The Hamilton Cycle Problem for Locally Traceable and Locally Hamiltonian Graphs

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Abstract

We say a graph G is locally $\mathcal{P}$ if for each vertex v in G the open neighbourhood of v induces a graph with property $\mathcal{P}$. The Hamilton Cycle Problem (HCP) is the problem of deciding whether a graph contains a Hamilton cycle. It is known that the HCP is NP-complete for locally traceable (LT) graphs with maximum degree 6. We extend that result to 1-tough graphs and to graphs with restricted degree sequences. If $\mathcal{R}$ is a set of nonnegative integers, we say a graph G is $\mathcal{R}$-regular if the degrees of all the vertices in $V(G)$ are elements of $\mathcal{R}$. We show that the HCP is NP-complete for $\mathcal{R}$-regular LT graphs if $\mathcal{R}$ is any set of natural numbers with $max(\mathcal{R}) \geq 6$, with the possible exception of $\{4, 6\}$ and $\{6\}$.

It is known that the HCP is NP-complete for LH graphs with maximum degree 10. We improve this result by showing that the HCP is NP-complete for 1-tough LH graphs with maximum degree 9 and for $\mathcal{R}$-regular LH graphs if $\mathcal{R}$ is any set of natural numbers with $\min(\mathcal{R}) = 3$ and $\max(\mathcal{R}) \in \{9, 10\}$, or if $\mathcal{R}$ is any set of natural numbers with $\max(\mathcal{R}) \geq 11$. Finally, we show that the HCP for k-connected LH graphs that are also locally $(k-1)$-connected is NP-complete for every $k \geq 3$.

Keywords: locally traceable, locally hamiltonian, Hamilton Cycle Problem

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1. Introduction

We say a graph G is *locally* $\mathcal{P}$ if for each vertex v in G the open neighbourhood of v induces a graph (denoted $\langle N(v) \rangle$) with property $\mathcal{P}$. In this paper we will be considering locally connected (LC), localy traceable (LT), and locally hamiltonian (LH) graphs. If $\mathcal{R}$ is a set of nonnegative integers, we say a graph G is *$\mathcal{R}$-regular* if the degrees of all the vertices in $V(G)$ are elements of $\mathcal{R}$. The *Hamilton Cycle Problem* (HCP) is the problem of determining whether a graph has a Hamilton cycle, and is known to be NP-complete for many classes of graphs - see for example [3, 11, 16].

LC, LT and LH graphs have been intensively studied - see for example [1, 2, 4, 6, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24]. Results from [8, 11, 13, 15] show that the HCP for $\mathcal{R}$-regular LC graphs is fully solved if $\max(\mathcal{R}) \leq 4$ and also if $\mathcal{R} \subseteq \{3, 4, 5\}$, but NP-complete if $\max(\mathcal{R}) \geq 7$. Gordon et al. [11] conjectured the HCP is polynomially solvable for LC graphs with maximum degree at most 6. It has since been shown [1, 14] that the HCP is NP-complete for $\mathcal{R}$-regular LC graphs if $\mathcal{R} = \{2, 5\}$ or $\mathcal{R}$ is any set of natural numbers with $\max(\mathcal{R}) \geq 6$.

Locally traceable graphs are of course locally connected, and as shown in [2], there are only three connected LT graphs with maximum degree 5 that are not hamiltonian. This implies that the HCP for LT graphs

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with maximum degree 5 can be solved in polynomial time. It is known [24] that the HCP for LT graphs with maximum degree 6 is NP-complete. We extend that result to 1-tough graphs and show that the HCP is NP-complete for $\mathcal{R}$-regular LT graphs if $\mathcal{R}$ is any set of natural numbers with $\max(\mathcal{R}) \geq 6$, with the possible exception of $\{4, 6\}$ and $\{6\}$.

A graph is *fully cycle extendable* if every vertex lies in a 3-cycle, and every nonhamiltonian cycle in the graph can be extended by having a vertex added to it. It is known that connected LH graphs with maximum degree at most 6 are fully cycle extendable [2], and therefore hamiltonian, and as shown in [19, 22], many nonhamiltonian connected LH graphs with maximum degree 8 exist.

Skupień [20] observed that maximal planar graphs are LH, and Chvátal [7] and Wigderson [25] independently proved that the HCP is NP-complete for maximal planar graphs. Neither author considered constraints on the maximum degree of such graphs, but it is easy to manipulate Chvátal's construction such that his proof is valid for maximal planar graphs with maximum degree 12. De Wet et. al [24] showed that the HCP for LH graphs with maximum degree 10 is NP-complete.

Here we show that the HCP for 1-tough LH graphs with maximum degree 9 is NP-complete (the proof does not apply to maximal planar graphs), and also that the HCP is NP-complete for LH graphs that are $\mathcal{R}$-regular where $\mathcal{R}$ is any set of natural numbers with $\min(\mathcal{R}) = 3$ and $\max(\mathcal{R}) \in \{9, 10\}$, or if $\mathcal{R}$ is any set of natural numbers with $\max(\mathcal{R}) \geq 11$.

Finally, we show that the HCP for k-connected LH graphs that are locally $(k-1)$-connected is NP-complete for every $k \geq 3$.

2. Locally Traceable Graphs

It has been shown that the HCP for LC graphs with maximum degree 5 is NP-complete [1, 14]. However, the same does not apply to LT graphs, as can be seen from the following theorem. The three exceptional graphs referred to in the theorem are shown in Figure 1. Graphs of this kind are referred to as magwheels.

Theorem 2.1. [2] *Suppose G is a connected LT graph with $n(G) \geq 3$ and $\Delta(G) \leq 5$. Then G is fully cycle extendable if and only if $G \notin \{M_3, M_4, M_5\}$.*

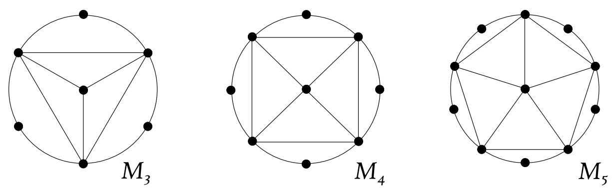

Figure 1: The graphs M_3, M_4 and M_5.

From Theorem 2.1 it is clear that the HCP for LT graphs with maximum degree 5 can be solved in polynomial time. However, as shown in [24], the HCP for LT graphs with maximum degree 6 is NP-complete. Here we present an alternative, simpler proof for this result and extend the result to 1-tough graphs. The new proof will be required for the the proof of the results relating to $\mathcal{R}$-regular graphs.

We will need the following earlier results (a graph is claw-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph):

Theorem 2.2. [16] *The Hamilton cycle problem is NP-complete for 3-connected cubic claw-free planar graphs.*

Theorem 2.3. [5] *Every 3-connected cubic graph is 1-tough.*

In [23] we developed the following technique for combining LT graphs to create larger LT graphs.

Construction 2.4. [23] (*Edge identification*) *Let G_1 and G_2 be two LT graphs such that $E(G_i)$ contains an edge u_iv_i so that there is a Hamilton path in $\langle N(u_i)\rangle$ that ends at v_i and a Hamilton path in $\langle N(v_i)\rangle$ that ends at u_i, $i = 1, 2$. Now create a larger graph G by identifying the edges u_1v_1 and u_2v_2 to a single edge uv.*

An edge in a LC graph that can be used in the above edge identification procedure will be called a *suitable edge.*

Theorem 2.5. [22, 23, 24] *Let G_1 and G_2 be two LT graphs that satisfy the conditions of Construction 2.4 and let G_1 and G_2 be combined by means of edge identification to create a graph G. Then*
(a) G is LT.
(b) If G_1 and G_2 are planar, then so is G.
(c) If at least one of G_1 and G_2 is nonhamiltonian, then G is nonhamiltonian.

The next corollary follows readily from Theorem 2.5.

Corollary 2.6. *Let G be an LT graph with two suitable edges xy and uv such that $\{x, y\}$ and $\{u, v\}$ have distinct neighbourhoods. Then if x is identified with u and y is identified with v, the resulting graph is still LT.*

We are now ready to prove the theorem.

Theorem 2.7. *The Hamilton Cycle Problem for 1-tough planar LT graphs with maximum degree 6 is NP-complete.*

Proof. By Theorem 2.2 the HCP for 3-connected cubic planar 1-tough graphs is NP-complete. Now consider any 1-tough 3-connected planar cubic graph G'. We shall show that G' can be transformed in polynomial time to a planar *LT* graph G with $\Delta(G) = 6$ such that G is hamiltonian if and only if G' is hamiltonian.

Each vertex in G' will be represented by a triangle in G, and will be referred to as a *node* in G.

The edges in G' will be represented by a more complicated structure in G to ensure that G is *LT* and also that G is hamiltonian if and only if G' is hamiltonian. Consider the smallest of the magwheels, M_3, and the graph S in Figure 2. The graph M_3 and two copies of the graph S are combined by means of edge identification to create the graph B in Figure 3. This graph will be used in G to represent the edges in G', and will be referred to as a *border*.

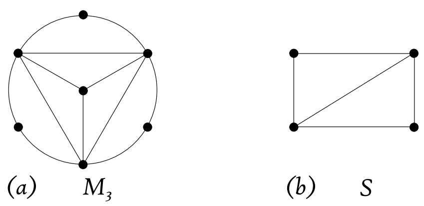

Figure 2: (a) The magwheel M_3 and (b) the graph S used in the proof of Theorem 2.7.

Figure 4 shows how the graph G' is transformed into graph G. A vertex z_i in G' becomes a triangle Z_i in G and an edge z_iz_j in G' becomes a border $B_{i,j}$ in G. All the combinations of different components are done by means of edge identification. It follows from Theorem 2.5 (a) and (b) and Corollary 2.6 that the resulting graph is *LT*, and since G' is planar, so is G.

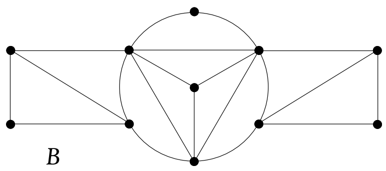

Figure 3: The border B used in the proof of Theorem 2.7.

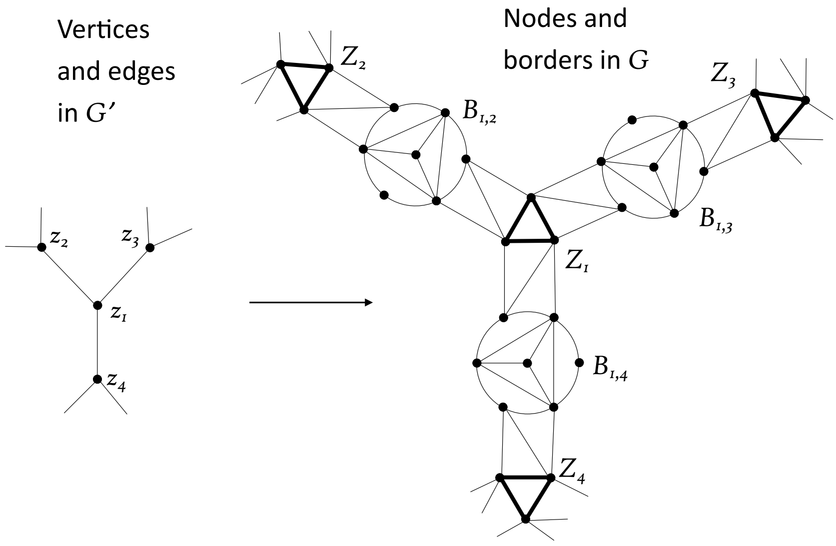

Figure 4: Translating graph G' into graph G in the proof of Theorem 2.7.

It remains to be shown that G is hamiltonian if and only if G' is hamiltonian. Figure 5 shows how a Hamilton cycle in G' translates to a Hamilton cycle in G. The heavy lines in the figure represent edges that are part of the Hamilton cycles. Now suppose C is a Hamilton cycle in G. Since the magwheel M_3 is nonhamiltonian, it follows that there does not exist a 2-path cover for M_3 for which the two pairs of end vertices are adjacent. Therefore C passes at most once through a given border from one node to another. Since each node has exactly three borders incident to it, C corresponds to a Hamilton cycle in G'.

Finally, to see that graph G is 1-tough, we note that since G' is 1-tough, removing vertices only from the nodes of G does not result in more components than vertices removed (the nodes are cliques). The magwheel M_3 used to construct the borders in G is not 1-tough: if the three vertices of degree 5 (labeled say v_1, v_2, v_3) are removed, the result is a graph consisting of four isolated vertices. If v_1, v_2, v_3 are removed from a border in G, the resulting graph contains two isolated vertices, and the border no longer connects the two nodes incident to it in G. We will now proceed to remove the vertices in the position of v_1, v_2, v_3 from borders in G. Let G_m be the graph $G_{m-1} - \{v_{m,1}, v_{m,2}, v_{m,3}\} - \{u_{m,1}, u_{m,2}\}$, $m \geq 1$, where m is the number of borders that have been broken in this way, $v_{m,1}, v_{m,2}, v_{m,3}$ are the vertices in border m in the same relative position as v_1, v_2, v_3 that have been removed and $u_{m,1}$ and $u_{m,2}$ are the two vertices that have been isolated by the removal of $v_{m,1}, v_{m,2}, v_{m,3}$ (note that $G_0 = G$). Removing an edge in any graph increases the number of components by at most one, so removing the vertices $v_{m,1}, v_{m,2}, v_{m,3}$ from a border in G_{m-1} increases the number of components by at most 3 ($u_{m,1}$, $u_{m,2}$ and possibly the number of components of G_m increases by one). Since G' is 3-connected, at least 3 borders in G have to be broken before G_m is disconnected. It follows that after two borders have been broken there are 4 isolated vertices and G_2 is still connected, and

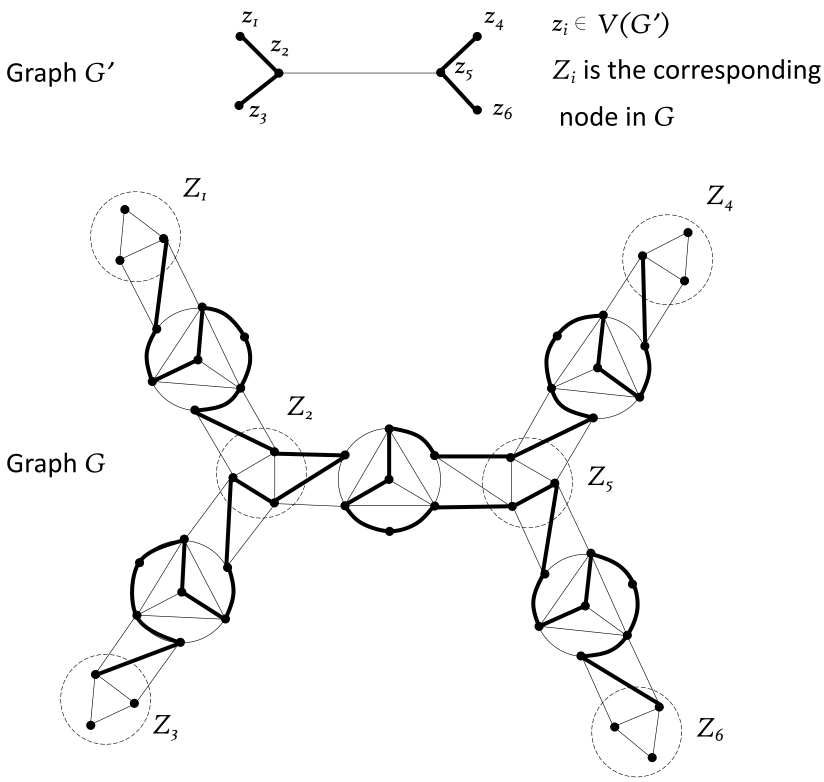

Figure 5: Translating a Hamilton cycle in G' into a Hamilton cycle in G in the proof of Theorem 2.7.

after m borders have been broken (by removing $3m$ vertices), the number of components in the resulting graph is at most $3+2+3+3+\cdots = 2+3(m-1) = 3m-1 < 3m$. The same argument applies to removing 2 vertices at a time from a border. It follows that G is 1-tough.

We will now use the construction in the proof of Theorem 2.7 to extend the result to $\mathcal{R}$-regular graphs.

Theorem 2.8. *The HCP is NP-complete for $\mathcal{R}$-regular LT graphs if $\mathcal{R}$ is any set of natural numbers with $max(\mathcal{R}) \geq 6$, with the possible exception of $\{4,6\}$ and $\{6\}$.*

Proof. It is sufficient to consider the cases $\mathcal{R} = \{2,6\}$, $\mathcal{R} = \{3,6\}$, $\mathcal{R} = \{5,6\}$ and $\mathcal{R} = \{d\}$, $d \geq 7$. In each case we will start with the basic construction G in the proof of Theorem 2.7 and modify the borders and/or nodes to create an $\mathcal{R}$-regular LT graph $G_{\mathcal{R}}$ that is hamiltonian if and only if G is hamiltonian.

Case 1: $\mathcal{R} = \{2,6\}$. Since it is not possible to manipulate the central vertex of a magwheel using edge identification (the edges incident to this vertex are not suitable for use in edge identification), we replace the graph M_3 with M_6 (a magwheel with six spokes). We then add three vertices to the graph, to create the graph M_6' shown in Figure 6 (a). This is done in order to avoid problems with vertex degree parity (since all the elements of $\mathcal{R}$ are even numbers, it is essential that there is an even number of edges between M_6 and each of the graphs that will be attached to it). This graph has the desired properties of being LT, nonhamiltonian, and traceable from each vertex in $\{v_1, v_2\}$ to each vertex in $\{v_3v_4\}$. The graph M_6' will form the core of the border and will be connected to the nodes (which are still triangles) using the graph F_1 in Figure 6 (b). This will be done by identifying the edge v_1v_2 in M_6' with the edge u_1u_2 in F_1. The edge u_3u_4 in F_1 is identified with one of the edges of the triangle representing the node. The same is done using the edge v_3v_4 to connect M_6' to the second node.

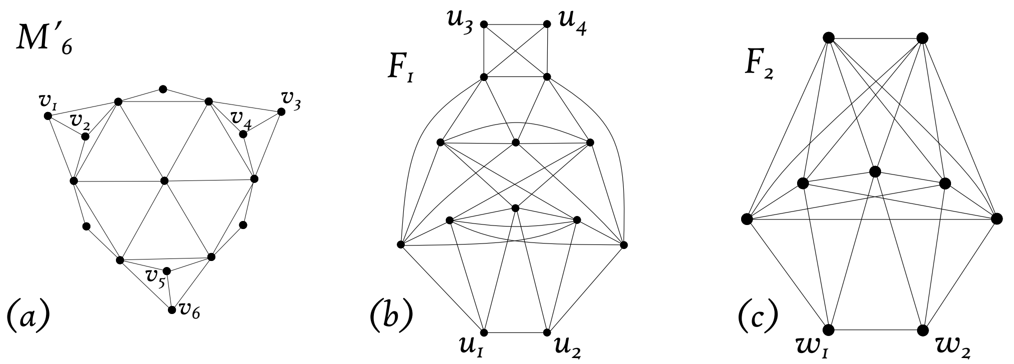

Figure 6: The graphs used to create a border of $G_{\{2,6\}}$.

At this point the vertices v_5 and v_6 in Figure 6 (a) are not yet of degree 2 or 6. This is addressed by identifying the edge v_5v_6 with the edge w_1w_2 in the graph F_2 in Figure 6 (c). The graphs F_1 and F_2 are LT and have Hamilton cycles that contain the edges u_1u_2 and u_3u_4, and w_1w_2, respectively. Hence by Theorem 2.5 (a) and Corollary 2.6, the resulting graph $G_{\{2,6\}}$ is LT. Moreover, $G_{\{2,6\}}$ is hamiltonian if and only if G is hamiltonian. A border in $G_{\{2,6\}}$ and the two nodes that it are attached to can be seen in Figure 7, where the heavy lines represent edges that result from edge identification. The heavy edges in Figure 8 illustrate how a Hamilton cycle in $G_{\{2,6\}}$ (a) passes through a given border and (b) does not pass through a given border.

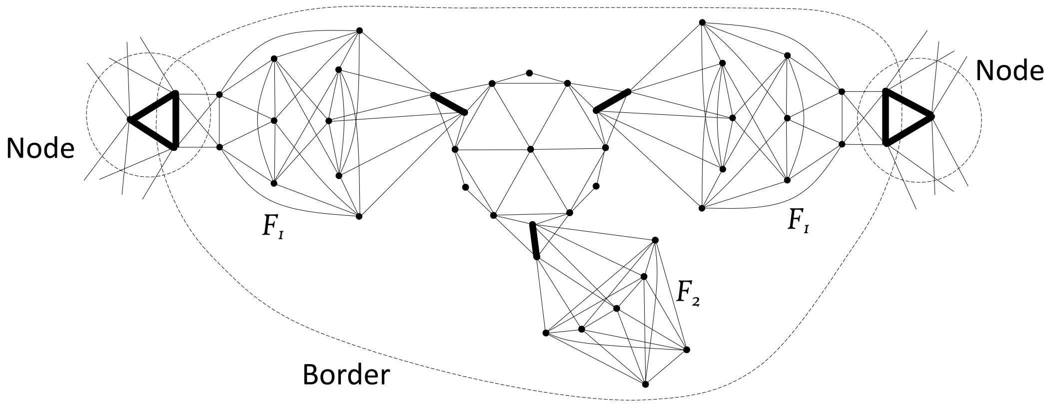

Figure 7: A border connecting two nodes in the graph $G_{\{2,6\}}$.

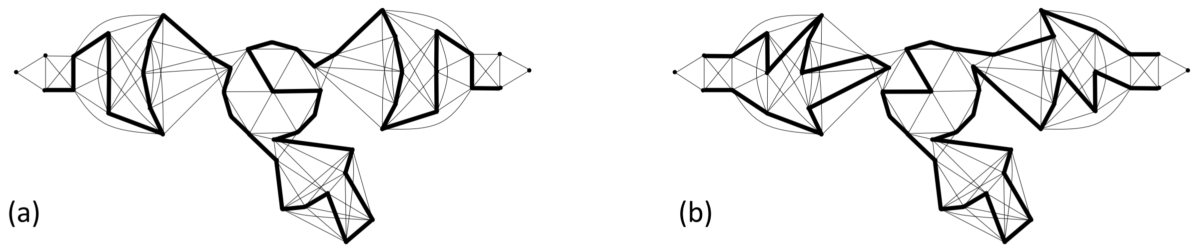

Figure 8: Part of a Hamilton cycle in $G_{\{2,6\}}$ that (a) passes through a given border and (b) does not pass through a given border.

Case 2: $\mathcal{R} = \{3, 6\}$. We use M_3 as node and also as the cores of the borders and combine these graphs using edge identification as shown in Figure 9. The edges v_1v_2 and u_1u_2 are identified, as are the edges v_3v_4 and u_3u_4.

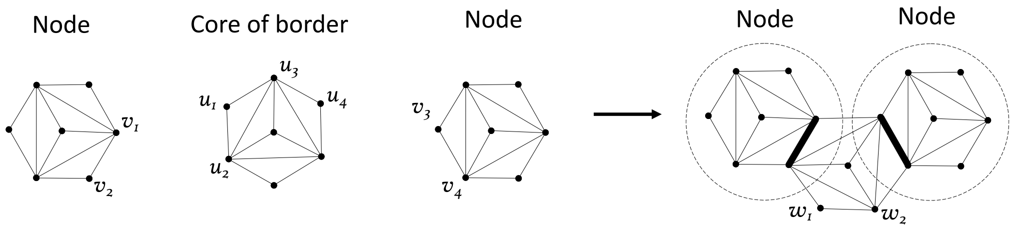

Figure 9: Combining copies of M_3 in the construction of $G_{\{3,6\}}$.

All that remains to be done is to address the degrees of the vertices w_1 and w_2 in Figure 9. This is done by identifying the edge w_1w_2 with the edge x_1x_2 in the LT graph F_3 in Figure 10. The resulting graph $G_{\{3,6\}}$ is LT and $\{3,6\}$-regular, and hamiltonian if and only if G is hamiltonian. Figure 11 shows a border connecting two nodes in $G_{\{3,6\}}$. The heavy lines in Figures 10 and 11 represent edges that are the result of edge identification.

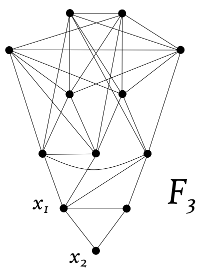

Figure 10: The graph used to change the degrees of vertices w_1 and w_2 in $G_{\{3,6\}}$ by means of edge identification.

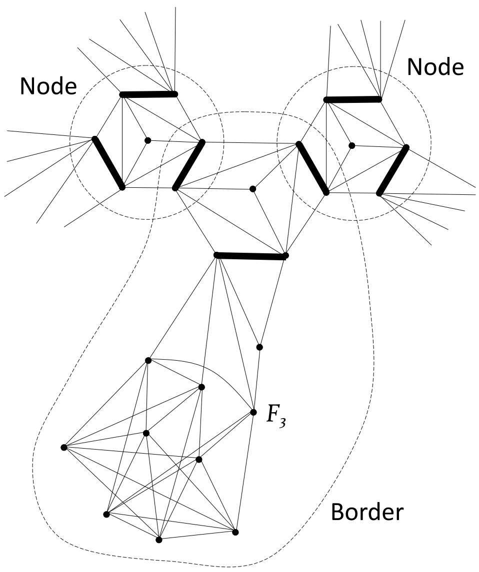

Figure 11: A border and two nodes in the graph $G_{\{3,6\}}$.

Case 3: $\mathcal{R} = \{5, 6\}$. In this case the border is constructed around M_5 (a magwheel with 5 spokes - Figure 12 (a)), and the nodes are again triangles. A node is linked to M_5 by identifying the edge u_3u_4 in the LT graph F_4 in Figure 12 (b) with one of the edges of the node, and the edge u_1u_2 is identified with either the edge v_1v_2 or the edge v_7v_8 in the magwheel. The vertices in the magwheel that are still of degree two are addressed by identifying the edges v_3v_4, v_5v_6 and v_9v_{10} with the edge w_1w_2 in three copies of the graph F_5 in Figure 12 (c). This creates the LT graph $G_{\{5,6\}}$ (the edges resulting from edge identifcation are shown as heavy lines in Figure 13). Again, $G_{\{5,6\}}$ is hamiltonian if and only if G is hamiltonian.

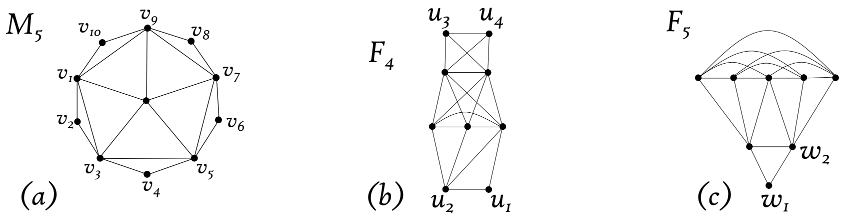

Figure 12: The graphs used in the construction of a border in $G_{\{5,6\}}$.

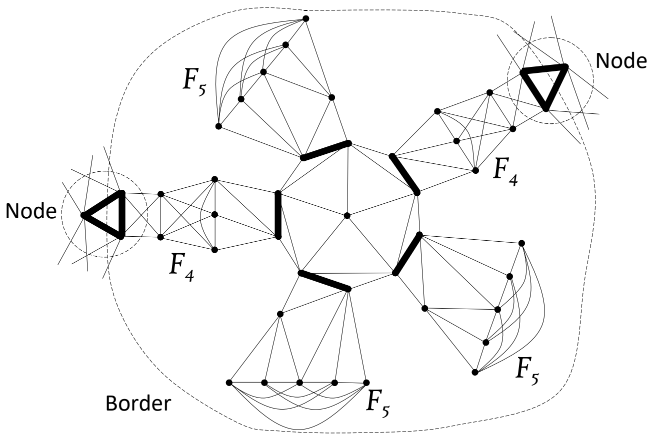

Figure 13: A border and two nodes in the graph $G_{\{5,6\}}$.

Case 4: r-Regular graphs. Due to constraints relating to vertex degree parity, we need different constructions for odd and even values of r. In each case, we start constructing the borders of G_r using a magwheel with r spokes. We first address the case where r is odd, and illustrate it with $r = 7$. As before, in constructing the graph G_7, we use hamiltonian LT graphs to connect the magwheel to the nodes, and also to address the vertices with degrees other than 7. We use the LT graph F_6 in Figure 14 (a) to connect the magwheels to the nodes (in this case, a node is the square of a 6-cycle) by means of edge identification (the edge u_1u_2 is identified with an edge on the rim of the magwheel, and the edge u_3u_4 is identified with an edge on the rim of the node - see Figure 15). Multiple copies of the LT graph F_7 in Figure 14 (b) are used to address the degrees of the remaining vertices by identifying the edge w_1w_2 with an edge on the rim of the magwheel. The complete structure of a border and two nodes can be seen in Figure 15 (the graphs in Figures 14 are represented by ovals).

For r an even number, we again add vertices to the outside of the magwheel as was done in the $\{2, 6\}$-regular case. In the case of $r = 8$, we end up with the graph M_8' in Figure 16 (a).

To construct the graph G_8, we connect M_8' to two nodes (which are again the squares of 6-cycles) using copies of the LT graph F_8 in Figure 16 (b). In particular we identify the edges v_9v_{17} and $v_{13}v_{21}$ with the edge u_1u_2, and the edge u_3u_4 is identified with an edge on the rim of the node. The degrees of the remaining vertices of G_8 are addressed using the graphs F_9 and F_{10} in Figure 16 (c) and (d). Using multiple copies

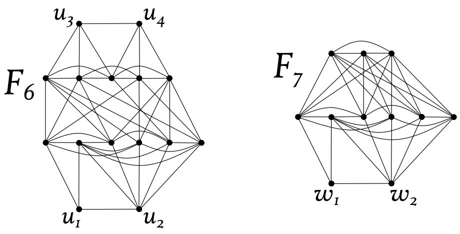

Figure 14: Auxiliary graphs used in the construction of the borders of G_7.

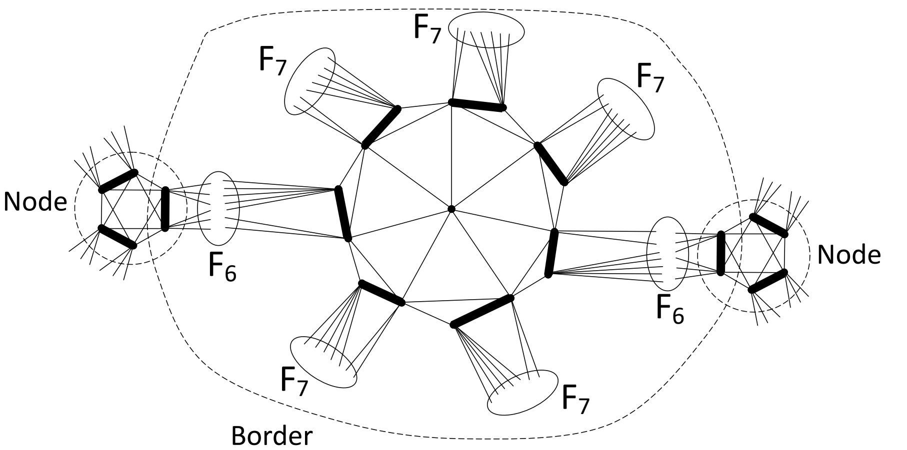

Figure 15: A border and two nodes in the graph G_7.

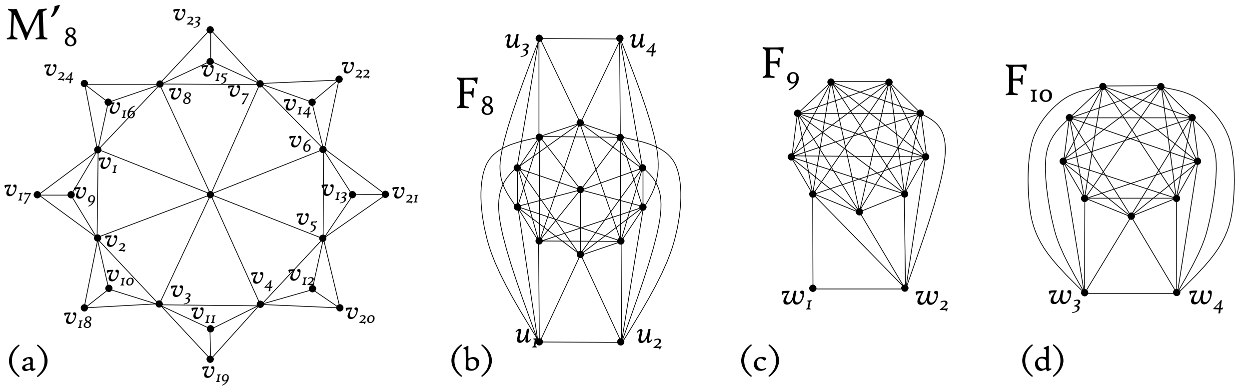

Figure 16: The graphs used in the construction of the borders of G_8.

of these graphs, we do the following: v_1v_{24}, v_8v_{16}, v_7v_{14}, v_5v_{20}, v_4v_{12}, v_3v_{18}, v_2v_{10}, and v_6v_{22} are identified with the edge w_1w_2 in distinct copies of F_9, and the edges $v_{23}v_{15}$ and $v_{11}v_{19}$ are identified with the edge w_3w_4 in distinct copies of the LT graph F_{10}. A border and the two nodes adjacent to it in the resulting graph G_8 are shown in Figure 17, where the graphs F_8, F_9 and F_{10} are represented by ovals.

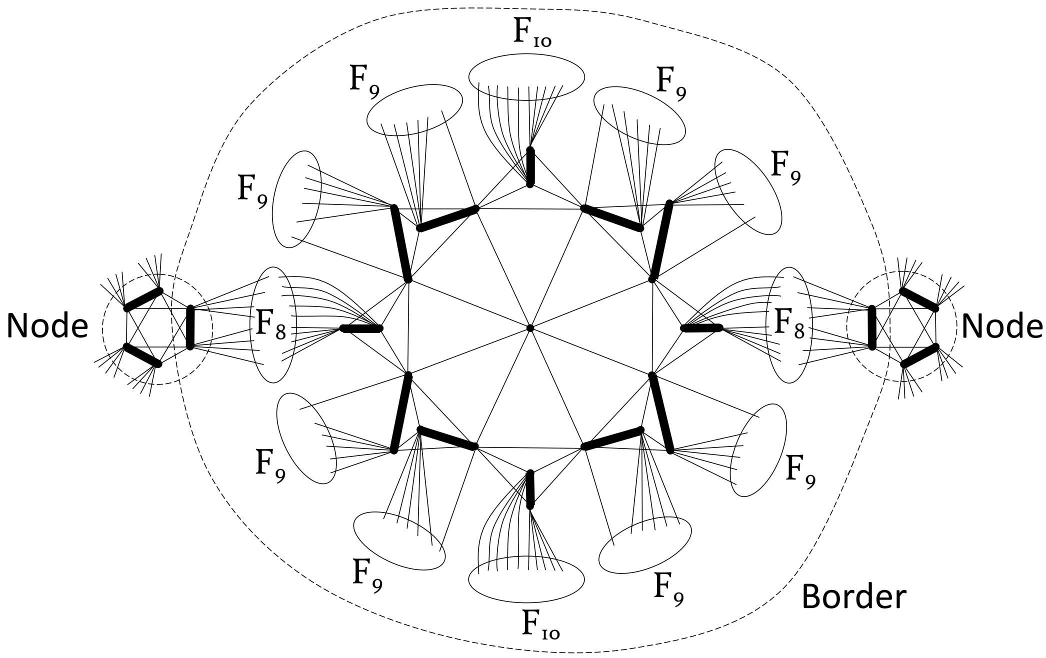

Figure 17: A border and two nodes in the graph G_8.

3. Locally Hamiltonian Graphs

It is known that connected LH graphs are hamiltonian if the maximum degree is at most 6, as can be seen from the following theorem.

Theorem 3.1. [2] *Let G be a connected LH graph with $n(G) \geq 3$ and $\Delta(G) \leq 6$. Then G is fully cycle extendable.*

There many known examples of connected LH graphs with maximum degree 8 that are not hamiltonian ([19, 22]), and there is a published "proof" for the claim that any connected LH graph with maximum degree 7 is hamiltonian [18]. However, there are serious gaps in the "proof", as discussed in [22, 24], and we consider this claim to be unproved.

As mentioned in Section 1, maximal planar graphs are LH, and the HCP is NP-complete for maximal planar graphs with maximum degree 12. For LH graphs, this number was improved to 10 in earlier work [24]. Here we improve this further by showing that the HCP is NP-complete for LH graphs with maximum degree 9, and we extend the result to 1-tough graphs.

Before we proceed with the proof, we will need some earlier results.

Construction 3.2. [23] *For $i = 1, 2$, let G_i be an LH graph that contains a triangle X_i such that for each vertex $x \in V(X_i)$, there is a Hamilton cycle of $\langle N(x) \rangle$ that contains the edge $X_i - x$. Suppose $V(X_i) = \{u_i, v_i, w_i\}$, $i = 1, 2$. Now create a graph G of order $n(G_1) + n(G_2) - 3$ by identifying the vertices u_i, $i = 1, 2$ to a single vertex u, and similarly the vertices v_i, $i = 1, 2$ to v and w_i, $i = 1, 2$ to w, while retaining all the edges present in the original two graphs.*

A triangle in a LH graph that can be used in the above triangle identification procedure will be called a *suitable triangle*.

Lemma 3.3. [23, 24] *Let G_1 and G_2 be two LH graphs, and let G be a graph obtained from G_1 and G_2 by identifying suitable triangles. Then G is LH.*

The corollary is fairly obvious:

Corollary 3.4. *Let G be an LH graph with two suitable triangles xyz and uvw such that $\{x, y, z\}$ and $\{u, v, w\}$ have distinct neighbourhoods. If we identify u with x, v with y, and w with z, then G is LH.*

We are now ready for our first new result.

Theorem 3.5. *The Hamilton Cycle Problem for 1-tough LH graphs with maximum degree 9 is NP-complete.*

Proof. Starting with a 3-connected cubic graph G', we will construct a connected LH graph G with $\Delta(G) = 9$ such that G is hamiltonian if and only if G' is hamiltonian.

The nodes of G (that replace the vertices in G') are copies of K_4 and the borders are copies of the graph B in Figure 19. The graph B is constructed by combining the nonhamiltonian LH Goldner-Harary graph H in Figure 18 (a) and two copies of the LH graph D in Figure 18 (b) using triangle identification in the following way: using the first copy of D, identify v_1 and x_1, v_2 and x_2, and v_3 and x_3, and using the second copy of D, identify u_1 and x_1, u_2 and x_2, and u_3 and x_3.

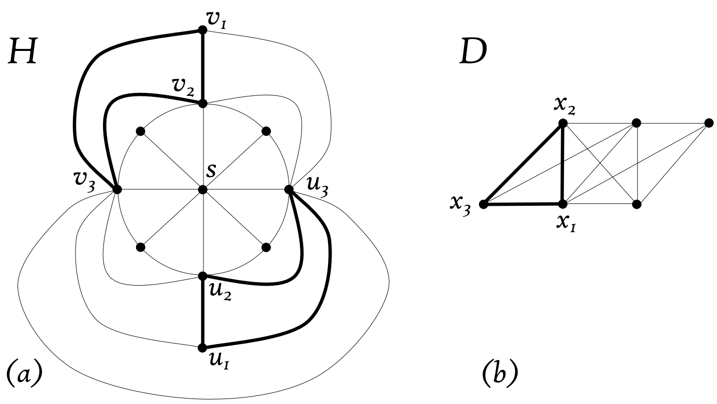

Figure 18: (a) The Goldner-Harary graph H and (b) the graph D used to construct the border B of Theorem 3.5.

The borders are connected to the nodes by means of triangle identification as shown in Figure 20, which shows a border in G and the two nodes adjacent to it (the heavy lines represent triangles that are the result of triangle identification).

Checking the degrees of the vertices that have been identified shows that $\Delta(G) = 9$ and by Lemma 3.3 and Corollary 3.4, G is LH.

Figure 21 shows how a Hamilton cycle in G' translates to a Hamilton cycle in G (the heavy lines represent paths in the Hamilton cycle).

Consider a copy of H in a border of G that connects two nodes, say Z_1 and Z_2. Assume that the edges between H and Z_1 are incident with vertices in $\{u_1, u_2, u_3\}$, and the edges between H and Z_2 are incident with vertices in $\{v_1, v_2, v_3\}$ (as labelled in Figure 18 (a)).

Now suppose C is a Hamilton cycle in G. Let $S = N(s) - \{v_2, v_3, u_2, u_3\}$ (i.e. the set of unlabelled neighbours of s in H in Figure 18 (a)). Then S is an independent set of cardinality four and $N(S) = \{v_2, v_3, u_2, u_3, s\}$. The intersection of C with $\langle N[s] \rangle$ is therefore a path with end vertices in $\{v_2, v_3, u_2, u_3\}$. Hence any path cover of H contains at most one path that has one end vertex in $\{u_1, u_2, u_3\}$ and the other in $\{v_1, v_2, v_3\}$.

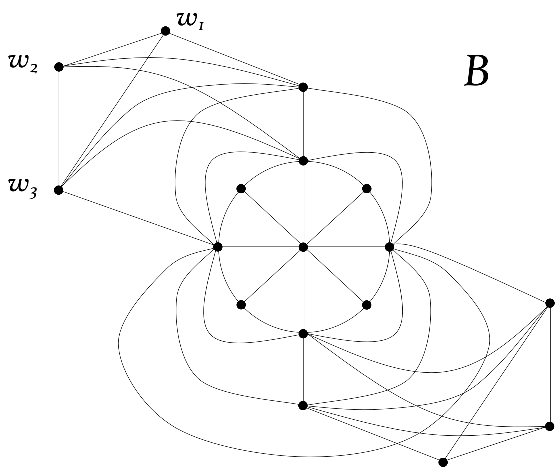

Figure 19: The graph B used in the proof of Theorem 3.5.

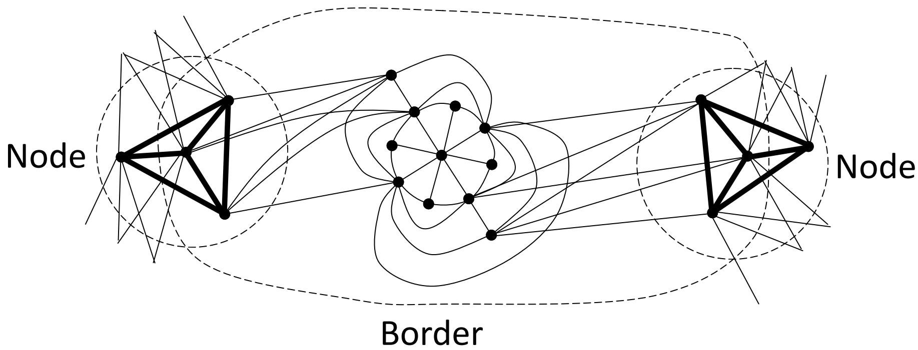

Figure 20: A border and two nodes in the graph G.

Thus every Hamilton cycle in G has at most one path from Z_1 to Z_2 that passes through the border between them. Therefore, C corresponds to a Hamilton cycle in G'. Since G' is 1-tough, it is not difficult to see that G is also 1-tough. The argument is similar to the one used for LT graphs in Theorem 2.7

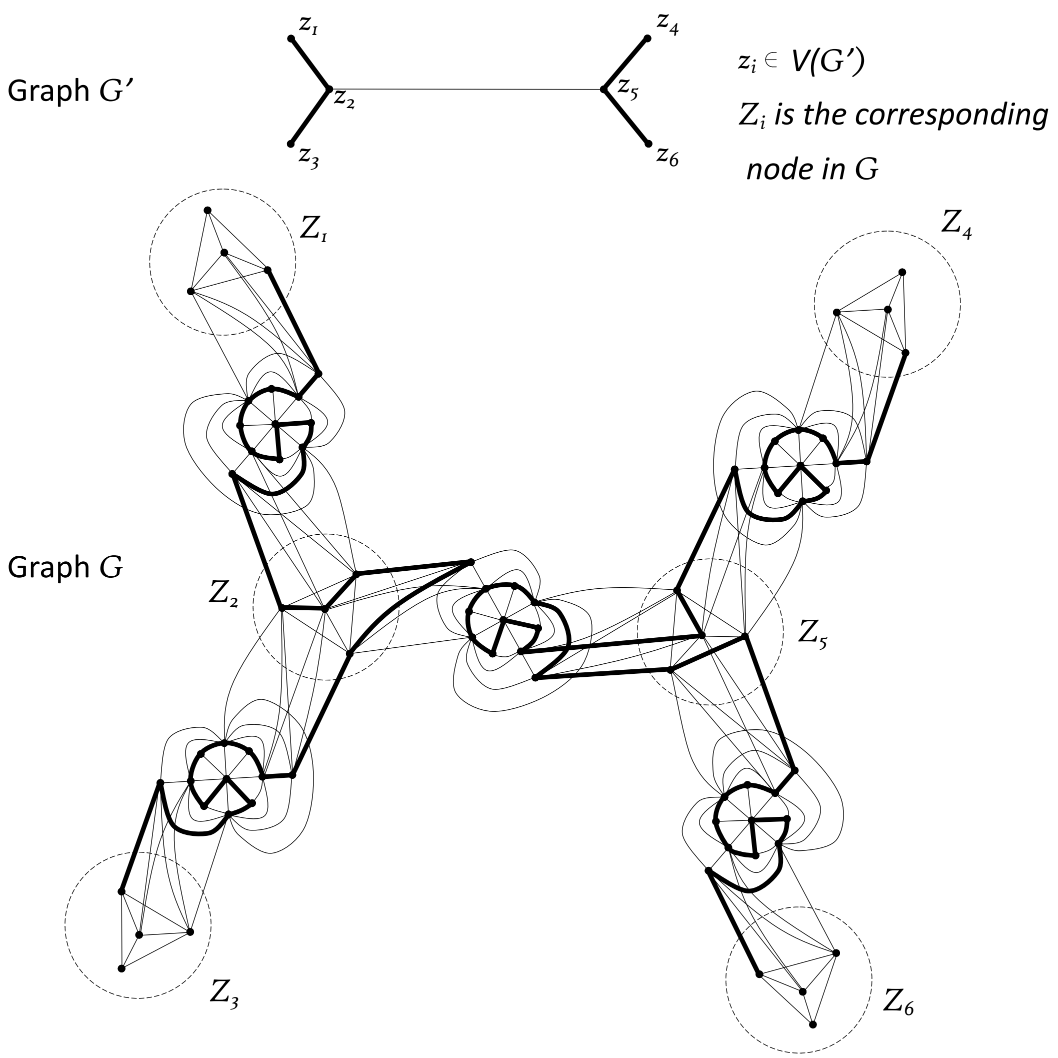

Figure 21: Translating a Hamilton cycle from G' to G.

We will now use the construction in the above proof for the results relating to $\mathcal{R}$-regular graphs.

Theorem 3.6. *The HCP is NP-complete for $\mathcal{R}$-regular LH graphs if $\mathcal{R}$ is any set of natural numbers with* $\min(\mathcal{R}) = 3$ *and* $\max(\mathcal{R}) \in \{9, 10\}$*, or if $\mathcal{R}$ is any set of natural numbers with* $\max(\mathcal{R}) \geq 11$.

Proof. It is sufficient to consider the cases $\mathcal{R} = \{3, 9\}$, $\mathcal{R} = \{3, 10\}$ and $\mathcal{R} = \{d\}$, where $d \geq 11$. For each case we start with the graph G constructed in the proof of Theorem 3.5 and modify the borders and/or nodes to create an $\mathcal{R}$-regular, LH graph $G_{\mathcal{R}}$ that is hamiltonian if and only if G is hamiltonian.

Case 1: $\mathcal{R} = \{3, 9\}$. The nodes of $G_{\{3,9\}}$ are copies of the graph in Figure 22. The borders in $G_{\{3,9\}}$ are based on the Goldner-Harary graph (with vertices as labeled in Figure 23 (a)), as was the case for G. However, the links between the Goldner-Harary graph and the nodes are now the two graphs F_{11} and F_{12} in Figure 23 (b) and (c). We use two different graphs so that we can modify the degrees of all the vertices in the Goldner-Harary graph that are not of degree 3. Specifically, we construct a border by identifying the pairs of vertices as listed in Table 1 (Note that the vertices labeled x_1, x_2 and x_3 are listed twice - this should be interpreted as the vertices with these labels in two separate copies of the node.). This yields the graph $G_{\{3,9\}}$, of which one border and the nodes attached to it are shown in Figure 24 (where the heavy lines represent edges between vertices that have been identified).

x_1	x_2	x_3	x_1	x_2	x_3	v_3	v_4	v_{11}	v_2	v_5	v_6	v_1
u_1	u_2	u_3	y_1	y_2	y_3	u_5	u_6	u_4	y_5	y_6	y_4	y_7

Table 1: Vertices identified in the proof of Case 1: $\mathcal{R} = \{3, 9\}$.

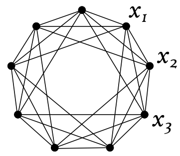

Figure 22: A node of $G_{\{3,9\}}$.

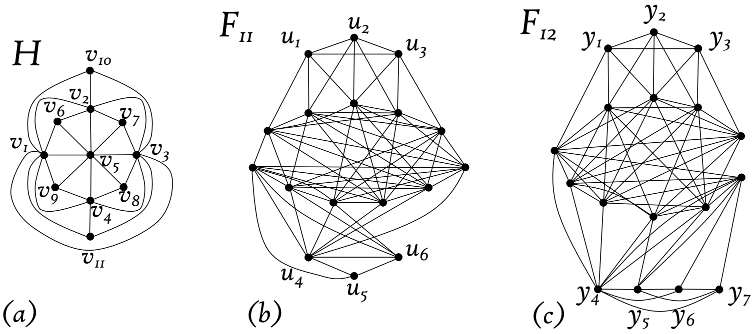

Figure 23: The graphs used to construct a border in $G_{\{3,9\}}$.

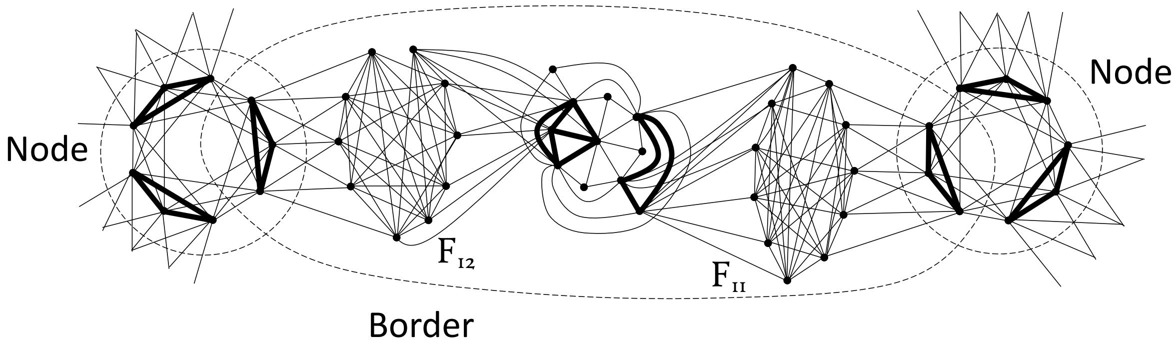

Figure 24: A border and two nodes in the graph $G_{\{3,9\}}$.

Note that the connection between the Goldner-Harary graph and the graph F_{12} in Figure 23 (c) was not done using triangle identification. This is clear from the fact that 4 vertices in each graph was identified with 4 vertices in the other graph. However, it is a simple matter to confirm that the resulting graph is still LH. As was the case in G, it is not possible for a Hamilton cycle in $G_{\{3,9\}}$ to pass through a border more than once. This is because, assuming after identifying vertices, the resulting vertices retain the labels from the Goldner-Harary graph, $N(\{v_7, v_8, v_9, v_{10}\}) = \{v_1, v_2, v_3, v_4, v_5\}$, and therefore all these vertices must occur consecutively (in some order) in any Hamilton cycle of $G_{\{3,9\}}$.

Case 2: $\mathcal{R} = \{3, 10\}$. The nodes of $G_{\{3,10\}}$ are the same as in $G_{\{3,9\}}$, but we use different graphs to connect the Goldner-Harary graph to the nodes, namely the graphs F_{13} and F_{14} in Figure 25 (a) and (b). Having linked up the nodes to the Goldner-Harary graph in this way, there are still some vertices that do not have degree 3 or 10. This is remedied by attaching the graph F_{15} in Figure 25 (c) to the Goldner-Harary graph. Table 2 lists the vertices that are identified with each other in the construction of a border in $G_{\{3,10\}}$ (refer to Figures 23 (a) and 25 to see which vertices the labels refer to). Note that v_2 occurs twice in the table. That is because it is first identified with u_6 and then the resulting vertex is identified with w_1. The vertices x_1, x_2 and x_3 are also listed twice in the table. This should be interpreted as the vertices with these labels in two separate copies of the node. A border and the nodes attached to it can be found in Figure 26, where the heavy lines represent edges resulting form triangle identification.

x_1	x_2	x_3	x_1	x_2	x_3	v_1	v_2	v_{10}	v_3	v_4	v_{11}	v_2	v_5	v_7
u_1	u_2	u_3	y_1	y_2	y_3	u_5	u_6	u_4	y_5	y_6	y_4	w_1	w_2	w_3

Table 2: Vertices identified in the proof of Case 2: $\mathcal{R} = \{3, 10\}$.

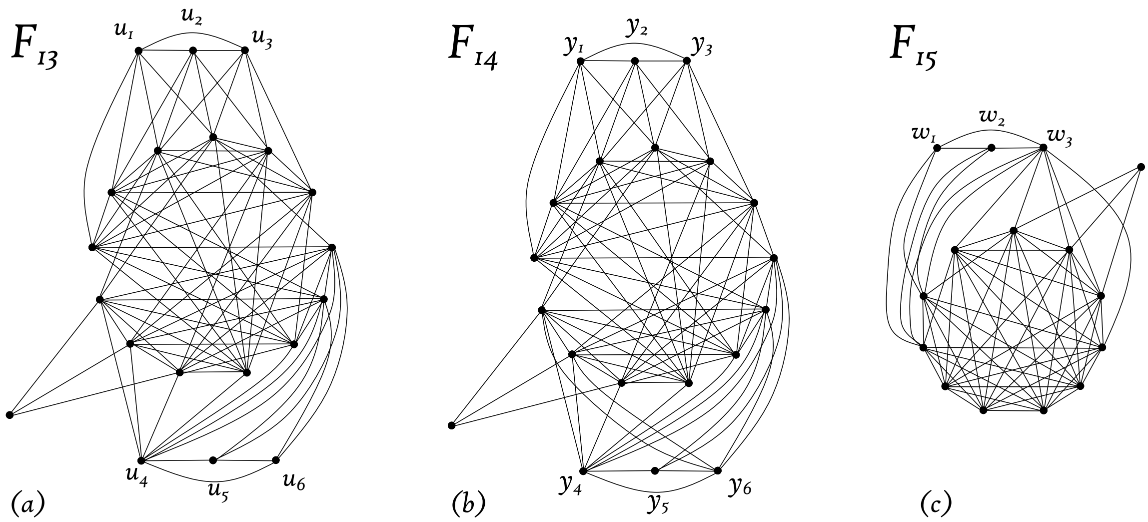

Figure 25: Graphs used in the construction of a border in $G_{\{3,10\}}$.

Case 3: r-regular graphs, where $d \geq 11$. We will construct the graph G_{11}. The constructions for larger values of d are similar. We use the graph in Figure 27 as nodes of G_{11}. The Goldner-Harary graph again forms the core of the border, and is connected to the nodes by means of the graph F_{16} in Figure 28 (a). In order to make the graph 11-regular, we attach four additional LH graphs to the Goldner-Harary graph using triangle identification. It is necessary to use 4 graphs because there are 4 remaining vertices of degree 3, and they form an independent set. The graphs attached to the Goldner-Harary graph are 3 copies of the graph F_{17} in Figure 28 (b) and 1 copy of the graph F_{18} in Figure 28 (c).

The vertices identified are shown in Table 3. Note that the vertices labeled x, u and w are listed more than once in the table. This should be interpreted as the same vertices in different copies of the graph to which they belong. Several vertices in the Goldner-Harary graph (labeled v_i for some i) are also listed more than once. In this case it means that these vertices are first identified with one other vertex, and then the resulting vertex is again identified with another vertex. A border in G_{11} and the two nodes attached to it are shown in Figure 29 (again, the heavy lines represent edges that are the result of triangle identification). Since G_{11} was constructed using triangle identification, it follows that it is LH. It is straightforward to confirm that

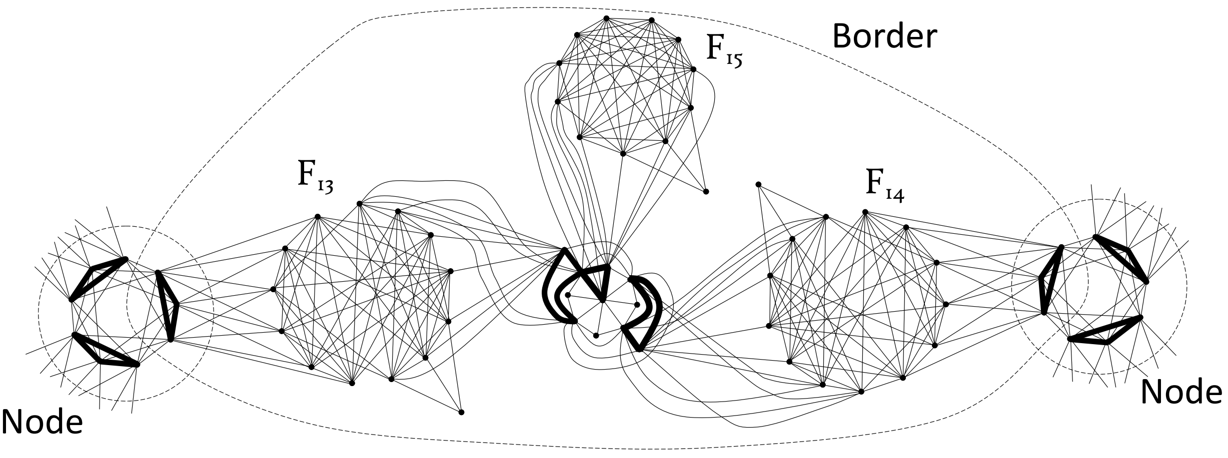

Figure 26: A border and two nodes of the graph $G_{\{3,10\}}$.

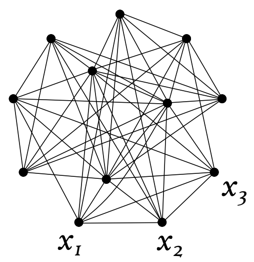

Figure 27: Node used in the construction of G_{11}.

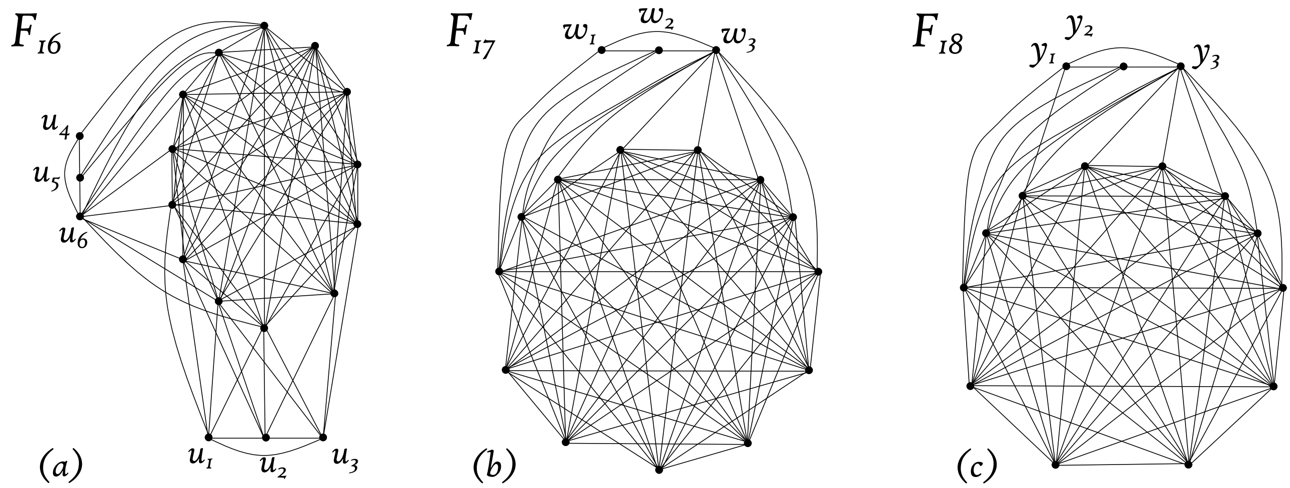

Figure 28: Graphs used in the construction of a border in G_{11}.

the conversion from G to G_{11} does not affect the hamiltonicity of the graph, that is, G_{11} is hamiltonian if and only if G is hamiltonian.

It should be noted that the construction for even values of d, although very similar, requires the use of the graph in Figure 30 instead of the Goldner-Harary graph in order to avoid problems with vertex degree parity

x_1 u_1	x_2 u_2	x_3 u_3	x_1 u_1	x_2 u_2	x_3 u_3	v_1 u_4	v_2 u_5	v_{10} u_6	v_3 u_4	v_4 u_5	v_{11} u_6
v_2 w_2	v_5 w_1	v_6 w_3	v_1 w_2	v_4 w_1	v_9 w_3	v_2 w_1	v_3 w_2	v_7 w_3	v_4 y_2	v_5 y_1	v_8 y_3

Table 3: Vertices identified in the proof of Case 2: $\mathcal{R} = \{11\}$.

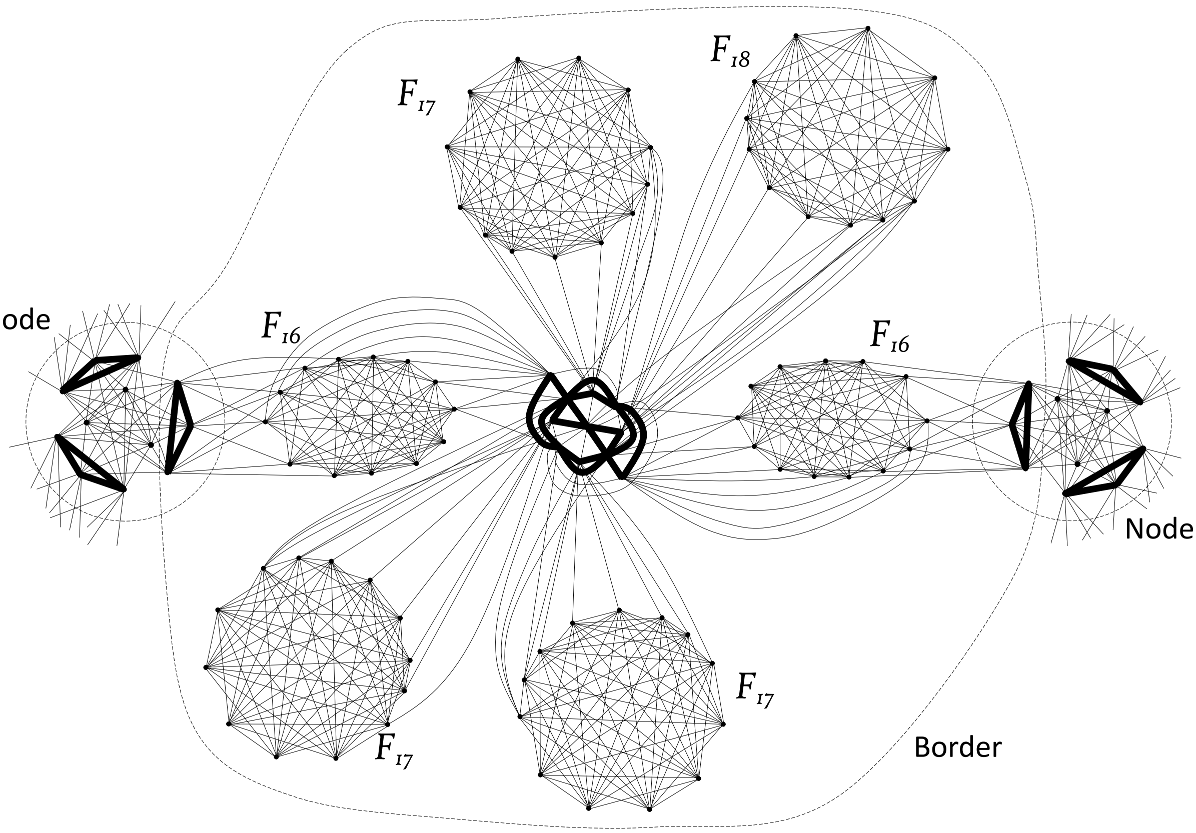

Figure 29: A border and the two nodes attached to it in the graph G_{11}.

(that is, having an odd number of edges connecting r-regular components, where r is an even number).

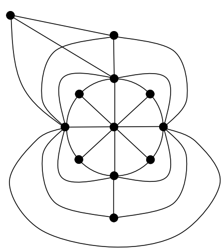

Figure 30: The graph that replaces the Goldner-Harary graph for even values of d.

Finally, we address the issue of connectivity. It is easy to construct a k-connected LH graph that is not hamiltonian, for any $k \geq 3$. We start with the complete graph K_{k+3} with vertices labeled $v_0, v_1, \ldots, v_{k+2}$. To this graph we add $k+2$ mutually independent vertices of degree k labeled $u_0, u_1, \ldots, u_{k+1}$ in such a way that $N(u_i) = \{v_i, v_{i+1}, \ldots, v_{i+(k-1)}\}$, subscripts taken modulo $(k+2)$. Call this graph H_k. The graph H_k is not hamiltonian since it is not 1-tough, because $\{v_0, v_1, \ldots, v_{k+1}\}$ is a cutset of order $k+2$, the removal of which results in a graph consisting of $k+3$ mutually independent vertices, $v_{k+2}, u_o, u_1, \ldots, u_{k+1}$. Since $\langle N(u_i)\rangle$, $i = 0, 1, \ldots, k+1$ and $\langle N(v_{k+2})\rangle$ are complete graphs, they are hamiltonian. For $i \in \{0, 1, \ldots, k+1\}$, $N(v_i) = \{v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k+2}, u_i, u_{i-1}, \ldots, u_{i-(k-1)}\}$, subscripts taken modulo $(k+2)$. Thus $v_{i-(k-1)}u_{i-(k-1)}v_{i-(k-2)}u_{i-(k-2)} \cdots v_{i-2}u_{i-2}v_{i-1}u_{i-1}v_{i+1}u_i v_{i+2}v_{k+2}v_{i-(k-1)}$ is a Hamilton cycle of $\langle N(v_i)\rangle$. It follows that H_k is LH. Since H_k consists of a clique of order $k+3$ and vertices of degree k whose neighbours are all in the clique, it is easy to see that H_k is k-connected. The graph H_4 can be seen in Figure 31.

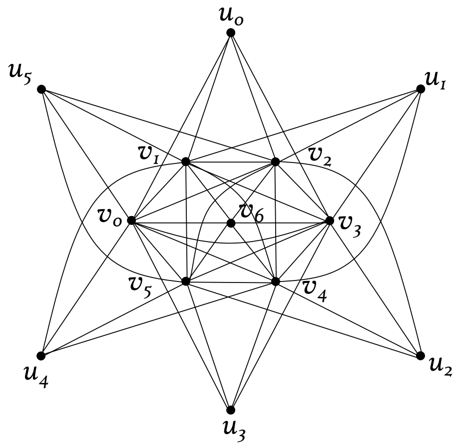

Figure 31: The graph H_4.

Theorem 3.7. *The HCP for k-connected LH graphs is NP-complete.*

PROOF. Again we base the proof on the construction used in the proof of Theorem 3.5, except that now the Goldner-Harary graph is replaced with the graph H_k and we do not use triangle identification. The nodes of G_k consist of copies of the complete graph K_{3k}. We create the graph G_k by connecting the graph H_k to two nodes, say N_1 and N_2, in the following way. Divide the vertices of each node into three equal subsets, each of which induces a clique of order k. The vertices in each of these cliques will only have neighbours in the same border of G_k. Let one of these subsets of $V(N_1)$ be W_1 with vertices $\{w_{1,1}, w_{1,2}, \ldots, w_{1,k}\}$ and let one of these subsets of $V(N_2)$ be W_2 with vertices $\{w_{2,1}, w_{2,2}, \ldots, w_{2,k}\}$. Edges are added between u_0 and each of the vertices in $\{w_{1,1}, w_{1,2}, \ldots, w_{1,k}\}$. Then for each v_j, where $j \in \{0, 1, \ldots, k-2\}$, edges are added between v_j and the vertices in $\{w_{1,1}, w_{1,2}, \ldots, w_{1,k-j-1}\}$. Similarly, edges are added between $u_{\lceil \frac{k+1}{2} \rceil}$ and the vertices in $\{w_{2,1}, w_{2,2}, \ldots, w_{2,k}\}$, and for each v_m, $m \in \{3, 4, \ldots, k+1\}$, edges are added between v_m and the vertices in $\{w_{2,1}, w_{2,2}, \ldots, w_{2,k-m+2}\}$. Figure 32 illustrates the construction for $k = 4$.

By symmetry, it is only necessary to confirm that the neighbourhoods of the vertices u_0, v_j, where $j \in \{0, 1, \ldots, k-2\}$ and $w_{1,m}$, where $m \in \{1, 2, \ldots, k\}$ induce hamiltonian graphs to show that G_k is LH. Note that $\langle N_{G_k}(u_0)\rangle$ consists of two cliques with at least three edges between the two cliques. It follows that $\langle N_{G_k}(u_0)\rangle$ is hamiltonian. For each v_j, where $j \in \{0, 1, \ldots, k-2\}$, there is a Hamilton cycle (described in the construction of the graph H_k) in $\langle N_{H_k}(v_j)\rangle$ which contains an edge between u_0 and a vertex v_p, where $p \in \{0, 1, \ldots, k-2\}$. In $\langle N_{G_k}(v_j)\rangle$, this edge can be replaced by the path $u_0 W v_p$, where W is a path in N_1 that includes all the vertices of $V(N_1) \cap N(v_j)$. Also, for each $m \in \{1, 2, \ldots, k\}$, $\langle N_{G_k}(w_{1,m})\rangle$ consists of two cliques with multiple edges between the cliques. It follows that $\langle N_{G_k}(w_{1,m})\rangle$ is hamiltonian, and therefore that G_k is LH.

Since the graph H_k is traceable between u_i and u_j for any distinct pair $\{u_i, u_j\} \subseteq \{u_0, u_1, \ldots, u_{k+1}\}$ and $H_k - u_i$ is hamiltonian for each $i \in \{0, 1, \ldots, k+1\}$, it follows that if G is hamiltonian then G_k is

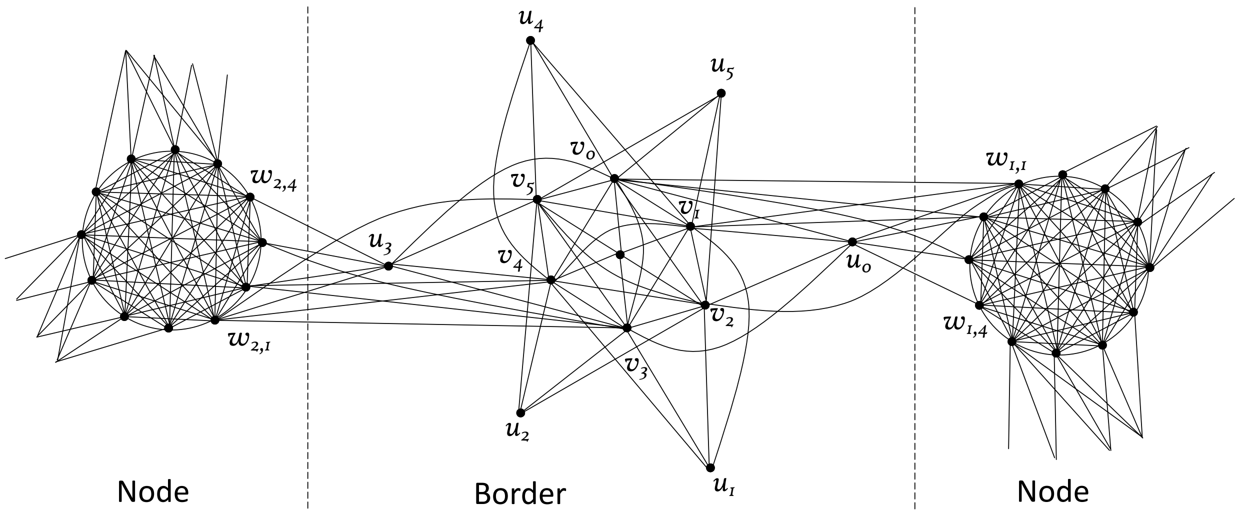

Figure 32: A border and the two nodes attached to it in the graph G_k for the case $k = 4$.

hamiltonian. To see that G is hamiltonian if G_k is hamiltonian, note that the vertices in the set $A = \{u_1, u_2, \ldots, u_{\lceil \frac{k+1}{2} \rceil - 1}, u_{\lceil \frac{k+1}{2} \rceil + 1}, \ldots, u_{k+1}, v_k\}$ only have neighbours in the set $B = \{v_0, v_1, \ldots, v_{k+1}\}$, and therefore any Hamilton cycle in G_k contains a path P such that $V(P) = A \cup B$. It follows that no Hamilton cycle can contain more than one path through any given border.

Corollary 3.8. *The HCP for k-connected LH graphs that are locally $(k-1)$-connected is NP-complete.*

Proof. It is sufficient to show that the graph G_k constructed in the proof of Theorem 3.7 is locally $(k-1)$-connected. Note that $\langle N(u_i) \rangle$, $i \in \{1, 2, \ldots, \lceil \frac{k+1}{2} \rceil - 1, \lceil \frac{k+1}{2} \rceil + 1, \ldots, k+1\}$, is a complete graph of order k and is therefore $(k-1)$-connected. Similarly, $\langle N(v_{k+2}) \rangle$ is a complete graph of order $k+2$ and is therefore $(k-1)$-connected. For $i \in \{0, 1, \ldots, k-1\}$, $\langle N(v_i) \rangle$ can be constructed in the following way: start with a clique, say F_i, of order $(k+2)$, where $V(F_i) = \{v_0, v_1, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_{k+2}\}$. There are either $k-2$ or $k-1$ vertices labeled u_j adjacent to v_i, where $j \in \{1, 2, \ldots, \lceil \frac{k+1}{2} \rceil - 1, \lceil \frac{k+1}{2} \rceil + 1, \ldots, k+1\}$ (u_0 and $u_{\lceil \frac{k+1}{2} \rceil}$ will be addressed later). Each of these vertices have $k-1$ neighbours in $V(F_i)$, and no other neighbours in $N(v_i)$.

Furthermore, each of the vertices in $\{v_0, v_1, \ldots, v_{i-1}\}$ is adjacent to each of the vertices in $\{w_1, w_2, \ldots, w_{k-i-1}\}$. Also, for $m \in \{1, 2, \ldots, k-i-2\}$, v_{i+m} is adjacent to $w_{k-i-1-m}$ (for example, v_{i+1} is adjacent to w_{k-i-2} and $v_{k-2} = v_{i+(k-i-2)}$ is adjacent to w_1). Finally, u_0 is adjacent to all the vertices in $\{v_0, v_1, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_{k+2}, w_1, w_2, \ldots, w_{k-i-1}\}$. Therefore, in $\langle N(v_i) \rangle$, for any vertex w_x, there are $(k-1)$ internally disjoint paths from w_x to $v_p \in V(F_i)$. See Figure 33 for a schematic representation. By symmetry, it follows that $\langle N(v_i) \rangle$, $i = 0, 1, \ldots, k+1$ is $(k-1)$-connected.

All that remains to be addressed are the neighbourhoods of u_0, $u_{\lceil \frac{k+1}{2} \rceil}$ and the vertices in the nodes. It follows from the above discussion that the graph induced by the neighbourhood of each of these vertices is $(k-1)$-connected. By symmetry, this exhausts the cases to consider, and it follows that G_k is locally $(k-1)$-connected.

4. Conclusions

In our view the most significant unresolved questions relating to the hamiltonicity of LT and LH graphs are the following:

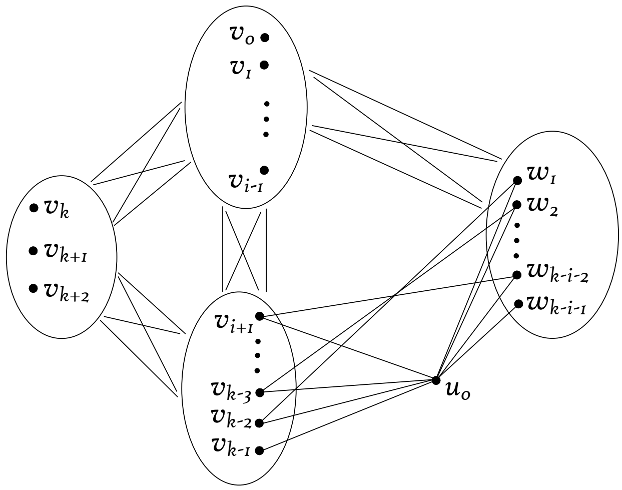

Figure 33: A schematic drawing of part of $\langle N(v_i)\rangle$. Only vertices and edges are shown that are necessary to see that there are $(k-1)$ internally disjoint paths from any vertex in a node (vertices labeled w) to any vertex in F_i (vertices labeled v). The ovals represent cliques.

- Do connected nonhamiltonian LT graphs that are $\{4,6\}$-regular or 6-regular exist, and if so, is the HCP for these classes of graphs NP-complete?
- Are all connected LH graphs with maximum degree at most 7 hamiltonian?
- Is the HCP for LH graphs with maximum degree 8 solvable in polynomial time?

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