THREE GEOMETRIC CONSTANTS FOR MORREY SPACES

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Abstract. In this paper we calculate three geometric constants, namely the von Neumann-Jordan constant, the James constant, and the Dunkl-Williams constant, for Morrey spaces and discrete Morrey spaces. These constants measure uniformly nonsquareness of the associated spaces. We obtain that the three constants are the same as those for L^1 and L^∞ spaces.

1. Introduction

The *von Neumann-Jordan constant* $\mathrm{C_{NJ}}(X)$ (see [8]), the *James constant* $\mathrm{C}_J(X)$ (see [6]) and the *Dunkl-Williams constant* $\mathrm{C_{DW}}(X)$ (see [3]) for a Banach space X are given by

$$\mathrm{C_{NJ}}(X) := \sup\left\{ \frac{\|x+y\|_X^2 + \|x-y\|_X^2}{2(\|x\|_X^2 + \|y\|_X^2)} \,:\, x,y \in X \setminus \{0\} \right\},$$

$$\mathrm{C_J}(X) := \sup\left\{ \min\{\|x+y\|_X, \|x-y\|_X\} \,:\, x,y \in X,\ \|x\|_X = \|y\|_X = 1 \right\},$$

and

$$\mathrm{C_{DW}}(X) := \sup\left\{ \frac{\|x\|_X + \|y\|_X}{\|x-y\|_X} \left\| \frac{x}{\|x\|_X} - \frac{y}{\|y\|_X} \right\|_X \,:\, x,y \in X,\ x \neq 0, y \neq 0, x \neq y \right\},$$

respectively. It is well known that $1 \leq C_{\mathrm{NJ}}(X) \leq 2$ for every Banach space X, and that $C_{\mathrm{NJ}}(X) = 1$ if and only if X is a Hilbert space. Meanwhile, $\sqrt{2} \leq C_{\mathrm{J}}(X) \leq 2$ holds for every Banach space X, and $C_{\mathrm{J}}(X) = \sqrt{2}$ if (but not only if) X is a Hilbert space (see [2, 4]). As for the Dunkl-Williams constant, we have $2 \leq \mathrm{C_{DW}}(X) \leq 4$ and $\mathrm{C_{DW}}(X) = 2$ if and only if X is a Hilbert space [3]. For Lebesgue spaces $L^p = L^p(\mathbb{R}^d)$ where $1 \leq p \leq \infty$, we have $C_{\mathrm{NJ}}(L^p) = \max\{2^{2/p-1}, 2^{1-2/p}\}$ and $C_{\mathrm{J}}(L^p) = \max\{2^{1/p}, 2^{1-1/p}\}$ [9]. Meanwhile, we know that $C_{\mathrm{DW}}(L^1) = C_{\mathrm{DW}}(L^\infty) = 4$ [7].

In this paper, we shall calculate the three constants for Morrey spaces and discrete Morrey spaces. Let $1 \leq p \leq q < \infty$. The *Morrey space* $\mathcal{M}_q^p = \mathcal{M}_q^p(\mathbb{R}^d)$ is the set of all the measurable functions f on $\mathbb{R}^d$ for which

$$\|f\|_{\mathcal{M}_q^p} := \sup_{B=B(a,r)} |B|^{\frac{1}{q}-\frac{1}{p}} \left(\int_B |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty,$$

where $B(a,r)$ denotes the ball centered at $a \in \mathbb{R}^d$ having radius $r > 0$ and Lebesgue measure $|B|$ (see, e.g., [1]). Since $\mathcal{M}_q^p$ is a Banach space, it follows from [2, 3, 4] that

$$C_{\mathrm{NJ}}(\mathcal{M}_q^p),\ C_{\mathrm{J}}(\mathcal{M}_q^p) \leq 2 \ \text{ and } \ C_{\mathrm{DW}}(\mathcal{M}_q^p) \leq 4.$$

Our result for Morrey spaces is the following:

Theorem 1.1. *If $1 \leq p < q < \infty$, then $C_{\mathrm{NJ}}(\mathcal{M}_q^p) = C_{\mathrm{J}}(\mathcal{M}_q^p) = 2$ and $C_{\mathrm{DW}}(\mathcal{M}_q^p) = 4$.*

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Note that $\mathcal{M}^p_p = L^p$ holds and that their norms are identical. The above theorem tells us that the case where $q > p$ is quite different from the case where $q = p$. When $q > p$, the three constants $C_{\rm J}(\mathcal{M}^p_q)$, $C_{\rm NJ}(\mathcal{M}^p_q)$, and $C_{\rm DW}(\mathcal{M}^p_q)$ take the same value as those for L^1 and L^∞ spaces.

Moving on to discrete Morrey spaces, let $\omega := \mathbb{N} \cup \{0\}$. For $m := (m_1, \dots, m_d) \in \mathbb{Z}^d$ and $N \in \omega$, let

$$S_{m,N} := \{k \in \mathbb{Z}^d \colon \|k - m\|_\infty \le N\},$$

where $\|(m_1, \dots, m_d)\|_\infty := \max\{|m_i| \colon 1 \le i \le d\}$ for $(m_1, \dots, m_d) \in \mathbb{Z}^d$. The cardinality of $S_{m,N}$, denoted by $|S_{m,N}|$, is $(2N+1)^d$, for every $m \in \mathbb{Z}^d$ and $N \in \omega$. Given $1 \le p \le q < \infty$, we define the *discrete Morrey space* $\ell^p_q = \ell^p_q(\mathbb{Z}^d)$ to be the space of all functions (sequences) $x \colon \mathbb{Z}^d \to \mathbb{R}$ for which

$$\|x\|_{\ell^p_q} := \sup_{m \in \mathbb{Z}^d, N \in \omega} |S_{m,N}|^{\frac{1}{q} - \frac{1}{p}} \left(\sum_{k \in S_{m,N}} |x(k)|^p \right)^{\frac{1}{p}} < \infty.$$

We note that ℓ^p_q, equipped with the above norm, is a Banach space (see [5]). Our result for discrete Morrey spaces is the following:

Theorem 1.2. *If $1 \le p < q < \infty$, then $C_{\rm NJ}(\ell^p_q) = C_{\rm J}(\ell^p_q) = 2$ and $C_{\rm DW}(\ell^p_q) = 4$.*

This theorem also tells us that the case where $q > p$ is quite different from the case where $q = p$ (where $\ell^p_p = \ell^p$).

2. Proof of Theorems

We prove both theorems by finding two elements in the space such that the associated expressions are equal to two, two, and four, respectively.

2.1. **Proof of Theorem 1.1.**

Proof. Let $1 \le p < q < \infty$, and let $f(x) := |x|^{-d/q}$, $x \in \mathbb{R}^d$, where $|x|$ denotes the Euclidean norm of x. Then $f \in \mathcal{M}^p_q(\mathbb{R}^d)$ (see [10, §2]). Define $g(x) := \chi_{(0,1)}(|x|)f(x)$, $h(x) := f(x) - g(x)$, and $k(x) := -f(x) + 2g(x)$, for $x \in \mathbb{R}^d$. By a change of variables, we see that

$$\|t^{d/q} g(t\cdot)\|_{\mathcal{M}^p_q} = \|g\|_{\mathcal{M}^p_q}$$

and

$$\|t^{d/q} h(t\cdot)\|_{\mathcal{M}^p_q} = \|h\|_{\mathcal{M}^p_q}$$

for all $t > 0$. Since

$$t^{d/q} g(tx) = \chi_{(0,1)}(t|x|) f(x)$$

and

$$t^{d/q} h(tx) = \chi_{(0,1)}(t|x|) f(x) - \chi_{[1,\infty)}(t|x|) f(x)$$

for $t > 0$ and $x \in \mathbb{R}^d$, by the monotone convergence property of Morrey spaces we have

$$\|f\|_{\mathcal{M}^p_q} = \|g\|_{\mathcal{M}^p_q} = \|h\|_{\mathcal{M}^p_q} = \|k\|_{\mathcal{M}^p_q} \in (0, \infty).$$

This implies that

$$\|f + k\|^2_{\mathcal{M}^p_q} + \|f - k\|^2_{\mathcal{M}^p_q} = 4(\|f\|^2_{\mathcal{M}^p_q} + \|k\|^2_{\mathcal{M}^p_q})$$

and

$$\min\{\|f + k\|_{\mathcal{M}^p_q}, \|f - k\|_{\mathcal{M}^p_q}\} = \min\{\|2g\|_{\mathcal{M}^p_q}, \|2h\|_{\mathcal{M}^p_q}\} = 2\|f\|_{\mathcal{M}^p_q} = 2\|k\|_{\mathcal{M}^p_q}.$$

By definition and the fact that both $C_{\rm NJ}(\mathcal{M}^p_q),\ C_{\rm J}(\mathcal{M}^p_q) \le 2$, we conclude that

$$C_{\rm NJ}(\mathcal{M}^p_q) = C_{\rm J}(\mathcal{M}^p_q) = 2,$$

as desired.

Finally, we calculate the Dunkl–Williams constant using the same ideas as in [7]. We consider f and $(1+r)g + (1-r)h$ for $r \in (0,1)$. We calculate

$$\begin{aligned}
&\frac{\|f\|_{\mathcal{M}^p_q} + \|(1+r)g + (1-r)h\|_{\mathcal{M}^p_q}}{\|f - (1+r)g - (1-r)h\|_{\mathcal{M}^p_q}} \left\| \frac{f}{\|f\|_{\mathcal{M}^p_q}} - \frac{(1+r)g + (1-r)h}{\|(1+r)g + (1-r)h\|_{\mathcal{M}^p_q}} \right\|_{\mathcal{M}^p_q} \\
&= \frac{\|f\|_{\mathcal{M}^p_q} + (1+r)\|f\|_{\mathcal{M}^p_q}}{r\|f\|_{\mathcal{M}^p_q}} \left\| \frac{f}{\|f\|_{\mathcal{M}^p_q}} - \frac{(1+r)g + (1-r)h}{(1+r)\|f\|_{\mathcal{M}^p_q}} \right\|_{\mathcal{M}^p_q} \\
&= \frac{\|f\|_{\mathcal{M}^p_q} + (1+r)\|f\|_{\mathcal{M}^p_q}}{r\|f\|_{\mathcal{M}^p_q}} \left\| \frac{2rh}{(1+r)\|f\|_{\mathcal{M}^p_q}} \right\|_{\mathcal{M}^p_q} \\
&= \frac{4+2r}{1+r}.
\end{aligned}$$

If we let $r \downarrow 0$, we obtain $C_{\rm DW}(\mathcal{M}^p_q) = 4$, as required. □

Before we conclude this subsection, a remark may be in order. Let $1 \le p \le q < \infty$. The *local Morrey space* $\mathrm{L}\mathcal{M}^p_q = \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^d)$ is the set of all the measurable functions f on $\mathbb{R}^d$ for which

$$\|f\|_{\mathrm{L}\mathcal{M}^p_q} := \sup_{B=B(0,r)} |B|^{\frac{1}{q}-\frac{1}{p}} \left(\int_B |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty.$$

Arguing similary as before, we see that $C_{\rm NJ}(\mathrm{L}\mathcal{M}^p_q) = C_{\rm J}(\mathrm{L}\mathcal{M}^p_q) = 2$ and $C_{\rm DW}(\mathrm{L}\mathcal{M}^p_q) = 4$ whenever $1 \le p < q < \infty$.

2.2. **Proof of Theorem 1.2.**

Proof. Let $1 \le p < q < \infty$, and let us first consider the case where $d = 1$. Let $n \in \mathbb{Z}$ be an even number with $n > 2^{\frac{q}{q-p}} - 1$, or equivalently

$$(n+1)^{\frac{1}{q}-\frac{1}{p}} < 2^{-\frac{1}{p}}.$$

Consider the sequence $(x_k)_{k\in\mathbb{Z}}$ defined by

$$x_0 = x_n = 1, \text{and } x_k = 0 \text{ for all } k \notin \{0, n\}$$

and the sequence $(y_k)_{k\in\mathbb{Z}}$ defined by

$$y_0 = 1,\ y_n = -1, \text{and } y_k = 0 \text{ for all } k \notin \{0, n\}.$$

Then, we have

$$\begin{aligned}
\|x\|_{\ell^p_q} &= \sup_{m\in\mathbb{Z}, N\in\omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}} \\
&= \max\left\{ 1, |S_{\frac{n}{2},\frac{n}{2}}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{\frac{n}{2},\frac{n}{2}}} |x_k|^p \right)^{1/p} \right\} \\
&= \max\left\{ 1, (n+1)^{\frac{1}{q}-\frac{1}{p}} 2^{\frac{1}{p}} \right\}.
\end{aligned}$$

With the choice of n above, we see that

$$(n+1)^{\frac{1}{q}-\frac{1}{p}}2^{\frac{1}{p}} < 1.$$

Therefore $\|x\|_{\ell^p_q} = 1$. Similarly, one may verify that $\|y\|_{\ell^p_q} = 1$. Moreover, we may observe that

$$\|x+y\|_{\ell^p_q} = 2 \quad \text{and} \quad \|x-y\|_{\ell^p_q} = 2.$$

Hence, we obtain

$$\frac{\|x+y\|^2_{\ell^p_q} + \|x-y\|^2_{\ell^p_q}}{2(\|x\|^2_{\ell^p_q} + \|y\|^2_{\ell^p_q})} = \frac{2^2+2^2}{2(1^2+1^2)} = 2.$$

Hence we conclude that $C_{\rm NJ}(\ell^p_q) = 2$. With the same choices of x and y, we have

$$C_{\rm J}(\ell^p_q) = \sup\{\min\{\|x+y\|_{\ell^p_q}, \|x-y\|_{\ell^p_q}\} : x, y \in X, \|x\|_{\ell^p_q} = \|y\|_{\ell^p_q} = 1\} = 2.$$

We shall now consider the general case where $d \geq 1$. Let $n \in \mathbb{Z}$ be an even number with $n > 2^{\frac{q}{d(q-p)}} - 1$, or equivalently

$$(n+1)^{d(\frac{1}{q}-\frac{1}{p})} < 2^{-\frac{1}{p}}.$$

Let $x \in \ell^p_q$ be the function $x \colon \mathbb{Z}^d \to \mathbb{R}$ where

$$x(k) := \begin{cases} 1, & \text{if } k = (0,0,\ldots,0), (n,0,\ldots,0) \\ 0, & \text{otherwise.} \end{cases}$$

and $y \in \ell^p_q$ be the function $y \colon \mathbb{Z}^d \to \mathbb{R}$ where

$$y(k) := \begin{cases} 1, & \text{if } k = (0,0,\ldots,0) \\ -1, & \text{if } k = (n,0,\ldots,0) \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$\begin{aligned} \|x\|_{\ell^p_q} &= \sup_{m\in\mathbb{Z}^d, N\in\omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}} \\ &= \max\left\{ 1, |S_{\frac{n}{2},\frac{n}{2}}|^{d(\frac{1}{q}-\frac{1}{p})} \left(\sum_{k\in S_{\frac{n}{2},\frac{n}{2}}} |x_k|^p \right)^{1/p} \right\} \\ &= \max\left\{ 1, (n+1)^{d(\frac{1}{q}-\frac{1}{p})} 2^{\frac{1}{p}} \right\}. \end{aligned}$$

Note that with the choice of n above, we have

$$(n+1)^{d(\frac{1}{q}-\frac{1}{p})}2^{\frac{1}{p}} < 1,$$

whence $\|x\|_{\ell^p_q} = 1$. Similarly $\|y\|_{\ell^p_q} = 1$. Moreover, we also have

$$\|x+y\|_{\ell^p_q} = 2 \quad \text{and} \quad \|x-y\|_{\ell^p_q} = 2.$$

Therefore, we obtain

$$\frac{\|x+y\|^2_{\ell^p_q} + \|x-y\|^2_{\ell^p_q}}{2(\|x\|^2_{\ell^p_q} + \|y\|^2_{\ell^p_q})} = \frac{2^2+2^2}{2(1^2+1^2)} = 2,$$

whence $C_{\rm NJ}(\ell^p_q) = 2$. The same choices of x and y give

$$C_{\rm J}(\ell^p_q) = \sup\{\min\{\|x+y\|_{\ell^p_q}, \|x-y\|_{\ell^p_q}\} : x, y \in X, \|x\|_{\ell^p_q} = \|y\|_{\ell^p_q} = 1\} = 2.$$

Finally as for the Dunkl–Williams constant, we use the couple $x + y$ and $(1 + r)x + (1 - r)y$ for $0 < r < 1$ and argue similarly to the case of Morrey spaces. □

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