Hindawi
Journal of Mathematics
Volume 2019, Article ID 8509494, 15 pages
https://doi.org/10.1155/2019/8509494

Research Article

Common Fixed Points of $(\alpha, \eta) - (\theta, \digamma)$ Rational Contractions with Applications

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Received 3 December 2018; Revised 6 February 2019; Accepted 17 February 2019; Published 26 March 2019

Academic Editor: Basil K. Papadopoulos

We present the notion of set valued $(\alpha, \eta) - (\theta, \digamma)$ rational contraction mappings and then some common fixed point results of such mappings in the setting of metric spaces are established. Some examples are presented to support the concepts introduced and the results proved in this paper. These results unify, extend, and refine various results in the literature. Some fixed point results for both single and multivalued $(\theta, \digamma)$ rational contractions are also obtained in the framework of a space endowed with partial order. As application, we establish the existence of solutions of nonlinear elastic beam equations and first-order periodic problem.

1. Introduction and Preliminaries

Let $(\mathcal{X}, d)$ be a metric space. A mapping $T : \mathcal{X} \longrightarrow \mathcal{X}$ is called a contraction if there exists a constant $c \in [0, 1)$ such that, for any $x, y \in \mathcal{X}$, we have

$$d(Tx, Ty) \leq c d(x, y). \tag{1}$$

The widely known Banach contraction theorem [1] states that a contraction mapping on a complete metric space $\mathcal{X}$ has a unique fixed point; that is, there exists a point x in $\mathcal{X}$ such that $x = Tx$.

In the last few decades, several authors have extended and generalized this principle in various directions.

Jleli and Samet [2] presented a new type of contractive mapping, namely θ-contraction mapping and established an interesting fixed point theorem for such mappings in a generalized metric space. The concept of generalized metric spaces was introduced by Branciari [3], where the triangle inequality is replaced by the inequality $d(w, z) \leq d(w, x) + d(x, y) + d(y, z)$ for all pairwise distinct points $w, x, y, z \in \mathcal{X}$.

Jleli and Samet [2] considered the set Θ of real valued functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ which satisfy the following conditions:

(θ_1) θ is nondecreasing;

(θ_2) for each sequence $\{c_n\} \subset (0, \infty)$, $\lim_{n \longrightarrow \infty} \theta(c_n) = 1$ if and only if $\lim_{n \longrightarrow \infty} c_n = 0^+$;

(θ_3) there exist $m \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{c \longrightarrow 0^+}((\theta(c) - 1)/c^m) = l$.

Example 1. Define $\theta_i : (0, \infty) \longrightarrow (1, \infty)$ where $i = 1, 2$ by

$$\theta_1(t) = e^{\sqrt{t}} \quad \text{and} \quad \theta_2(t) = e^{\sqrt{te^t}}. \tag{2}$$

Then $\theta_1, \theta_2 \in \Theta$.

A mapping $T : \mathcal{X} \longrightarrow \mathcal{X}$ on a metric space $(\mathcal{X}, d)$ is called a $\theta-$contraction if for any $x, y \in \mathcal{X}$ and $\theta \in \Theta$, we have

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^c \tag{3}$$

whenever $d(Tx, Ty) > 0$ and $0 \leq c < 1$.

Theorem 2 (see [2]). *Let $(\mathscr{X}, d)$ be a complete generalized metric space and $T : \mathscr{X} \longrightarrow \mathscr{X}$. If there exist $\theta \in \Theta$ and $0 \leq c < 1$ such that*

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^c \tag{4}$$

holds for any $x, y \in \mathscr{X}$ whenever $d(Tx, Ty) \neq 0$. Then T has a fixed point.

Ahmad *et al.* [4] modified the class Θ of mappings as follows:

$$\Omega = \left\{\theta : (0, \infty) \longrightarrow (1, \infty) \text{ satisfy } \theta_1, \theta_2 \text{ and } \theta_3'\right\} \tag{5}$$

where (θ_3') $\theta : (0, \infty) \longrightarrow (1, \infty)$ is continuous.

Example 3. Define $\theta_i : (0, \infty) \longrightarrow (1, \infty)$ for $i = 1, 2, 3$ by

$$\begin{gathered} \theta_1(t) = e^t, \\ \theta_2(t) = e^{\sqrt{t}} \\ \text{and } \theta_3(t) = e^{te^t}. \end{gathered} \tag{6}$$

Then, $\theta_1, \theta_2, \theta_3 \in \Omega$.

Authors in [4] considered the following result of Jleli and Samet [2] with the function $\theta \in \Omega$ instead of $\theta \in \Theta$:

Theorem 4. *Let $(\mathscr{X}, d)$ be a complete metric space and $T : \mathscr{X} \longrightarrow \mathscr{X}$ a θ-contraction, where $\theta \in \Omega$. Then T has a unique fixed point $\tilde{x} \in \mathscr{X}$ and for any $x_0 \in \mathscr{X}$, the sequence $\{T^n x_0\}$ converges to $\tilde{x}$.*

Note that the Banach contraction theorem immediately follows from the above theorem.

Let $\mathscr{X}$ be a nonempty set endowed with a metric d. Let $P(\mathscr{X})$ be the set of all nonempty subsets of $\mathscr{X}$, $K(\mathscr{X})$ denotes the set of all nonempty compact subsets of $\mathscr{X}$, and $CB(\mathscr{X})$ denotes the set all nonempty closed and bounded subsets. For $A, B \in CB(\mathscr{X})$ and $x \in \mathscr{X}$, define distance of a point x from the set A by

$$d(x, A) = \inf_{x \in A} d(x, A). \tag{7}$$

A mapping $H : CB(\mathscr{X}) \times CB(\mathscr{X}) \longrightarrow \mathbb{R}^+$ defined by

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\} \tag{8}$$

is called the generalized Pompeiu-Hausdorff distance induced by d.

Let $T : \mathscr{X} \longrightarrow CB(\mathscr{X})$. A point $x \in \mathscr{X}$ is called a fixed point of T if $x \in Tx$.

Nadler [5] obtained the following multivalued version of Banach contraction principle.

Theorem 5. *Let $(\mathscr{X}, d)$ be a complete metric space. If $T : \mathscr{X} \longrightarrow CB(\mathscr{X})$ satisfies*

$$H(Tx, Ty) \leq cd(x, y) \tag{9}$$

for any $x, y \in \mathscr{X}$ and $c \in (0, 1)$, then T has a fixed point.

Afterwards, many researchers have obtained fixed point results for multivalued mappings satisfying certain generalized contractive conditions. Hançer *et al.* [6] introduced multivalued θ– contraction mappings as follows.

Let $(\mathscr{X}, d)$ be a metric space, $\theta \in \Theta$, and $T : \mathscr{X} \longrightarrow P(\mathscr{X})$. Then, T is called a multivalued θ-contraction if, for any $x, y \in \mathscr{X}$,

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^c \tag{10}$$

holds whenever $H(Tx, Ty) > 0$ where $0 \leq c < 1$.

They established the following fixed point results for multivalued θ-contraction mappings on complete metric spaces.

Theorem 6. *Let $(\mathscr{X}, d)$ be a complete metric space and $T : \mathscr{X} \longrightarrow K(\mathscr{X})$ a multivalued θ-contraction. Then T has a fixed point.*

For further results in this direction, we refer to [7–10].

Another variation of contraction mapping that can be found in literature is α-contraction mapping.

Asl *et al.* [11] initiated the concept of α_*-admissibility in case of multivalued mappings, whereas Mohammadi *et al.* [12] presented the notion of α-admissibility in case of multifunctions.

Karapinar *et al.* [13] presented the idea of a triangular α-admissible mapping.

Definition 7. Let $\alpha, \eta : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}^+$. A pair (α, η) is called triangular if for any $x, y, z \in \mathscr{X}$, $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, z) \geq \eta(y, z)$ imply that $\alpha(x, z) \geq \eta(x, z)$.

Recently, Abbas *et al.* [14] proposed a concept of α -closed mappings for set valued mappings. We present the following generalization of the definition.

Definition 8. Let $T, S : \mathscr{X} \longrightarrow P(\mathscr{X})$ and $\alpha, \eta : \mathscr{X} \times \mathscr{X} \longrightarrow \mathbb{R}^+$. We say that a pair (T, S) is triangular (α, η)-closed if the pair (α, η) is triangular and for any $x, y \in \mathscr{X}$ with $\alpha(x, y) \geq \eta(x, y)$ we have $\alpha(u, v) \geq \eta(u, v)$ for all $u \in Tx$ and $v \in Su$.

If $S = T$ then a mapping T which is triangular (α, η) closed is referred to simply as a T triangular (α, η)-closed mapping.

Example 9. Let $\mathscr{X} = [0, 1]$. Let $T, S : \mathscr{X} \longrightarrow P(\mathscr{X})$ be defined by

$$\begin{gathered} Tx = \{1, x\}. \\ \text{and } Sx = \left\{0, x^2\right\} \end{gathered} \tag{11}$$

Define the mappings $\alpha, \eta : \mathscr{X} \times \mathscr{X} \longrightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} e^{x-y}, & \text{whenever } x \geq y \\ 0, & \text{whenever } x < y \end{cases} \tag{12}$$

and

$$\eta(x, y) = x^2 \quad \text{for all } x \in [0, 1]. \tag{13}$$

It is obvious that the pair (T, S) is triangular (α, η)-closed.

The following lemma is crucial in our results.

Lemma 10. *Let $T, S : \mathscr{X} \longrightarrow P(\mathscr{X})$. Suppose that the pairs (T, S) and (S, T) are a triangular (α, η)-closed. Assume that there exists $x_0 \in \mathscr{X}$ with $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ where $x_1 \in Tx_0$. Define sequences $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$, then $\alpha(x_m, x_n) \geq \eta(x_m, x_n)$ for all $m, n \in \mathbb{N}$ with $m > n$.*

Proof. By assumption, there exist $x_0 \in \mathscr{X}$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$. Since (T, S) is (α, η)-closed and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ we obtain $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ where $x_1 \in Tx_0$ and $x_2 \in Sx_1$. As (S, T) is (α, η)-closed, we have x_3 in Tx_2 such that $\alpha(x_2, x_3) \geq \eta(x_2, x_3)$. Continuing this way, we have sequences $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $\alpha(x_{n+1}, x_{n+2}) \geq \eta(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}$.

Since the pair (α, η) is triangular we obtain that

$$\alpha(x_n, x_{n+2}) \geq \eta(x_n, x_{n+2}). \tag{14}$$

Thus by induction, we have

$$\alpha(x_n, x_m) \geq \eta(x_n, x_m). \tag{15}$$

for any $m, n \in \mathbb{N}$ with $m > n$. □

Parvaneh *et al.* [15] introduced the concept of $\alpha - H\Theta$-contraction with respect to a family of functions H and obtained some $\theta-$contraction fixed point results in metric and ordered metric spaces.

They introduced the following family of functions:

Let $\digamma$ denote the set of functions $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^+$ satisfying condition (A^*):

For all $s_1, s_2, s_3, s_4 \in \mathbb{R}^+$ with $s_1.s_2.s_3.s_4 = 0$ there exists $c \in [0, 1)$ such that

$$F(s_1, s_2, s_3, s_4) = c \tag{16}$$

Following are some examples of such functions [15].

Example 11. $F(s_1, s_2, s_3, s_4) = L\min(s_1, s_2, s_3, s_4) + c$, where $L \in \mathbb{R}^+$ and $c \in [0, 1)$.

Example 12. $F(s_1, s_2, s_3, s_4) = ce^{L\min(s_1,s_2,s_3,s_4)}$, where $L \in \mathbb{R}^+$ and $c \in [0, 1)$.

The following definition which is a generalization of $\alpha-$continuity [16] is needed in the sequel.

Definition 13. Let $(\mathscr{X}, d)$ be a metric space, $\alpha, \eta : \mathscr{X} \times \mathscr{X} \longrightarrow \mathbb{R}^+$ and $T, S : \mathscr{X} \longrightarrow CB(\mathscr{X})$. A pair (T, S) is (α, η)-continuous at the point $x \in \mathscr{X}$ if, for any sequence $\{x_n\}$ in $\mathscr{X}$, $\lim_{n\longrightarrow\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ implies that $\lim_{n\longrightarrow\infty} H(Tx_n, Sx) = 0$. We say that pair (T, S) is (α, η)-continuous on $(CB(\mathscr{X}), H)$ if the pair (T, S) is (α, η)-continuous on each $x \in \mathscr{X}$.

In this paper, we introduce multivalued (α, η)-$(\theta, \digamma)$ rational contraction pair of multivalued mappings and prove the existence of common fixed points of the pair in a metric space. We also obtain some fixed point results for both single and multivalued $(\theta, \digamma)$ rational contraction mappings in a space endowed with a partial order. As application, we establish the existence of solutions of nonlinear elastic beam equations and first-order periodic problem.

2. Common Fixed Point Results

Throughout this section we assume that $(\mathscr{X}, d)$ is a metric space and $\theta \in \Omega$ where $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfies (θ_1), (θ_2) and (θ'_3). Let $\digamma$ be a family of continuous and nondecreasing functions where $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^+$ for $F \in \digamma$.

We now present the following definitions:

Definition 14. Let $\alpha, \eta : \mathscr{X} \times \mathscr{X} \longrightarrow \mathbb{R}^+$, $\theta \in \Omega$, $T, S : \mathscr{X} \longrightarrow P(\mathscr{X})$, and $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^+$.

(1) A pair (T, S) is called a multivalued $(\alpha, \eta) - (\theta, \digamma)$ rational contraction pair if, for any $x, y \in \mathscr{X}$ with $\alpha(x, y) \geq \eta(x, y)$ and $H(Tx, Sy) > 0$, the following condition holds:

$$\theta(H(Tx, Sy)) \leq \{\theta(M_{T,S}(x, y))\}^{F(N_{T,S}(x,y))} \tag{17}$$

where

$$M_{T,S}(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}d(x, Sy), \frac{d(y, Sy)(1 + d(x, Tx))}{1 + d(x, y)}\right\} \tag{18}$$

and

$$N_{T,S}(x, y) = \{d(x, Tx), d(y, Sy), d(x, Sy), d(y, Tx)\}. \tag{19}$$

(2) A pair (S, T) is called a multivalued $(\alpha, \eta) - (\theta, \digamma)$ rational contraction if, for any $x, y \in \mathscr{X}$ with $\alpha(x, y) \geq \eta(x, y)$ and $H(Sx, Ty) > 0$, the following condition holds:

$$\theta(H(Sx, Ty)) \leq \{\theta(M_{S,T}(x, y))\}^{F(N_{S,T}(x,y))} \tag{20}$$

where

$$M_{S,T}(x, y) = \max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}d(x, Ty), \frac{d(y, Ty)(1 + d(x, Sx))}{1 + d(x, y)}\right\} \tag{21}$$

and

$$N_{S,T}(x, y) = \{d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\}. \tag{22}$$

(3) A mapping $T : \mathcal{X} \longrightarrow P(\mathcal{X})$ is called a multivalued $(\alpha,\eta) - (\theta,\digamma)$ rational contraction if, for any $x, y \in \mathcal{X}$ with $\alpha(x,y) \geq \eta(x,y)$ and $H(Tx,Ty) > 0$, the following condition holds:

$$\alpha(x,y)\theta(H(Tx,Ty)) \leq \{\theta(M(x,y))\}^{F(N(x,y))} \tag{23}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}d(x,Ty), \frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}\right\} \tag{24}$$

and

$$N(x,y) = \{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{25}$$

Remark 15.

(1) If $\alpha,\eta : \mathcal{X}\times\mathcal{X} \longrightarrow \mathbb{R}^+$ are defined as $\alpha = \eta(x,y) = 1$ for all $x, y \in \mathcal{X}$ in Definition 14, then the pairs of mappings (T,S) and (S,T) are multivalued $(\theta,\digamma)$ generalized rational contractions.

(2) If $\alpha,\eta : \mathcal{X}\times\mathcal{X} \longrightarrow \mathbb{R}^+$ are defined as $\alpha(x,y) = \eta(x,y) = 1$ for all $x,y \in \mathcal{X}$ in Definition 14, then T is a multivalued $(\theta,\digamma)$ generalized rational contraction mapping.

Theorem 16. *Let $T,S : \mathcal{X} \longrightarrow CB(\mathcal{X})$. Suppose that the pairs (T,S) and (S,T) are multivalued $(\alpha,\eta) - (\theta,\digamma)$ rational contractions such that*

(C1) *the pairs (T,S) and (S,T) are triangular (α,η)-closed;*

(C2) *there exist $x_0 \in \mathcal{X}$ and $x_1 \in Tx_0$ with $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$;*

(C3) *(T,S) and (S,T) are (α,η)-continuous.*

Then there exists $\tilde{x} \in \mathcal{X}$ such that $\tilde{x} \in T\tilde{x} \cap S\tilde{x}$.

Proof. If $M_{T,S}(x,y) = M_{S,T}(x,y) = 0$, for some $x,y \in \mathcal{X}$, then we have our conclusion. Assume that $M_{T,S}, M_{S,T} > 0$ for all $x,y \in \mathcal{X}$. By assumption there exist $x_0 \in \mathcal{X}$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$. If $x_0 = x_1$ or $d(x_1,S_1) = 0$ then the result follows. Assume that $x_0 \neq x_1$ and $x_1 \notin Sx_1$ then

$$0 < d(x_1,Sx_1) \leq H(Tx_0,Sx_1). \tag{26}$$

This implies

$$1 < \theta(d(x_1,Sx_1)) \leq \theta(H(Tx_0,Sx_1)). \tag{27}$$

Since $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$ and (T,S) is multivalued $(\alpha,\eta) - (\theta,\digamma)$ rational contraction, we obtain that

$$\begin{aligned} 1 < \theta(d(x_1,Sx_1)) &\leq (H(Tx_0,Sx_1)) \\ &\leq \{\theta(M_{T,S}(x_0,x_1))\}^{F(N_{T,S}(x_0,x_1))} \end{aligned} \tag{28}$$

where

$$\begin{aligned} N_{T,S}(x_0,x_1) &= \{d(x_0,Tx_0), d(x_1,Sx_1), d(x_0,Sx_1), d(x_1,Tx_0)\} \\ &\leq \{d(x_0,x_1), d(x_1,Sx_1), d(x_0,Sx_1), d(x_1,x_1)\} \\ &= \{d(x_0,x_1), d(x_1,Sx_1), d(x_0,Sx_1), 0\}. \end{aligned} \tag{29}$$

Since $d(x_0,x_1).d(x_1,Sx_1).d(x_0,Sx_1).0 = 0$, by (A^*) there exists $c \in [0,1)$ such that

$$F(d(x_0,x_1), d(x_1,Sx_1), d(x_0,Sx_1), 0) = c. \tag{30}$$

Therefore, from (29) and using the fact that F is nondecreasing we obtain

$$\begin{aligned} F(N_{T,S}(x_0,x_1)) \\ \leq F(d(x_0,x_1), d(x_1,Sx_1), d(x_0,Sx_1), 0) = c. \end{aligned} \tag{31}$$

Also,

$$\begin{aligned} M_{T,S}(x_0,x_1) &= \max\Big\{d(x_0,x_1), d(x_0,Tx_0), d(x_1,Sx_1), \frac{1}{2}d(x_0,Sx_1), \frac{d(x_1,Sx_1)(1+d(x_0,Tx_0))}{1+d(x_0,x_1)}\Big\} \\ &\leq \max\Big\{d(x_0,x_1), d(x_0,x_1), d(x_1,Sx_1), \frac{1}{2}d(x_0,Sx_1), \frac{d(x_1,Sx_1)(1+d(x_0,x_1))}{1+d(x_0,x_1)}\Big\} \\ &= \max\Big\{d(x_0,x_1), d(x_0,x_1), d(x_1,Sx_1), \frac{1}{2}d(x_0,Sx_1), \frac{d(x_1,Sx_1)(1+d(x_0,x_1))}{1+d(x_0,x_1)}\Big\}, \end{aligned} \tag{32}$$

since

$$\begin{aligned} \frac{1}{2}d(x_0,Sx_1) &\leq \frac{1}{2}d(x_0,x_1) + \frac{1}{2}d(x_1,Sx_1) \\ &\leq \max\{d(x_0,x_1), d(x_1,Sx_1)\}. \end{aligned} \tag{33}$$

Thus,

$$M_{T,S}(x_0,x_1) \leq \max\{d(x_0,x_1), d(x_1,Sx_1)\}. \tag{34}$$

Replacing (31) and (34) in (28) we get

$$\begin{aligned} \theta(d(x_1,Sx_1)) \\ \leq \{\theta(\max\{d(x_0,x_1), d(x_1,Sx_1)\})\}^c. \end{aligned} \tag{35}$$

If $\max\{d(x_0,x_1), d(x_1,Sx_1))\} = d(x_1,Sx_1)$, then

$$\theta(d(x_1,Sx_1)) \leq (d(x_1,Sx_1)))^c, \tag{36}$$

a contradiction. Hence $\max\{d(x_0,x_1), d(x_1,Sx_1))\} = d(x_0,x_1)$. Therefore,

$$1 \leq \theta(d(x_1,Sx_1)) \leq (\theta(d(x_0,x_1)))^c. \quad (37)$$

By (θ'_3) we have

$$\theta(d(x_1,Sx_1)) = \inf_{y\in Sx_1} \theta(d(x_1,y)). \quad (38)$$

Thus, there exists $x_2 \in Sx_1$ such that

$$\theta(d(x_1,Sx_1)) = \theta(d(x_1,x_2)). \quad (39)$$

Then from (37) we have

$$\theta(d(x_1,x_2)) \leq (\theta(d(x_0,x_1)))^c. \quad (40)$$

Since $x_1 \in Tx_0$, $x_2 \in Sx_1$, $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$, and (T,S) is (α,η)-closed, we have $\alpha(x_1,x_2) \geq \eta(x_1,x_2)$. If $x_1 = x_2$ or $d(x_2,Tx_2) = 0$ then the result follows. Suppose that $x_1 \neq x_2$ and $x_2 \notin Tx_2$. Thus

$$1 < \theta(d(x_2,Tx_2)) \leq \theta(H(Sx_1,Tx_2)). \quad (41)$$

As $\alpha(x_1,x_2) \geq \eta(x_1,x_2)$, we have

$$\begin{aligned} 1 &< \theta(d(x_2,Tx_2)) \leq \theta(H(Sx_1,Tx_2)) \\ &\leq \{\theta(M_{S,T}(x_1,x_2))\}^{F(N_{T,S}(x_1,x_2))} \end{aligned} \quad (42)$$

where

$$\begin{aligned} &N_{T,S}(x_1,x_2) \\ &= \{d(x_1,Tx_1), d(x_2,Sx_2), d(x_1,Sx_2), d(x_2,Tx_1)\} \\ &\leq \{d(x_1,x_2), d(x_2,Sx_2), d(x_1,Sx_2), d(x_2,x_2)\} \\ &= \{d(x_1,x_2), d(x_2,Sx_2), d(x_1,Sx_2), 0\}. \end{aligned} \quad (43)$$

Since $d(x_1,x_2).d(x_2,Sx_2).d(x_1,Sx_2).0 = 0$, by (A^*) there exists $c \in [0,1)$ such that

$$F(d(x_1,x_2), d(x_2,Sx_2), d(x_1,Sx_2), 0) = c \quad (44)$$

Therefore, from (43) and using the fact that F is nondecreasing we have

$$\begin{aligned} &F(N_{T,S}(x_0,x_1)) \\ &\leq F(d(x_1,x_2), d(x_2,Sx_2), d(x_1,Sx_2), 0) = c. \end{aligned} \quad (45)$$

Further,

$$\begin{aligned} M_{S,T}(x_1,x_2) &= \max\Big\{d(x_1,x_2), d(x_1,Sx_1), \\ &d(x_2,Tx_2), \frac{1}{2}d(x_1,Tx_2), \\ &\frac{d(x_2,Tx_2)(1+d(x_1,Sx_1))}{1+d(x_1,x_2)}\Big\} \leq \max\Big\{d(x_1,x_2), \\ &d(x_1,x_2), d(x_2,Tx_2), \frac{1}{2}d(x_1,Tx_2), \\ &\frac{d(x_2,Tx_2)(1+d(x_1,x_2))}{1+d(x_1,x_2)}\Big\} = \max\Big\{d(x_1,x_2), \\ &d(x_1,x_2), d(x_2,Tx_2), \frac{1}{2}d(x_1,Tx_2), \\ &\frac{d(x_2,Tx_2)(1+d(x_1,x_2))}{1+d(x_1,x_2)}\Big\}. \end{aligned} \quad (46)$$

Since

$$\begin{aligned} \frac{1}{2}d(x_1,Tx_2) &\leq \frac{1}{2}d(x_1,x_2) + \frac{1}{2}d(x_2,Tx_2) \\ &\leq \max\{d(x_1,x_2), d(x_2,Tx_2)\}, \end{aligned} \quad (47)$$

we get

$$M_{S,T}(x_1,x_2) \leq \max\{d(x_1,x_2), d(x_2,Tx_2)\}. \quad (48)$$

Replacing (45) and (48) in (42) we have

$$\begin{aligned} &\theta(d(x_2,Tx_2)) \\ &\leq \{\theta(\max\{d(x_1,x_2), d(x_2,Tx_2)\})\}^c. \end{aligned} \quad (49)$$

If $\max\{d(x_1,x_2), d(x_2,Tx_2))\} = d(x_2,Tx_2)$, then

$$\theta(d(x_2,Tx_2)) \leq (\theta(d(x_2,Tx_2)))^c, \quad (50)$$

a contradiction. Hence $\max\{d(x_1,x_2), d(x_2,Tx_2))\} = d(x_1,x_2)$. Therefore,

$$1 \leq (d(x_2,Tx_2)) \leq (\theta d(x_1,x_2)))^c. \quad (51)$$

By (θ'_3) we have

$$\theta(d(x_2,Tx_2)) = \inf_{y\in Tx_2} \theta(d(x_2,y)). \quad (52)$$

Thus, there exists $x_3 \in Tx_2$ such that

$$\theta(d(x_2,Tx_2)) = (d(x_2,x_3)). \quad (53)$$

Therefore, from (51) we have

$$\theta(d(x_2,x_3)) \leq (\theta(d(x_1,x_2)))^c. \quad (54)$$

Furthermore, from (40) we have

$$\theta(d(x_2,x_3)) \leq (\theta(d(x_1,x_2)))^c \leq (\theta(d(x_0,x_1)))^{c^2}. \quad (55)$$

Proceeding in the same manner, we obtain a sequence $\{x_i\}$ in $\mathcal{X}$ such that $x_{2i} \neq x_{2i+1}$, $x_{2i} \notin Tx_{2i}$, $x_{2i+1} \notin Sx_{2i+1}$, $x_{2i+1} \in Tx_{2i}$, and $x_{2i+2} \in Sx_{2i+1}$ with $\alpha(x_{2i}, x_{2i+1}) \geq \eta(x_{2i}, x_{2i+1})$ and it satisfies

$$\theta\left(d\left(x_{2i+1}, x_{2i+2}\right)\right) \leq \left(\theta\left(d\left(x_0, x_1\right)\right)\right)^{c^{2i+1}}, \tag{56}$$

for each $i \in \mathbb{N}_0$. As $x_{2i+1} \in Tx_{2i}$, $x_{2i+2} \in Sx_{2i+1}$, and $\alpha(x_{2i}, x_{2i+1}) \geq \eta(x_{2i}, x_{2i+1})$, we have $\alpha(x_{2i+1}, x_{2i+2}) \geq \eta(x_{2i+1}, x_{2i+2})$. Then,

$$0 < d\left(x_{2i+2}, Tx_{2i+2}\right) \leq H\left(Sx_{2i+1}, Tx_{2i+2}\right). \tag{57}$$

Therefore,

$$\begin{aligned}\theta\left(d\left(x_{2i+2}, Tx_{2i+2}\right)\right) &\leq \theta\left(H\left(Sx_{2i+1}, Tx_{2i+2}\right)\right)\\ &\leq \alpha\left(x_{2i+1}, x_{2i+2}\right)\theta\left(H\left(Sx_{2i+1}, Tx_{2i+2}\right)\right)\\ &\leq \left(\theta\left(M_{S,T}\left(x_{2i+1}, x_{2i+2}\right)\right)\right)^{F(N_{S,T}(x_{2i+1}, x_{2i+2}))}.\end{aligned} \tag{58}$$

Thus,

$$\begin{aligned}&N_{S,T}\left(x_{2i+1}, x_{2i+2}\right)\left\{d\left(x_{2i+1}, Sx_{2i+1}\right), d\left(x_{2i+2}, Tx_{2i+2}\right),\right.\\ &\quad\left.d\left(x_{2i+1}, Tx_{2i+2}\right), d\left(x_{2i+2}, Sx_{2i+1}\right)\right\}\\ &\quad\leq \left\{d\left(x_{2i+1}, x_{2i+2}\right), d\left(x_{2i+2}, Tx_{2i+2}\right),\right.\\ &\quad\left.d\left(x_{2i+1}, Tx_{2i+2}\right), d\left(x_{2i+2}, x_{2i+2}\right)\right\}\\ &\quad= \left\{d\left(x_{2i+1}, x_{2i+2}\right), d\left(x_{2i+2}, Tx_{2i+2}\right),\right.\\ &\quad\left.d\left(x_{2i+1}, Tx_{2i+2}\right), 0\right\}.\end{aligned} \tag{59}$$

Since $d(x_{2i+1}, x_{2i+2}).d(x_{2i+2}, Tx_{2i+2}).d(x_{2i+1}, Tx_{2i+2}).0 = 0$ by (A^*) there exists $c \in [0, 1)$ such that $F(d(x_{2i+1}, x_{2i+2}), d(x_{2i+2}, Tx_{2i+2}), d(x_{2i+1}, Tx_{2i+2}), 0) = c$. Thus, from definition of F and (59) we obtain

$$F\left(N_{S,T}\left(x_{2i+1}, x_{2i+2}\right)\right) \leq c. \tag{60}$$

Also,

$$\begin{aligned}&\frac{1}{2}d\left(x_{2i+1}, Tx_{2i+2}\right)\\ &\quad\leq \frac{1}{2}d\left(x_{2i+1}, x_{2i+2}\right) + \frac{1}{2}d\left(x_{2i+2}, Tx_{2i+2}\right)\\ &\quad\leq \max\left\{d\left(x_{2i+1}, x_{2i+2}\right), d\left(x_{2i+2}, Tx_{2i+2}\right)\right\}.\end{aligned} \tag{61}$$

Then

$$\begin{aligned}&M_{S,T}\left(x_{2i+1}, x_{2i+2}\right) = \max\left\{d\left(x_{2i+1}, x_{2i+2}\right), d\left(x_{2i+1},\right.\right.\\ &\quad\left.Sx_{2i+1}\right), d\left(x_{2i+2}, Tx_{2i+2}\right), \frac{1}{2}d\left(x_{2i+1},\right.\\ &\quad\left.\left.Tx_{2i+2}\right)d\left(x_{2i+2}, Tx_{2i+2}\right)\frac{1 + d\left(x_{2i+1}, Sx_{2i+1}\right)}{1 + d\left(x_{2i+1}, x_{2i+2}\right)}\right\}\\ &\quad\leq \max\left\{d\left(x_{2i+1}, x_{2i+2}\right), d\left(x_{2i+1}, x_{2i+2}\right), d\left(x_{2i+2},\right.\right.\\ &\quad\left.Tx_{2i+2}\right)\frac{1}{2}d\left(x_{2i+1}, Tx_{2i+2}\right), d\left(x_{2i+2}, Tx_{2i+2}\right)\\ &\quad\left.\cdot\frac{1 + d\left(x_{2i+1}, x_{2i+2}\right)}{1 + d\left(x_{2i+1}, x_{2i+2}\right)}\right\} \leq \max\left\{d\left(x_{2i+1}, x_{2i+2}\right),\right.\\ &\quad\left.d\left(x_{2i+2}, Tx_{2i+2}\right)\right\}.\end{aligned} \tag{62}$$

Therefore, replacing (60) and (62) in (58) we obtain

$$\begin{aligned}&\theta\left(d\left(x_{2i+2}, Tx_{2i+2}\right)\right)\\ &\quad\leq \left(\theta\max\left\{d\left(x_{2i+1}, x_{2i+2}\right), d\left(x_{2i+2}, Tx_{2i+2}\right)\right\}\right)^c.\end{aligned} \tag{63}$$

If $\max\{d(x_{2i+1}, x_{2i+2}), d(x_{2i+2}, Tx_{2i+2})\} = d(x_{2i+2}, Tx_{2i+2})$ then

$$\theta\left(d\left(x_{2i+2}, Tx_{2i+2}\right)\right) \leq \left(\theta\left(d\left(x_{2i+2}, Tx_{2i+2}\right)\right)\right)^c \tag{64}$$

a contradiction. Further,

$$\theta\left(d\left(x_{2i+2}, Tx_{2i+2}\right)\right) \leq \left(\theta(d\left(x_{2i+1}, x_{2i+2}\right)\right)^c. \tag{65}$$

Again, using (θ'_3) we have

$$\theta\left(d\left(x_{2i+2}, Tx_{2i+2}\right)\right) = \inf_{y\in Tx_{2i+2}}\theta\left(d\left(x_{2i+2}, y\right)\right). \tag{66}$$

Therefore, there exists $x_{2i+3} \in Tx_{2i+2}$ such that

$$\theta\left(d\left(x_{2i+2}, Tx_{2i+2}\right)\right) = \theta\left(d\left(x_{2i+2}, x_{2i+3}\right)\right). \tag{67}$$

Thus, we have

$$\theta\left(d\left(x_{2i+2}, x_{2i+3}\right)\right) \leq \left(\theta\left(d\left(x_{2i+1}, x_{2i+2}\right)\right)\right)^c. \tag{68}$$

From (56), we have

$$\begin{aligned}\theta\left(d\left(x_{2i+2}, x_{2i+3}\right)\right) &\leq \theta(\left(d\left(x_{2i+1}, x_{2i+2}\right)\right)^c\\ &\leq \theta(\left(d\left(x_0, x_1\right)\right)^{c^{2i+2}}.\end{aligned} \tag{69}$$

Hence, we have a sequence $\{x_n\}$ in $\mathcal{X}$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ such that

$$\theta\left(d\left(x_{n+1}, x_{n+2}\right)\right) \leq \left(\theta\left(d\left(x_0, x_1\right)\right)\right)^{c^{n+1}}, \tag{70}$$

for all $n \in \mathbb{N}_0$. On taking limit as $n \longrightarrow \infty$, we obtain

$$1 < \lim_{n\longrightarrow\infty}\theta\left(d\left(x_{n+1}, x_{n+2}\right)\right) \leq 1 \tag{71}$$

which implies that $\lim_{n\longrightarrow\infty}\theta(d(x_{n+1},x_{n+2}))=1$. Then by (θ_2) we obtain

$$\lim_{n\longrightarrow\infty} d(x_{n+1},x_{n+2})=0. \tag{72}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not Cauchy, then there exist $\epsilon>0$ and $m(i)>n(i)>i$ for all $i\in\mathbb{N}_0$ such that

$$\begin{aligned} &d(x_{n(i)},x_{m(i)})\geq\epsilon\\ &\text{and } d(x_{n(i)},x_{m(i)-1})<\epsilon. \end{aligned} \tag{73}$$

Thus,

$$\begin{aligned} \epsilon&\leq d(x_{n(i)},x_{m(i)})\\ &\leq d(x_{n(i)},x_{m(k)-1})+d(x_{m(i)-1},x_{m(i)})\\ &<\epsilon+d(x_{m(i)-1},x_{m(i)}). \end{aligned} \tag{74}$$

Therefore, from the above inequality and (72), we obtain

$$\lim_{k\longrightarrow\infty} d(x_{n(i)},x_{m(i)})=\epsilon. \tag{75}$$

Also,

$$d(x_{n(i)},x_{m(i)})\leq d(x_{n(i)},x_{n(i)+1})+d(x_{n(i)+1},x_{m(i)}) \tag{76}$$

and

$$d(x_{n(i)+1},x_{m(i)})\leq d(x_{n(i)+1},x_{n(i)})+d(x_{n(i)},x_{m(i)}). \tag{77}$$

On taking limit as $i\longrightarrow\infty$ in (76) and using (75) (77), we have

$$\epsilon\leq\lim_{i\longrightarrow\infty} d(x_{n(i)+1},x_{m(i)})\leq\epsilon. \tag{78}$$

Therefore,

$$\lim_{i\longrightarrow\infty} d(x_{n(i)+1},x_{m(i)})=\epsilon. \tag{79}$$

Similarly, we obtain that

$$\begin{aligned} \lim_{i\longrightarrow\infty} d(x_{n(i)+1},x_{m(i)+1})&=\lim_{k\longrightarrow\infty} d(x_{n(i)},x_{m(k)+1})\\ &=\lim_{i\longrightarrow\infty} d(x_{n(i)+1},Tx_{m(i)})=\epsilon. \end{aligned} \tag{80}$$

By Lemma 10, we have

$$\alpha(x_{n(i)},x_{m(i)})\geq\eta(x_{n(i)},x_{m(i)}). \tag{81}$$

If $d(x_{n(i)+1},Tx_{m(i)})=0$, we have

$$\begin{aligned} d(x_{n(i)},x_{m(i)})&\leq d(x_{n(i)},x_{n(i+1)})+d(x_{n(i+1)},Tx_{m(i)})\\ &\quad+d(Tx_{m(i)},x_{m(i)})\\ &\leq d(x_{n(i)},x_{n(i+1)})+d(x_{n(i+1)},Tx_{m(i)})\\ &\quad+d(x_{m(i)+1},x_{m(i)})\\ &\leq d(x_{n(i)},x_{n(i)+1})+d(x_{m(i)+1},x_{m(i)}). \end{aligned} \tag{82}$$

Taking limit $i\longrightarrow\infty$ we have

$$\lim_{i\longrightarrow\infty} d(x_{n(i)},x_{m(i)})=0 \tag{83}$$

a contradiction to our assumption. Thus, assume that $d(x_{n(i)+1},Tx_{m(i)})>0$. Therefore, we have

$$\begin{aligned} \theta(d(x_{n(i)+1},Tx_{m(i)}))&\leq\theta(d(Sx_{n(i)},Tx_{m(i)}))\\ &\leq(\theta(M_{S,T}(x_{n(i)},x_{m(i)})))^{F(N_{S,T}(x_{n(i)},x_{m(i)}))} \end{aligned} \tag{84}$$

where

$$\begin{aligned} N_{S,T}(x_{n(i)},x_{m(i)})&=\{d(x_{n(i)},Sx_{n(i)}),d(x_{m(i)},Tx_{m(i)}),\\ &\quad d(x_{n(i)},Tx_{m(i)}),d(x_{m(i)},Sx_{n(i)})\}\\ &\leq\{d(x_{n(i)},x_{n(i)+1}),d(x_{m(i)},x_{m(i)+1}),\\ &\quad d(x_{n(i)},x_{m(i)+1}),d(x_{m(i)},x_{n(i)+1})\}. \end{aligned} \tag{85}$$

Taking limit $i\longrightarrow\infty$ in (85),

$$\begin{aligned} \lim_{i\longrightarrow\infty} N_{S,T}(x_{n(i)},x_{m(i)})&\leq\Big\{\lim_{i\longrightarrow\infty}(d(x_{n(i)},x_{n(i)+1})),\\ &\quad\lim_{i\longrightarrow\infty}(d(x_{m(i)},x_{m(k)+1})),\\ &\quad\lim_{i\longrightarrow\infty}(d(x_{n(i)},x_{m(i)+1})),\lim_{i\longrightarrow\infty}(d(x_{m(i)},x_{n(i)+1}))\Big\}\\ &=\{0,0,0,0\}. \end{aligned} \tag{86}$$

Now, by (A^*) there exists $c\in[0,1)$ such that $F(0,0,0,0)=c$. Thus, using the continuity of F and (86),

$$F\left(\lim_{i\longrightarrow\infty} N_{S,T}(x_{n(i)},x_{m(i)})\right)\leq c. \tag{87}$$

Moreover,

$$\begin{aligned} M_{S,T}(x_{n(i)},x_{m(i)})&=\max\Big\{d(x_{n(i)},x_{m(i)}),\\ &\quad d(x_{n(i)},Sx_{n(i)}),d(x_{m(i)},Tx_{m(i)}),\frac{1}{2}d(x_{n(i)},Tx_{m(i)}),\\ &\quad\frac{d(x_{m(i)},Tx_{m(i)})(1+d(x_{n(i)},Sx_{n(i)}))}{1+d(x_{n(i)},x_{m(i)})}\Big\}\\ &\leq\max\Big\{d(x_{n(i)},x_{m(i)}),d(x_{n(i)},x_{n(i)+1}),\\ &\quad d(x_{m(i)},x_{m(i)+1}),\frac{1}{2}d(x_{n(i)},x_{m(k)+1}),\\ &\quad\frac{d(x_{m(i)},x_{m(i)+1}))(1+d(x_{n(i)},x_{n(i)+1})}{1+d(x_{n(i)},x_{m(i)})}\Big\}. \end{aligned} \tag{88}$$

Taking limits as $i\longrightarrow\infty$ and using (72) and (80) we obtain that

$$\lim_{i\longrightarrow\infty} M_{S,T}(x_{n(i)},x_{m(i)})\leq\epsilon. \tag{89}$$

Thus, using (84) and the continuity of θ we have

$$\theta\left(\lim_{i\longrightarrow\infty} d\left(x_{n(k)+1}, Tx_{m(i)}\right)\right) \leq \left(\theta \cdot \lim_{i\longrightarrow\infty}\left(M_{S,T}\left(x_{n(i)}, x_{m(i)}\right)\right)\right)^{F(\lim_{i\longrightarrow\infty} N_{S,T}(x_{n(i)}, x_{m(i)}))}. \tag{90}$$

From (89), we obtain

$$\theta(\epsilon) \leq (\theta(\epsilon))^c < \theta(\epsilon) \tag{91}$$

a contradiction. Hence, $\{x_n\}$ is a Cauchy. Since $\mathcal{X}$ is a complete metric space, there exists $\tilde{x} \in \mathcal{X}$ such that $\lim_{n\longrightarrow\infty} d(x_n, \tilde{x}) = 0$. As the pair (T, S) is (α, η)-continuous, we have $\lim_{n\longrightarrow\infty} H(Tx_{2n}, S\tilde{x}) = 0$.

Note that

$$\begin{aligned} d(\tilde{x}, S\tilde{x}) &\leq d(\tilde{x}, x_{2n+1}) + d(x_{2n+1}, S\tilde{x}) \\ &\leq d(\tilde{x}, x_{2n+1}) + H(Tx_{2n}, S\tilde{x}). \end{aligned} \tag{92}$$

On taking limit as $n \longrightarrow \infty$ on both sides of the above inequality, we obtain that $d(\tilde{x}, S\tilde{x}) = 0$ and hence $\tilde{x} \in S\tilde{x}$. As (S, T) is (α, η)-continuous, we have $\lim_{n\longrightarrow\infty} H(Sx_{2n+1}, T\tilde{x}) = 0$. Also,

$$\begin{aligned} d(\tilde{x}, T\tilde{x}) &\leq d(\tilde{x}, x_{2n+2}) + d(x_{2n+2}, T\tilde{x}) \\ &\leq d(\tilde{x}, x_{2n+2}) + H(Sx_{2n+1}, T\tilde{x}). \end{aligned} \tag{93}$$

On taking limit as $n \longrightarrow \infty$ on both sides of the above inequality, we obtain that $d(\tilde{x}, T\tilde{x}) = 0$ and hence $\tilde{x} \in T\tilde{x}$. Thus there exists $\tilde{x}$ such that $\tilde{x} \in T\tilde{x} \cap S\tilde{x}$. □

We may omit the (α, η)-continuity condition in the above theorem by condition (H).

If $\{x_n\}$ is a sequence in $\mathcal{X}$ with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $\lim_{n\longrightarrow\infty} d(x_n, x) = 0$ for some $x \in \mathcal{X}$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$.

Theorem 17. *Let $T, S : \mathcal{X} \longrightarrow CB(\mathcal{X})$. Suppose that the pairs (T, S) and (S, T) are multivalued $(\alpha, \eta) - (\theta, \digamma)$ rational contractions such that*

- **(C1)** *(T, S) and (S, T) are triangular (α, η)-closed mapping;*
- **(C2)** *there exist $x_0 \in \mathcal{X}$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;*
- **(C3)** *the pair (α, η) satisfies condition (H).*

Then there exists $\tilde{x} \in \mathcal{X}$ such that $\tilde{x} \in T\tilde{x} \cap S\tilde{x}$.

Proof. As in Theorem 16, we obtain a Cauchy sequence $\{x_n\}$ in the complete metric space $\mathcal{X}$ with $\lim_{n\longrightarrow\infty} d(x_{2n}, \tilde{x}) = 0$ where $\tilde{x} \in \mathcal{X}$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$. As the pair (α, η) satisfies condition (H), $\alpha(x_n, \tilde{x}) \geq \eta(x_n, \tilde{x})$ for all $n \in \mathbb{N}_0$. We need to show that $\tilde{x}$ is the common fixed point. Suppose on the contrary that $\tilde{x} \notin S\tilde{x}$.

From condition (H), we obtain $\alpha(x_{2n}, \tilde{x}) \geq \eta(x_{2n}, \tilde{x})$ as $\alpha(x_{2n}, x_{2n+1}) \geq \eta(x_{2n}, x_{2n+1})$ and $\lim_{n\longrightarrow\infty} d(x_{2n}, \tilde{x}) = 0$. If $d(x_{2n+1}, S\tilde{x}) = 0$ then

$$d(\tilde{x}, S\tilde{x}) \leq d(\tilde{x}, x_{2n+1}) + d(x_{2n+1}, S\tilde{x}) \leq d(\tilde{x}, x_{2n+1}) \tag{94}$$

Taking limit $k \longrightarrow \infty$ in the above equation we obtain

$$d(\tilde{x}, S\tilde{x}) = 0 \tag{95}$$

a contradiction to our assumption. Thus, we assume $d(x_{2n+1}, S\tilde{x}) > 0$. Then,

$$0 < d(x_{2n+1}, S\tilde{x}) \leq H(T_{2n}, S\tilde{x}). \tag{96}$$

Further,

$$\begin{aligned} \theta(d(x_{2n+1}, S\tilde{x})) &\leq \theta(H(T_{2n}, S\tilde{x})) \\ &\leq \alpha(x_{2n}, \tilde{x})\,\theta(H(T_{2n}, S\tilde{x})) \\ &\leq (\theta(M_{T,S}(x_{2n}, \tilde{x})))^{F(N_{T,S}(x_{2n}, \tilde{x})} \end{aligned} \tag{97}$$

where

$$\begin{aligned} N_{T,S}(x_{2n}, \tilde{x}) = \{&d(x_{2n}, Tx_{2n}), d(\tilde{x}, S\tilde{x}), d(\tilde{x}, Tx_{2n}), \\ &d(x_{2n}, S\tilde{x})\} \leq \{d(x_{2n}, x_{2n+1}), d(\tilde{x}, S\tilde{x}), d(\tilde{x}, x_{2n+1}), \\ &d(x_{2n}, S\tilde{x})\}. \end{aligned} \tag{98}$$

Taking $n \longrightarrow \infty$ in the above inequality we have

$$\begin{aligned} \lim_{n\longrightarrow\infty} N_{T,S}(x_{2n}, \tilde{x}) \leq \Big\{&\lim_{n\longrightarrow\infty}(d(x_{2n}, x_{2n+1})), \\ &\lim_{n\longrightarrow\infty}(d(\tilde{x}, S\tilde{x})), \\ &\lim_{n\longrightarrow\infty}(d(\tilde{x}, x_{2n+1}))), \lim_{n\longrightarrow\infty}(d(x_{2n}, S\tilde{x}))\Big\} = \Big\{0, \\ &d(\tilde{x}, S\tilde{x}), 0, \lim_{n\longrightarrow\infty}(d(x_{2n}, S\tilde{x}))\Big\}. \end{aligned} \tag{99}$$

Thus, as previously shown, by (A^*) and continuity of F we have

$$\lim_{n\longrightarrow\infty} F(N_{T,S}(x_{2n}, \tilde{x})) \leq c. \tag{100}$$

Also,

$$\begin{aligned} M_{T,S}(x_{2n}, \tilde{x}) = \max\Big\{&d(x_{2n}, \tilde{x}), d(x_{2n}, Tx_{2n}), \\ &d(\tilde{x}, S\tilde{x}), \frac{1}{2}d(x_{2n}, S\tilde{x}), \\ &\frac{d(\tilde{x}, S\tilde{x})(1 + d(x_{2n}, Tx_{2n}))}{1 + d(x_{2n}, \tilde{x})}\Big\} \leq \max\Big\{d(x_{2n}, \tilde{x}), \\ &d(x_{2n}, x_{2n+1}), d(\tilde{x}, S\tilde{x}), \frac{1}{2}d(x_{2n}, S\tilde{x}), \\ &\frac{d(\tilde{x}, S\tilde{x})(1 + d(x_{2n}, x_{2n+1}))}{1 + d(x_{2n}, \tilde{x})}\Big\}. \end{aligned} \tag{101}$$

Hence

$$\lim_{n\longrightarrow\infty} M_{T,S}(x_{2n}, \tilde{x}) \leq d(\tilde{x}, S\tilde{x}). \tag{102}$$

Moreover, from (97) we have

$$\theta\left(d\left(x_{2n+1}, S\tilde{x}\right)\right) \leq \left\{\theta\left(M_{T,S}\left(x_{2n}, \tilde{x}\right)\right)\right\}^{F(N_{T,S}(x_{2n},\tilde{x}))} \quad (103)$$

On taking limit as $n \longrightarrow \infty$ in the above inequality and using (θ_3'), we obtain that

$$\begin{aligned}&\theta\left(\lim_{n\longrightarrow\infty} d\left(x_{2n+1}, S\tilde{x}\right)\right)\\&\quad\leq \left(\theta\left(\lim_{n\longrightarrow\infty} M_{T,S}\left(x_{2n}, \tilde{x}\right)\right)\right)^{\lim_{n\longrightarrow\infty} F(N_{T,S}(x_{2n},\tilde{x}))}\\&\quad\leq \left(\theta\left(d\left(\tilde{x}, S\tilde{x}\right)\right)\right)^{c} < \theta\left(d\left(\tilde{x}, S\tilde{x}\right)\right).\end{aligned} \quad (104)$$

It follows from (θ_1) that

$$\lim_{n\longrightarrow\infty} d\left(x_{2n+1}, S\tilde{x}\right) < d\left(\tilde{x}, S\tilde{x}\right). \quad (105)$$

That is,

$$d\left(\tilde{x}, S\tilde{x}\right) < d\left(\tilde{x}, S\tilde{x}\right), \quad (106)$$

a contradiction. Hence, $d(\tilde{x}, S\tilde{x}) = 0$. Similarly, we can show $d(\tilde{x}, T\tilde{x}) = 0$. Hence, $\tilde{x} \in S\tilde{x} \cap T\tilde{x}$ □

Example 18. Let $\mathcal{X} = \mathbb{R}^+$ and

$$d(x, y) = \begin{cases}\max\{x, y\}, & x \neq y\\ 0, & x = y.\end{cases} \quad (107)$$

Define the mappings $S, T : \mathcal{X} \longrightarrow CB(\mathcal{X})$ by

$$Sx = \begin{cases}\{x-1, x+1\}, & x > 1\\ \left\{0, \dfrac{x}{16}\right\}, & 0 \leq x \leq 1.\end{cases} \quad (108)$$

and

$$Tx = \begin{cases}\{1, x+1\}, & x > 1\\ \left\{0, \dfrac{x}{8}\right\}, & 0 \leq x \leq 1.\end{cases} \quad (109)$$

Define $\alpha, \eta : \mathcal{X} \times \mathcal{X}$ and $H : \mathbb{R}^4 \longrightarrow \mathbb{R}^+$ by

$$\alpha(x, y) = \begin{cases}1, & 0 \leq x, y \leq 1\\ 0, & \text{either } x \text{ or } y \notin [0, 1],\end{cases} \quad (110)$$

$$\eta(x, y) = x^2 \quad \text{for all } x \in \mathcal{X} \quad (111)$$

and

$$F(s_1, s_2, s_3, s_4) = \sqrt{\frac{1}{8}}. \quad (112)$$

It is evident that both the pairs (T, S) and (S, T) are triangular $\alpha{-}\eta$-closed. Now, we show that the pair (T, S) is a $(\alpha, \eta){-}(\theta, \digamma)$ rational contraction for $\theta(t) = e^{\sqrt{t}}$. That is, we need to show that

$$\sqrt{\frac{H(Tx, Sy)}{M_{T,S}(x, y)}} \leq F\left(N_{T,S}(x, y)\right) \quad (113)$$

for all $x, y \in \mathcal{X}$.

Let $x, y \in [0, 1]$. Without any loss of generality, we may assume that $y \leq x$. Thus,

$$\sqrt{\frac{H(Tx, Sy)}{M_{T,S}(x, y)}} = \sqrt{\frac{x/8}{x}} = \sqrt{\frac{x/8}{x}} = \sqrt{\frac{1}{8}}. \quad (114)$$

Thus, the pair (T, S) is a $(\alpha, \eta) - (\theta, \digamma)$. Similarly, (S, T) is $(\alpha, \eta) - (\theta, \digamma)$ rational for F as defined above. If $x_0 = 1$, $T1 = \{0, 1/8\}$ then $\alpha(1, 0) = 1 = \eta(1, 0)$. For any sequence $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, we have $\{x_n\} \in [0, 1]$ and $\{x_n\}$ converges to some $x \in [0, 1]$. Thus $\alpha(x_n, x) \geq \eta(x_n, x)$. All the conditions of Theorem 17 are satisfied and $\tilde{x} = 0$ is the common fixed point of S and T.

Corollary 19. *Let $T, S : \mathcal{X} \longrightarrow CB(\mathcal{X})$. Suppose that the pairs (T, S) and (S, T) are continuous multivalued $(\theta, \digamma)$ rational contractions, then there exists $\tilde{x} \in \mathcal{X}$ such that $\tilde{x} \in T\tilde{x} \cap S\tilde{x}$.*

Proof. Define $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in \mathcal{X}$. Then the result follows from Theorem 16. □

Theorem 20. *Let $T : \mathcal{X} \longrightarrow CB(\mathcal{X})$ be a multivalued $(\alpha, \eta) - (\theta, \digamma)$ rational contraction mapping such that*

- **(C1)** *T is a triangular (α, η)-closed mapping;*
- **(C2)** *there exist $x_0 \in \mathcal{X}$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;*
- **(C3)** *T is (α, η)-continuous.*

Then there exists $\tilde{x} \in \mathcal{X}$ such that $\tilde{x} \in T\tilde{x}$.

Proof. By choosing $T = S$ in Theorem 16 the result follows. □

Theorem 21. *Let $T : \mathcal{X} \longrightarrow CB(\mathcal{X})$ be a multivalued $(\alpha, \eta) - (\theta, \digamma)$ rational contraction mapping such that*

- **(C1)** *T is a triangular (α, η)-closed mapping;*
- **(C2)** *there exist $x_0 \in \mathcal{X}$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;*
- **(C3)** *the pair (α, η) satisfies condition (H).*

Then there exists $\tilde{x} \in \mathcal{X}$ such that $\tilde{x} \in T\tilde{x}$.

Proof. By choosing $T = S$ in Theorem 17 the result follows. □

Theorem 22. *Let $T : \mathcal{X} \longrightarrow CB(\mathcal{X})$ be a multivalued mapping. Suppose that T satisfies the following conditions:*

- **(C1)** *For any $x, y \in \mathcal{X}$ such that $\alpha(x, y) \geq \eta(x, y)$ and $0 \leq c < 1$, we have*

$$H(Tx, Ty) \leq cM(x, y) \quad (115)$$

where

$$\begin{aligned}M(x, y) = \max\Bigg\{&d(x, y), d(x, Tx), d(y, Ty),\\&\frac{1}{2}d(x, Ty), \frac{d(y, Ty)\left(1 + d(x, Tx)\right)}{1 + d(x, y)}\Bigg\};\end{aligned} \quad (116)$$

(C2) *T is a triangular (α,η)-closed mapping;*

(C3) *there exist $x_0 \in \mathcal{X}$ and $x_1 \in Tx_0$ with $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$;*

(C4) *T is (α,η)-continuous or (α,η) satisfies condition (H).*

Then there exists $\tilde{x} \in \mathcal{X}$ such that $\tilde{x} \in T\tilde{x}$.

Proof. Taking $\theta(t) = e^t$ and $F(s_1,s_2,s_3,s_4) = c$ where $c \in [0,1)$ in Theorems 20 and 21, the result follows. □

Example 23. Let $\mathcal{X} = [0,\infty)$ and

$$d(x,y) = \begin{cases} x+y, & x \neq y \\ 0, & x = y \end{cases} \tag{117}$$

Define the mapping $T : \mathcal{X} \longrightarrow CB(\mathcal{X})$ by

$$Tx = \begin{cases} \left\{\dfrac{x}{6}, \dfrac{x}{9}\right\}; & 0 \leq x \leq 1 \\ \left\{\dfrac{2x}{x+1}\right\}; & x > 1 \end{cases} \tag{118}$$

Define $\alpha, \eta : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+$ as

$$\alpha(x,y) = \begin{cases} 1; & 0 \leq x, y \leq 1 \\ \dfrac{1}{8}; & \text{otherwise} \end{cases} \tag{119}$$

and

$$\eta(x,y) = \frac{1}{4} \quad \text{for all } x, y \in \mathcal{X}. \tag{120}$$

Clearly, T is a triangular (α,η)-closed mapping. If $x_0 = 1$ and $x_1 = 1/6$, then $\alpha(x_0,x_1) = 1 > 1/4 = \eta(x,y)$. Note that (C(3)) is also satisfied. Let $x, y \in \mathcal{X}$, then $\alpha(x,y) \geq 1$ if $0 \leq x, y \leq 1$. Assume that $x \neq y$, then

$$\begin{aligned} e^{\sqrt{H(Tx,Ty)}} &= e^{\sqrt{\max\{\delta(Tx,Ty),\delta(Ty,Tx)\}}} \\ &= e^{\sqrt{\max\{x/6+y/9,x/9+y/6\}}} \leq e^{\sqrt{(x+y)/4}} \\ &\leq e^{\sqrt{M(x,y)/4}} = \left(e^{\sqrt{M(x,y)}}\right)^{1/2}. \end{aligned} \tag{121}$$

Therefore, T is a multivalued $(\alpha,\eta)-(\theta,\digamma)$ rational contraction with $\theta(t) = e^{\sqrt{t}}$ and $F(s_1,s_2,s_3,s_4) = 1/2$. Thus, all the conditions of Theorem 21 are satisfied and $\tilde{x} = 0$ is a fixed point of T.

Corollary 24. *Let $T, S : \mathcal{X} \longrightarrow CB(\mathcal{X})$. Suppose that, for any $x, y \in \mathcal{X}$ such that $\alpha(x,y) \geq \eta(x,y)$, and $0 \leq c < 1$, the pairs (T,S) and (S,T) satisfy*

$$\theta(H(Tx,Sy)) \leq (\theta(d(x,y)))^c \tag{122}$$

whenever $(H(Tx,Sy)) > 0$ and

$$\theta(H(Sx,Ty)) \leq (\theta(d(x,y)))^c \tag{123}$$

whenever $(H(Sx,Ty)) > 0$. If the following conditions also hold:

(C1) *(T,S) and (S,T) are triangular (α,η)-closed mappings;*

(C2) *there exist $x_0 \in \mathcal{X}$ and $x_1 \in Tx_0$ with $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$;*

(C3) *(T,S) and (S,T) are (α,η)-continuous.*

Then there exists $\tilde{x} \in \mathcal{X}$ such that $\tilde{x} \in T\tilde{x} \cap S\tilde{x}$.

Now, we apply the results for the existence of common fixed points of single valued mappings on a complete metric space.

Definition 25. Let $f, g : \mathcal{X} \longrightarrow \mathcal{X}$ be two mappings on a nonempty set $\mathcal{X}$ and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+$. A pair (f,g) is called (α,η) -admissible if for any $x, y \in \mathcal{X}$, with $\alpha(x,y) \geq \eta(x,y)$, we have $\alpha(fx,gy) \geq \eta(fx,gy)$.

Denote the set of fixed points of f and g by $F(f)$ and $F(g)$, respectively.

Theorem 26. *Let $f, g : \mathcal{X} \longrightarrow \mathcal{X}$. Suppose that the pairs (f,g) and (g,f) are $(\alpha,\eta)-(\theta,\digamma)$ rational contractions such that*

(C1) *the pairs (f,g) and (g,f) are triangular (α,η)-admissible mappings;*

(C2) *there exists $x_0 \in \mathcal{X}$ such that $\alpha(x_0,fx_0) \geq \eta(x_0,fx_0)$;*

(C3) *(f,g) and (g,f) are (α,η)-continuous.*

Then $F(f) \cap F(g) \neq \phi$.

Proof. Define $T, S : \mathcal{X} \longrightarrow CB(\mathcal{X})$ as $Tx = \{fx\}$ and $Sx = \{gx\}$. Then Theorem 16 implies the result. □

Theorem 27. *Let $f, g : \mathcal{X} \longrightarrow \mathcal{X}$. Suppose that the pairs (f,g) and (g,f) are $(\alpha,\eta)-(\theta,\digamma)$ rational contractions such that*

(C1) *(f,g) and (g,f) are triangular (α,η)-admissible mapping;*

(C2) *there exists $x_0 \in \mathcal{X}$ such that $\alpha(x_0,fx_0) \geq \eta(x_0,fx_0)$;*

(C3) *the pair (α,η) satisfies condition (H).*

Then $F(f) \cap F(g) \neq \phi$.

Proof. Define $T, S : \mathcal{X} \longrightarrow CB(\mathcal{X})$ as $Tx = \{fx\}$ and $Sx = \{gx\}$. Then Theorem 17 implies the result. □

Example 28. Let $\mathcal{X} = [0,\infty)$ and

$$d(x,y) = \begin{cases} x+y, & x \neq y \\ 0, & x = y \end{cases} \tag{124}$$

Define $f, g : \mathcal{X} \longrightarrow \mathcal{X}$ by

$$\begin{aligned} fx &= \begin{cases} \dfrac{x}{10}, & 0 \leq x \leq 1 \\ \dfrac{(1-x)^2+1}{10}, & x > 1 \end{cases} \\ \text{and } gx &= \begin{cases} \dfrac{x}{10}, & 0 \leq x \leq 1 \\ \dfrac{(2x-1)^2}{10}, & x > 1 \end{cases} \end{aligned} \tag{125}$$

Define $\alpha, \eta : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & 0 \le x, y \le 1 \\ \dfrac{1}{10}, & x > 1 \end{cases} \tag{126}$$

and

$$\eta(x, y) = 1 \quad \text{for all } x, y \in \mathcal{X}. \tag{127}$$

Clearly, (f, g) and (g, f) are triangular (α, η)-admissible mappings.

Define $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^+$ as

$$F(s_1, s_2, s_3, s_4) = \frac{1}{2}. \tag{128}$$

Note that pairs (f, g) and (g, f) are (α, η)-continuous. Also, (f, g) and (g, f) are $(\alpha, \eta) - (\theta, \digamma)$ rational contractions for $\theta(t) = \sqrt{t}$. If $x_0 = 0$ and $fx_0 = 0$, then $\alpha(0, 0) = 1 = \eta(0, 0)$. Thus all the conditions of Theorem 26 are satisfied. Thus, f and g have a common fixed point $\tilde{x} = 0$ in $\mathcal{X}$.

Corollary 29. *Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be an (α, η)-continuous $(\alpha, \eta) - (\theta, \digamma)$ rational contraction. Then f has a fixed point in $\mathcal{X}$ if there exists $x_0 \in \mathcal{X}$ such that $\alpha(x_0, fx_0) \ge \eta(x_0, fx_0)$ and f is triangular (α, η)-admissible. Furthermore, the fixed point is unique if $\alpha(x, y) \ge \eta(x, x)$.*

Corollary 30. *Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be a triangular (α, η)-admissible $(\alpha, \eta) - (\theta, \digamma)$ rational contraction. Then f has a fixed point in $\mathcal{X}$ provided that there exists $x_0 \in \mathcal{X}$ such that $\alpha(x_0, fx_0) \ge \eta(x_0, fx_0)$ and (α, η) satisfies condition (H). Furthermore, the fixed point is unique if $\alpha(x, y) \ge \eta(x, x)$.*

Example 31. Let $\mathcal{X} = \{0, 1/4, 1/2\}$. Note that

$$d(x, y) = \begin{cases} \dfrac{x + y}{2}, & x \ne y \\ 0, & x = y \end{cases} \tag{129}$$

defines the metric on $\mathcal{X}$. Define $T : \mathcal{X} \longrightarrow \mathcal{X}$ by

$$Tx = \begin{cases} x^2, & x = 0, \dfrac{1}{2} \\ 0, & x = \dfrac{1}{4}. \end{cases} \tag{130}$$

Define $\alpha, \eta : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+$ and

$$\alpha(x, y) = \begin{cases} \cosh\left(\dfrac{x + y}{4}\right), & \text{if } (x, y) \in \left\{(0, 0), \left(\dfrac{1}{4}, \dfrac{1}{4}\right), \left(\dfrac{1}{2}, \dfrac{1}{2}\right), \left(0, \dfrac{1}{4}\right), \left(0, \dfrac{1}{2}\right)\right\} \\ \dfrac{1}{4}, & \text{otherwise,} \end{cases} \tag{131}$$

and

$$\eta(x, y) = 1 \quad \text{for all } x, y \in \mathcal{X}. \tag{132}$$

Clearly, T is (α, η)-continuous. Define $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^+$ as

$$F(s_1, s_2, s_3, s_4) = \frac{9}{10} e^{4 \min\{s_1, s_2, s_3, s_4\}}. \tag{133}$$

Let $\theta(t) = 1 + t$. We only consider the case where $(x, y) = (0, 1/2)$; all other cases are trivial. Note that

$$\begin{aligned} \theta(d(Tx, Ty)) &= (1 + d(Tx, Ty)) = \left(1 + \frac{1}{8}\right) \\ &\le \left(1 + \frac{3}{8}\right)^{9/10} \le (1 + M(x, y))^{9/10} \\ &\le \theta(M(x, y))^{H(N(x,y))}. \end{aligned} \tag{134}$$

Thus, T is an $(\alpha, \eta) - (\theta, \digamma)$ rational contraction. Also, T is triangular (α, η)-admissible. Let $x_0 = 0$, then $\alpha(x_0, Tx_0) \ge 1 = \eta(x_0, Tx_0)$. All the conditions of Corollary 29 are satisfied and $\tilde{x} = 0$ is a fixed point of T.

3. Application to Nonlinear Elastic Beam Equations

We study the existence of solutions of fourth-order two-point boundary value problem given by

$$\begin{aligned} \mu^{(4)}(t) &= f(t, \mu(t)), \quad t \in [0, 1], \\ \mu(0) &= \mu(1) = \mu'(0) = \mu'(1) = 0, \end{aligned} \tag{135}$$

which represents the bending of an elastic beam clamped at both ends. The boundary value problem in (135) can be written as [17]

$$\mu(t) = \int_0^1 G(t, s) f(s, \mu(s)) \, ds \quad \text{for } t \in [0, 1], \tag{136}$$

where the Green function associated with the given boundary value problem is given by

$$G(t, s) = \frac{1}{6} \cdot \begin{cases} t^2 (1 - s)^2 [(s - t) + 2(1 - t) s], & 0 \le t \le s \le 1 \\ s^2 (1 - t)^2 [(t - s) + 2(1 - s) t], & 0 \le s \le t \le 1 \end{cases} \tag{137}$$

where $\sup_{t \in I} \int_0^1 G(t, s) = 1/384$ (see [18]).

Let $\mathcal{X} = (\mathbb{C}[0,1],[0,\infty))$ be the space of all continuous functions defined on $[0,1]$. The metric on $\mathcal{X}$ is given by

$$d(x,y) = \max_{t\in[0,1]} |x(t) - y(t)| \tag{138}$$

for all $t \in [0,1]$. Note that the space $\mathcal{X} = (\mathbb{C}([0,1]), d)$ is complete metric space.

Theorem 32. *Suppose that the following hypotheses are satisfied:*

(1) $f : [0,1]\times[0,\infty) \longrightarrow [0,\infty)$ *is continuous;*

(2) $f(s,.) : [0,\infty) \longrightarrow [0,\infty)$ *is nondecreasing for each* $s \in [0,1]$*;*

(3) *for* $\mu(s) \le \nu(s)$ *for* $\mu,\nu \in \mathcal{X}$*, we have*

$$|f(s,\mu(s)) - f(s,\nu(s))| \le M(\mu(s),\nu(s)), \tag{139}$$

for any $s \in [0,1]$*, where*

$$\begin{aligned} M(\mu(s),\nu(s)) = \max \Big\{ & d(\mu(s),\nu(s)), \\ & d(\mu(s), f(s,\mu(s))), d(\nu(s), f(s,\nu(s))), \\ & \frac{1}{2} d(\mu(s), f(s,\nu(s))), \\ & \frac{d(\nu(s), f(s,\nu(s)))(1 + d(\mu(s), f(s,\mu(s))))}{1 + d(\mu(s),\nu(s))} \Big\}; \end{aligned} \tag{140}$$

(4) *there exists* $\mu_0 \in \mathcal{X}$ *such that, for all* $t \in [0,1]$,

$$\mu_0(t) \le \int_0^1 G(t,s) f(s,\mu_0(s))\, ds. \tag{141}$$

Then problem (135) has a solution in $\mathcal{X}$.

Proof. Let $\mu \in \mathcal{X}$. Define the operator $T : \mathcal{X} \longrightarrow \mathcal{X}$ by

$$(T\mu)(t) = \int_0^1 G(t,s) f(s,\mu(s))\, ds \quad \text{for } t \in [0,1]. \tag{142}$$

Clearly, T is continuous. Define $\alpha,\eta : \mathcal{X}\times\mathcal{X} \longrightarrow [0,\infty)$ as

$$\alpha(\mu,\nu) = \begin{cases} 1, & \mu(t) \le \nu(t),\ t \in [0,1] \\ 0, & \textit{otherwise}. \end{cases} \tag{143}$$

and

$$\eta(\mu,\nu) = 1 \quad \text{for all } \mu,\nu \in \mathcal{X}. \tag{144}$$

Clearly, (α,η) is triangular. Also, since $f(s,.)$ is nondecreasing, then for any $\mu,\nu \in \mathcal{X}$ such that $\mu(t) \le \nu(t)$ for all $t \in [0,1]$ we obtain

$$\begin{aligned} (T\mu)(t) &= \int_0^1 G(t,s) f(s,\mu(s))\, ds \\ &\le \int_0^1 G(t,s) f(s,\nu(s))\, ds = (T\nu)(t). \end{aligned} \tag{145}$$

Hence, $\alpha(T\mu,T\nu) \ge 1$. Since $\alpha(\mu,\nu) \ge 1$, T is (α,η)-admissible. Now, for all $\mu,\nu \in \mathcal{X}$ such that $\mu(s) \le \nu(s)$ for all $s \in [0,1]$ we have

$$\begin{aligned} |(T\mu)(t) - (T\nu)(t)| &= \Big| \int_0^1 G(t,s) f(s,\mu(s))\, ds \\ &\quad - \int_0^1 G(t,s) f(s,\nu(s))\, ds \Big| \\ &\le \int_0^1 G(t,s) |f(s,\mu(s)) - f(s,\nu(s))|\, ds \\ &\le \int_0^1 G(t,s) M(\mu(s),\nu(s))\, ds \\ &= M(\mu(s),\nu(s)) \int_0^1 G(t,s)\, ds \end{aligned} \tag{146}$$

where $t \in [0,1]$. Using the fact that $\sup_{t\in[0,1]} \int_0^1 G(t,s)ds = 1/384$ we have

$$d(T\mu,T\nu) \le \frac{1}{384} M(\mu,\nu). \tag{147}$$

Now,

$$\begin{aligned} d(T\mu,T\nu) e^{d(T\mu,T\nu)} &\le \frac{1}{384} M(\mu,\nu) e^{(1/384)M(\mu,\nu)} \\ &\le \frac{1}{384} M(\mu,\nu) e^{M(\mu,\nu)} \end{aligned} \tag{148}$$

Now, passing through exponential we obtain

$$\begin{aligned} e^{d(T\mu,T\nu)e^{d(T\mu,T\nu)}} &\le e^{(1/384)M(\mu,\nu)e^{M(\mu,\nu)}} \\ &\le \left(e^{M(\mu,\nu)(e^{M(\mu,\nu)})}\right)^{1/384}. \end{aligned} \tag{149}$$

Thus, T satisfies

$$\theta(d(T\mu,T\nu)) \le (\theta) M(\mu,\nu)))^{F(N(\mu,\nu))} \tag{150}$$

with $\theta(t) = e^{te^t}$ and $F(N(\mu,\nu)) = 1/384 < 1$. Since all the conditions of Corollary 29 are satisfied, then problem (135) has a solution in $\mathcal{X}$. □

4. An Application to First-Order Periodic Problem

In this section, we establish the necessary conditions for existence of a fixed point of a mapping in the setting of a partially ordered metric space. Throughout this section, we assume that $(\mathcal{X}, d, \preccurlyeq)$ is a partially ordered metric space.

Definition 33. A sequence $\{x_n\} \subset \mathcal{X}$ is called $\preccurlyeq$-preserving if $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}_0$.

Definition 34. A mapping $T : \mathcal{X} \longrightarrow P(\mathcal{X})$ is called $\preccurlyeq$-closed if, for any $x, y \in \mathcal{X}$ with $x \preccurlyeq u$, $u \preccurlyeq v$ for all $u \in Tx$ and $v \in Ty$.

Definition 35. A mapping $T : \mathcal{X} \longrightarrow CB(\mathcal{X})$ is called $\preccurlyeq$-continuous at a point $x \in \mathcal{X}$ if, for any sequence $\{x_n\}$ in $\mathcal{X}$, $x_n \preccurlyeq x_{n+1}$ and $\lim_{n\longrightarrow\infty} d(x_n, x) = 0$ implies that $\lim_{n\longrightarrow\infty} H(Tx_n, Tx) = 0$ for all $n \in \mathbb{N}_0$. We say that the mapping T is $\preccurlyeq$-continuous on $\mathcal{X}$ if $T \preccurlyeq$ -continuous at every $x \in \mathcal{X}$.

Corollary 36. *Let $T : \mathcal{X} \longrightarrow CB(\mathcal{X})$ be $\preccurlyeq -(\theta, \digamma)$ rational contraction such that T is a $\preccurlyeq$-closed mapping and T is $\preccurlyeq$-continuous. If there exists x_0 such that $x_0 \preccurlyeq Tx_0$, then there exists $\widetilde{x} \in \mathcal{X}$ such that $\widetilde{x} \in T\widetilde{x}$.*

Proof. Define $\alpha, \eta : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+$ by $\alpha(x, y) = 1$ whenever $x \preccurlyeq y$ and $\alpha(x, y) = 0$ whenever $x \npreceq y$ and $\eta(x, y) = 1$ whenever $x \preccurlyeq y$. Thus, the result follows from Theorem 21. □

Example 37. Let $\mathcal{X} = \{0, 1, 2\}$. Define a metric d on $\mathcal{X}$ by

$$\begin{aligned} d(0,1) &= d(1,0) = 7, \\ d(0,2) &= d(2,0) = 3, \\ d(1,2) &= d(2,1) = 4 \end{aligned} \tag{151}$$

Define $x \preccurlyeq y$ by

$$\preccurlyeq := \{(0,0), (1,1), (2,2), (0,1), (0,2)\}. \tag{152}$$

Note that $\mathcal{X}$ is a partially ordered metric space. Define the mapping $T : \mathcal{X} \longrightarrow CB(\mathcal{X})$ by $T0 = T2 = \{0\}$ and $T1 = \{2\}$. It can easily be shown that T is $\preccurlyeq$-closed, $\preccurlyeq$-continuous, and $\preccurlyeq -(\theta, \digamma)$ rational contraction for $\theta(t) = e^{\sqrt{t}}$ and $c = 9/10$. If $x_0 = 0$ and $x_1 = 1 \in Tx_0$, then we have $x_0 \preccurlyeq x_1$. Note that all the conditions of Corollary 36 are satisfied and $\{0, 2\}$ is the set of fixed points of T.

Corollary 38. *Suppose $f : \mathcal{X} \longrightarrow \mathcal{X}$ is a $\preccurlyeq -(\theta, \digamma)$ rational contraction. If f is $\preccurlyeq$-closed and $\preccurlyeq$-continuous and there exists $x_0 \in \mathcal{X}$ such that $x_0 \preccurlyeq fx_0$, then f has a fixed point in $\mathcal{X}$.*

Corollary 39. *Suppose $f : \mathcal{X} \longrightarrow \mathcal{X}$ is a $\preccurlyeq$-closed and $\preccurlyeq$-continuous mapping that satisfies*

$$\theta(d(Tx, Ty)) \leq (\theta(d(x, y)))^{H(N(x,y))} \tag{153}$$

for any $x, y \in \mathcal{X}$ with $x \preccurlyeq y$. Then f has a fixed point in $\mathcal{X}$.

We now apply the Corollary 38 in proving the existence of solution of the first-order periodic problem.

Let $\mathcal{X} = (\mathbb{C}[0, W], \mathbb{R})$ be the space of all continuous functions defined on $[0, W]$. The metric on $\mathcal{X}$ is given by

$$d(x, y) = \max_{t\in[0,W]} |x(t) - y(t)|. \tag{154}$$

Define the partial order on $\mathcal{X}$ by

$$x \preccurlyeq y \quad \text{if and only if } x(t) \leq y(t) \tag{155}$$

for all $t \in [0, W]$. Note that the space $\mathcal{X} = (\mathbb{C}([0, W]), d)$ is partially ordered complete metric space. The following first-order periodic problem is given by

$$\begin{aligned} u'(t) &= f(t, u(t)), \quad t \in [0, W] \\ \mu(0) &= \mu(W), \end{aligned} \tag{156}$$

where $W > 0$ and $f : [0, W] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. The problem in (156) can be written as

$$\begin{aligned} \mu'(t) + \lambda\mu(t) &= f(t, \mu(t)) + \lambda\mu(t), \\ & \qquad t \in [0, W] \text{ and } \lambda > 0 \\ \mu(0) &= \mu(W). \end{aligned} \tag{157}$$

Problem (157) is equivalent to

$$\mu(t) = \int_0^W G(t, s)[f(s, \mu(s) + \lambda\mu(s))]\, ds \tag{158}$$

where $G : [0, W] \times [0, W] \longrightarrow \mathbb{R}$ is defined by

$$G(t, s) = \begin{cases} \dfrac{e^{\lambda(W+s-t)}}{e^{\lambda W - 1}}, & 0 \leq s \leq t \leq W \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda W - 1}}, & 0 \leq t \leq s \leq W. \end{cases} \tag{159}$$

Note that

$$\int_0^W G(t, s)\, ds = \frac{1}{\lambda}. \tag{160}$$

The following definition will be used in our theorem.

Definition 40. A lower solution for (156) is a function $\beta \in C([0, W], \mathbb{R})$ differentiable on $[0, W]$ such that

$$\begin{aligned} \beta'(t) &\leq f(t, \beta(t)), \quad \text{for all } t \in [0, W], \\ \beta(0) &\leq \beta(W). \end{aligned} \tag{161}$$

Theorem 41. *Suppose that following conditions hold:*

(1) *$f : [0, W] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function;*

(2) *$f(s, .) : \mathbb{R} \longrightarrow \mathbb{R}$ is a nondecreasing function for each $s \in [0, W]$;*

(3) *there exists $k \in \mathbb{R}^+$ where $k/\lambda < 1$ such that*

$$\begin{aligned} &|f(s, \mu(s)) + \lambda\mu(s) - f(s, \nu(s)) + \lambda\nu(s)| \\ &\quad \leq kM(\mu(s), \nu(s)) \end{aligned} \tag{162}$$

where

$$\begin{aligned} M(\mu(s), \nu(s)) = \max \Bigg\{ & d(\mu(s), \nu(s)), \\ & d(\mu(s), f(s, \mu(s))), d(\nu(s), f(s, \nu(s))), \\ & \frac{1}{2} d(\mu(s), f(s, \nu(s))), \\ & \frac{d(\nu(s), f(s, \nu(s)))(1 + d(\mu(s), f(s, \mu(s))))}{1 + d(\mu(s), \nu(s))} \Bigg\} \end{aligned} \tag{163}$$

holds for all $\mu \preccurlyeq \nu$ and $s \in [0, W]$;

(4) *there exists a lower solution of problem (156).*

Then problem (156) has a solution in $\mathcal{X}$.

Proof. Let $\mu \in \mathcal{X}$. Define the integral operator $T : \mathcal{X} \longrightarrow \mathcal{X}$ as

$$T\mu(t) = \int_0^W G(t,s)\left[f(s,\mu(s)) + \lambda\mu(s)\right] ds \quad \text{for } t \in [0,W]. \tag{164}$$

Clearly, T is continuous. For any $\mu, \nu \in \mathcal{X}$ such that $\mu \preccurlyeq \nu$ we obtain

$$\begin{aligned} d(T\mu, T\nu) &= \sup_{t\in[0,W]} \left|T\mu(t) - T\nu(t)\right| \\ &\le \sup_{t\in[0,W]} \int_0^W G(t,s) \\ &\cdot \left|f(s,\mu(s)) - \lambda\mu(s) - f(s,\nu(s)) - \lambda\nu(s)\right| ds \\ &\le \sup_{t\in[0,W]} \int_0^W G(t,s)\, kM(\mu(s),\nu(s))\, ds \\ &\le kM(\mu,\nu) \sup_{t\in[0,W]} \int_0^W G(t,s)\, ds \le \frac{kM(\mu,\nu)}{\lambda}. \end{aligned} \tag{165}$$

Since $k/\lambda < 1$ we obtain

$$\begin{aligned} d(T\mu, T\nu)\, e^{d(T\mu,T\nu)} &\le \frac{kM(\mu,\nu)}{\lambda} e^{kM(\mu,\nu)/\lambda} \\ &\le \frac{kM(\mu,\nu)}{\lambda} e^{M(\mu,\nu)}. \end{aligned} \tag{166}$$

Passing through exponential, we have

$$e^{d(T\mu,T\nu)e^{d(T\mu,T\nu)}} \le e^{(kM(\mu,\nu)/\lambda)e^{M(\mu,\nu)}} = \left(e^{M(\mu,\nu)e^{M(\mu,\nu)}}\right)^{k/\lambda}. \tag{167}$$

Setting $k/\lambda = c$, we obtain that

$$e^{d(T\mu,T\nu)e^{d(T\mu,T\nu)}} \le \left(e^{M(\mu,\nu))e^{M(\mu,\nu)}}\right)^{c}. \tag{168}$$

Thus,

$$\theta(d(T\mu,T\nu)) \le (\theta(M(\mu,\nu)))^{F(N(\mu,\nu))} \tag{169}$$

for $\theta(t) = e^{te^t}$, $F(s_1,s_2,s_3,s_4) = c$ where $c < 1$ and $\mu \preccurlyeq \nu$. Since $f(s,.)$ is nondecreasing then for any $\mu,\nu \in \mathcal{X}$ such that $\mu(t) \le \nu(t)$ for all $t \in [0,W]$, we have

$$\begin{aligned} T\mu(t) &= \int_0^T G(t,s)\left[f(s,\mu(s)) + \lambda\mu(s)\right] ds \\ &\le \int_0^W G(t,s)\left[f(s,\nu(s)) + \lambda\nu(s)\right] ds \\ &= T\nu(t). \end{aligned} \tag{170}$$

Therefore, T is $\preccurlyeq$-closed. If $\beta \in \mathcal{X}$ is a lower solution of (156), then by simple calculations we have

$$\begin{aligned} \beta(t) &\le \int_0^W G(t,s)\left[f(s,\beta(s)) + \lambda\beta(s)\right] ds \\ &\le (T\beta)(t). \end{aligned} \tag{171}$$

Hence, by the Corollary 38, problem (156) has a solution in $\mathcal{X}$. □

Remark 42. Our results generalize, extend, and refine several results in the literature.

(1) Our results dealing with single valued mappings can be viewed as an extension and generalization of Banach fixed point theorem [1]. It is worth mentioning that the results in [15] are not a generalization of the Banach fixed point theorem.

(2) Theorems 20 and 21 extend Nadler's theorem [5], Bianchini's Theorem [19], and Hancer's theorem [6].

(3) Corollaries 29 and 30 generalize Theorems 2.3 and 2.4 in [4] and refine Theorems 2.5 and 2.7 in [15].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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