Particle Swarm Optimization: Understanding Order-2 Stability Guarantees

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Abstract. This paper's primary aim is to provide clarity on which guarantees about particle stability can actually be made. The particle swarm optimization algorithm has undergone a considerable amount of theoretical analysis. However, with this abundance of theory has come some terminological inconstancies, and as a result it is easy for a practitioner to be misguided by overloaded terminology. Specifically, the criteria for both order-1 and order-2 stability are well studied, but the exact definition of order-2 stability is not consistent amongst researchers. A consequence of this inconsistency in terminology is that the existing theory may in fact misguide practitioners instead of assisting them. In this paper it is theoretically and empirically demonstrated which practical guarantees can in fact be made about particle stability. Specifically, it is shown that the definition of order-2 stability which accurately reflects PSO behavior is that of convergence in second order moment to a constant, and not to zero.

Keywords: Particle Swarm Optimization · Stability Analysis · Stability Criteria

1 Introduction

Particle swarm optimization (PSO), originally developed by Kennedy and Eberhart [1], has become a widely used optimization technique [2]. Given PSO's success, a substantial amount of theoretical work has been performed on the stochastic search algorithm to try and predict and understand its underlying behavior [3–9].

One aspect of PSO that has attracted a great deal of theoretical research is that of particle convergence. Given the complexity of PSO's internal dynamics, much of the early research was performed under a number of modeling assumptions. The original works on particle convergence [3, 4, 10, 11] considered a deterministic model where the stochastic terms in PSO's update equations were replaced with constants. In a deterministic context the concept of particle convergence in $\mathbb{R}^n$ is well defined. Newer research has focused on modeling PSO's dynamics with the stochasticity still present. In a stochastic context simply stating a sequence is convergent is ambiguous. This ambiguity arises from the fact that there exist numerous types of stochastic convergence, ranging from convergence in a *nth* order moment, convergence in probability, to almost sure convergence, to name a few.

In current theoretical PSO research, two different definitions for order-2 stability of particle positions have arisen. The first definition is in line with the concept of convergence of a second order moment to a constant, whereas the second definition makes the stronger requirement that the second order moment of the sequence must not only converge, but rather that is must converge to zero. This overloading is potentially problematic as existing stability research provides criteria on the control parameters of PSO that will ensure order-2 stability. However, in practice, which type of order-2 stability can be actually expected may be unclear.

The concept of particle stability is vital for effective use of PSO, as it was shown by Cleghorn and Engelbrecht [12] that parameter configurations that resulted in particle instability almost always caused PSO to perform worse than random search. Given the relationship between particle stability and performance it is important to fully understand the practical implications of particle stability on the swarm. The concept of stability is also vital to PSO variants. Many popular PSO variants have also undergone stability analysis [6, 13–17].

In this paper, guided by existing theory, it is empirically demonstrated which practical guarantees can in fact be made about particle stability. Specifically, which definition of order-2 stability accurately reflects these guarantees is demonstrated.

A description of PSO is given in section 2. Section 3 briefly summarizes the existing theoretical results on PSO stability, and shows why convergence to a zero second order moment cannot be guaranteed by the theory. The experimental setup is presented in section 4, followed by the experimental results and a discussion thereof in section 5. Section 6 presents a summary of the findings of this paper.

2 Particle Swarm Optimization

Particle swarm optimization (PSO) was originally inspired by the complex movement of birds in a flock. The variant of PSO this section focuses on uses the inertia coefficient, as proposed by Shi and Eberhart [18], which is referred to as PSO in this paper.

The PSO algorithm is defined as follows: Let $f : \mathbb{R}^d \to \mathbb{R}$ be the objective function that the PSO algorithm aims to find an optimum for, where d is the dimensionality of the objective function. For the sake of simplicity, a minimization problem is assumed from this point onwards. Specifically, an optimum $\boldsymbol{o} \in \mathbb{R}^d$ is defined such that, for all $\boldsymbol{x} \in \mathbb{R}^d$, $f(\boldsymbol{o}) \leq f(\boldsymbol{x})$. In this paper the analysis focus is on objective functions where the optima exist. Let $\Omega(t)$ be a set of N particles in $\mathbb{R}^d$ at a discrete time step t. Then $\Omega(t)$ is said to be the particle swarm at time t. The position $\boldsymbol{x}_i$ of particle i is updated using

$$\boldsymbol{x}_i(t+1) = \boldsymbol{x}_i(t) + \boldsymbol{v}_i(t+1), \tag{1}$$

where the velocity update, $\boldsymbol{v}_i(t+1)$, is defined as

$$\boldsymbol{v}_i(t+1) = w\boldsymbol{v}_i(t) + c_1\boldsymbol{r}_1(t) \otimes (\boldsymbol{y}_i(t) - \boldsymbol{x}_i(t)) + c_2\boldsymbol{r}_2(t) \otimes (\hat{\boldsymbol{y}}_i(t) - \boldsymbol{x}_i(t)), \quad (2)$$

where $r_{1,k}(t), r_{2,k}(t) \sim U(0,1)$ for all t and $1 \leq k \leq d$. The operator $\otimes$ is used to indicate component-wise multiplication of two vectors. The position $\boldsymbol{y}_i(t)$ represents the "best" position that particle i has visited, where "best" means the location where the particle had obtained the lowest objective function evaluation. The position $\hat{\boldsymbol{y}}_i(t)$ represents the "best" position that the particles in the neighborhood of the i-th particle have visited. The coefficients c_1, c_2, and w are the cognitive, social, and inertia weights, respectively. A full algorithm description is presented in algorithm 1.

A primary feature of the PSO algorithm is social interaction, specifically the way in which knowledge about the search space is shared amongst the particles in the swarm. In general, the social topology of a swarm can be viewed as a graph, where nodes represent particles, and the edges are the allowable direct communication routes. The fixed topologies star, ring, and Von Neumann, are frequently used in PSO [19]. A number of dynamic topologies have also been proposed. The interested reader is referred to the work of [20] for an in-depth discussion on dynamic topologies.

Algorithm 1 PSO algorithm

Create and initialize a swarm, $\Omega(0)$, of N particles uniformly within a predefined hypercube of dimension d.
Let f be the objective function.
Let $\boldsymbol{y}_i$ represent the personal best position of particle i, initialized to $\boldsymbol{x}_i(0)$.
Let $\hat{\boldsymbol{y}}_i$ represent the neighborhood best position of particle i, initialized to $\boldsymbol{x}_i(0)$.
Initialize $\boldsymbol{v}_i(0)$ to $\mathbf{0}$.
Let $t = 0$
repeat
    for all particles $i = 1, \cdots, N$ **do**
        if $f(\boldsymbol{x}_i) < f(\boldsymbol{y}_i)$ **then**
            $\boldsymbol{y}_i = \boldsymbol{x}_i$
        end if
        for all particles $\hat{i}$ with particle i in their neighborhood **do**
            if $f(\boldsymbol{y}_i) < f(\hat{\boldsymbol{y}}_{\hat{i}})$ **then**
                $\hat{\boldsymbol{y}}_{\hat{i}} = \boldsymbol{y}_i$
            end if
        end for
    end for
    $t = t + 1$
    for all particles $i = 1, \cdots, N$ **do**
        update the velocity of particle i using equation (2)
        update the position of particle i using equation (1)
    end for
until stopping condition is met

3 Theoretical Results on Particle Stability

This section briefly discusses the state of PSO particle stability analysis. Specific focus is placed on the research pertaining to the second order moment of particles.

The PSO algorithm has undergone a considerable amount of theoretical research. In particular the study of long term particle dynamics has attracted a lot of attention. Despite PSO's simplicity as an optimizer, accurately modeling the algorithm in a tractable fashion has been non trivial, with much of the early research requiring multiple simplifications in the modeling of PSO. The two primary aspects of PSO that have been subject to this simplification are the removal of stochasticity (deterministic assumption) [3, 4, 7, 21] and the fixing of the personal and neighborhood best positions (stagnation assumption) [22–25]. A detailed discussion on when and where each modeling assumption was used in PSO literature can be found in [9].

Recent works on particle stability have catered for the stochasticity of PSO and focused on the weakening of the stagnation assumption [8, 9, 26]. Currently, the weakest assumption used in the study of particle stability is the non-stagnate distribution assumption [9]. Under the non-stagnate distribution assumption, personal and neighborhood best positions are modeled as sequences of random variables that are convergent in first and second order moments to a constant (specifically in expectation and variance).

Research on PSO stability has focused on deriving criteria for PSO's control parameters that will ensure order-1 and order-2 stability of particle positions; where order-1 and order-2 stability of a stochastic sequence are defined as follows:

Definition 1. *Order*-1 *stability*
The sequence $(\boldsymbol{s}_t)$ *in* $\mathbb{R}^d$ *is order*-1 *stable if there exists an* $\boldsymbol{s}_E \in \mathbb{R}^d$ *such that*

$$\lim_{t\to\infty} E[\boldsymbol{s}_t] = \boldsymbol{s}_E \tag{3}$$

where $E[\boldsymbol{s}_t]$ *is the expectation of* $\boldsymbol{s}_t$*.*

Definition 2. *Order*-2 *stability*
The sequence $(\boldsymbol{s}_t)$ *in* $\mathbb{R}^d$ *is order*-2 *stable if there exists a* $\boldsymbol{s}_V \in \mathbb{R}^d$ *such that*

$$\lim_{t\to\infty} V[\boldsymbol{s}_t] = \boldsymbol{s}_V \tag{4}$$

where $V[\boldsymbol{s}_t]$ *is the variance of* $\boldsymbol{s}_t$*.*

Specifically, under varying degrees of model simplification, the authors Poli [24], Liu [8], Bonyadi and Michalewicz [26], and Cleghorn and Engelbrecht [9], have derived the following necessary and sufficient criteria for order-1 and order-2 stability of PSO's particles with update equations (1) and (2):

$$0 < c_1 + c_2 < \frac{24\left(1 - w^2\right)}{7 - 5w} \quad \text{and} \quad |w| < 1. \tag{5}$$

where $c_1 = c_2$.

However, in the works of Liu [8] and Bonyadi and Michalewicz [26] order-2 stability is defined in a more restrictive manner. Specifically, the limit of the second order moment, $\boldsymbol{s}_V$, is required to be zero. At first glance the result of Liu [8] and Bonyadi and Michalewicz [26] appears stronger. However, under closer inspection it becomes clear that the only way to ensure convergence of variance to zero is to impose additional requirements of the personal and neighborhood best positions. Specifically, Poli [27] showed that the component-wise limit point for particle variance is defined as

$$V[x_{ij}] = \frac{c(w+1)}{4c(5w-7) - 48w^2 + 48}(\hat{y}_{ij} - y_{ij})^2, \qquad (6)$$

where $c = c_1 = c_2$. It is clear from equation (6) that, when the criteria of equation (5) are satisfied, the only way for the variance of particle positions to become zero is if $\hat{y}_{ij} = y_{ij}$ for each $1 \leq j \leq d$. This implies that for every particle i in the swarm to be have a zero variance the personal and neighborhood best positions must become the same position during the run. A similar argument for PSOs with arbitrary distributions can be found in [9]. In practice the distance between the personal and neighborhood best positions cannot be guaranteed to approach zero, which will be demonstrated in this paper.

For the sake of clarity in the field, it is proposed that the restrictive version of order-2 stability where $\boldsymbol{s}_V$ must be zero, should be referred to as order-2* stability, with the aim of providing clearer guidance for PSO practitioners. For the rest of this paper this new terminology will be utilized.

It is worth mentioning that convergence to a non-zero variance can be seen in a positive light. If the variance of particle positions collapses to zero, the swarm has ceased searching, as any newly generated particle positions can only be sampled from a fixed point. Whereas, if the positional variance is non-zero, it is still possible for a swarm to be usefully searching, as newly visited particle position are sampled from an area of non-zero measure. Said informally, a non-zero positional variance means that the swarm never gives up.

In order to show that the criteria of equation (5), for order-2 stability, cannot ensure order-2* stability the following proposition is proved

Proposition 1. *There exists no control coefficients, w, c_1, c_2 satisfying equation* (5) *that ensure order-*2* *stability for any objective function f.*

Proof: All that is required is the construction of an f that demonstrates that for any w, c_1, c_2 satisfying equation (5) the particle variance as defined in equation (6) cannot become $\mathbf{0}$ for all particles. From this point on the proof focuses on a particle which does not have its personal best positions equal to the global best position yet. Consider the 1-dimensional objective function $f(x) = x^4 - x^2$, which has two global minimum at $x = \frac{-1}{\sqrt{2}}$ and $x = \frac{1}{\sqrt{2}}$ and is illustrated in figure 1. Let $\hat{y}_i = \frac{-1}{\sqrt{2}}$, $y_i = \frac{1}{\sqrt{2}}$, and Γ_i be the distribution, parameterize by w, c_1, c_2, $v_i(t-1)$, $x_i(t-1)$, $\hat{y}_i$, and y_i, that describes the possible next positions of particle i, namely $x_i(t)$. For the new positions $x_i(t) \sim \Gamma_i$ to be accepted as a new global best position $f(x_i(t))$ would need to be strictly less than $f(\frac{-1}{\sqrt{2}}) = -\frac{1}{4}$, which is not possible since $x = \frac{-1}{\sqrt{2}}$ is a global minimum. As a results $\hat{y}_{ij}$ cannot change. Following the same line of argument it is clear that y_{ij} can also not change, as such $(\hat{y}_{ij} - y_{ij})^2$ will remain constant and equal to $(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}})^2 = 2$. Since $\frac{c(w+1)}{4c(5w-7)-48w^2+48}$ cannot be zero if the criteria of equation (5) are satisfied [27], it follows that order-2* stability cannot occur. □

It is conceivable that while order-2* stability cannot be is theoretical guaranteed for all objective functions, it may occur for non-artificially designed ones like that used in proposition 1. To demonstrate that order-2* stability cannot be guaranteed in practice either it is empirically shown in sections 4 and 5 that order-2* stability cannot be guaranteed by selecting control parameter that satisfy equation (5) for an array of well known objective function.

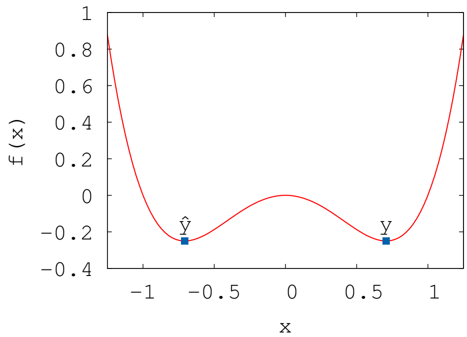


Fig. 1: $f(x) = x^4 - x^2$, with $\hat{y} = \frac{-1}{\sqrt{2}}$ and $y = \frac{1}{\sqrt{2}}$

4 Experimental Setup

This section summarizes the experimental procedure used in this paper.

The experiment utilized a population size of 20 and the gbest topology. The optimization is allowed to run for 5000 iterations, so as to provide a fair indication of long term particle behavior. Particle positions were initialized within $[-100, 100]^d$ and velocities were initialized to $\mathbf{0}$ [28]. Particle personal and neighborhood best positions were only updated with feasible solutions. The analysis is done in 5, 10, 20, and 30 dimensions.

The experiment is conducted over the parameter region corresponding to the criteria of order-1 and order-2 stability as defined in equation (5). A total of 504 stable parameter configurations were tested with spacing of 0.1 between each considered $c_1+c_2 > 0$ and each considered $w > -1$. The experiment is conducted using the 11 base objective functions from the CEC 2014 problem set [29]. The functions are as follows: Ackley (F1), Bent Cigar (F2), Discus (F3), Expanded Griewank plus Rosenbrock (F4), Griewank (F5), HappyCat (F6), HGBat (F7), High Conditioned Elliptic (F8), Katsuura (F9), Rastrigin (F10), and Rosenbrock (F11).

For each considered parameter configuration the following measurement is made:

$$\mathit{\Gamma}\left(t+1\right)=\frac{1}{N}\sum_{i=1}^{N}\|\hat{\boldsymbol{y}}_{i}\left(t+1\right)-\boldsymbol{y}_{i}\left(t\right)\|_{\infty}, \tag{7}$$

where $\|\mathbf{x}\|_{\infty} = \max_{1\leq i\leq N}|x_i|$ is the infinity norm. This norm is selected as it makes direct comparison of results across different dimensionality easy. The choice of the infinity norm is done without loss of generality, since the convergence of a sequence under any arbitrary norm in $\mathbb{R}^n$ implies convergence under all other norms, as all norms are equivalent in a finite dimensional Banach space. [30]. If $\mathit{\Gamma}$ does not go to zero, it implies that each particle cannot be order-2* stable, since equation (6) could not be zero for all particles. Furthermore, it should be noted that if the criteria of equation (5) for order-1 and order-2 stability are satisfied, the term $\frac{c(w+1)}{4c(5w-7)-48w^2+48}$ is never zero. It is known that for all parameter configurations tested, particles will present as order-1 and order-2 since the criteria of equation (5) has been empirically verified without the presence of simplifying assumptions by Cleghorn and Engelbrecht [31].

The results reported in section 5 are derived from 50 independent runs for each objective function, each tested parameter configuration, and each considered dimensionality.

5 Experimental Results and Discussion

This section presents the results of the experiments described in section 4.

It is known that each tested parameter configuration is both order-1 and order-2 stable as they satisfy equation (5). Given this knowledge all particles in a swarm are classified as order-2* stable only if $\mathit{\Gamma}(5000) = \mathit{\Gamma} \leq \epsilon$, where $\epsilon = 0.001$. The allowance for an ϵ of 0.001 is relatively relaxed. However, if a configuration does not even ensure that $\mathit{\Gamma} \leq 0.01$ then it is very clearly not order-2* stable.

For each tested dimension a table is presented to summarize the recored $\mathit{\Gamma}$ measurements across the 11 base objective functions of the CEC 2014 problem set [29]. Specifically, the number of parameter configurations that resulted in $\mathit{\Gamma} \leq \epsilon$ (order-2* stable) and that resulted in $\mathit{\Gamma} > \epsilon$ (not order-2* stable) are presented. Additionally, the average and maximum value for $\mathit{\Gamma}$ are presented from the 504 parameter configurations. The reported values for $\mathit{\Gamma}$ are comparable across different dimensionalities, given the use of the infinity norm. Furthermore, the resulting $\mathit{\Gamma}$ can be easily related to the initialized component-wise size of the search space. Namely, the largest possible value for $\|\hat{\boldsymbol{y}}_{i}\left(t+1\right)-\boldsymbol{y}_{i}\left(t\right)\|_{\infty}$ is

Table 1: Γ Measurements in 5-Dimensions

Function	Number of $\Gamma \leq \epsilon$	Number of $\Gamma > \epsilon$	Average Γ	Max Γ
F1	0	504	11.4421	39.3982
F2	504	0	0.0000	0.0007
F3	504	0	0.0000	0.0001
F4	213	291	0.0352	0.3289
F5	71	433	7.6169	34.1461
F6	150	354	0.1320	1.1043
F7	155	349	0.1161	1.1066
F8	504	0	0.0000	0.0000
F9	306	198	11.6192	61.9315
F10	248	256	0.8043	3.7139
F11	297	207	1.4899	32.1101

200. The results for 5, 10, 20, and 30 dimensions are reported in tables 1, 2, 3, and 4 respectively.

It is immediately apparent from tables 1 through 4 to that there are numerous parameter configurations that are both order-1 and order-2 stable but are not order-2* stable. It is also clear from table 1 that there is also not a subset of the region defined in equation (5) that can guarantee order-2* stability. Specifically, in the case of the Ackley (F1) objective function there are 0 parameter configurations that are order-2* stable in 5, 10, or 30 dimensions. In the 20 dimensional case only 1 configuration is classified as order-2* as illustrated in table 3.

Table 2: Γ Measurements in 10-Dimensions

Function	Number of $\Gamma \leq \epsilon$	Number of $\Gamma > \epsilon$	Average Γ	Max Γ
F1	0	504	25.1646	130.9834
F2	504	0	0.0000	0.0035
F3	504	0	0.0000	0.0033
F4	451	53	0.0089	0.3157
F5	249	255	2.9112	16.2970
F6	364	140	0.0926	1.1898
F7	376	128	0.0844	1.3799
F8	504	0	0.0000	0.0012
F9	290	214	10.0291	56.4448
F10	275	229	0.6743	3.6561
F11	457	47	0.4510	11.8847

In 5, 10, and 20 dimensions all parameter configurations are order-2* stable for the Bent Cigar (F2) and High Conditioned Elliptic (F8) objective functions as illustrated in tables 1, 2, and 3 respectively. However, in the 30 dimensional case, only for the Griewank (F5) function were all parameter configurations order-2* stable as illustrated in table 4. Interestingly, for some objective functions, such as Griewank (F5), an increase in dimensionality makes order-2* stability more obtainable, where as the opposite was true for some objective functions, such as the Katsuura (F9) objective function.

It should also be noted that in the case where order-2* stability does not occur the actual particle variance may be relatively large, given the relationship between particle variance and equation (6). The most extreme case occurs with the Ackley (F1) objective function which had the largest maximum and average Γ measurements across all dimensions. Specifically, in 10 dimensions the average Γ value was 25.1646, which is 12.58% of the maximum component-wise distance, indicating that on average, particle behavior is far from order-2* stable. In 30 dimensions the average Γ value increased to 30.0851, a substantial 15% of the maximum component-wise distance as illustrated in table 4. A similar, though less extreme situation occurs with the Katsuura (F9) objective function which has a relatively large average Γ value across 5, 10, 20, and 30 dimensions.

Table 3: Γ Measurements in 20-Dimensions

Function	Number of $\Gamma \leq \epsilon$	Number of $\Gamma > \epsilon$	Average Γ	Max Γ
F1	1	503	28.0691	154.5342
F2	504	0	0.0000	0.0012
F3	503	1	0.0001	0.0242
F4	474	30	0.0084	0.4301
F5	488	16	0.0044	0.2557
F6	430	74	0.1214	2.3300
F7	440	64	0.0648	1.2652
F8	504	0	0.0000	0.0050
F9	289	215	7.2412	35.7945
F10	326	178	0.5773	3.4318
F11	470	34	0.1958	7.0119

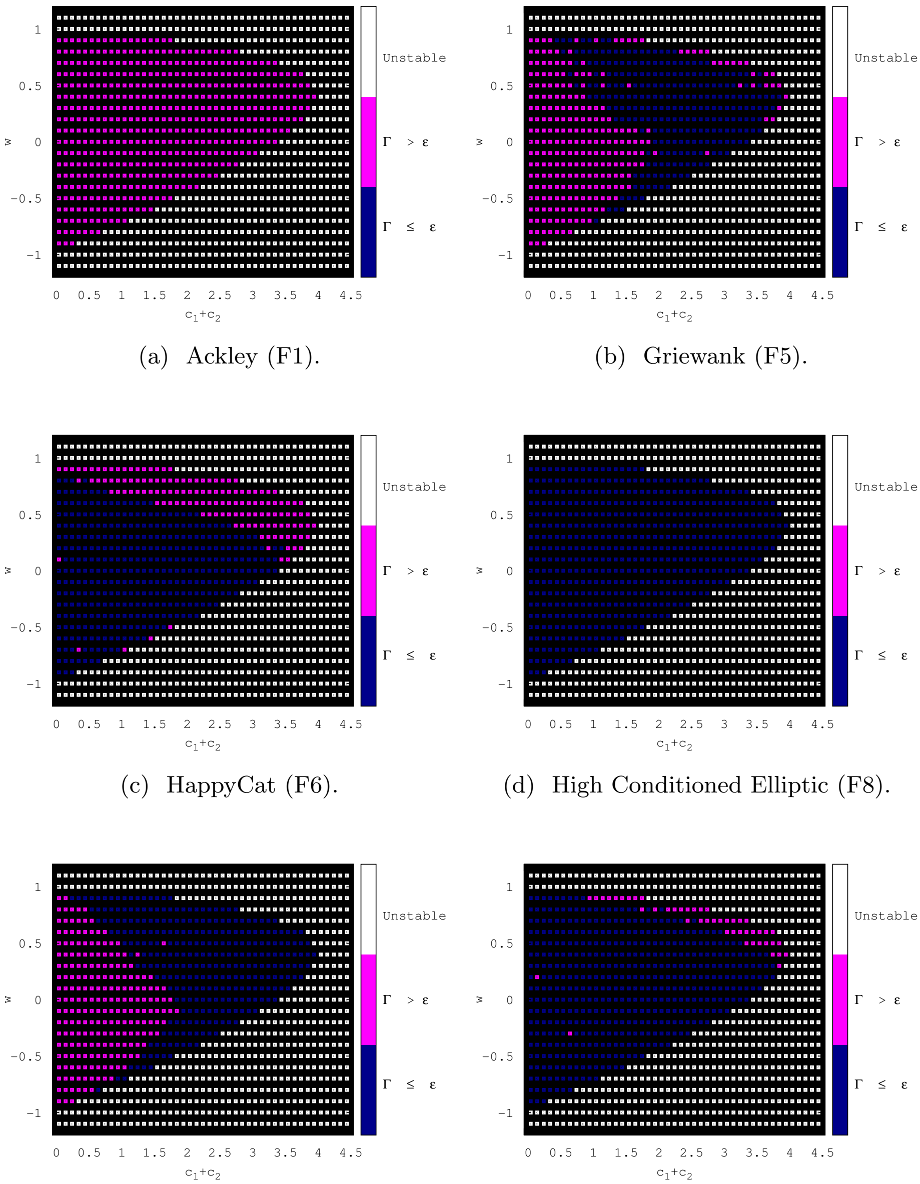
A snapshot of the parameter configurations for which order-2* stability occurred is provided for Ackley (F1), Griewank (F5), HappyCat (F6), High Conditioned Elliptic (F8), Katsuura (F9), and Rosenbrock (F11) in figures 2(a) to 2(f) for the 10 dimensional case and in figures 3(a) to 3(f) for the 30 dimensional case.

In the 10-dimensional case the Ackley (F1) and High Conditioned Elliptic (F8) objective functions present the most extreme behavior, as illustrated in figures 2(a) and 2(d) respectively. Under the Ackley (F1) objective function the particles are never order-2* stable whereas under the High Conditioned Elliptic (F8) objective function the particles are order-2* stable for all configurations satisfying equation (5). In all other snapshots, of the 10 dimensional case, there are some parameter configurations that do present as order-2* stable and some that do not, despite being both order-1 and order-2 stable. It is interesting to note that there is not a clear pattern present in figures 2(a) to 2(f) with regards to when order-2* stability will occur. Specifically, in case of the Katsuura (F9) and Griewank (F5) objective functions the region of order-2* stable parameter configurations excludes a curved region excluding many parameter configurations with low cognitive and social weight values as illustrated in figures 2(e) and 2(b). Whereas in the case of the HappyCat (F6) and Rosenbrock (F11) objective functions the region of order-2* stable parameter configurations excludes a number of configurations with large positive inertia values, while configuration with low cognitive and social coefficient weights are included as illustrated in figures 2(c) and 2(f). One of the most striking difference between the 10 and the 30 dimensional cases is that of Griewank (F5). In the 10 dimensions there are numerous configuration which do not exhibit order-2* stability, while in the 30 dimensions all configurations exhibit order-2* stability as illustrated in figured 2(b) and 3(b) respectively.

Table 4: Γ Measurements in 30-Dimensions

Function	Number of $\Gamma \leq \epsilon$	Number of $\Gamma > \epsilon$	Average Γ	Max Γ
F1	0	504	30.0851	181.9311
F2	502	2	0.0001	0.0328
F3	501	3	0.0068	3.3226
F4	477	27	0.0068	0.3774
F5	504	0	0.0000	0.0035
F6	452	52	0.1015	2.5492
F7	459	45	0.0593	4.5566
F8	503	1	0.0001	0.0250
F9	276	228	6.0534	27.9352
F10	339	165	0.5417	3.4578
F11	475	29	0.1412	5.5532

It should be noted that providing PSO with considerably more iterations than 5000 does not lead to order-2* stability presenting itself. A fact that is theoretically evident from proposition 1. However, for a more practical illustration, consider PSO optimizing the Ackley (F1) objective function, with coefficients $w = 0.729$, $c_1 = 1.494$, and $c_2 = 1.494$, that satisfy the stability criteria of equation (5). The Γ measurement is plotted across 10^6 iteration, averaged over 50 runs, in figure 4. The $\Gamma(t)$ measurement clearly decreases as the iterations, t, increase. However, the rate of decrease is rapidly slowing, with $\Gamma(t)$ still over

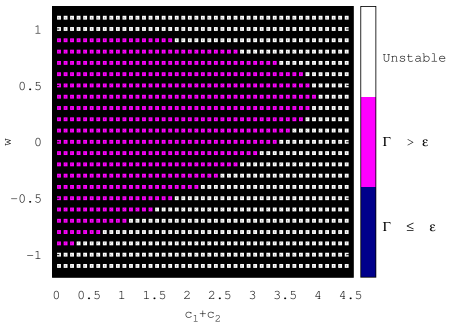


(a) Ackley (F1). (b) Griewank (F5).

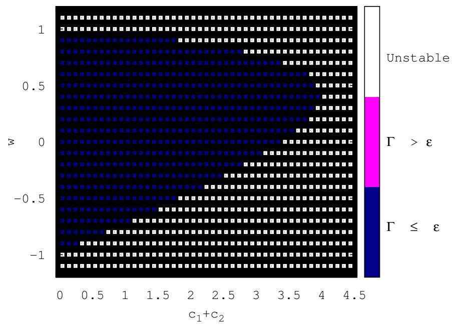
(c) HappyCat (F6). (d) High Conditioned Elliptic (F8).

(e) Katsuura (F9). (f) Rosenbrock (F11).

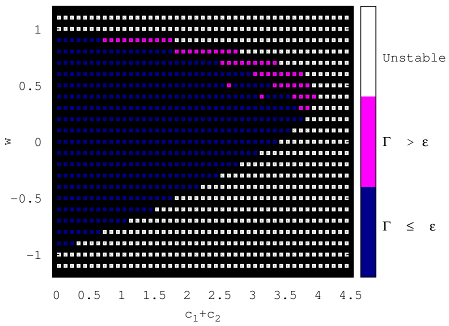
Fig. 2: Summary of parameter configurations that lead to order-2* stability in 10-dimensions. If a parameter configuration has $\mathit{\Gamma} \leq \epsilon$ it is said to be order-2* stable. Parameter configurations that violate the criteria for order-1 and order-2 stability as defined in equation (5) are marked as unstable. When $\mathit{\Gamma} > \epsilon$ and equation (5) is satisfied the parameter configuration is said to be order-1 and order-2 stable but not order-2* stable



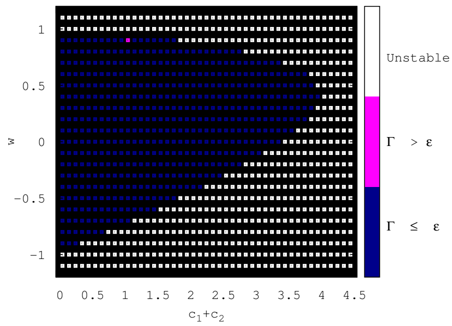
(a) Ackley (F1).



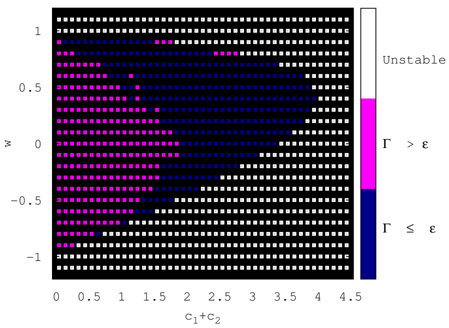
(b) Griewank (F5).



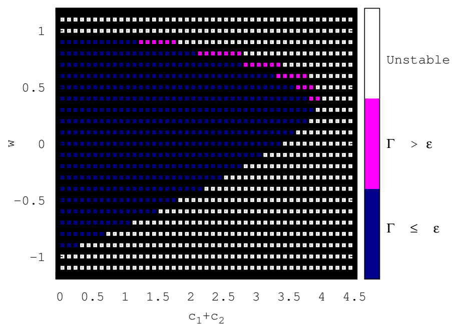
(c) HappyCat (F6).



(d) High Conditioned Elliptic (F8).



(e) Katsuura (F9).



(f) Rosenbrock (F11).

Fig. 3: Summary of parameter configurations that lead to order-2* stability in 30-dimensions. If a parameter configuration has $\mathit{\Gamma} \leq \epsilon$ it is said to be order-2* stable. Parameter configurations that violate the criteria for order-1 and order-2 stability as defined in equation (5) are marked as unstable. When $\mathit{\Gamma} > \epsilon$ and equation (5) is satisfied the parameter configuration is said to be order-1 and order-2 stable but not order-2* stable

8.7 after 10^6 iteration in the 10 dimensional case. While this is not proof that Γ cannot eventually become 0 on the Ackley (F1) objective function, there is certainly no guarantee of it given proposition 1 for an arbitrary objective function. From a practical perspective it is clear that one should not expect order-2* stability to occur even if order-2 stability criteria are met.

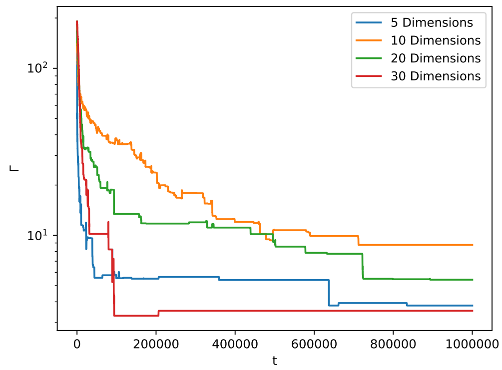


Fig. 4: Long term average $\Gamma(t)$ of a swarm on the Ackley (F1) objective function in 5, 10, 20, and 30 dimensions under the coefficients $w = 0.729$, $c_1 = 1.494$, and $c_2 = 1.494$.

What is clear from tables 1 to 4 and figures 2 and 3 is that whether or not a swarm's particles will be order-2* stable is problem dependent. Specifically, in practice utilizing the criteria of equation (5) can only guarantee order-1 and order-2 stability and not the more restrictive order-2* stability which required a zero positions variance.

6 Conclusion

In this paper it is theoretically and empirically demonstrated which practical guarantees can in fact be made about particle stability in the particle swarm optimization (PSO) algorithm. Specifically, it is shown that the definition of order-2 stability which accurately reflects PSO behavior is that of convergence in second order moment to a constant, and not to zero. It was empirically shown that convergence in second order moment to 0 (order-2* stability), cannot be guaranteed by using the criteria for order-1 and order-2 stability. Furthermore, the occurrence of order-2* stability is shown to be problem dependent, and not directly controllable by coefficient selection, excluding of course the trivial case of zero valued social and cognitive weights.

This paper provided clarification on the meaning of order-2 stability and the guarantees on particle stability that can be made. This clarification will allow PSO practitioners to be better guided by the wealth of existing PSO theory.

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