Discrete growth-decay-fragmentation equation – well-posedness and long term dynamics

J. Banasiak · L.O. Joel · S. Shindin

Abstract In this paper we consider discrete growth-decay-fragmentation equations that describe the size-distribution of clusters that can undergo splitting, growth and decay. The clusters can be for instance animal groups that can split but can also grow, or decrease in size due to birth or death of individuals in the group, or chemical particles where the growth and decay can be due to surface deposition or erosion. We prove that for a large class of such problems the solution semigroup is analytic and compact and thus has the asynchronous exponential growth property; that is, the long term behaviour of any solution is given by a scalar exponential function multiplied by a vector, called the stable population distribution, that are independent on the initial conditions.

J. Banasiak
Department of Mathematics and Applied Mathematics, University of Pretoria, South Africa
& Institute of Mathematics, Technical University of Łódź, Łódź, Poland
E-mail: jacek.banasiak@up.ac.za

L. O. Joel
School of Mathematics, Statistics and Computer Science
University of Kwazulu-Natal, Durban, South Africa
E-mail: oluwaseyejoel@gmail.com

S. Shindin
School of Mathematics, Statistics and Computer Science
University of Kwazulu-Natal, Durban, South Africa
E-mail: shindins@ukzn.ac.za

1 Introduction

Coagulation and fragmentation models that describe the processes of objects forming larger clusters or, conversely, splitting into smaller fragments, have received a lot of attention over several decades due to their importance in chemical engineering and other fields of science and technology, see e.g. [22,44]. One of the most efficient approaches to modelling dynamics of such processes is through the kinetic (rate) equation which describes the evolution of the distribution of interacting clusters with respect to their size/mass. The first model of this kind, consisting of an infinite system of ordinary differential equations, was derived by Smoluchowski, [42,43], to describe pure coagulation in the discrete case; that is, if the ratio of the mass of the basic building block (monomer) to the mass of a typical cluster is positive, and thus the size of a cluster is a finite multiple of the masses of the monomers. In many applications, however, it turned out to be advantageous to allow clusters to be composed of particles of any size $x > 0$. This leads to the continuous integro-differential equation that was derived by Müller in the pure coagulation case, [34], and extended to a coagulation–fragmentation version in [31].

In the last few decades it has been observed that also living organisms form clusters or split into subgroups depending on circumstances, see e.g. [18, 28,35,36] for modelling concerning larger animals, or [2,29] for phytoplankton models. It turns out that also the process of cell division may be modelled within the same framework, see e.g. [13,14,38,40]. What was not always fully recognized in some papers mentioned above was that the living matter has its own vital dynamics; that is, in addition to forming or breaking clusters, individuals within them are born or die, leading to the growth or decay of the clusters, and so the latter processes must be adequately represented in the models. In the continuous case, the birth and death processes are incorporated into the model by adding an appropriate first order transport term, analogously to the age or size structured McKendrick model, see [2,8,10,13, 38]. On the other hand, in the discrete case the vital processes are modelled by adding the classical birth-and-death terms to the Smoluchowski equation. Note that e.g. the pure birth terms (or pure death terms) can be obtained by the Euler discretization of the first order differential operator of the continuous case, while the full birth-and-death problem can be thought of as the discretization of the diffusion operator.

One of the most important problems in the analysis of dynamical systems is to determine their long term behaviour and hence this aspect of the theory of growth–fragmentation equations has received much attention. The first systematic mathematical study of the binary cell division model was carried out using semigroup theory in [19]; the semigroup approach was significantly extended to more general models in [33]. Recently a number of results have

been obtained by the General Relative Entropy (or related) methods that lead to convergence of solutions in spaces weighted by the eigenvector of the adjoint problem, see e.g. [6, 20, 21, 30, 32, 38, 39].

All above results concern growth–fragmentation models with continuous size distribution. Recently it has been observed, however, that a large class of discrete fragmentation equations has much better properties than their continuous counterparts, especially when considered in spaces where sufficiently high moments of solutions are finite. In particular, the fragmentation operator in such spaces generates a compact analytic semigroup. In this paper, we explore these ideas for the full growth-decay-fragmentation equation and show, in particular, that under natural assumptions on the coefficients of the problem the growth-decay-fragmentation semigroup is analytic, compact and irreducible and thus has the asynchronous exponential growth (AEG) property, see [4]; that is its long term behaviour of solutions is given by the scalar exponential function $e^{\lambda t}$, where λ is the dominant eigenvalue (and the spectral bound) of the fragmentation operator, multiplied by the strictly positive eigenvector belonging to this eigenvalue, called the stable population distribution. The word *asynchronous* comes from the fact that the behaviour of solutions is 'not synchronized' with the initial conditions.

The paper is organized as follows. In Section 2 we introduce the model. Sections 3 and 4 contain the results on the solvability of the problem. We use the perturbation method and thus the way in which the right hand side of the equation is split is of utmost importance. We found that first considering the subdiagonal part (Section 3) and treating the full equation as a perturbation of the subdiagonal part by remaining part, being a positive operator, by using the Kato perturbation theorem (Section 4) provides the best results. In the Appendix we provide a description of an alternative method, in which we split the equations in a seemingly more natural way, considering the processes of growth and decay separately from the independent process of fragmentation, and then using the Trotter–Kato representation formula to prove the existence of the solution semigroup. We find that this approach produces weaker results due to worse properties of the growth-decay semigroup. Finally, in Section 5 we show that the growth-decay-fragmentation semigroup has the AEG property and in Section 6 we provide some examples and numerical illustrations of the theoretical results obtained in the earlier sections.

2 The model

We consider a collection of clusters of sizes $n \in \mathbb{N}$; that is, consisting of n monomers (cells, individuals,...), described by their size specific density $f = (f_n)_{n \in \mathbb{N}}$. We assume that the number of monomers in each cluster can change by, say, a cell division (with the daughter cell staying in the cluster) or its death. In an inanimate scenario, this can happen by the deposition of a particle from the solute or, conversely, by its dissolution. If we assume that the probability of a more than one birth or death event in a cluster hap-

pening simultaneously is negligible, then the process can be modelled by the classical birth-and-death system of equations, see e.g. [29, p. 1199], though in this setting it should be remembered that the birth or death in a cluster leads to its growth or decay and the latter is the subject of the modelling process. We note that in the case of continuous size distribution the growth process is modelled by the first order differential operator with respect to size, $f \to -\partial_x(gf)$, where g is the growth rate, see e.g. [8], whose Euler discretization with step-size 1 at $x = n$ is $g(n)f(n) - g(n+1)f(n+1)$. Similarly, the decay operator $f \to \partial_x(df)$ can be discretized as $-d(n)f(n) + d(n+1)f(n+1)$ and, using a central difference scheme, the diffusion operator $f \to \partial_x(D\partial_x f)$ yields $D(n+1)f(n+1) - (D(n+1) + D(n))f(n) + D(n)f(n-1)$.

We further assume that the clusters can split into several smaller clusters. Combining both processes, we arrive at the following system of equations:

$$\frac{df_1}{dt} = -g_1 f_1 + d_2 f_2 + \sum_{i=2}^{\infty} a_i b_{1,i} f_i,$$

$$\frac{df_n}{dt} = g_{n-1} f_{n-1} - (a_n + g_n + d_n)f_n + d_{n+1}f_{n+1} + \sum_{i=n+1}^{\infty} a_i b_{n,i} f_i, \quad n \geq 2,$$

$$f_n(0) = f_n^{in}, \quad n \geq 1, \tag{1}$$

or

$$\frac{df}{dt} = \mathcal{G}^- f + (\mathcal{A} + \mathcal{G}^0 + \mathcal{D}^0)f + \mathcal{D}^+ f + \Xi \mathcal{A} f = \mathcal{G}f + \mathcal{D}f + \mathcal{A}f + \mathcal{B}f$$

$$\qquad = \mathcal{G}f + \mathcal{D}f + \mathcal{F}f,$$

$$f(0) = f^{in}, \tag{2}$$

where $f = (f_n)_{n \in \mathbb{N}}$ is the vector whose components f_n give the numbers of n-clusters; the operator

$$\mathcal{A} = \text{diag}(-a_n)_{n \geq 1}, \quad a_1 = 0, \quad a_n > 0, \quad n \geq 2,$$

gives the rates at which the clusters of mass n undergo splitting;

$$\mathcal{G}^0 = \text{diag}(-g_n)_{n \geq 1}, \quad g_n \geq 0, \quad n \geq 1,$$

is the growth rate;

$$\mathcal{D}^0 = \text{diag}(-d_n)_{n \geq 1}, \quad d_1 = 0, \quad d_n \geq 0, \quad n \geq 2,$$

is the death rate; $\mathcal{G}^-, \mathcal{D}^+$ are, respectively, the left and right shifts of $\mathcal{G}$ and $\mathcal{D}$; that is

$$\mathcal{G}^- f = (0, g_1 f_1, \ldots g_n f_n, \ldots), \quad \mathcal{D}^+ f = (d_2 f_2, \ldots d_n f_n, \ldots),$$

$\Xi = (b_{n,i})_{1 \leq n < i, i \geq 2}$ is the daughter distribution function, also called the fragmentation kernel, that gives the numbers of i-clusters resulting from splitting of a mass n parent and

$$\mathcal{G} = \mathcal{G}^- + \mathcal{G}^0, \quad \mathcal{D} = \mathcal{D}^+ + \mathcal{D}^0, \quad \mathcal{B} = \Xi \mathcal{A}, \quad \mathcal{F} = \mathcal{A} + \mathcal{B}.$$

Coefficients $b_{n,i}$, $1 \leq n < i$, $i \geq 2$, are nonnegative numbers satisfying

$$\sum_{n=1}^{i-1} n b_{n,i} = i. \tag{3}$$

The total mass of the ensemble is given by

$$M(t) = \sum_{n=1}^{\infty} n f_n(t), \qquad t \geq 0; \tag{4}$$

then it is known, see e.g. [7,11], that in the pure fragmentation case ($\mathcal{G} = \mathcal{D} = 0$) the mass is conserved, $M(t) = \sum_{i=1}^{\infty} n f_n(0), t \geq 0$.

Later, we shall use the fact that (1) can be written as the pure growth-fragmentation model

$$\frac{df_1}{dt} = -g_1 f_1 + \sum_{i=2}^{\infty} \mathsf{a}_i \mathsf{b}_{1,i} f_i,$$

$$\frac{df_n}{dt} = g_{n-1} f_{n-1} - (g_n + \mathsf{a}_n) f_n + \sum_{i=n+1}^{\infty} \mathsf{a}_i \mathsf{b}_{n,i} f_i, \quad n \geq 2,$$

$$f_n(0) = f_n^{in}, \quad n \geq 1, \tag{5}$$

where $\mathsf{a}_n = a_n + d_n$, $n \geq 2$, (with $\mathsf{a}_1 = 0$) and

$$\mathsf{b}_{n,i} = \begin{cases} \frac{a_{n+1} b_{n,n+1} + d_{n+1}}{a_{n+1} + d_{n+1}}, & i = n+1, \\ \frac{a_i b_{n,i}}{a_i + d_i}, & i \geq n+2. \end{cases} \tag{6}$$

We note that the fragmentation part of this model no longer is conservative as

$$\sum_{n=1}^{i-1} n \mathsf{b}_{n,i} = i \left(1 - \frac{d_i}{i(a_i + d_i)} \right), \qquad i \geq 2, \tag{7}$$

so it corresponds to the model with the so-called discrete mass-loss with mass-loss fraction $\lambda_n = d_n/n(a_n + d_n)$, see [15,24], mathematically analysed in [41].

The analysis of the pure fragmentation equation most often is carried out in the space $X_1 := \ell_1^1$ in which the norm of a nonnegative f gives the total mass of the ensemble. However, it is much better to consider (1) in the spaces with finite higher moments, $X_m := \ell_m^1$, with the norm

$$\|f\|_{[m]} = \sum_{n=1}^{\infty} n^m |f_n|, \qquad m \geq 1. \tag{8}$$

In the sequel, for any infinite diagonal matrix $\mathcal{P} = \mathrm{diag}(p_n)_{n \geq 1}$, we define the operator P_m in X_m by $P_m f = \mathcal{P} f$ on $D(P_m) = \{ f \in X_m; \ \mathcal{P} f \in X_m \}$.

3 Analysis of the subdiagonal part

In this section, we shall consider the simplified problem corresponding to the subdiagonal part of (5),

$$\frac{df}{dt} = \mathcal{K}f = \mathcal{G}^- f + (\mathcal{A} + \mathcal{G}^0 + \mathcal{D}^0)f, \quad f(0) = f^{in}. \tag{9}$$

Denote for brevity $\mathcal{T} = \mathcal{A} + \mathcal{G}^0 + \mathcal{D}^0$ and consider the operator $(T_m, D(T_m))$ defined, as above, by $T_m f = \mathcal{T}f$ on $D(T_m) = \{f \in X_m; \ \mathcal{T}f \in X_m\}$. Then $G_m^- = \mathcal{G}^-|_{D(T_m)}$ is a well defined positive operator in X_m and we can apply the substochastic semigroup theory, [9], to $\mathcal{K}|_{D(T_m)} = T_m + G_m^-$. Let $K_{m,\max}$ denote the maximal extension of K_m; that is, $K_{m,\max}f = \mathcal{T}f + \mathcal{G}^- f$ on

$$D(K_{m,\max}) = \Big\{f \in X_m; \ \sum_{n=2}^{\infty} n^m |a_n f_n + d_n f_n + g_n f_n - g_{n-1} f_{n-1}| < \infty\Big\}.$$

Theorem 1 *1. If*

$$\liminf_{n \to \infty} \Big(a_n + d_n - g_n \frac{(n+1)^m - n^m}{n^m} \Big) > -\infty, \tag{10}$$

then there is an extension K_m of $T_m + G_m^-$ that generates a quasicontractive (of type $\mathcal{G}(1, \omega)$ for some $\omega \in \mathbb{R}$) positive semigroup on X_m and, moreover, $K_m = K_{m,\max}$. The resolvent $R(\lambda, K_m)$ for $\lambda > \omega$ is given by

$$[R(\lambda, K_m)f]_n = \sum_{i=1}^{n} \frac{f_i}{\lambda + \theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j}, \quad n \geq 1, \tag{11}$$

where $\theta_1 = g_1$ and $\theta_n = g_n + d_n + a_n, n \geq 1$.
2. If there is $m' > m$ such that[1]

$$\liminf_{n \to \infty} \frac{n(a_n + d_n)}{g_n} \geq m', \tag{12}$$

then (10) is satisfied. Moreover, $D(K_m) = D(A_m) \cap D(D_m^0) \cap D(G_m)$, where $G_m = \mathcal{G}|_{D(G_m)}$ with

$$D(G_m) = \Big\{f \in X_m; \ \sum_{n=2}^{\infty} |g_n f_n - g_{n-1} f_{n-1}| < \infty\Big\}, \tag{13}$$

and $(K_m, D(K_m)) = \overline{(T_m + G_m^-, D(T_m))}$.

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[1] In (12) and everywhere below, it is assumed that the numerical coefficients like a_n, d_n, g_n, e.t.c. take their values in the extended half line $\overline{\mathbb{R}}_+ = [0, \infty) \cup \{+\infty\}$, where for any $a \in (0, \infty)$, we let $\frac{a}{0} = \infty$. In particular, according to this convention for a nonnegative sequence $(g_n)_{n \geq 0}$, with $g_n = 0$, $n \geq n_0$, (12) holds for any finite $m' \geq 0$.

3. *If (12) is satisfied, then $R(\lambda, K_m)$, $\lambda > \omega$, is compact provided*

$$\liminf_{n \to \infty}(a_n + d_n) = \infty \tag{14}$$

and either

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_0 + \theta_n} < \infty, \quad or \quad \lim_{n \to \infty} \frac{n}{\lambda_0 + \theta_n} = 0, \tag{15}$$

for some fixed $\lambda_0 > \omega$.

4. *If*

$$\liminf_{n \to \infty} \frac{a_n + d_n}{g_n} > 0, \tag{16}$$

then $K_m = G_m^- + T_m = G_m^- + G_m^0 + D_m^0 + A_m$ and $(G_{K_m}(t))_{t \geq 0}$ is an analytic semigroup. If additionally (14) is satisfied, then $(G_{K_m}(t))_{t \geq 0}$ is compact.

Proof ad 1.) As in (5), we denote $\mathsf{a}_n = a_n + d_n, n \geq 1$. Note that by virtue of (10), $\mathsf{a}_n - n^{-m}g_n((n+1)^m - n^m) \geq \alpha > -\infty$ for $n \geq 1$. Then, if $\alpha < 0$, for $f \in D(T_m)_+$, we have

$$\sum_{n=1}^{\infty} n^m [(T_m + G_m^-)f]_n = -\sum_{n=1}^{\infty} n^m f_n \left(\mathsf{a}_n - g_n \frac{(n+1)^m - n^m}{n^m} \right)$$

$$= -\sum_{n=1}^{\infty} n^m f_n \left(\mathsf{a}_n - g_n \frac{(n+1)^m - n^m}{n^m} - \alpha \right) - \alpha \sum_{n=1}^{\infty} n^m f_n$$

$$= -c_1(f) + c_0(f), \tag{17}$$

where c_0 is a bounded functional on X_m and c_1 is nonnegative. If $\alpha \geq 0$, we set $\alpha = 0$ in the formulae above. Thus, as in [9, Proposition 9.29], there is an extension $K_m \supset G_m^- + T_m$ generating a smallest quasicontractive (with the growth rate $\omega \geq 0$ not exceeding $\|c_0\|$) positive semigroup. By [9, Theorem 6.20], $K_m \subset K_{m,\max}$. However, in view of the two-diagonal structure of $K_{m,\max}$, for $\lambda > \omega$ it is immediate that $\text{Ker}(\lambda I - K_{m,\max}) = \{0\}$, hence [9, Lemma 3.50 & Proposition 3.52] gives $K_m = K_{m,\max}$.

Let $\lambda > \omega$. We use the formula, see [9, Proposition 9.29],

$$R(\lambda, K_m)f = \sum_{k=0}^{\infty} R(\lambda, T_m)[G_m^- R(\lambda, T_m)]^k f, \quad f \in X_m, \quad \lambda > \omega. \tag{18}$$

Since $R(\lambda, T_m)$ is represented by the matrix $\mathcal{R}(\lambda) = \text{diag}\left(\frac{1}{\lambda + \theta_n}\right)_{n \geq 1}$, and G_m^- is represented by $\mathcal{G}^-$, we have $\mathcal{R}(\lambda)[\mathcal{G}^- \mathcal{R}(\lambda)]^k = (\gamma_{ij}^{(k)})_{i,j \in \mathbb{N}}$, where

$$\gamma_{ij}^{(k)} = \begin{cases} \frac{1}{\lambda + \theta_i} \prod_{l=j}^{i-1} \frac{g_l}{\lambda + \theta_l}, & i \geq k+1, \ j = i - k, \\ 0, & \text{otherwise.} \end{cases}$$

Since the convergence in X_m implies the coordinate-wise convergence, we see that for each n the component $[R(\lambda, K_m)f]_n$ of the series (18) terminates after n terms and hence the resolvent is given by (11).

ad 2.) Since c_1 extends to $D(K_m)_+$ and c_0 is bounded in X_m, for $f \in D(K_m)_+$ we can write

$$\sum_{n=1}^{\infty} n^m [(T_m + G_m^-)f]_n = -\lim_{l \to \infty} \left(\sum_{n=1}^{l} n^m f_n \left(\mathsf{a}_n - g_n \frac{(n+1)^m - n^m}{n^m} \right) - l^m g_l f_l \right)$$

$$= c_0(f) - c_1(f) + \lim_{l \to \infty} l^m g_l f_l \qquad (19)$$

and hence the last limit exists. Further, we have

$$c_1(f) = \sum_{n=1}^{\infty} n^m f_n \left(-\alpha + \mathsf{a}_n - g_n \frac{(n+1)^m - n^m}{n^m} \right)$$

$$\geq \sum_{n=1}^{\infty} n^m f_n \mathsf{a}_n \left(1 - \frac{g_n}{n\mathsf{a}_n} \left(m + O\left(\frac{1}{n} \right) \right) \right),$$

as $-\alpha > 0$ (otherwise we set $\alpha = 0$). If (12) is satisfied, then (possibly adjusting n_0 from the previous part of the proof) for $n \geq n_0$

$$1 - \frac{g_n}{n\mathsf{a}_n} \left(m + O\left(\frac{1}{n} \right) \right) \geq 1 - \frac{m}{m'} + \frac{g_n}{n\mathsf{a}_n} O\left(\frac{1}{n} \right) \geq c' > 0$$

on account of $m' > m$ and $g_n/n\mathsf{a}_n \leq 1/m'$. Since c_1 extends to $D(K_m)_+$ by monotonic limits, we argue as in [7, Theorem 2.1] that any $f \in D(K_m)$ is summable with the weights $(n^m \mathsf{a}_n)_{n \geq 1}$ and hence, by (12), it is also summable with the weight $(n^{m-1} g_n)_{n \geq 1}$. Therefore, in particular, $D(K_m) \subset D(A_m) \cap D(D_m^0)$ and hence also $D(K_m) \subset D(G_m)$ holds by the definition of $D(K_{\max,m})$. The converse inclusion is obvious. Further, by (19), $\lim_{l \to \infty} l^m g_l f_l$ exists, and thus it must be 0. Indeed, otherwise $l^m g_l f_l > c$ for some $c > 0$ and large l contradicting the summability of $(n^{m-1} g_n)_{n \geq 1}$. But then (19) implies that K_m is honest, hence $(K_m, D(K_m)) = \overline{(T_m + G_m^-, D(T_m))}$ by [9, Corollary 6.14].

ad 3.) Though not strictly necessary, the estimates of the norm of the resolvent are instructive and used also further down. To simplify the calculations, instead of $\| \cdot \|_{[m]}$, we employ the norm $\|f\|_* := \sum_{n=1}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} |f_n|$ that is equivalent to $\| \cdot \|_{[m]}$ by virtue of the Gautschi inequality

$$c_m n^m \leq \frac{\Gamma(n+m)}{\Gamma(n)} = C_m n^m, \quad n \geq 1, \qquad (20)$$

which holds uniformly for all $n \geq 1$, and a fixed $m > 0$, with absolute constants $0 < c_m < C_m$, see e.g. [27].

Let $f \in X_m$ and $\lambda > \omega$. Then, changing the order of summation,

$$\|R(\lambda, K_m)f\|_* \leq \sum_{i=1}^{\infty} |f_i| \sum_{n=i}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} \frac{1}{\lambda + \theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j}$$

$$= \frac{1}{\lambda - \omega} \sum_{i=1}^{\infty} |f_i| \sum_{n=i}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} \left(\frac{\lambda - \omega + g_n}{\lambda + \theta_n} - \frac{g_n}{\lambda + \theta_n} \right) \prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j}$$

$$\leq \frac{1}{\lambda - \omega} \sum_{i=1}^{\infty} |f_i| \sum_{n=i}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} \left(\prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j} - \prod_{j=i}^{n} \frac{g_j}{\lambda + \theta_j} \right).$$

$$(21)$$

The internal sum can be written as

$$\lim_{N \to \infty} \sum_{n=i}^{N} \frac{\Gamma(n+m)}{\Gamma(n)} \left(\prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j} - \prod_{j=i}^{n} \frac{g_j}{\lambda + \theta_j} \right) \qquad (22)$$

$$= \frac{\Gamma(i+m)}{\Gamma(i)} + \lim_{N \to \infty} \left(\sum_{n=i+1}^{N} \frac{\Gamma(n+m)}{\Gamma(n)} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j} - \sum_{n=i}^{N} \frac{\Gamma(n+m)}{\Gamma(n)} \prod_{j=i}^{n} \frac{g_j}{\lambda + \theta_j} \right)$$

and then the last term can be transformed as

$$\lim_{N \to \infty} \left(\sum_{n=i+1}^{N} \frac{\Gamma(n+m-1)}{\Gamma(n-1)} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j} \right.$$

$$\left. + m \sum_{n=i+1}^{N} \frac{\Gamma(n+m-1)}{\Gamma(n)} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j} - \sum_{n=i}^{N} \frac{\Gamma(n+m)}{\Gamma(n)} \prod_{j=i}^{n} \frac{g_j}{\lambda + \theta_j} \right)$$

$$= \lim_{N \to \infty} \left(m \sum_{n=i+1}^{N} \frac{\Gamma(n+m-1)}{\Gamma(n)} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j} - \frac{\Gamma(N+m)}{\Gamma(N)} \prod_{j=i}^{N} \frac{g_j}{\lambda + \theta_j} \right).$$

Using (12), for sufficiently large j we have

$$\frac{g_j}{\lambda + \theta_j} = \frac{g_j}{\lambda + g_j + (a_j + d_j)} \leq \frac{j g_j}{\lambda j + (j + m')g_j} \leq \frac{j}{m' + j}, \qquad (23)$$

hence

$$0 \leq \limsup_{N \to \infty} \frac{\Gamma(N+m)}{\Gamma(N)} \prod_{j=i}^{N} \frac{g_j}{\lambda + \theta_j} \leq \limsup_{N \to \infty} \frac{\Gamma(N+m)}{\Gamma(N)} \frac{\Gamma(N+1)\Gamma(i+m')}{\Gamma(N+m'+1)\Gamma(i)}$$

$$= \limsup_{N \to \infty} \frac{N}{N+m'} \frac{\Gamma(N+m)\Gamma(i+m')}{\Gamma(N+m')\Gamma(i)} = 0,$$

on account of the Gautschi inequality, see (20). Thus, (22) can be estimated from above by

$$\frac{\Gamma(i+m)}{\Gamma(i)} + m \sum_{n=i+1}^{\infty} \frac{\Gamma(n+m-1)}{\Gamma(n)} \prod_{j=i}^{n-1} \frac{j}{j+m'}$$

$$= \frac{\Gamma(i+m)}{\Gamma(i)} + m \sum_{n=i+1}^{\infty} \frac{\Gamma(n+m-1)\Gamma(i+m')}{\Gamma(i)\Gamma(n+m')}$$

$$= \frac{\Gamma(i+m)}{\Gamma(i)} + m \sum_{n=i+1}^{\infty} \frac{\Gamma(n+m-1)}{\Gamma(i)\Gamma(n-i)} B(n-i, m'+i), \qquad (24)$$

where B is the Beta function. The sum above can be computed explicitly. Indeed, using integral representation for the Beta function, the Taylor series expansion

$$(1-t)^{-m-i} = \sum_{n=i+1}^{\infty} \frac{\Gamma(n+m-1)}{\Gamma(n-i)\Gamma(m+i)} t^{n-i-1}, \quad |t| < 1,$$

that converges pointwise in the interval $(-1, 1)$, nonnegativity of the partial sums of the above series in $[0, 1)$ and the Lebesgue dominated convergence theorem, we obtain

$$\sum_{n=i+1}^{\infty} \frac{\Gamma(n+m-1)}{\Gamma(i)\Gamma(n-i)} \int_0^1 (1-t)^{m'+i-1} t^{n-i-1} dt$$

$$= \frac{\Gamma(m+i)}{\Gamma(i)} \int_0^1 \left((1-t)^{m'+i-1} \sum_{n=i+1}^{\infty} \frac{\Gamma(n+m-1)}{\Gamma(n-i)\Gamma(m+i)} t^{n-i-1} \right) dt$$

$$= \frac{\Gamma(m+i)}{\Gamma(i)} \int_0^1 (1-t)^{m'-m-1} dt = \frac{\Gamma(m+i)}{\Gamma(i)} \frac{1}{m'-m}.$$

Substituting the above into (24) and returning to (21), we obtain

$$\|R(\lambda, K_m)f\|_* \leq \frac{m'}{m'-m} \frac{1}{\lambda - \omega} \|f\|_*. \qquad (25)$$

To prove the compactness, we fix some $\lambda_0 > \omega$ and consider the projections

$$P_N f = (f_1, f_2, \ldots, f_N, 0, \ldots), \quad N \geq 1. \qquad (26)$$

Since $P_N R(\lambda_0, K_m)$ is an operator with finite dimensional range, it is compact. We consider

$$\|P_{N-1} R(\lambda_0, K_m)f - R(\lambda_0, K_m)f\|_* \leq \sum_{n=N}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} \sum_{i=1}^{n-1} \frac{|f_i|}{\lambda_0 + \theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0 + \theta_j}$$

$$= \sum_{i=1}^{N-1} |f_i| S_{N,i} + \sum_{i=N}^{\infty} |f_i| S_{i+1,i}, \qquad (27)$$

where

$$S_{l,i} = \sum_{n=l}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} \frac{1}{\lambda_0 + \theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0 + \theta_j}.$$

Now,

$$S_{i+1,i} = \sum_{n=i+1}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} \frac{1}{\lambda_0 + \theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0 + \theta_j}$$

$$\leq \left(\sup_{k \geq i+1} \mathsf{a}_k^{-1} \right) \sum_{n=i}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} \frac{\mathsf{a}_n}{\lambda_0 + \theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0 + \theta_j}$$

$$\leq \left(\sup_{k \geq i+1} \mathsf{a}_k^{-1} \right) \sum_{n=i}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} \left(1 - \frac{g_n}{\lambda_0 + \theta_n} \right) \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0 + \theta_j}$$

$$= \left(\sup_{k \geq i+1} \mathsf{a}_k^{-1} \right) \sum_{n=i}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} \left(\prod_{j=i}^{n-1} \frac{g_j}{\lambda_0 + \theta_j} - \prod_{j=i}^{n} \frac{g_j}{\lambda_0 + \theta_j} \right)$$

$$\leq \left(\sup_{k \geq i+1} \mathsf{a}_k^{-1} \right) \frac{m'}{m' - m} \frac{\Gamma(i+m)}{\Gamma(i)},$$

where we used the estimates for (21). Hence

$$\sum_{i=N}^{\infty} |f_i| S_{i+1,i} \leq \left(\sup_{k \geq N+1} \mathsf{a}_k^{-1} \right) \frac{m'}{m' - m} \sum_{i=N}^{\infty} |f_i| \frac{\Gamma(i+m)}{\Gamma(i)}$$

$$\leq \left(\sup_{k \geq N+1} \mathsf{a}_k^{-1} \right) \frac{m'}{m' - m} \|f\|_* \qquad (28)$$

and, by (14), this term tends to 0 as $N \to \infty$, uniformly on the unit ball of X_m.

Since, by (23),

$$\prod_{j=i}^{n} \frac{g_j}{\lambda_0 + \theta_j} \leq \prod_{j=i}^{n} \frac{j}{j+m'} \leq \prod_{j=i}^{n} \frac{j}{j+m} = \frac{\Gamma(n)\Gamma(i+m)}{\Gamma(i)\Gamma(n+m)} \frac{n}{n+m},$$

we have

$$S_{N,i} = \sum_{n=N}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} \frac{1}{\lambda_0 + \theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0 + \theta_j}$$

$$\leq \frac{\Gamma(i+m)}{\Gamma(i)} \sum_{n=N}^{\infty} \frac{1}{\lambda_0 + \theta_n} \frac{n-1}{n-1+m} \leq \frac{\Gamma(i+m)}{\Gamma(i)} \sum_{n=N}^{\infty} \frac{1}{\lambda_0 + \theta_n}.$$

Hence

$$\sum_{i=1}^{N-1} |f_i| S_{N,i} \leq \left(\sum_{n=N}^{\infty} \frac{1}{\lambda_0 + \theta_n} \right) \sum_{i=1}^{\infty} |f_i| \frac{\Gamma(i+m)}{\Gamma(i)}$$

and, using the first option of (15) and combining the above estimate with (28), we see that

$$\lim_{N \to \infty} P_{N-1} R(\lambda_0, K_m) = R(\lambda_0, K_m)$$

in the uniform operator norm. Therefore $R(\lambda_0, K_m)$ is compact.

To use the second option of (15), first we re-write the formula for $S_{N,i}$ as

$$S_{N,i} = \sum_{n=N}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n)} \frac{1}{\lambda_0 + \theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0 + \theta_j}$$

$$\leq \frac{\Gamma(i+m')}{\Gamma(i)} \sum_{n=N}^{\infty} \frac{1}{\lambda_0 + \theta_n} \frac{\Gamma(n+m)}{\Gamma(n+m')}.$$

Then, using again the Gautschi inequality, for large i and $N > i$ we can write

$$\frac{\Gamma(i+m')}{\Gamma(i)} \sum_{n=N}^{\infty} \frac{1}{\lambda_0 + \theta_n} \frac{\Gamma(n+m)}{\Gamma(n+m')} \leq C i^m N^{m'-m} \sum_{n=N}^{\infty} \frac{n^{m-m'}}{\lambda_0 + \theta_n},$$

for some constant C, since $m' - m > 0$. Now, by assumption, $\frac{n^{m-m'}}{\lambda_0 + \theta_n}$ is summable (as $\lambda + \theta_n \geq cn$, for some $c > 0$ and large values of n, while $m - m' < 0$), hence $\sum_{n=N}^{\infty} \frac{n^{m-m'}}{\lambda_0 + \theta_n}$ converges to 0 as $N \to \infty$. Since $N^{m-m'}$ monotonically converges to 0, we can use the Stolz–Cesáro theorem. We have

$$\lim_{N \to \infty} N^{m'-m} \sum_{n=N}^{\infty} \frac{n^{m-m'}}{\lambda_0 + \theta_n} = \frac{1}{m' - m} \lim_{N \to \infty} \frac{N+1}{\lambda_0 + \theta_N} = 0,$$

by assumption, hence we see that

$$\sum_{i=1}^{N-1} |f_i| S_{N,i} \leq \left(N^{m'-m} \sum_{n=N}^{\infty} \frac{n^{m-m'}}{\lambda_0 + \theta_n} \right) \sum_{i=1}^{\infty} |f_i| i^m$$

and for the fixed λ_0 the thesis follows as above. The general case $\lambda > \omega$, follows from this and the standard resolvent identity $R(\lambda, K_m) = R(\lambda_0, K_m) + (\lambda_0 - \lambda) R(\lambda_0, K_m) R(\lambda, K_m)$.

ad 4.) By (16), $g_n \leq C(a_n + d_n)$ for large n and some $C > 0$, hence (12) holds and thus also the thesis of 2. Moreover, (16) implies

$$D(A_m) \cap D(D_m^0) \subset D(G_m^0) \subset D(T_m)$$

and hence, by 2.), $D(K_m) \subset D(T_m)$. Since K_m is an extension of $(T_m + G_m^-, D(T_m))$, we see that $K_m = T_m + G_m^-$, but then we also have $K_m = A_m + D_m^0 + G_m^0 + G_m^-$. Further, since $(T_m, D(T_m))$ is a diagonal operator, it generates an analytic semigroup and hence $(K_m, D(T_m))$ also generates an analytic semigroup by the Arendt–Rhandi theorem, [3].

Now, the assumption on g_n allows for a simpler proof of the compactness without the need for (15). By virtue of the above and [9, Theorem 4.3], $I - G_m^- R(\lambda, T_m)$ is invertible and

$$R(\lambda, T_m + G_m^-) = R(\lambda, T_m)[I - G_m^- R(\lambda, T_m)]^{-1}.$$

In view of the last identity, it suffices to show that $R(\lambda, T_m)$ is compact for some $\lambda > 0$. However, the latter is immediate, for by virtue of (14), $R(\lambda, T_m)$ is the uniform limit of finite rank operators. □

Remark 1 We note that (16) also allows to apply the Miyadera perturbation theorem, see e.g. [25, Theorem III.3.16]. Indeed, if (16) is satisfied, then we can find n_0 such that for $n \geq n_0 + 1$

$$\frac{(n+1)^m g_n}{n^m(g_n + \mathsf{a}_n)} \leq q < 1$$

and then $\omega > 0$ such that

$$\max_{1 \leq n \leq n_0} \frac{(n+1)^m g_n}{n^m(g_n + \mathsf{a}_n + \omega)} \leq q < 1.$$

Since the generation for $T_m + G_m^-$ is equivalent to that for $T_m + G_m^- - \omega I$, [9, Lemma 4.15], the Miyadera condition for $T_m + G_m^- - \omega I$ and $f \in D(T_m)_+$ reads

$$\int_0^\delta \|G_m^- G_{T_m}(t)f\|_{[m]} dt = \sum_{n=1}^\infty \frac{(n+1)^m g_n(1 - e^{-(g_n + \mathsf{a}_n + \omega)\delta})}{n^m(g_n + \mathsf{a}_n + \omega)} n^m f_n$$

$$\leq \sum_{n=1}^{n_0} \frac{(n+1)^m g_n}{n^m(g_n + \mathsf{a}_n + \omega)} n^m f_n + \sum_{n=n_0+1}^\infty \frac{(n+1)^m g_n}{n^m(g_n + \mathsf{a}_n)} n^m f_n$$

$$\leq q\|f\|_{[m]}.$$

At the same time, if $\mathsf{a}_n/g_n \to 0$ as $n \to \infty$, then $g_n/(\mathsf{a}_n + g_n) \to 1$ and the above estimate is not available.

4 Growth-fragmentation equation

We introduce the following notation, see [7],

$$\Delta_n^{(m)} := n^m - b_n^{(m)} := n^m - \sum_{k=1}^{n-1} k^m b_{k,n}, \quad n \geq 2, \quad m \geq 0. \tag{29}$$

Then, for $n \geq 2$,

$$\Delta_n^{(0)} = 1 - b_1^0 \leq 0, \quad \Delta_n^{(1)} = 0, \quad \Delta_n^{(m)} \geq 0, \quad m > 1. \tag{30}$$

Further, let us recall the notation $\theta_1 = g_1$ and $\theta_n = a_n + g_n + d_n, n \geq 2$.

Theorem 2 *1. Let (12) be satisfied. If for some $m' > m > 1$*

$$\liminf_{n \to \infty} \frac{a_n}{\mathsf{a}_n} \frac{\Delta_n^{(m)}}{n^m} > \frac{m}{m'} \tag{31}$$

holds, where as before $\mathsf{a}_n = a_n + d_n$, then

$$(Y_m, D(K_m)) = (K_m + D_m^+ + B_m, D(K_m))$$
$$= \overline{(T_m + G_m^- + D_m^+ + B_m, D(T_m))} \tag{32}$$

generates a positive semigroup in X_m. If additionally (14) and (15) are satisfied, $R(\lambda, Y_m)$ is compact for sufficiently large λ.

2. If for some $m > 1$

$$\liminf_{n \to \infty} \frac{a_n}{\theta_n} \frac{\Delta_n^{(m)}}{n^m} > 0 \tag{33}$$

holds, then

$$(U_m, D(T_m)) = (A_m + G_m + D_m + B_m, D(T_m)) = (Y_m, D(Y_m)), \quad (34)$$

where $D(T_m) = D(A_m + G_m^0 + D_m^0)$, generates a positive, analytic semigroup in X_k for any $k \geq m$.

Proof ad 1.) Using the same ideas as in the proof of Theorem 1, part 1, for the full operator with the aid of $f \in D(K_m)_+ \subset D(A_m)_+ \cap D(D_m^0)_+$ and (13), we obtain

$$\sum_{n=1}^{\infty} n^m [(K_m + D_m^+ + B_m)f]_n = \sum_{n=1}^{\infty} n^m [(A_m + D_m + B_m)f]_n + \sum_{n=1}^{\infty} n^m [G_m f]_n.$$

Now, using the convention that $g_0 f_0 = 0$,

$$\sum_{n=1}^{\infty} n^m [G_m f]_n = \lim_{l \to \infty} \sum_{n=1}^{l} n^m (g_{n-1} f_{n-1} - g_n f_n)$$

$$= \lim_{l \to \infty} \left(\sum_{n=1}^{l} ((n+1)^m - n^m) g_n f_n - (l+1)^m g_{l+1} f_{l+1} \right)$$

$$= \sum_{n=1}^{\infty} ((n+1)^m - n^m) g_n f_n,$$

by the proof of Theorem 1, part 2. Hence

$$\sum_{n=1}^{\infty} n^m [(K_m + D_m^+ + B_m)f]_n = \sum_{n=1}^{\infty} \mathsf{a}_n n^m f_n \left(\left(\left(1 + \frac{1}{n} \right)^m - 1 \right) \frac{g_n}{\mathsf{a}_n} \right.$$

$$\left. - \frac{a_n}{\mathsf{a}_n} \left(1 - \frac{1}{n^m} \sum_{k=1}^{n-1} k^m b_{k,n} \right) - \frac{d_n}{\mathsf{a}_n} \left(1 - \left(1 - \frac{1}{n} \right)^m \right) \right)$$

$$=: -\sum_{n=1}^{\infty} \Lambda_n \mathsf{a}_n n^m f_n. \tag{35}$$

Thus, if $\Lambda_n \geq 0$ for large n, then there is an extension $(Y_m, D(Y_m))$ of $(K_m + D_m^+ + B_m, D(A_m) \cap D(D_m^0) \cap D(G_m))$ generating a positive semigroup. Since

$$\Lambda_n = \frac{a_n}{\mathsf{a}_n} \frac{\Delta_n^{(m)}}{n^m} + \frac{d_n}{\mathsf{a}_n} O\left(\frac{1}{n}\right) - \frac{g_n}{n\mathsf{a}_n}\left(m + O\left(\frac{1}{n}\right)\right),$$

where both $\frac{d_n}{\mathsf{a}_n} O\left(\frac{1}{n}\right)$ and $\frac{g_n}{n\mathsf{a}_n} O\left(\frac{1}{n}\right)$ converge to zero due to the boundedness of d_n/a_n and $\frac{g_n}{n\mathsf{a}_n} \leq \frac{1}{m'} + \varepsilon_n$, with $\varepsilon_n \to 0^+$ as $n \to \infty$ (see (12)), we have $\Lambda_n \geq \Lambda > 0$ for some Λ and large n, provided (31) is satisfied. We observe that, in view of (29) and the trivial inequality $\frac{a_n}{\mathsf{a}_n} \leq 1$, $\Lambda \leq 1$ so that $1 < m < m'$ is a necessary condition for (31) to hold. Hence, if (31) holds, $D(Y_m) \subset D(A_m) \cap D(D_m^0)$. Then, since $D(B_m), D(D_m^+) \subset D(A_m) \cap D(D_m^0)$ and Y_m is a restriction of the maximal operator, $D(Y_m) \subset D(G_m)$ and hence the first part of (32) is proved. To prove the second part, we note that $(K_m, D(K_m)) = \overline{(T_m + G_m^-, D(T_m))}$. Since $K_m + D_m^+ + B_m$ is the generator, it is closed and thus

$$\overline{(T_m + G_m^- + D_m^+ + B_m, D(T_m))} \subset \overline{(K_m + D_m^+ + B_m, D(K_m))}$$
$$= (K_m + D_m^+ + B_m, D(K_m)). \qquad (36)$$

On the other hand, $D(K_m) \subset D(D_m^0) \cap D(A_m) = D(D_m^+) \cap D(B_m)$, hence $D_m^+ + B_m$ is K_m-bounded by [9, Lemma 4.1 & Theorem 2.65]. Let $f \in D(K_m)$. Then $f = \lim_{n\to\infty} f_n$ with $f_n \in D(T_m)$ and $\lim_{n\to\infty} K_m f_n = \lim_{n\to\infty}(T_m + G_m^-)f_n = K_m f$. By K_m-boundedness, $((D_m^+ + B_m)f_n)_{n\in\mathbb{N}}$ converges. By (36), $T_m + G_m^- + D_m^+ + B_m$ is closable and hence

$$K_m f + D_m^+ f + B_m f = \lim_{n\to\infty}(T_m + G_m^- + D_m^+ + B_m)f_n = \overline{(T_m + G_m^- + D_m^+ + B_m)}f.$$

Thus

$$K_m + D_m^+ + B_m \subset \overline{T_m + G_m^- + D_m^+ + B_m}$$

and (32) follows.

The compactness of $R(\lambda, Y_m)$ follows from

$$R(\lambda, Y_m) = R(\lambda, K_m)[I - (B_m + D_m^+)R(\lambda, K_m)]^{-1},$$

where the second term on the right hand side is a bounded operator by $D(B_m + D_m^+) = D(A_m) \cap D(D_m^0) \supset D(K_m)$. Thus the proof of the compactness of $R(\lambda, Y_m)$ follows as in item 4.) of Theorem 1.

ad 2.) By [7, Theorem 2.1], if (33) holds for some m_0, then it holds for any $m \geq m_0$. Hence, we can fix an m for which (33) holds. Then, for $f \in D(T_m) =$

$D(A_m + G_m^0 + D_m^0) = D(A_m) \cap D(G_m^0) \cap D(D_m^0)$, we obtain

$$\sum_{n=1}^{\infty} n^m [(G_m + D_m + A_m + B_m)f]_n = \sum_{n=1}^{\infty} \theta_n n^m f_n \left(\left(\left(1 + \frac{1}{n}\right)^m - 1 \right) \frac{g_n}{\theta_n} \right.$$
$$- \frac{a_n}{\theta_n} \left(1 - \frac{1}{n^m} \sum_{k=1}^{n-1} k^m b_{k,n} \right) - \frac{d_n}{\theta_n} \left(1 - \left(1 - \frac{1}{n} \right)^m \right) \right)$$
$$=: -\sum_{n=1}^{\infty} \theta_n n^m f_n \Theta_n.$$

Then we proceed as above. Since

$$\Theta_n = \frac{a_n}{\theta_n} \frac{\Delta_n^{(m)}}{n^m} + \frac{d_n}{\theta_n} O\left(\frac{1}{n}\right) - \frac{g_n}{\theta_n} O\left(\frac{1}{n}\right),$$

where the terms $\frac{g_n}{\theta_n} O\left(\frac{1}{n}\right)$ and $\frac{d_n}{\theta_n} O\left(\frac{1}{n}\right)$ converge to zero due to the boundedness of g_n/θ_n and d_n/θ_n, $\Theta_n \geq c > 0$ for large n if and only if (33) is satisfied. Hence $(U_m, D(T_m)) := (G_m + D_m + A_m + B_m, D(A_m + G_m^0 + D_m^0))$ generates an analytic semigroup as the positive perturbation of the diagonal operator $(T_m, D(T_m))$. However, by the closedness,

$$(A_m + G_m + D_m + B_m, D(A_m + G_m^0 + D_m^0))$$
$$= (T_m + G_m^- + D_m^+ + B_m, D(T_m))$$
$$= \overline{(T_m + G_m^- + D_m^+ + B_m, D(T_m))} = (Y_m, D(Y_m)).$$

$\square$

Remark 2 We note that (33) implies that both $\frac{a_n}{\theta_n}$ and $\frac{\Delta_n^{(m)}}{n^m}$ must be bounded away from 0 and thus, in particular, (16) is satisfied so that $(G_{K_m}(t))_{t \geq 0}$ and the pure fragmentation semigroup is analytic and, provided (14) holds, compact in its own right.

5 Asynchronous exponential growth

Proposition 1 *If assumptions (14) and (33) are satisfied, then the semigroup $(G_{U_m}(t))_{t \geq 0}$ is analytic and compact.*

Proof By (34), $(G_{U_m}(t))_{t \geq 0}$ can be considered to be generated as the perturbation $U_m = T_m + (G_m^- + D_m^+ + B_m)$ so, as in the proof of Theorem 1.4, $R(\lambda, T_m)X_m \subset D(U_m)$ implies

$$R(\lambda, U_m) = R(\lambda, T_m)(I - (G_m^- + D_m^+ + B_m)R(\lambda, T_m))^{-1},$$

where, by

$$(\lambda I - T_m)R(\lambda, U_m) = (I - (G_m^- + D_m^+ + B_m)R(\lambda, T_m))^{-1},$$

the operator $(I-(G_m^-+D_m^++B_m)R(\lambda,T_m))^{-1}$ is bounded. Hence, $R(\lambda,U_m)$ is compact, provided $R(\lambda,T_m)$ is compact and that was proved in Theorem 1.4. Since $(G_{U_m}(t))_{t\geq 0}$ is analytic, its compactness follows from [25, Theorem II.4.29]. $\qquad\square$

Proposition 2 *The semigroup $(G_{U_m}(t))_{t\geq 0}$ is irreducible if and only if $g_n > 0$ for all $n \geq 1$.*

Proof Necessity. If $g_{n_0} = 0$ for some $n_0 \geq 1$, then the closed n_0-dimensional subspace of X_m, spanned by the canonical vectros $e_n = (\delta_{i,n})_{i\geq 1}$, $1 \leq n \leq n_0$ is invariant under the action of the generator $(U_m, D(T_m))$ and hence the semigroup $(G_{U_m}(t))_{t\geq 0}$ is not irreducible.

Sufficiency. By [16, Proposition 7.6] it suffices to show that $R(\lambda, U_m)e_n > 0$ for all $n \geq 1$ and for some fixed $\lambda > s(U_m)$. To simplify the calculations, we use the representation

$$(U_m, D(T_m)) = (K_m + D_m^+ + B_m, D(T_m)) =: (K_m + \mathsf{B}_m, D(T_m)),$$

(see (34), corresponding to (5) and (6)), which combined with the resolvent formula from [9, Proposition 9.29] (compare (18)), gives

$$R(\lambda, U_m)f = \sum_{k=0}^{\infty} R(\lambda, K_m)[\mathsf{B}_m R(\lambda, K_m)]^k f, \quad f \in X_m, \quad \lambda > s(U_m). \quad (37)$$

Let $n_0 \geq 1$ be fixed. From (37) and (11), we infer

$$\left[R(\lambda, U_m)e_{n_0}\right]_n \geq \left[R(\lambda, K_m)(I + \mathsf{B}_m R(\lambda, K_m))e_{n_0}\right]_n$$
$$= \sum_{i=1}^{n} \frac{1}{\lambda + \theta_n} \prod_{k=i}^{n-1} \frac{g_k}{\lambda + \theta_k} \left(\delta_{i,n_0} + \sum_{j=i+1}^{\infty} \mathsf{a}_j \mathsf{b}_{i,j} \left(\sum_{s=1}^{j} \frac{\delta_{s,n_0}}{\lambda + \theta_j} \prod_{l=s}^{j-1} \frac{g_l}{\lambda + \theta_l}\right)\right).$$

In view of our assumptions, the last formula indicates that $\left[R(\lambda, U_m)e_{n_0}\right]_n > 0$, provided $n \geq n_0$. Moreover, for $n_0 > 1$,

$$\left[R(\lambda, K_m)\mathsf{B}_m R(\lambda, K_m))e_{n_0}\right]_{n_0-1}$$
$$= \sum_{i=1}^{n_0-1} \frac{1}{\lambda + \theta_{n_0-1}} \prod_{k=i}^{n_0-2} \frac{g_k}{\lambda + \theta_k} \left(\sum_{j=i+1}^{\infty} \mathsf{a}_j \mathsf{b}_{i,j} \frac{1}{\lambda + \theta_j} \prod_{l=n_0}^{j-1} \frac{g_l}{\lambda + \theta_l}\right) > 0 \quad (38)$$

for, if not, then
$$\mathsf{b}_{i,n_0} = 0, \quad 1 \leq i \leq n_0 - 1,$$

as all other terms and multipliers are positive. This contradicts (7) that requires

$$\sum_{i=1}^{n_0-1} i \mathsf{b}_{i,n_0} = n_0 - \frac{d_{n_0}}{a_{n_0} + d_{n_0}} > 0, \qquad n_0 > 1.$$

Thus, if $n_0 = 2$, then $\left[R(\lambda, U_m)e_{n_0}\right]_n > 0$ for $n \geq 1$. If $n_0 > 2$, then we consider the third term in (37), evaluated at e_{n_0},

$$
\begin{aligned}
R(\lambda, K_m)[\mathsf{B}_m R(\lambda, K_m)]^2 e_{n_0} &= R(\lambda, K_m)\mathsf{B}_m[R(\lambda, K_m)\mathsf{B}_m R(\lambda, K_m)e_{n_0}] \\
&= R(\lambda, K_m)\mathsf{B}_m \Phi,
\end{aligned}
$$

where $\Phi = (\phi_n)_{n \geq 1}$ is a sequence with $\phi_n > 0$ for $n \geq n_0 - 1$. Since in the proof of (38) we only used the fact that $R(\lambda, K_m)\mathsf{B}_m$ acted on a sequence with a positive n_0 entry, we see in the same way that

$$
[R(\lambda, K_m)[\mathsf{B}_m R(\lambda, K_m)]^2 e_{n_0}]_{n_0 - 2} > 0.
$$

Repeating the argument, for $n_0 > k$ we have

$$
\left[R(\lambda, K_m)[B_m R(\lambda, K_m)]^k e_{n_0}\right]_{n_0 - k} > 0, \quad 1 \leq k \leq n_0 - 1
$$

and hence $R(\lambda, U_m)e_{n_0} > 0$. Since any $f \in X_{m,+}$, $f \neq 0$, is bounded from below by a finite linear combination of elements from $\{e_n\}_{n \geq 1}$, we conclude that $R(\lambda, U_m)$, and hence $(G_{U_m}(t))_{t \geq 0}$, are irreducible. $\square$

Thus [26, Theorem VI.3.5] yields the following result.

Theorem 3 *Assume that $g_n > 0$, $n \geq 1$ and (14) and (33) are satisfied. Then there exist a strictly positive $e \in X_m$, a strictly positive $h \in X_m^*$, $M \geq 1$ and $\epsilon > 0$ such that for any $f^{in} \in X_n$ and $t \geq 0$*

$$
\|e^{-s(U_m)t}G_{U_m}(t)f^{in} - \langle h, f^{in}\rangle e\|_{[m]} \leq Me^{-\epsilon t}. \tag{39}
$$

6 Examples

Example 1 To illustrate the above result, consider the growth-fragmentation problem

$$
\begin{aligned}
\frac{df_1}{dt} &= -g_1 f_1 + \sum_{i=2}^{\infty} a_i b_{1,i} f_i, \\
\frac{df_n}{dt} &= g_{n-1} f_{n-1} - (a_n + g_n) f_n, \quad n \geq 2 \\
f_n(0) &= f_n^{in}, \quad n \geq 1,
\end{aligned} \tag{40}
$$

where

$$
b_{n,i} = \begin{cases} i & \text{for} \quad n = 1, \\ 0 & \text{otherwise}; \end{cases}
$$

that is, any particle breaks down into monomers. Since $d_n = 0$ for all n, we take any unbounded $(a_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ satisfying (14) and

$$
\gamma a_n \leq g_n \leq g a_n, \quad n \geq 2 \tag{41}
$$

for some $\gamma \leq g$. We note that in this settings $\Delta_n^{(m)} = n^m - n$ and (33) holds for any $m > 1$. Hence, the semigroup $(G_{U_m}(t))_{t \geq 0}$ that solves (40) is analytic and compact in X_m for any $m > 1$ and Theorem 3 holds. Moreover,

$$\lambda f_1 = -g_1 f_1 + \sum_{i=2}^{\infty} a_i b_{1,i} f_i,$$

$$\lambda f_n = g_{n-1} f_{n-1} - (a_n + g_n) f_n, \quad n \geq 2 \tag{42}$$

can be explicitly solved. Indeed, let $\lambda \geq 0$ and, starting from the second equation, we get

$$f_{n,\lambda} = \frac{g_1 f_{1,\lambda}}{\lambda + g_n + a_n} \prod_{j=2}^{n-1} \frac{g_j}{\lambda + g_j + a_j}, \quad n \geq 2$$

and

$$\sum_{n=2}^{\infty} a_n b_{1,n} f_{n,\lambda} = g_1 f_{1,\lambda} \sum_{n=2}^{\infty} \frac{a_n n}{\lambda + a_n + g_n} \prod_{j=2}^{n-1} \frac{g_j}{\lambda + g_j + a_j}.$$

Now, by (41) (see (16)), $\frac{g_j}{\lambda + g_j + a_j} \leq c = \frac{g}{1+g} < 1$ and thus,

$$\sum_{n=2}^{\infty} \frac{a_n n}{\lambda + a_n + g_n} \prod_{j=2}^{n-1} \frac{g_j}{\lambda + g_j + a_j} \leq \sum_{n=2}^{\infty} n c^{n-2} < \infty. \tag{43}$$

Hence, after dividing by $g_1 f_{1,\lambda} \neq 0$, the first equation in (42) takes the form

$$\psi(\lambda) := \frac{\lambda + g_1}{g_1} = \sum_{n=2}^{\infty} \frac{a_n n}{\lambda + a_n + g_n} \prod_{j=2}^{n-1} \frac{g_j}{\lambda + g_j + a_j} =: \phi(\lambda)$$

and the eigenvalue problem reduces to the algebraic equation $\psi(\lambda) = \phi(\lambda)$. By (43), the series defining ϕ is uniformly convergent on $[0, \infty)$, hence ϕ is continuous there and

$$\phi(0) = \sum_{n=2}^{\infty} \frac{a_n n}{g_n} \prod_{j=2}^{n} \frac{g_j}{g_j + a_j}.$$

Using (41), we have, for $q = \frac{\gamma}{1+\gamma}$,

$$\phi(0) \geq \frac{1}{g} \sum_{n=2}^{\infty} n q^{n-1} = \frac{1}{g} \frac{d}{dq} \sum_{n=2}^{\infty} q^n = \frac{1}{g} \frac{d}{dq} \frac{q^2}{1-q} = \frac{1}{g} \frac{2q - q^2}{(1-q)^2} = \frac{1}{g} \left(\frac{1}{(1-q)^2} - 1 \right);$$

that is,

$$\phi(0) \geq \frac{(\gamma + 1)^2 - 1}{g} > 1$$

provided

$$g + 1 < (\gamma + 1)^2 \leq (g + 1)^2, \tag{44}$$

where the second inequality follows from $\gamma \leq g$, implied by (41). We see that, in particular, if $\gamma = g$; that is, $g_n = ga_n$, then (44) is satisfied. Also, $\lim_{\lambda \to \infty} \phi(\lambda) = 0$. On the other hand, $\psi(0) = 1$ and $\lim_{\lambda \to \infty} \psi(\lambda) = +\infty$. Since ϕ is decreasing and ψ is increasing, there is exactly one $\lambda_0 > 0$ for which (42) has a solution (with arbitrary f_1 that can be set to 1). Moreover, we see that

$$\sum_{n=1}^{\infty} a_n n^m f_{n,\lambda} = g_1 \sum_{n=2}^{\infty} \frac{a_n n^m}{\lambda + a_n + g_n} \prod_{j=2}^{n-1} \frac{g_j}{\lambda + g_j + a_j} \leq g_1 \sum_{n=2}^{\infty} n^m c^{n-2} < \infty,$$

and thus $f_{\lambda_0} = (f_{n,\lambda_0})_{n \in \mathbb{N}}$ is the Perron eigenvector of the generator U_m.

Example 2 The dominant eigenvalue λ_0 can be explicitly found in certain cases. Let us consider general problem (1) with $g_n = rn, d_n = 0$ for all $n \in \mathbb{N}$ and some $r > 0$ and with other coefficients satisfying the assumptions of Theorem 3. Let $f_\lambda = (f_{n,\lambda})_{n \in \mathbb{N}} \in D(U_m)$ satisfy

$$\lambda f_{1,\lambda} = -r f_{1,\lambda} + \sum_{i=2}^{\infty} a_i b_{1,i} f_{i,\lambda},$$

$$\lambda f_{n,\lambda} = r(n-1) f_{n-1,\lambda} - (a_n + rn) f_{n,\lambda} + \sum_{i=n+1}^{\infty} a_i b_{n,i} f_{i,\lambda}, \quad n \geq 2. \quad (45)$$

Multiplying the n-th equation by n and summing them, we obtain

$$\lambda \sum_{n=1}^{\infty} n f_{n,\lambda} = r \sum_{n=1}^{\infty} n f_{n,\lambda}.$$

The above is satisfied if either $\lambda = r$ or $\sum_{n=1}^{\infty} n f_{n,\lambda} = 0$. Since we know that the Perron eigenvector must be positive, we obtain that $\lambda_0 = r$ is the Perron eigenvalue. As a byproduct, we see that any eigenvector f_λ belonging to an eigenvalue $\lambda \neq r$ must satisfy $\sum_{n=1}^{\infty} n f_{n,\lambda} = 0$.

To conclude, let us consider the transposed matrix

$$\mathcal{U}^T = \begin{pmatrix} -g_1 & g_1 & 0 & 0 & 0 & \ldots \\ a_2 b_{1,2} & -(g_2 + a_2) & g_2 & 0 & 0 & \ldots \\ a_3 b_{1,3} & a_3 b_{2,3} & -g_3 + a_3 & g_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n b_{1,n} & a_n b_{2,n} & \ldots & -(g_n + a_n) & g_n & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Let U_m^* be the adjoint to U_m acting in $X_m^* = \{(v_n)_{n \in \mathbb{N}} ; \sup_{n \in \mathbb{N}} n^{-m} |v_n| < \infty\}$ and let $f^* \in D(U_m^*)$. Then, by definition

$$\langle U_m^* f^*, f \rangle = \langle f^*, U_m f \rangle, \quad f \in D(U_m).$$

Taking $f = (\delta_{n,N})_{n \in \mathbb{N}}$, we see that

$$[U_m^* f^*]_N = \langle f^*, U_m f \rangle = \sum_{n=1}^{N-1} f_n^* a_N b_{n,N} - (g_N + a_N) f_N^* + g_N f_{N+1}^* = [\mathcal{U}^T f^*]_N$$

hence U_m^* is a restriction of $\mathcal{U}^T$ to $D(U_m^*) \subset D(U_{m,\max}^*) = \{f \in X_m^* : \mathcal{U}^T f \in X_m^*\}$. On the other hand, let $f^* \in D(U_{m,\max}^*)$, $f \in D(U_m)$. Then, since $D(U_m)$ is a weighted l^1 space, $\bigcup_{N=1}^{\infty} P_N D(U_m)$, where P_N is the projection defined in (26), is a core for U_m. Using the fact that $U_m P_N D(U_m)$ is finite dimensional, for each N

$$\langle \mathcal{U}^T f^*, P_N f \rangle = \langle f^*, U_m P_N f \rangle = \langle U_m^* f^*, P_N f \rangle$$

and hence, passing to the limit with $N \to \infty$, $f^* \in D(U_m^*)$. Thus $U_m^* = \mathcal{U}^T$ with $D(U_m^*) = D(U_{m,\max}^*)$.

Using the assumption that $g_n = rn$, we see that $h = (1, 2, \ldots, n, \ldots) \in D(U_m^*)$ for any $m \geq 1$ and

$$U_m^* h = rh.$$

Thus, by Theorem 3,

$$U_m(t) f^{in} = e^{rt} \left(\sum_{n=1}^{\infty} n f_n^{in} \right) e + O(e^{r't})$$

for some $r' < r$, where e is the Perron eigenvector with unit mass; that is $e = f_{\lambda_0} / \sum_{n=1}^{\infty} n f_{n,\lambda_0}$.

To illustrate the formulas derived in the last two examples, we let $m = 2$, $r = 1$, $a_n = 2n$, $f_n^{in} = \delta_{n,10} 10$ and integrate (40) numerically in the time interval $t \in [0, 20]$. As evident from Fig. 1, the solution $f(t)$ very quickly settles to its asymptotic limit $\langle h, f^{in} \rangle e$ (see the top-right diagram), while in complete agreement with Theorem 3, the deviation $\| e^{-rt} G_{U_m}(t) f^{in} - \langle h, f^{in} \rangle e \|_{[m]}$ decreases exponentially as t increases (see the bottom-left diagram).

A crucial role in the analysis is played by (33). It ensures that most of the mass of the daughter particles is concentrated in smaller particles, [7]. A large class of fragmentation kernels, that can be considered to be a discrete equivalent of the homogeneous kernels in continuous fragmentation, satisfying (33) is presented in the next example.

Example 3 Assume that $b_{k,n}$ can be written as

$$b_{k,n} = \zeta(n) h\left(\frac{k}{n}\right), \quad 1 \leq k \leq n-1, \quad n \in \mathbb{N}, \tag{46}$$

where h is a Riemann integrable function on $[0, 1]$ and $\zeta(n)$ is an appropriate sequence that ensures that (3) is satisfied. By (3), we have

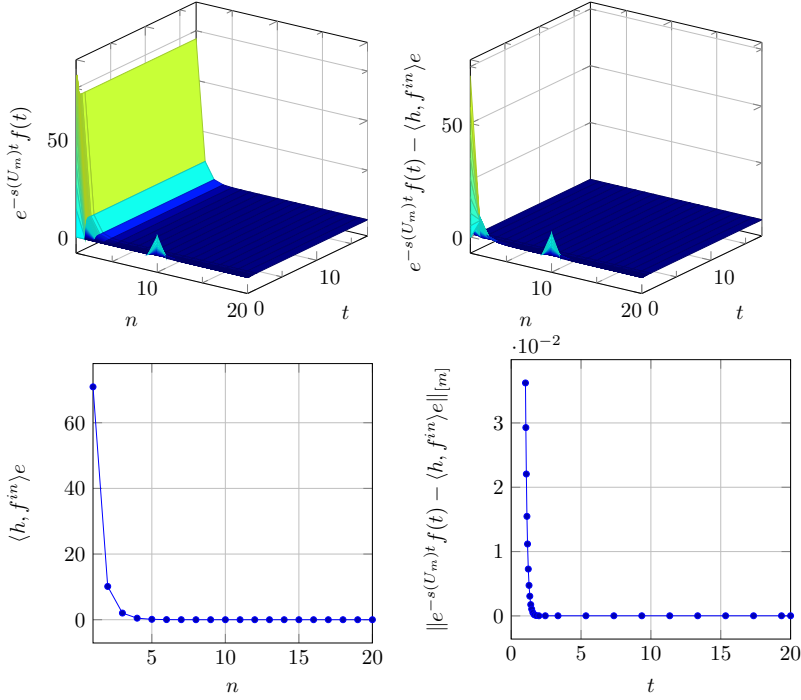$$1 = \zeta(n)(n-1) \sum_{k=1}^{n-1} \frac{k}{n} h\left(\frac{k}{n}\right) \frac{1}{n-1}.$$

Fig. 1 The long time behavior of (40). The semigroup solution $f(t) = G_{U_m}(t)f^{in}$ of (40) (top-left); the asymptotic error $e^{-s(U_m)t}f(t) - \langle h, f^{in}\rangle e$ (top-right); the asymptotic mass distribution $\langle h, f^{in}\rangle e$ (bottom-left) and the evolution of the asymptotic error $\|e^{-s(U_m)t}f(t) - \langle h, f^{in}\rangle e\|_{[m]}$, for $t \geq 1$ (bottom-right).

Since

$$\frac{k-1}{n-1} \leq \frac{k}{n} \leq \frac{k}{n-1}$$

for $1 \leq k \leq n$, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n} h\left(\frac{k}{n}\right) \frac{1}{n-1} = \int_0^1 zh(z)dz$$

and thus

$$\lim_{n\to\infty}(n-1)\zeta(n) = \frac{1}{\int_0^1 zh(z)dz}.$$

Therefore

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} \left(\frac{k}{n}\right)^p b_{k,n} = \lim_{n\to\infty} \zeta(n)(n-1) \sum_{k=1}^{n-1} \left(\frac{k}{n}\right)^p h\left(\frac{k}{n}\right) \frac{1}{n-1}$$

$$= \frac{\int_0^1 z^p h(z)dz}{\int_0^1 zh(z)dz} < 1.$$

Thus

$$\liminf_{n\to\infty} \frac{\Delta_n^{(p)}}{n^p} = \lim_{n\to\infty} \frac{n^p - \sum\limits_{k=1}^{n-1} k^p b_{k,n}}{n^p} > 0$$

and hence (33) is satisfied.

We note that (46) is obviously satisfied by the binary uniform fragmentation

$$b_{n,i} = \frac{2}{i-1}, \quad n = 1, \ldots, i-1.$$

Another example is offered by the binary fragmentation written in terms of a symmetric infinite matrix $(\psi_{i,j})_{i,j\geq 1}$ as

$$\frac{df_n}{dt} = -\frac{1}{2} f_n \sum_{i=1}^{n-1} \psi_{i,n-i} + \sum_{i=n+1}^{\infty} \psi_{n,i-n} f_i, \quad n \geq 1,$$

see [5,17,44]. Translating into our notation, we get

$$b_{n,i} = \frac{\psi_{n,i-n}}{a_i}, \quad a_n = \frac{1}{2} \sum_{i=1}^{n-1} \psi_{i,n-i}, \quad i \geq 2, \quad 1 \leq n \leq i-1.$$

Typical cases in the polymer degradation are $\psi_{i,j} = (i+j)^\beta$ and $\psi_{i,j} = (ij)^\beta$. The first case gives $a_n = \frac{1}{2} n^\beta (n-1)$ and $b_{n,i} = \frac{2}{i-1}$ and hence it is a uniform binary fragmentation (see the long time behavior of $G_{U_m}(t) f^{in}$, with $m = 2$, $\beta = \frac{1}{10}$, $g_n = d_n = n^{1+\beta}$ and $f_n^{in} = \delta_{10,n} 10$, in Fig. 2). In the second case, we have

$$b_{n,i} = \frac{n^\beta (i-n)^\beta}{a_i} = \frac{i^{2\beta}}{a_i} \left(\frac{n}{i}\right)^\beta \left(1 - \frac{n}{i}\right)^\beta$$

and (46) is satisfied with $\zeta(n) = \frac{n^{2\beta}}{a_n}$ and $h(z) = z^\beta (1-z)^\beta$.

A typical qualitative behavior of $G_{U_m}(t) f^{in}$, with $m = 2$, $\beta = \frac{1}{10}$, $d_n = g_n = n^{1+\beta}$ and $f_n^{in} = \delta_{10,n} 10$, is shown in Fig. 3.

Example 4 On the other hand, the fragmentation process given by

$$\begin{aligned} b_{1,2} = 2, \quad &\text{and} \quad b_{1,i} = b_{i-1,i} = 1, \\ b_{n,i} = 0, \quad &i \geq 2, \quad 2 \leq n \leq i-2, \end{aligned} \tag{47}$$

obviously does not satisfy (33) and, in fact, the corresponding semigroup is neither analytic, nor compact, see [7].
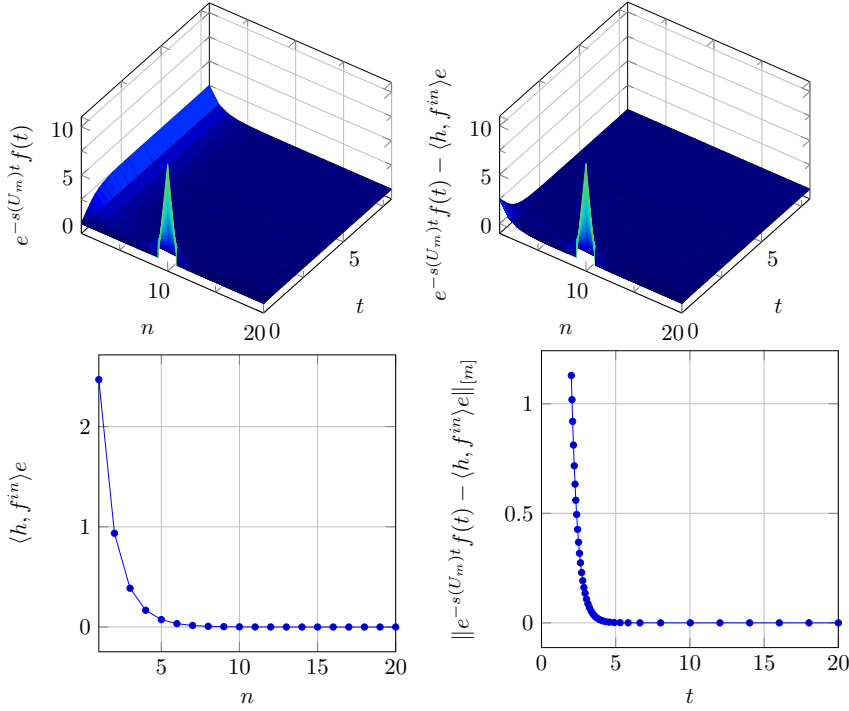
Fig. 2 The long time behavior of (1), $\psi_{i,j} = (i+j)^{\beta}$. The semigroup solution $f(t) = G_{U_m}(t)f^{in}$ of (1) (top-left); the asymptotic error $e^{-s(U_m)t}f(t) - \langle h, f^{in}\rangle e$ (top-right); the asymptotic mass distribution $\langle h, f^{in}\rangle e$ (bottom-left) and the evolution of the asymptotic error $\|e^{-s(U_m)t}f(t) - \langle h, f^{in}\rangle e\|_{[m]}$, for $t \geq 1$ (bottom-right).

7 Appendix: an alternative view at the model

In Theorem 1, we have seen a regularizing role played by the diagonal operator induced by $\mathcal{A}$ even in the case not involving the full fragmentation operator. In many applications, however, (1) models a combination of two independent processes – the birth-and-death process and the fragmentation process and it is important to investigate when they exist irrespective of each other. In other words, we consider (1) as

$$\frac{d}{dt}f = \mathcal{G}f + \mathcal{D}f + \mathcal{F}f, \qquad f(0) = f^{in}. \tag{48}$$

The pure birth-and-death problem

$$\frac{d}{dt}f = \mathcal{V}f = \mathcal{G}f + \mathcal{D}f, \qquad f(0) = f^{in} \tag{49}$$

has been extensively analysed in the space X_0, see e.g. [9, Chapter 7]. Its behaviour in X_m creates, however, unexpected challenges.
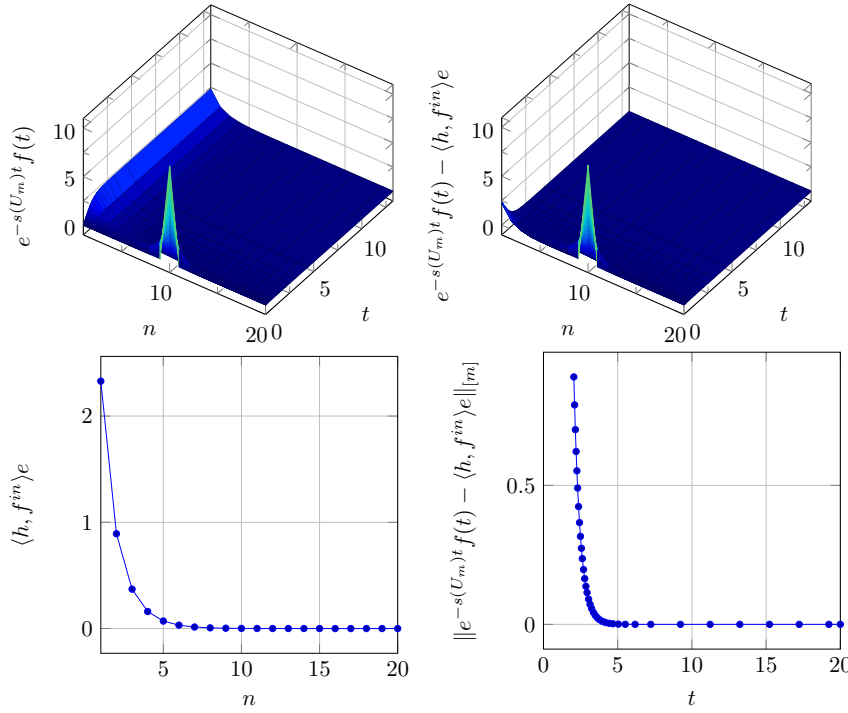
Fig. 3 The long time behavior of (1), $\psi_{i,j} = (ij)^\beta$. The semigroup solution $f(t) = G_{U_m}(t)f^{in}$ of (1) (top-left); the asymptotic error $e^{-s(U_m)t}f(t) - \langle h, f^{in}\rangle e$ (top-right); the asymptotic mass distribution $\langle h, f^{in}\rangle e$ (bottom-left) and the evolution of the asymptotic error $\|e^{-s(U_m)t}f(t) - \langle h, f^{in}\rangle e\|_{[m]}$, for $t \geq 1$ (bottom-right).

Example 5 If there is C such that

$$g_n \leq Cn, \quad n \geq 1, \tag{50}$$

then there is a realization of the growth expression $\mathcal{G}$ that generates a C_0-semigroup in X_m. Indeed, this again follows from the Kato–Voigt theorem. We consider $\mathcal{G}$ as the perturbation of $\mathcal{G}^0$ by $\mathcal{G}^-$; that is, we introduce $G_m^0 = \mathcal{G}^0|_{D(G_m^0)}$, with

$$D(G_m^0) = \{f \in X_m; \mathcal{G}^0 f \in X_m\}.$$

Then, as in (17), for $f \in D(G_m^0)$,

$$\sum_{n=1}^{\infty} n^m [(G_m^0 + G_m^-)f]_n = \sum_{n=0}^{\infty} n^m f_n \left(g_n \frac{(n+1)^m - n^m}{n^m}\right) \leq C'\|f\|_{[m]}, \tag{51}$$

for some constant C'. Hence, there is an extension of $G_m^0 + G_m^-$ generating a C_0-semigroup in X_m. On the other hand, if for some $c, C > 0$

$$cn^q \leq g_n \leq Cn^q, \quad n \geq 1, q > 1, \tag{52}$$

then there is no realisation of $\mathcal{G}$ with resolvent bounded in X_m with $q \leq m+1$. Indeed, the resolvent of the generator, if it exists, must be given by (11),

$$[R_\lambda f]_n = \sum_{i=1}^{n} \frac{f_i}{\lambda + g_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + g_j}, \quad n \geq 1. \tag{53}$$

Let us fix λ. Then

$$\prod_{j=i}^{n-1} \frac{g_j}{\lambda + g_j} \geq g_\lambda := \prod_{j=1}^{\infty} \frac{g_j}{\lambda + g_j},$$

where $g_\lambda \neq 0$, and, for $f \in X_{m,+}$,

$$\|R_\lambda f\|_{[m]} = \sum_{n=1}^{\infty} n^m \sum_{i=1}^{n} \frac{f_i}{\lambda + g_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + g_j} = \sum_{i=1}^{\infty} f_i \sum_{n=i}^{\infty} \frac{n^m}{\lambda + g_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + g_j}$$

$$\geq g_\lambda \sum_{i=1}^{\infty} f_i \sum_{n=i}^{\infty} \frac{n^m}{(\lambda + g_n)} \geq g_\lambda C^{-1} \sum_{i=1}^{\infty} f_i \sum_{n=i}^{\infty} \frac{1}{n^{q-m}}.$$

Hence R_λ is not bounded if (52) is satisfied and hence, in particular, there is no realisation of $\mathcal{G}$ generating a C_0-semigroup in X_m. We note that for $q = 2$ and $m = 1$ we have a discrete version of the nonexistence result obtained in [12, Remark 2].

Let us return to the full birth-and-death model (48). As before, we introduce $V_m^0 + V_m^1 := G_m^0 + D_m^0 + G_m^- + D_m^+$ on

$$D(V_m^0) = \{f \in X_m;\ (\mathcal{G}^0 + \mathcal{D}^0)f \in X_m\}.$$

We have

Theorem 4 *1. If*

$$\limsup_{n \to \infty} \Gamma_n \leq C \tag{54}$$

for some constant $C \in \mathbb{R}$, where

$$\Gamma_n = g_n \left(\left(1 + \frac{1}{n}\right)^m - 1 \right) - d_n \left(1 - \left(1 - \frac{1}{n}\right)^m \right),$$

then there is an extension V_m of $V_m^0 + V_m^1$ that generates a quasicontractive semigroup $(G_{V_m}(t))_{t \geq 0}$ on X_m.
2. Condition (54) is satisfied if either
    a) (50) holds, or
    b) $\limsup\limits_{n \to \infty} \dfrac{d_n}{g_n} > 1$ and $d_n = O(n^2)$, or
    c) $\dfrac{d_n}{g_n} \geq 1 + \dfrac{m'-1}{n}$ for sufficiently large n and $m' > m$.
3. If any of the conditions of point 2. is satisfied, then $V_m = \overline{V_m^0 + V_m^1}$.

Proof Statement 1. of the theorem follows in a standard way as an application of the Kato–Voigt theorem as for $f \in D(V_m^0)_+$ we have

$$\sum_{n=1}^{\infty} n^m [(G_m^0 + D_m^0 + G_m^- + D_m^+)f]_n = \sum_{n=1}^{\infty} n^m f_n \Gamma_n.$$

For statement 2. we observe that

$$\left(1 + \frac{1}{n}\right)^m - 1 = \frac{m}{n} + \frac{m(m-1)}{2n^2} + O\left(\frac{1}{n^3}\right),$$
$$1 - \left(1 - \frac{1}{n}\right)^m = \frac{m}{n} - \frac{m(m-1)}{2n^2} + O\left(\frac{1}{n^3}\right).$$

Thus, if 2a) is satisfied, then the positive part of Γ_n is bounded. If 2b) is satisfied, then

$$\Gamma_n \le d_n \left(\frac{m(m-1)}{n^2} + O\left(\frac{1}{n^3}\right)\right)$$

for sufficiently large n and hence again Γ_n is bounded from above. Finally, if 2c) is satisfied, then

$$\Gamma_n \le g_n \left(\left(\frac{m(m-1)}{n^2} + O\left(\frac{1}{n^3}\right)\right) - \frac{m'-1}{n}\left(\frac{m}{n} - \frac{m(m-1)}{2n^2} + O\left(\frac{1}{n^3}\right)\right)\right)$$
$$= \frac{g_n}{n^2}\left(m(m-m') + O\left(\frac{1}{n}\right)\right)$$

and hence Γ_n is negative for large n and thus also bounded from above.

To prove the last statement, we use the approach of [9, Theorem 7.11], based on the extension technique, see [9, Theorem 6.22]. Let $f \in D(V_m)_+$. Then

$$\sum_{n=1}^{\infty} n^m (-(g_n + d_n)f_n + g_{n-1}f_{n-1} + d_{n+1}f_{n+1})$$
$$= \sum_{k=1}^{\infty} k^m f_k \Gamma_k + \lim_{n\to\infty} (-g_n f_n + d_{n+1}f_{n+1})n^m, \qquad (55)$$

where the limit exists. For honesty, it suffices to prove that for any $f \in D(V_m)_+$

$$\lim_{n\to\infty} (-g_n f_n + d_{n+1}f_{n+1})n^m \ge 0.$$

Assume, to the contrary, that for some $0 \le f \in D(V_m)_+$, the limit is negative so that there exists $b > 0$ such that

$$(-g_n f_n + d_{n+1}f_{n+1})n^m \le -b, \qquad (56)$$

for all $n \ge n_0$ with large enough n_0. Thus, for $n \ge n_0$ we obtain by induction that for arbitrary k

$$f_n \ge \frac{b}{g_n}\left(\sum_{i=0}^{k} \frac{1}{(n+i)^m} \prod_{j=1}^{i} \frac{d_{n+j}}{g_{n+j}}\right).$$

Because k is arbitrary, we obtain

$$f_n \geq \frac{b}{g_n} \left(\sum_{i=0}^{\infty} \frac{1}{(n+i)^m} \prod_{j=1}^{i} \frac{d_{n+j}}{g_{n+j}} \right), \quad n \geq n_0.$$

Thus, if

$$\sum_{n=1}^{\infty} \frac{n^m}{g_n} \left(\sum_{i=0}^{\infty} \frac{1}{(n+i)^m} \prod_{j=1}^{i} \frac{d_{n+j}}{g_{n+j}} \right) = +\infty \tag{57}$$

(where we put $\prod_{j=1}^{0} \cdot = 1$) is satisfied, then $\sum_{n=0}^{\infty} n^m f_n = +\infty$ which contradicts $f \in D(V_m)_+$.

Now, if (50) is satisfied, we have

$$\sum_{n=1}^{\infty} \frac{n^m}{g_n} \left(\sum_{i=0}^{\infty} \frac{1}{(n+i)^m} \prod_{j=1}^{i} \frac{d_{n+j}}{g_{n+j}} \right) \geq \sum_{n=1}^{\infty} \frac{1}{g_n} = +\infty.$$

Similarly, if assumption 2.b) is satisfied, we have

$$\sum_{n=1}^{\infty} \frac{n^m}{g_n} \left(\sum_{i=0}^{\infty} \frac{1}{(n+i)^m} \prod_{j=1}^{i} \frac{d_{n+j}}{g_{n+j}} \right) \geq \sum_{n=1}^{\infty} \frac{n^m}{g_n} \left(\sum_{i=n}^{\infty} \frac{1}{i^m} \right) \geq C \sum_{n=1}^{\infty} \frac{1}{n} = +\infty,$$

where we used the integral estimate for the inner sum. Finally, if 2.c) is satisfied, we can write

$$\sum_{n=1}^{\infty} \frac{n^m}{g_n} \left(\sum_{i=0}^{\infty} \frac{1}{(n+i)^m} \prod_{j=1}^{i} \frac{d_{n+j}}{g_{n+j}} \right) \geq \sum_{n=1}^{\infty} \frac{n^m}{g_n} \left(\sum_{i=0}^{\infty} \frac{1}{(n+i)^m} \prod_{j=1}^{i} \left(1 + \frac{m'-1}{n+j} \right) \right).$$

Now, as in the proof of Theorem 1, by the Stirling formula,

$$\prod_{j=1}^{i} \left(1 + \frac{m'-1}{n+j} \right) = \frac{\Gamma(n+i+m')\Gamma(n+1)}{\Gamma(n+m')\Gamma(n+i+1)} = O\left(\left(\frac{n+i+m'}{n+m'} \right)^{m'-1} \right)$$

and the inner series diverges if the second condition of 2.c) is satisfied. $\qquad \square$

By [7, Theorem 2.1], under standard assumptions on the fragmentation coefficients $F_m = \overline{A_m + B_m}$ generates a quasicontractive $(G_{F_m}(t))_{t \geq 0}$.

Theorem 5 *Assume the conditions of Theorem 2, item 1. and of Theorem 4, item 3. are satisfied. Then $Y_m = \overline{V_m + F_m}$ and*

$$G_{Y_m}(t)f = \lim_{n \to \infty} \left(G_{V_m}\left(\frac{t}{n} \right) G_{F_m}\left(\frac{t}{n} \right) \right)^n f, \quad f \in X_m, \tag{58}$$

uniformly on bounded time intervals.

Proof First, we observe that $D(V_m) \cap D(F_m) \supset D(G_m^0) \cap D(D_m^0) \cap D(A_m^0)$ and the latter is dense in X_m. Next, we see that

$$[\lambda I - (V_m + F_m)]D(V_m) \cap D(A_m)$$
$$\supset [\lambda I - (V_m + F_m)]D(G_m^0) \cap D(D_m^0) \cap D(A_m)$$
$$= [\lambda I - (T_m + G_m^- + D_m^+ + B_m)]D(T_m).$$

Since $\overline{(T_m + G_m^- + D_m^+ + B_m, D(T_m))}$ is the generator a semigroup, $[\lambda I - (T_m + G_m^- + D_m^+ + B_m)]D(T_m)$ is dense in X_m for sufficiently large λ. Indeed, if $f \in X_m$, then $f = (\lambda I - \overline{T_m + G_m^- + D_m^+ + B_m})u$ for some $u \in D(\overline{T_m + G_m^- + D_m^+ + B_m})$ and $u = \lim_{n \to \infty} u_n$ with $u_n \in D(T_m)$ and

$$\lim_{n \to \infty}(T_m + G_m^- + D_m^+ + B_m)u_n = \overline{T_m + G_m^- + D_m^+ + B_m}u.$$

But then $f = \lim_{n \to \infty}(\lambda u_n - (T_m + G_m^- + D_m^+ + B_m)u_n)$; that is, $f \in \overline{[\lambda I - (T_m + G_m^- + D_m^+ + B_m)]D(T_m)}$.

Since both $(G_{V_m}(t))_{t \geq 0}$ and $(G_{F_m}(t))_{t \geq 0}$ are quasicontractive, [37, Corollary 3.5.5] implies that $\overline{V_m + F_m}$ is the generator of a quasicontractive semigroup. Now

$$\lambda I - Y_m = \lambda I - \overline{T_m + G_m^- + D_m^+ + B_m}$$
$$= \lambda I - \overline{(D_m^0 + D_m^+ + G_m^- + G_m^0) + (A_m + B_m)} \subset \lambda I - \overline{V_m + F_m}$$

and, since both Y_m and $\overline{V_m + F_m}$ are generators, we must have $Y_m = \overline{V_m + F_m}$. Then (58) follows from [37, Corollary 3.5.5]. □

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