-product is of the form $\Delta(w_i) = w_i \otimes w_i$, no summation implied, is called *group-like.*

Just as '$\bullet$' was required to be associative so also Δ can be required to be co-associative which means

$$(\mathrm{id} \otimes \Delta) \circ \Delta(w) = (\Delta \otimes \mathrm{id}) \circ \Delta(w) \tag{4.7}$$

The co-unit ϵ is such that for any $w \in W$

$$(\epsilon \otimes \mathrm{id}) \circ \Delta(w) = w = (\mathrm{id} \otimes \epsilon) \circ \Delta(w) \tag{4.8}$$

4.1.3 Bialgebras and Hopf algebras

In the definitions of algebras and co-algebras different base vector spaces were used, V for the algebra and W for the co-algebra. One can insist that the vector space used to

[4] 'co-algebra' because it complements the algebra structure.

define an algebra also be used to define a co-algebra. Doing so results is an algebraically symmetric structure known as a *bialgebra*. Thus a bialgebra is a 6-tuple $(H, \bullet, \eta, \Delta, \epsilon; k)$ where H is a vector space over a field k with maps as defined above. Since the algebra and co-algebra maps of the vector space that makes up the bialgebra are to co-exist we require that they be compatible and the compatibility of $\bullet$ with Δ means

$$\Delta(h_i \bullet h_j) = \Delta(h_i) \bullet \Delta(h_j), \tag{4.9}$$

that of $\bullet$ with ϵ

$$\epsilon(h_i \bullet h_j) = \epsilon(h_i) \bullet \epsilon(h_j) \tag{4.10}$$

There also exists a subset of bialgebras whose feature of distinction is that they possess an antipodal map (usually denoted by S and called the *antipode*); these are known as *Hopf algebras*. This in effect implies that by appending S to a bialgebra we obtain a Hopf algebra, i.e. $(H, \bullet, \eta, \Delta, \epsilon, S; k)$. The antipode S is a linear anti-homomorphic map $S : H \to H$ which maps $h \mapsto h^{-1}$, hence its role is to invert elements of H with respect to the product $\bullet$ so that $S(h_i \bullet h_j) = S(h_j) \bullet S(h_i)$. Note that the order of composition reverses just as, for example, in matrix multiplication $(A.B)^{-1} = (B^{-1}).(A^{-1})$ where A and B are invertible matrices. Furthermore, the antipode must be compatible with the existing bialgebra maps and thus S must satisfy the relation

$$\bullet\,(\mathrm{id} \otimes S) \circ \Delta = \bullet(S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon \tag{4.11}$$

which is a summary of the compatibility of $\bullet$ with S and η with ϵ. For brevity it is customary to refer to a HA $(H, \bullet, \eta, \Delta, \epsilon, S; k)$ by its underlying vector space H. The action of coproduct and counit maps on the identity element, $\mathbb{1}_H$, of the Hopf algebra are defined as:

$$\Delta(\mathbb{1}_H) = \mathbb{1} \otimes \mathbb{1} \quad \text{and } \epsilon(\mathbb{1}_H) = 1 \tag{4.12}$$

thus $\mathbb{1}_H$ is group-like. Below is the commutative diagram of a Hopf algebra:

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "\mathrm{id} \otimes S"] & & H \otimes H \arrow[dr, "\bullet"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\epsilon"] & & k \arrow[rr, "\eta"] & & H \\
& H \otimes H \arrow[rr, "S \otimes \mathrm{id}"'] & & H \otimes H \arrow[ur, "\bullet"'] &
\end{tikzcd}$$

FIGURE 4.3: Hopf Algebra commutative diagram

4.1.4 Examples of Hopf algebras

Following the definitions above we construct examples of Hopf algebras [69].

4.1.4.1 The tensor algebra

Let V be a (finite) vector space over $\mathbb{C}$ and choose the tensor product $\otimes$ as the product map. Then T^nV denotes the n^{th} tensor power of V in the sense that

$$T^nV = \underbrace{V \otimes V \otimes \cdots \otimes V}_{\text{n-times}} = V^{\otimes n} \tag{4.13}$$

so that a tensor product of $v \in T^nV$ and $w \in T^mV$ is an element $z := v \otimes w$ belonging to $T^{n+m}V$. It is now clear that the space of all tensor polynomials of vector space V which we denote by $T(V)$ possesses an algebra structure, that is the triple $(T(V), \otimes; \mathbb{C})$ is a *tensor algebra.*

$$T(V) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} T^nV \tag{4.14}$$

is the formal definition of the space of tensor polynomials of V. The relations

$$\left.\begin{array}{l} \Delta(v) = v \otimes \mathbb{1} + \mathbb{1} \otimes v \quad , \quad S(v) = -v \\ \Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1} \quad , \quad \epsilon(v) = 0 \end{array}\right\} \forall \;\; v \in V \tag{4.15}$$

complete the co-algebra structure and thus $(T(V), \otimes, \mathbb{1}, \Delta, \epsilon, S; \mathbb{C})$ together with the relations in (4.15) is a Hopf algebra.

4.1.4.2 Universal Enveloping Algebra

A useful HA which can be defined using the tensor algebra $T(V)$ above is the Universal Enveloping Algebra [UEA]. For this we define $\mathcal{I}_L$ to be a proper invariant subalgebra of $T(V)$ [70][69]. $\mathcal{I}_L$ known as a *left ideal* then has a property that it absorbs all elements of $T(V)$ when multiplied from the left:

$$v \otimes w \;\in\; \mathcal{I}_L \;, \;\forall\; w \;\in\; \mathcal{I}_L \text{ and } v \;\in\; T(V) \tag{4.16}$$

A similar subset can be defined for the case of multiplication from the right, which is known as a *right ideal* but the scenario is special when a invariant subalgebra is simultaneously a left and a right ideal in which case the it known as a *two-sided ideal.*

The notion of a two-sided ideal enables us to construct a UEA as follows. Let $\mathcal{I}$ be the smallest possible two-sided ideal of the algebra $T(V)$ generated by elements of the form

$$v \otimes w - w \otimes v - [v, w] \tag{4.17}$$

then the quotient $T(V)/\mathcal{I}$ defines a UEA of V which we denote by $\mathcal{U}(V)$. As in (4.15), we recognize that the triple $(\mathcal{U}(V), \otimes; \mathbb{C})$ as an algebra structure and $(\mathcal{U}(V), \otimes, \mathbb{1}, \Delta, \epsilon, S; \mathbb{C})$ as the co-algebra structure, hence the UEA is HA. The Hopf algebras we shall be concerned with are UEA's of Lie algebras, i.e. V will be a Lie algebra. UEA's are useful because they contain all the representations of their underlying vector space, hence their 'universal' designation.

4.1.5 Properties of Hopf Algebras

4.1.5.1 Action of HAs

HAs have a plethora of interesting properties whose full exposition can fill volumes but we will highlight the ones pertinent to this work. What is worthy of note is that HAs can act on other mathematical 'sets' in the same way that regular groups in group theory act on others 'sets'. This means we can use H, a HA whose structure we know, to study the structure of 'set' A (and vice versa). The advantage here is that questions in/about A can be recast in terms of H and its properties (and vice versa). For this to work we must ascertain that the action of H on A preserves the structure of A. If 'set' A is an algebra isomorphic to an HA then the HA action (from the left), denoted by $\triangleright$, must satisfy

$$h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b) \quad \text{and} \quad h \triangleright \mathbb{1}_A = \epsilon(h)\mathbb{1}_A \tag{4.18}$$

and if 'set' A is a co-algebra then the (left) action of HA H on A must satisfy

$$\left.\begin{aligned} \epsilon(h \triangleright a) &= \epsilon(h)\epsilon(a) \text{ and} \\ \Delta(h \triangleright a) &= \sum (h_{(1)} \triangleright a_{(1)}) \otimes (h_{(2)} \triangleright a_{(2)}) \end{aligned}\right\} \forall\ h \in H \text{ and } \forall\ a \in A \tag{4.19}$$

The 'set' A is either called a (left) H-module algebra when (4.18) hold and A is an algebra or a (left) H-module co-algebra when (4.19) hold and A is a co-algebra.

4.1.5.2 Quasitriangularity

A HA H is called *co-commutative* if any element $h \in H$ satisfies

$$\Delta^{op}(h) = \Delta(h) \tag{4.20}$$

where $\Delta^{op} = \tau \circ \Delta$ and τ is a transposition map in a sense that $\tau(u \otimes v) = v \otimes u$. Thus HAs composed of only group-like elements are always co-commutative. What happens when co-commutativity does not hold, $\Delta^{op}(h) \neq \Delta(h)$? One can systematically relax (4.20) by proposing the existence of an invertible element $R \in H \otimes H$ which restores the equality in a sense that

$$\Delta^{op}(h) = R[\Delta(h)]R^{-1}\ \forall\ h \in H \tag{4.21}$$

and also obeys the *braidings*:

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23} \ , \ (\mathrm{id} \otimes \Delta)R = R_{13}R_{12} \tag{4.22}$$

The subscripts i, j in R_{ij} refer to the non-trivial sites of the vector space chain $H^{\otimes n}$ of length $n \in \mathbb{Z}^+$. If R does exist and satisfies (4.22) then the HA is said to be *quasi-triangular* and R is the quasitriangular structure. It is then clear that co-commutative HAs are trivially quasitriangular with $R = \mathbb{1} \otimes \mathbb{1}$. The quasitriangular structure, R, is actually the R-matrix we encountered before in Section 3.2; it solves the (quantum) Yang-Baxter equation [YBE]

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{4.23}$$

Mathematically the YBE arises from the fact that there are two equally valid ways of performing the operation $(\mathrm{id} \otimes \tau) \circ (\mathrm{id} \otimes \Delta)R$ and they must produce the same result ([71]) :

$$\begin{aligned}
(\mathrm{id} \otimes \tau) \circ (\mathrm{id} \otimes \Delta)R &= (\mathrm{id} \otimes \tau)[(\mathrm{id} \otimes \Delta)R] && (4.24)\\
&= (\mathrm{id} \otimes \tau)[R_{13}R_{12}] && (4.25)\\
&= R_{12}R_{13} && (4.26)\\
&\text{or} && \\
&= [(\mathrm{id} \otimes \tau \circ \Delta)R] && (4.27)\\
&= [(\mathrm{id} \otimes \Delta^{op})R] && (4.28)\\
&= R_{23}[(\mathrm{id} \otimes \Delta)R]R_{23}^{-1} && (4.29)\\
&= R_{23}R_{13}R_{12}R_{23}^{-1} && (4.30)
\end{aligned}$$

The relation (4.23) is used as a first simple test for signs of integrability and/or consistency. For a model whose S-matrix is factorizable[5] , the scattering of three particles is consistent if the S-matrix satisfies the YBE with R_{ij} replaced by S_{ij} the S-matrix element corresponding to the scattering of the i-th and j-th particles. By a consistent S-matrix we mean a many-body can reduced into many 2-body problems without the need to care for how the 2 bodies are chosen. This is usually represented by the YBE lattice diagram in Figure 4.4 where the intersection point of lines i and j represents R_{ij}. We have also seen the general form of (4.23), the spectral parameter dependent one, given by

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \tag{4.31}$$

[5]a scattering process involving N particles can be viewed as a sequence of 2-particle scattering

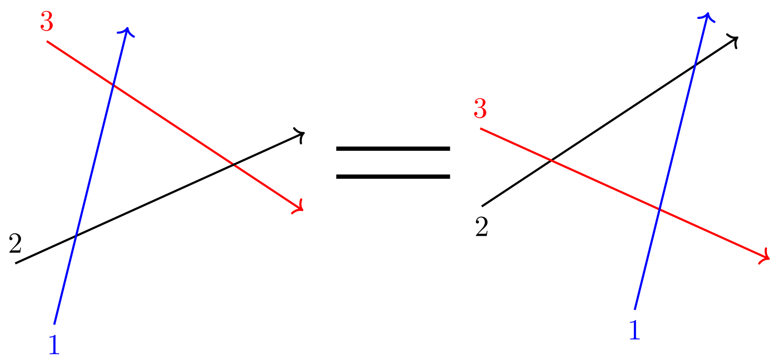


FIGURE 4.4: Diagram representing the Yang-Baxter equation

here u and v are additive spectral parameters. Momentum is an example of an additive spectral parameter. In this case if a process represented by R occurs so that (4.31) is satisfied then such a process is both consistent and preserves total momentum. For each choice of total momentum we obtain a new scenario where the process is consistent. This little detour is to help highlight the importance of the R-matrix and needless to say we will extensively use and exploit its connection to HAs and the study of integrability of models.

4.1.5.3 Twisting

Another property of HAs which will be useful is the notion of twisting HAs to obtain other HAs. Given a quasitriangular HA (H, R) one can construct a new quasitriangular HA provided there is an invertible element $F \in H \otimes H$ which obeys the relations

$$(F \otimes \mathbb{1}) \circ (\Delta \otimes \mathrm{id})F = (\mathrm{id} \otimes \Delta)F \circ (\mathbb{1} \otimes F) \tag{4.32}$$

$$\text{and } (\epsilon \otimes \mathrm{id})F = \mathbb{1} = (\mathrm{id} \otimes \epsilon)F \tag{4.33}$$

The new quasitriangular HA is $(H, \bullet, \eta, \Delta_F, \epsilon, S_F, R_F; k)$ where H is the underlying vector space and the co-product, antipode and quasitriangular structure are now defined as follows

$$\left.\begin{aligned} \Delta_F(h) &= F\big[\Delta(h)\big]F^{-1} \\ S_F(h) &= F\big[S(h)\big]F^{-1} \end{aligned}\right\} \ \forall\ h \in H \text{ and } R_F = F_{21}RF^{-1} \tag{4.34}$$

This method of generating new HAs from known ones is called *twisting* and the invertible element F, the *Drinfeld twist* [72]. Note that the algebra part of the HA is unchanged, only the co-algebra maps are affected by twisting. Fortunately for our purposes twisting preserves quasitriangularity since

$$\tau \circ \Delta_F(h) = \tau \circ \left(F\Delta(h)F^{-1}\right) = F_{21}\left[\Delta^{op}(h)\right]F^{-1} = F_{21}\left[R\Delta(h)R^{-1}\right]F_{21}^{-1} \tag{4.35}$$

$$= \left(F_{21}RF^{-1}\right)\left(F\Delta(h)F^{-1}\right)\left(FR^{-1}F_{21}^{-1}\right) = R_F\left[\Delta_F(h)\right]R_F^{-1} \tag{4.36}$$

Therefore if a quasitriangular Hopf algebra with quasitriangular structure R is Drinfeld twisted by F then the resulting twisted Hopf algebra will also be quasitriangular and its quasitriangular structure, R_F is

$$R_F = F_{21}RF^{-1} \tag{4.37}$$

This property of HAs is one of the essential ingredients used in this work because the types of HAs commonly found in physics are of the quasitriangular kind and twisting preserves this.

4.1.5.4 Quasitriangular HAs observed in Physics

Quasitriangular HAs are the type of HAs that are common in physics, being first observed in the context of quantum inverse scattering methods [QISM] to solve quantum integrable systems [73]. There they are known as quantum groups, a name whose source is clear in the light of the canonical quantization prescription. Recall that in classical mechanics the dynamics of a system can (in principle) be described by a Hamiltonian, $H\left(q(t), p(t), t\right)$, a function of time t and phase space coordinates: canonical positions $q(t)$ and momenta $p(t)$. This means the state of a system at time t corresponds to a point $(q(t), p(t))$ in phase space X. How the system advances from state to state is governed by Hamilton's equations quoted below:

$$\dot{q} \equiv \frac{dq}{dt} = \partial_p H = \{q, H\}_{P.B.} \tag{4.38}$$

$$\dot{p} \equiv \frac{dp}{dt} = -\partial_q H = \{p, H\}_{P.B.}\,. \tag{4.39}$$

Here $\{\cdot,\cdot\}_{P.B}$ is the *Poisson bracket*. The coordinates of phase space satisfy

$$\{q, p\}_{P.B.} = 1 \tag{4.40}$$

An observable $\mathcal{O}(q(t), p(t); t)$ is a function that belongs to $\mathcal{F}(X)$ – a space of functions that act on phase space X – and evolves with time according to

$$\frac{d}{dt}\mathcal{O} = \left(\frac{\partial \mathcal{O}}{\partial q}\right)\left(\frac{dq}{dt}\right) + \left(\frac{\partial \mathcal{O}}{\partial p}\right)\left(\frac{dp}{dt}\right) + \frac{\partial \mathcal{O}}{\partial t} \tag{4.41}$$

$$= \left(\frac{\partial \mathcal{O}}{\partial q}\right)\left(\frac{\partial H}{\partial p}\right) - \left(\frac{\partial \mathcal{O}}{\partial p}\right)\left(\frac{\partial H}{\partial q}\right) + \frac{\partial \mathcal{O}}{\partial t} \tag{4.42}$$

$$= \{\mathcal{O}, H\}_{P.B.} + \frac{\partial \mathcal{O}}{\partial t}. \tag{4.43}$$

And if the observable $\mathcal{O}$ does not explicitly depend on time then (4.43) reduces to

$$\frac{d}{dt}\mathcal{O} = \{\mathcal{O}, H\}_{P.B.}. \tag{4.44}$$

Hence $\{\mathcal{O}, H\}_{P.B}$ vanishes iff $\mathcal{O}(q(t), p(t))$ is conserved. The set of all $\mathcal{O}(q(t), p(t))$ that Poisson commute with the Hamiltonian, together with the Poisson bracket [6] constitute a Lie Algebra that describes the symmetries of the system. In canonical quantization the states are represented by vectors that live in a Hilbert space $\mathcal{H}$ which replaces phase space X. Observables $\mathcal{O}$ become operators $\widehat{\mathcal{O}}$ belonging to $Op(\mathcal{H})$, a space of operators that act on vectors from $\mathcal{H}$. The Poisson bracket is replaced with a commutator

$$\{\cdot, \cdot\}_{P.B.} \rightarrow -\frac{i}{\hbar}[\cdot, \cdot] \tag{4.45}$$

so that (4.40) becomes $[q, p] = i\hbar$ — the commutation relation for the Heisenberg algebra — and the quantum analogue of (4.44) is the Heisenberg equation of motion

$$i\hbar\frac{\partial}{\partial t}\widehat{\mathcal{O}} = [\widehat{\mathcal{O}}, \hat{H}] \tag{4.46}$$

The space, $Op(\mathcal{H})$, is made of operators $\mathcal{O}_i$ which generally do not commute but instead their commutativity is controlled by an operator, $R \in Op(\mathcal{H})^{\otimes 2}$, so that

$$\widehat{\mathcal{O}}_i\widehat{\mathcal{O}}_j = \widehat{R}_{i\,j}^{k\,l}\widehat{\mathcal{O}}_k\widehat{\mathcal{O}}_l \tag{4.47}$$

The relations (4.47) are the starting point for defining a quantum group/algebra [74]. equipped with the commutator is thus expected to defined a quantum version of a Lie group, hence the name quantum groups. Having discussed HAs, now is most opportune for the introduction of the notion of *quasi-Hopf Algebras* [qHAs] since they are generalizations of HAs.

[6] $\{.,.\}_{P.B.}$ satisfies all the axioms of a Lie algebra definition

4.2 quasi-Hopf Algebras

A quasi-Hopf algebra is in essence a Hopf algebra as defined above with the exception that the co-associativity condition (4.7) is now relaxed to

$$\big(\mathrm{id}\otimes\Delta\big)\circ\Delta(h)=\phi\big[\big(\Delta\otimes\mathrm{id}\big)\circ\Delta(h)\big]\phi^{-1}\ ,\ \forall\ h\in H \tag{4.48}$$

and the antipode is defined by the triple (S,α,β) whose details are described below [72]. Quasi-Hopf algebras are a natural progression from Hopf algebras, being the most general class of algebras that remain closed under arbitrary twisting [71][72]. Comparing (4.48) to (4.21) highlights that the role of ϕ is to control co-associativity just as R did co-commutativity. The object ϕ, known as a *co-associator*, belongs to $H\otimes H\otimes H$ and obeys

$$(\mathbb{1}\otimes\phi)\big[\mathrm{id}\otimes\Delta\otimes\mathrm{id})\phi\big](\phi\otimes\mathbb{1})=\big[(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)\phi\big]\big[(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\phi\big] \tag{4.49}$$

an equation known as the *pentagon relation*. This relation encodes the different placements of brackets on 4 objects, enclosing 2 objects. For instance, there are two possible pathways one can take when moving a left-justified bracketing of 4 letters to a right-justified bracketing

$$[(ab)c]d\longrightarrow a[b(cd)] \tag{4.50}$$

One pathway involves two steps and the other, three. The left and right hand side of (4.49) are the details of how to step-by-step perform bracketing for each pathway while exposing the role of the co-associator ϕ at each step. The diagrammatic representation of (4.49) is

$$\begin{tikzcd}
 & (a\otimes(b\otimes c))\otimes d \arrow[rr, "(\mathrm{id}\otimes\Delta\otimes\mathrm{id})\phi"] & & a\otimes((b\otimes c)\otimes d) \arrow[dr, "\mathbb{1}\otimes\phi"] & \\
((a\otimes b)\otimes c)\otimes d \arrow[ur, "\phi\otimes\mathbb{1}"] \arrow[drr, "(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\phi"'] & & & & a\otimes(b\otimes(c\otimes d)) \\
 & & (a\otimes b)\otimes(c\otimes d) \arrow[urr, "(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)\phi"'] & &
\end{tikzcd}$$

FIGURE 4.5: Role of the co-associator in the pentagon relation

What is worthy of highlight about the antipode of qHA is that it is now a triple (S, α, β) with α and β elements in H which together must satisfy

$$\bullet(\mathbb{1} \otimes \alpha)(S \otimes \mathrm{id}) \circ \Delta(h) = \epsilon(h)\alpha \quad \text{and} \tag{4.51}$$

$$\bullet(\beta \otimes \mathbb{1})(\mathrm{id} \otimes S) \circ \Delta(h) = \epsilon(h)\beta, \quad \forall\ h \in H \tag{4.52}$$

This is nothing more than the generalization of (4.11) implying that the compatibility of S with $\bullet$ in qHAs would fail unless α and β are supplied. In addition, the antipode triple must satisfy

$$\bullet \circ (\bullet \otimes \mathrm{id})\big[(\mathbb{1} \otimes \beta \otimes \alpha)(\mathrm{id} \otimes S \otimes \mathrm{id})\phi\big] = \mathbb{1} \tag{4.53}$$

$$\bullet \circ (\bullet \otimes \mathrm{id})\big[(\mathbb{1} \otimes \alpha \otimes \beta)(S \otimes \mathrm{id} \otimes S)\phi^{-1}\big] = \mathbb{1} \tag{4.54}$$

in order to ascertain compatibility with the new object ϕ. So $(H, \bullet, \eta, \Delta, \epsilon, \phi, S, \alpha, \beta; k)$ is a qHA which is co-commutative defined over a field k. qHAs, being generalizations of HAs, inherit properties reminiscent of HA and thus can also admit a quasitriangular structure. So we can consider quasitriangular qHAs which have quasitriangular structures R that satisfy (4.21). In this case the quasitriangular structure R satisfies generalized braiding condition given by

$$(\Delta \otimes \mathrm{id})R = \phi_{312}R_{13}\phi^{-1}_{132}R_{23}\phi_{123}\ , \ (\mathrm{id} \otimes \Delta)R = \phi^{-1}_{231}R_{13}\phi_{213}R_{12}\phi^{-1}_{123} \tag{4.55}$$

which results to a generalized YBE [71][61]

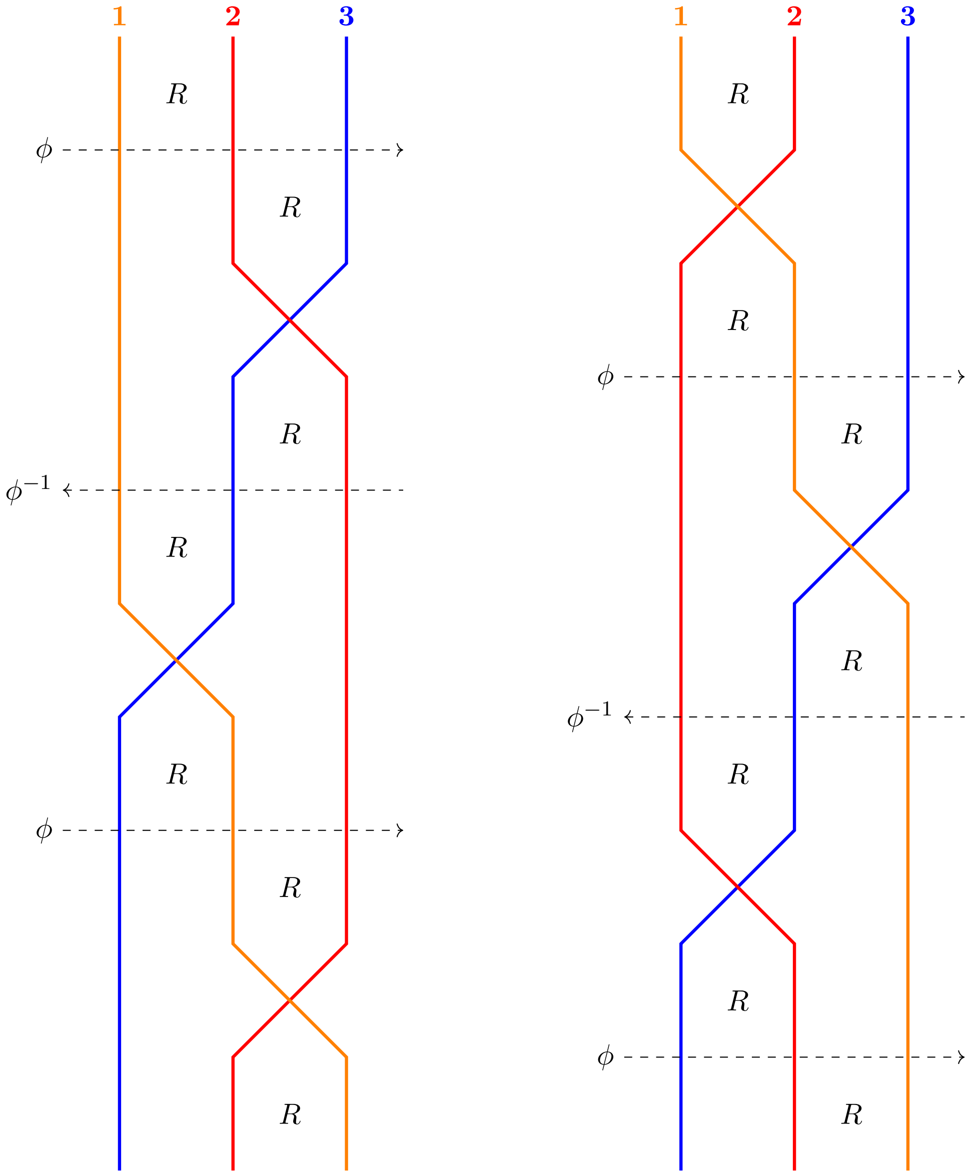
$$R_{12}\phi_{312}R_{13}\phi^{-1}_{132}R_{23}\phi_{123} = \phi_{321}R_{23}\phi^{-1}_{231}R_{13}\phi_{213}R_{12}. \tag{4.56}$$

For an intuitive feeling of this imagine that the LHS (or RHS) of (4.56) to be an "operator" which acts on a left-justified vector $(\mathbf{1} \otimes \mathbf{2}) \otimes \mathbf{3}$ from the left then its action is portrayed in Figure [4.6a] (or Figure [4.6b]). Figure [4.6] is read from top to bottom and the fact that R is initially between **1** and **2** indicates their association [left-justification], hence R acts trivially on **3** because R is a binary operator. The co-associator acts to change the association to where **2** and **3** are associated. The intersection of diagonal lines represents the action of R hence the first intersection on LHS is R_{23} while on the RHS it is R_{12}. The outcome of the LHS of (4.56) can be represented as an equation:

$$R_{12}\phi_{312}R_{13}\phi^{-1}_{132}R_{23}\phi_{123} \triangleright [(\mathbf{1} \otimes \mathbf{2}) \otimes \mathbf{3}] = \mathbf{3} \otimes (\mathbf{2} \otimes \mathbf{1}) \tag{4.57}$$

and the outcome of the RHS is given by:

$$\phi_{321}R_{23}\phi^{-1}_{231}R_{13}\phi_{213}R_{12} \triangleright [(\mathbf{1} \otimes \mathbf{2}) \otimes \mathbf{3}] = \mathbf{3} \otimes (\mathbf{2} \otimes \mathbf{1}) \tag{4.58}$$



(A) LHS of quasi-YBE

(B) RHS of quasi-YBE

FIGURE 4.6: Graphical representation of the quasi-YBE

qHAs also have the notion of twisting, albeit a generalized form. A given quasitriangular qHA H defined by the tuple $(H, \Delta, \epsilon, R, \phi, S, \alpha, \beta; k)$[7] can be twisted to obtain a new qHA H_F given by $(H, \Delta_F, \epsilon, R_F, \phi_F, S_F, \alpha_F, \beta_F; k)$ provided we can find an invertible $F \in H^{\otimes 2}$ for which $(\epsilon \otimes \mathrm{id})F = 1 = (\mathrm{id} \otimes \epsilon)F$ holds. The twisted structures are given by

$$\Delta_F(h) = F\big(\Delta(h)\big)F^{-1} \; , \; R_F = F_{21} R F^{-1} \tag{4.59}$$

$$\alpha_F = \bullet\big[(S \otimes \mathrm{id})(\mathbb{1} \otimes \alpha)F^{-1}\big] \; , \; \beta_F = \bullet\big[(\mathrm{id} \otimes S)(\mathbb{1} \otimes \beta)F\big] \tag{4.60}$$

$$\phi_F = F_{23}\big[(\mathrm{id} \otimes \Delta)F\big]\phi\big[(\Delta \otimes \mathrm{id})F^{-1}\big]F_{12}^{-1} \tag{4.61}$$

[7]For brevity we have suppressed the algebra part of the qHA since it is unaffected by twisting.

and only they are affected, thus everything else in H_F is the same as in the untwisted H i.e. $\epsilon, S_F = S$. We highlight that the twisted co-associator ϕ_F is defined using the untwisted co-product and also note that in the quasi-Hopf algebra setting the twist F need not be a 2-cocyle as was the case in the Hopf algebra setting.

We do acknowledge that, on the surface, the objects discussed in this section may seem to have nothing to do with physics, thus appealing only to mathematicians but the remaining sections are devoted to establishing their connection to the study of physics. This much is sufficient HA theory for the reader to appreciate the goal of the present work.

Chapter 5

The global quasi-Hopf symmetry in $\mathcal{N} = 4$ SYM

The intention in this chapter is to first associate a quasi-Hopf algebraic structure to $\mathcal{N} = 4$ SYM and then Drinfeld twist the said structure to arrive at the Leigh-Strassler deformations which were constructed in [49] and whose planar integrability was studied in [60]. These marginal deformations only affect the superpotential of $\mathcal{N} = 4$ SYM which means they will affect the internal $SO(6)$-symmetry which the 6 real-scalar fields ϕ^i possess with $i = 1, \ldots, 6$ and as we pointed out earlier, $SO(6) \simeq SU(4)$ [37]. Using the $\mathcal{N} = 1$ superspace formalism described in Section 2.3.3 we can express the $\mathcal{N} = 4$ supermultiplet in terms of $1 \times (\mathcal{N} = 1)$ gauge supermultiplet and $3 \times (\mathcal{N} = 1)$ chiral supermultiplets. The complex scalar component fields, φ^i, of the chiral superfields Φ^i corresponding to each of the 3 chiral supermultiplets are obtained by joining the real scalar fields ϕ^i's by pairs, i.e. $\varphi^j = \frac{1}{\sqrt{2}}(\phi^j + i\phi^{j+3})$ for $j = 1, 2, 3$ [75]. In this formalism the superpotential takes the form

$$\mathcal{W}_{\mathcal{N}=4} = g\mathrm{Tr}\left(\Phi^1[\Phi^2, \Phi^3]\right) \tag{5.1}$$

where g is the coupling constant, the Φ^i's are chiral superfields and now only the $SU(3) \times U(1)_R$ subgroup of the $SU(4)$ R-symmetry is explicitly manifest. For comparison we recall from Section 3.2 that the Leigh-Strassler deformed superpotential in $\mathcal{N} = 1$ superspace language is given by

$$\mathcal{W}_{LS} = \kappa\mathrm{Tr}\left[\Phi^1[\Phi^2, \Phi^3]_q + \frac{h}{3}\left(\sum_{i=1}^{3}(\Phi^i)^3\right)\right] \tag{5.2}$$

where $[X, Y]_q = XY - qYX$ is a deformed commutator and q and h are deformation parameters which can be complex. It is easy to see that $\mathcal{W}_{LS}$ remains invariant under

$U(1)$ transformations but, in general, not under $SU(3)$ transformations except for some specific values of q and h.

5.1 quasi-Hopf structure of $\mathcal{N}=4$ SYM

We shall focus on the $SU(3) =: H$ sector of $\mathcal{N}=4$ SYM as the vector space that constitutes the algebra part of the quasi-Hopf algebra structure in the sense of Chapter 4. Matrix multiplication is the group multiplication of $SU(3)$ and here it will serve as the multiplication map $\bullet : H \otimes H \to H$ and $\mathbb{C}$ will be the field over which vector space H is defined. We can also choose the unit map η to be $\eta(1) \to \mathbb{1}_H$. We know that scalar multiplication is compatible with matrix multiplication, so the field structure of $\mathbb{C}$ is compatible with the vector space structure of H. Thus half the work is done, all that remains is to associate an accompanying co-algebra structure to H.

Before we do so it is best to mention that in the literature of quasi-Hopf algebras there is usually no need to make a clear distinction between group and algebra. Since some maps are easier defined at the group level while others at the algebra level, we will seek to make that as clear as possible. Thus $SU(3) =: H$ will refer to the group while $\mathfrak{su}(3) =: \mathfrak{h}$ to the underlying algebra. This will hopefully make clear the level at which each map is defined. For the co-product we use the symmetric map

$$\Delta(a) = a \otimes \mathbb{1} + \mathbb{1} \otimes a \ , \ \forall \ a \in \mathfrak{h} \ \text{ and } \ \Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1} \tag{5.3}$$

Here we have defined the co-product at the algebra level hence the exponentiation of its result at this level elevates to the group level. In order to apply this co-product on a group element U then U must first be expressed as an exponential of a linear combination of the generators, λ_i, of $\mathfrak{h}$

$$U = \exp(c^i \lambda_i) \equiv \sum_{n=0} \frac{1}{n!} \left(c^i \lambda_i\right)^n \tag{5.4}$$

with implied summation for some $\{c^i\} \subset \mathbb{C}$. By recollecting the result of the action of Δ at the algebra level, we are then able to see its effect at the group level. There is no guarantee that this will result in a closed expression. The antipode, being a triple (S, α, β) is easy to define at either level, but we will prefer to do so at the algebra level. We define $S : \mathfrak{h} \to \mathfrak{h}$ as

$$S(a) = -a \ , \ \forall \ a \in \mathfrak{h} \tag{5.5}$$

so that at the level of the group, the antipode S maps an element e^a to its inverse e^{-a}. To complete the definition of our antipode we set $\beta = \mathbb{1} = \alpha$. This is possible only because for $\mathcal{N}=4$ SYM the co-associator is $\mathbb{1}^3$. In general we will not be able to simultaneously set α and β to $\mathbb{1}$ because α and β need to also satisfy (4.53) [71]. So in the general case we will set $\beta = \mathbb{1}$ and this will restrict our choice of α to

$$\alpha = [\bullet(\mathrm{id} \otimes S \otimes \mathrm{id})\phi]^{-1} = \left[\phi^{(1)}(S(\phi^{(2)})\phi^{(3)})\right]^{-1} \tag{5.6}$$

Next we employ the unit determinant of any element of H to define a co-unit map $\epsilon : H \to \mathbb{C}$ as $\epsilon(U) := \det(U)$,$\forall\ U \in H$. The R-matrix corresponding to the scalar sector of $\mathcal{N} = 4$ SYM is a 9×9 identity matrix, $R_{\mathbb{1}}$, which trivially satisfies the braiding relations (4.22). Putting everything together we conclude that $(H, \Delta, \epsilon, S, R_{\mathbb{1}})$ is the quasitriangular Hopf algebra structure associated with $\mathcal{N} = 4$ SYM. Recall that Hopf algebras are trivially quasi-Hopf with an identity for a co-associator since Δ is co-associative.

5.2 Twisting $\mathcal{N} = 4$ SYM

In this section we consider the Leigh-Strassler deformations of $\mathcal{N} = 4$ SYM and find the twist which performs the deformation. To make our approach as transparent as possible, we first focus our discussion on a specific deformation and then generalize it to the full Leigh-Strassler deformations. This is due to the fact that the expressions are cumbersome and more so for the full Leigh-Strassler deformations. Our treatment will focus on twisting R-matrices at the quantum limit, so there will be no spectral parameter. The spectral parameter can easily be restored and in any case the Drinfeld twist only twist the identity part of the spectral parameter dependent R-matrix. We shall demonstrate this fact in Chapter 7.

5.2.1 The twist for the real β-deformation

For now we start with the real β-deformation of $\mathcal{N} = 4$ SYM, a model that has been studied much in the context of the AdS-CFT correspondence. It is among the earliest tests of the correspondence, being the first example of a field theory not maximally symmetric to have a known gravity dual which has geometric interpretation [52]. It is not entirely clear if every deformation has a string theory interpretation under the AdS-CFT correspondence. It turns out that for deformations which are achievable by Drinfeld twisting more can be said about the deformed theory. It is known that abelian Drinfeld twists give rise to deformed gauge theories that have a gravity dual with a geometric interpretation. A Drinfeld twist F is said to be abelian if it can be expressed as

$$F = e^{-ir} \text{ where } r = \alpha^{ij} a_i \wedge a_j \tag{5.7}$$

where r is the classical R-matrix and a_i's are Cartan generators of (the sub-sector of) the theory. The real β-deformation of the scalar sector of $\mathcal{N}=4$ SYM can be performed by Drinfeld twisting the R-matrix which corresponds to the $SU(3)$ sector of $\mathcal{N}=4$ SYM. The Drinfeld twist suitable to execute the real β-deformation was found to be abelian, being expressed in terms of the $SU(3)$ Cartan generators [76]. Thus the real β-deformed $\mathcal{N}=4$ SYM has a geometric gravity dual. The key to this connection is that abelian Drinfeld twists are defined in terms of classical r-matrices where classical r-matrices are solutions to the equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{5.8}$$

which is known as the classical Yang-Baxter Equation [CYBE]. In (5.8) a lowercase r is used to denote the classical R-matrix in order to distinguish it from the quantum R-matrix. Twists that are defined in terms of classical R-matrices have been shown to be TsT transformations in disguise [77] [78]. The R-matrix for the real β-deformed theory is given by

$$R_{\beta_r} = \mathrm{diag}(1, q^{-1}, q, q, 1, q^{-1}, q^{-1}, q, 1) \tag{5.9}$$

where $q = \exp(i\beta)$ with $\beta \in \mathbb{R}$. By Taylor expanding this R-matrix we can isolate the first-order term. This term is the real β deformed classical R-matrix and it can used to construct an abelian Drinfeld twist according to (5.7). In [79] an equivalent theory known as the w-deformed theory was shown to have a gravity dual using the languages of Hopf algebras and generalized geometry [80]. The algebraic structure of the w-deformed theory turned out to be a (quasitriangular) Hopf algebra rather than a (quasitriangular) quasi-Hopf algebra. This is to say the abelian twist produced a co-associative co-product, hence the co-associator is trivial.

5.2.2 The twist for the imaginary β-deformation

Next we consider the imaginary β-deformation as the pilot model on which we can show explicit results of qHA treatment since it is not feasible for the full Leigh-Strassler theory. The R-matrix of the imaginary β-deformed theory is obtained by setting $h = 0 = \bar{h}$ and

$\bar{q} = q$ in (3.26), and the result is

$$R_{\beta_i} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2q}{q^2+1} & 0 & \frac{1-q^2}{q^2+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2q}{q^2+1} & 0 & 0 & 0 & \frac{q^2-1}{q^2+1} & 0 & 0 \\ 0 & \frac{q^2-1}{q^2+1} & 0 & \frac{2q}{q^2+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2q}{q^2+1} & 0 & \frac{1-q^2}{q^2+1} & 0 \\ 0 & 0 & \frac{1-q^2}{q^2+1} & 0 & 0 & 0 & \frac{2q}{q^2+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q^2-1}{q^2+1} & 0 & \frac{2q}{q^2+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.10}$$

here q is a number given by $q = e^{i\beta}$ and it is real since β is imaginary. By calculation, we noted that R_{β_i} is a triangular R-matrix because it satisfies the triangular relation:

$$R_{21} = \tau(R_{12}) = (R_{12})^{-1} \tag{5.11}$$

$R_{\mathbb{1}}$ is triangular and the fact that R_{β_i} is also triangular may serve as evidence to the existence of a suitable Drinfeld twist because twisting preserves triangularity [71].

The objective is to demonstrate that there is a Drinfeld twist, F_{β_i}, which deforms the R-matrix of $\mathcal{N} = 4$ SYM into R_{β_i} in accord with the Drinfeld twisting prescription (4.34). The R-matrix for $\mathcal{N} = 4$ SYM is

$$R_{\mathbb{1}} = \mathbb{1}_3 \otimes \mathbb{1}_3 \tag{5.12}$$

and the existence of a twist, F_{β_i}, that performs the imaginary β-deformation amounts to

$$R_{\beta_i} = (F_{\beta_i})_{21} R_{\mathbb{1}} F_{\beta_i}^{-1} = (F_{\beta_i})_{21} F_{\beta_i}^{-1} \tag{5.13}$$

The features of the twist are encoded onto the R-matrix, R_{β_i}, it being constructed according to (5.13), thus in order to find the twist we highlight a few properties of R_{β_i} and then use them as a guide to calculate the twist. Firstly note that R_{β_i} is orthogonal and has unit determinant. We also shall impose that F_{β_i} also have unit determinant.

The fact that R_{β_i} is orthogonal complies with the twist also being orthogonal because

$$R_{\beta_i}.R_{\beta_i}^T = \mathbb{1} \tag{5.14}$$

$$\left[(F_{\beta_i})_{21}F_{\beta_i}^{-1}\right].\left[(F_{\beta_i})_{21}F_{\beta_i}^{-1}\right]^T = \mathbb{1} \tag{5.15}$$

$$\left[(F_{\beta_i})_{21}F_{\beta_i}^{-1}\right].\left[(F_{\beta_i}^{-1})^T(F_{\beta_i})_{21}^T\right] = \mathbb{1} \tag{5.16}$$

The final equality holds if $(F_{\beta_i}^{-1})^{-1} = (F_{\beta_i}^{-1})^T$. It is not clear if this condition is necessary but it is sufficient that we impose it as a simplifying condition. We can conclude then that F_{β_i} can be written as exponential of a skew-symmetric matrix, a fact which will prove to be useful later.

Another property of interest is that the entries of R_{β_i} are related to one another by $\mathbb{Z}_3$-symmetry. Recall that the R-matrix acts on a basis given by:

$$\{|1\ 1\rangle\,,|1\ 2\rangle\,,|1\ 3\rangle\,,|2\ 1\rangle\,,|2\ 2\rangle\,,|2\ 3\rangle\,,|3\ 1\rangle\,,|3\ 2\rangle\,,|3\ 3\rangle\}$$

then the $\mathbb{Z}_3$-symmetry of the entries of R_{β_i} means $[R_{\beta_i}]^{i\ j}_{k\ l} = [R_{\beta_i}]^{i+1\ j+1}_{k+1\ l+1}$ for $i,j,k,l \in \{1,2,3\}$. After imposing these properties as constraints on the twist we obtained the following:

$$F_{\beta_i} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{q+1}{\sqrt{2}\sqrt{q^2+1}} & 0 & \frac{q-1}{\sqrt{2}\sqrt{q^2+1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{q+1}{\sqrt{2}\sqrt{q^2+1}} & 0 & 0 & 0 & -\frac{q-1}{\sqrt{2}\sqrt{q^2+1}} & 0 & 0 \\ 0 & -\frac{q-1}{\sqrt{2}\sqrt{q^2+1}} & 0 & \frac{q+1}{\sqrt{2}\sqrt{q^2+1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q+1}{\sqrt{2}\sqrt{q^2+1}} & 0 & \frac{q-1}{\sqrt{2}\sqrt{q^2+1}} & 0 \\ 0 & 0 & \frac{q-1}{\sqrt{2}\sqrt{q^2+1}} & 0 & 0 & 0 & \frac{q+1}{\sqrt{2}\sqrt{q^2+1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{q-1}{\sqrt{2}\sqrt{q^2+1}} & 0 & \frac{q+1}{\sqrt{2}\sqrt{q^2+1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.17}$$

This twist is orthogonal and has unit determinant from construction but in addition the twist is triangular, a property we did not insist on. Its presence means that the imaginary beta deformed R-matrix can be simplified to $R_{\beta_i} = (F_{\beta_i}^{-1})^2$. Thus we have a Drinfeld twist which deforms the $\mathcal{N} = 4$ SYM R-matrix, $R_{\mathbb{1}}$, to the imaginary β-deformed one R_{β_i}. Furthermore this twist is NOT a 2-cocycle (4.32). In order to prove this we first must write it as an exponential. In principle one can Taylor expand the twist in orders of the q parameter as follows

$$F_{\beta_i} = \mathbb{1} + q f^{(1)}_{\beta_i} + \frac{q^2}{2} f^{(2)}_{\beta_i} + \mathcal{O}(q^3) + \dots \tag{5.18}$$

and use the first-order term, $f^{(1)}_{\beta_i}$ – which we call the *classical twist*– as an ansatz from which to construct an exponential form of the twist. We will often use f_{β_i} to denote the classical twist, suppressing the label for the order in expansion. This means the twist can then be written as

$$F_{\beta_i} = e^{i\alpha_{\beta_i} f_{\beta_i}} \tag{5.19}$$

where α_{β_i} was *a priori* an unknown function of q which we later determined to be

$$\alpha_{\beta_i} = \arccos\left(\frac{q+1}{\sqrt{2}\sqrt{q^2+1}}\right) \tag{5.20}$$

The classical twist here is given by

$$f_{\beta_i} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.21}$$

We can go a step further and express f_{β_i} in a more useful form, which form is in terms of the generators of $\mathfrak{su}(3)$. The classical twist is now

$$f_{\beta_i} = -\frac{1}{2}\left(\lambda_1 \wedge \lambda_2 + \lambda_5 \wedge \lambda_4 + \lambda_6 \wedge \lambda_7\right) \tag{5.22}$$

The generators of $\mathfrak{su}(3)$ known as the Gell-Mann matrices are :

$$\begin{aligned} &\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \end{aligned} \tag{5.23}$$

and in our convention the wedge is understood as $A \wedge B = A \otimes B - B \otimes A$. On this form we can apply the co-product on the twist, compute $(\Delta \otimes \mathrm{id})F_{\beta_i}$ and $(\mathrm{id} \otimes \Delta)F_{\beta_i}$ and then confirm that

$$F_{12}.(\Delta \otimes \mathrm{id})F_{\beta_i} \neq (\mathrm{id} \otimes \Delta)F_{\beta_i}.F_{23} \tag{5.24}$$

This ascertains that we are indeed in a quasi-Hopf algebra setting rather than the Hopf algebra one, otherwise there would be no need to compute the co-associator as it would be trivial. According to Drinfeld [72] the twisted co-associator is given by

$$\phi_{F_{\beta_i}} = (F_{\beta_i})_{23} \cdot [(\mathrm{id} \otimes \Delta)\, F_{\beta_i}] \cdot \left[(\Delta \otimes \mathrm{id})F_{\beta_i}^{-1}\right] \cdot (F_{\beta_i}^{-1})_{12} \tag{5.25}$$

$$\phi_{F_{\beta_i}}^{-1} = (F_{\beta_i})_{12} \cdot [(\Delta \otimes \mathrm{id})F_{\beta_i}] \cdot \left[(\mathrm{id} \otimes \Delta)\, F_{\beta_i}^{-1}\right] \cdot (F_{\beta_i}^{-1})_{23} \tag{5.26}$$

and functions to re-associate different copies of some module, $\mathcal{A}$, of the qHA, that is, it maps $\phi_{F_{\beta_i}} : (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} \to \mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A})$ and its inverse does the opposite map $\phi_{F_{\beta_i}}^{-1} : \mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A}) \to (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A}$. As it shall be clear in the subsequent chapters, the action of qHA on a vector space $\mathcal{A}$ will induce non-associativity on the product structure of $\mathcal{A}$. This is a manifestation of the non-associativity of the qHA.

Unfortunately the co-associator, being a 27×27 matrix with cumbersome expressions for entries, cannot be presented in the usual array notation. So we rather refer the reader to the Mathematica notebook associated with [11] in which it was calculated and to Appendix [B.1] where some entries are quoted. See Appendix [A.3] for the details involved in computing the co-associator. The imaginary β-twisted quasi-Hopf structure is thus

$$\mathcal{H}_{\beta_i} := \left(H, \Delta_{F_{\beta_i}}, \epsilon, R_{\beta_i}, \phi_{F_{\beta_i}}, S_{F_{\beta_i}}, \alpha_{F_{\beta_i}}, \beta_{F_{\beta_i}}; \mathbb{C}\right) \tag{5.27}$$

with the twisting performed in accord with the prescription in (4.59).

5.2.3 Twist for the general Leigh-Strassler Deformation

In the same fashion as in the imaginary β-deformed case we will compute the twist for the full Leigh-Strassler deformation. We begin with the R-matrix that corresponds to the general Leigh-Strassler deformation of $\mathcal{N} = 4$ SYM which is

$$R_{qh} = \frac{1}{2d^2}\begin{bmatrix} t_1 & 0 & 0 & 0 & 0 & -2\bar{h} & 0 & 2\bar{h}q & 0 \\ 0 & 2\bar{q} & 0 & t_3 & 0 & 0 & 0 & 0 & 2h\bar{q} \\ 0 & 0 & 2q & 0 & -2h & 0 & t_2 & 0 & 0 \\ 0 & t_2 & 0 & 2q & 0 & 0 & 0 & 0 & -2h \\ 0 & 0 & 2\bar{h}q & 0 & t_1 & 0 & -2\bar{h} & 0 & 0 \\ 2h\bar{q} & 0 & 0 & 0 & 0 & 2\bar{q} & 0 & t_3 & 0 \\ 0 & 0 & t_3 & 0 & 2h\bar{q} & 0 & 2\bar{q} & 0 & 0 \\ -2h & 0 & 0 & 0 & 0 & t_2 & 0 & 2q & 0 \\ 0 & -2\bar{h} & 0 & 2\bar{h}q & 0 & 0 & 0 & 0 & t_1 \end{bmatrix} \tag{5.28}$$

with $d^2 = (1+q\bar{q}+h\bar{h})/2$, $t_1 = 1-h\bar{h}+q\bar{q}$, $t_2 = -1+h\bar{h}+q\bar{q}$ and $t_3 = 1+h\bar{h}-q\bar{q}$ [1]. R_{qh} depends on complex deformation parameters (q, h) along with their complex conjugates $\bar{q}$ and $\bar{h}$. We will generally treat the deformation parameters and their complex conjugates as independent. In this case when $q = 1 = \bar{q}$ and $h = 0 = \bar{h}$ we recover $R_{\mathbb{1}}$ and R_{β_i} when $h = 0 = \bar{h}$ with $q = \bar{q}$.

To find the twist, F_{qh}, that performs the deformation

$$R_{qh} = (F_{qh})_{21} \cdot R_{\mathbb{1}} \cdot (F_{qh})^{-1} = (F_{qh})_{21} \cdot F_{qh}^{-1} \tag{5.29}$$

we first note that R_{qh} is unitary with unit determinant, where unitarity means

$$R_{qh}^{-1} = \left[(F_{qh})_{21} \cdot F_{qh}^{-1}\right]^{\dagger} = \left[(F_{qh}^{-1})^{\dagger}\right] \cdot [(F_{qh})_{21}]^{\dagger} \tag{5.30}$$

again is imposed at the level of the twist for simplification. As it was in the case of imaginary β-deformation, so also in the (q, h)-deformation case: the entries of R_{qh} enjoy a $\mathbb{Z}_3$-symmetry. The twist that solves (5.29) after imposing the above conditions is

$$F_{qh} = \begin{bmatrix} a & 0 & 0 & 0 & 0 & e & 0 & f & 0 \\ 0 & b & 0 & c & 0 & 0 & 0 & 0 & g \\ 0 & 0 & i & 0 & j & 0 & d & 0 & 0 \\ 0 & d & 0 & i & 0 & 0 & 0 & 0 & j \\ 0 & 0 & f & 0 & a & 0 & e & 0 & 0 \\ g & 0 & 0 & 0 & 0 & b & 0 & c & 0 \\ 0 & 0 & c & 0 & g & 0 & b & 0 & 0 \\ j & 0 & 0 & 0 & 0 & d & 0 & i & 0 \\ 0 & e & 0 & f & 0 & 0 & 0 & 0 & a \end{bmatrix} \tag{5.31}$$

[1] The functions t_i have been introduced for brevity of the notation

The entries of F_{qh} are functions of q and h (and c.c.) whose explicit forms are supplied in Appendix [A.2]. Thus far we have two ingredients of a qHA: the quasitriangular structure R_{qh} and the invertible twist F_{qh}. In order to construct the rest we need to apply the co-product on the twist. The next objective is to write the twist as an exponential of a combination of algebra generators because the co-product action is defined over algebra generators. Begin with the ansatz

$$F_{qh} = e^{f_{qh}} \tag{5.32}$$

where f_{qh} is a linear combination of generators. The unitarity of F_{qh} guarantees that such an expression exists and the matrix function f_{qh} must be anti-hermitian and contains all the dependence on q and h (and c.c.) . The task at hand is to compute f_{qh} and to do so it useful to first note that F_{qh} smoothly reduces to unity when we take the undeformed limit: $q = 1 = \bar{q}$ (alternatively $\beta = 0$) and $h = 0 = \bar{h}$. By Taylor expanding F_{qh} (on the RHS of (5.32)) around the undeformed limit one can isolate the first-order terms in the respective deformation parameter limits and use them to reconstruct the appropriate form of f_{qh}. Since there are 4 deformation parameters – q,$\bar{q}$,h and $\bar{h}$– we then expect that a minimum of four functions is required to recover f_{qh}. And in order to obtain these functions we use the four different well-defined deformation limits, namely:

1. real β (β_r), $\bar{q} = 1/q$
2. imaginary β (β_i), $\bar{q} = q$
3. real h (h_r), $\bar{h} = h$ and
4. imaginary h (h_i) , $\bar{h} = -h$.

The outcome of this procedure is that f_{qh} can be written as

$$f_{qh} = i\left(\alpha_{\beta_r} f_{\beta_r} + \alpha_{\beta_i} f_{\beta_i} + \alpha_{h_r} f_{h_r} + \alpha_{h_i} f_{h_i}\right) \tag{5.33}$$

where the α's are implicitly functions of q and h (and c.c.) labelled by the limit at which they were obtained and their forms are presented below:

$$\begin{aligned}
\alpha_{\beta_r} &= \frac{i\left(b-\bar{b}\right)(-1+c)\,r}{\sqrt{1+b-c}\sqrt{1+\bar{b}-c}\sqrt{-b\bar{b}+(1-c)(3-b-\bar{b}+c)}}\\
\alpha_{\beta_i} &= \frac{(b\bar{b}-(-1+c)^2)r}{\sqrt{1+b-c}\sqrt{1+\bar{b}-c}\sqrt{-b\bar{b}+(1-c)(3-b-\bar{b}+c)}}\\
\alpha_{h_r} &= -\frac{\left[(1+b-c)f+(1+\bar{b}-c)\bar{f}\right]r}{2\sqrt{1+b-c}\sqrt{1+\bar{b}-c}\sqrt{-b\bar{b}+(1-c)(3-b-\bar{b}+c)}}\\
\alpha_{h_i} &= -\frac{i\left[(1+b-c)f-(1+\bar{b}-c)\bar{f}\right]r}{2\sqrt{1+b-c}\sqrt{1+\bar{b}-c}\sqrt{-b\bar{b}+(1-c)(3-b-\bar{b}+c)}} \qquad (5.34)\\
\text{and } r &= \cos^{-1}\left[\frac{1}{2}\left(b+\bar{b}+\frac{f\bar{f}}{c-1}\right)\right].
\end{aligned}$$

The f's with subscripts are 9×9 matrices whose explicit form appears in Appendix [A.1]. At present we are interested in their decomposition in terms of the Gell-Mann $\mathfrak{su}(3)\times\mathfrak{su}(3)$ basis

$$\begin{aligned}
f_{\beta_r} &= \frac{\sqrt{3}}{2}\left(\lambda_3\wedge\lambda_8\right)\\
f_{h_r} &= \frac{1}{2}\left(\lambda_1\wedge\lambda_5+\lambda_7\wedge\lambda_1+\lambda_2\wedge\lambda_4+\lambda_2\wedge\lambda_6+\lambda_7\wedge\lambda_4+\lambda_6\wedge\lambda_5\right) \qquad (5.35)\\
f_{\beta_i} &= -\frac{1}{2}\left(\lambda_1\wedge\lambda_2+\lambda_5\wedge\lambda_4+\lambda_6\wedge\lambda_7\right)\\
f_{h_i} &= \frac{1}{2}\left(\lambda_4\wedge\lambda_1+\lambda_1\wedge\lambda_6+\lambda_2\wedge\lambda_5+\lambda_2\wedge\lambda_7+\lambda_6\wedge\lambda_4+\lambda_5\wedge\lambda_7\right) \qquad (5.36)
\end{aligned}$$

The λ's are the Gell-Mann 3×3 matrices in (5.23). We are now in the position to discuss the action of the co-product on F_{qh}. Recall from (4.9) that one of the implications of the compatibility of the algebra and co-algebra structures of a qHA H is

$$\Delta(a\bullet b)=\Delta(a)\bullet\Delta(b) \text{ where } a,b\in H. \qquad (5.37)$$

Employing this property allows us to compute the action of the co-product on the twist by first performing a series expansion of the twist

$$F_{qh}=\mathrm{e}^{f_{qh}}=\sum_{n=0}^{\infty}\frac{(f_{qh})^n}{n!} \qquad (5.38)$$

and then apply the co-product to each term and finally recollect the outcome into an exponential form again to obtain

$$\begin{aligned}
(\Delta\otimes\mathrm{id})F_{qh} &= (\Delta\otimes\mathrm{id})(\mathbb{1}\otimes\mathbb{1})+(\Delta\otimes\mathrm{id})(f_{qh})+\frac{1}{2}(\Delta\otimes\mathrm{id})(f_{qh}\cdot f_{qh})+\ \ldots \qquad (5.39)\\
&= e^{\left[(\Delta\otimes\mathrm{id})f_{qh}\right]}. \qquad (5.40)
\end{aligned}$$

By the same argument $(\mathrm{id} \otimes \Delta)F_{qh} = \exp\left[(\mathrm{id} \otimes \Delta)f_{qh}\right]$ is obtained. To reduce clutter, we suppress the qh subscript and define $f_{12} := f_{qh} \otimes \mathbb{1}$, $f_{23} := \mathbb{1} \otimes f_{qh}$ and $f_{13} := P_{23} \cdot f_{12} \cdot P_{23} \equiv P_{12} \cdot f_{23} \cdot P_{12}$ where P_{ij}'s are permutation matrices that interchange the i-th and j-th positions/spaces. In this notation we have

$$(\Delta \otimes \mathrm{id})F_{qh} = \mathrm{e}^{(f_{23}+f_{13})} \qquad (\mathrm{id} \otimes \Delta)F_{qh} = \mathrm{e}^{(f_{13}+f_{12})} \tag{5.41}$$

$$(\Delta \otimes \mathrm{id})F^{-1}_{qh} = \mathrm{e}^{-(f_{23}+f_{13})} \qquad (\mathrm{id} \otimes \Delta)F^{-1}_{qh} = \mathrm{e}^{-(f_{13}+f_{12})} \tag{5.42}$$

which means the co-associator, ϕ and its inverse, ϕ^{-1} are given by

$$\phi_{123} = F_{23} \cdot \left[(\mathrm{id} \otimes \Delta)F_{qh}\right] \cdot \left[(\Delta \otimes \mathrm{id})F^{-1}_{qh}\right] \cdot F^{-1}_{12} \tag{5.43}$$

$$= \mathrm{e}^{f_{23}} \cdot \left[\mathrm{e}^{(f_{13}+f_{12})}\right] \cdot \left[\mathrm{e}^{-(f_{13}+f_{23})}\right] \cdot \mathrm{e}^{-f_{12}} \tag{5.44}$$

$$\phi^{-1}_{123} = \mathrm{e}^{f_{12}} \cdot \left[\mathrm{e}^{(f_{13}+f_{23})}\right] \cdot \left[\mathrm{e}^{-(f_{13}+f_{12})}\right] \cdot \mathrm{e}^{-f_{23}} \tag{5.45}$$

In this form it becomes apparent that a co-associator obtained by Drinfeld twisting will satisfy the pentagon relation (4.49) (See Appendix [B.2]), braiding conditions (4.55) and consequently the generalized YBE (4.56). The latter was also verified by explicit computer calculation [11]. Thus the qHA structure of the qh-deformed theory is

$$\mathcal{H}_{qh} := \left(H, \Delta_{F_{qh}}, \epsilon, R_{qh}, S, \alpha_{F_{qh}}, \beta_{F_{qh}}, \phi_{F_{qh}}\right) \tag{5.46}$$

In summary we conclude that F_{qh} is an appropriate twist that deforms the trivial qHA structure of $\mathcal{N} = 4$ SYM to the qHA that corresponds to the Leigh-Strassler deformed field theory.

Chapter 6

The Star product and deformations

Having uncovered the qHA structure of marginally deformed $\mathcal{N}=4$ SYM we would like to realize it at the level of the Lagrangian of the theory and discuss its implications. As we have discussed in Chapter 4 that the 3 chiral superfields Φ^i of $\mathcal{N}=4$ SYM possess $SU(3)$-symmetry hence they form a module or representation space of $SU(3)$. Here we shall be concerned with the fundamental representation of $SU(3)$ which we denote by $\mathcal{A}$ and the chiral fields are vectors. The module $\mathcal{A}$, being a representation space, has a product map $m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ which we will interchangeably write as $\cdot$ to prevent clutter. Thus for vectors a, b and c in $\mathcal{A}$ we have

$$m(a \otimes b) \equiv a \cdot b \tag{6.1}$$

and an important note is that in $\mathcal{N}=4$ SYM the product m is associative so that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. This is closely related to the fact that the algebraic structure associated with the $SU(3)$ internal symmetry group of $N=4$ SYM is a trivially quasi-Hopf, hence the co-associator ϕ is $\mathbb{1}^{\otimes 3}$. The effect is that an $SU(3)$ transformation U on a product of vectors is

$$\begin{aligned}
U[(a \cdot b) \cdot c] &= U\left[m \circ (m \otimes \mathrm{id})(a \otimes b \otimes c)\right] && (6.2)\\
&= m\left[(\Delta \otimes \mathrm{id}) \circ \Delta(U) \triangleright (a \otimes b \otimes c)\right] && (6.3)\\
&= m\left[(\mathrm{id} \otimes \Delta) \circ \Delta(U) \triangleright (a \otimes b \otimes c)\right] && (6.4)\\
&= U\left[m \circ (\mathrm{id} \otimes m)(a \otimes b \otimes c)\right] = U[a \cdot (b \cdot c)] && (6.5)
\end{aligned}$$

The third equality is a result of the co-associativity condition simplifying to

$$(\Delta \otimes \mathrm{id}) \circ \Delta(x) = \phi \left[(\mathrm{id} \otimes \Delta) \circ \Delta(x)\right] \phi^{-1} = (\mathrm{id} \otimes \Delta) \circ \Delta(x) \tag{6.6}$$

due to the trivial co-associator. We confess the woeful notation used in (6.3) and (6.4) since three vectors are contracted with only one product map. Strictly speaking it should be $m \circ (m \otimes \mathrm{id})$ and $m \circ (\mathrm{id} \otimes m)$ respectively. This abuse of notation will be employed from here onward unless the situation calls for clarity or there is potential for misunderstanding.

6.1 Definition of the star product

There is considerable difference when the qHA structure is not trivial. If the non-trivial qHA can be arrived at by the Drinfeld twisting of a trivial qHA using a twist F then the twisted co-product Δ_F will in general not be compatible with the module product m unless it also is twisted [81]. In [82] and [83] the twisted module product on two vectors is defined as

$$m_F(a \otimes b) := m(F^{-1} \triangleright a \otimes b) \equiv a \star b \tag{6.7}$$

which they refer to as a *star product*. According to (6.7) the algorithm of the twisted module product is to first apply the inverse of the twist on the vectors and then contract them using the original, untwisted module product m. In the case of three vectors the $\star$-product can be performed in two ways:

$$\begin{aligned}(a \star b) \star c &= m\left[\left((\Delta \otimes \mathrm{id})F^{-1}\right) \cdot (F^{-1} \otimes \mathbb{1}) \triangleright [a \otimes b \otimes c]\right] \\ &= m\left[(\Delta_1 F^{-1}) \cdot (F_{12}^{-1}) \triangleright [a \otimes b \otimes c]\right] \end{aligned}\tag{6.8}$$

$$\begin{aligned}a \star (b \star c) &= m\left[\left((\mathrm{id} \otimes \Delta)F^{-1}\right) \cdot (\mathbb{1} \otimes F^{-1})\right) \triangleright [a \otimes b \otimes c]\right] \\ &= m\left[(\Delta_2 F^{-1}) \cdot (F_{23}^{-1}) \triangleright [a \otimes b \otimes c]\right] \end{aligned}\tag{6.9}$$

where $(\Delta \otimes \mathrm{id}) \equiv \Delta_1$ and $(\mathrm{id} \otimes \Delta) \equiv \Delta_2$. These are generally not equal since the co-associator of the qHA associated with marginally deformed $\mathcal{N} = 4$ SYM is not trivial. Moreover, in the context of qHA, the Drinfeld twist F_{qh} is not required to be a 2-cocycle (4.32). This is very important since F_{qh} being 2-cocycle would trivialize the co-associator thus limiting us to an HA setting.

Note that it is the untwisted co-product that appears in the cubic star product expressions (6.8, 6.9). For ease of reference we introduce the following shorthand notation:

$$(\Delta_1 F^{-1}) \cdot (F_{12}^{-1}) =: [F_{3,L}] \text{ and } (\Delta_2 F^{-1}) \cdot (F_{23}^{-1}) =: [F_{3,R}] \tag{6.10}$$

The unfortunate fact is that the expressions in (6.8) and (6.9) are counter-intuitive. For example, we usually think of the co-associator ϕ as a map which when applied from the left shifts the parentheses to the right in a sense that $\phi\,[(a\cdot b)\cdot c] = a\cdot(b\cdot c)$. The inverse of the co-associator is expected to do the opposite, that is, $\phi^{-1}\,[a\cdot(b\cdot c)] = (a\cdot b)\cdot c$. This intuition is not compatible with (6.8) and (6.9); the following demonstrates this point. Observe that with $\phi = F_{23}.[\Delta_2 F].[\Delta_1 F^{-1}].F_{12}^{-1}$ and $\phi^{-1} = F_{12}.[\Delta_1 F^{-1}].[\Delta_1 F].F_{23}^{-1}$ we have

$$\begin{aligned} a\star(b\star c)\triangleleft\phi &= \left[((\Delta_2 F^{-1})\cdot(F_{23}^{-1}))(F_{23}.[\Delta_2 F].[\Delta_1 F^{-1}].F_{12}^{-1})\triangleright(a\otimes b\otimes c)\right] \\ &= (a\star b)\star c \qquad (6.11) \\ (a\star b)\star c\triangleleft\phi^{-1} &= \left[(\Delta_1 F^{-1})\cdot(F_{12}^{-1})(F_{12}.[\Delta_1 F^{-1}].[\Delta_1 F].F_{23}^{-1})\triangleright(a\otimes b\otimes c)\right] \\ &= a\star(b\star c) \qquad (6.12) \end{aligned}$$

thus the co-associator maps the expression with right-justified parentheses to that with left-justified parentheses from the right and vice versa. There are contexts in which these expressions are useful: one is in opposite quasi-Hopf algebra structures, denoted by qHA$^{\text{op}}$ and the other is in right modules of the qHA. It is nonetheless true that qHA$^{\text{op}}$'s are qHA in their own right and there the relations above are not unorthodox. We intend to work with qHA structures and their left action on modules thus hold to the convention of the left action.

The predicament can be overcome by viewing the module space as a metric vector space with an inner product such that $\langle a^i\otimes b^j|a_k\otimes b_l\rangle = \delta^i_k\delta^j_l$. An example of this is the n-dimensional Euclidean metric space where the row vectors and column vectors serve the roles of $a^i\otimes b^j$ and $a_k\otimes b_l$ respectively. If we declare that (6.7), (6.8) and (6.9) are expressions that pertain to the module vector space $V\otimes V\otimes V$ spanned by the basis $\{z_i\otimes z_j\otimes z_k\}$ then the inner product implies that the $\star$-product on the dual vector space $V'\otimes V'\otimes V'$ is spanned by the basis $\{z^i\otimes z^j\otimes z^k\}$ must act as $z^i\star z^j = m[F\triangleright[z^i\otimes z^j]]$, thus $\langle z^i\otimes z^j\otimes z^k|z_l\otimes z_m\otimes z_n\rangle = \delta^i{}_l\delta^j{}_m\delta^k{}_n$. This amounts to an inversion and such an inversion on the cubic expressions gives

$$\begin{aligned} (z^i\star z^j)\star z^k &= m\left[(F\otimes \mathbb{1})\cdot(\Delta\otimes\mathrm{id})F\triangleright[z^i\otimes z^j\otimes z^k]\right] \\ &= m\left[(F_{12})\cdot(\Delta_1 F)\triangleright(z^i\otimes z^j\otimes z^k)\right] \\ &= m\left[[F_{3,L}^{-1}]\triangleright(z^i\otimes z^j\otimes z^k)\right] \\ z^i\star(z^j\star z^k) &= m\left[(\mathbb{1}\otimes F)\cdot(\mathrm{id}\otimes\Delta)F\triangleright[z^i\otimes z^j\otimes z^k]\right] \\ &= m\left[(F_{23})\cdot(\Delta_2 F)\triangleright(z^i\otimes z^j\otimes z^k)\right] \\ &= m\left[[F_{3,R}^{-1}]\triangleright(z^i\otimes z^j\otimes z^k)\right] \end{aligned} \qquad (6.13)$$

where we have invoked (6.10):

$$[F^{-1}_{3,L}] := (F_{12}) \cdot (\Delta_1 F) \quad \text{and} \quad [F^{-1}_{3,R}] := (F_{23}) \cdot (\Delta_2 F) \tag{6.14}$$

The L (or R) in $[F_{3,L}]$ (or $[F_{3,R}]$) refers to the parentheses being left-justified (or right-justified). Now we have restored the intuitive action of the co-associator

$$\phi \triangleright (a \star b) \star c = m\left[(F_{23}) \cdot (\Delta_2 F) \cdot (\Delta_1 F^{-1}) \cdot (F^{-1}_{12})((F_{12}) \cdot (\Delta_1 F) \triangleright [a \otimes b \otimes c]\right] \tag{6.15}$$

$$= m\left[(F_{23}) \cdot (\Delta_2 F) \triangleright [a \otimes b \otimes c]\right] = a \star (b \star c) \tag{6.16}$$

thus the co-associator ϕ moves the brackets to the right and while, as it can now be easily shown, the inverse ϕ^{-1} moves the brackets to the left.

Beyond the conveniences that this view of the $\star$-product affords is the fact that it is consistent with the pentagon relation (4.49) which is represented in Figure (4.5) and it makes the quasi-YBE, as quoted in (4.56), understandable. Taking this as motivation we resume the $\star$-product discussion and write (6.7) in index form over coordinates $\{z^i\}$

$$z^i \star z^j = F^{i\ j}_{k\ l}\ z^k z^l \tag{6.17}$$

Note that the $\star$-product is generally non-commutative when F is not an identity even though the coordinates $\{z^i\}$ themselves are commutative. The inverse relation of (6.17) is given by

$$z^i z^j = (F^{-1})^{i\ j}_{k\ l}\ z^k \star z^l \tag{6.18}$$

and the combination of these two relations instills confidence on the definition of the star product because, with a little "trickery"[1], we have that

$$\begin{aligned} z^i \star z^j &= F^{i\ j}_{k\ l}\ z^k z^l &(6.19)\\ &= F^{i\ j}_{k\ l}\ z^l z^k &(6.20)\\ &= F^{i\ j}_{k\ l}(F^{-1})^{l\ k}_{m\ n} z^m \star z^n &(6.21)\\ &= R^{i\ j}_{m\ n}\ z^m \star z^n &(6.22)\end{aligned}$$

where the substitution[2] $F^{i\ j}_{k\ l}(F^{-1})^{l\ k}_{m\ n} = R^{i\ j}_{m\ n}$ was made in order to obtain the last equality, a quantum plane relation. The interpretation of the last equality is that the $\star$ commutativity of the z^i's is controlled by R. This is exactly the starting point of the Fadeev-Reshetikhin-Takhtajan [FRT] construction of a quantum linear algebra. In this construction the elements of the algebra are viewed as quantum objects that no longer commute unless an object R is introduced to mediate the commutativity [84][74]. In

[1] the commutativity of the coordinate z^i

[2] This is the index form of $F_{21}(F_{12})^{-1} = R$

(6.19) we have a similar case where the non-commutativity arises from introduction of a $\star$-product on commutative coordinates z^i. In brief, the $\star$-product definition (6.17) is compliant with the RTT relations of the FRT construction.

In this confidence we revisit the cubic $\star$-product expressions (6.13) and express them in index form in the $\{z^i\}$ basis:

$$\begin{aligned}(z^i \star z^j) \star z^k &= [(F_{12}) \cdot (\Delta_1 F)]^{ijk}_{lmn} z^l z^m z^n =: [F^{-1}_{3,L}]^{ijk}_{lmn} z^l z^m z^n \\ z^i \star (z^j \star z^k) &= [(F_{23}) \cdot (\Delta_2 F)]^{ijk}_{lmn} z^l z^m z^n =: [F^{-1}_{3,R}]^{ijk}_{lmn} z^l z^m z^n\end{aligned} \tag{6.23}$$

With this we have every ingredient necessary to explicitly compute the cubic $\star$-product relation. Because these computations produce large expressions, here we shall present the outcome for the imaginary β case and then give an outline for the general case of marginal deformations since they are similar in spirit.

6.2 Imaginary β deformed star product

Recall from (5.19) and (5.22) that the imaginary β-deformed twist is given by $F_{\beta_i} = \exp(i\alpha_{\beta_i} f_{\beta_i})$ where $f_{\beta_i} = -\frac{1}{2}(\lambda_1 \wedge \lambda_2 + \lambda_5 \wedge \lambda_4 + \lambda_6 \wedge \lambda_7)$. It thus follows that

$$(F_{\beta_i} \otimes \mathbb{1}) \equiv (F_{\beta_i})_{12} = \exp[i\alpha_{\beta_i}(f_{\beta_i})_{12}] \text{ with } (f_{\beta_i})_{12} = f_{\beta_i} \otimes \mathbb{1} \tag{6.24}$$

$$(\mathbb{1} \otimes F_{\beta_i}) \equiv (F_{\beta_i})_{23} = \exp[i\alpha_{\beta_i}(f_{\beta_i})_{23}] \text{ with } (f_{\beta_i})_{23} = \mathbb{1} \otimes f_{\beta_i} \tag{6.25}$$

and since the co-product exponentiates, we obtain that

$$(\Delta \otimes \mathrm{id})F_{\beta_i} = \exp\left[i\alpha_{\beta_i}(\Delta \otimes \mathrm{id})f_{\beta_i}\right] \text{ and } (\mathrm{id} \otimes \Delta)F_{\beta_i} = \exp\left[i\alpha_{\beta_i}(\mathrm{id} \otimes \Delta)f_{\beta_i}\right] \tag{6.26}$$

We conclude that for the imaginary β-deformation the cubic product relations are

$$[F^{-1}_{3,L}]_{\beta_i} = \exp[i\alpha_{\beta_i}(f_{\beta_i})_{12}] \cdot \exp\left[i\alpha_{\beta_i}(\Delta \otimes \mathrm{id})f_{\beta_i}\right] \tag{6.27}$$

$$[F^{-1}_{3,R}]_{\beta_i} = \exp[i\alpha_{\beta_i}(f_{\beta_i})_{23}] \cdot \exp\left[i\alpha_{\beta_i}(\mathrm{id} \otimes \Delta)f_{\beta_i}\right] \tag{6.28}$$

and the explicit forms of the non-zero elements of $[F_{3,L}]$ and $[F_{3,R}]$ are contained in Appendix [A.4]. It is very easy to confirm that $[F^{-1}_{3,R}] = \phi[F^{-1}_{3,L}]$. The details of how the cubic terms $[F_{3,L}]$ and $[F_{3,R}]$ calculated are in Appendix [A.3].

6.3 General (q,h)-deformed star product

The principle is the same for defining the $\star$-product for the full (q,h) deformed theory except the elements $[F_{3,L}]$ and $[F_{3,R}]$ are too large to be written out on a page. Here we once again refer the reader to the Mathematica notebook associated to [11]. Evidence of the $\star$-product being sensible and compatible with quantum algebra relation is seen in the fact that

$$\begin{aligned} z^1 \star z^2 - qz^2 \star z^1 + hz^3 \star z^3 &= \frac{\sqrt{q+1}\sqrt{h\bar{h}+q\bar{q}+1}}{\sqrt{2}\sqrt{\bar{q}+1}}\left(z^1 z^2 - z^2 z^1\right) \\ &= 0\,, \quad \text{since} \quad [z^1, z^2] = 0 \end{aligned} \tag{6.29}$$

So the quantum plane relations found in [57] are here recovered by the star product as defined in (6.19). This also holds true for the cyclic permutations of the expression (6.29).

The field theory version of this expression will have z^i replaced by the chiral superfields Φ^i. This is made by recalling the AdS-CFT duality conjecture that type IIB String theory in AdS$_5\times$ S^5 is dual to $\mathcal{N} = 4$ SYM in flat 4-dimensional Minkowski space [21] [85] [86][87]. The setup for the AdS-CFT conjecture is IIB superstring theory (9+1)-dimensional flat Minkowski space with a stack made up of N coincident D3-branes. On one hand D3-branes are viewed as (3+1)-dimensional surfaces on which open strings can attach and on the other, open strings can be considered as excitations of D3-branes. We focus on the dynamics of an open string whose ends are attached to different D3-branes in the stack. For energies much less than the string length, hence only focus on the massless excitations of the open string, the dynamics of the open string are described by a theory on the world volume of the D3-brane on which they attach. The string excitations that are parallel to the D3-brane are described by a gauge field A_μ and the transverse excitations by 6 real scalar fields ϕ^i corresponding to 6 directions transverse to the D3-brane stack. This field content matches the bosonic part of $\mathcal{N} = 4$ SYM. And since supersymmetry is present and the strings are to be SUSY invariant, it follows that the superpartners (Weyl spinors) are also included. Thus at low energies, massless excitations of open strings are described by $\mathcal{N} = 4$ SYM. The 6 real scalar fields ϕ^i can be used to parameterize the transverse 6-dimensional space. In the language of $\mathcal{N} = 1$ superspace and superfields, these 6 real scalar fields with $SO(6)$ are combined into 3 chiral superfields Φ^i's with $SU(4)$. The relations involving z^i are to be cast in terms of Φ^i. So then we have

$$\Phi^1 \star \Phi^2 - q\Phi^2 \star \Phi^1 + h\Phi^3 \star \Phi^3 = \frac{\sqrt{q+1}\sqrt{h\bar{h}+q\bar{q}+1}}{\sqrt{2}\sqrt{\bar{q}+1}}[\Phi^1, \Phi^2] \tag{6.30}$$

The superfields Φ^i's are $N \times N$ matrices, hence they do not always commute. Simply replacing z's with Φ will not give the chiral superfield analogue of (6.29). We must demand that the RHS of (6.30) vanish. Such a demand on products of chiral superfields will place constraints on the F-terms of these chiral superfields; so in order to guarantee the vanishing of RHS of (6.30), the Φ^i as matrices must be diagonal. So via $\star$-product on Leigh-Strassler relations one obtains constraints on the F-terms of $\mathcal{N}=4$ SYM [88] and in [89] [90] these constraints are collected in a non-commutativity matrix Θ.

Introducing the $\star$-product to the Leigh-Strassler superpotential led to a pleasant surprise. Naively speaking the superpotential is cubic in the chiral fields thus we expect that the left-justified superpotential is different from the right-justified superpotential unless multiplied (from the left) by the (q,h) co-associator. However, from calculation, we noted that, after writing out the cyclically related terms explicitly, the left-justified Leigh-Strassler superpotential with the $\star$-product gives

$$\begin{aligned}
&(\Phi^1\star\Phi^2)\star\Phi^3+(\Phi^2\star\Phi^3)\star\Phi^1+(\Phi^3\star\Phi^1)\star\Phi^2-q((\Phi^1\star\Phi^3)\star\Phi^2+(\Phi^2\star\Phi^1)\star\Phi^3+(\Phi^3\star\Phi^2)\star\Phi^1)+\\
&\qquad h((\Phi^1\star\Phi^1)\star\Phi^1+(\Phi^2\star\Phi^2)\star\Phi^2+(\Phi^3\star\Phi^3)\star\Phi^3)\\
&=\frac{\sqrt{q+1}\sqrt{h\bar{h}+q\bar{q}+1}}{\sqrt{2}\sqrt{\bar{q}+1}}\left((\Phi^1\Phi^2)\Phi^3+(\Phi^2\Phi^3)\Phi^1-(\Phi^1\Phi^3)\Phi^2-(\Phi^2\Phi^1)\Phi^3-(\Phi^3\Phi^2)\Phi^1\right)
\end{aligned} \tag{6.31}$$

and so does the right-justified Leigh-Strassler superpotential with the star product. This means in the superpotential the $\star$-product is insensitive to the bracketing even though the $\star$-product itself is not associative. We attribute this insensitivity to bracketing of the Leigh-Strassler superpotential to its cyclicity in the chiral fields.

6.4 Inverse star product

We would now like to automate the Leigh-Strassler deformations. More specifically, given some quantity, $\mathcal{O}_{SYM}$, of $\mathcal{N}=4$ SYM we would like to compute the Leigh-Strassler deformed counterpart, $\mathcal{O}_{LS}$, by simply promoting to a "star" product every product involved in the expression that defines $\mathcal{O}_{SYM}$. For this purpose we define an inverse star product and denote it with an asterisk $*$ and we desire that $\mathcal{O}_{*SYM}=\mathcal{O}_{LS}$. We then define inverse star product on vectors a and b as follows

$$a*b=m_{F^{-1}}(a\otimes b)=m[F\triangleright(a\otimes b)] \tag{6.32}$$

and on the dual vector space, the space of our interest, the $*$-product acts as

$$z^i*z^j=m[F^{-1}\triangleright(z^i\otimes z^j)] \quad\Rightarrow\quad z^i*z^j=(F^{-1})^{i\ j}_{\ k\ l}z^kz^l \tag{6.33}$$

The $*$-product allows us to recover Leigh-Strassler deformed relations from the undeformed $\mathcal{N}=4$ SYM relation. Note how the deformed version of (6.29) is automatically obtained by promoting the regular multiplication to $*$-product

$$z^1 * z^2 - z^2 * z^1 = \frac{\sqrt{2}\sqrt{\bar{q}+1}}{\sqrt{q+1}\sqrt{h\bar{h}+q\bar{q}+1}}\left[z^1 z^2 - q z^2 z^1 + h(z^3)^2\right] \tag{6.34}$$

On three coordinates the $*$-product gives the same form as (6.23) and since $[F^{-1}_{3,L}] \neq [F^{-1}_{3,R}]$ it is clear that the $*$-product is not associative. The cubic $*$-product terms are given by

$$(z^i * z^j) * z^k = m_F\left[(z^i \otimes z^j) \otimes z^k\right] = m\left[[F^{-1}_{3,L}] \triangleright (z^i \otimes z^j \otimes z^k)\right] \tag{6.35}$$

$$z^i * (z^j * z^k) = m_F\left[z^i \otimes (z^j \otimes z^k)\right] = m\left[[F^{-1}_{3,R}] \triangleright (z^i \otimes z^j \otimes z^k)\right] \tag{6.36}$$

6.5 The Leigh-Strassler Superpotential

The $*$-product has so far been successful at automating deformations of quantum algebra relations when dealing with a product of two fields[3]. This however is not enough, we must test it at the cubic level. When three fields are considered the question of associativity is unavoidable because the product map is binary. So we have to choose which two fields to multiply first: either the left-justified bracketing or the right-justified bracketing. However it turns out that in the $\mathcal{N} = 4$ SYM superpotential the fields appear in a cyclic way making this choice immaterial, that is,

$$\mathcal{W}_{*SYM_L} = \mathcal{W}_{*SYM_R} \tag{6.37}$$

where $\mathcal{W}_{*SYM_L} := \mathrm{Tr}\left[(\Phi^1 * \Phi^2) * \Phi^3 - (\Phi^1 * \Phi^3) * \Phi^2\right]$ is the left-justified superpotential and the right-justified one is $\mathcal{W}_{*SYM_R} := \mathrm{Tr}\left[\Phi^1 * (\Phi^2 * \Phi^3) - \Phi^1 * (\Phi^3 * \Phi^2)\right]$. This is due to fact that the $*$-product is invariant, as far as associativity is concerned, under cyclic permutation. Explicit computation shows that

$$(z^1 * z^2) * z^3 + (z^3 * z^1) * z^2 + (z^2 * z^3) * z^1 = z^1 * (z^2 * z^3) + z^3 * (z^1 * z^2) + z^2 * (z^3 * z^1) \tag{6.38}$$

We emphasize that this observation does *not* mean the $*$-product is associative, rather it shows that in certain cyclic expressions evidence of non-associativity is absent. This is to our advantage because we can exploit this property and show that, with a little

[3] or coordinates, so also throughout

trick, $\mathcal{W}_{SYM}$ can be written in a form that exposes its cyclic property:

$$
\begin{aligned}
W_{SYM} &= g\,\mathrm{Tr}\left(\Phi^1\Phi^2\Phi^3 - \Phi^1\Phi^3\Phi^2\right) \\
&= \frac{3g}{3}\,\mathrm{Tr}\left(\Phi^1\Phi^2\Phi^3 - \Phi^1\Phi^3\Phi^2\right) \\
&= \frac{g}{3}\,\mathrm{Tr}\left(\Phi^1\Phi^2\Phi^3 + \Phi^1\Phi^2\Phi^3 + \Phi^1\Phi^2\Phi^3 - \Phi^1\Phi^3\Phi^2 - \Phi^1\Phi^3\Phi^2 - \Phi^1\Phi^3\Phi^2\right) \\
&= \frac{g}{3}\,\mathrm{Tr}\left(\Phi^1\Phi^2\Phi^3 + \Phi^3\Phi^1\Phi^2 + \Phi^2\Phi^3\Phi^1 - \Phi^1\Phi^3\Phi^2 - \Phi^2\Phi^1\Phi^3 - \Phi^3\Phi^2\Phi^1\right)
\end{aligned}
\tag{6.39}
$$

Here the cyclicity of the trace is pivotal. If the $*$-product is now introduced in the final equality of (6.39) then we obtain

$$
\begin{aligned}
W_{*SYM} = \frac{g}{3}\,\mathrm{Tr}\big[&\Phi^1 * \Phi^2 * \Phi^3 + \Phi^3 * \Phi^1 * \Phi^2 + \Phi^2 * \Phi^3 * \Phi^1 \\
&-\Phi^1 * \Phi^3 * \Phi^2 - \Phi^2 * \Phi^1 * \Phi^3 - \Phi^3 * \Phi^2 * \Phi^1\big] \\
= \frac{\kappa}{3}\,\mathrm{Tr}\big[&\Phi^1\Phi^2\Phi^3 + \Phi^3\Phi^1\Phi^2 + \Phi^2\Phi^3\Phi^1 - q(\Phi^1\Phi^3\Phi^2 + \Phi^2\Phi^1\Phi^3 + \Phi^3\Phi^2\Phi^1) \\
&+h(\Phi^1\Phi^1\Phi^1 + \Phi^2\Phi^2\Phi^2 + \Phi^3\Phi^3\Phi^3)\big] \\
= \kappa\,\mathrm{Tr}\bigg[&\Phi^1\Phi^2\Phi^3 - q(\Phi^1\Phi^3\Phi^2) + \frac{h}{3}\left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right)\bigg]
\end{aligned}
\tag{6.40}
$$

where $\kappa = \frac{g\sqrt{2}\sqrt{1+\bar{q}}}{\sqrt{1+q}\sqrt{1+h\bar{h}+q\bar{q}}}$. This exactly matches (5.2) and we conclude that $\mathcal{W}_{*SYM} = \mathcal{W}_{LS}$ and in writing $\mathcal{W}_{*SYM}$ we did not have to care about how we associate the fields except that there be no mixed associations i.e. expressions of the form $\Phi^{(i)} * (\Phi^{(j)} * \Phi^{(k)}) - (\Phi^{(l)} * \Phi^{(m)}) * \Phi^{(n)}$ must be forbidden.

In [91] a similar work was done, treating the w-deformed $\mathcal{N} = 4$ SYM which is a theory unitarily equivalent to the real β-deformed $\mathcal{N} = 4$ SYM [92]. The R-matrix of the real β-deformed theory is obtained from the general Leigh-Strassler R-matrix in the limit where $\bar{q} = q^{-1}$, $h = 0 = \bar{h}$. There is a group of deformations, parameterized by w, which are related to the real β-deformation via a unitary transformation [93][92]. These are known as w-deformations and their R-matrix is obtained by the substitutions

$$
\bar{q} = q\ ,\ \ \bar{h} = h \text{ and } \ q \to 1 + w\ ,\ \ h \to w \text{ where } w \text{ is real} \tag{6.41}
$$

The unitary transformation that relates the β-deformed and w-deformed theories is simplify a redefinition of the fields in the theories. In [79], the Hopf algebraic structure of the w-deformed theory was studied using a twist given by

$$F_w = \begin{bmatrix} \frac{C}{w+1} & 0 & 0 & 0 & 0 & \frac{Cw}{(w+1)^2} & 0 & 0 & 0 \\ 0 & \frac{C}{w+1} & 0 & \frac{Cw}{(w+1)^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{C}{w+1} & 0 & \frac{Cw}{(w+1)^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C}{w+1} & 0 & 0 & 0 & 0 & \frac{Cw}{(w+1)^2} \\ 0 & 0 & 0 & 0 & \frac{C}{w+1} & 0 & \frac{Cw}{(w+1)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C}{w+1} & 0 & \frac{Cw}{(w+1)^2} & 0 \\ 0 & 0 & \frac{Cw}{(w+1)^2} & 0 & 0 & 0 & \frac{C}{w+1} & 0 & 0 \\ \frac{Cw}{(w+1)^2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{C}{w+1} & 0 \\ 0 & \frac{Cw}{(w+1)^2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{C}{w+1} \end{bmatrix} \tag{6.42}$$

with $C = (1+w)\,/\sqrt[3]{(1+2w)(1+w+w^2)}$. Since F_{qh} is more general than F_w, it is natural to expect F_{qh} to reduce to F_w in the limit (6.41). However an explicit check shows that $F_{qh} \neq F_w$ in the limit $q \to 1+w$, $h \to w$. One way to understand this is to recall that according to [49] the fixed points of $\mathcal{N}=1$ superconformal theories which are marginal deformations of $\mathcal{N}=4$ SYM can define a manifold. The implication of this is that the space of Leigh-Strassler deformed theories is parameterized by a function, γ, with a constraint $\gamma(\kappa, g, q, h) = 0$. By this we see that specifying the parameters q and h does not define a unique point in the manifold, i.e. κ can still vary [4]. We thus understand the discrepancy $F_{qh} \neq F_w$ in the above limit to be a manifestation of being at points which have the same q and h but different κ.

6.6 Mixed plane relations

In the definition of the $\star$-product (or $*$-product) the focus has been primarily on z^i, the holomorphic coordinates. Now we can extend the discussion to include both the antiholomorphic and mixed sectors of the theory, thence define the $\star$-product (or $*$-product) in that context. The antiholomorphic sector is related to the holomorphic by hermitian conjugation, making it similar to holomorphic sector. For this reason we focus our attention more on the mixed sector. To define the $\star$-product in these sectors we adopt the Leigh-Strassler deformed quantum relations in [57] and require that our $\star$-product definition be consistent with those relations. When adapted for our case these quantum relations take the form:

$$\begin{aligned} z^i \star z^j &= R^{j}{}_{k}{}^{i}{}_{l} z^k \star z^l \;, \quad & \bar{z}_i \star \bar{z}_j &= \bar{z}_k \star \bar{z}_l R^{k}{}_{j}{}^{l}{}_{i} \\ z^i \star \bar{z}_j &= R^{i}{}_{l}{}^{k}{}_{j} \bar{z}_k \star z^l \;, \quad & \bar{z}_i \star z^j &= \tilde{R}^{j}{}_{l}{}^{k}{}_{i} z^l \star \bar{z}_k \end{aligned} \tag{6.43}$$

[4] g is fixed by the constraint on γ

R is the R-matrix and $\tilde{R}$ is the so-called *second inverse* of R whose defining relation is

$$R^{i\ k}_{\ j\ l}\tilde{R}^{j\ n}_{\ m\ k} = \delta^i_m\delta^n_l = \tilde{R}^{i\ k}_{\ j\ l}R^{j\ n}_{\ m\ k} \tag{6.44}$$

It is worth noting that the indices are contracted in a way different to the regular matrix multiplication. As shown in [71] the second inverse, $\tilde{R}$, can be obtained more directly from R by a simple algorithm. Recall that $R \in H \otimes H$ and the recipe to compute its second inverse is by transposing in the second copy of H, inverting the result and then transposing again we obtain $\tilde{R}$. If T_i is an operator that transposes in the ith space then in pseudo-Sweedler notation the second inverse of R is given by

$$\tilde{R} = T_2\left[T_2\left(R_{(1)} \otimes R_{(2)}\right)\right]^{-1}. \tag{6.45}$$

The $\star$-product consistent with (6.43) at the level of the twist can be defined as

$$z^i \star \bar{z}_j = z^k F^{l\ i}_{\ j\ k}\bar{z}_l \quad , \quad \bar{z}_i \star z^j = \bar{z}_l G^{l\ i}_{\ i\ k} z^k \tag{6.46}$$

where there has been the introduction of a new object G, which we call a G-tensor. The second inverse of the G-tensor, denoted by $\tilde{G}$, provides a second factorization of the R-matrix, that means it satisfies

$$R^{i\ k}_{\ j\ l} = \tilde{G}^{k\ m}_{\ n\ j}F^{n\ i}_{\ l\ m} \tag{6.47}$$

Explicit calculation shows that just as $\tilde{G}$ provides a second factorization for R so G provides a second factorization for $\tilde{R}$, that is

$$\tilde{R}^{i\ j}_{\ m\ n} = \tilde{F}^{j\ l}_{\ k\ m}G^{k\ i}_{\ n\ l} \tag{6.48}$$

The mixed $\star$-product definitions (6.46) imply

$$z^j \star \bar{z}_i = z^l\bar{z}_k F^{k\ j}_{\ i\ l} = \bar{z}_k z^l F^{k\ j}_{\ i\ l} = \bar{z}_m \star z^n \tilde{G}^{m\ l}_{\ k\ n}F^{k\ j}_{\ i\ l} = \bar{z}_m \star z^n R^{j\ m}_{\ n\ i} \tag{6.49}$$

and these are consistent with (6.43). Drawing inspiration from (6.43) we define the mixed plane relations for the $*$-product as

$$z^i * \bar{z}_j = z^k \tilde{F}^{l\ i}_{\ j\ k}\bar{z}_l \text{ and } \bar{z}_i * z^j = \bar{z}_k \tilde{G}^{k\ j}_{\ i\ j} z^l \tag{6.50}$$

These relations will be used in 7.1.1 when we consider the effect of the $*$-product on the kinetic terms of the Lagrangian. As seen in (3.19), the scalar kinetic terms are essentially $\bar{\Phi}^i\Phi^i$. Applying the $*$-product on these terms we found that they are not deformed. This is in accord with the fact that Leigh-Strassler deformations only affect the superpotential. In Appendix [A.47] there are the non-zero entries of the G-tensor and from these the rest can be recovered by the use of $\mathbb{Z}_3$-symmetry that the G-tensor inherits from its definition.

Having arrived at the correct $*$-product definition to automate marginal deformations and confirmed the consistency of the definition with known quantum relations we then conclude this chapter.

Chapter 7

Final Remarks and Conclusions

In this chapter we highlight and make remarks about the qHA structure of marginally deformed $\mathcal{N} = 4$ SYM and conclude this work with an outlook on future work.

7.1 quasi-Hopf Invariance

We recall that in the undeformed $\mathcal{N} = 4$ SYM in superspace notation, the chiral fields possessed a $SU(3)$ symmetry which we called H for ease when referencing. From the discussion in Chapter 4 the Lie group H is trivially qHA. The current claim is that in the marginally deformed theory this symmetry deformed to $\mathcal{H}_{qh}$. That is by relaxing the requirements of invariance one is able to find traces of $SU(3)$ in the deformed theory.

Let $U = e^{iT}$ be a member of $SU(3)$ and z^i a vector belonging to the vector space $\mathcal{A}$, hence $\mathcal{A}$ is a module of $SU(3)$. For our case $\mathcal{A}$ is three dimensional which means z^i is a three-dimensional vector or 3×1 matrix and $(z^i \otimes z^j)$, a 9×1 matrix. Let us first invoke the notation of maps, making the product map explicit, and write the expression $z_i z_j$ in the undeformed theory as

$$z_i z_j = m(z_i \otimes z_j) \tag{7.1}$$

The effect of the $SU(3)$ transformation $z^i \longrightarrow z^{i'} = U^{i'}_j z^j$ on expressions such as $z_1 z_2$ is

$$(z_1 z_2)' = m[(z_1 \otimes z_2)'] = m[\Delta(U) \triangleright z_1 \otimes z_2] \tag{7.2}$$

and since $SU(3)$ (or H) is trivially qHA with a symmetric coproduct Δ which exponentiates with ease, it follows that

$$\Delta(U) = \Delta(e^{iT}) = e^{\Delta(iT)} = e^{i(T \otimes \mathbb{1} + \mathbb{1} \otimes T)} = (e^{iT} \otimes \mathbb{1})(\mathbb{1} \otimes e^{iT}) = U \otimes U \tag{7.3}$$

thus we have $(z_1 z_2)' = m[(U \otimes U) \triangleright z_1 \otimes z_2] = (U \triangleright z_1) \cdot (U \triangleright z_2)$ which in component form is

$$z^{i'} z^{j'} = U^{i'}_{i} z^{i} U^{j'}_{j} z^{j}. \tag{7.4}$$

The extension of this argument to a left-justified[1] cubic product gives

$$\begin{aligned} ((z_1 z_2) z_3)' = m[((z_1 \otimes z_2) \otimes z_3)'] &= m[\Delta(U) \triangleright ((z_1 z_2) \otimes z_3)] & (7.5)\\ &= m[(\Delta \otimes \mathrm{id}) \circ \Delta(U) \triangleright (z_1 \otimes z_2 \otimes z_3)] & (7.6) \end{aligned}$$

and applying the co-product leads to

$$\begin{aligned} (\Delta \otimes \mathrm{id}) \circ \Delta(U) &= (\Delta \otimes \mathrm{id}) \circ \Delta(e^{iT}) & (7.7)\\ &= \exp[(\Delta \otimes \mathrm{id}) \circ \Delta(iT)] & (7.8)\\ &= \exp[(\Delta \otimes \mathrm{id})(iT \otimes \mathbb{1} + \mathbb{1} \otimes iT)] & (7.9)\\ &= \exp[i(T \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes T \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes T)] & (7.10)\\ &= e^{i(T \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1})} \cdot e^{i(\mathbb{1} \otimes T \otimes \mathbb{1})} \cdot e^{i(\mathbb{1} \otimes \mathbb{1} \otimes T)} & (7.11)\\ &= (U \otimes \mathbb{1} \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes U \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes \mathbb{1} \otimes U) & (7.12) \end{aligned}$$

thus we conclude that $((z_1 z_2) z_3)' = [(U \triangleright z^1) \cdot (U \triangleright z^2)] \cdot (U \triangleright z^3)$ is invariant under H. The general cubic product in index notation is

$$((z_1 z_2) z_3)' = (U^{i'}_{i} z^{i} U^{j'}_{j} z^{j}) U^{k'}_{k} z^{k} \tag{7.13}$$

Admittedly this treatment is a nuclear warhead to a gun fight, not necessary. However it does help to introduce the reader to the inner workings of the qHA structure on familiar ground.

We now consider the deformed case where the qHA structure is $\mathcal{H}_{qh}$ and the module $\mathcal{A}$ has $*$-product as its multiplication map. Recall that $\mathcal{H}_{qh}$ and H are related by twisting then the co-product of $\mathcal{H}_{qh}$ is $\Delta_{F_{qh}}(\cdot) = F_{qh}\,[\Delta(\cdot)]\,F_{qh}^{-1}$. The action of U on the quadratic

[1] the right-justified case works in a similar way

term is

$$(z_1 * z_2)' = m_F\left(\Delta_F(U) \triangleright [z_1 \otimes z_2]\right) \tag{7.14}$$
$$= m_F\left(F\Delta(U)F^{-1} \triangleright [z_1 \otimes z_2]\right) \tag{7.15}$$
$$= m\left(F^{-1}F\Delta(U)F^{-1} \triangleright [z_1 \otimes z_2]\right) \tag{7.16}$$
$$= m\left(\Delta(U)F^{-1} \triangleright [z_1 \otimes z_2]\right) \tag{7.17}$$
$$= m\left(\Delta(U)\left(\sum F^{-1}_{(1)} \otimes F^{-1}_{(2)}\right) \triangleright [z_1 \otimes z_2]\right) \tag{7.18}$$
$$= m\left(\sum (U \otimes U)(F^{-1}_{(1)} \otimes F^{-1}_{(2)}) \triangleright [z_1 \otimes z_2]\right) \tag{7.19}$$
$$= \left(UF^{-1}_{(1)} \triangleright z_1\right) \cdot \left(UF^{-1}_{(2)} \triangleright z_2\right) \tag{7.20}$$

and we observe that the quadratic term transforms as before; this fact will be useful in the discussion of kinetic terms of the Lagrangian. The expression (7.20) when written in component form is

$$\begin{aligned} z^{i'} * z^{j'} &= \sum (UF^{-1}_{(1)})^{i'}_{\ i} z^i (UF^{-1}_{(2)})^{j'}_{\ j} z^j \\ &= \sum U^{i'}_{\ i''}[F^{-1}_{(1)}]^{i''}_{\ i} z^i U^{j'}_{\ j''}[F^{-1}_{(2)}]^{j''}_{\ j} z^j \\ &= U^{i'}_{\ i''}U^{j'}_{\ j''}[F^{-1}]^{i''j''}_{\ i\ j} z^i z^j \\ &= U^{i'}_{\ i}U^{j'}_{\ j}(z^i * z^j) \end{aligned} \tag{7.21}$$

Next is the investigation of the cubic terms since the superpotential is cubic in the chiral scalar fields. The expression considered is $((z_1 * z_2) * z_3)$ here the left-justified cubic term was chosen but the same can be said for the right-justified expression. Using (6.10) we find the action of U on this expression to be

$$((z_1 * z_2) * z_3)' = U\Big((z_1 * z_2) * z_3\Big) = U\Big(m_F[(z_1 \otimes z_2) \otimes z_3]\Big) \tag{7.22}$$
$$= m_F\Big([(\Delta_F \otimes \mathrm{id}) \circ \Delta_F(U) \triangleright (z_1 \otimes z_2) \otimes z_3]\Big) \tag{7.23}$$
$$= m\Big(\left[\Delta_1 F^{-1}\right] \cdot \left[F^{-1}_{12}\right] \cdot [(\Delta_F \otimes \mathrm{id}) \circ \Delta_F(U)] \triangleright [(z_1 \otimes z_2) \otimes z_3]\Big) \tag{7.24}$$
$$= m\Big([F_{3,L}] \cdot [(\Delta_F \otimes \mathrm{id}) \circ \Delta_F(U)] \triangleright [(z_1 \otimes z_2) \otimes z_3]\Big) \tag{7.25}$$

for clarity we extract, expand and simplify $(\Delta_F \otimes \mathrm{id}) \circ \Delta_F(U)$ to obtain

$$\begin{aligned}
(\Delta_F \otimes \mathrm{id}) \circ \Delta_F(U) &= (\Delta_F \otimes \mathrm{id})\Big[F(U \otimes U)F^{-1}\Big] \\
&= \Big[(\Delta_F \otimes \mathrm{id})F\Big]\Bigg[(\Delta_F \otimes \mathrm{id})(U \otimes U)\Bigg]\Big[(\Delta_F \otimes \mathrm{id})F^{-1}\Big] \\
&= \Big[F_{12}\Big(\Delta_1 F\Big)F_{12}^{-1}\Big]\Bigg[F_{12}\Big(\Delta_1(U \otimes U)\Big)F_{12}^{-1}\Bigg]\Big[F_{12}\Big(\Delta_1 F^{-1}\Big)F_{12}^{-1}\Big] \\
&= F_{12}\Big[\Delta_1 F\Big]\Bigg[U \otimes U \otimes U\Bigg]\Big[\Delta_1 F^{-1}\Big]F_{12}^{-1} \\
&= [F_{3,L}^{-1}] \cdot \Bigg[U \otimes U \otimes U\Bigg] \cdot [F_{3,L}]
\end{aligned} \tag{7.26}$$

Substituting this back into (7.22) gives

$$((z_1 * z_2) * z_3)' = m\Big([F_{3,L}] \cdot [F_{3,L}^{-1}] \cdot \Bigg[U \otimes U \otimes U\Bigg] \cdot [F_{3,L}] \triangleright [(z_1 \otimes z_2) \otimes z_3]\Big) \tag{7.27}$$

$$= m\Big([U \otimes U \otimes U] \cdot [F_{3,L}] \triangleright [(z_1 \otimes z_2) \otimes z_3]\Big) \tag{7.28}$$

$$= \sum \big(U[F_{3,L}]_{(1)} \triangleright z_1\big) \cdot \big(U[F_{3,L}]_{(2)} \triangleright z_2\big) \cdot \big(U[F_{3,L}]_{(3)} \triangleright z_3\big) \tag{7.29}$$

in component form this amounts to

$$(z^{i'} * z^{j'}) * z^{k'} = U^{i'}_{i} U^{j'}_{j} U^{k'}_{k} ((z^i * z^j) * z^k) \tag{7.30}$$

So the cubic $*$-product expressions under $U \in H$ transform as they did before the introduction of $*$-product and the importance of this fact is made apparent in the next section.

7.1.1 Leigh-Strassler Potential and Kinetic terms

Knowing the above, it is straight forward to prove the quasi-Hopf invariance of the superpotential

$$\mathcal{W}'_{LS} = \frac{\epsilon_{i'j'k'}}{3} \mathrm{Tr}\left[(\Phi^i * \Phi^j) * \Phi^k\right]' \tag{7.31}$$

$$= \frac{\epsilon_{i'j'k'}}{3} U^{i'}_{i} U^{j'}_{j} U^{k'}_{k} \mathrm{Tr}\left[(\Phi^i * \Phi^j) * \Phi^k\right] \tag{7.32}$$

$$= det(U) \frac{\epsilon_{ijk}}{3} \mathrm{Tr}\left[(\Phi^i * \Phi^j) * \Phi^k\right] = \mathcal{W}_{LS} \tag{7.33}$$

The kinetic terms are, in an anticlimactic fashion, verified to be invariant under U because from explicit calculation based on (6.50) we find that

$$\bar{z}_1 * z^1 + \bar{z}_2 * z^2 + \bar{z}_3 * z^3 = \bar{z}_1 z^1 + \bar{z}_2 z^2 + \bar{z}_3 z^3 \tag{7.34}$$

$$z^1 * \bar{z}_1 + z^2 * \bar{z}_2 + z^3 * \bar{z}_3 = z^1 \bar{z}_1 + z^2 \bar{z}_2 + z^3 \bar{z}_3 \tag{7.35}$$

making the invariance of the deformed Kähler potential, $\mathcal{K}_{LS}$, under U a trivial matter:

$$\mathcal{K}_{LS}(\Phi, \bar{\Phi}) = \bar{\Phi}_i * \Phi^i = \bar{\Phi}_i \Phi^i = \mathcal{K}_{SYM}(\Phi, \bar{\Phi}) \tag{7.36}$$

The success of the $*$-product is more impressive in the superpotential $\mathcal{W}_{*SYM}$ than in the Kähler potential. The invariance of $\mathcal{K}_{LS}$ is indicative of the fact that Leigh-Strassler deformation only affect the superpotential. We do not expect this success to persist in automating the deformation of more complicated operators of $\mathcal{N} = 4$ SYM.

7.1.2 quasi-Yang-Baxter Equation with spectral parameter dependence

We also briefly considered this work in the context of integrability. Here the Yang-Baxter equation [YBE] is useful as it provides a simple check for early signs of integrability. This is because integrability is closely related to factorizability of the R-matrix and the YBE encodes factorizability. It is simple to check that the real β-deformation of $\mathcal{N} = 4$ SYM (i.e the deformation parameter choice: $q = e^{i\beta}$, $\bar{q} = q^{-1}$ and $h = 0 = \bar{h}$) is integrable; its R-matrix satisfies the YBE without a spectral parameter. An equivalent integrable model with deformation parameter choice given by $q = 0$ $h = e^{i\theta}$ was found in [60]. In general marginal deformations of $\mathcal{N} = 4$ SYM do not satisfy the YBE. It thus would be interesting to use qHA structure to understand the interplay between deformation and integrability since with qHAs we have the quasi-Hopf YBE (4.23) an analog of YBE. From an explicit check we found that all (q, h)-deformations satisfy (4.23). Upon the re-introduction of the spectral parameter into the R_{qh}-matrix we learned that the spectral parameter dependent R-matrix corresponding to undeformed $\mathcal{N} = 4$ SYM is

$$R(u) = \frac{1}{u+i}(u\mathbb{1} \otimes \mathbb{1} + iP) \quad \text{with } P- \text{ a permutation matrix} \tag{7.37}$$

Drinfeld twisting $R(u)$ produces the deformed R-matrix with spectral parameter dependence

$$\begin{aligned} R_{qh}(u) &= F_{21} R(u) F^{-1} \\ &= F_{21} \left[\frac{1}{u+i} \left(u\mathbb{1} \otimes \mathbb{1} + iP \right) \right] F^{-1} \\ &= \frac{1}{u+i} \left(uR_{qh} + iF_{21} \cdot P \cdot F^{-1} \right) \end{aligned} \tag{7.38}$$

$$
\begin{aligned}
R_{qh}(u) &= \frac{1}{u+i}\left(uR_{qh} + iP\cdot P\cdot F_{21}\cdot P\cdot F^{-1}\right) \\
&= \frac{1}{u+i}\left(uR_{qh} + iP\cdot F_{12}\cdot F^{-1}\right) \\
&= \frac{1}{u+i}\left(uR_{qh} + iP\right)
\end{aligned}
\tag{7.39}
$$

It is evident that twisting by F_{qh} only affects the identity part of $R(u)$, making it similar to the work in [94]. Moreover it was found that $R_{qh}(u)$ also satisfies the quasi-Hopf YBE; introducing the spectral parameter has not compromised this relation.

7.2 Conclusion and Outlook

In this work we have found a Drinfeld twist F_{qh} that twists the trivial one-loop R-matrix corresponding to $\mathcal{N}=4$ SYM into R_{qh}, the R-matrix that corresponds to marginally deformed $\mathcal{N}=4$ SYM. This was done in the quantum limit, where the spectral parameter is taken to infinity. From this twist the full co-associator was computed [11] and then it was checked for consistency via the generalized YBE. The present report however explicitly only contains results from the imaginary β-deformed case. The analysis of the general (q,h) deformed case produces cumbersome expressions for the co-associator which are too large to reasonably contain in a page. The general co-associator has nonetheless been calculated explicitly and, by series expansion, checked for consistency numerically. Then contact was made with gauge field theory by the definition of the star product. The introduction of this star product in the superpotential of $\mathcal{N}=4$ SYM was shown to reproduce the Leigh-Strassler superpotential, effectively automating the process of deforming. Finally we discussed invariance under quasi-Hopf transformation and demonstrated the quasi-Hopf invariance of the Leigh-Strassler superpotential.

We hope that this approach could shed light concerning the loss of integrability as one deforms away from points that correspond to integrable models and possibly provide reasons or argument of the absence of integrability in a spirit similar to that in [95][2]. As the introduction of the spectral parameter was simple and without complications we think performing a Bethe ansatz procedure will inform as to where the barrier to constructing an infinity of conserved charges is. Upon reaching this barrier, qHAs can potentially be useful in defining a generalized/relaxed notion of integrability i.e. 'quasi-integrability'. Even further we can use qHAs to learn more about the nature of the deformation parameter space as a manifold [96]. Then we can begin to understand why do some points on the space give rise to integrable theories while others do not. Another possible direction of research is to study how quasi-Hopf symmetry is related to Yangian symmetry and use their relation to further investigate unknown aspects of theories [97].

[2]Our case is concerned with quantum integrability, while [95] focused on classical integrability

In context of the AdS/CFT correspondence, following [91], we would like to understand how the star product could manifest itself in the gravity side, thus possibly obtain a (q,h)-deformed background dual to Leigh-Strassler theory, perhaps obtain more insight concerning methods on how to generate gravity duals for a given CFT. Our analysis was focused on Leigh-Strassler theories, thus on the $\mathrm{AdS}_5\times\mathrm{S}^5$ / $\mathcal{N}=4$ SYM version of the duality. A long term pursuit could be to consider models other than $\mathcal{N}=4$ SYM, that is other versions of the AdS/CFT duality and understand what role quasi-Hopf symmetry plays there. The ABJM model [98] is a possible candidate for the 3-dimensional case and so is the 2-dimensional in [99]. These are a few examples which can be studied from the qHA perspective. These are open questions worth further investigating. Whatever the direction, quasi-Hopf algebras have great potential in their use in physics research.

Appendix A

Details: The Twist

A.1 The classical twist matrices

These are the first-order matrix terms –which we called classical twists– of the (q,h)-twist, F_{qh}, in specific limits of the deformation parameters. Included is a table which summarizes the limits of the deformation parameters at which these classical twists are recovered.

$$f_{\beta_r} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}, \qquad f_{\beta_i} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&-i&0&0&0&0&0\\ 0&0&0&0&0&0&i&0&0\\ 0&i&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-i&0\\ 0&0&-i&0&0&0&0&0&0\\ 0&0&0&0&0&i&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$f_{h_r} = \begin{bmatrix} 0&0&0&0&0&-i&0&i&0\\ 0&0&0&0&0&0&0&0&i\\ 0&0&0&0&-i&0&0&0&0\\ 0&0&0&0&0&0&0&0&-i\\ 0&0&i&0&0&0&-i&0&0\\ i&0&0&0&0&0&0&0&0\\ 0&0&0&0&i&0&0&0&0\\ -i&0&0&0&0&0&0&0&0\\ 0&-i&0&i&0&0&0&0&0 \end{bmatrix}, \quad f_{h_i} = \begin{bmatrix} 0&0&0&0&0&-1&0&1&0\\ 0&0&0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&-1&0&0\\ -1&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&-1&0&1&0&0&0&0&0 \end{bmatrix}$$

TABLE A.1: Deformation parameter limits for classical twists

Classical Twist	Parameters
f_{β_r}	$\bar{q} = {}^1\!/_q,\ h \ = 0 = \bar{h}$
f_{β_i}	$\bar{q} = q\ ,\ h \ = 0 = \bar{h}$
f_{h_r}	$\bar{h} = h = h_r\ ,\ \bar{q} = 1 = q$
f_{h_i}	$\bar{h} = -h = ih_i\ ,\ \bar{q} = 1 = q$

A.2 Entries of the general (q, h) twist

Below are the entries of the twist, F_{qh} in (5.31), corresponding to the full (q, h) deformed theory. These entries are not entirely independent of one another, they can be expressed in terms of one another in the sense that

$$a = 1 - d - c \qquad e = \bar{h}(b - c) + f \tag{A.1}$$

$$e = \bar{h}(b - c) + f \qquad j = \bar{e} \tag{A.2}$$

$$i = \bar{b} \qquad g = \bar{f} \tag{A.3}$$

$$d = c + \frac{\sqrt{2}(1 + q\bar{q})}{\sqrt{(1 + q)(1 + \bar{q})}\sqrt{1 + h\bar{h} + q\bar{q}}} \qquad b = c + \frac{\sqrt{2(1 + q)}}{\sqrt{1 + h\bar{h} + q\bar{q}}} \tag{A.4}$$

From the above one can then choose the set of entries, $\{b, \bar{b}, c, f, \bar{f}\}$, and use it as a basis in terms of which the other entries. It is not altogether clear if this choice is the best for transparency or the most minimal for computation, nonetheless the entries that form the basis in terms of the deformation parameters q and h are given by

$$b = \frac{h\bar{h}q\bar{q} + 3h\bar{h}q - h\bar{h}\bar{q} + h\bar{h} + q^2\bar{q}^2 + q^2\bar{q} - 2q^2 - q\bar{q}^2 + q - \bar{q} + 1}{\sqrt{2}\sqrt{q + 1}\sqrt{\bar{q} + 1}\sqrt{h\bar{h} + q\bar{q} + 1}(2h\bar{h} + q\bar{q} - q - \bar{q} + 1)} + \frac{h\bar{h}}{2h\bar{h} + q\bar{q} - q - \bar{q} + 1} \tag{A.5}$$

$$c = \frac{h\bar{h}q\bar{q} - h\bar{h}q - h\bar{h}\bar{q} - 3h\bar{h} + q^2\bar{q}^2 - q^2\bar{q} - q\bar{q}^2 + q + \bar{q} - 1}{\sqrt{2}\sqrt{q + 1}\sqrt{\bar{q} + 1}\sqrt{h\bar{h} + q\bar{q} + 1}(2h\bar{h} + q\bar{q} - q - \bar{q} + 1)} + \frac{h\bar{h}}{2h\bar{h} + q\bar{q} - q - \bar{q} + 1} \tag{A.6}$$

$$d = \frac{h\bar{h}}{2h\bar{h} + q\bar{q} - q - \bar{q} + 1} - \frac{3h\bar{h}q\bar{q} + h\bar{h}q + h\bar{h}\bar{q} - h\bar{h} + q^2\bar{q}^2 - q^2\bar{q} - q\bar{q}^2 + q + \bar{q} - 1}{\sqrt{2}\sqrt{q + 1}\sqrt{\bar{q} + 1}\sqrt{h\bar{h} + q\bar{q} + 1}(2h\bar{h} + q\bar{q} - q - \bar{q} + 1)} \tag{A.7}$$

$$f = \frac{\bar{h}(q - 1)}{2h\bar{h} + q\bar{q} - q - \bar{q} + 1} - \frac{\sqrt{2}\bar{h}\sqrt{q + 1}(h\bar{h} + q\bar{q} - \bar{q})}{\sqrt{\bar{q} + 1}\sqrt{h\bar{h} + q\bar{q} + 1}(2h\bar{h} + q\bar{q} - q - \bar{q} + 1)} \tag{A.8}$$

$\bar{b}$ and $\bar{f}$ are complex conjugates of b and f respectively.

A.3 Cubic product: Technical details

In computing the cubic $\star$ and $*$ products, one was unavoidably involved with either $[F_{3,L}]$ or $[F_{3,R}]$ and their inverses. Recall that

$$\begin{aligned} [F_{3,L}] &:= (\Delta_1 F^{-1}) \cdot (F_{12}^{-1}) \quad & [F_{3,R}] &:= (\Delta_2 F^{-1}) \cdot (F_{23}^{-1}) \\ [F_{3,L}^{-1}] &:= (F_{12}) \cdot (\Delta_1 F) & [F_{3,R}^{-1}] &:= (F_{23}) \cdot (\Delta_2 F) \end{aligned} \tag{A.9}$$

The action of the co-product Δ was defined at the level of the algebra (See Chapter 4) while the twist is an object that extends over two copies of the group. This has necessitated that the twist be expressed as an exponential of a linear combination of the generators of the group. Having exponentiated the twist, one applies the co-product to obtain for example

$$\Delta_1 F = \Delta_1 e^{f_{qh}} = e^{\Delta_1 f_{qh}} = e^{(f_{13}+f_{23})} \text{ and } \Delta_2 F = e^{\Delta_2 f_{qh}} = e^{(f_{12}+f_{13})} \tag{A.10}$$

This implies that the action of the co-product on the twist is an exponential of a linear combination of various classical twists which are extended over three copies of the algebra i.e. these objects $\Delta_1 f_{qh}$, $\Delta_2 f_{qh}$ are matrices of size 27×27. The exponentiation of these is computationally expensive. Below is a sketch of the procedure used to overcome/avoid this. Call to mind that if V is a matrix that diagonalizes another matrix M, that is $V^\dagger D V = M$ then

$$M = V^\dagger D V \;\longrightarrow\; e^M = V^\dagger e^D V \tag{A.11}$$

where D is a diagonal matrix. Let $M_L := \Delta_1 f_{qh}$ and $M_R := \Delta_2 f_{qh}$ then the eigen equations will be

$$E_i^{(L)} M_L = \lambda_i E_i^{(L)} \quad \text{and} \quad E_j^{(R)} M_R = \lambda_j E_j^{(R)} \tag{A.12}$$

where E_i^L is the i-th eigenvector of M_L whose corresponding eigenvalue is λ_i[1],so also for M_R. Thus it follows that

$$\Delta_1 F = e^{\Delta_1 f_{qh}} = e^{M_L} = (V^{(L)})^\dagger e^{D_L} V^{(L)} \tag{A.13}$$

$$\Delta_2 F = e^{\Delta_2 f_{qh}} = e^{M_R} = (V^{(R)})^\dagger e^{D_R} V^{(R)} \tag{A.14}$$

where the diagonalizing matrix $V^{(L)}$ is made from the eigenvectors $E^{(L)}$, that is $E_i^{(L)}$ is the i-th row of $V^{(L)}$. The diagonal matrices D_L and D_R are equal because $M_L = P_{13} M_R P_{13}$ which also implies that the eigenvectors are related to one another. Consolidating everything, the conclusion is

[1] a number, not to be confused with the Gell-Mann matrices

$$\begin{aligned} \Delta_1 F = e^{M_L} = (V^{(L)})^\dagger e^{D_L} V^{(L)} \ , \qquad \Delta_2 F = e^{M_R} = (V^{(R)})^\dagger e^{D_R} V^{(R)} \\ \Delta_1 F^{-1} = e^{-M_L} = (V^{(L)})^\dagger e^{-D_L} V^{(L)} \ , \quad \Delta_2 F^{-1} = e^{-M_R} = (V^{(R)})^\dagger e^{-D_R} V^{(R)} \end{aligned} \tag{A.15}$$

where the inverse case is also shown as the same arguments hold. The diagonal matrices have the eigenvalues of either of the M's. This is the procedure that was used to compute the cubic terms $[F_{3,X}]$, $[F^{-1}_{3,X}]$, $X = \{L, R\}$ and the co-associator ϕ.

A.4 Cubic terms: imaginary β case

Contained in this section are the non-zero entries of the cubic relations $[F_{3,L}]$ for the case where β is imaginary hence q is real and $h = 0 = \bar{h}$. In order to contain these entries the substitution $\rho = \arccos\left(\frac{1+q}{\sqrt{2(1+q^2)}}\right)$ was made. The entries displayed are those with the first index fixed to 1 and the rest are recoverable via the $\mathbb{Z}_3$ symmetry which entries enjoy

$$[F_{3,L}]^{1\ 1\ 1}_{1\ 1\ 1} = 1 \tag{A.16}$$

$$[F_{3,L}]^{1\ 1\ 2}_{1\ 1\ 2} = \cos\left(\sqrt{2}\rho\right) \tag{A.17}$$

$$[F_{3,L}]^{1\ 1\ 3}_{1\ 1\ 3} = \cos\left(\sqrt{2}\rho\right) \tag{A.18}$$

$$[F_{3,L}]^{1\ 1\ 2}_{1\ 2\ 1} = -\frac{q\sin\left(\sqrt{2}\rho\right)}{\sqrt{q^2+1}} \tag{A.19}$$

$$[F_{3,L}]^{1\ 1\ 3}_{1\ 3\ 1} = \frac{\sin\left(\sqrt{2}\rho\right)}{\sqrt{q^2+1}} \tag{A.20}$$

$$[F_{3,L}]^{1\ 2\ 1}_{1\ 1\ 2} = \frac{\sin\left(\sqrt{2}\rho\right)}{\sqrt{2}} \tag{A.21}$$

$$[F_{3,L}]^{1\ 2\ 1}_{1\ 2\ 1} = \frac{q\cos\left(\sqrt{2}\rho\right)+1}{\sqrt{2}\sqrt{q^2+1}} \tag{A.22}$$

$$[F_{3,L}]^{1\ 2\ 2}_{1\ 2\ 2} = \frac{q\cos\left(\sqrt{2}\rho\right)+1}{\sqrt{2}\sqrt{q^2+1}} \tag{A.23}$$

$$[F_{3,L}]^{1\ 2\ 3}_{1\ 2\ 3} = \frac{(q+1)(2\cos\left(\sqrt{3}\rho\right)+1)}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.24}$$

$$[F_{3,L}]^{1\ 2\ 3}_{1\ 3\ 2} = \frac{-\sqrt{3}(q+1)\sin\left(\sqrt{3}\rho\right)+(q-1)\cos\left(\sqrt{3}\rho\right)-q+1}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.25}$$

$$[F_{3,L}]^{1\ 3\ 1}_{1\ 1\ 3} = -\frac{\sin\left(\sqrt{2}\rho\right)}{\sqrt{2}} \tag{A.26}$$

$$[F_{3,L}]^{1\ 3\ 2}_{1\ 2\ 3} = \frac{\sqrt{3}(q+1)\sin\big(\sqrt{3}\rho\big) - (q-1)\cos\big(\sqrt{3}\rho\big) + q - 1}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.27}$$

$$[F_{3,L}]^{1\ 3\ 1}_{1\ 3\ 1} = \frac{q + \cos\big(\sqrt{2}\rho\big)}{\sqrt{2}\sqrt{q^2+1}} \tag{A.28}$$

$$[F_{3,L}]^{1\ 3\ 2}_{1\ 3\ 2} = \frac{(q+1)(2\cos\big(\sqrt{3}\rho\big) + 1)}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.29}$$

$$[F_{3,L}]^{1\ 3\ 3}_{1\ 3\ 3} = \frac{q + \cos\big(\sqrt{2}\rho\big)}{\sqrt{2}\sqrt{q^2+1}} \tag{A.30}$$

$$[F_{3,L}]^{1\ 1\ 2}_{2\ 1\ 1} = -\frac{\sin\big(\sqrt{2}\rho\big)}{\sqrt{q^2+1}} \tag{A.31}$$

$$[F_{3,L}]^{1\ 2\ 1}_{2\ 1\ 1} = \frac{\cos\big(\sqrt{2}\rho\big) - q}{\sqrt{2}\sqrt{q^2+1}} \tag{A.32}$$

$$[F_{3,L}]^{1\ 2\ 2}_{2\ 1\ 2} = \frac{\cos\big(\sqrt{2}\rho\big) - q}{\sqrt{2}\sqrt{q^2+1}} \tag{A.33}$$

$$[F_{3,L}]^{1\ 2\ 3}_{2\ 1\ 3} = -\frac{(q-1)(2\cos\big(\sqrt{3}\rho\big) + 1)}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.34}$$

$$[F_{3,L}]^{1\ 2\ 2}_{2\ 2\ 1} = -\frac{\sin\big(\sqrt{2}\rho\big)}{\sqrt{2}} \tag{A.35}$$

$$[F_{3,L}]^{1\ 2\ 3}_{2\ 3\ 1} = \frac{\sqrt{3}(q-1)\sin\big(\sqrt{3}\rho\big) - (q+1)\cos\big(\sqrt{3}\rho\big) + q + 1}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.36}$$

$$[F_{3,L}]^{1\ 3\ 2}_{2\ 1\ 3} = \frac{-\sqrt{3}(q-1)\sin\big(\sqrt{3}\rho\big) - (q+1)\cos\big(\sqrt{3}\rho\big) + q + 1}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.37}$$

$$[F_{3,L}]^{1\ 3\ 2}_{2\ 3\ 1} = \frac{-\sqrt{3}(q+1)\sin\big(\sqrt{3}\rho\big) - (q-1)\cos\big(\sqrt{3}\rho\big) + q - 1}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.38}$$

$$[F_{3,L}]^{1\ 1\ 3}_{3\ 1\ 1} = \frac{q\sin\big(\sqrt{2}\rho\big)}{\sqrt{q^2+1}} \tag{A.39}$$

$$[F_{3,L}]^{1\ 2\ 3}_{3\ 1\ 2} = \frac{-\sqrt{3}(q-1)\sin\big(\sqrt{3}\rho\big) - (q+1)\cos\big(\sqrt{3}\rho\big) + q + 1}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.40}$$

$$[F_{3,L}]^{1\ 2\ 3}_{3\ 2\ 1} = \frac{\sqrt{3}(q+1)\sin\big(\sqrt{3}\rho\big) + (q-1)\cos\big(\sqrt{3}\rho\big) - q + 1}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.41}$$

$$[F_{3,L}]^{1\ 3\ 1}_{3\ 1\ 1} = \frac{q\cos\big(\sqrt{2}\rho\big) - 1}{\sqrt{2}\sqrt{q^2+1}} \tag{A.42}$$

$$[F_{3,L}]^{1\ 3\ 2}_{3\ 1\ 2} = \frac{(q-1)(2\cos\big(\sqrt{3}\rho\big) + 1)}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.43}$$

$$[F_{3,L}]^{1\ 3\ 3}_{3\ 1\ 3} = \frac{q\cos\big(\sqrt{2}\rho\big) - 1}{\sqrt{2}\sqrt{q^2+1}} \tag{A.44}$$

$$[F_{3,L}]^{1\ 3\ 2}_{3\ 2\ 1} = \frac{\sqrt{3}(q-1)\sin\big(\sqrt{3}\rho\big) - (q+1)\cos\big(\sqrt{3}\rho\big) + q + 1}{3\sqrt{2}\sqrt{q^2+1}} \tag{A.45}$$

$$[F_{3,L}]^{1\ 3\ 3}_{3\ 3\ 1} = \frac{\sin\big(\sqrt{2}\rho\big)}{\sqrt{2}} \tag{A.46}$$

A.5 G-tensor

Below is a list of the non-zero entries of G-tensor whose defining relation is in (6.47) and the rest of the entries are recoverable by the use of the $\mathbb{Z}_3$ symmetry which the G-tensor inherits from the R-matrix.

$$G^{1\ 1}_{1\ 1} = \frac{-q\bar{q}(h\bar{h}(a+d)+c+d)+q^2\bar{q}^2(a+c)-ah\bar{h}+a+ch^2\bar{h}^2-ch\bar{h}+dh^2\bar{h}^2+d}{2(h^2\bar{h}^2-h(\bar{h}q\bar{q}+\bar{h})+q\bar{q}(q\bar{q}-1)+1)} \tag{A.47}$$

$$G^{1\ 2}_{1\ 2} = \frac{(h\bar{h}+q\bar{q}+1)(bh\bar{h}\bar{q}+bq^2+f(\bar{h}^2-hq\bar{q})+h^2j\bar{q}^2+\bar{h}jq)}{2(h^3\bar{q}^3+3h\bar{h}q\bar{q}-\bar{h}^3+q^3)} \tag{A.48}$$

$$G^{1\ 3}_{1\ 3} = -\frac{(h\bar{h}+q\bar{q}+1)(\bar{h}q(e\bar{h}q+hi)+eh\bar{q}+g(h^2-\bar{h}q\bar{q})+i\bar{q}^2)}{2(h^3-3h\bar{h}q\bar{q}-\bar{h}^3q^3-\bar{q}^3)} \tag{A.49}$$

$$G^{1\ 1}_{2\ 3} = -\frac{(h\bar{h}+q\bar{q}+1)(eh\bar{h}q+e\bar{q}^2+gh\bar{q}+g\bar{h}^2q^2+h^2i-\bar{h}iq\bar{q})}{2(h^3-3h\bar{h}q\bar{q}-\bar{h}^3q^3-\bar{q}^3)} \tag{A.50}$$

$$G^{1\ 2}_{2\ 1} = \frac{-q\bar{q}(h\bar{h}(a+c)+a+d)+ah^2\bar{h}^2+a-h\bar{h}(c+d)+q^2\bar{q}^2(c+d)+c+dh^2\bar{h}^2}{2(h^2\bar{h}^2-h(\bar{h}q\bar{q}+\bar{h})+q\bar{q}(q\bar{q}-1)+1)} \tag{A.51}$$

$$G^{1\ 3}_{2\ 2} = \frac{(h\bar{h}+q\bar{q}+1)(b(\bar{h}^2-hq\bar{q})+fh^2\bar{q}^2+f\bar{h}q+h\bar{h}j\bar{q}+jq^2)}{2(h^3\bar{q}^3+3h\bar{h}q\bar{q}-\bar{h}^3+q^3)} \tag{A.52}$$

$$G^{1\ 1}_{3\ 2} = \frac{(h\bar{h}+q\bar{q}+1)(\bar{h}(bq+fh\bar{q})+h\bar{q}(bh\bar{q}-jq)+fq^2+\bar{h}^2j)}{2(h^3\bar{q}^3+3h\bar{h}q\bar{q}-\bar{h}^3+q^3)} \tag{A.53}$$

$$G^{1\ 2}_{3\ 3} = -\frac{(h\bar{h}+q\bar{q}+1)(e(h^2-\bar{h}q\bar{q})+\bar{h}q(gh+\bar{h}iq)+g\bar{q}^2+hi\bar{q})}{2(h^3-3h\bar{h}q\bar{q}-\bar{h}^3q^3-\bar{q}^3)} \tag{A.54}$$

$$G^{1\ 3}_{3\ 1} = \frac{-q\bar{q}(a+h\bar{h}(c+d)+c)+q^2\bar{q}^2(a+d)+ah^2\bar{h}^2-ah\bar{h}+ch^2\bar{h}^2+c-dh\bar{h}+d}{2(h^2\bar{h}^2-h(\bar{h}q\bar{q}+\bar{h})+q\bar{q}(q\bar{q}-1)+1)} \tag{A.55}$$

The functions a, d, c, f of q and h are as defined in Appendinx A.2.

Appendix B

Details: The co-associator

B.1 Co-associator: imaginary β case

As in the previous Appendix, here is a presentation of the non-zero terms of the imaginary β deformed co-associator with the first index fixed to 1.

$$\phi^{111}_{111} = 1 \tag{B.1}$$

$$\phi^{112}_{112} = \frac{q(\sin(2\sqrt{2}\rho) + 2\sqrt{2}) - 2\sin(\sqrt{2}\rho) + 2\sqrt{2}\cos(\sqrt{2}\rho)}{4\sqrt{q^2+1}} \tag{B.2}$$

$$\phi^{113}_{113} = \frac{2q(\sin(\sqrt{2}\rho) + \sqrt{2}\cos(\sqrt{2}\rho)) - \sin(2\sqrt{2}\rho) + 2\sqrt{2}}{4\sqrt{q^2+1}} \tag{B.3}$$

$$\phi^{112}_{121} = \frac{q^2\cos(2\sqrt{2}\rho) + q^2 - 4\sqrt{2}q\sin(\sqrt{2}\rho) - 2}{4(q^2+1)} \tag{B.4}$$

$$\phi^{113}_{131} = \frac{-2q^2 + 4\sqrt{2}q\sin(\sqrt{2}\rho) + \cos(2\sqrt{2}\rho) + 1}{4q^2+4} \tag{B.5}$$

$$\phi^{121}_{112} = \frac{2q\sin(\sqrt{2}\rho) - 2\sqrt{2}q\cos(\sqrt{2}\rho) + \sin(2\sqrt{2}\rho) + 2\sqrt{2}}{4\sqrt{q^2+1}} \tag{B.6}$$

$$\phi^{121}_{121} = \frac{2\sqrt{2}(q^2-1)\sin(\sqrt{2}\rho) + 2(q^2+1)\cos(\sqrt{2}\rho) + q\cos(2\sqrt{2}\rho) + 3q}{4(q^2+1)} \tag{B.7}$$

$$\phi^{122}_{122} = \frac{q(\sin(2\sqrt{2}\rho) + 2\sqrt{2}) - 2\sin(\sqrt{2}\rho) + 2\sqrt{2}\cos(\sqrt{2}\rho)}{4\sqrt{q^2+1}} \tag{B.8}$$

$$\phi^{123}_{123} = \frac{\sqrt{3}(q^2-1)\sin(2\sqrt{3}\rho) + (q^2+1)\cos(2\sqrt{3}\rho) + 2q(q+3) + 2}{6(q^2+1)} \tag{B.9}$$

$$\phi^{123}_{132} = \frac{1}{2} - \frac{\sqrt{3}q\sin(2\sqrt{3}\rho) + 3}{3(q^2+1)} \tag{B.10}$$

$$\phi^{131}_{\ 113} = -\frac{q(\sin(2\sqrt{2}\rho) - 2\sqrt{2}) + 2\sin(\sqrt{2}\rho) + 2\sqrt{2}\cos(\sqrt{2}\rho)}{4\sqrt{q^2+1}} \tag{B.11}$$

$$\phi^{132}_{\ 123} = \frac{\sqrt{3}q\sin(2\sqrt{3}\rho) + 3}{3q^2+3} - \frac{1}{2} \tag{B.12}$$

$$\phi^{131}_{\ 131} = \frac{2\sqrt{2}(q^2-1)\sin(\sqrt{2}\rho) + 2(q^2+1)\cos(\sqrt{2}\rho) + q\cos(2\sqrt{2}\rho) + 3q}{4(q^2+1)} \tag{B.13}$$

$$\phi^{132}_{\ 132} = \frac{\sqrt{3}(q^2-1)\sin(2\sqrt{3}\rho) + (q^2+1)\cos(2\sqrt{3}\rho) + 2q(q+3) + 2}{6(q^2+1)} \tag{B.14}$$

$$\phi^{133}_{\ 133} = \frac{2q(\sin(\sqrt{2}\rho) + \sqrt{2}\cos(\sqrt{2}\rho)) - \sin(2\sqrt{2}\rho) + 2\sqrt{2}}{4\sqrt{q^2+1}} \tag{B.15}$$

$$\phi^{112}_{\ 211} = \frac{2\sqrt{2}(q^2-1)\sin(\sqrt{2}\rho) - 2(q^2+1)\cos(\sqrt{2}\rho) + q\cos(2\sqrt{2}\rho) + 3q}{4(q^2+1)} \tag{B.16}$$

$$\phi^{121}_{\ 211} = \frac{-2q^2 + 4\sqrt{2}q\sin(\sqrt{2}\rho) + \cos(2\sqrt{2}\rho) + 1}{4q^2+4} \tag{B.17}$$

$$\phi^{122}_{\ 212} = \frac{2q\sin(\sqrt{2}\rho) - 2\sqrt{2}q\cos(\sqrt{2}\rho) + \sin(2\sqrt{2}\rho) + 2\sqrt{2}}{4\sqrt{q^2+1}} \tag{B.18}$$

$$\phi^{123}_{\ 213} = \frac{\sqrt{3}q\sin(2\sqrt{3}\rho) + 3}{3q^2+3} - \frac{1}{2} \tag{B.19}$$

$$\phi^{122}_{\ 221} = -\frac{1}{2}\sin^2(\sqrt{2}\rho) \tag{B.20}$$

$$\phi^{123}_{\ 231} = \frac{2}{3}\sin^2(\sqrt{3}\rho) \tag{B.21}$$

$$\phi^{132}_{\ 213} = \frac{-\sqrt{3}(q^2-1)\sin(2\sqrt{3}\rho) + (q^2+1)\cos(2\sqrt{3}\rho) + 2(q-3)q + 2}{6(q^2+1)} \tag{B.22}$$

$$\phi^{113}_{\ 311} = \frac{2\sqrt{2}(q^2-1)\sin(\sqrt{2}\rho) - 2(q^2+1)\cos(\sqrt{2}\rho) + q\cos(2\sqrt{2}\rho) + 3q}{4(q^2+1)} \tag{B.23}$$

$$\phi^{123}_{\ 312} = \frac{-\sqrt{3}(q^2-1)\sin(2\sqrt{3}\rho) + (q^2+1)\cos(2\sqrt{3}\rho) + 2(q-3)q + 2}{6(q^2+1)} \tag{B.24}$$

$$\phi^{131}_{\ 311} = \frac{q^2\cos(2\sqrt{2}\rho) + q^2 - 4\sqrt{2}q\sin(\sqrt{2}\rho) - 2}{4(q^2+1)} \tag{B.25}$$

$$\phi^{132}_{\ 312} = \frac{1}{2} - \frac{\sqrt{3}q\sin(2\sqrt{3}\rho) + 3}{3(q^2+1)} \tag{B.26}$$

$$\phi^{133}_{\ 313} = -\frac{q(\sin(2\sqrt{2}\rho) - 2\sqrt{2}) + 2\sin(\sqrt{2}\rho) + 2\sqrt{2}\cos(\sqrt{2}\rho)}{4\sqrt{q^2+1}} \tag{B.27}$$

$$\phi^{132}_{\ 321} = \frac{2}{3}\sin^2(\sqrt{3}\rho) \tag{B.28}$$

$$\phi^{133}_{\ 331} = -\frac{1}{2}\sin^2(\sqrt{2}\rho) \tag{B.29}$$

As we mentioned in the Chapter 5 that the expressions for the general (q, h) are large and cumbersome; therefore we refer the reader to the Mathematica file that associated to [11] for specific details.

B.2 Pentagon relation

A technical detail which the co-associator must satisfy as part of the qHA definition is the pentagon relation (4.49). Here we supply a proof of this for the general (q,h)-deformation qHA. It is useful to note that the trivial co-associator $\phi = \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}$ under the undeformed co-product Δ satisfies the Pentagon relation. Since $\Delta(\mathbb{1}) = \mathbb{1}\otimes\mathbb{1}$ then it follows that

$$\begin{aligned}(\mathbb{1}\otimes\phi)\cdot\big[\Delta_2\phi\big]\cdot(\phi\otimes\mathbb{1}) &= \big[\Delta_3\phi\big]\cdot\big[\Delta_1\phi\big]\\ \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} &= \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\end{aligned} \tag{B.30}$$

The twisted qHA has a co-associator ϕ_F and co-product $\Delta_F := \tilde{\Delta}$ which are explicitly given by

$$\phi_F = F_{23}\cdot\big[\Delta_2 F\big]\cdot\big[\Delta_1 F^{-1}\big]\cdot F_{12}^{-1}\quad,\quad \Delta_F(\cdot) = F[\Delta(\cdot)]F^{-1} = \tilde{\Delta}(\cdot) \tag{B.31}$$

Using the expressions in (5.43) we can compute the different parts of the pentagon relation:

$$(\mathbb{1}\otimes\phi_F) = e^{f_{34}}\cdot e^{f_{24}+f_{23}}\cdot e^{-f_{24}-f_{34}}\cdot e^{-f_{23}} \tag{B.32}$$

$$(\phi_F\otimes\mathbb{1}) = e^{f_{23}}\cdot e^{f_{13}+f_{12}}\cdot e^{-f_{13}-f_{23}}\cdot e^{-f_{12}} \tag{B.33}$$

These expressions live in the vector space made up of four copies of the algebra i.e. $H^{\otimes 4}$, thus (B.33) is not identical to the second equality of (5.43). The computation of the remaining parts of the pentagon result in:

$$\tilde{\Delta}_1\phi_F = F_{12}\cdot\big[\Delta_1\phi_F\big]F_{12}^{-1} \tag{B.34}$$

$$= e^{f_{12}}\cdot\Big[e^{f_{34}}\cdot e^{(f_{13}+f_{23}+f_{14}+f_{24})}\cdot e^{-(f_{14}+f_{24}+f_{34})}\cdot e^{-(f_{13}+f_{23})}\Big]\cdot e^{-f_{12}} \tag{B.35}$$

$$\tilde{\Delta}_2\phi_F = F_{23}\big[\Delta_2\phi_F\big]\cdot F_{23}^{-1} \tag{B.36}$$

$$= e^{f_{23}}\cdot e^{(f_{24}+f_{34})}\cdot e^{(f_{12}+f_{13}+f_{14})}\cdot e^{-(f_{14}+f_{24}+f_{34})}\cdot e^{-(f_{12}+f_{13})}\cdot e^{-f_{23}} \tag{B.37}$$

$$\tilde{\Delta}_3\phi_F = F_{34}\cdot\big[\Delta_3\phi_F\big]\cdot F_{34}^{-1} \tag{B.38}$$

$$= e^{f_{34}}\cdot\Big[e^{(f_{23}+f_{24})}\cdot e^{(f_{12}+f_{13}+f_{14})}\cdot e^{-(f_{13}+f_{14}+f_{23}+f_{24})}\cdot e^{-f_{12}}\Big]\cdot e^{-f_{34}} \tag{B.39}$$

An extra necessary ingredient, noted by explicit calculation, is that $[f_{12}, f_{34}] = 0$. The RHS and LHS of (4.49) simplify to

$$\begin{aligned}\tilde{\Delta}_3\phi_F\cdot\tilde{\Delta}_1\phi_F &= e^{f_{34}}\cdot\Big[e^{(f_{23}+f_{24})}\cdot e^{(f_{12}+f_{13}+f_{14})}\cdot e^{-(f_{14}+f_{24}+f_{34})}\cdot e^{-(f_{13}+f_{23})}\Big]\cdot e^{-f_{12}}\\ &= (\mathbb{1}\otimes\phi_F)\cdot\tilde{\Delta}_2\phi_F\cdot(\phi_F\otimes\mathbb{1}) \qquad \square\end{aligned} \tag{B.40}$$

Appendix C

Discussion: Integrability

In this appendix we provide a short discussion, seeking to explain the connection between the R-matrix introduced in Chapter 3 and integrability.

One of the quantities of interest in QFT is the S-matrix. The elements of the S-matrix correspond to scattering amplitudes and for multi-particle processes the computation of these elements is complicated. The ability to view a multi-particle scattering process as a series of 2-particles scattering processes provides a major simplification. S-matrices for which this can be done are called factorizable. To uncover the root of this property we need to consider the S-matrix operator which we designate with S.

The S-matrix operator S is usually split into non-interacting and interacting part, $S = \mathbb{1} + it$ where t is the transfer matrix which 'transfers' a system from an incoming/input state $|a_i\rangle$ to an outgoing/output state $|a_o\rangle$ [100]. Since QFTs generally have an infinite number of degrees of freedom, computing $\langle a_0| \, t \, |a_i\rangle$ is a complicated task. It is useful to rather carry this out on discretized spacetime, hence a lattice (approximation) of the continuous QFT [3][101].

At this junction having a spin chain picture according to the discussion in Section 3.1.1.2 in mind. This is helpful because discretizing effectively makes spacetime a lattice and each point would then be a site of the spin chain. For simplicity, consider a 1-dimensional periodic spin chain. An interesting object to define is a transition matrix $T^m_{na}(u)$ which transports us from site n to site m of the lattice for a fixed time and is parameterized by u, a spectral parameter. The index a of the transition matrix means it acts on both the site vector space and on an auxiliary vector space as each site has other degrees of freedom. In order to see the effect of going around the full spin chain one defines the monodromy matrix $\hat{T}$ as the product of transition matrices from adjacent sites i.e. for a spin chain of length L, the monodromy matrix is

$$\hat{T}(u) = \prod_{n=0}^{L-1} T_n^{n+1}(u) \tag{C.1}$$

where the auxiliary index a is suppressed. The trace of the monodromy matrix is equal to the transfer matrix t. The nature of the theory being quantum means the product of monodromy matrices do not commute. Here enters the R-matrix to describe/define the commutation relations of monodromy matrices [102] [103] [104]

$$R(u,v)\hat{T}_a(u)\hat{T}_b(v) = \hat{T}_b(v)\hat{T}_a(u)R(u,v) \tag{C.2}$$

In this way is a quantum model defined by its RTT relations and thus the R-matrix becomes the object of study. Moreover if the R-matrix of a (sector of a) given model satisfies the Yang-Baxter equation

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \tag{C.3}$$

then that (sector of the) model is integrable. In order to obtain the conserved charges that commute with the Hamiltonian of the model one can employ algebraic Bethe Ansatz methods [63] [104].

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