RELATIVE WEAK MIXING OF W*-DYNAMICAL SYSTEMS VIA JOININGS

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ABSTRACT. A characterization of relative weak mixing in W*-dynamical systems in terms of a relatively independent joining is proven.

1. Introduction

This paper studies relative weak mixing for W*-dynamical systems in terms of joinings. Here a W*-dynamical system refers to a von Neumann algebra with a faithful normal tracial state which is invariant under the dynamics, given by iteration of a fixed $*$-automorphism of the von Neumann algebra (i.e. we focus exclusively on actions of the group $\mathbb{Z}$). The main result is a characterization of relative weak mixing in terms of relative ergodicity of the relative product of the system with its mirror image on the commutant (in the cyclic representation). The relative product system is defined using the relatively independent joining obtained from the conditional expectation onto the von Neumann subalgebra relative to which we are working. Generalizing the classical case, the subalgebra in question is always taken to be globally invariant under the dynamics of the W*-dynamical system.

The proof involves a careful analysis of the interplay between the von Neumann algebra, its commutant, and the conditional expectation. Some results of independent interest obtained on the way to the main result, do not require the state to be tracial. In this case, we need to restrict ourselves to subalgebras which are globally invariant under the modular group, to ensure the existence of the conditional expectation.

In classical ergodic theory it is well known that a dynamical system is weakly mixing if and only if its product with itself is ergodic. Our main result is essentially noncommutative and relative version of this.

A noncommutative theory of joinings has been developed in [7], [8] and [9], generalizing some aspects of the classical theory (see [17] for a thorough treatment, and [13] as well as [23] for the origins). It included a study of weak mixing, relative ergodicity and compact subsystems. Subsequent work was done in [4], which among other things developed

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various characterizations of joinings and also obtained a more complete theory for weak mixing, building on an approach to noncommutative joinings outlined in [21, Section 5]. Also see [3] for connected results. Earlier work related to noncommutative joinings appeared in [24], connected to entropy, and [11], regarding ergodic theorems.

An investigation of relative weak mixing is a natural next step in the development of the theory of noncommutative joinings. Relative weak mixing has already been studied and used very effectively in the noncommutative context in [22] and [2], but not from a joining point of view.

In particular, the authors of [2] proved quite a remarkable structure theorem, namely that an asymptotically abelian W*-dynamical system is weakly mixing relative to the center of the von Neumann algebra. This allowed them to apply classical ergodic results to the system on the center, and then extend these results to the noncommutative system. They defined relative weak mixing in terms of a certain ergodic limit, which is the approach taken in this paper as well. However, we adapt their definition to a form which is more convenient in the proof of our main result. The two definitions are nevertheless equivalent when the invariant state is tracial. To prove this, we make use of the semi-finite trace obtained in the basic construction from the von Neumann algebra and the subalgebra relative to which we are working.

Since systems which are not asymptotically abelian do occur, we do not assume asymptotic abelianness in this paper.

Furthermore, systems can be weakly mixing relative to nontrivial subalgebras other than the center. This includes cases where the von Neumann algebra of the system is a factor (i.e. when the center is trivial). Therefore we work relative to more general von Neumann subalgebras.

In the classical case, relative weak mixing is often defined in terms of a relatively independent joining, or relative product, illustrating the importance of this characterization in the classical case. However, it is in many cases just stated for ergodic systems, since any system can be decomposed into ergodic parts. See for example [14, Theorem 7.5], [30, Definition 7.9] and [17, Definition 9.22]. But we note that in [16] and [15, Definition 6.2], on the other hand, ergodicity is not assumed.

In the noncommutative case the assumption of ergodicity becomes problematic, as typically some form of asymptotic abelianness is required to do an ergodic decomposition. See for example [5, Subsection 4.3.1] for an exposition. Therefore we study the joining characterization of relative weak mixing without the assumption of ergodicity. In particular the proof of our main result has to deal with the difficulty of the system not being ergodic.

A number of other noncommutative relative ergodic properties have already been studied in the literature, for example in [10], building on ideas from [12], which was based in turn on variations of unique ergodicity as studied in [1]. Those properties, however, are more of a topological nature, rather than purely measure theoretic in origin, if one thinks in terms of classical ergodic theory, and the techniques involved are quite different from those in this paper.

The required background on relatively independent joinings is reviewed in Section 2, which also sets out much of the notation used later in the paper. The definition of relative weak mixing is formulated in Section 3. Some relevant characterizations in terms of ergodic limits are then derived. A noncommutative example is subsequently presented to illustrate the points made above regarding asymptotic abelianness, the center, and ergodicity. The main result of the paper, and its proof, appear in Section 4.

2. Relatively independent joinings

For convenience we summarize the special case of relatively independent joinings that we need here, along with some additional definitions. Simultaneously this fixes notation that will be used throughout the paper. We use the same setup as in [7, 8, 9]. Please refer in particular to [9, Sections 2 and 3] for further discussion. Also note that we use the convention where inner products are linear in the right and conjugate linear in the left.

In the remainder of this paper W*-dynamical systems are referred to simply as "systems" and they are defined as follows:

Definition 2.1. A *system* $\mathbf{A} = (A, \mu, \alpha)$ consists of a faithful normal state μ on a (necessarily σ-finite) von Neumann algebra A, and a $*$-automorphism α of A, such that $\mu \circ \alpha = \mu$.

In the rest of the paper, the symbols $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{F}$ denote systems (A, μ, α), (B, ν, β) and (F, λ, φ). For $\mathbf{A}$ we assume without loss that A is a von Neumann algebra on the Hilbert space H, with μ given by a cyclic and separating vector $\Omega \in H$, i.e.

$$\mu(a) = \langle \Omega, a\Omega \rangle$$

for all $a \in A$.

Definition 2.2. A *joining* of $\mathbf{A}$ and $\mathbf{B}$ is a state ω on the algebraic tensor product $A \odot B$ such that $\omega(a \otimes 1_B) = \mu(a)$, $\omega(1_A \otimes b) = \nu(b)$ and $\omega \circ (\alpha \odot \beta) = \omega$ for all $a \in A$ and $b \in B$.

The modular conjugation associated to the state μ, will be denoted by J, and we let

$$j : B(H) \to B(H) : a \mapsto Ja^*J.$$

The dynamics α of a system $\mathbf{A}$ can be represented by a unitary operator U on H defined by extending

$$Ua\Omega := \alpha(a)\Omega.$$

It satisfies

$$UaU^* = \alpha(a)$$

for all $a \in A$.

Definition 2.3. We call $\mathbf{F}$ a *subsystem* of $\mathbf{A}$ if F is a von Neumann subalgebra of A (containing the unit of A) such that $\mu|_F = \lambda$ and $\alpha|_F = \varphi$. If F is globally invariant under modular group associated to μ, then $\mathbf{F}$ is called a *modular subsystem* of $\mathbf{A}$.

Throughout the rest of the paper, $\mathbf{F}$ will be a modular subsystem of $\mathbf{A}$. Note that if the state μ of the system $\mathbf{A}$ is a trace (i.e. $\mu(ab) = \mu(ba)$ for all $a, b \in A$), then all of its subsystems are modular. Much of our work, in particular our main result, Theorem 4.2, is for the case where μ is tracial.

Given a system $\mathbf{A}$, carry the state and dynamics of $\mathbf{A}$ over to A' in a natural way using j, by defining a state μ' and $*$-automorphism α' on A' by

$$\mu'(b) := \mu \circ j(b) = \langle \Omega, b\Omega \rangle$$

and

$$\alpha'(b) := j \circ \alpha \circ j(b) = UbU^*$$

for all $b \in A'$, since $UJ = JU$. This defines the system

$$\mathbf{A}' := (A', \mu', \alpha').$$

Since $\mathbf{F}$ is a modular subsystem of $\mathbf{A}$, we obtain a modular subsystem $\tilde{\mathbf{F}} = \left(\tilde{F}, \tilde{\lambda}, \tilde{\varphi}\right)$ of $\mathbf{A}'$ as follows: Set

$$\tilde{F} := j(F) \subset A'$$

(note that by the symbol $\subset$ we mean inclusion, with equality allowed), and let

$$\tilde{\lambda} := \mu'|_{\tilde{F}}$$

and

$$\tilde{\varphi} := \alpha'|_{\tilde{F}}.$$

We can now construct the relatively independent joining of $\mathbf{A}$ and $\mathbf{A}'$ over $\mathbf{F}$:

Since $\mathbf{F}$ is a modular subsystem of $\mathbf{A}$, we know by Tomita-Takesaki theory (see for example [28, Theorem IX.4.2]) that we have a unique conditional expectation

$$D : A \to F$$

such that $\lambda \circ D = \mu$. Then

$$D' := j \circ D \circ j : A' \to \tilde{F}$$

is the unique conditional expectation such that $\tilde{\lambda} \circ D' = \mu'$.

Let P be the projection of H onto

$$H_F := \overline{F\Omega} = \overline{\tilde{F}\Omega},$$

where the last equality follows from $JH_F = H_F$. Then

$$D(a)\Omega = Pa\Omega$$

for all $a \in A$. This follows from the general construction of such conditional expectations; see for example [27, Section 10.2]. Similarly,

$$D'(b)\Omega = Pb\Omega$$

for all $b \in A'$. Also note that

$$D \circ \alpha = \alpha \circ D = \varphi \circ D,$$

and analogously for D', since

$$PU = UP,$$

as is easily verified from $\alpha(F) = F$.

Define the unital $*$-homomorphism

$$\delta : F \odot \tilde{F} \to B(H),$$

to be the linear extension of $F \times \tilde{F} \to B(H) : (a, b) \mapsto ab$. Defining the *diagonal* state

$$\Delta_\lambda : F \odot \tilde{F} \to \mathbb{C}$$

of λ by

$$\Delta_\lambda(c) := \langle \Omega, \delta(c)\Omega \rangle$$

for all $c \in F \odot \tilde{F}$, allows us to define a state $\mu \odot_\lambda \mu'$ on $A \odot A'$ by

$$\mu \odot_\lambda \mu' := \Delta_\lambda \circ E \tag{1}$$

where

$$E := D \odot D'.$$

Note that $\mu \odot_\lambda \mu'$ is indeed a joining of $\mathbf{A}$ and $\mathbf{A}'$, with the property that $(\mu \odot_\lambda \mu')|_{F \odot \tilde{F}} = \Delta_\lambda$, and it is called the *relatively independent joining of* $\mathbf{A}$ *and* $\mathbf{A}'$ *over* $\mathbf{F}$. We also denote this joining by

$$\omega := \mu \odot_\lambda \mu'.$$

Remark 2.4. Notice that the framework we have set up so far, in particular the relatively independent joining, fits in very naturally with the modular theory of von Neumann algebras:

Note firstly that similar to the fact that ω is a joining of $\mathbf{A}$ and $\mathbf{A}'$, we also have

$$\omega \circ (\sigma_t^\mu \odot (\sigma_t^\mu)') = \omega$$

where σ_t^μ denotes the modular group associated to μ, and $(\sigma_t^\mu)'$ is defined analogously to α'. This follows, since $D \circ \sigma_t^\mu = \sigma_t^\mu \circ D$ and $D' \circ \sigma_t^{\mu'} = \sigma_t^{\mu'} \circ D'$, and where we also note that $(\sigma_t^\mu)' = \sigma_{-t}^{\mu'}$. From the point of view of von Neumann algebras (i.e. noncommutative measure theory), this is a very natural property for a joining to have, and indeed in [4, Definition 3.1] it is included as part of the definition of joinings more generally, even though here we have not required it in Definition 2.2.

Secondly, by [18, Lemma 1 of Section 1] (or see [28, Corollary VIII.1.4]) it follows that $\alpha^{-1} \circ \sigma_t^{\mu} \circ \alpha = \sigma_t^{\mu \circ \alpha} = \sigma_t^{\mu}$, so

$$\sigma_t^{\mu} \circ \alpha = \alpha \circ \sigma_t^{\mu}$$

and analogously for α' and $\sigma_t^{\mu'}$, again showing that the framework used here fits in very neatly with of modular theory.

We write

$$\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}' := (A \odot A', \mu \odot_{\lambda} \mu', \alpha \odot \alpha')$$

and call $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ the *relative product system* (of $\mathbf{A}$ and $\mathbf{A}'$ over $\mathbf{F}$). It is an example of a $*$-*dynamical system*, namely it consists of a state $\omega = \mu \odot_{\lambda} \mu'$ on a unital $*$-algebra $A \odot A'$, and a $*$-automorphism $\alpha \odot \alpha'$ of $A \odot A'$ such that $\omega \circ (\alpha \odot \alpha') = \omega$. However, this is typically not a (W*-dynamical) system as given by Definition 2.1.

The cyclic representation of $A \odot A'$ obtained from ω by the GNS construction will be denoted by $(H_{\omega}, \pi_{\omega}, \Omega_{\omega})$. Since ω can be extended to a state on the maximal C*-algebraic tensor product $A \otimes_m A'$ (see for example [8, Proposition 4.1]), we know that π_{ω} is a $*$-homomorphism from $A \odot A'$ into the bounded operators $B(H_{\omega})$. Let

$$\gamma_{\omega} : A \odot A' \to H_{\omega} : t \mapsto \pi_{\omega}(t)\Omega_{\omega}.$$

Furthermore, let W denote the unitary representation of

$$\tau := \alpha \odot \alpha'$$

on H_{ω}, i.e. it is defined as the extension of

$$W\gamma_{\omega}(t) := \gamma_{\omega}(\tau(t))$$

for all $t \in A \odot A'$.

The cyclic representation obtained from ω, allows us to construct cyclic representations $(H_{\mu}, \pi_{\mu}, \Omega_{\omega})$ and $(H_{\mu'}, \pi_{\mu'}, \Omega_{\omega})$ of (A, μ) and (A', μ') respectively, which are naturally embedded into H_{ω} (as in [7, Construction 2.3]), by setting

$$H_{\mu} := \overline{\gamma_{\omega}(A \otimes 1)} \quad \text{and} \quad \pi_{\mu}(a) := \pi_{\omega}(a \otimes 1)|_{H_{\mu}}$$

for every $a \in A$, and similarly for $H_{\mu'}$ and $\pi_{\mu'}$.

The representation $(H_{\mu}, \pi_{\mu}, \Omega_{\omega})$ is unitarily equivalent to our initial representation $(H, \mathrm{id}_A, \Omega)$ of (A, μ), but we make use of both representations later on. I.e., whereas $a \in A$ is in the initial cyclic representation, we always write it explicitly as $\pi_{\mu}(a)$ when using the cyclic representation $(H_{\mu}, \pi_{\mu}, \Omega_{\omega})$.

Now we consider cyclic representations of (F, λ) and $(\tilde{F}, \tilde{\lambda})$:

Note that $(H_F, \delta, \Omega_{\omega})$ is a cyclic representation of $(F \odot \tilde{F}, \Delta_{\lambda})$, since $H_F = \overline{\delta(F \odot \tilde{F})\Omega}$. However, $(\overline{\gamma_{\omega}(F \odot \tilde{F})}, \pi_{\omega}|_{F \odot \tilde{F}}, \Omega_{\omega})$ is also a cyclic representation of $(F \odot \tilde{F}, \Delta_{\lambda})$, so these two representations are unitarily equivalent via the unitary operator $V : H_F \to \overline{\gamma_\omega(F \odot \tilde{F})}$ defined as the extension of $\delta(t)\Omega \mapsto \gamma_\omega(t)$ for $t \in F \odot \tilde{F}$. Therefore

$$H_\lambda := \overline{\gamma_\omega(F \otimes 1)} = V\overline{\delta(F \otimes 1)\Omega} = VH_F = V\overline{\delta(1 \otimes \tilde{F})\Omega} = \overline{\gamma_\omega(1 \otimes \tilde{F})},$$

which means that (F, λ) and $(\tilde{F}, \tilde{\lambda})$ are cyclicly represented on the same subspace H_λ of H_ω by

$$\pi_\lambda(f) := \pi_\mu(f)|_{H_\lambda} \quad \text{and} \quad \pi_{\tilde{\lambda}}(\tilde{f}) := \pi_{\mu'}(\tilde{f})|_{H_\lambda}$$

for all $f \in F$ and $\tilde{f} \in \tilde{F}$.

3. Relative weak mixing

This section presents the definition and two closely related characterizations of relative weak mixing in terms of ergodic averages. These characterizations do not yet involve the relative independent joining. An example of relative weak mixing is also given.

In terms of the notation in the previous section, our main definition is the following:

Definition 3.1. We call a system $\mathbf{A}$ *weakly mixing relative to* the modular subsystem $\mathbf{F}$ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda\left(|D(b\alpha^n(a))|^2\right) = 0 \tag{2}$$

for all $a, b \in A$ with $D(a) = D(b) = 0$.

In the classical case this is often also expressed by saying that $\mathbf{A}$ is a *weakly mixing extension* of $\mathbf{F}$.

Remark 3.2. We recover the absolute case of weak mixing from this definition, by using $F = \mathbb{C}1_A$. Indeed, in this case we have $D(a) = \mu(a)1_A$ for all $a \in A$. Thus, Eq. (2) becomes

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\mu(ba^n(a))|^2 = 0,$$

or equivalently,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\mu(ba^n(a))| = 0, \tag{3}$$

for all $a, b \in A$ such that $\mu(a) = \mu(b) = 0$.

The reason for this equivalence is that for any bounded sequence (c_n) of non-negative real numbers, bounded by $c > 0$, say, we have

$$\frac{1}{N} \sum_{n=1}^{N} c_n^2 \leq \frac{c}{N} \sum_{n=1}^{N} c_n$$

and, using the Cauchy-Schwarz inequality,

$$\frac{1}{N}\sum_{n=1}^{N} c_n \leq \left(\frac{1}{N}\sum_{n=1}^{N} c_n^2\right)^{\frac{1}{2}}. \tag{4}$$

Therefore,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} c_n^2 = 0 \Leftrightarrow \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} c_n = 0.$$

Condition (3) in turn is easily seen to be equivalent to the following:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} |\mu(ba^n(a)) - \mu(b)\mu(a)| = 0$$

for all $a, b \in A$ (simply replace a and b by $a - \mu(a)$ and $b - \mu(b)$ respectively in Eq. (3)). This is the standard definition of weak mixing.

Our first simple characterization of relative weak mixing, which will also be used in the proof of this paper's main theorem in the next section, is the following:

Proposition 3.3. *The system* $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$ *if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} \lambda\left(|D\left(b\alpha^n(a)\right) - D(b)D(\alpha^n(a))|^2\right) = 0 \tag{5}$$

for all $a, b \in A$.

Proof. Assume that $\mathbf{A}$ is weakly mixing relative to $\mathbf{F}$. For any $a, b \in A$, setting $a_0 := a - D(a)$ and $b_0 := b - D(b)$, we have $D(a_0) = D(b_0) = 0$ and

$$D(b_0\alpha^n(a_0)) = D\left(b\alpha^n(a)\right) - D(b)D(\alpha^n(a)).$$

Hence Eq. (5) follows from Definition 3.1. The converse is trivial by assuming either $D(a) = 0$ or $D(b) = 0$. □

This gives us variations of this characterization as well, for example, $\mathbf{A}$ is weakly mixing relative to $\mathbf{F}$ if and only if Eq. (2) holds for all $a, b \in A$ with $D(a) = 0$.

Next we are going to show that when μ is a trace, Definition 3.1 is equivalent to [2, Definition 3.7]. To do this, we use the basic construction in a similar way to how it was used in [2, Sections 3 and 4] to prove their structure theorem.

The von Neumann algebra generated by A and $e_F = P$ (the projection of H onto $\overline{F\Omega}$) will be denoted by

$$\langle A, e_F\rangle$$

and is referred to as the *basic construction.* When μ is a trace, we obtain from it a faithful semifinite normal tracial weight $\bar{\mu} : \langle A, e_F\rangle^+ \to$

$[0, \infty]$. It is also defined and tracial on the strongly dense $*$-subalgebra $Ae_FA := \mathrm{span}\{ae_Fb : a, b \in A\}$ of $\langle A, e_F\rangle$ via the equation

$$\bar{\mu}(ae_Fb) = \mu(ab).$$

For more on the basic construction and the trace $\bar{\mu}$, see [25, Chapter 4]. Some of the early literature on this topic can be found in [26], [6] and [19].

We can extend the dynamics of α to $\langle A, e_F\rangle$ using the equation

$$\bar{\alpha}(a) = UaU^*$$

for $a \in \langle A, e_F\rangle$. That $\bar{\mu} \circ \bar{\alpha} = \bar{\mu}$, is not elementary, but it is a simple consequence of the following result:

Theorem 3.4. [25, Theorem 4.3.11] *Let φ be a weight on $\langle A, e_F\rangle$ with $\varphi = \bar{\mu}$ on $(Ae_FA)^+$. If φ is normal, then $\varphi = \bar{\mu}$.*

Next, similar to the case of U, we have a unitary operator $\bar{U} : \bar{H} \to \bar{H}$ representing $\bar{\alpha}$ on the Hilbert space $\bar{H}$ arising from the GNS construction for $(\langle A, e_F\rangle, \bar{\mu})$, which is described for example in [20, Section 7.5]. We denote the quotient map of this construction as

$$\mathcal{N}_{\bar{\mu}} \to \bar{H} : x \mapsto \hat{x},$$

where $\mathcal{N}_{\bar{\mu}} := \{x \in \langle A, e_F\rangle : \bar{\mu}(x^*x) < \infty\}$. Since $\bar{\mu} \circ \bar{\alpha} = \bar{\mu}$, we can define the unitary $\bar{U} : \bar{H} \to \bar{H}$ via

$$\bar{U}\hat{x} = \widehat{\alpha(x)}.$$

In order to prove the equivalence with [2, Definition 3.7], we need three lemmas which we present now. The first is just a slight variation of the calculations that appear at the beginning of the proof of [2, Proposition 3.8]:

Lemma 3.5. *Assume that μ is a trace. Let $a, b \in A$. Then*

$$\bar{\mu}(b^*e_Fb\bar{\alpha}^n(ae_Fa^*)) = \lambda(|D(b\alpha^n(a))|^2).$$

Proof. $\bar{\mu}(b^*e_Fb\bar{\alpha}^n(ae_Fa^*)) = \bar{\mu}(D(c)e_FD(c^*)) = \mu(D(c)D(c^*))$ in terms of $c := b\alpha^n(a)$. □

The following is a version of the van der Corput lemma:

Lemma 3.6. [29, Lemma 2.12.7] *Let (v_n) be a bounded sequence of vectors in a Hilbert space $\mathfrak{H}$ such that*

$$\lim_{M\to\infty} \frac{1}{M} \sum_{h=1}^{M} \left(\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle v_n, v_{n+h}\rangle \right) = 0. \tag{6}$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} v_n = 0.$$

Putting these two lemmas together, we obtain the following:

Lemma 3.7. *Assume μ is a trace. Let $a \in A$ satisfy*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(|D(a^* \alpha^n(a))|^2) = 0. \tag{7}$$

Then, for all $b \in A$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(|D(b \alpha^n(a))|^2) = 0. \tag{8}$$

Proof. Let $x := a e_F a^*$ and $y := b^* e_F b$. Observe that $\bar{\mu}(y\bar{\alpha}^n(x)) \geq 0$ by Lemma 3.5, and

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^{N} \bar{\mu}(y\bar{\alpha}^n(x))^2 &= \frac{1}{N} \sum_{n=1}^{N} \langle \hat{y}, \bar{U}^n \hat{x} \rangle^2 = \left\langle \hat{y}, \frac{1}{N} \sum_{n=1}^{N} \langle \hat{y}, \bar{U}^n \hat{x} \rangle \bar{U}^n \hat{x} \right\rangle \\ &\leq \|\bar{y}\| \left\| \frac{1}{N} \sum_{n=1}^{N} \langle \hat{y}, \bar{U}^n \hat{x} \rangle \bar{U}^n \hat{x} \right\| \end{aligned} \tag{9}$$

Let $v_n := \langle \hat{y}, \bar{U}^n \hat{x} \rangle \bar{U}^n \hat{x}$, for every $n \in \mathbb{N}$. Clearly, the sequence (v_n) is bounded. We can estimate, for every $n, h \in \mathbb{N}$,

$$|\langle v_n, v_{n+h} \rangle| \leq \|\hat{x}\|^2 \|\hat{y}\|^2 \bar{\mu}(x\bar{\alpha}^h(x)).$$

This, together with Lemma 3.5 and our assumption Eq. (7), imply Eq. (6). Thus, from Lemma 3.6, we have $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} v_n = 0$. Therefore, from (9), we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \bar{\mu}(y\bar{\alpha}^n(x))^2 = 0.$$

Consequently, from (4),

$$\lim_{N\to\infty} \sum_{n=1}^{N} \bar{\mu}(y\bar{\alpha}^n(x)) = 0.$$

Again by Lemma 3.5, we are done. □

This finally implies the following characterization of relative weak mixing (which in [2] was used as the definition):

Proposition 3.8. *Assume that μ is a trace. Then* **A** *is weakly mixing relative to the subsystem* **F** *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(|D(a^* \alpha^n(a))|^2) = 0,$$

for all $a \in A$ such that $D(a) = 0$.

Remark 3.9. Essential to the proof of the commutative version of Lemma 3.7 (outlined in [29, Exercise 2.14.1]), is a conditional version of the Cauchy-Schwarz inequality in terms of the conditional expectation $\mathbb{E}$:

$$|\mathbb{E}(\bar{f}g|Y)| \leq \|\mathbb{E}(|f|^2|Y)\|_{L^2(X|Y)}\|\mathbb{E}(|g|^2|Y)\|_{L^2(X|Y)}$$

where f, g belong to the $L^\infty(Y)$-module $L^2(X|Y) = \{h \in L^2(X) : \mathbb{E}(|h|^2|Y) \in L^\infty(Y)\}$ ([29, Section 2.13]). In the noncommutative case, however, our approach above allows us to simplify the argument and avoid some snags. We essentially used a noncommutative translation of the proof of the absolute case [29, Corollary 2.12.8], but in terms of the basic construction, to prove Lemma 3.7.

Before we get to an example, we note a few simple general facts:

Firstly, $D(a) = 0$ for $a \in A$, if and only if a is of the form $a = c - D(c)$ for some $c \in A$.

Secondly,

$$\lambda(D(\alpha^n(a^*)b^*)D(b\alpha^n(a))) = \|PbU^n a\Omega\|^2$$

for all $a, b \in A$, by a straightforward calculation. If, in addition λ is a trace, then we have

$$\|PbU^n a\Omega\| = \left\|PU^n a^* U^{-n} b^* \Omega\right\| \tag{10}$$

for all $a, b \in A$, by a similar calculation for $\lambda(D(b\alpha^n(a))D(\alpha^n(a^*)b^*))$.

To show that relative weak mixing is indeed relevant in noncommutative W*-dynamical systems, in particular for non-ergodic systems which are not asymptotically abelian, we provide the following example:

Example 3.10. Let G be any discrete group, and let A be the group von Neumann algebra obtained from it. In other words, A is the von Neumann algebra on $H = l^2(G)$ generated by the following set of unitary operators:

$$\{l(g) : g \in G\}$$

where l is the left regular representation of G, i.e. the unitary representation of G on H with each $l(g) : H \to H$ given by

$$[l(g)f](h) = f(g^{-1}h)$$

for all $f \in H$ and $g, h \in G$. Equivalently,

$$l(g)\delta_h = \delta_{gh}$$

for all $g, h \in G$, where $\delta_g \in H$ is defined by $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $h \neq g$. Setting

$$\Omega := \delta_1$$

where $1 \in G$ denotes the identity of G, we can define a faithful normal trace μ on A by

$$\mu(a) := \langle \Omega, a\Omega \rangle$$

for all $a \in A$. Then $(H, \mathrm{id}_A, \Omega)$ is the cyclic representation of (A, μ).

Given any automorphism T of G, we define a unitary operator on H by

$$Uf := f \circ T^{-1}$$

for all $f \in H$. From this we obtain a $*$-automorphism of A by setting

$$\alpha(a) := UaU^*$$

for all $a \in A$, which satisfies $\alpha(l(g)) = l(T(g))$ for all $g \in G$.

Then $\mathbf{A} = (A, \mu, \alpha)$ is a system which we call the *dual system* of (G, T). (See [8, Section 3] for more background on this type of system in the context of quantum groups, W*-algebraic ergodic theory and joinings.)

Define a subsystem $\mathbf{F} = (F, \lambda, \varphi)$ of $\mathbf{A}$ by letting F be the von Neumann subalgebra of A generated by

$$\{l(g) : g \in K\}$$

where $K := \{g \in G : T^{\mathbb{N}}(g) \text{ is finite}\}$. Here $T^{\mathbb{N}}(g) := \{T(g), T^2(g), T^3(g), ...\}$ is the *orbit* of g. Furthermore $\lambda := \mu|_F$ and $\varphi := \alpha|_A$.

We call $\mathbf{F}$ the *finite orbit subsystem* of $\mathbf{A}$.

We can find D explicitly in this case: The projection P above is now the projection of H onto the Hilbert subspace spanned by $\{\delta_g : g \in K\}$. Therefore we have

$$D(l(g)) = \begin{cases} l(g) \text{ for } g \in K \\ 0 \text{ for } g \notin K \end{cases} \tag{11}$$

for all $g \in G$.

Note that the unital $*$-algebra generated by $\{l(g) : g \in G\}$ is exactly $A_0 = \mathrm{span}\{l(g) : g \in G\}$.

Suppose that for any $g, h \in G$ with $g \notin K$, it is true that

$$D(l(hT^n(g))) = 0 \tag{12}$$

for n large enough, i.e. for $n > n_0$ for some n_0. Then, for any $c_0, b_0 \in A_0$, and $a_0 := c_0 - D(c_0)$, we have

$$Pb_0U^n a_0\Omega = 0$$

for n large enough. Since A_0 is strongly dense in A, it follows that

$$\lim_{n\to\infty} Pb_0U^n a\Omega = 0$$

for all $a \in A$ such that $D(a) = 0$, by simply considering any $c \in A$ and some $c_0 \in A_0$ such that $\|c_0\Omega - c\Omega\| < \varepsilon$ for an $\varepsilon > 0$ of our choosing, and setting $a := c - D(c)$.

Since λ is a trace, we can apply a similar argument to $\|Pb_0U^n a\Omega\| = \|PU^n a^* U^{-n} b_0^* \Omega\|$ (see Eq. (10)) to show that

$$\lim_{n\to\infty} PbU^n a\Omega = 0$$

and therefore

$$\lim_{n\to\infty} \lambda(D(\alpha^n(a^*)b^*)D\,(b\alpha^n(a))) = 0$$

for all $a, b \in A$ such that $D(a) = 0$. It follows easily from this that $\mathbf{A}$ is weakly mixing relative to $\mathbf{F}$. The limit above could be interpreted as $\mathbf{A}$ having a stronger property, namely that $\mathbf{A}$ is "strongly mixing relative to $\mathbf{F}$".

What remains is to show specific cases for which Eq. (12) holds and which illustrate the points made above about noncommutative systems.

A simple case is when G is the free group on a countably infinite set of symbols S. We then consider any bijection $T : S \to S$ which has both finite and infinite orbits in S, say T is a permutation when restricted to some finite non-empty subset, or to each of infinitely many finite non-empty subsets, while it shifts the remaining infinite subset of S. We obtain a automorphism T of G from this bijection. Then Eq. (12) follows from Eq. (11).

But at the same time, F is then not trivial, i.e. F strictly contains the subalgebra $\mathbb{C}1$, and is in general not abelian. In fact, F is $*$-isomorphic to the group von Neumann algebra of the free group K on the symbols with finite orbits. That $F \neq \mathbb{C}1$, also implies that $\mathbf{A}$ is not ergodic (see [8, Theorem 3.4]). Furthermore,

$$\|[\alpha^n(l(g)), l(h)]\Omega\| = \sqrt{2}$$

if $T^n(g)h \neq hT^n(g)$, which is the case if g and h are in two separate orbits, or if $g = h$ has an infinite orbit. Hence $\mathbf{A}$ is not asymptotically abelian in the sense of [2, Definition 1.10]. Furthermore, A is a factor.

We summarize the key conclusions from this example, as they concretely illustrate a number of remarks made in Section 1, motivating this paper:

Proposition 3.11. *Let* $\mathbf{A}$ *be the dual system of* (G, T)*, where* G *is the free group on a countably infinite set of symbols* S*, and* T *is an automorphism of* G *induced by a bijection* $T|_S : S \to S$ *which has both finite and infinite orbits (the former on non-empty subsets of* S*). Then* $\mathbf{A}$ *is weakly mixing relative to its non-trivial finite orbit subsystem (which in general consists of a noncommutative von Neumann subalgebra), but* $\mathbf{A}$ *is neither ergodic, nor asymptotically abelian, and furthermore its von Neumann algebra* A *is a factor.*

4. The joining characterization

This section presents the main result of the paper, still using the notation from Section 2.

Let H_ω^W denote the fixed point space of W. The relative independent joining (or the relative product system) will connect to relative weak mixing via the following notion:

Definition 4.1. We say that $\mathbf{A}\odot_{\mathbf{F}}\mathbf{A}'$ *is ergodic relative to* the modular subsystem $\mathbf{F}$ of $\mathbf{A}$, if $H_\omega^W \subset H_\lambda$.

Our main goal in this paper is to prove the following characterization of relative weak mixing:

Theorem 4.2. *Assume that μ is a trace. Then $\mathbf{A}$ is weakly mixing relative to $\mathbf{F}$ if and only if $\mathbf{A}\odot_{\mathbf{F}}\mathbf{A}'$ is ergodic relative to $\mathbf{F}$.*

The rest of this section is devoted to the proof of this theorem. We break the proof into a sequence of smaller results. Some of these are of independent interest (in particular Propositions 4.5, 4.9 and 4.10, and Remark 4.8), and do not require μ to be tracial.

The following lemma and proposition proves one direction of Theorem 4.2. In the classical case, this direction is also proven in [15, Proposition 6.2] and [16, Lemma 1.3], but using different arguments.

Lemma 4.3. *Consider a modular subsystem $\mathbf{F}$ of the system $\mathbf{A}$. For any $a \in A$ with $D(a) = 0$ and any $b \in A'$, we have*

$$\pi_\omega(a \otimes b)\Omega_\omega \perp H_\lambda.$$

Proof. For any $c \in F$,

$$\begin{aligned}\langle \pi_\lambda(c)\Omega_\omega, \pi_\mu(a)\Omega_\omega\rangle &= \langle \Omega_\omega, \pi_\mu(c^*a)\Omega_\omega\rangle = \mu(c^*a)\\ &= \lambda(D(c^*a)) = \lambda(c^*D(a))\\ &= 0.\end{aligned}$$

Hence, $\pi_\mu(a)\Omega_\omega \in H_\mu \ominus H_\lambda$. So $\pi_\mu(a)\Omega_\omega \perp H_{\mu'}$ by [9, Proposition 3.6]. On the other hand, $\pi_{\mu'}(b^*f)\Omega_\omega \in H_{\mu'}$ for any $f \in \tilde{F}$, so $\langle \pi_{\mu'}(b^*f)\Omega_\omega, \pi_\mu(a)\Omega_\omega\rangle = 0$. Therefore,

$$\begin{aligned}\langle \pi_{\tilde{\lambda}}(f)\Omega_\omega, \pi_\omega(a\otimes b)\Omega_\omega\rangle &= \langle \pi_\omega(1\otimes b^*)\pi_{\mu'}(f)\Omega_\omega, \pi_\omega(a\otimes 1)\Omega_\omega\rangle\\ &= \langle \pi_{\mu'}(b^*f)\Omega_\omega, \pi_\mu(a)\Omega_\omega\rangle\\ &= 0,\end{aligned}$$

proving the lemma, since $\pi_{\tilde{\lambda}}(\tilde{F})\Omega_\omega$ is dense in H_λ. □

Using this lemma we can show one direction of Theorem 4.2:

Proposition 4.4. *Assume that μ is a trace and that $\mathbf{A}\odot_{\mathbf{F}}\mathbf{A}'$ is ergodic relative to $\mathbf{F}$. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda\left(\left|D\left(b\alpha^n(a)\right)\right|^2\right) = 0$$

for all $a, b \in A$ such that $D(a) = 0$ or $D(b) = 0$.

Proof. Let Q be the projection of H_ω onto the fixed point space H_ω^W of W. By the mean ergodic theorem we then have

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\omega(\tau^n(s)t) &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle W^n\pi_\omega(s^*)\Omega_\omega,\pi_\omega(t)\Omega_\omega\rangle \\ &= \langle Q\pi_\omega(s^*)\Omega_\omega,\pi_\omega(t)\Omega_\omega\rangle\end{aligned}$$

for all $s,t\in A\odot A'$. This holds in particular for $s=a^*\otimes j(a)$ and $t=b^*\otimes j(b)$, where $a,b\in A$, and $D(a)=0$ or $D(b)=0$.

Suppose $D(a)=0$ (the case $D(b)=0$ is similar, by taking Q to the other side in the inner product above). Then $\pi_\omega(s^*)\Omega_\omega\perp H_\omega^W$ by Lemma 4.3, so $Q\pi_\omega(s^*)\Omega_\omega=0$. This means, by the definition of $\omega=\mu\odot_\lambda\mu'$ in Eq. (1), that

$$\begin{aligned}0 &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle\Omega, D(\alpha^n(a^*)b^*)D'(\alpha'^n(j(a))j(b))\Omega\rangle \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(\alpha^n(a^*)b^*)D(b\alpha^n(a))),\end{aligned}$$

as required, since

$$D'(\alpha'^n(j(a))j(b))\Omega = PJ\alpha^n(a^*)b^*\Omega = D(b\alpha^n(a))\Omega,$$

where we have used the fact that μ is a trace (so $Jc\Omega=c^*\Omega$ for all $c\in A$). □

Next we consider the other direction of Theorem 4.2. We don't have a reference to a proof of the classical case of this direction. Our first step is the following:

Proposition 4.5. $\mathbf{A}\odot_{\mathbf{F}}\mathbf{A}'$ *is ergodic relative to* $\mathbf{F}$ *if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\omega(t\tau^n(s)) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\omega(E(t)\tau^n(E(s))) \tag{13}$$

for all $s,t\in A\odot A'$. *Both limits exist, whether* $\mathbf{A}\odot_{\mathbf{F}}\mathbf{A}'$ *is ergodic relative to* $\mathbf{F}$ *or not.*

Proof. Let Q be the projection of H_ω onto the fixed point space H_ω^W of W. Let R be the projection of H_ω onto H_λ.

By the mean ergodic theorem, for all $s,t\in A\odot A'$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\omega(t\tau^n(s)) = \langle\gamma_\omega(t^*),Q\gamma_\omega(s)\rangle$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\omega(E(t)\tau^n(E(s))) = \langle\gamma_\omega(E(t^*)),Q\gamma_\omega(E(s))\rangle\,.$$

Let P_μ be the projection of H_μ onto H_λ, and $P_{\mu'}$ the projection of $H_{\mu'}$ onto H_λ. Consider $s = a \otimes b$, where $a \in A$ and $b \in A'$. Then, because $\pi_{\mu'}(D'(b))\Omega_\omega \in H_\lambda$, we know by the construction of D (see for example [27, Section 10.2]) that

$$\begin{aligned}\gamma_\omega(E(s)) &= \pi_\mu(D(a))\pi_{\mu'}(D'(b))\Omega_\omega = \pi_\lambda(D(a))\pi_{\mu'}(D'(b))\Omega_\omega \\ &= P_\mu \pi_\mu(a)\pi_{\mu'}(D'(b))\Omega_\omega = P_\mu \pi_\mu(a) P_{\mu'}\pi_{\mu'}(b)\Omega_\omega \\ &= R\pi_\mu(a)R\pi_{\mu'}(b)\Omega_\omega.\end{aligned}$$

since $R|_{H_\mu} = P_\mu$ and $R|_{H_{\mu'}} = P_{\mu'}$.

For $y \in H_{\mu'} \ominus H_\lambda$ and $f \in F$, we have

$$\langle \pi_\lambda(f)\Omega_\omega, \pi_\omega(a \otimes 1)y\rangle = \langle \pi_\mu(a^* f)\Omega_\omega, y\rangle = 0,$$

since $\pi_\mu(a^* f)\Omega_\omega \in H_\mu \perp (H_{\mu'} \ominus H_\lambda)$ by [9, Proposition 3.6]. So $\pi_\omega(a \otimes 1)y \perp H_\lambda$, which means that

$$\gamma_\omega(E(s)) = R\pi_\omega(a \otimes 1)\pi_{\mu'}(b)\Omega_\omega = R\pi_\omega(a \otimes b)\Omega_\omega.$$

So

$$\gamma_\omega(E(s)) = R\gamma_\omega(s)$$

for all $s \in A \odot A'$. Hence,

$$\langle \gamma_\omega(E(t^*)), Q\gamma_\omega(E(s))\rangle = \langle R\gamma_\omega(t^*), QR\gamma_\omega(s)\rangle$$

for all $s, t \in A \odot A'$.

Now, if $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$, i.e. $Q \leq R$, it follows that

$$\langle \gamma_\omega(E(t^*)), Q\gamma_\omega(E(s))\rangle = \langle \gamma_\omega(t^*), Q\gamma_\omega(s)\rangle$$

from which we see that Eq. (13) holds for all $s, t \in A \odot A'$.

Conversely, if Eq. (13) holds for all $s, t \in A \odot A'$, then we have

$$\langle R\gamma_\omega(t^*), QR\gamma_\omega(s)\rangle = \langle \gamma_\omega(t^*), Q\gamma_\omega(s)\rangle$$

for all $s, t \in A \odot A'$. It follows that $RQR = Q$, so $Q \leq R$, meaning $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$. □

As a consequence of this proposition, we have the following lemma towards the proof of Theorem 4.2:

Lemma 4.6. *Assume that μ is a trace. Then $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$ if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda(|D(b\alpha^n(a))|^2) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda(|D(b)D(\alpha^n(a))|^2) \tag{14}$$

for all $a, b \in A$. Both limits exist, whether $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$ or not.

Proof. Suppose $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$, then Eq. (13) holds. Applying it to $s = a \otimes c$ and $t = b \otimes d$, for $a, b \in A$ and $c, d \in A'$, we obtain

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega([b\alpha^n(a)] \otimes [d\alpha'^n(c)]) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega([D(b)D(\alpha^n(a))] \otimes [D'(d)D'(\alpha'^n(c))]). \end{aligned}$$

Using the definition of ω, this is equivalent to

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b\alpha^n(a))D'(d\alpha'^n(c))\Omega \rangle \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b)D(\alpha^n(a))D'(d)D'(\alpha'^n(c))\Omega \rangle \, . \end{aligned}$$

Setting $c = j(a^*) = JaJ$ and $d = j(b^*) = JbJ$, we have in particular

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b\alpha^n(a))JD(b\alpha^n(a))\Omega \rangle \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b)D(\alpha^n(a))JD(b)D(\alpha^n(a))\Omega \rangle \, . \end{aligned}$$

Since μ is a trace, this is equivalent to

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b\alpha^n(a))D(\alpha^n(a^*)b^*)\Omega \rangle \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b)D(\alpha^n(a))D(\alpha^n(a^*))D(b^*)\Omega \rangle \, . \end{aligned}$$

Since λ is a trace, this is equivalent to Eq. (14).

Note that from the manipulations above we also see that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(|D(b\alpha^n(a))|^2) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(t\tau^n(s))$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(|D(b)D(\alpha^n(a))|^2) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(E(t)\tau^n(E(s))$$

exist by Proposition 4.5, whether $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$ or not, where $s = a \otimes (JaJ)$ and $t = b \otimes (JbJ)$.

Now, suppose Eq. (14) holds, then we have by the equivalences above, that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\omega([b\alpha^n(a)]\otimes[JbJ\alpha'^n(JaJ)])$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\omega([D(b)D(\alpha^n(a))]\otimes[D'(JbJ)D'(\alpha'^n(JaJ))]),$$

i.e.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\omega((b\otimes(JbJ))\tau^n(a\otimes(JaJ)))$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\omega(E(b\otimes(JbJ))\tau^n(E(a\otimes(JaJ)))).$$

Because of the polarization identity, applied in turn to the two appearances of the sesquilinear form $A\times A\ni(a,c)\mapsto a\otimes(JcJ)$ above (once inside τ^n and once outside), Eq. (13) then follows, so $\mathbf{A}\odot_{\mathbf{F}}\mathbf{A}'$ is ergodic relative to $\mathbf{F}$ by Proposition 4.5. □

In order to proceed, we need the notion of relative ergodicity for a system itself:

Definition 4.7. We say that $\mathbf{A}$ *is ergodic relative to* $\mathbf{F}$ if $H^U\subset H_F$, where H^U is the fixed point space of $U:H\to H$, and $H_F=\overline{F\Omega}$.

This generalizes ergodicity of $\mathbf{A}$, which is the special case $H^U=\mathbb{C}\Omega$.

Remark 4.8. In [9, Definition 4.1] an alternative condition was used instead of $H^U\subset H_F$ to define relative ergodicity, namely

$$A^\alpha\subset F,$$

where $A^\alpha:=\{a\in A:\alpha(a)=a\}$. For our purposes here, Definition 4.7 is the more convenient definition, but the question nevertheless arises whether the two conditions are equivalent. From [9, Proposition 4.2] we know that $H^U=\overline{A^\alpha\Omega}$, so if $A^\alpha\subset F$, then $H^U\subset H_F$. This fact is used in Proposition 4.9.

We do not need the converse. However, it does hold, since $\mathbf{F}$ is a modular subsystem, as we now explain. The conditional expectation D is determined by

$$D(a)|_{H_F}=Pa|_{H_F}$$

for all $a\in A$; see for example [27, Section 10.2]. The subalgebra A^α is easily seen to be globally invariant under the modular group as well (see [9, Proposition 4.2]), hence we also have a unique conditional expectation $D_{A^\alpha}:A\to A^\alpha$ such that $\mu\circ D_{A^\alpha}=\mu$, which is similarly determined by

$$D_{A^\alpha}(a)|_{H^U}=Qa|_{H^U}$$

where Q is the projection of H onto H^U. Assuming $H^U \subset H_F$, it follows that

$$D(D_{A^\alpha}(a))|_{H^U} = PD_{A^\alpha}(a)|_{H^U} = Qa|_{H^U} = D_{A^\alpha}(a)|_{H^U}$$

and therefore $D(D_{A^\alpha}(a)) = D_{A^\alpha}(a)$, since $\Omega \in H^U$ is separating for A. So, for $a \in A^\alpha$, we have

$$a = D_{A^\alpha}(a) = D(D_{A^\alpha}(a)) \in F$$

which means that $A^\alpha \subset F$.

To summarize: $\mathbf{A}$ is ergodic relative to $\mathbf{F}$, if and only if $A^\alpha \subset F$.

The following generalizes the standard fact that weak mixing implies ergodicity:

Proposition 4.9. *If* $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$*, then* $\mathbf{A}$ *is ergodic relative to* $\mathbf{F}$.

Proof. From Proposition 3.3, we have $\lambda(|D(ba) - D(b)D(a)|^2) = 0$ for $a \in A^\alpha$ and all $b \in A$. Since λ is faithful, it follows that $D(b(a - D(a))) = D(ba) - D(b)D(a) = 0$. In particular, setting $b = (a - D(a))^*$, we conclude that $a = D(a) \in F$, since μ is faithful and $\lambda \circ D = \mu$. So $A^\alpha \subset F$, hence $H^U \subset H_F$ by the first part of Remark 4.8. ☐

Next we consider a version of Proposition 4.5 for a system itself.

Proposition 4.10. $\mathbf{A}$ *is ergodic relative to* $\mathbf{F}$ *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(b\alpha^n(a)) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(D(b)\alpha^n(D(a))) \tag{15}$$

for all $a, b \in A$. *Both limits exist, whether* $\mathbf{A}$ *is ergodic relative to* $\mathbf{F}$ *or not.*

Proof. Essentially the same argument, using the mean ergodic theorem, as in the proof of Proposition 4.5, but with Q now the projection of H onto H^U, and with R replaced by P. ☐

Using the last three results, we can now prove the remaining direction of Theorem 4.2:

Proposition 4.11. *Assume that* μ *is tracial and that* $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$. *Then* $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ *is ergodic relative to* $\mathbf{F}$.

Proof. Note that for all $a, b \in A$,

$$\begin{aligned}\lambda\left(|D(b\alpha^n(a)) - D(b)D(\alpha^n(a))|^2\right) &= \lambda(|D(b\alpha^n(a))|^2) \\ &\quad - \lambda(D(\alpha^n(a^*)b^*)D(b)D(\alpha^n(a))) \\ &\quad - \lambda(D(\alpha^n(a^*))D(b^*)D(b\alpha^n(a))) \\ &\quad + \lambda(D(\alpha^n(a^*))D(b^*)D(b)D(\alpha^n(a))).\end{aligned}$$

Consider the second term and use the trace property of μ:

$$\begin{aligned}\lambda(D(\alpha^n(a^*)b^*)D(b)D(\alpha^n(a))) &= \lambda(D(\alpha^n(a^*)b^*D(b)D(\alpha^n(a))))\\ &= \mu(\alpha^n(a^*)b^*D(b)D(\alpha^n(a)))\\ &= \mu(b^*D(b)\alpha^n(D(a)a^*)).\end{aligned}$$

Since **A** is ergodic relative to **F** by Proposition 4.9, we now have by Proposition 4.10 that

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(\alpha^n(a^*)b^*)D(b)D(\alpha^n(a)))\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(b^*)D(b)\alpha^n(D(a)D(a^*)))\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(|D(b)D(\alpha^n(a))|^2)\end{aligned}$$

Similarly

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(\alpha^n(a^*))D(b^*)D(b\alpha^n(a)))\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(|D(b)D(\alpha^n(a))|^2)\end{aligned}$$

and

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(\alpha^n(a^*))D(b^*)D(b)D(\alpha^n(a)))\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(D(b^*)D(b)\alpha^n(aD(a^*)))\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(|D(b)D(\alpha^n(a))|^2)\end{aligned}$$

Keep in mind that all these limits exist by Proposition 4.10. Then by Proposition 3.3,

$$\begin{aligned}0 &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda\left(|D\left(b\alpha^n(a)\right) - D(b)D(\alpha^n(a))|^2\right)\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}[\lambda(|D(b\alpha^n(a))|^2) - \lambda(|D(b)D(\alpha^n(a))|^2)],\end{aligned}$$

so

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(|D(b\alpha^n(a))|^2) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(|D(b)D(\alpha^n(a))|^2),$$

since both limits exist (see Lemma 4.6). By Lemma 4.6 we are done. □

This completes the proof of Theorem 4.2. To summarize: the one direction is given by Proposition 4.4, the other by Proposition 4.11.

To connect this to the structure theorem in [2], we mention the following: Suppose that we have an asymptotically abelian W*-dynamical system $\mathbf{A}$ with a tracial invariant state, as defined in [2, Definition 1.10]. According to [2, Theorem 1.14] (and Proposition 3.8), such a system is weakly mixing relative to the *central system* $\mathbf{C} := (A \cap A', \mu|_{A\cap A'}, \alpha|_{A\cap A'})$. Theorem 4.2 then shows that $\mathbf{A} \odot_{\mathbf{C}} \mathbf{A}'$ is ergodic relative to $\mathbf{C}$.

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