Perpendicular Lines: Four Proofs of the Negative Reciprocal Relationship

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INTRODUCTION

One of the great joys of mathematics is finding multiple ways of arriving at a solution or proving a result. Some approaches can be messy and longwinded, while others can be elegant and succinct. But the joy lies in arriving at a common endpoint, however circuitous the route may have been. In this article we explore four different ways of proving the negative reciprocal relationship between the gradients of perpendicular lines. Each proof uses elementary ideas from other topics encountered in the high school Mathematics curriculum.

STATING THE THEOREM

Suppose $y = m_1x + c_1$ and $y = m_2x + c_2$ are two straight lines with $m_1, m_2 \neq 0$. If the two lines are perpendicular to one another then $m_1 \times m_2 = -1$.

PROOF 1 – THEOREM OF PYTHAGORAS

Let L_1 and L_2 be perpendicular lines with equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively. Without loss of perpendicularity we can translate these two lines so that they both pass through the origin. Let us call these translated lines L_3 and L_4 respectively. Now draw vertical line $x = k$ $(k \neq 0)$ as illustrated in Figure 1. The vertical line cuts L_3 at $A(k; m_1k)$ and L_4 at $B(k; m_2k)$.

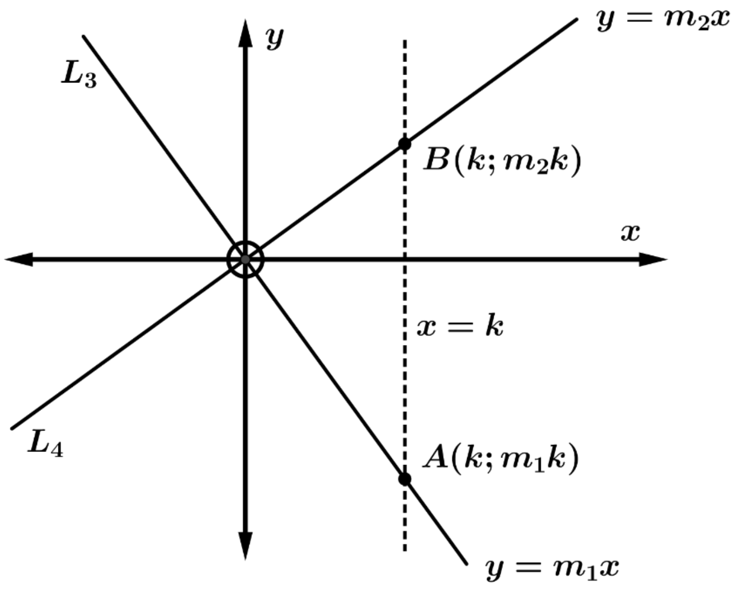

FIGURE 1: Lines $y = m_1x$, $y = m_2x$ and $x = k$.

Since lines L_3 and L_4 are perpendicular, triangle BOA is right-angled. By applying the Pythagorean theorem we obtain:

$$\begin{aligned} AB^2 &= OA^2 + OB^2 \\ (m_2k - m_1k)^2 &= (k^2 + {m_1}^2k^2) + (k^2 + {m_2}^2k^2) \\ (m_2 - m_1)^2k^2 &= 2k^2 + {m_1}^2k^2 + {m_2}^2k^2 \\ {m_2}^2k^2 - 2m_1m_2k^2 + {m_1}^2k^2 &= 2k^2 + {m_1}^2k^2 + {m_2}^2k^2 \\ -2m_1m_2k^2 &= 2k^2 \\ m_1m_2 &= -1 \end{aligned}$$

PROOF 2 – SIMILARITY

Suppose lines L_1 and L_2 intersect at point C. Let points A and B be the x-intercepts of lines L_1 and L_2 respectively. Now draw a line parallel to the y-axis passing through C and intersecting the x-axis at D as illustrated in Figure 2.

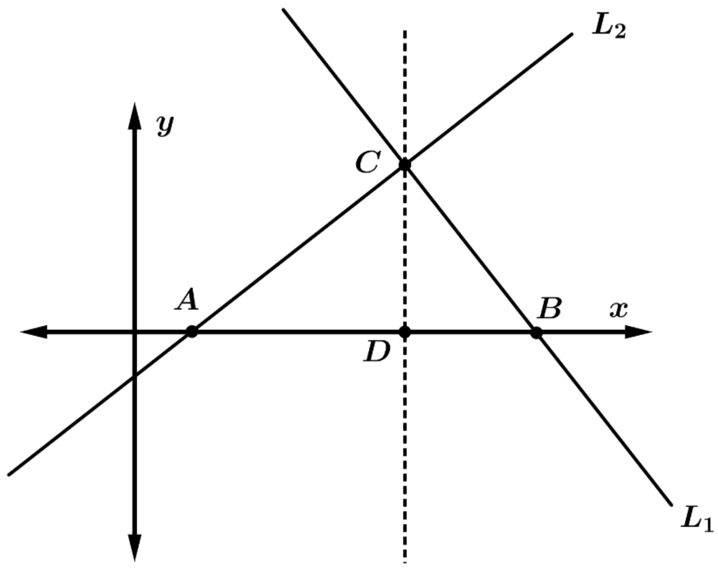

FIGURE 2: Forming similar triangles.

Noting that $A\hat{C}B = A\hat{D}C = C\hat{D}B = 90°$, it follows that triangles ADC and CDB are similar. We thus have:

$$\frac{AD}{DC} = \frac{CD}{DB} \quad \Rightarrow \quad AD.DB = CD^2$$

Using 'rise over run' we see that the gradients of lines L_1 and L_2 are $-\frac{CD}{DB}$ and $\frac{CD}{DA}$ respectively. Thus:

$$m_1 \times m_2 = -\frac{CD}{DB} \times \frac{CD}{DA} = -\frac{CD^2}{AD.DB} = -\frac{CD^2}{CD^2} = -1$$

PROOF 3 – TRIGONOMETRY

Let L_1 and L_2 be perpendicular lines with equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively. Let the angles of inclination of L_1 and L_2 be α_1 and α_2 respectively, as illustrated in Figure 3.

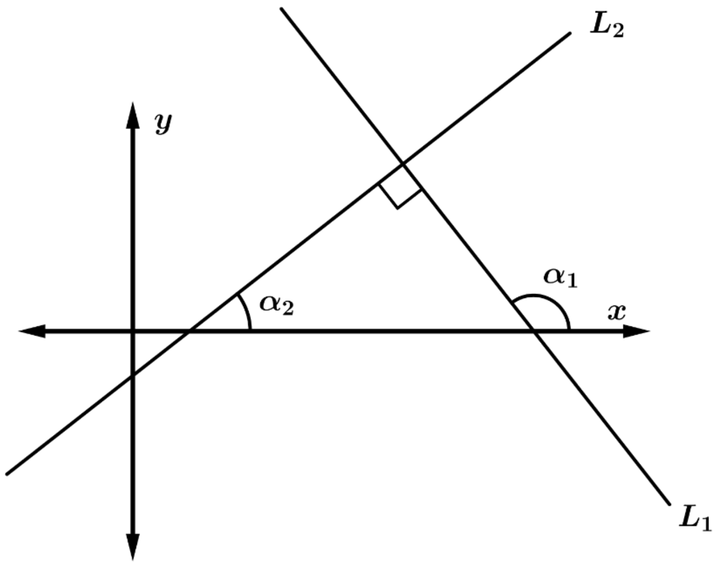

FIGURE 3: Angles of inclination.

From the angles of inclination we have $\tan\alpha_1 = m_1$ and $\tan\alpha_2 = m_2$. Note that $\alpha_1 = 90° + \alpha_2$, from which it follows that $\alpha_1 - \alpha_2 = 90°$. Using the compound angle formula for tan, we have:

$$\tan(\alpha_1 - \alpha_2) = \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1\tan\alpha_2} = \frac{m_1 - m_2}{1 + m_1m_2}$$

Since $\alpha_1 - \alpha_2 = 90°$, and $\tan 90°$ is undefined, this means that the denominator of the above fraction must be zero when the two lines are perpendicular. Thus $1 + m_1m_2 = 0$, from which it follows that $m_1m_2 = -1$.

PROOF 4 – CIRCLE GEOMETRY

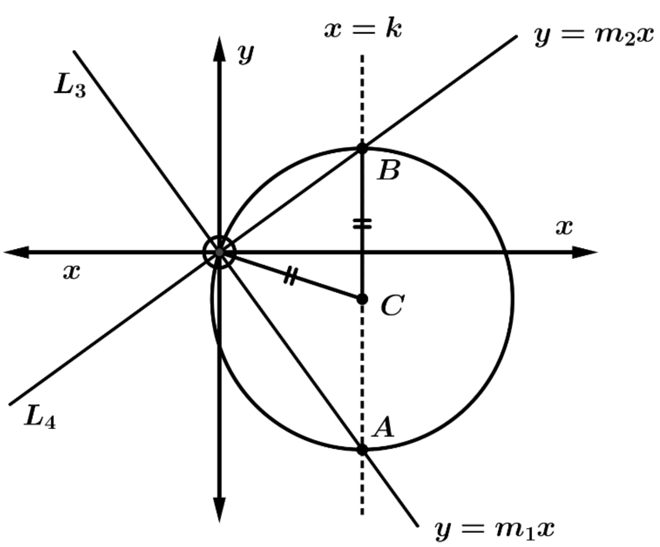

FIGURE 4: Circle with centre C and diameter AB.

Let L_1 and L_2 be perpendicular lines with equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively. Without loss of perpendicularity we can translate these two lines so that they both pass through the origin. Let us call these translated lines L_3 and L_4 respectively. Now draw vertical line $x = k$ $(k \neq 0)$ as illustrated in Figure 4. The vertical line cuts L_3 at $A(k; m_1k)$ and L_4 at $B(k; m_2k)$. The midpoint of AB is $C\left(k; \frac{m_1k + m_2k}{2}\right)$. Since the angle subtended by a semicircle is 90°, and lines L_3 and L_4 are perpendicular to one another, a circle with centre C and diameter AB will pass through the origin O. We thus have CO and CB equal in length:

$$
\begin{aligned}
CO &= CB \\
\sqrt{(k-0)^2 + \left(\frac{m_1k + m_2k}{2} - 0\right)^2} &= \sqrt{(k-k)^2 + \left(\frac{m_1k + m_2k}{2} - m_2k\right)^2} \\
\sqrt{k^2 + \frac{k^2(m_1 + m_2)^2}{4}} &= \sqrt{\left(\frac{m_1k - m_2k}{2}\right)^2} \\
k^2 + \frac{k^2(m_1 + m_2)^2}{4} &= \frac{k^2(m_1 - m_2)^2}{4} \\
4 + {m_1}^2 + 2m_1m_2 + {m_2}^2 &= {m_1}^2 - 2m_1m_2 + {m_2}^2 \\
4m_1m_2 &= -4 \\
m_1m_2 &= -1
\end{aligned}
$$

CONCLUDING COMMENTS

In this article we have explored four different ways of proving the negative reciprocal relationship between the gradients of perpendicular lines, each proof drawing on different ideas encountered in the high school Mathematics curriculum. The first proof made use of the Pythagorean theorem. The second made us of similar triangles while the third utilised the compound angle formula for tan. The final proof makes use of the fact that the angle subtended by a semicircle is 90°. In each of these proofs the role of the right-angled triangle is pivotal.

All four of these proofs could be used for classroom exploration. The educational value is twofold. Not only will students develop a deeper appreciation for the negative reciprocal relationship between the gradients of perpendicular lines, but they will be able to make connections between seemingly different parts of mathematics, thereby building a more interconnected sense of the subject.