ON COMPACT PACKINGS OF THE PLANE WITH CIRCLES OF THREE RADII

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Abstract. A compact circle-packing P of the Euclidean plane is a set of circles which bound mutually disjoint open discs with the property that, for every circle $S \in P$, there exists a maximal indexed set $\{A_0, \ldots, A_{n-1}\} \subseteq P$ so that, for every $i \in \{0, \ldots, n-1\}$, the circle A_i is tangent to both circles S and $A_{i+1 \mod n}$.

We show that there exist at most 13617 pairs (r, s) with $0 < s < r < 1$ for which there exist a compact circle-packing of the plane consisting of circles with radii s, r and 1.

We discuss computing the exact values of such $0 < s < r < 1$ as roots of polynomials and exhibit a selection of compact circle-packings consisting of circles of three radii. We also discuss the apparent infeasibility of computing *all* these values on contemporary consumer hardware.

1. Introduction

By a *circle-packing* (or just *packing*) P we mean a set of circles in the Euclidean plane, so that the open discs bounded by the circles are pairwise disjoint. We define $\text{radii}(P) := \{\text{radius}(S) \mid S \in P\}$. If $|\text{radii}(P)| < \infty$, we will assume that P is maximal and that it is scaled so that $\max \text{radii}(P) = 1$. For $n \in \mathbb{N}$, we will say P is an *n-packing* if $|\text{radii}(P)| = n$. We say a circle-packing P is *compact* if, for every circle $S \in P$, there exists some $m \in \mathbb{N}$ and a maximal indexed set of circles $\{A_0, \ldots, A_{m-1}\} \subseteq P$ so that all the circles $A_0, \ldots, A_{m-1}$ are tangent to S and, for every $i \in \{0, \ldots, m-1\}$, the circle A_i is tangent to $A_{i+1 \mod m}$. The circles $A_0, \ldots, A_{m-1}$ are called the *neighbors* of S. For $n \in \mathbb{N}$, we define the sets

$$\Delta_n := \left\{ (r_i)_{i=1}^{n-1} \in (0,1)^{n-1} \;\middle|\; 0 < r_{n-1} < \ldots < r_1 < 1 \right\}$$

and

$$\Pi_n := \left\{ (r_i) \in \Delta_n \;\middle|\; \begin{array}{c} \text{There exists a compact } n\text{-packing } P \\ \text{with } \text{radii}(P) = \{r_1, \ldots, r_{n-1}, 1\}. \end{array} \right\}.$$

In [7], Kennedy proved that $|\Pi_2| = 9$, i.e., that there exist exactly nine values of $r_0 \in (0,1)$ for which there exist compact 2-packings P with $\text{radii}(P) = \{r_0, 1\}$. Eight of these nine values were known previously: seven values appear in [5] and a further one in [8]. Kennedy computed the remaining value[1] $r_0 \approx 0.545151$ and demonstrated the existence of a compact 2-packing P with $\text{radii}(P) = \{r_0, 1\}$.

In this paper we will concern ourselves with compact 3-packings.

Of course, one may construct a compact 3-packing by packing circles into the interstitial gaps of a compact 2-packing, hence $|\Pi_3| \geq |\Pi_2| = 9$. Therefore the first

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[1]A root of the polynomial $r^8 - 8r^7 - 44r^6 - 232r^5 - 482r^4 - 24r^3 + 388r^2 - 120r + 9$.

question to ask is whether there exists a compact 3-packing that does not arise in this way. By merely guessing, it is possible to construct such a packing, cf. Figure 1.1. Another such packing appears in [5, Fig. 15. 27/1, p.187]. Hence $|\Pi_3| > |\Pi_2| = 9$ and since not all compact 3-packings arise from compact 2-packings by filling interstitial gaps, we are motivated to ask:

Question. *What is the cardinality of* Π_3*?*

The first goal of this paper is to answer this question by proving that Π_3 is finite (cf. Theorem 5.5). The second goal is to obtain the bound $|\Pi_3| \leq 13617$ (cf. Sections 7 and 8).

We briefly describe the analysis leading up to this result.

The main idea follows the arguments presented in [7] quite closely in spirit, but does become more technical and relies significantly on searches performed by computer. The majority of the work concerns analysis of the functions $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} : \Delta_3 \to (0, \pi)^6$ (these functions are defined explicitly in Section 2), which parameterize the possible sizes of angles formed by connecting the centers of mutually tangent circles of radii s, r or 1 with $0 < s < r < 1$. By construction (cf. Section 2), a necessary condition for $(r_0, s_0) \in \Delta_3$ to be an element of Π_3 is that there exist specific tuples $\eta, \zeta, \xi \in \mathbb{Z}^6$ with non-negative coordinates satisfying

$$\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = \zeta \cdot \boldsymbol{\beta}(r_0, s_0) = \xi \cdot \boldsymbol{\gamma}(r_0, s_0) = 2\pi.$$

I.e., the 2π-contours of the three functions $\Delta_3 \ni (r, s) \mapsto \eta \cdot \boldsymbol{\alpha}(r, s)$, $\Delta_3 \ni (r, s) \mapsto \zeta \cdot \boldsymbol{\beta}(r, s)$ and $\Delta_3 \ni (r, s) \mapsto \xi \cdot \boldsymbol{\beta}(r, s)$ intersect in (r_0, s_0).

Theorem 3.7 establishes necessary conditions that such tuples $\eta, \zeta, \xi \in \mathbb{Z}^6$ must satisfy, and one easily computes that there exist only 55 tuples η for which it is possible to have $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ for some $(r_0, s_0) \in \Pi_3$, (cf. Proposition 3.9). In Section 4, by a careful and rather technical analysis of the 2π-contours of the functions $\Delta_3 \ni (r, s) \mapsto \eta \cdot \boldsymbol{\alpha}(r, s)$ and $\Delta_3 \ni (r, s) \mapsto \zeta \cdot \boldsymbol{\beta}(r, s)$ and using the 55 tuples η computed earlier, we establish a final necessary condition on $\zeta \in \mathbb{Z}^6$ for $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$ to hold for some $(r_0, s_0) \in \Pi_3$. This final condition shows that there can exist only finitely many such $\zeta \in \mathbb{Z}^6$, and allows for the exact computation of all 248395 elements of a certain set $K \subseteq \mathbb{Z}^{6\times 2}$ consisting of all tuples η and $\zeta \in \mathbb{Z}^6$ which satisfy the necessary conditions that we established (cf. Proposition 5.2).

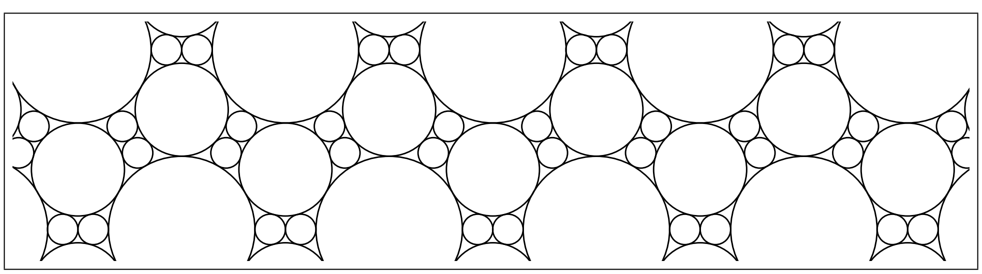

FIGURE 1.1. A compact circle-packing P with $\text{radii}(P) = \{s_0, r_0, 1\}$ where $s_0 \approx 0.208266$ and $r_0 \approx 0.635671$ are respective roots of the polynomials $1 + 2s - 27s^2 - 28s^3 + 4s^4$ and $-1 - 12r - 18r^2 + 60r^3 + 3r^4$. We have $(r_0, s_0) \in \Pi_3$, but there exists no compact circle-packing Q with $\text{radii}(Q) = \{r_0, 1\}$, [7].

By observing that each element of Π_3 is determined by an element from K (cf. Proposition 5.4), and that each element of K determines at most one element of Π_3 (cf. Proposition 5.3), we conclude that the set Π_3 is finite and that $|\Pi_3|\leq 248395$ (cf. Theorem 5.5).

A further analysis of the 2π-contours of the functions $\Delta_3 \ni (r,s) \mapsto \eta\cdot\boldsymbol{\alpha}(r,s)$ and $\Delta_3 \ni (r,s) \mapsto \zeta\cdot\boldsymbol{\beta}(r,s)$ in Section 6, provides necessary and sufficient conditions for such contours to intersect and allow for determining the sharper bound $|\Pi_3|\leq 13617$ with methods described in Sections 7 and 8.

The results of our computations are included as a dataset.

From here, further excluding elements that are not in Π_3 exactly seems to be infeasible on contemporary consumer hardware. Firstly, for a candidate element $(r_0, s_0) \in \Delta_3$ to be an element of Π_3, there necessarily must exist a certain tuple $\xi \in \mathbb{Z}^6$ that satisfies $\xi \cdot \boldsymbol{\gamma}(r_0, s_0) = 2\pi$. However, the search space of all $\xi \in \mathbb{Z}^6$ that might satisfy $\xi \cdot \boldsymbol{\gamma}(r_0, s_0) = 2\pi$ can sometimes be very large (containing up to 7×10^{21} elements), and is hence very time-consuming to sift through (cf. Section 7). Secondly, given a numerical approximation of a candidate element (r_0, s_0) of Π_3, it is possible to compute polynomials which have the exact values of r_0 and s_0 as roots (cf. Section 8). Although computing these polynomials proceeds through a simple algorithm (Algorithm 8.1) together with computing standard Gröbner bases, performing the actual computation can be very time-consuming and RAM intensive depending on the input.

It is however possible to compute certain elements of Π_3 exactly, and we display an arbitrary (but far from exhaustive) selection of compact 3-packings in the final section.

The fact that both Π_2 and Π_3 are finite, and that not every compact 3-packing arises from a compact 2-packing by filling interstitial gaps, motivates the following conjecture:

Conjecture. *For every $n \in \mathbb{N}$, the set Π_n is finite and the sequence $(|\Pi_n|)_{n\in\mathbb{N}}$ is strictly increasing.*

Furthermore, for every $n \in \mathbb{N}$, there exists an element $(r_{n-1}, r_{n-2}, \ldots, r_1) \in \Pi_n$ with $(r_{n-2}, \ldots, r_1) \notin \Pi_{n-1}$, i.e., not every compact n-packing arises from filling interstitial gaps of a compact $(n-1)$-packing.

The computational nature of the problem might be a hindrance to proving this conjecture. Proposition 3.9, which is established purely by exhaustion performed by computer, is required to bootstrap the proofs of Propositions 4.1(6) and 4.2(7) which together are crucial in our proof that $|\Pi_3|< \infty$. This suggests that establishing $|\Pi_n|< \infty$ for specific values of $n \in \mathbb{N}$ might be easier, but perhaps less interesting, than a proof of the above conjecture in its full generality.

2. Preliminary definitions, results and notation

We define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{T} := \mathbb{N}_0^6$. Let three mutually tangent circles A, B and C have respective radii a, b and c. By the cosine rule, the angle $\theta(a, b, c)$ formed at the center of A by the line segments connecting the center of A with the centers of B and C is given by

$$\theta(a,b,c) = \arccos\left(\frac{(a+b)^2 + (a+c)^2 - (b+c)^2}{2(a+c)(a+b)}\right).$$

We define the functions $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} : \Delta_3 \to (0, \pi)^6$, for $(r, s) \in \Delta_3$ by

$$\begin{aligned}
\boldsymbol{\alpha}(r,s) &:= (\theta(s,1,1), \theta(s,r,r), \theta(s,s,s), \theta(s,1,r), \theta(s,1,s), \theta(s,r,s)),\\
\boldsymbol{\beta}(r,s) &:= (\theta(r,1,1), \theta(r,r,r), \theta(r,s,s), \theta(r,1,r), \theta(r,1,s), \theta(r,r,s)),\\
\boldsymbol{\gamma}(r,s) &:= (\theta(1,1,1), \theta(1,r,r), \theta(1,s,s), \theta(1,1,r), \theta(1,1,s), \theta(1,r,s)).
\end{aligned}$$

Let $(r_0, s_0) \in \Pi_3$ be fixed and let P be a compact 3-packing with radii$(P) = \{s_0, r_0, 1\}$. For any $D \in P$ of radius $t \in \{s_0, r_0, 1\}$. We set

$$\boldsymbol{\tau} := \begin{cases} \boldsymbol{\alpha} & t = s_0 \\ \boldsymbol{\beta} & t = r_0 \\ \boldsymbol{\gamma} & t = 1. \end{cases}$$

Let $\{A_0, \ldots, A_{n-1}\} \subseteq P$ be the sequence of neighbors of D for some $n \in \mathbb{N}$. Connecting the center of D with the centers of $A_0, \ldots, A_{n-1}$, we denote the *angle-count for D* by $\xi^{(D)} \in \mathbb{T}$, which has, for $i \in \{1, 2, 3, 4, 5, 6\}$, its i'th coordinate defined as number of times the angle $\boldsymbol{\tau}_i(r_0, s_0)$ occurs around the center of D. Explicitly: We define

$$\begin{array}{rclcrclcl}
\kappa_{1,1} & := & (1,0,0,0,0,0) & \quad & \kappa_{1,r_0} & := & \kappa_{r_0,1} & := & (0,0,0,1,0,0)\\
\kappa_{r_0,r_0} & := & (0,1,0,0,0,0) & \quad & \kappa_{1,s_0} & := & \kappa_{s_0,1} & := & (0,0,0,0,1,0)\\
\kappa_{s_0,s_0} & := & (0,0,1,0,0,0) & \quad & \kappa_{r_0,s_0} & := & \kappa_{s_0,r_0} & := & (0,0,0,0,0,1),
\end{array}$$

and define $\sigma(j) := \text{radius}(A_j) \in \{s_0, r_0, 1\}$ for $j \in \{0, \ldots, n-1\}$. Then

$$\xi^{(D)} := \sum_{j=0}^{n-1} \kappa_{\sigma(j), \sigma(j+1 \mod n)}.$$

Since $\{A_0, \ldots, A_{n-1}\}$ is the sequence of neighbors of D, it is clear that

$$\xi^{(D)} \cdot \boldsymbol{\tau}(r_0, s_0) = 2\pi.$$

Hence, for all circles A, B and C in P with respective radii s_0, r_0 and 1, we have

$$\xi^{(A)} \cdot \boldsymbol{\alpha}(r_0, s_0) = \xi^{(B)} \cdot \boldsymbol{\beta}(r_0, s_0) = \xi^{(C)} \cdot \boldsymbol{\gamma}(r_0, s_0) = 2\pi.$$

Therefore, a necessary condition for (r_0, s_0) to be an element of Π_3 is that the 2π-contours of the functions

$$\begin{aligned}
\Delta_3 \ni (r,s) &\mapsto \xi^{(A)} \cdot \boldsymbol{\alpha}(r,s),\\
\Delta_3 \ni (r,s) &\mapsto \xi^{(B)} \cdot \boldsymbol{\beta}(r,s),\\
\Delta_3 \ni (r,s) &\mapsto \xi^{(C)} \cdot \boldsymbol{\gamma}(r,s),
\end{aligned}$$

intersect in (r_0, s_0).

3. Necessary conditions on angle-counts

Let $(s_0, r_0) \in \Pi_3$ and let P be a compact 3-packing with radii$(P) = \{s_0, r_0, 1\}$. In the current section we show that there necessarily exist circles A,B and C in P of respective radii s_0,r_0 and 1, whose angle counts $\xi^{(A)}$, $\xi^{(B)}$ and $\xi^{(C)}$ must necessarily satisfy certain conditions. These conditions are collected in Theorem 3.7. This theorem prompts the definition of a number of predicates in Definition 3.8 which can be easily implemented in any programming language. Finally, Proposition 3.9 lists all 55 possible values that the tuple $\xi^{(A)}$ can take on. This observation is crucial in later sections for showing that the tuple $\xi^{(B)}$ may also only take on finitely many values.

We begin by observing that for each radius $t \in \{s_0, r_0, 1\}$, there must exist a circle $D \in P$ of t that is not fully surrounded by 6 neighbors with radius t.

Proposition 3.1. *Let $(r_0, s_0) \in \Pi_3$ and let P be any compact 3-packing with radii$(P) = \{s_0, r_0, 1\}$.*

(1) *There exists a circle $A \in P$ with radius s_0 so that $\xi^{(A)} \cdot (1,1,0,1,1,1) \neq 0$ and $\xi^{(A)} \cdot (0,0,1,0,0,0) < 6$.*
(2) *There exists a circle $B \in P$ with radius r_0 so that $\xi^{(B)} \cdot (1,0,1,1,1,1) \neq 0$ and $\xi^{(B)} \cdot (0,1,0,0,0,0) < 6$.*
(3) *There exists a circle $C \in P$ with radius 1 so that $\xi^{(C)} \cdot (0,1,1,1,1,1) \neq 0$ and $\xi^{(C)} \cdot (1,0,0,0,0,0) < 6$.*

Proof. We prove (1). Suppose, for all $A \in P$ of radius s_0, that $\xi^{(A)} \cdot (1,1,0,1,1,1) = 0$. Then we must have $\xi^{(A)} \cdot (0,0,1,0,0,0) = 6$ for all $A \in P$ of radius s_0, and hence P either cannot contain circles of radius s_0, or cannot contain circles of radii 1 or r_0. This contradicts radii$(P) = \{s_0, r_0, 1\}$. The other assertions follow similarly. □

Next, we observe that there must exist a pair of circles of respective radii s_0 or r_0, so that at least one of these circles has a neighbor of radius 1.

Proposition 3.2. *Let $(r_0, s_0) \in \Pi_3$ and let P be any compact 3-packing with radii$(P) = \{s_0, r_0, 1\}$. There exist circles A and B from P with respective radii s_0 and r_0, so that $\xi^{(A)} \cdot (1,0,0,1,1,0) \neq 0$ or $\xi^{(B)} \cdot (1,0,0,1,1,0) \neq 0$.*

Proof. If it were the case that for every pair of circles A and B from P with respective radii s_0 and r_0 that $\xi^{(A)} \cdot (1,0,0,1,1,0) = 0$ and $\xi^{(B)} \cdot (1,0,0,1,1,0) = 0$, then P could not contain any circles of radius 1, or consists only of circles of radius 1. This contradicts radii$(P) = \{s_0, r_0, 1\}$. □

Propositions 3.3 through 3.6, establishes general necessary conditions that circles in P must satisfy.

Proposition 3.3. *Let P be any compact 3-packing. For every circle $D \in P$, we have*

(1) $\xi^{(D)} \cdot (2,0,0,1,1,0) = 0 \mod 2$.
(2) $\xi^{(D)} \cdot (0,2,0,1,0,1) = 0 \mod 2$.
(3) $\xi^{(D)} \cdot (0,0,2,0,1,1) = 0 \mod 2$.

Proof. Let $(r_0, s_0) \in \Pi_3$ be such that radii$(P) = \{s_0, r_0, 1\}$. Consider any circle C in the packing P of radius 1. The line segments connecting the center of C to the centers of its neighboring circles in P, can only have lengths 2, $1 + r_0$ or $1 + s_0$. Since each such line segment is a leg of exactly two angles formed around the center of C, the angle-count with a given leg-length must be even. I.e.,

$$\xi^{(C)} \cdot (2,0,0,1,1,0) = 0 \mod 2,$$
$$\xi^{(C)} \cdot (0,2,0,1,0,1) = 0 \mod 2,$$
$$\xi^{(C)} \cdot (0,0,2,0,1,1) = 0 \mod 2.$$

A similar argument will establish the result if C has radius r_0 or s_0. □

Proposition 3.4. *Let* $(r_0, s_0) \in \Pi_3$ *and let* P *be any compact* 3*-packing with radii*$(P) = \{s_0, r_0, 1\}$*. For any circle* $A \in P$ *of radius* s_0 *with* $\xi^{(A)} \cdot (1,1,0,1,1,1) \neq 0$*, we have* $\xi^{(A)} \cdot (1,1,1,1,1,1) < 6$ *and* $\xi^{(A)} \cdot (6,6,2,6,3,3) > 12$.

Proof. Since $0 < s_0 < r_0 < 1$, we have $\boldsymbol{\alpha}_i(r_0, s_0) > 3^{-1}\pi$ for all $i \in \{1,2,4,5,6\}$ and $\boldsymbol{\alpha}_3(r_0, s_0) = 3^{-1}\pi$. Therefore, since $\xi^{(A)} \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$, we have $\xi^{(A)} \cdot (1,1,1,1,1,1) < 3\pi^{-1}(\xi^{(A)} \cdot \boldsymbol{\alpha}(r_0, s_0)) = 6$.

On the other hand, since $0 < s_0 < r_0 < 1$, we have

$$\boldsymbol{\alpha}_1(r_0, s_0),\ \boldsymbol{\alpha}_2(r_0, s_0),\ \boldsymbol{\alpha}_4(r_0, s_0),\ 2\boldsymbol{\alpha}_5(r_0, s_0),\ 2\boldsymbol{\alpha}_6(r_0, s_0) < \pi$$

and $\boldsymbol{\alpha}_3(r_0, s_0) = 3^{-1}\pi$. Therefore, since $\xi^{(A)} \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$, we obtain $12 = 6\pi^{-1}(\xi^{(A)} \cdot \boldsymbol{\alpha}(r_0, s_0)) < \xi^{(A)} \cdot (6,6,2,6,3,3)$. ◻

Proposition 3.5. *Let* $(r_0, s_0) \in \Pi_3$ *and let* P *be any compact* 3*-packing with radii*$(P) = \{s_0, r_0, 1\}$*. For any circle* $B \in P$ *of radius* r_0 *with* $\xi^{(B)} \cdot (1,0,1,1,1,1) \neq 0$*, we have* $\xi^{(B)} \cdot (1,1,0,1,0,0) < 6$ *and* $\xi^{(B)} \cdot (6,2,2,3,3,2) > 12$.

Proof. Since $0 < s_0 < r_0 < 1$, we have $\boldsymbol{\beta}_1(r_0, s_0),\ \boldsymbol{\beta}_4(r_0, s_0) > 3^{-1}\pi$, with

$$\boldsymbol{\beta}_3(r_0, s_0),\ \boldsymbol{\beta}_5(r_0, s_0),\ \boldsymbol{\beta}_6(r_0, s_0) > 0$$

and $\boldsymbol{\beta}_2(r_0, s_0) = 3^{-1}\pi$. Therefore, since $\xi^{(B)} \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$, we obtain $\xi^{(B)} \cdot (1,1,0,1,0,0) < 3\pi^{-1}(\xi^{(B)} \cdot \boldsymbol{\beta}(r_0, s_0)) = 6$.

On the other hand, since $0 < s_0 < r_0 < 1$, we have

$$\boldsymbol{\beta}_1(r_0, s_0),\ 3\boldsymbol{\beta}_3(r_0, s_0),\ 2\boldsymbol{\beta}_4(r_0, s_0),\ 2\boldsymbol{\beta}_5(r_0, s_0),\ 3\boldsymbol{\beta}_6(r_0, s_0) < \pi$$

and $\boldsymbol{\beta}_2(r_0, s_0) = 3^{-1}\pi$. Therefore, since $\xi^{(B)} \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$, we obtain $\xi^{(B)} \cdot (6,2,2,3,3,2) > 6\pi^{-1}(\xi^{(B)} \cdot \boldsymbol{\beta}(r_0, s_0)) = 12$. ◻

Proposition 3.6. *Let* $(r_0, s_0) \in \Pi_3$ *and let* P *be any compact* 3*-packing with radii*$(P) = \{s_0, r_0, 1\}$*. For every circle* $B \in P$ *with radius* r_0*, we have*

$$\xi^{(B)} \cdot (2,2,0,2,1,1) \leq 12,$$

with the inequality strict if $\xi^{(B)} \cdot (2,0,0,1,1,0) > 0$.

Proof. Let B be any circle in P with radius r_0 and let N be the set of neighbors of B. Then

$$\xi^{(B)} \cdot (2,0,0,1,1,0) = 2|\{C \in N | \text{radius}(C) = 1\}|,$$
$$\xi^{(B)} \cdot (0,2,0,1,0,1) = 2|\{C \in N | \text{radius}(C) = r_0\}|.$$

However, all the angles formed at the center of B by connecting the center of B with the centers of circles from $\{C \in N | \text{radius}(C) \in \{r_0, 1\}\}$ are greater or equal to $\pi/3$ and add up to 2π. Therefore $|\{C \in N | \text{radius}(C) \in \{r_0, 1\}\}| \leq 6$, and hence

$$\begin{aligned}
\xi^{(B)} \cdot (2,2,0,2,1,1) &= \xi^{(B)} \cdot (0,2,0,1,0,1) + \xi^{(B)} \cdot (2,0,0,1,1,0) \\
&= 2|\{C \in N | \text{radius}(C) = r_0\}| + 2|\{C \in N | \text{radius}(C) = 1\}| \\
&= 2|\{C \in N | \text{radius}(C) \in \{r_0, 1\}\}| \\
&\leq 12.
\end{aligned}$$

If $\xi^{(B)} \cdot (2,0,0,1,1,0) > 0$, then $\{C \in N | \text{radius}(C) = 1\} \neq \emptyset$. Hence at least one of the angles formed at the center of B by connecting the center of B with the centers of circles from $\{C \in N | \text{radius}(C) \in \{r_0, 1\}\}$ is greater than $\pi/3$, with all of them still adding up to 2π. Therefore we must have $|\{C \in N | \text{radius}(C) \in \{r_0, 1\}\}| < 6$, and the result follows. □

By collecting the previous propositions into the following theorem, we note, for every $(r_0, s_0) \in \Pi_3$ and compact 3-packing P with radii$(P) = \{s_0, r_0, 1\}$, that there must necessarily exist three circles in P with respective radii s_0, r_0 and 1, whose angle-counts satisfy the stated conditions.

Theorem 3.7. *Let $(r_0, s_0) \in \Pi_3$ and let P be any compact 3-packing with radii$(P) = \{s_0, r_0, 1\}$. There exist circles A, B and C in P with respective radii s_0, r_0 and 1 so that:*

(1) $\xi^{(A)} \cdot (1,0,0,1,1,0) \neq 0$ *or* $\xi^{(B)} \cdot (1,0,0,1,1,0) \neq 0$.
(2) $\xi^{(A)} \cdot (1,1,0,1,1,1) \neq 0$.
(3) $\xi^{(A)} \cdot (1,1,1,1,1,1) < 6$ *and* $\xi^{(A)} \cdot (6,6,2,6,3,3) > 12$.
(4) $\xi^{(B)} \cdot (1,0,1,1,1,1) \neq 0$.
(5) $\xi^{(B)} \cdot (1,1,0,1,0,0) < 6$ *and* $\xi^{(B)} \cdot (6,2,2,3,3,2) > 12$.
(6) $\xi^{(B)} \cdot (2,2,0,2,1,1) \leq 12$.
(7) *If* $\xi^{(B)} \cdot (2,0,0,1,1,0) > 0$, *then* $\xi^{(B)} \cdot (2,2,0,2,1,1) < 12$.
(8) $\xi^{(C)} \cdot (0,1,1,1,1,1) \neq 0$ *and* $\xi^{(C)} \cdot (1,0,0,0,0,0) < 6$.
(9) *For* $D \in \{A, B, C\}$ *we have*

$$\xi^{(D)} \cdot (2,0,0,1,1,0) = 0 \mod 2,$$
$$\xi^{(D)} \cdot (0,2,0,1,0,1) = 0 \mod 2,$$
$$\xi^{(D)} \cdot (0,0,2,0,1,1) = 0 \mod 2.$$

(10) *For* $D \in \{A, B, C\}$ *there exists some* $n \in \mathbb{N}$ *and* $\sigma \in \{s_0, r_0, 1\}^{\{0,\ldots,n-1\}}$ *with*

$$\xi^{(D)} = \sum_{j=0}^{n-1} \kappa_{\sigma(j),\sigma(j+1 \mod n)}.$$

Proof. Proposition 3.2 yields circles A and B in P with respective radii s_0 and r_0 so that

$$\xi^{(A)} \cdot (1,0,0,1,1,0) \neq 0 \quad \text{or} \quad \xi^{(B)} \cdot (1,0,0,1,1,0) \neq 0.$$

If $\xi^{(A)} \cdot (1,1,0,1,1,1) = 0$, then we also have $\xi^{(A)} \cdot (1,0,0,1,1,0) = 0$. Since the above disjunction is true, we must then have $\xi^{(B)} \cdot (1,0,0,1,1,0) \neq 0$, which implies $\xi^{(B)} \cdot (1,0,1,1,1,1) \neq 0$. Proposition 3.1 yields some $A' \in P$ of radius s_0, so that $\xi^{(A')} \cdot (1,1,0,1,1,1) \neq 0$ and $\xi^{(A')} \cdot (0,0,1,0,0,0) < 6$ and we redefine A as A'. We may employ a similar argument to redefine B if $\xi^{(B)} \cdot (1,0,1,1,1,1) = 0$. This establishes (1), (2) and (4)

By Proposition 3.1 there exists a circle $C \in P$ satisfying $\xi^{(C)} \cdot (0,1,1,1,1,1) \neq 0$ and $\xi^{(C)} \cdot (1,0,0,0,0,0) < 6$, establishing (8).

The remaining assertions (3), (5), (6), (7), (9) and (10) follow immediately from Propositions 3.3–3.6 and the definition of the angle-counts $\xi^{(A)}$, $\xi^{(B)}$ and $\xi^{(C)}$. □

Motivated by the previous result, we will define a number of predicates on $\mathbb{T}$ which will hopefully improve readability of the subsequent sections. These predicates are named in what is hoped to be a meaningful manner (even if some of their meanings might only become apparent in the next section). These predicates can easily be implemented on a computer.

Definition 3.8. Let $\eta, \zeta \in \mathbb{T}$ and let $(r_0, s_0) \in \Delta_3$ (here we regard r_0 and s_0 purely as distinct index symbols). We define the following predicates:

$$\mathrm{Seq}\,(\eta) := \exists n \in \mathbb{N},\ \exists \sigma \in \{s_0, r_0, 1\}^{\{0,\ldots,n-1\}},\ \eta = \sum_{j=0}^{n-1} \kappa_{\sigma(j),\sigma(j+1 \mod n)},$$

$$\begin{aligned}\mathrm{Mod2}\,(\eta) := & (\eta \cdot (2,0,0,1,1,0) = 0 \mod 2)\\ & \wedge (\eta \cdot (0,2,0,1,0,1) = 0 \mod 2)\\ & \wedge (\eta \cdot (0,0,2,0,1,1) = 0 \mod 2),\end{aligned}$$

$$\begin{aligned}\text{s-Bounds}\,(\eta) &:= (\eta \cdot (1,1,1,1,1,1) < 6) \wedge (\eta \cdot (6,6,2,6,3,3) > 12),\\ \text{s-NonHex}\,(\eta) &:= (\eta \cdot (1,1,0,1,1,1) \neq 0),\\ \text{s-Necessary}\,(\eta) &:= \text{s-Bounds}\,(\eta) \wedge \text{s-NonHex}\,(\eta) \wedge \mathrm{Seq}\,(\eta) \wedge \mathrm{Mod2}\,(\eta)\,,\end{aligned}$$

$$\begin{aligned}\text{r-Bounds}\,(\zeta) &:= (\zeta \cdot (1,1,0,1,0,0) < 6) \wedge (\zeta \cdot (6,2,2,3,3,2) > 12),\\ &\quad \wedge (\zeta \cdot (2,2,0,2,1,1) \leq 12),\\ \text{r-NonHex}\,(\zeta) &:= (\zeta \cdot (1,0,1,1,1,1) \neq 0),\\ \text{r-FewLargeNeighbors}\,(\zeta) &:= (\zeta \cdot (2,0,0,1,1,0) > 0) \Rightarrow (\zeta \cdot (2,2,0,2,1,1) < 12),\\ \text{r-Necessary}\,(\zeta) &:= \text{r-Bounds}\,(\zeta) \wedge \text{r-NonHex}\,(\zeta) \wedge \mathrm{Seq}\,(\zeta) \wedge \mathrm{Mod2}\,(\zeta)\,,\\ &\quad \wedge \text{r-FewLargeNeighbors}\,(\zeta)\\ \text{r-VerticalContour}\,(\zeta) &:= \zeta \cdot (0,0,1,0,1,1) = 0,\end{aligned}$$

$$\text{sr-Disjunct}\,(\eta,\zeta) := (\eta \cdot (1,0,0,1,1,0) \neq 0) \vee (\zeta \cdot (1,0,0,1,1,0) \neq 0),$$

$$\begin{aligned}\text{1-NonHex}\,(\eta) &:= (\eta \cdot (0,1,1,1,1,1) \neq 0) \wedge (\eta \cdot (1,0,0,0,0,0) < 6),\\ \text{1-Necessary}\,(\eta) &:= \text{1-NonHex}\,(\eta) \wedge \mathrm{Seq}\,(\eta) \wedge \mathrm{Mod2}\,(\eta)\,.\end{aligned}$$

Now, a straightforward brute-force search by computer can establish that the set $\{\eta \in \mathbb{T} \mid \text{s-Necessary}\,(\eta)\}$ is finite and has exactly 55 elements, see Proposition 3.9 below.

We note that the 3rd coordinate of elements from $\{\xi \in \mathbb{T} \mid \text{r-Necessary}\,(\xi)\}$ is not bounded above, and hence this set may be infinite. The next two sections will address this issue.

Proposition 3.9. *The set $\{\eta \in \mathbb{T} \mid \textit{s-Necessary}\,(\eta)\}$ has exactly 55 elements, and its members are listed in Table 1.*

4. Contour analysis

Theorem 3.7 provides no upper bound on the 3rd coordinate of angle-counts for midsize circles in a compact 3-packing. In this section, for arbitrary elements $\eta \in \{\xi \in \mathbb{T} \mid \text{s-Necessary}\,(\xi)\}$ and $\zeta \in \{\xi \in \mathbb{T} \mid \text{r-Necessary}\,(\xi)\}$, we will analyze the properties of the 2π-contours of the functions

$$\begin{aligned}\Delta_3 \ni (r,s) &\mapsto \eta \cdot \boldsymbol{\alpha}(r,s),\\ \Delta_3 \ni (r,s) &\mapsto \zeta \cdot \boldsymbol{\beta}(r,s).\end{aligned}$$

$(0,0,0,1,1,3)$	$(0,0,1,2,2,0)$	$(0,1,2,0,0,2)$	$(1,0,0,0,4,0)$	$(1,2,0,2,0,0)$
$(0,0,0,1,3,1)$	$(0,0,2,1,1,1)$	$(0,2,0,0,0,2)$	$(1,0,0,1,1,1)$	$(2,0,0,0,2,0)$
$(0,0,0,2,0,2)$	$(0,1,0,0,0,4)$	$(0,2,0,1,1,1)$	$(1,0,0,2,0,0)$	$(2,0,0,1,1,1)$
$(0,0,0,2,2,0)$	$(0,1,0,0,2,2)$	$(0,2,0,2,0,0)$	$(1,0,0,2,0,2)$	$(2,0,0,2,0,0)$
$(0,0,0,3,1,1)$	$(0,1,0,1,1,1)$	$(0,2,1,0,0,2)$	$(1,0,0,2,2,0)$	$(2,0,1,0,2,0)$
$(0,0,0,4,0,0)$	$(0,1,0,2,0,0)$	$(0,3,0,0,0,0)$	$(1,0,0,4,0,0)$	$(2,1,0,2,0,0)$
$(0,0,1,0,0,4)$	$(0,1,0,2,0,2)$	$(0,3,0,0,0,2)$	$(1,0,1,0,2,0)$	$(3,0,0,0,0,0)$
$(0,0,1,0,2,2)$	$(0,1,0,2,2,0)$	$(0,3,0,2,0,0)$	$(1,0,1,1,1,1)$	$(3,0,0,0,2,0)$
$(0,0,1,0,4,0)$	$(0,1,0,4,0,0)$	$(0,4,0,0,0,0)$	$(1,0,2,0,2,0)$	$(3,0,0,2,0,0)$
$(0,0,1,1,1,1)$	$(0,1,1,0,0,2)$	$(0,5,0,0,0,0)$	$(1,1,0,1,1,1)$	$(4,0,0,0,0,0)$
$(0,0,1,2,0,2)$	$(0,1,1,1,1,1)$	$(1,0,0,0,2,2)$	$(1,1,0,2,0,0)$	$(5,0,0,0,0,0)$.

Table 1. The 55 members of the set $\{\eta \in \mathbb{T} \mid \text{s-Necessary}\,(\eta)\,\}$.

The main goal in this section is establishing an upper bound for the 3rd coordinate of angle-counts for midsize circles in a compact 3-packing, through this contour analysis.

We begin with an analysis of the 2π-contours of $\Delta_3 \ni (r,s) \mapsto \eta \cdot \boldsymbol{\alpha}(r,s)$ in Proposition 4.1. A crucial part of Proposition 4.1 is (6), which explicitly describes a region in Δ_3 containing the 2π-contours of $\Delta_3 \ni (r,s) \mapsto \eta \cdot \boldsymbol{\alpha}(r,s)$ for all $\eta \in \{\xi \in \mathbb{T} \mid \text{s-Necessary}\,(\xi)\,\}$.

Subsequently, in Proposition 4.2(7), we prove that if the 3rd coordinate of $\zeta \in \{\xi \in \mathbb{T} \mid \text{r-Necessary}\,(\xi)\,\}$ is too large, then the 2π-contour of $\Delta_3 \ni (r,s) \mapsto \zeta \cdot \boldsymbol{\beta}(r,s)$ lies in a region that is disjoint from the region containing the 2π-contours of $\Delta_3 \ni (r,s) \mapsto \eta \cdot \boldsymbol{\alpha}(r,s)$ for all $\eta \in \{\xi \in \mathbb{T} \mid \text{s-Necessary}\,(\xi)\,\}$. Therefore, these contours cannot intersect, while such an intersection is a necessary condition for all angle-counts for circles in a compact 3-packing, as described in Section 2. This allows us to establish a bound on the 3rd coordinate of angle-counts for midsize circles in a compact 3-packing and lays the groundwork for showing that Π_3 is finite in the next section.

Proposition 4.1. *Let $\eta \in \{\xi \in \mathbb{T} \mid \text{s-Necessary}\,(\xi)\,\}$. For any $m \in (0,1)$, we define the functions $f_\eta : \Delta_3 \to \mathbb{R}$ and $g_{\eta,m} : (0,1) \to \mathbb{R}$ by*

$$f_\eta(r,s) := \eta \cdot \boldsymbol{\alpha}(r,s) \quad (r,s) \in \Delta_3$$
$$g_{\eta,m}(r) := \eta \cdot \boldsymbol{\alpha}(r,mr) \quad r \in (0,1).$$

Then:

(1) *There exists some $(r_0,s_0) \in \Delta_3$ with $f_\eta(r_0,s_0) = 2\pi$.*

(2) *For any $m \in (0,1)$ the function $g_{\eta,m}$ is monotone decreasing and, if $\eta \cdot (1,0,0,1,1,0) \neq 0$, then $g_{\eta,m}$ is strictly decreasing. If $\eta \cdot (1,0,0,1,1,0) = 0$ then $g_{\eta,m}$ is a constant function.*

(3) *We have $(\partial_2 f_\eta)(r,s) < 0$ for all $(r,s) \in \Delta_3$.*

(4) *There exists some $a \in [0,1)$ and a differentiable function $\phi : (a,1) \to (0,1)$ so that $f_\eta(r,\phi(r)) = 2\pi$ for all $r \in (a,1)$. We may choose $a \in [0,1)$ so that the graph of ϕ equals the whole contour $\{(r,s) \in \Delta_3 \mid f_\eta(r,s) = 2\pi\}$.*

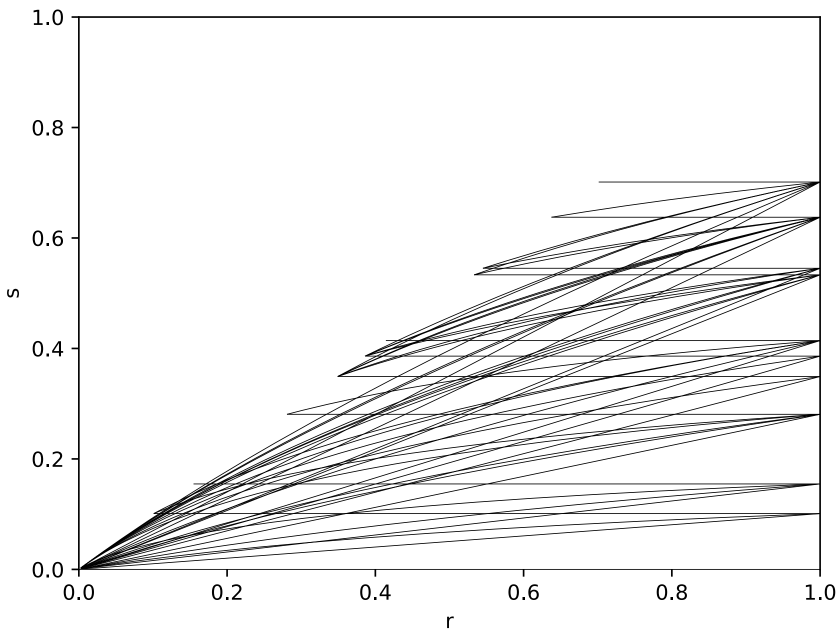

FIGURE 4.1. The contours $\{(r,s) \in \Delta_3 \mid \eta \cdot \boldsymbol{\alpha}(r,s) = 2\pi\}$ for the 55 elements $\eta \in \{\xi \in \mathbb{T} \mid \text{s-Necessary}\,(\xi)\,\}$.

(5) *Let $(r_0, s_0) \in \Delta_3$ be any point satisfying $f_\zeta(r_0, s_0) = 2\pi$ and let $\phi : (a, 1) \to (0, 1)$ be as yielded by (4). With $m_0 := s_0/r_0$, we have*
   (a) *If $\eta \cdot (1, 0, 0, 1, 1, 0) \neq 0$, then $\phi(r) > m_0 r$ for $r \in (a, r_0)$ and $\phi(r) < m_0 r$ for $r \in (r_0, 1)$.*
   (b) *If $\eta \cdot (1, 0, 0, 1, 1, 0) = 0$, then $\phi(r) = m_0 r$ for $r \in (0, 1)$.*

(6) *We have $f_\eta(r, s) > 2\pi$ for all $(r, s) \in \Delta_3$ satisfying $s \leq 10^{-1} r$.*

Proof. We prove (1). On $\{(r, s) \mid 0 \leq s \leq r \leq 1\}$, each of the functions $(r, s) \mapsto \boldsymbol{\alpha}_i(r, s)$ for $i \in \{1, \ldots, 6\}$ attains its minimum and maximum respectively at $(1, 1)$ and $(1, 0)$. Since s-Necessary$\,(\eta)$ is true, we have $\eta \cdot (1, 1, 1, 1, 1, 1) < 6$ and $\eta \cdot (6, 6, 2, 6, 3, 3) > 12$, and therefore there exists some $(r_1, s_1) \in \Delta_3$ (close to $(1, 1)$) with $f_\eta(r_1, s_1) < 2\pi$ and there exists some $(r_2, s_2) \in \Delta_3$ (close to $(1, 0)$) with $f_\eta(r_2, s_2) > 2\pi$. By the Intermediate Value Theorem, there exists some $(r_0, s_0) \in \Delta_3$ with $f_\eta(r_0, s_0) = 2\pi$.

The assertion (2) can be verified by using a computer algebra system. Explicitly, for $m \in (0, 1)$, where, for $r \in (0, 1)$,

$$U(r) := \frac{2m}{(mr + 1)\sqrt{mr(mr + 2)}},$$
$$V(r) := \frac{m}{(mr + 1)\sqrt{m(mr + r + 1)}},$$

$$W(r) := \frac{m}{(mr+1)\sqrt{2mr+1}},$$

the function $g_{\eta,m}$ has derivative

$$g'_{\eta,m}(r) = -\eta \cdot (U(r), 0, 0, V(r), W(r), 0) \quad (r \in (0,1)).$$

Furthermore, this derivative is seen to be everywhere non-positive, and everywhere strictly negative if $\eta \cdot (1,0,0,1,1,0) \neq 0$ and zero when $\eta \cdot (1,0,0,1,1,0) = 0$, establishing (2).

We prove (3). For $i \in \{1,2,3,4,5,6\}$, the functions $\Delta_3 \ni (r,s) \mapsto \boldsymbol{\alpha}_i(r,s)$ all have non-positive (strictly negative for $i \in \{1,2,4,5,6\}$) partial derivatives with respect to the second parameter everywhere on Δ_3. Since s-NonHex(η) is true, we have that$(\partial_2 f_\eta)(r,s) < 0$ for all $(r,s) \in \Delta_3$, establishing (3).

We prove (4). We define $G := \{r \in (0,1) \mid \exists s \in (0,r),\ f_\eta(r,s) = 2\pi\}$, which is non-empty by (1). By (3), for every $r \in G$, there exists a unique $\phi(r) \in (0,r)$ satisfying $f_\eta(r,\phi(r)) = 2\pi$. It is clear that the graph of ϕ equals the contour $\{(r,s) \in \Delta_3 \mid f_\eta(r,s) = 2\pi\}$. By (3) and the Implicit Function Theorem, the set G is open and $\phi : G \to (0,1)$ is differentiable. Since $(\partial_1 f_\eta)(r,s) \geq 0$ for all $(r,s) \in \Delta_3$, the set G is connected and hence is an open interval (a,b). Furthermore, it can be verified (by computer) that we may always choose $b = 1$ (See Figure 4.1).

We prove (5). If $\eta \cdot (1,0,0,1,1,0) \neq 0$, by (2), the function g_{η,m_0} is strictly decreasing. Therefore $f_\eta(r, rm_0) = g_{\eta,m_0}(r) > 2\pi$ for all $r \in (a, r_0)$ and $f_\eta(r, rm_0) = g_{\eta,m_0}(r) < 2\pi$ for all $r \in (r_0, 1)$. But, by (3), we have $(\partial_2 f_\eta)(r,s) < 0$ for all $(r,s) \in \Delta_3$, so that we must have $\phi(r) > m_0 r$ for $r \in (a, r_0)$, and $\phi(r) < m_0 r$ for $r \in (r_0, 1)$. If $\eta \cdot (1,0,0,1,1,0) = 0$, by (2), the function g_{η,m_0} constant, and since $g_{\eta,m_0}(r_0) = 2\pi$, we have that ϕ must equal the function $(0,1) \ni r \mapsto m_0 r$.

We prove (6). Figure 4.1 may be a helpful visual aid. It can be verified (by computer) that $\lim_{r\to 1} f_\eta\left(r, 10^{-1}r\right) > 2\pi$. Then, by (2), we obtain $f_\eta\left(r, 10^{-1}r\right) > 2\pi$ for all $r \in (0,1)$. But, by (3), we have $(\partial_2 f_\eta)(r,s) < 0$ for all $(r,s) \in \Delta_3$, so that $f_\eta(r,s) > 2\pi$ for all $(r,s) \in \Delta_3$ satisfying $s \leq 10^{-1}r$, establishing (6). □

Proposition 4.2. *Let $\zeta \in \{\xi \in \mathbb{T} \mid$ r-Necessary$(\xi)\}$ be arbitrary. For any $m \in (0,1)$, we define the functions $f_\zeta : F \to \mathbb{R}$ and $g_{\zeta,m} : (0,1) \to \mathbb{R}$ by*

$$f_\zeta(r,s) := \zeta \cdot \boldsymbol{\beta}(r,s) \quad (r,s) \in \Delta_3$$
$$g_{\zeta,m}(r) := \zeta \cdot \boldsymbol{\beta}(r, mr) \quad r \in (0,1).$$

Then:

(1) *There exists some $(r_0, s_0) \in \Delta_3$ with $f_\zeta(r_0, s_0) = 2\pi$.*

(2) *For any $m \in (0,1)$, the function $g_{\zeta,m}$ is monotone decreasing and, if $\zeta \cdot (1,0,0,1,1,0) \neq 0$, then $g_{\zeta,m}$ is strictly decreasing. If $\zeta \cdot (1,0,0,1,1,0) = 0$, then $g_{\zeta,m}$ is a constant function.*

(3) *If r-VerticalContour(ζ) is false, then $(\partial_2 f_\zeta)(r,s) > 0$ for all $(r,s) \in \Delta_3$.*

(4) *If r-VerticalContour(ζ) is true, then there exists $r_0 \in (0,1)$ for which $f_\zeta(r_0, s) = 2\pi$ for all $s \in (0, r_0)$.*

(5) *If r-VerticalContour(ζ) is false, then there exists some differentiable function $\psi : (c,d) \to (0,1)$ so that $f_\zeta(r, \psi(r)) = 2\pi$ for all $r \in (c,d)$. We may choose the interval (c,d) so that the graph of ψ equals the entire contour $\{(r,s) \in \Delta_3 \mid f_\zeta(r,s) = 2\pi\}$.*

(6) *Let r-VerticalContour(ζ) be false and $(r_0, s_0) \in \Delta_3$ be any point satisfying $f_\zeta(r_0, s_0) = 2\pi$. With $\psi : (c, d) \to \mathbb{R}$ as yielded by (5) and $m_0 := s_0/r_0$,*

(a) *If $\zeta \cdot (1, 0, 0, 1, 1, 0) \neq 0$, then $\psi(r) < m_0 r$ for $r \in (c, r_0)$ and $\psi(r) > m_0 r$ for $r \in (r_0, d)$.*

(b) *If $\zeta \cdot (1, 0, 0, 1, 1, 0) = 0$, then $\psi(r) = m_0 r$ for all $r \in (0, 1)$.*

(7) *If $\zeta_3 \geq 35$, then $f_\zeta(r, s) > 2\pi$ for all $(r, s) \in \Delta_3$ satisfying $s \geq 10^{-1} r$.*

Proof. We prove (1). On $\{(r, s) \mid 0 \leq s \leq r \leq 1\}$, each of the functions $(r, s) \mapsto \boldsymbol{\beta}_i(r, s)$ for $i \in \{1, \ldots, 6\}$ attains its minimum and maximum respectively at $(1, 0)$ and $(0, 0)$ (or approached near $(0, 0)$, if the function is not defined at $(0, 0)$). Since r-Necessary(ζ) is true, we have $\zeta \cdot (1, 1, 0, 1, 0, 0) < 6$ and $\zeta \cdot (6, 2, 2, 3, 3, 2) > 12$, and hence there exists some $(r_1, s_1) \in \Delta_3$ (close to $(1, 0)$) with $f_\zeta(r_1, s_1) < 2\pi$ and there exists some $(r_2, s_2) \in \Delta_3$ (close to $(0, 0)$) with $f_\zeta(r_2, s_2) > 2\pi$. By the Intermediate Value Theorem, there exists some $(r_0, s_0) \in \Delta_3$ with $f_\zeta(r_0, s_0) = 2\pi$.

The assertion (2) can be verified with a computer algebra system. Explicitly, for $r \in (0, 1)$ and $m \in (0, 1)$, defining

$$U(r) := \frac{2}{(r+1)\sqrt{r(r+2)}},$$
$$V(r) := \frac{1}{(r+1)\sqrt{2r+1}},$$
$$W(r) := \frac{m}{(r+1)\sqrt{m(mr+r+1)}},$$

the function $g_{\zeta,m}$ has derivative

$$g'_{\zeta,m}(r) = -\zeta \cdot (U(r), 0, 0, V(r), W(r), 0) \quad (r \in (0, 1)),$$

which is easily seen to be non-positive, and strictly negative if $\zeta \cdot (1, 0, 0, 1, 1, 0) \neq 0$, and zero when $\zeta \cdot (1, 0, 0, 1, 1, 0) = 0$.

We prove (3). For $i \in \{1, 2, 3, 4, 5, 6\}$, the functions $\Delta_3 \ni (r, s) \mapsto \boldsymbol{\beta}_i(r, s)$ all have non-negative (strictly positive for $i \in \{3, 5, 6\}$) partial derivatives with respect to the second parameter everywhere on Δ_3. Therefore, if $\zeta \cdot (0, 0, 1, 0, 1, 1) \neq 0$, then $(\partial_2 f_\zeta)(r, s) > 0$ for all $(r, s) \in \Delta_3$, establishing (3).

The assertion (4) follows from (1) when we notice that, for $i \in \{1, 2, 4\}$, the functions $\Delta_3 \ni (r, s) \mapsto \boldsymbol{\beta}_i(r, s)$ are all independent of the second parameter s.

We prove (5). Define $G := \{r \in (0, 1) \mid \exists s \in (0, r),\ f_\zeta(r, s) = 2\pi\}$, which is non-empty by (1). By (3), for every $r \in G$, there exists a unique $\psi(r) \in (0, r)$ such that $f_\zeta(r, \psi(r)) = 2\pi$. It is clear that graph of the function $\psi : G \to (0, 1)$ equals the contour $\{(r, s) \in \Delta_3 \mid f_\zeta(r, s) = 2\pi\}$. By (3), and the Implicit Function Theorem, G is open and the function $\psi : G \to (0, 1)$ is differentiable. Since $(\partial_1 f_\zeta)(r, s) \leq 0$ for all $(r, s) \in \Delta_3$, the set G is connected, and hence must be some open interval (c, d).

We prove (6). If $\eta \cdot (1, 0, 0, 1, 1, 0) \neq 0$, by (2), the function g_{ζ,m_0} is strictly decreasing, and hence $f_\zeta(r, rm_0) = g_{\zeta,m_0}(r) > 2\pi$ for $r \in (c, r_0)$ and $f_\zeta(r, rm_0) = g_{\zeta,m_0}(r) < 2\pi$ for $r \in (r_0, d)$. By (3), we have $(\partial_2 f_\zeta)(r, s) > 0$ for all $(r, s) \in \Delta_3$, and therefore $\psi(r) < m_0 r$ for $r \in (c, r_0)$ and $\psi(r) > m_0 r$ for $r \in (r_0, d)$. On the other hand, if $\eta \cdot (1, 0, 0, 1, 1, 0) = 0$, the function g_{ζ,m_0} is constant, and since $g_{\zeta,m_0}(r_0) = 2\pi$, we have that ψ equals $(0, 1) \ni r \mapsto m_0 r$.

We prove (7). We assume that $\zeta_3 \geq 35$. A straightforward computation shows that, for all $r \in (0,1)$, we have $35\boldsymbol{\beta}_3(r, 10^{-1}r) > 2\pi$. Therefore $f_\zeta(r, 10^{-1}r) = \zeta \cdot \boldsymbol{\beta}(r, 10^{-1}r) > 2\pi$ for all $r \in (0,1)$. Since $\zeta_3 \geq 35$, we have $\zeta \cdot (0,0,1,0,1,1) \neq 0$, so that, by (3), we have $(\partial_2 f_\zeta)(r,s) > 0$ for all $(r,s) \in \Delta_3$. Hence $f_\zeta(r,s) > 2\pi$ for all $(r,s) \in \Delta_3$ with $s \geq 10^{-1}r$. $\square$

5. The set Π_3 is finite

We are now in a position to prove one of our main results, Theorem 5.5, in this section. We begin by defining the following predicate:

Definition 5.1. For $\zeta \in \mathbb{T}$ we define the predicate

$$\text{r-BoundsExtra}\,(\zeta) := (\zeta \cdot (0,0,1,0,0,0) < 35).$$

We define the set

$$K := \left\{ (\eta, \zeta) \in \mathbb{T}^2 \;\middle|\; \begin{array}{c} \text{s-Necessary}\,(\eta) \\ \wedge\text{r-Necessary}\,(\zeta) \\ \wedge\text{r-BoundsExtra}\,(\zeta) \\ \wedge\text{sr-Disjunct}\,(\eta,\zeta) \end{array} \right\}.$$

We will argue in this section that $|K| < \infty$ and that $|\Pi_3| \leq |K|$.

A straightforward computer search will establish the cardinality of K. All elements of K are provided in the attached dataset.

Proposition 5.2. *The set K is finite and has exactly* 248395 *elements.*

The next proposition shows that every element of K determines at most one point of Δ_3.

Proposition 5.3. *For any pair $(\eta, \zeta) \in K$, there exists at most one (perhaps no) point $(r_0, s_0) \in \Delta_3$ for which $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$.*

Proof. Let $(\eta, \zeta) \in K$ be arbitrary. If there exists no point $(r,s) \in \Delta_3$ for which $\eta \cdot \boldsymbol{\alpha}(r,s) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r,s) = 2\pi$, then we are done.

Let $(r_0, s_0) \in \Delta_3$ be such that $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$. We claim that there exists no other point in Δ_3 for which this is true.

By Proposition 4.1(4), there exists some $\phi : (a,1) \to \mathbb{R}$ so that $\phi(r_0) = s_0$ and $\eta \cdot \boldsymbol{\alpha}(r, \phi(r)) = 2\pi$ for all $r \in (a,1)$.

We now distinguish between the two cases where r-VerticalContour (ζ) is true and r-VerticalContour (ζ) is false.

If it is the case that r-VerticalContour (ζ) is true, then, by Proposition 4.2(4), we have $\{(r,s) \in \Delta_3 \mid \zeta \cdot \boldsymbol{\beta}(r,s) = 2\pi\} = \{(r_0, s) \in \Delta_3 \mid s \in (0, r_0)\}$, and hence the pair (r_0, s_0) is the only point in Δ_3 for which $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$.

On the other hand, if r-VerticalContour (ζ) is false, then, by Proposition 4.2(5), there exists some function $\psi : (c,d) \to (0,1)$ satisfying $\psi(r_0) = s_0$ and $\zeta \cdot \boldsymbol{\beta}(r, \psi(r)) = 2\pi$ for all $r \in (c,d)$. Since sr-Disjunct (η, ζ) is true, by Propositions 4.1(2) and 4.2(2) we cannot have that both functions ϕ and ψ are equal to the function $(0,1) \ni r \mapsto mr$ for any $m \in (0,1)$. Then, by Propositions 4.1(5) and 4.2(6), we have that (r_0, s_0) is the only point in Δ_3 for which $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$. $\square$

Now, for any $(r_0, s_0) \in \Pi_3$ and compact 3-packing P with $\text{radii}(P) = \{s_0, r_0, 1\}$, in the following result we will prove that there must exist circles A and B of respective radii s_0 and r_0, so that $(\xi^{(A)}, \xi^{(B)}) \in K$.

Proposition 5.4. *Let $(r_0, s_0) \in \Pi_3$ and let P be any compact 3-packing with radii$(P) = \{s_0, r_0, 1\}$. There exists circles $A, B \in P$ of respective radii s_0 and r_0 so that the following is true:*

$$\begin{aligned}&\textit{s-Necessary}\left(\xi^{(A)}\right) \wedge \textit{r-Necessary}\left(\xi^{(B)}\right)\\ &\quad\wedge \textit{sr-Disjunct}\left(\xi^{(A)}, \xi^{(B)}\right) \wedge \textit{r-BoundsExtra}\left(\xi^{(B)}\right)\\ &\quad\quad\wedge \quad \xi^{(A)} \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi \quad \wedge \quad \xi^{(B)} \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi.\end{aligned}$$

In particular, we have $(\xi^{(A)}, \xi^{(B)}) \in K$.

Proof. By Theorem 3.7 there exist circles $A, B \in P$ so that

$$\text{s-Necessary}\left(\xi^{(A)}\right) \wedge \text{r-Necessary}\left(\xi^{(B)}\right) \wedge \text{sr-Disjunct}\left(\xi^{(A)}, \xi^{(B)}\right)$$

is true. By definition of the angle-counts $\xi^{(A)}$ and $\xi^{(B)}$, we have that $\xi^{(A)} \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\xi^{(B)} \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$.

Suppose that r-BoundsExtra $\left(\xi^{(B)}\right)$ is false. Then, since $\xi^{(B)} \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$, by Proposition 4.2(7), we have that $s_0 < 10^{-1} r_0$. However, since $\xi^{(A)} \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$, Proposition 4.1(6) yields the contradictory inequality $s_0 > 10^{-1} r_0$. Therefore r-BoundsExtra $\left(\xi^{(B)}\right)$ is true. □

Finally we are able to prove one of our main results:

Theorem 5.5. *The set Π_3 is finite and $|\Pi_3| \leq |K| = 248395$.*

Proof. We define

$$L := \left\{ (r, s) \in \Delta_3 \;\middle|\; \begin{array}{c} \exists (\eta, \zeta) \in K, \\ \eta \cdot \boldsymbol{\alpha}(r, s) = 2\pi, \\ \zeta \cdot \boldsymbol{\beta}(r, s) = 2\pi. \end{array} \right\}.$$

By Proposition 5.2, the set K has 248395 elements and for each $(\eta, \zeta) \in K$, by Proposition 5.3, there exists at most one point $(r_0, s_0) \in \Delta_3$ for which $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$. Therefore, we have $|L| \leq |K|$.

We claim that $\Pi_3 \subseteq L$. Let $(r_0, s_0) \in \Pi_3$ and P be a compact 3-packing with radii$(P) = \{s_0, r_0, 1\}$. By Proposition 5.4, there exist circles A and B in P so that $(\xi^{(A)}, \xi^{(B)}) \in K$ and $\xi^{(A)} \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\xi^{(B)} \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$. Therefore $(r_0, s_0) \in L$ and hence $\Pi_3 \subseteq L$. We conclude that $|\Pi_3| \leq |L| \leq |K| \leq 248395$. □

6. Necessary and sufficient conditions for contour intercepts

With K as defined in Section 5, in the current section we will provide necessary and sufficient conditions on elements $(\eta, \xi) \in K$ for there to exist some $(r_0, s_0) \in \Delta_3$ satisfying $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$. These necessary and sufficient conditions allow for computing a sharper bound on $|\Pi_3|$ in the next section.

Proposition 6.1. *Let $(\eta, \zeta) \in K$ with $\phi : (a, 1) \to (0, 1)$ and $\psi : (c, d) \to (0, 1)$ as yielded by applying Propositions 4.1(4) and 4.2(5) to η and ζ respectively. The statements (1) and (2) are equivalent:*

(1) *There exists a unique point $(r_0, s_0) \in \Delta_3$ for which $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$.*

(2) *Either r-VerticalContour* (ζ) *is true and there exists some* $r_0 \in (a,1)$ *so that* $\eta \cdot \boldsymbol{\alpha}(r_0, \phi(r_0)) = 2\pi$ *and* $\zeta \cdot \boldsymbol{\beta}(r_0, \phi(r_0)) = 2\pi$*;* ***or*** *r-VerticalContour* (ζ) *is false, and all of the following hold:*

  (a) $a < d$.

  (b) *If* $a = c = 0$*, then* $\lim_{r\downarrow 0} \frac{\psi(r)}{r} < \lim_{r\downarrow 0} \frac{\phi(r)}{r}$.

  (c) *If* $d = 1$*, then* $\lim_{r\uparrow 1} \phi(r) < \lim_{r\uparrow 1} \psi(r)$.

Proof. We prove that (1) implies (2). Let $(r_0, s_0) \in \Delta_3$ be the unique point for which $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$.

If r-VerticalContour (ζ) is true then, since $s_0 = \phi(r_0)$, we immediately have that $\eta \cdot \boldsymbol{\alpha}(r_0, \phi(r_0)) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, \phi(r_0)) = 2\pi$.

On the other hand, if r-VerticalContour (ζ) is false, with $m_0 := s_0/r_0$ we immediately note that, since sr-Disjunct (η, ζ) is true, by Propositions 4.1(6) and 4.2(6) we cannot have that both ψ and ϕ are equal to the function $(0,1) \ni r \mapsto m_0 r$.

We prove (2)(a). Noting that $\phi(r_0) = \psi(r_0) = s_0$ we have $r_0 \in (c,d) \cap (a,1)$ so that $a < r_0 < d$, establishing (2)(a).

We prove (2)(b). Assume $a = c = 0$ and let $(x_n) \subseteq (0, r_0)$ be any strictly decreasing sequence that converges to zero. Assuming ϕ does not equal the function $r \mapsto m_0 r$, by repeatedly applying Proposition 4.1(5), we notice that

$$\phi(x_{n+1}) > \frac{\phi(x_n)}{x_n} x_{n+1}$$

which implies that $(\phi(x_n)/x_n)$ is strictly increasing. Since $(\phi(x_n)/x_n)$ is bounded above by 1, the limit $\lim_{r\downarrow 0} \frac{\phi(r)}{r}$ exists by The Monotone Convergence Theorem and is strictly greater than m_0 since $r_0^{-1}\phi(r_0) = m_0$. Similarly, by Proposition 4.2(6) and assuming ψ does not equal the function $r \mapsto m_0 r$, we obtain

$$\psi(x_{n+1}) < \frac{\psi(x_n)}{x_n} x_{n+1}.$$

The sequence is strictly $(\psi(x_n)/x_n)$ is strictly decreasing and bounded below by zero, hence the limit $\lim_{r\downarrow 0} \frac{\psi(r)}{r}$ exists and strictly less than m_0 since $r_0^{-1}\psi(r_0) = m_0$. Hence $\lim_{r\downarrow 0} \frac{\psi(r)}{r} \le m_0 \le \lim_{r\downarrow 0} \frac{\phi(r)}{r}$ and one of the inequalities must be strict, since not both ψ and ϕ are equal to the function $(0,1) \ni r \mapsto m_0 r$, establishing (2)(b).

We prove (2)(c). Assume $d = 1$. By Propositions 4.1(5) and 4.2(6) we have

$$\lim_{r\uparrow 1} \phi(r) \le m_0 \le \lim_{r\uparrow 1} \psi(r).$$

Since sr-Disjunct (η, ζ) is true, Propositions 4.1(5) and 4.2(6) imply that one of these inequalities must be strict[2]. This establishes (2)(c).

We prove (2) implies (1).

Assume r-VerticalContour (ζ) is true and there exists some $r_0 \in (a,1)$ so that $\eta \cdot \boldsymbol{\alpha}(r_0, \phi(r_0)) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, \phi(r_0)) = 2\pi$. With $s_0 := \phi(r_0)$, by Proposition 5.3, $(r_0, s_0) \in \Delta_3$ is the unique point for which $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$.

[2]With $m_0 := s_0/r_0$ and any fixed $r_1 \in (r_0, 1)$ define $m_1 := \phi(r_1)/r_1$. By Proposition 4.1(2), if ϕ does not equal the function $(0,1) \ni r \mapsto m_0 r$, then $\phi(r_1) < m_0 r_1$ and for all $r \in (r_1, 1)$, we have $\phi(r) < m_1 r < m_0 r_1 < m_0 r$. So that $\lim_{r\uparrow 1} \phi(r) \le m_1 < m_0$. A similar argument holds for ψ through application of Proposition 4.2(6).

Assume r-VerticalContour(ζ) is false and all the statements (2)(a), (2)(b) or (2)(c) are true.

If $d < 1$, then Proposition 4.2(6) implies that $\lim_{r\uparrow d} \psi(r) = d$, so that, regardless of the value of d (using (2)(c) when $d = 1$), we have $\lim_{r\uparrow d} \phi(r) < \lim_{r\uparrow d} \psi(r)$. Hence there exists some $r_1 \in (\max\{a, c\}, d)$ so that $\phi(r_1) < \psi(r_1)$.

Also, Proposition 4.1(5) implies that $\lim_{r\downarrow a} \phi(r) = a$ and Proposition 4.2(6) implies that $\lim_{r\downarrow c} \psi(r) = 0$. Therefore, if $\max\{a, c\} > 0$, then there exists some $r_2 \in (\max\{a, c), r_1)$ so that $\psi(r_2) < \phi(r_2)$. On the other hand if $\max\{a, c\} = 0$, then by (2)(b) there also exists some $r_2 \in (0, r_1)$ so that $\psi(r_2)/r_2 < \phi(r_2)/r_2$, and hence we also have $\psi(r_2) < \phi(r_2)$.

Now, by the Intermediate Value Theorem, there exists some $r_0 \in (r_2, r_1)$ so that $s_0 := \phi(r_0) = \psi(r_0)$ and hence $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$. By Proposition 5.3, we may conclude this point (r_0, s_0) is unique. □

7. Computation of an upper bound for $|\Pi_3|$

With K as defined in Section 5, for any $(\eta, \zeta) \in K$, the quantities $\lim_{r\downarrow 0} \frac{\psi(r)}{r}$, $\lim_{r\downarrow 0} \frac{\phi(r)}{r}$, $\lim_{r\uparrow 1} \phi(r)$, $\lim_{r\uparrow 1} \psi(r)$ etc. mentioned in Proposition 6.1(2) can easily be computed numerically to arbitrary precision. We computed these quantities to precision 10^{-300} using *Sympy* [9] in conjunction with *mpmath* [6]. In many cases these approximations are sufficiently accurate to determine whether or not the inequalities in Proposition 6.1(2) are strict, and hence whether elements in $(\eta, \zeta) \in K$ determine a unique point $(r_0, s_0) \in \Delta_3$ for which $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$.

The remaining cases where strict inequality between the quantities in Proposition 6.1(2) could not be definitively determined by high precision numerical approximations, the quantities were each determined exactly as a root of a polynomial. We briefly describe how we determine these polynomials. In Section 8 we show how we may determine bivariate polynomials p and q whose solution sets necessarily contain the graphs of the functions ϕ and ψ as yielded by Propositions 4.1(4) and 4.2(5). The limit $\lim_{r\downarrow 0} \frac{\phi(r)}{r}$ is then a solution in m of the polynomial equation $\lim_{r\to 0} p(r, mr) = 0$. The values a and $\lim_{r\to 1} \phi(r)$ appearing in Proposition 6.1(2) are solutions of the polynomial equations $p(r, r) = 0$ and $p(1, s) = 0$ respectively. Similarly for q and ψ.

Now, where strict inequality between the quantities in Proposition 6.1(2) could not be determined by numerics, they can be confirmed as equal by being the roots of identical polynomials in all remaining cases. Hence, by Proposition 6.1, such $(\eta, \zeta) \in K$ do not admit a solution in Δ_3 to the equations $\eta \cdot \boldsymbol{\alpha}(r, s) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r, s) = 2\pi$.

Finally, the results of these computations together with Proposition 6.1, allow for determining all the elements $(\eta, \zeta) \in K$ satisfying

$$\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = \zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$$

for a unique $(r_0, s_0) \in \Delta_3$. This, in turn, allows for determining an upper bound on the cardinality of the set

$$L := \left\{ (r, s) \in \Delta_3 \;\middle|\; \begin{array}{c} \exists(\eta, \zeta) \in K, \\ \eta \cdot \boldsymbol{\alpha}(r, s) = 2\pi, \\ \zeta \cdot \boldsymbol{\beta}(r, s) = 2\pi. \end{array} \right\},$$

which contains Π_3 (cf. Theorem 5.5). This establishes the sharper bound

$$|\Pi_3| \leq |L| \leq 13617.$$

We note that excluding certain (numerically approximated) elements from L as not actually being in Π_3 seems to be computationally infeasible on consumer hardware. By Theorem 3.7, a necessary condition for $(r_0, s_0) \in L$ to be an element of Π_3 is that there exists some $\xi \in \mathbb{T}$ for which 1-Necessary (ξ) is true and $\xi \cdot \boldsymbol{\gamma}(r_0, s_0) = 2\pi$. The closer a point $(r_0, s_0) \in L$ is to the origin, the larger the search space of elements $\xi \in \mathbb{T}$ becomes for which one must verify 1-Necessary (ξ) and $\xi \cdot \boldsymbol{\gamma}(r_0, s_0) = 2\pi$. E.g., $(r_1, s_1) = (0.0000581261\ldots, 0.0000125188\ldots) \in \Delta_3$ approximates an element[3] in L and by computing $\boldsymbol{\gamma}(r_1, s_1)$, a naive bound on the number of elements $\xi \in \mathbb{T}$ in the search space can be seen to be roughly 7×10^{21} which will require a considerable length of time to sift through. Some experiments with programs written in *Cython* [2] indicate that 12 months of computation on a modest quad-core desktop PC, utilizing all four cores, is an extremely optimistic estimate for how long such a search might take. Hence, a reasonably sized compute cluster is required to perform such searches within a reasonable time. Searches might also be further sped up by utilizing graphics processing units.

Furthermore, to confirm that an element L is in Π_3, it is of course required to construct a compact packing with the specified radii.

Still, for some elements of L (which are not too close to the origin) one is able to verify within a reasonable amount of time whether or not they satisfy the mentioned necessary condition, and ultimately, whether they are elements of Π_3 by constructing packings. We display some of them in the last section.

8. Exact computation of elements from L.

With L as defined in the previous section, we describe how we may compute exact values of elements of L (and hence of elements of Π_3) as roots of polynomials.

With $(\eta, \zeta) \in K$ and $(r_0, s_0) \in L$ satisfying $\eta \cdot \boldsymbol{\alpha}(r_0, s_0) = 2\pi$ and $\zeta \cdot \boldsymbol{\beta}(r_0, s_0) = 2\pi$, we consider the equations

$$\cos(\eta \cdot \boldsymbol{\alpha}(r, s)) - 1 = 0 \quad \text{and} \quad \cos(\zeta \cdot \boldsymbol{\beta}(r, s)) - 1 = 0.$$

---

[3] The point $(0.0000581261602\ldots, 0.0000125188787\ldots) \in \Delta_3$ approximates the point $(r_0, s_0) \in L$ which is defined as the solution of

$$(0, 0, 1, 1, 1, 1) \cdot \boldsymbol{\alpha}(r, s) = (1, 0, 4, 0, 2, 0) \cdot \boldsymbol{\beta}(r, s) = 2\pi.$$

The exact values of r_0 and s_0 are as roots of the 16th degree polynomials

$$\begin{aligned} &471537r^{16} - 41484960r^{15} - 659124096r^{14} + 58464363120r^{13} + 1743725080084r^{12} \\ &\quad + 17900565761408r^{11} + 80565633090512r^{10} + 135832773328592r^{9} \\ &\quad - 55749863701666r^{8} - 312172905934624r^{7} - 79130757636960r^{6} + 18998456541200r^{5} \\ &\quad + 5684720044996r^{4} - 232167452096r^{3} - 4432749936r^{2} - 23293776r + 1369 \end{aligned}$$

and

$$\begin{aligned} &9s^{16} - 2952s^{15} + 297624s^{14} - 9490392s^{13} + 146307340s^{12} - 1264707784s^{11} \\ &\quad + 6454982728s^{10} - 19303597784s^{9} + 30925167782s^{8} - 17475748952s^{7} - 13037319960s^{6} \\ &\quad + 14055271864s^{5} + 4034895724s^{4} - 2996664152s^{3} - 1151616584s^{2} - 109340424s + 1369 \end{aligned}$$

respectively (cf. Section 8).

A simple algorithm (Algorithm 8.1 below) can be used to manipulate the above system into a system of two-variable polynomial equations, for certain $p, q \in \mathbb{Z}[r, s]$,

$$p(r, s) = 0$$
$$q(r, s) = 0$$

that necessarily has (r_0, s_0) as a solution.

With ϕ and ψ as yielded by Propositions 4.1 and 4.2, by construction, the polynomials p and q will necessarily satisfy $p(r, \phi(r)) = 0$ and $q(r, \psi(r)) = 0$ for all r in the respective domains of ϕ and ψ. This observation allows for exactly computing the quantities mentioned in Proposition 6.1(2) as discussed in Section 7.

Although Algorithm 8.1 is very simple and easily implemented in a computer algebra system, for certain values in $\mathbb{T}$ the computation may be slow and very RAM intensive yielding large[4] results.

Furthermore, by computing appropriate Gröbner bases for the ideal generated by p and q (cf. [1, Section 2.3] or [3, Chapter 3]) we may eliminate a variable from each polynomial and hence express the coordinates of $(r_0, s_0) \in L$ as a roots of univariate polynomials. Again, for certain inputs, computing these Gröbner bases can sometimes be very RAM intensive. Some of our computations required more than 200GB of RAM, at which point they were halted.

We implemented Algorithm 8.1 below in *Sympy* [9] and *SymEngine* [10], and used *Singular* [4] for further factoring of results and for computing Gröbner bases.

Algorithm 8.1. (1) ***Input:*** *A* **given expression** *which is of the form*

$$\cos(\eta \cdot \boldsymbol{\tau}(r, s)) - 1$$

where $\eta \in \mathbb{T}$ *and* $\boldsymbol{\tau} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$.

(2) *Expand the* **given expression** *and convert all terms in it to have a common denominator. Define the* **partial result** *as the numerator of this expression (Note that, by inspection of the relevant trigonometric identities, the denominator can be seen to be non-zero for all* $(r, s) \in \Delta_3$*, and may hence be disregarded. Furthermore, all radicals that occur are square-roots).*

(3) *While the* **partial result** *has terms with square-roots of expressions in variables* r *and/or* s *as factors, we repeatedly do the following:*
   - *Let* **rad** *be any square-root of an expression in* r *and/or* s *occurring as a factor to a term in the* **partial result**.
   - *Let* **left** *be the sum of all terms in the* **partial result** *which contains* **rad** *as a factor*
   - *Let* **right** *be the sum of all terms in the* **partial result** *which do not contain* **rad** *as a factor.*
   - *Fully expand both sides of the equation* $\mathbf{left}^2 = \mathbf{right}^2$ *and redefine the* **partial result** *as* $\mathbf{left}^2 - \mathbf{right}^2$.

(4) ***Return:*** **partial result**.

The following example displays the result of applying Algorithm 8.1.

[4]The polynomial yielded by applying Algorithm 8.1 to the expression $\cos((1, 1, 12, 1, 1, 1) \cdot \boldsymbol{\beta}(r, s)) - 1$ is 33.6 megabytes when written unfactorized to a plain text file.

Example 8.2. We apply Algorithm 8.1 to the expression

$$\cos((0,0,0,1,1,3)\cdot\boldsymbol{\alpha}(r,s))-1.$$

This expression expands to

$$\frac{1}{2\,(r+s)^4\,(s+1)^2}\times$$
$$\left(\begin{array}{l}
-2r^4s^2-4r^4s-2r^4-14r^3s^3-10r^3s^2-8r^3s-30r^2s^4-18r^2s^3\\
-12r^2s^2-18rs^5-30rs^4-8rs^3-2s^5-2s^4\\
+r^2s\sqrt{r^2+2rs}\sqrt{16r^2s+16rs^2+16rs}-2rs^3\sqrt{r^2+2rs}\sqrt{2s+1}\\
+3r^2s\sqrt{2s+1}\sqrt{16r^2s+16rs^2+16rs}+10rs^2\sqrt{r^2+2rs}\sqrt{2s+1}\\
+2rs^2\sqrt{r^2+2rs}\sqrt{16r^2s+16rs^2+16rs}-6s^3\sqrt{r^2+2rs}\sqrt{2s+1}\\
+6rs^2\sqrt{2s+1}\sqrt{16r^2s+16rs^2+16rs}-6s^4\sqrt{r^2+2rs}\sqrt{2s+1}\\
-3s^3\sqrt{r^2+2rs}\sqrt{16r^2s+16rs^2+16rs}-2r^3\sqrt{r^2+2rs}\sqrt{2s+1}\\
-s^3\sqrt{2s+1}\sqrt{16r^2s+16rs^2+16rs}+2r^3s\sqrt{r^2+2rs}\sqrt{2s+1}\\
+6r^2s^2\sqrt{r^2+2rs}\sqrt{2s+1}-2r^2s\sqrt{r^2+2rs}\sqrt{2s+1}
\end{array}\right).$$

We eliminate the denominator, which is never zero on Δ_3, and eliminating the radicals as described in Algorithm 8.1 yields the following two-variable polynomial (factorized for the sake of expressing it more compactly):

$$-16s^2\,(r+s)^4\,(s+1)^4\left(\begin{array}{l}
r^6s-4r^6+12r^5s^2-26r^5s-4r^5+54r^4s^3\\
\quad-40r^4s^2-7r^4s+108r^3s^4+28r^3s^3+12r^3s^2\\
\qquad+81r^2s^5+60r^2s^4+14r^2s^3-18rs^5-16rs^4+s^5
\end{array}\right)^2.$$

By construction, this polynomial has the property that, if $(r_0,s_0)\in\Delta_3$ is such that $(0,0,0,1,1,3)\cdot\boldsymbol{\alpha}(r_0,s_0)=2\pi$, then (r_0,s_0) necessarily is also a root of the above polynomial.

9. Examples of compact 3-packings

In this section we display an arbitrary selection of compact 3-packings. We stress that this list is far from exhaustive.

Example 9.1. The values $s_0\approx 0.299248$ and $r_0\approx 0.438405$ are approximations to the unique solution in Δ_3 of the equations

$$\begin{aligned}
(0,0,0,1,1,3)\cdot\boldsymbol{\alpha}(r,s)&=2\pi,\\
(1,0,3,0,2,0)\cdot\boldsymbol{\beta}(r,s)&=2\pi,\\
(0,0,2,4,0,4)\cdot\boldsymbol{\gamma}(r,s)&=2\pi.
\end{aligned}$$

Exact values of s_0 and r_0 are as roots of the respective polynomials

$$s^6-54s^5+175s^4-68s^3+15s^2-6s+1$$

and

$$5r^6+38r^5+39r^4-28r^3+19r^2-10r+1.$$

Figure 9.1 displays a compact 3-packing P with $\mathrm{radii}(P)=\{s_0,r_0,1\}$.

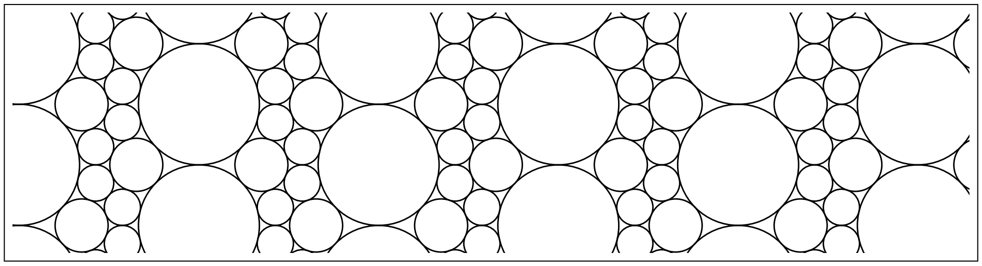

FIGURE 9.1. A compact circle-packing P with radii$(P) = \{s_0, r_0, 1\}$ where $s_0 \approx 0.299248$ and $r_0 \approx 0.438405$ are roots of the polynomials as given in Example 9.1.

Example 9.2. The values $s_0 \approx 0.468169$ and $r_0 \approx 0.822210$ are approximations to the unique solution in Δ_3 of the equations

$$(0,0,0,1,1,3) \cdot \boldsymbol{\alpha}(r,s) = 2\pi,$$
$$(0,0,3,2,2,0) \cdot \boldsymbol{\beta}(r,s) = 2\pi,$$
$$(0,2,2,0,0,4) \cdot \boldsymbol{\gamma}(r,s) = 2\pi.$$

Exact values of s_0 and r_0 are as roots of the respective polynomials

$$49s^9 - 340s^8 + 1200s^7 - 1600s^6 - 378s^5 + 560s^4 + 64s^3 - 64s^2 - 7s + 4$$

and

$$2r^9 + 17r^8 + 120r^7 + 56r^6 + 60r^5 - 2r^4 - 88r^3 - 40r^2 + 2r + 1.$$

Figure 9.2 displays a compact 3-packing P with radii$(P) = \{s_0, r_0, 1\}$.

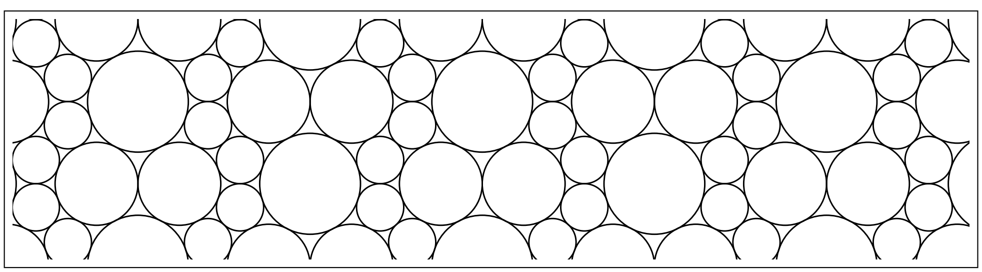

FIGURE 9.2. A compact circle-packing P with radii$(P) = \{s_0, r_0, 1\}$ where $s_0 \approx 0.468169$ and $r_0 \approx 0.822210$ are roots of the polynomials as given in Example 9.2.

Example 9.3. The values $s_0 \approx 0.484497$ and $r_0 \approx 0.865150$ are approximations to the unique solution in Δ_3 of the equations

$$(0,0,0,1,1,3) \cdot \boldsymbol{\alpha}(r,s) = 2\pi,$$
$$(2,0,3,0,2,0) \cdot \boldsymbol{\beta}(r,s) = 2\pi,$$
$$(0,0,1,4,0,2) \cdot \boldsymbol{\gamma}(r,s) = 2\pi.$$

Exact values of s_0 and r_0 are as roots of the respective polynomials

$$s^{11} - 824s^{10} + 5452s^9 - 14096s^8 + 24438s^7 - 20688s^6$$

$$+\,15404s^5 - 13520s^4 - 3375s^3 + 5480s^2 + 192s - 512$$

and

$$r^{11} + 18r^{10} + 132r^9 + 568r^8 + 1454r^7 + 1788r^6 \\ + 308r^5 - 680r^4 + 121r^3 - 670r^2 - 1120r + 128.$$

Figure 9.3 displays a compact 3-packing P with $\mathrm{radii}(P) = \{s_0, r_0, 1\}$.

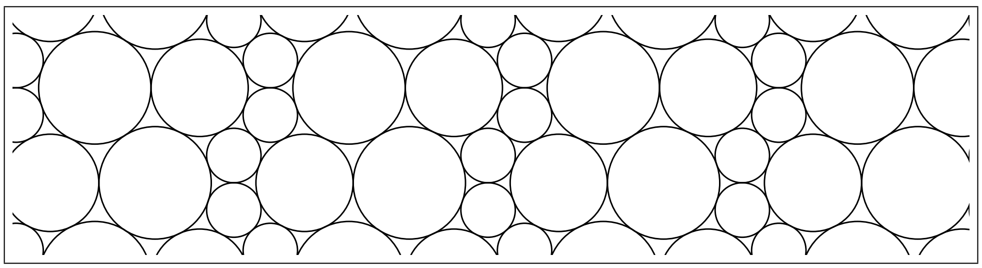

FIGURE 9.3. A compact circle-packing P with $\mathrm{radii}(P) = \{s_0, r_0, 1\}$ where $s_0 \approx 0.484497$ and $r_0 \approx 0.865150$ are roots of the polynomials as given in Example 9.3.

Example 9.4. The values $s_0 \approx 0.275178$ and $r_0 \approx 0.948799$ are approximations to the unique solution in Δ_3 of the equations

$$(0,0,0,2,2,0)\cdot\boldsymbol{\alpha}(r,s) = 2\pi,$$
$$(0,0,0,2,6,0)\cdot\boldsymbol{\beta}(r,s) = 2\pi,$$
$$(0,1,3,0,0,6)\cdot\boldsymbol{\gamma}(r,s) = 2\pi.$$

Exact values of s_0 and r_0 are as roots of the respective polynomials

$$20s^4 - 36s^3 + 13s^2 + 6s - 2$$

and

$$5r^4 + 24r^3 + 15r^2 - 38r - 2.$$

Figure 9.4 displays a compact 3-packing P with $\mathrm{radii}(P) = \{s_0, r_0, 1\}$.

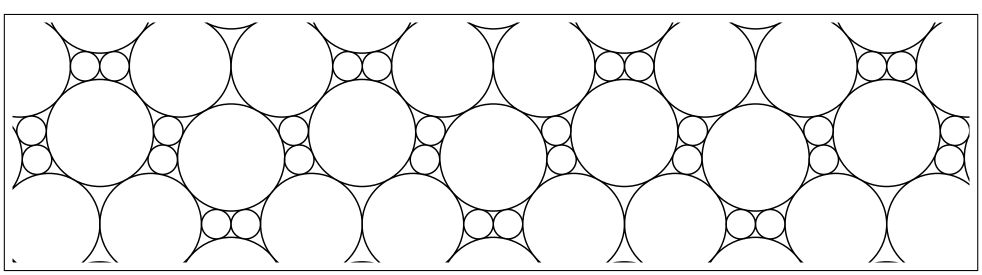

FIGURE 9.4. A compact circle-packing P with $\mathrm{radii}(P) = \{s_0, r_0, 1\}$ where $s_0 \approx 0.275178$ and $r_0 \approx 0.948799$ are roots of the polynomials as given in Example 9.4.

Example 9.5. The values $s_0 \approx 0.237538$ and $r_0 \approx 0.667499$ are approximations to the unique solution in Δ_3 of the equations

$$(0,0,0,2,2,0) \cdot \boldsymbol{\alpha}(r,s) = 2\pi,$$
$$(1,1,0,2,2,0) \cdot \boldsymbol{\beta}(r,s) = 2\pi,$$
$$(2,1,1,2,0,2) \cdot \boldsymbol{\gamma}(r,s) = 2\pi.$$

Exact values of s_0 and r_0 are as roots of the respective polynomials

$$64s^{12} - 704s^{11} + 15792s^{10} - 33536s^9 + 29964s^8 - 4540s^7 - 4859s^6 + 3322s^5 - 1757s^4 + 136s^3 + 307s^2 - 102s + 9$$

and

$$r^{12} - 4r^{11} + 66r^{10} - 3324r^9 + 727r^8 + 56696r^7 + 81500r^6 - 29400r^5 - 46657r^4 + 332r^3 + 5314r^2 + 276r + 9.$$

Figure 9.5 displays a compact 3-packing P with $\text{radii}(P) = \{s_0, r_0, 1\}$.

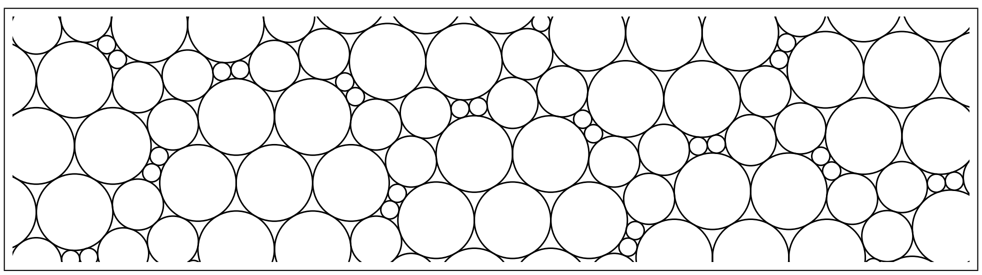

FIGURE 9.5. A compact circle-packing P with $\text{radii}(P) = \{s_0, r_0, 1\}$ where $s_0 \approx 0.237538$ and $r_0 \approx 0.667499$ are roots of the polynomials as given in Example 9.5.

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