TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 372, Number 7, 1 October 2019, Pages 4553–4583
https://doi.org/10.1090/tran/7821
Article electronically published on June 21, 2019

THE MAXIMUM DIMENSION OF A LIE NILPOTENT SUBALGEBRA OF $\mathbb{M}_n(F)$ OF INDEX m

J. SZIGETI, J. VAN DEN BERG, L. VAN WYK, AND M. ZIEMBOWSKI

ABSTRACT. The main result of this paper is the following: if F is any field and R any F-subalgebra of the algebra $\mathbb{M}_n(F)$ of $n \times n$ matrices over F with Lie nilpotence index m, then

$$\dim_F R \leqslant M(m+1,n),$$

where $M(m+1,n)$ is the maximum of $\frac{1}{2}\left(n^2 - \sum_{i=1}^{m+1} k_i^2\right) + 1$ subject to the constraint $\sum_{i=1}^{m+1} k_i = n$ and $k_1, k_2, \ldots, k_{m+1}$ nonnegative integers. This answers in the affirmative a conjecture by the first and third authors. The case $m = 1$ reduces to a classical theorem of Schur (1905), later generalized by Jacobson (1944) to all fields, which asserts that if F is an algebraically closed field of characteristic zero and R is any commutative F-subalgebra of $\mathbb{M}_n(F)$, then $\dim_F R \leqslant \left\lfloor \frac{n^2}{4} \right\rfloor + 1$. Examples constructed from block upper triangular matrices show that the upper bound of $M(m+1,n)$ cannot be lowered for any choice of m and n. An explicit formula for $M(m+1,n)$ is also derived.

CONTENTS

Received by the editors July 11, 2017.

2010 *Mathematics Subject Classification.* Primary 16S50, 16U80; Secondary 16R40, 17B99.

Key words and phrases. Lie nilpotent, matrix algebra, Lie algebra, commutative subalgebra, dimension.

The first author was partially supported by the National Research, Development and Innovation Office of Hungary (NKFIH) K119934.

The second and third authors were supported by the National Research Foundation of South Africa under grant numbers UID 85784 and UID 72375, respectively. All opinions, findings and conclusions or recommendations expressed in this publication are those of the authors and therefore the National Research Foundation does not accept any liability in regard thereto.

The fourth author was supported by the Polish National Science Centre grant UMO-2017/25/B/ST1/00384.

The second author is the corresponding author.

1. Introduction

In 1905 Schur [15, Satz I, p. 67] proved that the dimension over the field of complex numbers $\mathbb{C}$ of any commutative subalgebra of $\mathbb{M}_n(\mathbb{C})$ is at most $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$, where $\lfloor\ \rfloor$ denotes the integer floor function. Some forty years later, Jacobson [8, Theorems 1 and 2, p. 434] extended Schur's result by showing that the upper bound holds for commutative subalgebras of $\mathbb{M}_n(F)$ for all fields F.

In a subsequent further improvement, Gustafson [7, Section 2, p. 558] showed that Schur's theorem in its most general form could be proved with much greater efficiency using module theoretic methods. We record here that Gustafson's elegant arguments are the inspiration for a key proposition in this paper.

There have also appeared in the literature a number of papers offering alternative proofs of Schur's theorem and its subsequent extensions. In this regard, we refer the reader to [18], [11] and [9].

In response to a question posed in [7, Section 5, Open problem (a), p. 562] Cowsik [2] has proved a version of Schur's theorem for artinian rings that are not algebras, in which the module length of a faithful module substitutes for the dimension of the F-space on which the matrices act.

The common approach to establishing Schur's upper bound has been to show that if F is a field and R a commutative F-subalgebra of $\mathbb{M}_n(F)$, then there exist positive integers k_1 and k_2 such that $k_1 + k_2 = n$ and

$$\dim_F R \leqslant k_1 k_2 + 1.$$

An application of rudimentary calculus then shows that

$$\max\{k_1 k_2 + 1 : (k_1, k_2) \in \mathbb{N} \times \mathbb{N} \text{ and } k_1 + k_2 = n\} = \left\lfloor \frac{n^2}{4} \right\rfloor + 1,$$

whence $\dim_F R \leqslant \left\lfloor \frac{n^2}{4} \right\rfloor + 1$.

The upper bound of $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ is, moreover, easily seen to be optimal. Indeed, let F be any field and (k_1, k_2) any pair of positive integers satisfying $k_1 + k_2 = n$. Define rectangular array B by

$$B \stackrel{\text{def}}{=} \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leqslant i \leqslant k_1 < j \leqslant n\}$$

and subset J of $\mathbb{M}_n(F)$ by

$$J \stackrel{\text{def}}{=} \left\{ \sum_{(i,j)\in B} b_{ij} E_{(i,j)} \ : \ b_{ij} \in F \ \ \forall (i,j) \in B \right\}, \tag{1}$$

where $E_{(i,j)}$ denotes the matrix unit in $\mathbb{M}_n(F)$ associated with position (i,j). Observe that J comprises the set of all block upper triangular matrices that correspond with B; it has the following illuminating pictorial representation (the unshaded region in Figure 1 below corresponds with zero entries):
Denote by

$$FI_n \stackrel{\text{def}}{=} \{aI_n : a \in F\} \ (I_n \text{ is the } n \times n \text{ identity matrix})$$

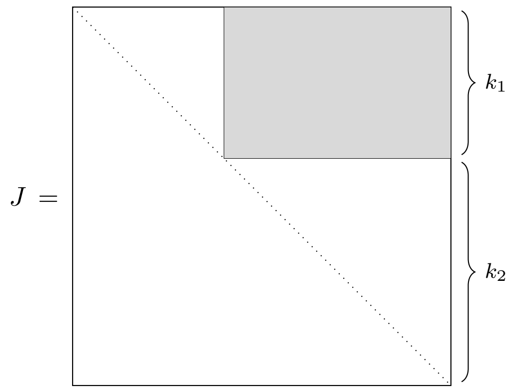


FIGURE 1

the set of all $n \times n$ scalar matrices over F, and define

$$R \overset{\text{def}}{=} FI_n + J. \tag{2}$$

It is easily seen that R is a local F-subalgebra of $\mathbb{M}_n(F)$ with (Jacobson) radical $J(R) = J$ such that $J^2 = 0$. This entails that R is commutative. It is clear too that

$$\dim_F R = k_1 k_2 + 1.$$

The above simple construction shows that the upper bound

$$\left\lfloor \frac{n^2}{4} \right\rfloor + 1 = \max\{k_1 k_2 + 1 : (k_1, k_2) \in \mathbb{N} \times \mathbb{N} \text{ and } k_1 + k_2 = n\}$$

cannot be lowered for any $n \geqslant 2$ and is thus optimal, as claimed.

We construct now an F-subalgebra R of $\mathbb{M}_n(F)$ similar to the one constructed above, but whose radical J comprises m blocks rather than a single block. We require first a compact notation for the description of such rings. To this end, let $k_1, k_2, \ldots, k_{m+1}$ be a sequence of positive integers such that $k_1+k_2+\cdots+k_{m+1} = n$. For each $p \in \{1, 2, \ldots, m\}$, define the rectangular array

$$B_p \overset{\text{def}}{=} \begin{cases} \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leqslant i \leqslant k_1 < j \leqslant n\} & \text{if } p = 1, \\[2ex] \{(i,j) \in \mathbb{N} \times \mathbb{N} : k_1 + k_2 + \cdots + k_{p-1} < i \leqslant k_1 + k_2 + \cdots + k_p \\ \qquad\qquad < j \leqslant n\} \quad \text{if } p > 1. \end{cases}$$

Put

$$B \overset{\text{def}}{=} \bigcup_{p=1}^{m} B_p. \tag{3}$$

Define J as in (1) but with B defined as in (3) above. The pictorial representation of J shown in Figure 2 reveals a stack of m blocks.

$J =$

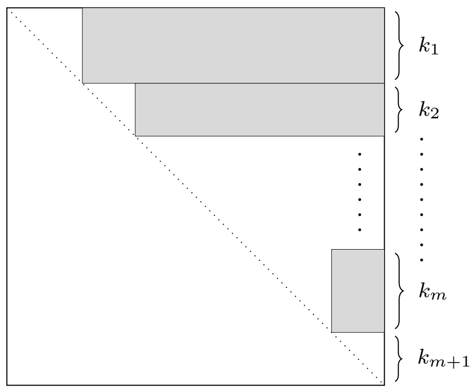


FIGURE 2

We shall call the F-algebra R defined as in (2) the *algebra of* $n \times n$ *matrices over* F *of type* $(k_1, k_2, \dots, k_{m+1})$. We see that R is again a local F-subalgebra of $\mathbb{M}_n(F)$ with radical $J(R) = J$ such that $J^{m+1} = 0$ and

$$\begin{aligned}\dim_F R &= k_1(n-k_1) + k_2(n-k_1-k_2) + \cdots \\ &\quad + k_m(n-k_1-k_2-\cdots-k_m) + 1 \\ &= \sum_{j=1}^{m} k_j \left(n - \sum_{i=1}^{j} k_i\right) + 1. \end{aligned} \tag{4}$$

A routine inductive argument shows that the expression (less 1) appearing on the right-hand side of (4) simplifies as

$$\sum_{j=1}^{m} k_j \left(n - \sum_{i=1}^{j} k_i\right) = \frac{1}{2}\left(n^2 - \sum_{i=1}^{m+1} k_i^2\right) = \sum_{i,j=1,\, i<j}^{m+1} k_i k_j,$$

so that (4) becomes

$$\dim_F R = \frac{1}{2}\left(n^2 - \sum_{i=1}^{m+1} k_i^2\right) + 1 = \sum_{i,j=1,\, i<j}^{m+1} k_i k_j + 1. \tag{5}$$

The algebra of $n \times n$ matrices over F of type $(k_1, k_2, \dots, k_{m+1})$ is clearly not commutative (unless $m = 1$), but it does satisfy a weak form of commutativity called Lie nilpotence. To put this notion in context, we first recall some basic facts about Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra[1] and $x_1, x_2, \ldots, x_m$ a finite sequence of elements in $\mathfrak{g}$. We define element $[x_1, x_2, \ldots, x_m]^*$ of $\mathfrak{g}$ recursively as follows:

$$[x_1]^* \stackrel{\text{def}}{=} x_1, \quad \text{and}$$

$$[x_1, x_2, \ldots, x_m]^* \stackrel{\text{def}}{=} [[x_1, x_2, \ldots, x_{m-1}]^*, x_m], \quad \text{for } m > 1.$$

Recall that if $\mathfrak{h}$ is any ideal of $\mathfrak{g}$, then the *Lower Central Series* $\{\mathfrak{h}_{[m]}\}_{m \in \mathbb{N}}$ of $\mathfrak{h}$ is defined by

$$\mathfrak{h}_{[m]} \stackrel{\text{def}}{=} \{[x_1, x_2, \ldots, x_m]^* : x_i \in \mathfrak{h} \quad \text{for } 1 \leqslant i \leqslant m\}.$$

We say $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}_{[m]} = 0$ for some $m \in \mathbb{N}$, $m > 1$, and more specifically, *nilpotent of index m* if $\mathfrak{g}_{[m+1]} = 0$.

Every ring R may be endowed with the structure of a Lie algebra (over the center of R), by choosing as bracket the *commutator* defined by

$$\forall r, s \in R, \qquad [r, s] \stackrel{\text{def}}{=} rs - sr.$$

Following [17, p. 4785], we call a ring R *Lie nilpotent* [resp. *Lie nilpotent of index m*] if R, considered as a Lie algebra via the commutator, is nilpotent [resp. nilpotent of index m]. The reader will observe that the commutative rings are precisely the rings that are Lie nilpotent of index 1.

A ring R is said to satisfy the *Engel condition of index m* if the identity

$$[x, \overbrace{y, \ldots, y}^{m \text{ times}}]^* = 0$$

holds in R. A ring is said to satisfy the *Engel condition* if it satisfies the Engel condition of index m for some $m \in \mathbb{N}$. Clearly a ring that is Lie nilpotent of index m satisfies the Engel condition of index m. The following result of Riley and Wilson [14, p. 974] establishes a partial converse.

Proposition 1. *If F is any field and R is an F-algebra that is generated by a finite number d of elements and R satisfies the Engel condition of index m, then R is Lie nilpotent of index $f(d, m) \geqslant m$, where the index $f(d, m)$ depends only on d and m.*

Lie nilpotent rings have been shown to play an important role in the proofs of certain classical results about polynomial and trace identities in the F-algebra $\mathbb{M}_n(F)$ (see [5] and [6]). For fields F of characteristic zero, Kemer's [10] pioneering work on the T-ideals of associative algebras has revealed the importance of identities satisfied by $n \times n$ matrices over the Grassmann (exterior) algebra

$$E = F\langle \{x_i : i \in \mathbb{N}\} : x_i x_j + x_j x_i = 0, \quad \leqslant i \leqslant j \rangle$$

generated by an infinite family $\{x_i : i \in \mathbb{N}\}$ of anticommutative indeterminates. For $n \times n$ matrices over a Lie nilpotent ring of index m, a Cayley-Hamilton identity of degree n^m (with left- or right-sided scalar coefficients) was found in [16]. Since the Grassmann algebra E is Lie nilpotent of index $m = 2$, the aforementioned Cayley-Hamilton identity for matrices in $\mathbb{M}_n(E)$ is of degree n^2. In [3], Domokos presents a slightly modified version of this identity in which the coefficients are invariant under the conjugation action of $\mathrm{GL}_n(F)$.

[1]Our Lie algebras are over a commutative ring that is not necessarily a field. No harm shall come of this more general interpretation since in the few instances where results about standard Lie algebras are used, the underlying commutative ring is a field.

This paper is an attempt to answer a conjecture posed in [17, p. 4785]. The statement of this conjecture is rendered less cumbersome if expressed in terms of a function $M(\ell, n)$ of positive integer arguments ℓ and n, defined as follows:

$$M(\ell,n) \overset{\text{def}}{=} \max\left\{\frac{1}{2}\left(n^2-\sum_{i=1}^{\ell}k_i^2\right)+1 : k_1,k_2,\dots,k_\ell \quad \text{are} \right.$$

$$\left. \text{nonnegative integers such that} \sum_{i=1}^{\ell}k_i=n\right\}. \tag{6}$$

Conjecture. *Let F be any field, m and n positive integers, and R an F-subalgebra of $\mathbb{M}_n(F)$ with Lie nilpotence index m. Then*

$$\dim_F R \leqslant M(m+1,n). \tag{7}$$

We shall henceforth refer to the above as *"the Conjecture"*. More specifically, if F is any fixed field, we shall say that *"the Conjecture holds in respect of F"* if (7) holds for all positive integers m and n and F-subalgebras R of $\mathbb{M}_n(F)$ with Lie nilpotence index m.

If R is any algebra over a field F, then a module V over R is precisely a representation of R via action on the underlying F-space structure on V. If the module is faithful, then this representation is faithful, thus yielding an embedding of R into $\operatorname{End}_F V$, the F-algebra of F-space endomorphisms on V. If V is also finite dimensional over F, say $\dim_F V = n$, then $\operatorname{End}_F V$ is isomorphic to $\mathbb{M}_n(F)$, and so we have an F-algebra embedding of R into $\mathbb{M}_n(F)$. (We point out that such a finite dimensional V is certain to exist if R is finite dimensional, for V can always be chosen to be R itself.) Thus, seen through a representation theoretic lens, inequality (7) sheds light on a possible *lower* bound for the dimension of a faithful module over a given Lie nilpotent algebra.

In the same spirit, Domokos [4, Theorem 1, p. 156] derives a lower bound for the dimension of a faithful module over a finite dimensional algebra satisfying the polynomial identity $[x_1, y_1][x_2, y_2]\cdots[x_m, y_m] = 0$, in terms of m.

Our initial task, which is easily accomplished, shall be to argue that the upper bound (7) is optimal for all choices of m and n.

Suppose first that $m+1 \leqslant n$. It is proven in Corollary 27(a) that for such m and n, $M(m+1,n) = \frac{1}{2}\left(n^2 - \sum_{i=1}^{m+1} k_i^2\right) + 1$ for some sequence of *positive* integers $k_1, k_2, \dots, k_{m+1}$ satisfying $\sum_{i=1}^{m+1} k_i = n$. Let F be any field and R the algebra of $n \times n$ matrices over F of type $(k_1, k_2, \dots, k_{m+1})$. As noted earlier, R has the form $R = FI_n + J$ with radical J satisfying $J^{m+1} = 0$. Since the set FI_n of scalar matrices is central in R, it can be shown that the kth terms of the Lower Central Series for R (interpreted as a Lie algebra via the commutator) and J coincide, that is to say, $R_{[k]} = J_{[k]}$ for $k > 1$. It is also evident that $J_{[k]} \subseteq J^k$ for every $k \in \mathbb{N}$. Thus $R_{[m+1]} = J_{[m+1]} \subseteq J^{m+1} = 0$, so R is Lie nilpotent of index m. It follows from (5) that $\dim_F R = \frac{1}{2}\left[n^2 - \sum_{i=1}^{m+1} k_i^2\right] + 1 = M(m+1,n)$.

Now suppose $m+1 > n$. No generality is lost if we suppose $n > 1$. It is proven in Corollary 27(b) that for such m and n, $M(m+1,n) = M(n,n) = \frac{1}{2}(n^2-n)+1$, and this, by (5), is equal to $\dim_F R$ where R is the algebra of $n \times n$ matrices over field F of type $(k_1, k_2, \dots, k_n)$ with $k_1 = k_2 = \cdots = k_n = 1$. (The reader will see that in this instance, R is just the algebra of all upper triangular matrices over F with constant main diagonal.) As shown in the previous paragraph, such an algebra R is Lie nilpotent of index $n-1$ and thus Lie nilpotent of index m, since $m \geqslant n-1$.

The theorem below collects together the conclusions drawn above.

Theorem 2. *Let F be any field, and m and n arbitrary positive integers. Then there exists an F-subalgebra R of $\mathbb{M}_n(F)$ with Lie nilpotence index m such that*

$$\dim_F R = M(m+1, n).$$

The main body of theory in this paper is developed in Sections 5 and 6 with module theoretic methods our primary tools. Sections 3 and 4 show that the Conjecture reduces to a consideration of local subalgebras of upper triangular matrix rings over an algebraically closed field. Section 7, which can be read independently of earlier sections, establishes important properties of the function $M(\ell, n)$ required in earlier theory. An explicit formula for $M(\ell, n)$ is also derived which is then shown to have a more simplified form for small values of ℓ. In Section 8 the algebra of $n \times n$ matrices of type $(d_1, d_2, \ldots, d_\ell)$ is used to provide a pictorial representation of the objects introduced in earlier theory. The content of Section 9, which is titled “Open questions”, is self-evident.

2. Preliminaries

The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets.

If X is any set, then X^n denotes the cartesian product of n copies of X.

$\mathbb{N}$ and $\mathbb{N}_0$ will denote the sets of positive integers and nonnegative integers, respectively.

All rings are associative and possess identity, and all modules are unital.

Let R be a ring and V a right R-module. We write $W \leqslant V$ to indicate that W is a submodule of V. If X is a nonempty subset of V and I is a right ideal of R, then

$$(0 :^I X) \overset{\text{def}}{=} \{a \in I : Xa = 0\} = I \cap (0 :^R X).$$

Observe that $(0 :^I X)$ is always a right ideal of R.

Let F be a field. For each $n \in \mathbb{N}$, $\mathbb{M}_n(F)$ (resp. $\mathbb{U}_n(F)$) (resp. $\mathbb{U}_n^*(F)$) shall denote the F-algebra of all $n \times n$ matrices over F (resp. upper triangular $n \times n$ matrices over F) (resp. upper triangular $n \times n$ matrices over F with constant main diagonal).

3. The passage to local algebras over an algebraically closed field

In this section we show that the Conjecture reduces to a consideration of local algebras over an algebraically closed field.

Lemma 3. *Let F be a subfield of field K and R an F-algebra. Let $r_1 \otimes b_1, r_2 \otimes b_2, \ldots, r_m \otimes b_m \in R \otimes_F K$ with $r_i \in R$ and $b_i \in K$ for $i \in \{1, 2, \ldots, m\}$. Then*

$$[r_1 \otimes b_1, r_2 \otimes b_2, \ldots, r_m \otimes b_m]^* = [r_1, r_2, \ldots, r_m]^* \otimes (b_1 b_2 \cdots b_m).$$

Proof. We provide only a proof of the inductive step.

Putting $r = [r_1, r_2, \dots, r_m]^*$ and $b = b_1 b_2 \cdots b_m$ we see that

$$\begin{aligned}
&[r_1 \otimes b_1, r_2 \otimes b_2, \dots, r_{m+1} \otimes b_{m+1}]^* \\
&= [r \otimes b, r_{m+1} \otimes b_{m+1}] \quad \text{[by the inductive hypothesis]} \\
&= (r \otimes b)(r_{m+1} \otimes b_{m+1}) - (r_{m+1} \otimes b_{m+1})(r \otimes b) \\
&= (rr_{m+1}) \otimes (bb_{m+1}) - (r_{m+1}r) \otimes (b_{m+1}b) \\
&= (rr_{m+1}) \otimes (bb_{m+1}) - (r_{m+1}r) \otimes (bb_{m+1}) \quad \text{[because } K \text{ is a field} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{so } b_{m+1}b = bb_{m+1}] \\
&= (rr_{m+1} - r_{m+1}r) \otimes (bb_{m+1}) \\
&= [r_1, r_2, \dots, r_{m+1}]^* \otimes (b_1 b_2 \cdots b_{m+1}).
\end{aligned}$$

$\square$

Proposition 4. *Let F be a subfield of field K and R an F-subalgebra of $\mathbb{M}_n(F)$. Then:*

(a) $\dim_F R = \dim_K (R \otimes_F K)$.
(b) *$R \otimes_F K$ is isomorphic to a K-subalgebra of $\mathbb{M}_n(K)$.*
(c) *If R is Lie nilpotent of index m, then so is $R \otimes_F K$.*

Proof. (a) is standard theory; see for example [1, Exercise 19.3, p. 231].

(b) The hypothesis entails that $R \otimes_F K$ is a K-subalgebra of $\mathbb{M}_n(F) \otimes_F K$. The result follows noting that $\mathbb{M}_n(F) \otimes_F K \cong \mathbb{M}_n(K)$ as K-algebras (see [12, Chapter 9, Exercise 10, p. 94]).

(c) Suppose R is Lie nilpotent of index m. Take $x_1, x_2, \dots, x_{m+1} \in R \otimes_F K$. Since the expression $[x_1, x_2, \dots, x_{m+1}]^*$ is additive in each of its $m+1$ arguments, $[x_1, x_2, \dots, x_{m+1}]^*$ is expressible as a sum of elements of the form $[r_1 \otimes b_1, r_2 \otimes b_2, \dots, r_{m+1} \otimes b_{m+1}]^*$ where $r_i \in R$ and $b_i \in K$ for $i \in \{1, 2, \dots, m+1\}$. By Lemma 3

$$\begin{aligned}
&[r_1 \otimes b_1, r_2 \otimes b_2, \dots, r_{m+1} \otimes b_{m+1}]^* \\
&= [r_1, r_2, \dots, r_{m+1}]^* \otimes (b_1 b_2 \cdots b_{m+1}) \\
&= 0 \otimes (b_1 b_2 \cdots b_{m+1}) \quad \text{[because } R \text{ is Lie nilpotent of index } m] \\
&= 0.
\end{aligned}$$

It follows that $[x_1, x_2, \dots, x_{m+1}]^* = 0$, so $R \otimes_F K$ is Lie nilpotent of index m. $\square$

Theorem 5. *Let $\mathcal{C}$ be a nonempty class of fields and $\overline{\mathcal{C}}$ the class of all subfields of fields in $\mathcal{C}$. The following statements are equivalent:*

(a) *The Conjecture holds in respect of all fields in $\mathcal{C}$;*
(b) *The Conjecture holds in respect of all fields in $\overline{\mathcal{C}}$.*

Proof. (b)$\Rightarrow$(a) is obvious since $\mathcal{C} \subseteq \overline{\mathcal{C}}$.

(a)$\Rightarrow$(b): Let m and n be positive integers, $F \in \overline{\mathcal{C}}$, and R an F-subalgebra of $\mathbb{M}_n(F)$ with Lie nilpotence index m. We must show that $\dim_F R \leqslant M(m+1, n)$.

Choose field extension K of F such that $K \in \mathcal{C}$. By Proposition 4(b) and (c), the K-algebra $R \otimes_F K$ is Lie nilpotent of index m and is isomorphic to a K-subalgebra of $\mathbb{M}_n(K)$. By part (a) of this theorem, $\dim_K(R \otimes_F K) \leqslant M(m+1, n)$. Hence by Proposition 4(a),

$$\dim_F R = \dim_K(R \otimes_F K) \leqslant M(m+1, n),$$

as required. □

It follows from Theorem 5 that the Conjecture will hold for a given field F if it can be shown to hold for the algebraic closure of F. We shall exploit this fact in the next section.

Proposition 6. *Every idempotent in a ring satisfying the Engel condition is central.*

Proof. If R is an arbitrary ring and $e = e^2 \in R$, then a routine calculation shows that for each $a \in R$,

$$[[1-e, (1-e)a], e] = (1-e)ae.$$

Putting $\alpha = (1-e)ae$ we see that $\alpha e = \alpha$ and $e\alpha = 0$, from which it follows that $[\alpha, e] = \alpha$. Iterating, we obtain

$$[[\alpha, e], e] = \alpha,$$

$$[[[\alpha, e], e], e] = \alpha, \quad \text{and in general}$$

$$[\alpha, \overbrace{e, e, \ldots, e}^{m}]^* = \alpha, \quad \text{for each } m \in \mathbb{N}.$$

If R satisfies the Engel condition of index m, then we have

$$\alpha = [\alpha, \overbrace{e, e, \ldots, e}^{m}]^* = 0,$$

and so

$$(1-e)ae = 0. \tag{8}$$

Interchanging the roles of e and $1-e$ in the above argument yields

$$ea(1-e) = 0. \tag{9}$$

Equations (8) and (9) imply that $ae - eae = 0$ and $ea - eae = 0$, whence $ea = ae$. We conclude that e is central. □

Proposition 7. *Every right artinian ring satisfying the Engel condition is isomorphic to a finite direct product of local rings.*

Proof. It is known (see [1, Theorem 27.6, p. 304] or [12, Theorem 5.9, p. 49]) that every right artinian ring R contains a complete set of primitive orthogonal idempotents $\{e_1, e_2, \ldots, e_k\}$ such that R decomposes as

$$R_R \cong e_1R \oplus e_2R \oplus \cdots \oplus e_kR,$$

where each e_iR has unique maximal proper submodule $e_iJ(R)$. If R satisfies the Engel condition, then each idempotent e_i is central by Proposition 6, so the above decomposition is a decomposition of (two-sided) ideals with each $e_iR = e_iRe_i$ a local ring. □

Lemma 8. *Let F be a field and e an idempotent of $\mathbb{M}_n(F)$. If* $\operatorname{rank} e = r$, *then $e\mathbb{M}_n(F)e \cong \mathbb{M}_r(F)$ as F-algebras.*

Proof. Since rank $e = r$, $F^{(n)}e$ has dimension r as an F-space, so $F^{(n)}e \cong F^{(r)}$ as F-spaces. Then

$$\begin{aligned} e\mathbb{M}_n(F)e &\cong \mathrm{End}_F\left(F^{(n)}e\right) \\ &\cong \mathrm{End}_F\left(F^{(r)}\right) \\ &\cong \mathbb{M}_r(F). \end{aligned}$$

□

The following theorem tells us that for a given field F, the Conjecture will hold for all F-subalgebras of $\mathbb{M}_n(F)$ if it can be shown to hold for all local F-subalgebras of $\mathbb{M}_n(F)$.

Theorem 9. *The following statements are equivalent for a field F:*

(a) *The Conjecture holds in respect of F;*
(b) *for all positive integers m and n, if R is any* local *F-subalgebra of $\mathbb{M}_n(F)$ with Lie nilpotence index m, then*

$$\dim_F R \leqslant M(m+1, n).$$

Proof. (a)⇒(b) is obvious.

(b)⇒(a): Let m and n be positive integers and R an F-subalgebra of $\mathbb{M}_n(F)$ with Lie nilpotence index m. Note that R satisfies the Engel condition of index m. Since R is a finite dimensional F-algebra, it is right (and left) artinian, and so by Proposition 7, $R \cong R_1 \times R_2 \times \cdots \times R_k$ where each R_i is a local right artinian ring. This entails the existence of a complete set of central primitive orthogonal idempotents $\{e_1, e_2, \ldots, e_k\}$ in R such that

$$R_R \cong e_1R \oplus e_2R \oplus \cdots \oplus e_kR \tag{10}$$

with $e_iR = e_iRe_i \cong R_i$ for each $i \in \{1, 2, \ldots, k\}$. For each $i \in \{1, 2, \ldots, k\}$ put

$$r_i \stackrel{\text{def}}{=} \text{rank } e_i. \tag{11}$$

The equation $1_R = I_n = e_1 + e_2 + \cdots + e_k$ induces the F-space decomposition

$$F^{(n)} = F^{(n)}e_1 \oplus F^{(n)}e_2 \oplus \cdots \oplus F^{(n)}e_k.$$

Thus

$$\begin{aligned} n &= \dim_F F^{(n)} \\ &= \dim_F(F^{(n)}e_1) + \dim_F(F^{(n)}e_2) + \cdots + \dim_F(F^{(n)}e_k) \\ &= r_1 + r_2 + \cdots + r_k \quad \text{[by (11)]}. \end{aligned} \tag{12}$$

Observe that each local ring e_iR is an F-subalgebra of $e_i\mathbb{M}_n(F)e_i$ and that $e_i\mathbb{M}_n(F)e_i \cong \mathbb{M}_{r_i}(F)$ for each $i \in \{1, 2, \ldots, k\}$ by Lemma 8. It is clear too that each e_iR must be Lie nilpotent of index m, since R has the same property and $e_iR \subseteq R$.

The aforementioned facts, together with (b), imply that

$$\dim_F(e_iR) \leqslant M(m+1, r_i)$$

for each $i \in \{1, 2, \ldots, k\}$.

Then

$$\begin{aligned}
\dim_F R &= \sum_{i=1}^{k} \dim_F(e_i R) \quad \text{[by (10)]} \\
&\leqslant \sum_{i=1}^{k} M(m+1, r_i) \\
&\leqslant M\left(m+1, \sum_{i=1}^{k} r_i\right) \quad \text{[by Proposition 28]} \\
&= M(m+1, n) \quad \text{[by (12)].}
\end{aligned}$$

□

4. Simultaneous triangularization and the passage to upper triangular matrix rings

The main result of this section (Theorem 12) shows that for algebraically closed fields F, the Conjecture reduces to a consideration of F-subalgebras of $\mathbb{U}_n^*(F)$, the algebra of upper triangular matrices over F with constant main diagonal.

Recall that an F-subalgebra R of $\mathbb{M}_n(F)$ is said to be *simultaneously upper triangularizable* in $\mathbb{M}_n(F)$ if there exists an invertible $U \in \mathbb{M}_n(F)$ such that $U^{-1}RU \subseteq \mathbb{U}_n(F)$.

A key result is the following. Although implicit in [8, Theorem 1, p. 434] we shall provide a proof in the absence of a suitable reference.

Proposition 10. *Let F be an algebraically closed field.*

(a) *If R is a finite dimensional local F-algebra, then R has F-space decomposition $R = F \cdot 1_R \oplus J(R)$.*

(b) *If R is a local F-subalgebra of $\mathbb{M}_n(F)$, then there exists an invertible $U \in \mathbb{M}_n(F)$ such that $U^{-1}RU \subseteq \mathbb{U}_n^*(F)$. Thus, R is isomorphic to an F-subalgebra of $\mathbb{U}_n^*(F)$.*

Proof. (a) Since R is local, it follows that $R/J(R)$ is a division algebra that is finite dimensional over F. Since F is algebraically closed this implies that $R/J(R) \cong F$. Inasmuch as $F \cdot 1_R \cap J(R) = 0$, the equation

$$\begin{aligned}
\dim_F(F \cdot 1_R + J(R)) &= 1 + \dim_F J(R) \\
&= \dim_F(R/J(R)) + \dim_F J(R) \\
&= \dim_F R
\end{aligned}$$

entails that $R = F \cdot 1_R \oplus J(R)$.

(b) It is known (see [13, Theorem 1.4.6, p. 12]) that for an algebraically closed field F, a necessary and sufficient condition for an F-subalgebra R of $\mathbb{M}_n(F)$ to be simultaneously upper triangularizable in $\mathbb{M}_n(F)$ is that $R/J(R)$ is commutative, a condition that is clearly met in our case. Hence $U^{-1}RU \subseteq \mathbb{U}_n(F)$ for some invertible $U \in \mathbb{M}_n(F)$. Putting $S = U^{-1}RU$ we note that since S is local, $S = FI_n \oplus J(S)$ by (a). Since every element of $J(S)$ is a nilpotent matrix in $\mathbb{U}_n(F)$, and a nilpotent upper triangular matrix is strictly upper triangular, we have

$$U^{-1}RU = S = FI_n \oplus J(S) \subseteq \mathbb{U}_n^*(F).$$

□

Remark 11.

(a) The observation that the factor ring $R/J(R)$ is commutative is key in the proof of Proposition 10(b). We point out that this property is possessed by all Lie nilpotent rings. Indeed, [17, Proposition 3.1(3), p. 4790] asserts that if $\operatorname{rad}(R)$ denotes the prime radical of a Lie nilpotent ring R, then $R/\operatorname{rad}(R)$ is commutative. Since $\operatorname{rad}(R) \subseteq J(R)$, the commutativity of $R/J(R)$ follows.

(b) If F is an algebraically closed field of characteristic zero (the latter assumption is not made in Proposition 10) and R is any Lie nilpotent F-subalgebra of $\mathbb{M}_n(F)$, then R can be shown to be simultaneously upper triangularizable in $\mathbb{M}_n(F)$ as a consequence of Lie's theorem, which asserts that if $\mathfrak{g}$ is a finite dimensional solvable Lie algebra with representation $\mathbb{M}_n(F)$, then $\mathfrak{g}$ is simultaneously upper triangularizable in $\mathbb{M}_n(F)$. Lie's theorem applies inasmuch as every Lie nilpotent ring is a nilpotent Lie algebra with respect to the commutator, and nilpotent Lie algebras are solvable. (This latter fact is explained in the second open question of Section 9.)

Theorem 12. *The following statements are equivalent for an algebraically closed field F:*

(a) *The Conjecture holds in respect of F;*
(b) *for all positive integers m and n, if R is any* local *F-subalgebra of $\mathbb{M}_n(F)$ with Lie nilpotence index m, then*

$$\dim_F R \leqslant M(m+1,n);$$

(c) *for all positive integers m and n, if R is any F-subalgebra of $\mathbb{U}_n^*(F)$ with Lie nilpotence index m, then*

$$\dim_F R \leqslant M(m+1,n).$$

Proof. (a) and (b) are equivalent by Theorem 9 without any restriction on the field F.

The equivalence of (b) and (c) is a consequence of Proposition 10(b), which tells us that up to isomorphism, the local F-subalgebras of $\mathbb{M}_n(F)$ are precisely the F-subalgebras of $\mathbb{U}_n^*(F)$. □

5. Subalgebras of $\mathbb{U}_n^*(F)$

The main body of theory is developed in this section.

Throughout this section and unless otherwise stated, F shall denote a field and R an F-subalgebra of $\mathbb{U}_n^*(F)$.

Let V be a faithful right R-module. We define a sequence $\{R_k\}_{k\in\mathbb{N}}$ of F-subalgebras of R, a sequence $\{J_k\}_{k\in\mathbb{N}}$ where each J_k is an ideal of R_k, and a sequence $\{U_k\}_{k\in\mathbb{N}}$ of F-subspaces of V as follows:

$$\begin{cases} R_1 \overset{\text{def}}{=} R, \\ J_1 \overset{\text{def}}{=} J(R_1), \quad \text{and} \\ U_1 \overset{\text{def}}{=} \text{any } F\text{-subspace complement of } VJ_1 \text{ in } V. \end{cases}$$

For $k \in \mathbb{N}$, $k \geqslant 2$, define

$$
\begin{cases}
R_k \overset{\text{def}}{=} FI_n + (0 :^{R_{k-1}} U_{k-1}), \\
J_k \overset{\text{def}}{=} J(R_k), \quad \text{and} \\
U_k \overset{\text{def}}{=} \text{ any } F\text{-subspace complement of } VJ_1J_2\cdots J_k \text{ in} \\
\qquad\quad VJ_1J_2\cdots J_{k-1}.
\end{cases}
\tag{13}
$$

It follows from the definition of U_k that

$$
VJ_1J_2\cdots J_{k-1} = U_k \oplus VJ_1J_2\cdots J_k \tag{14}
$$

as F-spaces.

For convenience we put $J_0 = R$.

Since $(0 :^{R_{k-1}} U_{k-1}) \subseteq R_{k-1}$ and since every F-subalgebra of $\mathbb{U}_n^*(F)$ contains FI_n, it is clear from the definition of R_k in (13) that $R_{k-1} \supseteq R_k$ for every $k \in \mathbb{N}$, $k \geqslant 2$. We thus have

$$
R_1 \supseteq R_2 \supseteq \cdots. \tag{15}
$$

It is easily shown that if S and T are any F-subalgebras of $\mathbb{U}_n^*(F)$, then $S \subseteq T$ if and only if $J(S) \subseteq J(T)$. In the light of this observation, (15) implies that

$$
J_1 \supseteq J_2 \supseteq \cdots. \tag{16}
$$

Since $J_k \subseteq J_1$ for all $k \in \mathbb{N}$ and J_1 is nilpotent, we must have $J_0J_1\cdots J_k = 0$ for k sufficiently large. Define

$$
\ell \overset{\text{def}}{=} \min\{k \in \mathbb{N} : J_0J_1\cdots J_k = 0\}. \tag{17}
$$

It follows from (16) that $J_0J_1\cdots J_{k-1} \supseteq J_0J_1\cdots J_k$ for each $k \in \mathbb{N}$. We thus have the descending chain

$$
R = J_0 \supseteq J_0J_1 \supseteq \cdots \supseteq J_0J_1\cdots J_{\ell-1} \supseteq J_0J_1\cdots J_\ell = 0.
$$

This, in turn, induces a descending chain

$$
V = VJ_0 \supseteq VJ_0J_1 \supseteq \cdots \supseteq VJ_0J_1\cdots J_{\ell-1} \supseteq 0. \tag{18}
$$

Note that $VJ_0J_1\cdots J_{\ell-1} \neq 0$ since $J_0J_1\cdots J_{\ell-1} \neq 0$ and V is a faithful right R-module.

Recall that if R is an arbitrary ring, then a submodule N of a right R-module M is said to be *superfluous* if

$$
\forall L \leqslant M, \qquad N + L = M \Rightarrow L = M.
$$

Lemma 13. *If I is a nilpotent ideal of an arbitrary ring R and M is any right R-module, then MI is a superfluous submodule of M.*

Proof. Suppose $MI + L = M$ with $L \leqslant M$. Multiplying by I we obtain $MI^2 + LI = MI$, so $MI^2 + LI + L = MI + L = M$. Continuing in this way, we obtain $MI^k + L = M$ for all $k \in \mathbb{N}$. Since I is nilpotent this yields, for k sufficiently large, the equation $MI^k + L = M \cdot 0 + L = L = M$. □

Important properties of the chain (18) are established in the next lemma.

Lemma 14. *Let the sequences $\{R_k\}_{k\in\mathbb{N}}$, $\{J_k\}_{k\in\mathbb{N}}$ and $\{U_k\}_{k\in\mathbb{N}}$ be defined as in* (13)*, and let positive integer ℓ be defined as in* (17)*. Let $k \in \{1, 2, \ldots, \ell\}$. Then:*

(a) *$VJ_0J_1\cdots J_k$ is a superfluous R_k-submodule of $VJ_0J_1\cdots J_{k-1}$;*
(b) *$U_kR_k = VJ_0J_1\cdots J_{k-1} = U_k \oplus\cdots\oplus U_\ell$;*
(c) *$J_{k+1} = (0 :^{R_k} U_k)$;*
(d) *$VJ_0J_1\cdots J_{k-1}$ is a faithful right R_k-module.*

Proof. (a) That $VJ_0J_1\cdots J_k$ is a right R_k-module is a consequence of the fact that $J_0J_1\cdots J_k$ is an F-subspace of R that is closed under right multiplication by elements from R_k.

Since $R_k \subseteq R_{k-1}$, every right R_{k-1}-module is canonically a right R_k-module. In particular, $VJ_0J_1\cdots J_{k-1}$ is a right R_k-module.

It remains to show that $VJ_0J_1\cdots J_k$ is superfluous in $VJ_0J_1\cdots J_{k-1}$. Put $U = VJ_0J_1\cdots J_{k-1}$. Since $J_k \subseteq J_1$ and J_1 is nilpotent, J_k must also be nilpotent. It follows from Lemma 13 that UJ_k is a superfluous submodule of U, as required.

(b) Since $U_kR_k \supseteq U_k$, it follows from (14) that

$$VJ_0J_1\cdots J_{k-1} = U_kR_k + VJ_0J_1\cdots J_k,$$

where the right-hand side of the above equation is a sum of R_k-submodules of $VJ_0J_1\cdots J_{k-1}$. Since $VJ_0J_1\cdots J_k$ is a superfluous R_k-submodule of $VJ_0J_1\cdots J_{k-1}$ by (a), we must have $U_kR_k = VJ_0J_1\cdots J_{k-1}$.

To establish the equation $VJ_0J_1\cdots J_{k-1} = U_k \oplus\cdots\oplus U_\ell$, we note first that the U_i constitute an independent family of F-subspaces of V. This is clear from the definition of the U_i in (13). This means that the sum $U_k \oplus\cdots\oplus U_\ell$ is indeed a direct sum of F-subspaces. It remains to establish equality.

Since, by (13), U_ℓ is an F-subspace complement of $VJ_0J_1\cdots J_\ell$ in $VJ_0J_1\cdots J_{\ell-1}$, and since $VJ_0J_1\cdots J_\ell = 0$ by definition of ℓ, we must have

$$VJ_0J_1\cdots J_{\ell-1} = U_\ell.$$

Repeated application of the formula for U_k in (13) shows that

$$VJ_0J_1\cdots J_{\ell-2} = U_{\ell-1} \oplus U_\ell$$

and, more generally, that

$$VJ_0J_1\cdots J_{k-1} = U_k \oplus\cdots\oplus U_\ell,$$

as required.

(c) Since $k-1 < \ell$, it follows from (b) and the minimality of ℓ that $U_kR_k = VJ_0J_1\cdots J_{k-1} \neq 0$, whence $U_k \neq 0$. This means that $(0 :^{R_k} U_k)$ must be a proper right ideal of R_k and so cannot contain any units of R_k. Inasmuch as R_k is an F-subalgebra of $\mathbb{U}_n^*(F)$, $(0 :^{R_k} U_k)$ must therefore comprise strictly upper triangular matrices. Since, by (13), $R_{k+1} = FI_n + (0 :^{R_k} U_k)$, we must have $J_{k+1} = J(R_{k+1}) = (0 :^{R_k} U_k)$.

(d) We use induction on k. Take $k = 1$. Then $VJ_0J_1\cdots J_{k-1} = VJ_0 = V$, which is a faithful R_1-module by hypothesis. This establishes the base case.

To establish the inductive step, take $t \in R_k$ with $k \geqslant 2$ and suppose that

$$(VJ_0J_1\cdots J_{k-1})t = 0. \tag{19}$$

Since $VJ_0J_1\cdots J_{k-1}\neq 0$, t cannot be a unit of R_k, and since R_k is local, we must have $t\in J_k$. By (c), $J_k=(0:^{R_{k-1}}U_{k-1})$, so

$$U_{k-1}t=0. \tag{20}$$

We thus have

$$\begin{aligned}(VJ_0J_1\cdots J_{k-2})t &= (U_{k-1}+VJ_0J_1\cdots J_{k-1})t \quad \text{[by (14)]}\\ &= 0 \ \text{ [by (19) and (20)]}.\end{aligned}$$

By the inductive hypothesis, $VJ_0J_1\cdots J_{k-2}$ is a faithful right R_{k-1}-module. Since $t\in J_k\subseteq R_k\subseteq R_{k-1}$, the above equation entails $t=0$. We conclude that $VJ_0J_1\cdots J_{k-1}$ is a faithful R_k-module. □

Remark 15.

(a) Taking $k=1$ in Lemma 14(b) yields the F-subspace decomposition

$$V=U_1\oplus U_2\oplus\cdots\oplus U_\ell. \tag{21}$$

Substituting the equation $VJ_0J_1\cdots J_{k-1}=U_k\oplus\cdots\oplus U_\ell$ of Lemma 14(b) into (21) yields

$$V=U_1\oplus\cdots\oplus U_{k-1}\oplus VJ_0J_1\ldots J_{k-1}. \tag{22}$$

(b) The faithfulness of $VJ_0J_1\cdots J_{k-1}$ proved in Lemma 14(d) means that $(VJ_0J_1\cdots J_{k-1})J_k=0$ if and only if $J_k=0$. Moreover, since V is faithful as a right R-module, we have that

$$VJ_0J_1\cdots J_k=0 \Leftrightarrow J_0J_1\cdots J_k=0.$$

It follows that

$$J_0J_1\cdots J_k=0 \Leftrightarrow J_k=0.$$

This has the consequence that

$$\ell=\min\{k\in\mathbb{N}: J_0J_1\cdots J_k=0\}=\min\{k\in\mathbb{N}: J_k=0\}.$$

Proposition 16. *Let the sequences $\{R_k\}_{k\in\mathbb{N}}$, $\{J_k\}_{k\in\mathbb{N}}$ and $\{U_k\}_{k\in\mathbb{N}}$ be defined as in* (13)*, and let positive integer ℓ be defined as in* (17)*. Then*

$$R=R_1\supset R_2\supset\cdots\supset R_\ell=R_{\ell+1}=\cdots$$

is a strictly descending chain of F-subalgebras of $\mathbb{U}_n^*(F)$ that stabilizes at R_ℓ. Moreover, $J_\ell=0$, so that $R_\ell=FI_n$.

Proof. Suppose $R_k=R_{k+1}$ for some $k\leqslant\ell$. Note that we cannot have $U_k=0$ since this would imply, by Lemma 14(b), that $VJ_0J_1\cdots J_{k-1}=0$, which contradicts the fact that $VJ_0J_1\cdots J_{k-1}\supseteq VJ_0J_1\cdots J_{\ell-1}\neq 0$. Now

$$\begin{aligned}0 &= U_kJ_{k+1} \quad \text{[because } J_{k+1}=(0:^{R_k}U_k) \text{ by Lemma 14(c)]}\\ &= U_kJ_k \quad \text{[because } J_k=J(R_k)=J(R_{k+1})=J_{k+1} \text{ by hypothesis]}\\ &= (U_kR_k)J_k \quad \text{[because } J_k \text{ is an ideal of } R_k\text{]}\\ &= (VJ_0J_1\cdots J_{k-1})J_k \quad \text{[by Lemma 14(b)]}.\end{aligned}$$

Since $k\leqslant\ell$ it follows from the minimality of ℓ that $k=\ell$. We have thus proven that $R_k\supset R_{k+1}$ for $k\in\{1,2,\ldots,\ell-1\}$.

In Remark 15(b) we noted that $J_\ell = 0$. Since $R_\ell \subseteq \mathbb{U}_n^*(F)$, this entails that $R_\ell = FI_n$. However, since every F-subalgebra of $\mathbb{U}_n^*(F)$ contains FI_n, the descending chain of F-subalgebras must stabilize at R_ℓ. □

Let the sequences $\{R_k\}_{k\in\mathbb{N}}$, $\{J_k\}_{k\in\mathbb{N}}$ and $\{U_k\}_{k\in\mathbb{N}}$ be defined as in (13), and let positive integer ℓ be defined as in (17). For each $k \in \{1, 2, \dots, \ell\}$ define

$$d_k \overset{\text{def}}{=} \dim_F U_k. \tag{23}$$

A key step in the proof of Theorem 17(c) below is inspired by [7, Section 2. Proof of Schur's Inequality, p. 558].

Theorem 17. *Let the sequences $\{R_k\}_{k\in\mathbb{N}}$, $\{J_k\}_{k\in\mathbb{N}}$ and $\{U_k\}_{k\in\mathbb{N}}$ be defined as in* (13)*, positive integer ℓ defined as in* (17)*, and $\{d_k : 1 \leqslant k \leqslant \ell\}$ defined as in* (23)*. Then:*

(a) $\dim_F(U_kJ_k) = \dim_F V - \sum_{i=1}^{k} d_i$ *for each* $k \in \{1, 2, \dots, \ell\}$.

(b) $\dim_F V = \sum_{i=1}^{\ell} d_i$.

(c) $\dim_F R \leqslant M(\ell, \dim_F V)$.

Proof. (a) Inasmuch as

$$\begin{aligned} V &= U_1 \oplus \cdots \oplus U_k \oplus VJ_0J_1\cdots J_k \quad \text{[by (22)]} \\ &= U_1 \oplus \cdots \oplus U_k \oplus (VJ_0J_1\cdots J_{k-1})J_k \\ &= U_1 \oplus \cdots \oplus U_k \oplus (U_kR_k)J_k \quad \text{[by Lemma 14(b)]} \\ &= U_1 \oplus \cdots \oplus U_k \oplus U_kJ_k \quad \text{[because } J_k \text{ is an ideal of } R_k], \end{aligned}$$

we have $\dim_F V = d_1 + \cdots + d_k + \dim_F(U_kJ_k)$, from which (a) follows.

(b) is an immediate consequence of (21) and (23).

(c) If $\ell = 1$, then $J = J_\ell = 0$, so $\dim_F R = 1 = M(1, \dim_F V)$, and there is nothing further to prove. Suppose $\ell \geqslant 2$.

We next derive the recursive formula

$$\dim_F J_k \leqslant d_k \left(\dim_F V - \sum_{i=1}^{k} d_i\right) + \dim_F J_{k+1} \qquad (1 \leqslant k \leqslant \ell). \tag{24}$$

To this end, take $k \in \{1, 2, \dots, \ell\}$, $X \in J_k$, and let $\rho_X : U_k \to U_kJ_k$ be the right multiplication by X map. Observe that ρ_X is an F-linear map and thus a member of $\mathrm{Hom}_F(U_k, U_kJ_k)$.

Define the map $\Theta : J_k \to \mathrm{Hom}_F(U_k, U_kJ_k)$ by $\Theta(X) = \rho_X$. It is also easily seen that Θ is an F-linear map. Note that

$$
\begin{aligned}
\operatorname{Ker}\Theta &= \{X \in J_k : \rho_X = 0\} \\
&= \{X \in J_k : U_kX = 0\} \\
&= (0 :^{J_k} U_k) \\
&= J_k \cap (0 :^{R_k} U_k) \\
&= J_k \cap J_{k+1} \quad \text{[by Lemma 14(c)]} \\
&= J_{k+1} \quad [\text{because } J_k \supseteq J_{k+1}]. \qquad (25)
\end{aligned}
$$

We thus have

$$
\begin{aligned}
\dim_F J_k &= \operatorname{rank}\Theta + \operatorname{nullity}\Theta \\
&\leqslant \dim_F\left(\mathrm{Hom}_F(U_k, U_kJ_k)\right) + \dim_F J_{k+1} \quad \text{[by (25)]} \\
&= \dim_F U_k \cdot \dim_F(U_kJ_k) + \dim_F J_{k+1} \\
&= d_k\left(\dim_F V - \sum_{i=1}^{k} d_i\right) + \dim_F J_{k+1} \quad \text{[by (a)]},
\end{aligned}
$$

which is (24).

Letting k take on the values from 1 to $\ell - 1$ in (24), we see that

$$
\begin{aligned}
\dim_F J_1 &\leqslant \sum_{j=1}^{\ell-1} d_j\left(\dim_F V - \sum_{i=1}^{j} d_i\right) + \dim_F J_\ell \\
&= \sum_{j=1}^{\ell-1} d_j\left(\dim_F V - \sum_{i=1}^{j} d_i\right) \quad [\text{because } J_\ell = 0] \\
&= \frac{1}{2}\left((\dim_F V)^2 - \sum_{i=1}^{\ell} d_i^2\right) \quad \left[\text{because } \dim_F V = \sum_{i=1}^{\ell} d_i \text{ by (b)}\right] \\
&\leqslant M(\ell, \dim_F V) - 1 \quad \text{[by the definition of } M(\ell, \dim_F V) \\
&\qquad\qquad \text{noting that } \dim_F V = \textstyle\sum_{i=1}^{\ell} d_i]. \qquad (26)
\end{aligned}
$$

Since R has F-space decomposition $R = FI_n \oplus J$, we have

$$
\begin{aligned}
\dim_F R &= 1 + \dim_F J \\
&= 1 + \dim_F J_1 \quad [\text{because } J = J_1] \\
&\leqslant 1 + M(\ell, \dim_F V) - 1 \quad \text{[by (26)]} \\
&= M(\ell, \dim_F V).
\end{aligned}
$$

□

In Proposition 29 it is shown that $M(\ell, n)$ is an increasing function in both arguments. This means, with reference to Theorem 17(c), that the smaller the value of ℓ, the lower the upper bound $M(\ell, \dim_F V)$ for $\dim_F R$.

We shall show presently that if the F-subalgebra R of $\mathbb{U}_n^*(F)$ has radical J satisfying $J^m = 0$ for some $m \in \mathbb{N}$, then the value of ℓ cannot exceed m, and so

$$\dim_F R \leqslant M(m, \dim_F V).$$

In the next section we shall strengthen the above by proving that if R has Lie nilpotence index m (this is the case if $J^{m+1} = 0$), then the value of ℓ cannot exceed $m+1$, from which we may deduce that

$$\dim_F R \leqslant M(m+1, \dim_F V).$$

Since the d_i are positive in Theorem 17(b), we must have $\ell \leqslant \dim_F V$. A combination of Theorem 17(c), the fact that $M(\ell, n)$ is increasing in its first argument (Proposition 29), and the formula for $M(n,n)$ derived in Corollary 27(a) yields:

Corollary 18. *If R is an F-subalgebra of $\mathbb{U}_n^*(F)$ and V is any faithful right R-module, then*

$$\dim_F R \leqslant M(\dim_F V, \dim_F V) = \tfrac{1}{2}\left((\dim_F V)^2 - \dim_F V\right) + 1.$$

Remark 19. If $V = F^n = \overbrace{F \times F \times \cdots \times F}^{n \text{ times}}$ is interpreted as a $1 \times n$ matrix over F, then it has the canonical structure of a faithful right module with respect to any F-subalgebra of the matrix algebra $\mathbb{M}_n(F)$. For such a module V, we have

$$\dim_F V = n.$$

This allows us to replace $\dim_F V$ with n in each of the results in this, and subsequent, sections. In particular, taking $\dim_F V = n$ in the previous corollary yields the upper bound

$$\dim_F R \leqslant \tfrac{1}{2}(n^2 - n) + 1,$$

an observation that has little value, since the expression $\frac{1}{2}(n^2 - n) + 1$ coincides with the dimension of the overlying F-algebra $\mathbb{U}_n^*(F)$.

Proposition 20. *Let the sequences $\{R_k\}_{k \in \mathbb{N}}$, $\{J_k\}_{k \in \mathbb{N}}$ and $\{U_k\}_{k \in \mathbb{N}}$ be defined as in* (13)*, and let positive integer ℓ be defined as in* (17)*. If $J^m = 0$ for some $m \in \mathbb{N}$, then $\ell \leqslant m$.*

Proof. Inasmuch as $J_0 J_1 \cdots J_m \subseteq J^m = 0$, it follows from the definition of ℓ in (17) that $\ell \leqslant m$. □

Corollary 21. *If R is an F-subalgebra of $\mathbb{U}_n^*(F)$ satisfying $J^m = 0$ and V is any faithful right R-module, then*

$$\dim_F R \leqslant M(m, \dim_F V).$$

Proof. It follows from Theorem 17(c) and Proposition 20 that there exists a positive integer $\ell \leqslant m$ such that $\dim_F R \leqslant M(\ell, \dim_F V)$. By Proposition 29, $M(\ell, \dim_F V) \leqslant M(m, \dim_F V)$, whence $\dim_F R \leqslant M(m, \dim_F V)$. □

6. Lie nilpotent subalgebras of $\mathbb{U}_n^*(F)$: The main theorem

A routine inductive argument establishes the following.

Lemma 22. *Let R be an arbitrary ring and $\{r_i : 1 \leqslant i \leqslant m\} \subseteq R$. Then*

$$[r_1, r_2, \ldots, r_m]^* = \sum_{\sigma \in S_m} c_\sigma r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(m)},$$

where $c_\sigma \in \{-1, 0, 1\}$ for all $\sigma \in S_m$ and $\{\sigma \in S_m : c_\sigma \neq 0$ and $\sigma(1) = 1\}$ is a singleton comprising the identity permutation.

Proposition 23. *Let the sequences $\{R_k\}_{k\in\mathbb{N}}$, $\{J_k\}_{k\in\mathbb{N}}$ and $\{U_k\}_{k\in\mathbb{N}}$ be defined as in (13), and positive integer ℓ defined as in (17). If R is Lie nilpotent of index m, then $\ell \leqslant m+1$.*

Proof. By Lemma 14(c), we have $J_2 = (0 :^R U_1)$. Pick arbitrary $r \in R$ and $b_k \in J_k$ for each $k \in \{2, \ldots, m+1\}$. Since $U_1 J_2 = 0$ and $J_2 \supseteq J_3 \supseteq \cdots \supseteq J_{m+1}$, we have $U_1 b_k = 0$ for all $k \in \{2, \ldots, m+1\}$. Thus, using Lemma 22, we see that $U_1[r, b_2, \ldots, b_{m+1}]^* = U_1 r b_2 \cdots b_{m+1}$. But R is Lie nilpotent of index m, so $[r, b_2, \ldots, b_{m+1}]^* = 0$, whence $U_1 r b_2 \cdots b_{m+1} = 0$. Since r is arbitrary, we get

$$\begin{aligned} 0 &= (U_1 R) b_2 \cdots b_{m+1} \\ &= V b_2 \cdots b_{m+1} \quad \text{[because } U_1 R = V \text{ by Lemma 14(b)]}, \end{aligned}$$

from which we infer that $b_2 \cdots b_{m+1} = 0$ since V is faithful. It follows that $J_2 \cdots J_{m+1} = 0$, so $\ell \leqslant m+1$ by definition of ℓ. □

Theorem 24. *For all positive integers m and n and fields F, if R is any F-subalgebra of $\mathbb{U}_n^*(F)$ with Lie nilpotence index m, then*

$$\dim_F R \leqslant M(m+1, n).$$

Proof. Let m and n be arbitrary positive integers, and F an arbitrary field. Let R be an F-subalgebra of $\mathbb{U}_n^*(F)$ with Lie nilpotence index m. If sequences $\{R_k\}_{k\in\mathbb{N}}$, $\{J_k\}_{k\in\mathbb{N}}$ and $\{U_k\}_{k\in\mathbb{N}}$ are defined as in (13) and positive integer ℓ is defined as in (17), then it follows from Theorem 17(c) that

$$\dim_F R \leqslant M(\ell, \dim_F V).$$

Choose V to be F^n, so that $\dim_F V = n$ (see Remark 19). By Proposition 23, $\ell \leqslant m+1$. Since $M(\ell, n)$ is increasing in its first argument by Proposition 29, we have

$$\dim_F R \leqslant M(\ell, \dim_F V) = M(\ell, n) \leqslant M(m+1, n).$$

□

Remark 25. Let R be any F-subalgebra of $\mathbb{U}_n^*(F)$ satisfying the polynomial identity

$$f(x_1, x_2, \ldots, x_m) = \sum_{\sigma \in S_m} c_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)} = 0,$$

where $c_\sigma \in F$ for all $\sigma \in S_m$ and $\{\sigma \in S_m : c_\sigma \neq 0$ and $\sigma(1) = 1\}$ is a singleton comprising the identity permutation.

Arguments similar to those used earlier in this section show that

$$\dim_F R \leqslant M(m+1, n).$$

We are finally in a position to complete the proof of the Conjecture.

Proof of Conjecture. Let F be any field with algebraic closure K. Taking the field F of Theorems 12 and 24 to be K, we see that the latter is just statement (c) of the former. It thus follows from Theorem 12 (c)⇒(a) that the Conjecture holds in respect of field K.

Taking the class of fields $\mathcal{C}$ in Theorem 5 to be the singleton $\mathcal{C} = \{K\}$ and noting that F is a subfield of K, we conclude that the Conjecture holds in respect of field F. Since F was chosen arbitrarily, the proof is complete. □

7. The function $M(\ell, n)$

The purpose of this section is twofold: First, to establish a number of important properties of the function $M(\ell, n)$ that are required in earlier theory, and second to obtain an explicit description of $M(\ell, n)$; without which, the important results of this paper remain somewhat opaque. This task will involve the solution of an integer-variable optimization problem. Our methods, however, are first principled and require no background knowledge of integer optimization techniques.

We shall make use of the following notation: if $\mathbf{k} = (k_1, k_2, \dots, k_\ell) \in \mathbb{N}_0^\ell$, then:

- ▷ $\operatorname{supp} \mathbf{k} \stackrel{\text{def}}{=} \{i \in \{1, 2, \dots, \ell\} : k_i > 0\}$ and
- ▷ $|\mathbf{k}| \stackrel{\text{def}}{=} \left(\sum_{i=1}^{\ell} k_i^2\right)^{1/2}$ so that $|\mathbf{k}|^2 = \sum_{i=1}^{\ell} k_i^2$.

Proposition 26. *Let ℓ and n be positive integers. The following statements are equivalent for $\mathbf{k} = (k_1, k_2, \dots, k_\ell) \in \mathbb{N}_0^\ell$ such that $\sum_{i=1}^{\ell} k_i = n$:*

(a) $M(\ell, n) = \frac{1}{2}\left(n^2 - |\mathbf{k}|^2\right) + 1$;

(b) $|k_i - k_j| \leqslant 1$ *for all* $i, j \in \{1, 2, \dots, \ell\}$.

Proof. (a)⇒(b): Suppose (a) holds, but $|k_p - k_q| \geqslant 2$ for some $p, q \in \{1, 2, \dots, \ell\}$. Without loss of generality, we may suppose that $k_p \geqslant k_q + 2$. Define $\mathbf{k}' = (k_1', k_2', \dots, k_\ell') \in \mathbb{N}_0^\ell$ by

$$k_i' \stackrel{\text{def}}{=} \begin{cases} k_i \text{ if } i \notin \{p, q\}, \\ k_p - 1 \text{ if } i = p, \\ k_q + 1 \text{ if } i = q. \end{cases}$$

Note that $\sum_{i=1}^{\ell} k_i' = \sum_{i=1}^{\ell} k_i = n$. Then

$$\begin{aligned} &\tfrac{1}{2}\left(n^2 - |\mathbf{k}'|^2\right) + 1 - M(\ell, n) \\ &= \tfrac{1}{2}\left(n^2 - |\mathbf{k}'|^2\right) + 1 - \left(\tfrac{1}{2}\left(n^2 - |\mathbf{k}|^2\right) + 1\right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{\ell}\left(k_i^2 - (k_i')^2\right)\right) \\ &= \tfrac{1}{2}\left(k_p^2 + k_q^2 - (k_p')^2 - (k_q')^2\right) \\ &= \tfrac{1}{2}\left(k_p^2 + k_q^2 - (k_p - 1)^2 - (k_q + 1)^2\right) \\ &= \tfrac{1}{2}\left(2k_p - 2k_q - 2\right) \\ &= k_p - k_q - 1 > 0 \quad [\text{because } k_p \geqslant k_q + 2]. \end{aligned}$$

This implies that $\frac{1}{2}\left(n^2 - |\mathbf{k}'|^2\right) + 1 > M(\ell, n)$, a contradiction.

(b)⇒(a): Suppose $\mathbf{k} = (k_1, k_2, \ldots, k_\ell) \in \mathbb{N}_0^\ell$ is such that $\sum_{i=1}^{\ell} k_i = n$ and $|k_i - k_j| \leqslant 1$ for all $i, j \in \{1, 2, \ldots, \ell\}$. Inasmuch as each k_i is nonnegative this implies the existence of some $r \in \mathbb{N}$ such that

$$k_i \in \{r-1, r\} \quad \forall i \in \{1, 2, \ldots, \ell\}. \tag{27}$$

Now suppose that $M(\ell, n) = \frac{1}{2}\left(n^2 - |\mathbf{k}'|^2\right) + 1$ with $\mathbf{k}' = (k'_1, k'_2, \ldots, k'_\ell) \in \mathbb{N}_0^\ell$ such that $\sum_{i=1}^{\ell} k'_i = n$. It follows from implication (a)⇒(b) that $|k'_i - k'_j| \leqslant 1$ for all $i, j \in \{1, 2, \ldots, \ell\}$, so there must exist some $s \in \mathbb{N}$ such that

$$k'_i \in \{s-1, s\} \quad \forall i \in \{1, 2, \ldots, \ell\}. \tag{28}$$

If $r < s$, then it follows from (27) and (28) that

$$k_i \leqslant r \leqslant s - 1 \leqslant k'_i \quad \forall i \in \{1, 2, \ldots, \ell\}.$$

Since $\sum_{i=1}^{\ell} k_i = \sum_{i=1}^{\ell} k'_i$, the above inequalities can only be satisfied if $k_i = k'_i$ for all $i \in \{1, 2, \ldots, \ell\}$, whence $\mathbf{k} = \mathbf{k}'$.

A similar argument shows that $\mathbf{k} = \mathbf{k}'$ whenever $r > s$. Thus if $r \neq s$, then $\mathbf{k} = \mathbf{k}'$, whence $\frac{1}{2}\left(n^2 - |\mathbf{k}|^2\right) + 1 = \frac{1}{2}\left(n^2 - |\mathbf{k}'|^2\right) + 1 = M(\ell, n)$, and the proof is complete.

Now suppose that $r = s$. Since $k_i, k'_i \in \{r, r-1\}$ for each $i \in \{1, 2, \ldots, \ell\}$ and since $\sum_{i=1}^{\ell} k_i = \sum_{i=1}^{\ell} k'_i$, it is easily seen that $\mathbf{k}$ and $\mathbf{k}'$ are equal to within permutation of their coordinates; that is to say, there exists a permutation $\sigma \in S_\ell$ such that $k'_i = k_{\sigma(i)}$ for all $i \in \{1, 2, \ldots, \ell\}$. Clearly, in such a situation $|\mathbf{k}| = |\mathbf{k}'|$ and $\frac{1}{2}\left(n^2 - |\mathbf{k}|^2\right) + 1 = \frac{1}{2}\left(n^2 - |\mathbf{k}'|^2\right) + 1 = M(\ell, n)$. □

Corollary 27. *Let ℓ and n be positive integers. Then:*

(a) *If $\ell \leqslant n$, then $M(\ell, n) = \frac{1}{2}\left(n^2 - |\mathbf{k}|^2\right) + 1$ for some $\mathbf{k} = (k_1, k_2, \ldots, k_\ell) \in \mathbb{N}_0^\ell$ with $k_i \geqslant 1$ for all $i \in \{1, 2, \ldots, \ell\}$. In particular, $M(n, n) = \frac{1}{2}\left(n^2 - n\right) + 1$.*
(b) *If $\ell > n$, then $M(\ell, n) = M(n, n) = \frac{1}{2}\left(n^2 - n\right) + 1$.*

Proof. By Proposition 26, we can choose $\mathbf{k} = (k_1, k_2, \ldots, k_\ell) \in \mathbb{N}_0^\ell$ such that $\sum_{i=1}^{\ell} k_i = n$, $M(\ell, n) = \frac{1}{2}\left(n^2 - |\mathbf{k}|^2\right) + 1$, and $|k_i - k_j| \leqslant 1$ for all $i, j \in \{1, 2, \ldots, \ell\}$.

(a) Suppose $\ell \leqslant n$. If $k_j = 0$ for some $j \in \{1, 2, \ldots, \ell\}$, then $k_i \in \{0, 1\}$ for all $i \in \{1, 2, \ldots, \ell\}$, whence $n = \sum_{i=1}^{\ell} k_i < \ell \leqslant n$, a contradiction.

If $\ell = n$, then clearly $k_i = 1$ for all $i \in \{1, 2, \ldots, \ell\}$, so $|\mathbf{k}|^2 = n$ and $M(\ell, n) = M(n, n) = \frac{1}{2}\left(n^2 - n\right) + 1$.

(b) Suppose $\ell > n$. Since $n = \sum_{i=1}^{\ell} k_i$, we must have $k_j = 0$ for some $j \in \{1, 2, \ldots, \ell\}$. Thus $k_i \in \{0, 1\}$ for all $i \in \{1, 2, \ldots, \ell\}$, so $|\mathbf{k}|^2 = n$ and $M(\ell, n) = M(n, n)$. □

Proposition 28. *Let ℓ be an integer satisfying $\ell \geqslant 2$ and $n_1, n_2, \ldots, n_k$ any sequence of positive integers. Then*

$$M\left(\ell, \sum_{i=1}^{k} n_i\right) \geqslant \sum_{i=1}^{k} M(\ell, n_i).$$

Proof. We provide a proof in the case $k = 2$; the arguments used can be applied mutatis-mutandis to establish the inductive step in a proof by induction on k. Choose $\mathbf{k} = (k_1, k_2, \ldots, k_\ell) \in \mathbb{N}_0^\ell$ such that

$$\sum_{i=1}^{\ell} k_i = n_1 \tag{29}$$

and

$$M(\ell, n_1) = \tfrac{1}{2}\left(n_1^2 - |\mathbf{k}|^2\right) + 1, \tag{30}$$

and choose $\overline{\mathbf{k}} = (\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_\ell) \in \mathbb{N}_0^\ell$ such that

$$\sum_{i=1}^{\ell} \bar{k}_i = n_2 \tag{31}$$

and

$$M(\ell, n_2) = \tfrac{1}{2}\left(n_2^2 - |\overline{\mathbf{k}}|^2\right) + 1. \tag{32}$$

If $|\operatorname{supp} \mathbf{k}| = |\operatorname{supp} \overline{\mathbf{k}}| = 1$, then it follows from (29) that $M(\ell, n_1) = \frac{1}{2}\left(n_1^2 - n_1^2\right) + 1 = 1$ and from (31) that $M(\ell, n_2) = \frac{1}{2}\left(n_2^2 - n_2^2\right) + 1 = 1$. Since ℓ, $n_1 + n_2 \geqslant 2$, it is clear that we can choose $\mathbf{k}^* = (k_1^*, k_2^*, \ldots, k_\ell^*) \in \mathbb{N}_0^\ell$ such that $|\operatorname{supp} \mathbf{k}^*| \geqslant 2$ and $\sum_{i=1}^{\ell} k_i^* = n_1 + n_2$. Then

$$\begin{aligned}
M(\ell, n_1 + n_2) &\geqslant \tfrac{1}{2}\left((n_1 + n_2)^2 - |\mathbf{k}^*|^2\right) + 1 \\
&= \frac{1}{2}\left(\left(\sum_{i=1}^{\ell} k_i^*\right)^2 - \sum_{i=1}^{\ell} (k_i^*)^2\right) + 1 \\
&= \sum_{i,j=1,\, i<j}^{\ell} k_i^* k_j^* + 1 \\
&\geqslant 2 = M(\ell, n_1) + M(\ell, n_2),
\end{aligned}$$

as required.

Now suppose $|\operatorname{supp} \mathbf{k}| \geqslant 2$ or $|\operatorname{supp} \overline{\mathbf{k}}| \geqslant 2$.

Put $\overline{\overline{\mathbf{k}}} = (\bar{\bar{k}}_1, \bar{\bar{k}}_2, \ldots, \bar{\bar{k}}_\ell) = \mathbf{k} + \overline{\mathbf{k}}$. By (29) and (31)

$$\sum_{i=1}^{\ell} \bar{\bar{k}}_i = n_1 + n_2. \tag{33}$$

Then

$$
\begin{aligned}
M(\ell, n_1+n_2) &\geqslant \frac{1}{2}\left((n_1+n_2)^2 - |\bar{\bar{\mathbf{k}}}|^2\right) + 1 \quad \text{[by (33) and the definition of } M(\ell, n_1+n_2)\text{]} \\
&= \tfrac{1}{2}\left(n_1^2 + n_2^2 - |\mathbf{k}|^2 - |\bar{\mathbf{k}}|^2\right) + n_1 n_2 - \sum_{i=1}^{\ell} k_i \bar{k}_i + 1 \\
&= M(\ell, n_1) + M(\ell, n_2) + n_1 n_2 - \sum_{i=1}^{\ell} k_i \bar{k}_i - 1 \quad \text{[by (30), (32)]} \\
&= M(\ell, n_1) + M(\ell, n_2) + \left(\sum_{i=1}^{\ell} k_i\right)\left(\sum_{i=1}^{\ell} \bar{k}_i\right) - \sum_{i=1}^{\ell} k_i \bar{k}_i - 1 \quad \text{[by (29), (31)]} \\
&= M(\ell, n_1) + M(\ell, n_2) + \sum_{i,j=1,\, i\neq j}^{\ell} k_i \bar{k}_j - 1. \qquad (34)
\end{aligned}
$$

Since, by hypothesis, $|\operatorname{supp} \mathbf{k}| \geqslant 2$ or $|\operatorname{supp} \bar{\mathbf{k}}| \geqslant 2$, we must have $\sum_{i,j=1,\, i\neq j}^{\ell} k_i \bar{k}_j \geqslant 1$. Hence by (34), $M(\ell, n_1+n_2) \geqslant M(\ell, n_1) + M(\ell, n_2)$, as required. □

Proposition 29. *The function $M(\ell, n)$ is increasing in both its arguments.*

Proof. That $M(\ell, n)$ is increasing in its second argument is an immediate consequence of Proposition 28.

To show that $M(\ell, n)$ is increasing in its first argument, it suffices to show that $M(\ell, n) \leqslant M(\ell+1, n)$. Choose $\mathbf{k} = (k_1, k_2, \ldots, k_\ell) \in \mathbb{N}_0^\ell$ such that $\sum_{i=1}^{\ell} k_i = n$ and $M(\ell, n) = \frac{1}{2}\left(n^2 - |\mathbf{k}|^2\right) + 1$. Putting $\mathbf{k}' = (k_1, k_2, \ldots, k_\ell, 0) \in \mathbb{N}_0^{\ell+1}$, we see that $M(\ell, n) = \frac{1}{2}\left(n^2 - |\mathbf{k}|^2\right) + 1 = \frac{1}{2}\left(n^2 - |\mathbf{k}'|^2\right) + 1 \leqslant M(\ell+1, n)$, as required. □

We attempt now an explicit description of the function $M(\ell, n)$. This is achieved in Theorem 31. If ℓ and n are positive integers with $\ell \geqslant n$, then Corollary 27 exhibits the simple formula $M(\ell, n) = \frac{1}{2}\left(n^2 - n\right) + 1$. We shall therefore restrict our attention to the case $\ell \leqslant n$. For such integers ℓ and n we denote by $n \,(\mathrm{mod}\ \ell)$ the nonnegative remainder on dividing n by ℓ, that is, the unique integer $r < \ell$ that satisfies

$$n = \left\lfloor \frac{n}{\ell} \right\rfloor \ell + r.$$

Let $r = n \,(\mathrm{mod}\ \ell)$ and define $\mathbf{d} = (d_1, d_2, \ldots, d_\ell) \in \mathbb{N}_0^\ell$ by

$$d_i \stackrel{\text{def}}{=} \begin{cases} \left\lfloor \frac{n}{\ell} \right\rfloor & \text{for } 1 \leqslant i \leqslant \ell - r, \\ \left\lfloor \frac{n}{\ell} \right\rfloor + 1 & \text{for } \ell - r < i \leqslant \ell. \end{cases} \qquad (35)$$

We omit the proof of the following routine lemma.

Lemma 30. *Let ℓ and n be positive integers with $\ell \leqslant n$ and $r = n \,(\mathrm{mod}\ \ell)$. If $\mathbf{d}$ is defined as in (35), then*

$$|\mathbf{d}|^2 = (\ell - r)\left\lfloor \frac{n}{\ell} \right\rfloor^2 + r\left(\left\lfloor \frac{n}{\ell} \right\rfloor + 1\right)^2 = \frac{n^2 - r^2}{\ell} + r.$$

Theorem 31. *Let ℓ and n be positive integers with $\ell \leqslant n$ and $r = n \,(\mathrm{mod}\ \ell)$. If* $\mathbf{d}$ *is defined as in* (35)*, then*

$$\begin{aligned} M(\ell,n) &= \tfrac{1}{2}\left(n^2 - |\mathbf{d}|^2\right) + 1 \\ &= \frac{1}{2}\left(n^2 - (\ell - r)\left\lfloor \frac{n}{\ell} \right\rfloor^2 - r\left(\left\lfloor \frac{n}{\ell} \right\rfloor + 1\right)^2\right) + 1 \\ &= \frac{n^2(\ell-1)}{2\ell} + \frac{1}{2}\left(\frac{r^2}{\ell} - r\right) + 1. \end{aligned}$$

Proof. It is clear from the definition of $\mathbf{d}$ in (35) that $\sum_{i=1}^{\ell} d_i = n$ and $|d_i - d_j| \leqslant 1$ for all $i, j \in \{1, 2, \ldots, \ell\}$. Hence, by Proposition 26 (b)$\Rightarrow$(a),

$$\begin{aligned} M(\ell,n) &= \tfrac{1}{2}\left(n^2 - |\mathbf{d}|^2\right) + 1 \\ &= \frac{1}{2}\left(n^2 - (\ell - r)\left\lfloor \frac{n}{\ell} \right\rfloor^2 - r\left(\left\lfloor \frac{n}{\ell} \right\rfloor + 1\right)^2\right) + 1 \quad \text{[by Lemma 30]} \\ &= \frac{1}{2}\left(n^2 - \left(\frac{n^2 - r^2}{\ell} + r\right)\right) + 1 \quad \text{[by Lemma 30]} \\ &= \frac{1}{2}\left(n^2 - \frac{n^2}{\ell} + \frac{r^2}{\ell} - r\right) + 1 \\ &= \frac{n^2(\ell-1)}{2\ell} + \frac{1}{2}\left(\frac{r^2}{\ell} - r\right) + 1. \end{aligned}$$

$\square$

Suppose F is any field and R is the algebra of $n \times n$ matrices over F of type

$$\mathbf{d} = (\overbrace{\left\lfloor \tfrac{n}{\ell} \right\rfloor, \left\lfloor \tfrac{n}{\ell} \right\rfloor, \ldots, \left\lfloor \tfrac{n}{\ell} \right\rfloor}^{(\ell - r)\text{ times}}, \overbrace{\left\lfloor \tfrac{n}{\ell} \right\rfloor + 1, \left\lfloor \tfrac{n}{\ell} \right\rfloor + 1, \ldots, \left\lfloor \tfrac{n}{\ell} \right\rfloor + 1}^{r\text{ times}}),$$

with $n \geqslant \ell \geqslant 2$.

Figure 3 is a pictorial representation of the radical J of R. Inasmuch as R has the form $R = FI_n + J$ with J satisfying $J^\ell = 0$, it follows that R is Lie nilpotent of index $\ell - 1$. (This assertion is explained in more detail in the discussion following the statement of the Conjecture (7).) Moreover,

$$\begin{aligned} \dim_F R &= \sum_{j=1}^{\ell-1} d_j \left(n - \sum_{i=1}^{j} d_i\right) + 1 \\ &= \tfrac{1}{2}\left(n^2 - |\mathbf{d}|^2\right) + 1 \\ &= M(\ell, n) \quad \text{[by Theorem 31]}. \end{aligned}$$

Thus R is an F-subalgebra of $\mathbb{M}_n(F)$ whose dimension is maximal amongst F-subalgebras of $\mathbb{M}_n(F)$ with Lie nilpotence index $\ell - 1$.

If $\frac{1}{2}\left(n^2 - |\mathbf{k}|^2\right) + 1$ is interpreted as a real-valued function of real variables $\mathbf{k} = (k_1, k_2, \ldots, k_\ell) \in \mathbb{R}^\ell$, the methods of multivariable calculus show that the function $\frac{1}{2}\left(n^2 - |\mathbf{k}|^2\right) + 1$, subject to the constraint $\sum_{i=1}^{\ell} k_i = n$, attains a maximum of

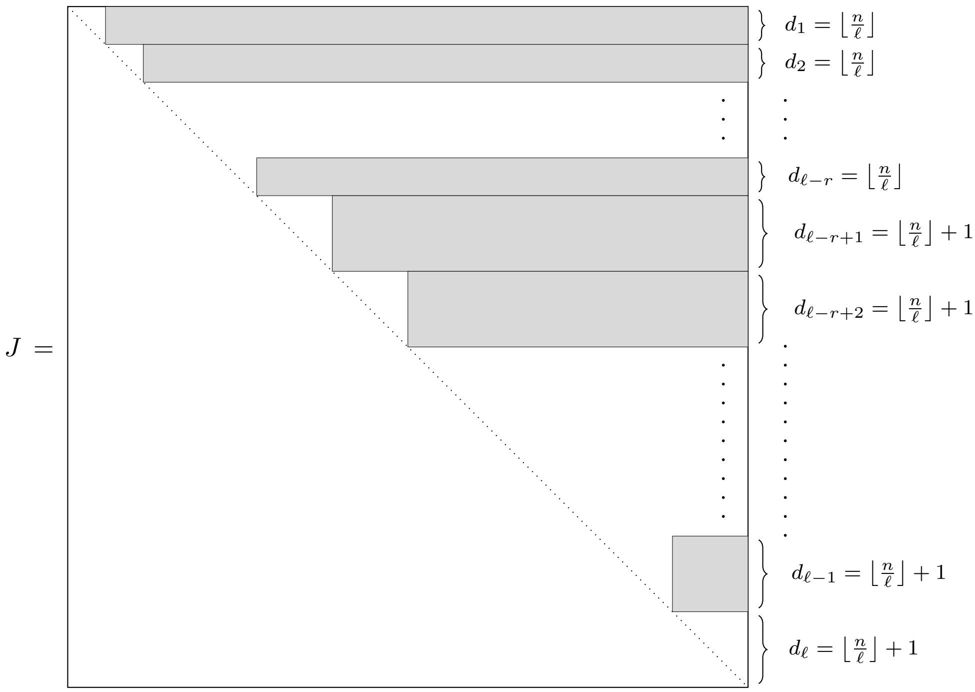


FIGURE 3. Pictorial representation of the radical J of R

$\frac{n^2(\ell-1)}{2\ell} + 1$ at $\mathbf{k} = (\frac{n}{\ell}, \frac{n}{\ell}, \dots, \frac{n}{\ell}) \in \mathbb{R}^\ell$. Thus

$$\left\lfloor \frac{n^2(\ell-1)}{2\ell} \right\rfloor + 1 \geq M(\ell, n). \tag{36}$$

We explore now instances in which (36) is an equation, a situation that arises precisely when $D < 1$, where

$$D \overset{\text{def}}{=} \frac{n^2(\ell-1)}{2\ell} + 1 - M(\ell, n).$$

It follows from Theorem 31 that

$$D = \frac{1}{2}\left(r - \frac{r^2}{\ell}\right), \tag{37}$$

where $r = n \pmod{\ell}$. Observe that $D = D(r, \ell)$ is a function only of r and ℓ.

Figure 4 is a sketch of the level curve $D(r, \ell) = 1$ in the $r\ell$-plane, interpreting r and ℓ as real-valued variables. A simple calculation shows that the curve has equation

$$\ell = \frac{r^2}{r-2}.$$

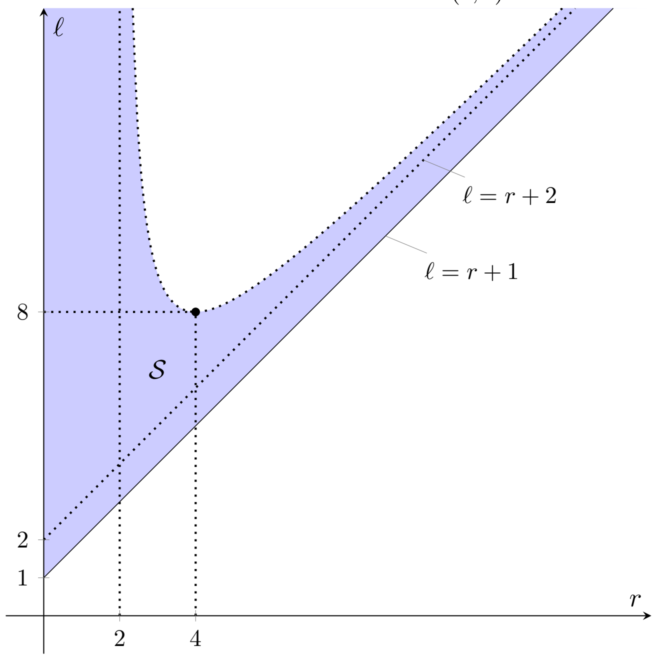
Its essential features are obtained using elementary calculus.

The shaded region is

$$\mathcal{S} \overset{\text{def}}{=} \{(r, \ell) \in \mathbb{R}^2 : 0 \leqslant r \leqslant \ell - 1 \quad \text{and} \quad D(r, \ell) < 1\}.$$

The content of Theorem 32 below is easily gleaned from Figure 4 by assembling together points (r, ℓ) belonging to $\mathcal{S}$ that have integral coordinates.

FIGURE 4. The level curve $D(r,\ell) = 1$



Theorem 32. *Let ℓ and n be positive integers with $\ell \leqslant n$ and $r = n \,(\mathrm{mod}\ \ell)$. Then the following statements are equivalent:*

(a) $M(\ell, n) = \left\lfloor \dfrac{n^2(\ell-1)}{2\ell} \right\rfloor + 1$;

(b) (r,ℓ) *belongs to one of the following (disjoint) sets:*

(i) $\{(r,\ell) : 0 \leqslant r \leqslant \ell - 1 \quad \textit{and} \quad 1 \leqslant \ell \leqslant 7\}$;

(ii) $\{(r,\ell) : 0 \leqslant r \leqslant 2 \quad \textit{and} \quad \ell \geqslant 8\}$;

(iii) $\{(r, r+1) : r \geqslant 7\} \cup \{(r, r+2) : r \geqslant 7\}$;

(iv) $\{(3,8), (5,8)\}$.

Remark 33. The reader will observe with reference to Theorem 32(b)(i) that if, amongst others, $1 \leqslant \ell \leqslant 7$, we have the simplified formula

$$M(\ell, n) = \left\lfloor \frac{n^2(\ell-1)}{2\ell} \right\rfloor + 1.$$

In particular, if $\ell = 2$, then

$$M(2, n) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1,$$

which corresponds with the upper bound in Schur's classical result.

8. An illustrative example

The main body of theory developed in Section 5 is based on the triple of sequences $\{R_k\}_{k\in\mathbb{N}}$, $\{J_k\}_{k\in\mathbb{N}}$ and $\{U_k\}_{k\in\mathbb{N}}$ defined in (13). In this section we show that the terms in these sequences are easily visualized in the case where R is the algebra of

$n \times n$ matrices over field F of type $(d_1, d_2, \ldots, d_\ell)$. Indeed, this special case provides the germ for our proof strategy.

Let F be any field and $(d_1, d_2, \ldots, d_\ell)$ any sequence of positive integers satisfying $\sum_{i=1}^{\ell} d_i = n$ with $\ell \geqslant 2$. Let R be the algebra of $n \times n$ matrices over F of type $(d_1, d_2, \ldots, d_\ell)$. We saw in Section 1 that the radical J of R has pictorial representation as shown in Figure 5.

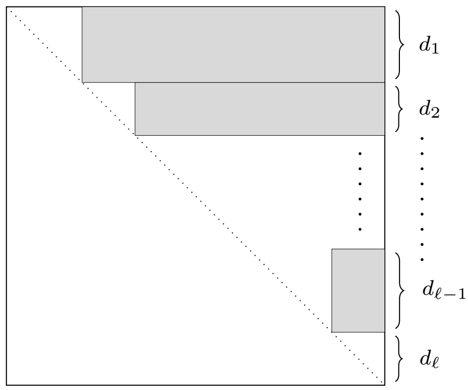
$J_1 = J =$ 

FIGURE 5

Observe that $\dim_F J_1$ corresponds with the sum of the dimensions (to be visualized as areas) of each of the $\ell - 1$ blocks that make up J_1. With this perspective we see that

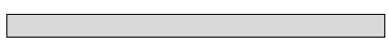
$$\dim_F J_1 = \overbrace{d_1(n-d_1)}^{\text{1st block}} + \overbrace{d_2(n-d_1-d_2)}^{\text{2nd block}} + \cdots + \overbrace{d_{\ell-1}(n-d_1-\cdots-d_{\ell-1})}^{(\ell-1)\text{th block}}.$$

Note also that

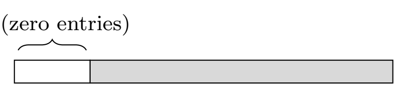
$$J_1^\ell = 0,$$

from which we infer that R_1 is Lie nilpotent of index $\ell - 1$. (This inference is explained in the discussion following the statement of the Conjecture (7).)

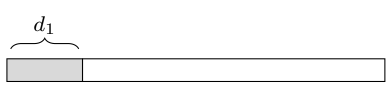
Take $V = F^n$, which in this context is to be visualized as a $1 \times n$ block thus:

$V =$ 

We infer from the pictorial representations of V and J_1 (Figure 5), that

$VJ_1 =$ 

Choosing

$U_1 =$ 

yields the representation shown in Figure 6.

Here:

- ▷ $\dim_F J_2 = \overbrace{d_2(n-d_1-d_2)}^{\text{2nd block}} + \cdots + \overbrace{d_{\ell-1}(n-d_1-\cdots-d_{\ell-1})}^{(\ell-1)\text{th block}}$;
- ▷ $J_2^{\ell-1} = 0$;

$J_2 = (0 :^{R_1} U_1) =$

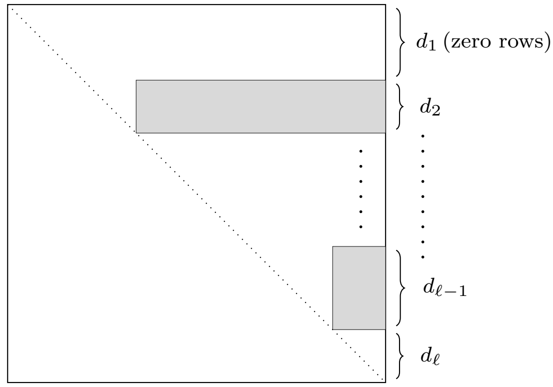
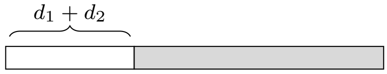
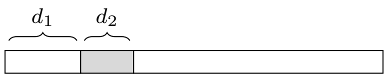


FIGURE 6

- R_2 is Lie nilpotent of index $\ell - 2$;
- $VJ_1J_2 =$  ;

- $U_2 =$  .

Continuing in this manner, we arrive at a smallest F-subalgebra of R properly containing FI_n, namely $R_{\ell-1}$, and this has radical comprising a single block:

$J_{\ell-1} =$

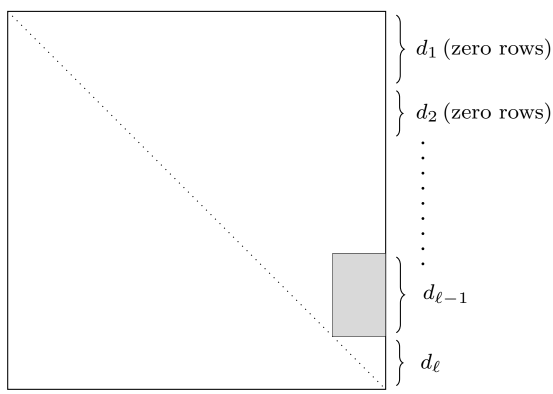
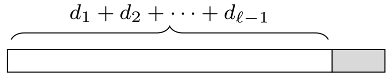
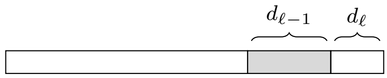


FIGURE 7

Here:

- $\dim_F J_{\ell-1} = \overbrace{d_{\ell-1}(n - d_1 - \cdots - d_{\ell-1})}^{(\ell-1)\text{th block}}$;
- $J_{\ell-1}^2 = 0$;

- ▷ $R_{\ell-1}$ is Lie nilpotent of index 1 and is thus commutative;
- ▷ $VJ_1J_2\cdots J_{\ell-1} =$  ;

- ▷ $U_{\ell-1} =$  .

9. Open questions

(1) The sequence $\{U_k\}_{k\in\mathbb{N}}$ of F-subspace complements defined in (13) is not unique. This has the consequence that the sequence $\{R_k\}_{k\in\mathbb{N}}$ of F-sub-algebras of R is not uniquely determined by R. Are the R_k unique to within isomorphism perhaps? Or failing this, are the dimensions (over F) of the R_k unique?

(2) Recall that if $\mathfrak{g}$ is a Lie algebra, then the *Derived Series* $\{\mathfrak{g}^{[m]}\}_{m\in\mathbb{N}}$ for $\mathfrak{g}$ is defined recursively as follows:

$$\mathfrak{g}^{[1]} \stackrel{\text{def}}{=} \mathfrak{g}, \quad \text{and}$$

$$\mathfrak{g}^{[m]} \stackrel{\text{def}}{=} [\mathfrak{g}^{[m-1]}, \mathfrak{g}^{[m-1]}] \quad \text{for } m > 1.$$

We say $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{[m]} = 0$ for some $m \in \mathbb{N}$, $m > 1$, and more specifically, *solvable of index m* if $\mathfrak{g}^{[m+1]} = 0$. We call a ring R *Lie solvable* (resp. *Lie solvable of index m*) if R, considered as a Lie algebra via the commutator, is solvable (resp. solvable of index m). If $\{\mathfrak{g}_{[m]}\}_{m\in\mathbb{N}}$ denotes the Lower Central Series for $\mathfrak{g}$, it is easily seen that $\mathfrak{g}^{[m]} \subseteq \mathfrak{g}_{[m]}$ for all $m \in \mathbb{N}$, from which it follows that every ring R that is Lie nilpotent of index m is also Lie solvable of index m. This being so, it is natural to ask whether the main theorems of this paper remain valid if the condition "Lie nilpotent of index m" is substituted with the weaker "Lie solvable of index m".

(3) Expressed in terms that make no explicit reference to the overlying matrix ring, a key result in this paper asserts that if R is an F-algebra with Lie nilpotence index m and V is any faithful right R-module, then $\dim_F R \leqslant M(m+1, \dim_F V)$. (This is Theorem 24 with $\dim_F V$ in place of n.) We ask whether the same inequality holds if the requirement that R be a finite dimensional F-algebra is weakened to R being merely a (two-sided) artinian ring. In such a situation, "R-module length" takes the place of "F-dimension", thus yielding the conjecture

> *If R is a (two-sided) artinian ring with Lie nilpotence index m and V is any faithful right R-module with finite composition length, then*
>
> $$\text{length } R_R \leqslant M(m+1, \text{length } V_R).$$

In the case where $m = 1$, the above reduces to the question [7, Section 5, Open problem (a), p. 562] that is answered in [2].[2]

Acknowledgment

The authors thank K. C. Smith for fruitful discussions on the topic of this paper.

[2]Although the ring R is assumed to be local in Gustafson's formulation, Cowsik's proof does not assume localness.

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Institute of Mathematics, University of Miskolc, 3515 Miskolc-Egyetemváros, Hungary

Email address: matjeno@uni-miskolc.hu

Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20, Hatfield, Pretoria 0028, South Africa

Email address: john.vandenberg@up.ac.za

Department of Mathematical Sciences, Stellenbosch University, Private Bag X1, Matieland 7602, Stellenbosch, South Africa

Email address: lvw@sun.ac.za

Faculty of Mathematics and Information Science, Technical University of Warsaw, 00-661 Warsaw, Poland

Email address: m.ziembowski@mini.pw.edu.pl