JENSEN-OSTROWSKI INEQUALITIES AND INTEGRATION SCHEMES VIA THE DARBOUX EXPANSION

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Abstract. Using Darboux's formula, which is a generalisation of Taylor's formula, we derive some Jensen-Ostrowski type inequalities. Applications for quadrature rules and f-divergence measures (specifically, for higher-order χ-divergence) are also given.

1. Introduction

In 1938, Ostrowski proved the following inequality [14]: Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x \in [a,b]$ and the constant $1/4$ is the best possible. In particular, when $x = (a+b)/2$, this inequality gives an error estimate to the midpoint rule: $\int_a^b f(t)\,dt \approx (b-a)f\left((a+b)/2\right)$.

The midpoint rule is the simplest form of quadrature rules. Derivative-based quadrature rules are of interest due to the larger number of parameters which increases the precision and order of accuracy (cf. Burg [2]). Wiersma [18] introduced a derivative-based quadrature rule that is similar to the Euler-Maclaurin formula.

In Wang and Guo [17], the Euler-Maclaurin formula, or simply Euler's formula, is derived from Darboux's formula.

Proposition 1 (Darboux's formula)**.** *Let $f(z)$ be an analytic function along the straight line from a point a to the point z, and let $\varphi(t)$ be an arbitrary polynomial of degree n. Then,*

$$\begin{aligned} &\varphi^{(n)}(0)\left[f(z) - f(a)\right] \\ &= \sum_{m=1}^{n} (-1)^{m-1}(z-a)^m \left[\varphi^{(n-m)}(1) f^{(m)}(z) - \varphi^{(n-m)}(0) f^{(m)}(a)\right] \\ &\quad + (-1)^n (z-a)^{n+1} \int_0^1 \varphi(t) f^{(n+1)}[(1-t)a + tz]\,dt. \end{aligned} \tag{1.2}$$

Taylor's formula is a special case, with $\varphi(t) = (t-1)^n$ [17].

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In [5], some inequalities are derived by utilising Taylor's formula (with integral remainder) :

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)\, dt.$$

These inequalities both generalise Ostrowski's and Jensen's inequalities for general integrals (and are referred to as Jensen-Ostrowski type inequalities). In particular, an Ostrowski type inequality in [5, p. 68] gives the following quadrature rule

$$\int_a^b f(t)\,dt \approx (b-a)f(\zeta) + \sum_{k=1}^{n} f^{(k)}(\zeta) \frac{(b-\zeta)^{k+1} - (a-\zeta)^{k+1}}{(k+1)!}, \tag{1.3}$$

for $\zeta \in [a,b]$ and the error estimate is given by

$$\|f^{(n+1)}\|_{[a,b],\infty} \frac{(\zeta-a)^{n+2} + (b-\zeta)^{n+2}}{(n+2)!}.$$

For further reading on this type of inequalities, we refer the readers to [3], [4], [5], [8], [9], and [10].

In this paper, we provide further, wider, and fuller treatment of our earlier work in [5] by considering Darboux's formula in place of Taylor's formula. The work also develops broader and more general application in areas such as derivative-based quadrature rules and divergence measures (specifically for the higher-order χ-divergence) as demonstrated in Sections 4 and 5, respectively.

2. Preliminaries

2.1. **Euler's formula.** This subsection serves as a reference point for the facts concerning Euler's formula. The explicit expression for the Bernoulli polynomial is

$$\varphi_n(x) = \sum_{k=0}^{n} \binom{n}{k} \varphi_k x^{n-k} \tag{2.1}$$

where

$$\varphi_0 = 1, \quad \text{and} \quad \sum_{k=0}^{n-1} \frac{1}{k!(n-k)!} \varphi_k = 0 \quad (n \geq 2).$$

The Bernoulli numbers are given by

$$\varphi_0 = 1, \quad \varphi_1 = -\frac{1}{2}, \quad \varphi_{2k} = (-1)^{k-1} B_k, \text{ and } \varphi_{2k+1} = 0 \quad (k \geq 2). \tag{2.2}$$

The first five Bernoulli numbers and polynomials are given in the following:

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66},$$

$$\varphi_0(x) = 1, \quad \varphi_1(x) = x - \frac{1}{2}, \quad \varphi_2(x) = x^2 - x + \frac{1}{6},$$

$$\varphi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \quad \varphi_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.$$

Choosing the Bernoulli polynomial $\varphi_n(t)$ in place of $\varphi(t)$ and replacing n with $2n$ and the polynomial φ_n with φ_{2n} in Darboux's formula (1.2) gives Euler's formula:

$$\begin{aligned} f(z)-f(a) &= \frac{z-a}{2}[f'(z)+f'(a)]+\sum_{k=1}^{n}(-1)^k\frac{(z-a)^{2k}}{(2k)!}B_k[f^{(2k)}(z)-f^{(2k)}(a)] \\ &\quad +\frac{(z-a)^{2n+1}}{(2n)!}\int_0^1 \varphi_{2n}(t)f^{(2n+1)}((1-t)a+tz)\,dt. \end{aligned} \tag{2.3}$$

2.2. **Identities.** Throughout the paper, let $(\Omega,\mathcal{A},\mu)$ be a measurable space with $\int_\Omega d\mu=1$, consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω, and a countably additive and positive measure μ on $\mathcal{A}$ with values in the set of extended real numbers. Throughout this subsection, let I be an interval in $\mathbb{R}$.

Lemma 1. *Let $f: I\to\mathbb{C}$ be such that $f^{(n)}$ is absolutely continuous on I and $a\in\mathring{I}$. Let $\varphi(t)$ be an arbitrary polynomial of degree exactly n. If $g:\Omega\to I$ is Lebesgue μ-measurable on Ω, $f\circ g$, $(g-a)^m$, $(g-a)^m(f^{(m)}\circ g)\in L(\Omega,\mu)$ for all $m\in\{1,...,n+1\}$, then we have*

$$\int_\Omega f\circ g\,d\mu-f(a)=P_{n,\varphi}(a,\lambda)+R_{n,\varphi}(a,\lambda), \tag{2.4}$$

for all $\lambda\in\mathbb{C}$, where $P_{n,\varphi}(a,\lambda)=P_{n,\varphi}(a,\lambda;f,g)$ is defined by

$$\begin{aligned} &P_{n,\varphi}(a,\lambda) \\ &= \frac{1}{\varphi^{(n)}(0)}\sum_{m=1}^{n}(-1)^{m-1}\left\{\varphi^{(n-m)}(1)\int_\Omega (g-a)^m(f^{(m)}\circ g)\,d\mu\right. \\ &\quad \left.-\varphi^{(n-m)}(0)f^{(m)}(a)\int_\Omega (g-a)^m\,d\mu\right\} \\ &\quad +\frac{(-1)^n\lambda}{\varphi^{(n)}(0)}\int_0^1\varphi(t)\,dt\int_\Omega (g-a)^{n+1}\,d\mu, \end{aligned} \tag{2.5}$$

and $R_{n,\varphi}(a,\lambda)=R_{n,\varphi}(a,\lambda;f,g)$ is defined by

$$\begin{aligned} &R_{n,\varphi}(a,\lambda)) \\ &= \frac{(-1)^n}{\varphi^{(n)}(0)}\int_\Omega (g-a)^{n+1}\left(\int_0^1\varphi(t)\left[f^{(n+1)}[(1-t)a+tg]-\lambda\right]dt\right)d\mu \\ &= \frac{(-1)^n}{\varphi^{(n)}(0)}\int_0^1\varphi(t)\int_\Omega (g-a)^{n+1}\left(\left[f^{(n+1)}[(1-t)a+tg]-\lambda\right]d\mu\right)dt. \end{aligned} \tag{2.6}$$

Proof. Since $f^{(n)}$ is absolutely continuous on I, $f^{(n+1)}$ exists almost everywhere on I and is Lebesgue integrable on I. By Proposition 1, we have

$$\begin{aligned} f(z)-f(a) &= \frac{1}{\varphi^{(n)}(0)}\sum_{m=1}^{n}(-1)^{m-1}(z-a)^m\{\varphi^{(n-m)}(1)f^{(m)}(z) \\ &\quad -\varphi^{(n-m)}(0)f^{(m)}(a)\}+\frac{\lambda(-1)^n(z-a)^{n+1}}{\varphi^{(n)}(0)}\int_0^1\varphi(t)\,dt \\ &\quad +\frac{(-1)^n(z-a)^{n+1}}{\varphi^{(n)}(0)}\int_0^1\varphi(t)\left[f^{(n+1)}[(1-t)a+tz]-\lambda\right]dt. \end{aligned}$$

By replacing z with $g(t)$ and integrating on Ω, we have

$$\begin{aligned}
&\int_\Omega f\circ g\,d\mu - f(a)\\
= &\frac{1}{\varphi^{(n)}(0)}\sum_{m=1}^{n}(-1)^{m-1}\left\{\varphi^{(n-m)}(1)\int_\Omega (g-a)^m (f^{(m)}\circ g)\,d\mu\right.\\
&\left.-\varphi^{(n-m)}(0)f^{(m)}(a)\int_\Omega (g-a)^m\,d\mu\right\} + \frac{(-1)^n\lambda}{\varphi^{(n)}(0)}\int_0^1\varphi(t)\,dt\int_\Omega (g-a)^{n+1}\,d\mu\\
&+\frac{(-1)^n}{\varphi^{(n)}(0)}\int_\Omega (g-a)^{n+1}\left(\int_0^1\varphi(t)\left[f^{(n+1)}[(1-t)a+tg]-\lambda\right]dt\right)d\mu.
\end{aligned}$$

The last equality in (2.6) follows by Fubini's theorem. □

Lemma 2. *Let $f : I \to \mathbb{C}$ be such that $f^{(2n)}$ is absolutely continuous on I and $a \in \mathring{I}$. Let $\varphi_{2n}(t)$ be the Bernoulli polynomials. If $g : \Omega \to I$ is Lebesgue μ-measurable on Ω, $f\circ g$, $(g-a)^m$, $(g-a)^m(f^{(m)}\circ g) \in L(\Omega,\mu)$ for all $m \in \{1,\ldots,2n+1\}$, then we have*

$$\int_\Omega f\circ g\,d\mu - f(a) = P_n(a,\lambda) + R_n(a,\lambda), \tag{2.7}$$

for all $\lambda\in\mathbb{C}$, where $P_n(a,\lambda) = P_n(a,\lambda; f, g)$ is defined by

$$\begin{aligned}
P_n(a,\lambda) = &\int_\Omega \frac{(g-a)}{2}\left[f'(a)+f'\circ g\right]d\mu\\
&+\int_\Omega\sum_{k=1}^{n}\frac{(-1)^k B_k (g-a)^{2k}}{(2k)!}\left[f^{(2k)}\circ g - f^{(2k)}(a)\right]d\mu\\
&+\lambda\int_0^1\varphi_{2n}(t)\,dt\int_\Omega\frac{(g-a)^{2n+1}}{(2n)!}\,d\mu
\end{aligned} \tag{2.8}$$

and $R_n(a,\lambda) = R_n(a,\lambda; f, g)$ is defined by

$$\begin{aligned}
&R_n(a,\lambda)\\
= &\int_\Omega\frac{(g-a)^{2n+1}}{(2n)!}\left[\int_0^1\varphi_{2n}(t)[f^{(2n+1)}((1-t)a+tg)-\lambda]\,dt\right]d\mu,\\
= &\int_0^1\varphi_{2n}(t)\int_\Omega\frac{(g-a)^{2n+1}}{(2n)!}\left[f^{(2n+1)}((1-t)a+tg)-\lambda\right]d\mu\,dt.
\end{aligned} \tag{2.9}$$

The proof follows by the Euler's formula (2.3) and similar arguments to those in the proof of Lemma 1. We omit the proof.

Remark 1. Recall that $B_1 = 1/6$ and $\varphi_2(t) = t^2 - t + 1/6$; and note that $\int_0^1\varphi_2(t)\,dt = 0$. Taking $n = 1$ in Lemma 2, we have

$$\begin{aligned}
&\int_\Omega f\circ g\,d\mu - f(a)\\
&=\int_\Omega\frac{(g-a)}{2}\left[f'(a)+f'\circ g\right]d\mu - \frac{1}{12}\int_\Omega (g-a)^2[f''\circ g - f''(a)]\,d\mu\\
&+\int_\Omega\frac{(g-a)^3}{2}\left[\int_0^1\left(t^2-t+\frac{1}{6}\right)[f^{(3)}((1-t)a+tg)-\lambda]\,dt\right]d\mu.
\end{aligned} \tag{2.10}$$

3. Main results: Jensen-Ostrowski inequalities

In this section, we derive some inequalities of Jensen-Ostrowski inequalities using the lemmas obtain in Subsection 2.2. We use the notation

$$\|k\|_{\Omega,p} := \begin{cases} \left(\int_\Omega |k(t)|^p \, d\mu(t)\right)^{1/p}, & p \geq 1, \ k \in L_p(\Omega,\mu); \\ \operatorname*{ess\,sup}_{t\in\Omega} |k(t)|, & p = \infty, \ k \in L_\infty(\Omega,\mu); \end{cases}$$

and

$$\|f\|_{[0,1],p} := \begin{cases} \left(\int_0^1 |f(s)|^p \, ds\right)^{1/p}, & p \geq 1, \ f \in L_p([0,1]); \\ \operatorname*{ess\,sup}_{s\in[0,1]} |f(s)|, & p = \infty, \ f \in L_\infty([0,1]). \end{cases}$$

We also denote by ℓ, the identity function on $[0,1]$, namely $\ell(t) = t$, for $t \in [0,1]$.

Throughout this section, let I be an interval in $\mathbb{R}$. We note that I is not necessarily a finite interval and therefore we make the following assumptions for functions f and g for a fixed $n \in \mathbb{N}$:

(A1) Let $f : I \to \mathbb{C}$ be such that $f^{(n)}$ is locally absolutely continuous on I, i.e. it is locally absolutely continuous on each closed subinterval $[a,b]$ on I, and $a \in \mathring{I}$.
(A2) Let $g : \Omega \to I$ be Lebesgue μ-measurable on Ω and $f \circ g$, $(g-a)^m$, $(g-a)^m \left(f^{(m)} \circ g\right) \in L(\Omega,\mu)$ for all $m \in \{1,...,n+1\}$.
(A3) We assume that $\left\|f^{(n+1)}\left[(1-\ell)\,a + \ell g\right] - \lambda\right\|_{[0,1],\infty} < \infty$ for all $t \in \Omega$ and $\lambda \in \mathbb{C}$.

Furthermore, the following cases are considered for a given $n \in \mathbb{N}$:

(C1) $\left\| |g-a|^{n+1} \right\|_{\Omega,\infty} < \infty$ and $\left\| \left\|f^{(n+1)}\left[(1-\ell)\,a + \ell g\right] - \lambda\right\|_{[0,1],\infty} \right\|_{\Omega,1} < \infty$;
(C2) $\left\| |g-a|^{n+1} \right\|_{\Omega,p} < \infty$ and $\left\| \left\|f^{(n+1)}\left[(1-\ell)\,a + \ell g\right] - \lambda\right\|_{[0,1],\infty} \right\|_{\Omega,q} < \infty$, where $p > 1$ with $1/p + 1/q = 1$;
(C3) $\left\| |g-a|^{n+1} \right\|_{\Omega,1} < \infty$ and $\left\| \left\|f^{(n+1)}\left[(1-\ell)\,a + \ell g\right] - \lambda\right\|_{[0,1],\infty} \right\|_{\Omega,\infty} < \infty$.

Theorem 1. *Let f and g be functions that satisfy (A1)-(A3) and $\varphi(t)$ be an arbitrary polynomial of degree n. Then,*

$$\begin{aligned} &\left| \int_\Omega f \circ g \, d\mu - f(a) - P_{n,\varphi}(a,\lambda) \right| \\ &\leq \int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|} dt \left(\int_\Omega |g-a|^{n+1} \|f_{n+1,g}(a,\lambda)\|_{[0,1],\infty} \, d\mu \right) \\ &\leq \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|} dt \right) \begin{cases} \left\| |g-a|^{n+1} \right\|_{\Omega,\infty} \left\| \|f_{n+1,g}(a,\lambda)\|_{[0,1],\infty} \right\|_{\Omega,1}, & \text{if (C1) holds;} \\ \left\| |g-a|^{n+1} \right\|_{\Omega,p} \left\| \|f_{n+1,g}(a,\lambda)\|_{[0,1],\infty} \right\|_{\Omega,q}, & \text{if (C2) holds;} \\ \left\| |g-a|^{n+1} \right\|_{\Omega,1} \left\| \|f_{n+1,g}(a,\lambda)\|_{[0,1],\infty} \right\|_{\Omega,\infty}, & \text{if (C3) holds;} \end{cases} \end{aligned} \tag{3.1}$$

for any $\lambda \in \mathbb{C}$, where $f_{n+1,g}(a,\lambda) = f^{(n+1)}\left[(1-\ell)\,a + \ell g\right] - \lambda$. Here, $P_{n,\varphi}(a,\lambda)$ is as defined in (2.5).

Proof. Taking the modulus in (2.4) for any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}&\left|\int_\Omega f\circ g\,d\mu - f(a) - P_{n,\varphi}(a,\lambda)\right|\\ &\le \int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\left(\int_\Omega |g-a|^{n+1}\left|f^{(n+1)}[(1-t)a+tg]-\lambda\right| d\mu\right) dt\\ &\le \int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\left(\int_\Omega |g-a|^{n+1}\left\|f^{(n+1)}[(1-\ell)a+\ell g]-\lambda\right\|_{[0,1],\infty} d\mu\right).\end{aligned}$$

We obtain the desired result by applying Hölder's inequality. □

Corollary 1. *Under the assumptions of Theorem 1, if* $\|f^{(n+1)}\|_{I,\infty} < \infty$*, then*

$$\begin{aligned}&\left|\int_\Omega f\circ g\,d\mu - f(a) - P_{n,\varphi}(a,0)\right|\\ &\le \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\right)\|f^{(n+1)}\|_{I,\infty}\left(\int_\Omega |g-a|^{n+1}\,d\mu\right).\end{aligned} \tag{3.2}$$

Here, $P_{n,\varphi}(a,\lambda)$ *is as defined in* (2.5).

Proof. Let $\lambda = 0$ in (2.4), and take the modulus to obtain

$$\begin{aligned}&\left|\int_\Omega f\circ g\,d\mu - f(a) - P_{n,\varphi}(a,0)\right|\\ &\le \int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\left(\int_\Omega |g-a|^{n+1}\left\|f^{(n+1)}[(1-\ell)a+\ell g]\right\|_{[0,1],\infty} d\mu\right).\end{aligned} \tag{3.3}$$

For any $t \in \Omega$ and almost every $s \in [0,1]$, we have

$$|f^{(n+1)}((1-s)a+sg(t))| \le \operatorname*{ess\,sup}_{u\in I} |f^{(n+1)}(u)| = \|f^{(n+1)}\|_{I,\infty}.$$

Therefore, we have

$$\begin{aligned}\left\|f^{(n+1)}((1-\ell)a+\ell g)\right\|_{[0,1],\infty} &\le \operatorname*{ess\,sup}_{s\in[0,1],\ t\in\Omega} \|f^{(n+1)}((1-s)a+sg(t))\|\\ &\le \|f^{(n+1)}\|_{I,\infty}.\end{aligned} \tag{3.4}$$

The desired inequality follows from (3.3) and (3.4). □

Utilising (2.3) and applying similar arguments to those in Theorem 1 and Corollary 1, we have the following results. We omit the proofs.

Theorem 2. *Let f and g be functions that satisfy (A1)-(A3) for* $2n$ *instead of* n*, and* $\varphi_{2n}(t)$ *be the Bernoulli polynomials. Then,*

$$\begin{aligned}&\left|\int_\Omega f\circ g\,d\mu - f(a) - P_{2n}(a,\lambda)\right|\\ &\le \int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\,dt\int_\Omega |g-a|^{2n+1}\|f_{2n+1,g}(a,\lambda)\|_{[0,1],\infty}\,d\mu\end{aligned} \tag{3.5}$$

$$\leq \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\,dt\right)\begin{cases} \left\||g-a|^{2n+1}\right\|_{\Omega,\infty}\left\|\|f_{2n+1,g}(a,\lambda)\|_{[0,1],\infty}\right\|_{\Omega,1}, \\ \qquad\qquad \text{if (C1) holds for } 2n, \\ \left\||g-a|^{2n+1}\right\|_{\Omega,p}\left\|\|f_{2n+1,g}(a,\lambda)\|_{[0,1],\infty}\right\|_{\Omega,q}, \\ \qquad\qquad \text{if (C2) holds for } 2n, \\ \left\||g-a|^{2n+1}\right\|_{\Omega,1}\left\|\|f_{2n+1,g}(a,\lambda)\|_{[0,1],\infty}\right\|_{\Omega,\infty}, \\ \qquad\qquad \text{if (C3) holds for } 2n, \end{cases}$$

for any $\lambda \in \mathbb{C}$, where $f_{2n+1,g}(a,\lambda) = f^{(2n+1)}\left((1-\ell)\,a + \ell g\right) - \lambda$. Here, $P_n(a,\lambda)$ is as defined in (2.8).

Corollary 2. *Under the assumptions of Theorem 2, if $\|f^{(2n+1)}\|_{I,\infty} < \infty$, then*

$$\left|\int_\Omega f\circ g\,d\mu - f(a) - P_{2n}(a,0)\right| \leq \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\,dt\right)\|f^{(2n+1)}\|_{I,\infty}\left(\int_\Omega |g-a|^{2n+1}\,d\mu\right). \tag{3.6}$$

Here, $P_n(a,\lambda)$ is as defined in (2.8).

Remark 2. Setting $n = 1$ in Corollary 2, we have

$$\left|\int_\Omega f\circ g\,d\mu - f(a) - \int_\Omega \frac{(g-a)}{2}\left[f'(a) + f'\circ g\right]d\mu + \frac{1}{12}\int_\Omega (g-a)^2[f''\circ g - f''(a)]\,d\mu\right| \leq \frac{\|f'''\|_{I,\infty}}{18\sqrt{3}}\int_\Omega |g-a|^3\,d\mu. \tag{3.7}$$

The following terminology introduced in [8] will be required for alternate Jensen-Ostrowski inequality results. For $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions [8]

$$U_{[a,b]}(\gamma,\Gamma) := \left\{h : [a,b] \to \mathbb{C} \middle| \mathrm{Re}\left[(\Gamma - h(t))(\overline{h(t)} - \overline{\gamma})\right] \geq 0 \text{ for a.e. } t \in [a,b]\right\}$$

and

$$\Delta_{[a,b]}(\gamma,\Gamma) := \left\{h : [a,b] \to \mathbb{C} \middle| \left|h(t) - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma| \text{ for a.e. } t \in [a,b]\right\}.$$

We recall some results in [8] concerning the above sets.

Proposition 2. *For any $\gamma, \Gamma \in \mathbb{C}$ and $\gamma \neq \Gamma$, we have*

(*i*) $U_{[a,b]}(\gamma,\Gamma) = \Delta_{[a,b]}(\gamma,\Gamma)$*; and*

(*ii*) $U_{[a,b]}(\gamma,\Gamma) = \big\{h : [a,b] \to \mathbb{C} \mid \big(\mathrm{Re}(\Gamma) - \mathrm{Re}(h(t))\big)\big(\mathrm{Re}(h(t)) - \mathrm{Re}(\gamma)\big) + \big(\mathrm{Im}(\Gamma) - \mathrm{Im}(h(t))\big)\big(\mathrm{Im}(h(t)) - \mathrm{Im}(\gamma)\big) \geq 0 \text{ for a.e. } t \in [a,b]\big\}.$

We refer to [8] for the proofs of these results. In a nutshell, they are consequences of the identity:

$$\frac{1}{4}|\Gamma - \gamma|^2 - \left|z - \frac{\gamma+\Gamma}{2}\right|^2 = \mathrm{Re}[(\Gamma - z)(\bar{z} - \bar{\gamma})], \quad \text{for all } z \in \mathbb{C}.$$

We have the following Jensen-Ostrowski inequality for functions with bounded higher $(n+1)$-th derivatives:

Theorem 3. *Let f and g be functions that satisfy (A1) and (A2) and $\varphi(t)$ be an arbitrary polynomial of degree n. For some $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, assume that $f^{(n+1)} \in U_{[a,b]}(\gamma,\Gamma) = \Delta_{[a,b]}(\gamma,\Gamma)$. Then,*

$$\begin{aligned}\left|\int_\Omega f \circ g \, d\mu - f(a) - P_{n,\varphi}\left(a, \frac{\gamma+\Gamma}{2}\right)\right| \\ \leq \frac{|\Gamma-\gamma|}{2} \int_\Omega |g-a|^{n+1} \, d\mu \int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|} \, dt.\end{aligned} \tag{3.8}$$

Here, $P_{n,\varphi}(a,\lambda)$ is as defined in (2.5).

Proof. Let $\lambda = (\gamma+\Gamma)/2$ in (2.4), we have

$$\begin{aligned}&\int_\Omega f \circ g \, d\mu - f(a) - P_{n,\varphi}\left(a, \frac{\gamma+\Gamma}{2}\right) \\ &= \frac{(-1)^n}{\varphi^{(n)}(0)} \int_\Omega (g-a)^{n+1} \left(\int_0^1 \varphi(t) \left[f^{(n+1)}[(1-t)a+tg] - \frac{\gamma+\Gamma}{2}\right] dt\right) d\mu.\end{aligned}$$

Since $f^{(n+1)} \in \Delta_{[a,b]}(\gamma,\Gamma)$, we have

$$\left|f^{(n+1)}\left((1-t)\,a+tg\right) - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma-\gamma|, \tag{3.9}$$

for almost every $t \in [0,1]$ and any $s \in \Omega$. Multiply (3.9) with $|\varphi(t)| > 0$ and integrate over $[0,1]$, we obtain

$$\int_0^1 |\varphi(t)| \left|f^{(n+1)}\left((1-t)\,a+tg\right) - \frac{\gamma+\Gamma}{2}\right| dt \leq \frac{1}{2}|\Gamma-\gamma| \int_0^1 |\varphi(t)| \, dt,$$

for any $s \in \Omega$. Now, we have

$$\begin{aligned}&\left|\int_\Omega f \circ g \, d\mu - f(a) - P_{n,\varphi}\left(a, \frac{\gamma+\Gamma}{2}\right)\right| \\ &\leq \int_\Omega |g-a|^{n+1} \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|} \left|f^{(n+1)}[(1-t)a+tg] - \frac{\gamma+\Gamma}{2}\right| dt\right) d\mu \\ &\leq \frac{|\Gamma-\gamma|}{2} \int_\Omega |g-a|^{n+1} \, d\mu \int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|} \, dt.\end{aligned}$$

This completes the proof. □

Similarly, we have the following via Euler's formula (2.3) and Lemma 2. We omit the proof.

Theorem 4. *Let f and g be functions that satisfy (A1) and (A2) for $2n$ instead of n and $\varphi_{2n}(t)$ be the Bernoulli polynomials. For some $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, assume that $f^{(2n+1)} \in U_{[a,b]}(\gamma,\Gamma) = \Delta_{[a,b]}(\gamma,\Gamma)$. Then,*

$$\begin{aligned}\left|\int_\Omega f \circ g \, d\mu - f(a) - P_{2n}\left(a, \frac{\gamma+\Gamma}{2}\right)\right| \\ \leq \frac{|\Gamma-\gamma|}{2(2n)!} \int_\Omega |g-a|^{2n+1} \, d\mu \int_0^1 |\varphi_{2n}(t)| \, dt.\end{aligned} \tag{3.10}$$

Here, $P_n(a,\lambda)$ is as defined in (2.8).

4. Applications: quadrature rules

In this section, we present quadrature rules based on the inequalities presented in Section 3. The associated composite rules may be stated in the usual manner by breaking up the interval $[a, b]$ into a number of subintervals, applying the quadrature rule for each subinterval, then adding up all the results. The precise statements for these composite rules are omitted.

Let $g : [a, b] \to [a, b]$ defined by $g(t) = t$ and $\mu(t) = t/(b-a)$ in Corollary 1. We have the following quadrature rule:

$$\begin{aligned}
\int_a^b f(t)\,dt &\approx (b-a)f(x) + \sum_{m=1}^{n} (-1)^{m-1} \left\{ \frac{\varphi^{(n-m)}(1)}{\varphi^{(n)}(0)} \int_a^b (t-x)^m f^{(m)}(t)\,dt \right. \\
&\quad \left. - \frac{\varphi^{(n-m)}(0)}{\varphi^{(n)}(0)} f^{(m)}(x) \int_a^b (t-x)^m\,dt \right\} \\
&= (b-a)f(x) + \sum_{m=1}^{n} (-1)^{m-1} \left\{ \frac{\varphi^{(n-m)}(1)}{\varphi^{(n)}(0)} \int_a^b (t-x)^m f^{(m)}(t)\,dt \right. \\
&\quad \left. - \frac{\varphi^{(n-m)}(0)}{\varphi^{(n)}(0)} f^{(m)}(x) \left(\frac{(b-x)^{m+1} - (a-x)^{m+1}}{m+1} \right) \right\},
\end{aligned}$$

(note that we also replace a in Corollary 1 by x) with the following error estimate:

$$\begin{aligned}
&\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt \left(\int_a^b |t-x|^{n+1}\,dt \right) \|f^{(n+1)}\|_{[a,b],\infty} \\
&= \int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt \left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{n+2} \right) \|f^{(n+1)}\|_{[a,b],\infty},
\end{aligned}$$

for $x \in [a, b]$.

Similarly, Corollary 2 gives us

$$\begin{aligned}
&\left| \frac{3}{2} \int_a^b f(t)\,dt - (b-a)f(x) - \frac{1}{2}\left[(b-x)f(b) - (a-x)f(a)\right] \right. \\
&- \frac{f'(x)}{4}\left[(b-x)^2 - (a-x)^2\right] \\
&\left. - \int_a^b \sum_{k=1}^{n} \frac{(-1)^k B_k (t-x)^{2k}}{(2k)!} \left[f^{(2k)}(t) - f^{(2k)}(x)\right] dt \right| \\
&\le \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\,dt \right) \|f^{(2n+1)}\|_{[a,b],\infty} \frac{(x-a)^{2n+2} + (b-x)^{2n+2}}{2n+2},
\end{aligned}$$

for all $x \in [a, b]$, thus we have the following quadrature rule:

$$\begin{aligned}
\int_a^b f(t)\,dt &\approx \frac{2}{3}(b-a)f(x) + \frac{1}{3}\left[(b-x)f(b) - (a-x)f(a)\right] \\
&\quad + \frac{f'(x)}{6}\left[(b-x)^2 - (a-x)^2\right] \\
&\quad + \frac{2}{3} \int_a^b \sum_{k=1}^{n} \frac{(-1)^k B_k (t-x)^{2k}}{(2k)!} \left[f^{(2k)}(t) - f^{(2k)}(x)\right] dt
\end{aligned}$$

for $x \in [a,b]$, with the following error estimate:

$$\frac{2}{3}\left(\int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\,dt\right)\|f^{(2n+1)}\|_{[a,b],\infty}\frac{(x-a)^{2n+2}+(b-x)^{2n+2}}{2n+2}$$

When $n=1$, we have

$$\begin{aligned}(4.1)\quad &\left|\frac{5}{3}\int_a^b f(t)\,dt-(b-a)f(x)-\frac{2}{3}\left[(b-x)f(b)-(a-x)f(a)\right]\right.\\ &-\frac{f'(x)}{4}\left[(b-x)^2-(a-x)^2\right]+\frac{1}{12}[(b-x)^2f'(b)-(a-x)^2f'(a)]\\ &\left.-\frac{f''(x)}{36}[(b-x)^3-(a-x)^3]\right|\le\frac{1}{72\sqrt{3}}\|f'''\|_{[a,b],\infty}[(x-a)^4+(b-x)^4],\end{aligned}$$

for $x \in [a,b]$, thus we have the following quadrature rule:

$$\begin{aligned}\int_a^b f(t)\,dt &\approx \frac{3f(x)}{5}(b-a)+\frac{2}{5}\left[(b-x)f(b)-(a-x)f(a)\right]\\ &-\frac{1}{20}[(b-x)^2f'(b)-(a-x)^2f'(a)]\\ &+\frac{3f'(x)}{20}\left[(b-x)^2-(a-x)^2\right]+\frac{f''(x)}{60}[(b-x)^3-(a-x)^3],\end{aligned}$$

for $x \in [a,b]$, with the following error estimate:

$$\frac{1}{120\sqrt{3}}\|f'''\|_{[a,b],\infty}[(x-a)^4+(b-x)^4].$$

5. Applications for f-Divergence

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be

$$\mathcal{P}:=\left\{p|p:\Omega\to\mathbb{R},\ p(t)\ge 0,\ \int_\Omega p(t)\,d\mu(t)=1\right\}.$$

We recall the definition of some divergence measures which we use in this text.

Definition 1. Let $p,q\in\mathcal{P}$ and $k\ge 2$.

1. The Kullback-Leibler divergence [12]:

$$D_{KL}(p,q):=\int_\Omega p(t)\log\left[\frac{p(t)}{q(t)}\right]d\mu(t),\quad p,q\in\mathcal{P}. \tag{5.1}$$

2. The χ^2-divergence:

$$D_{\chi^2}(p,q):=\int_\Omega p(t)\left[\left(\frac{q(t)}{p(t)}\right)^2-1\right]d\mu(t),\quad p,q\in\mathcal{P}. \tag{5.2}$$

3. Higher order χ-divergence [1]:

$$D_{\chi^k}(p,q):=\int_\Omega\frac{(q(t)-p(t))^k}{p^{k-1}(t)}d\mu(t)=\int_\Omega\left(\frac{q(t)}{p(t)}-1\right)^k p(t)\,d\mu(t), \tag{5.3}$$

$$D_{|\chi|^k}(p,q):=\int_\Omega\frac{|q(t)-p(t)|^k}{p^{k-1}(t)}d\mu(t)=\int_\Omega\left|\frac{q(t)}{p(t)}-1\right|^k p(t)\,d\mu(t). \tag{5.4}$$

Furthermore, (5.3) and (5.4) can be generalised as follows [13]:

$$D_{\chi^k,a}(p,q) := \int_\Omega \frac{(q(t)-ap(t))^k}{p^{k-1}(t)} d\mu(t) = \int_\Omega \left(\frac{q(t)}{p(t)} - a\right)^k p(t)\, d\mu(t), \tag{5.5}$$

$$D_{|\chi|^k,a}(p,q) := \int_\Omega \frac{|q(t)-ap(t)|^k}{p^{k-1}(t)} d\mu(t) = \int_\Omega \left|\frac{q(t)}{p(t)} - a\right|^k p(t)\, d\mu(t). \tag{5.6}$$

4. Csiszár f-divergence [6]:

$$I_f(p,q) := \int_\Omega p(t) f\left[\frac{q(t)}{p(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.7}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$.

Remark 3. (1) We note that when $k=2$, (5.3) coincides with (5.2).
(2) The Kullback-Leibler divergence and the χ^2-divergence are particular instances of Csiszár f-divergence. For the basic properties of Csiszár f-divergence, we refer the readers to [6], [7], and [16].

Example 1. (i) Let $f:(0,\infty)\to\mathbb{R}$ be defined by $f(t)=t\log(t)$. We have

$$I_f(p,q) = \int_\Omega p(t)\frac{q(t)}{p(t)} \log\left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_\Omega q(t)\log\left[\frac{q(t)}{p(t)}\right] d\mu(t) = D_{KL}(q,p).$$

(ii) Let $g:(0,\infty)\to\mathbb{R}$ be defined by $g(t)=-\log(t)$. We have

$$I_g(p,q) = -\int_\Omega p(t)\log\left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_\Omega p(t)\log\left[\frac{p(t)}{q(t)}\right] d\mu(t) = D_{KL}(p,q).$$

We obtain the next three results by choosing $g(t)=q(t)/p(t)$ in Corollary 1, Corollary 2, and (3.7). We also note that $\int_\Omega p(t)d\mu = 1$. The proofs are straightforward and therefore we omit the details.

Proposition 3. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function with the property that $f(1)=0$. Let $\varphi(t)$ be an arbitrary polynomial of degree n. Assume that $p,q\in\mathcal{P}$ and there exists constants $0<r<1<R<\infty$ such that*

$$r \le \frac{q(t)}{p(t)} \le R, \quad \text{for } \mu\text{-a.e. } t\in\Omega. \tag{5.8}$$

If $a\in[r,R]$ and $f^{(n)}$ is absolutely continuous on $[r,R]$, then we have the inequalities

$$\Bigg| I_f(p,q) - f(a) + \frac{1}{\varphi^{(n)}(0)} \sum_{m=1}^{n} (-1)^{m-1} \Big\{ \varphi^{(n-m)}(0) f^{(m)}(a) D_{\chi^m,a}(p,q)$$
$$-\varphi^{(n-m)}(1)\int_\Omega \frac{(q(t)-ap(t))^m}{p^{m-1}(t)} f^{(m)}\left(\frac{q(t)}{p(t)}\right) d\mu \Big\} \Bigg|$$
$$\le \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\, dt\right) \|f^{(n+1)}\|_{[r,R],\infty} D_{|\chi|^{n+1},a}(p,q).$$

Proposition 4. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function with the property that $f(1)=0$. Let $\varphi_{2n}(t)$ be the Bernoulli polynomials. Assume that $p,q\in\mathcal{P}$ and there exists constants $0<r<1<R<\infty$ such that*

$$r \le \frac{q(t)}{p(t)} \le R, \quad \text{for } \mu\text{-a.e. } t\in\Omega. \tag{5.9}$$

If $a \in [r,R]$ *and* $f^{(2n)}$ *is absolutely continuous on* $[r,R]$*, then we have the inequalities*

$$\begin{aligned}
&\Bigg| I_f(p,q) - f(a) - \frac{f'(a)}{2}(1-a) - \frac{1}{2}\int_\Omega [q(t)-ap(t)] f'\left(\frac{q(t)}{p(t)}\right) d\mu \\
&- \sum_{k=1}^{n} \frac{(-1)^k B_k}{(2k)!}\left[\int_\Omega \frac{(q(t)-ap(t))^{2k}}{p^{2k-1}(t)} f^{(2k)}\left(\frac{q(t)}{p(t)}\right) d\mu - f^{(2k)}(a) D_{\chi^{2k},a}(p,q)\right]\Bigg| \\
&\le \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\,dt\right)\|f^{(2n+1)}\|_{[r,R],\infty} D_{|\chi|^{2n+1},a}(p,q),
\end{aligned}$$

Corollary 3. *Let* $f : (0,\infty) \to \mathbb{R}$ *be a convex function with the property that* $f(1) = 0$*. Assume that* $p, q \in \mathcal{P}$ *and there exist constants* $0 < r < 1 < R < \infty$ *such that* $r \le \frac{q(t)}{p(t)} \le R$*, for* μ*-a.e.* $t \in \Omega$*. If* $a \in [r,R]$ *and* f'' *is absolutely continuous on* $[r,R]$*, then we have the inequalities*

$$\begin{aligned}
&\Bigg| I_f(p,q) - f(a) - \frac{f'(a)}{2}(1-a) - \frac{1}{2}\int_\Omega [q(t)-ap(t)] f'\left(\frac{q(t)}{p(t)}\right) d\mu \\
&+ \frac{1}{12}\int_\Omega \frac{(q(t)-ap(t))^2}{p(t)} f''\left(\frac{q(t)}{p(t)}\right) d\mu - \frac{f''(a)}{12} D_{\chi^2,a}(p,q)\Bigg| \\
&\le \frac{1}{18\sqrt{3}}\|f'''\|_{[r,R],\infty} D_{|\chi|^3,a}(p,q).
\end{aligned}$$

Example 2. We consider the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t\log(t)$. We have

$$f'(t) = \log(t) + 1 \quad \text{and} \quad f^{(k)}(t) = (-1)^k t^{-(k-1)}(k-2)!, \qquad \text{for } k \ge 2.$$

Thus, $\|f^{(k)}\|_{[r,R]} = r^{-(k-1)}(k-2)!$. Recall from Example 1 Part (i) that $I_f(p,q) = D_{KL}(q,p)$. We also have

$$\begin{aligned}
&\int_\Omega \frac{(q(t)-ap(t))^m}{p^{m-1}(t)} f^{(m)}\left(\frac{q(t)}{p(t)}\right) d\mu \\
=\ & (-1)^m (m-2)! \int_\Omega \frac{(q(t)-ap(t))^m}{p^{m-1}(t)}\left(\frac{p(t)}{q(t)}\right)^{m-1} d\mu \\
=\ & (-1)^m (-a)^m (m-2)! \int_\Omega \frac{(p(t)-\frac{1}{a}q(t))^m}{q^{m-1}(t)}\, d\mu \\
=\ & a^m (m-2)! D_{\chi^m,\frac{1}{a}}(q,p).
\end{aligned}$$

Therefore, Proposition 3 gives us:

$$\begin{aligned}
&\Bigg| D_{KL}(q,p) - a\log(a) - \frac{1}{\varphi^{(n)}(0)}\sum_{m=1}^{n}(m-2)!\left\{\frac{\varphi^{(n-m)}(0)}{a^{m-1}} D_{\chi^m,a}(p,q)\right. \\
&\left. + (-1)^{m-1} a^m \varphi^{(n-m)}(1) D_{\chi^m,\frac{1}{a}}(q,p)\right\}\Bigg| \qquad (5.10) \\
&\le \frac{(n-1)!}{r^n}\left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\right) D_{|\chi|^{n+1},a}(p,q).
\end{aligned}$$

In particular, when $a = 1$, we have

$$\begin{aligned}(5.11)\qquad &\Bigg|D_{KL}(q,p) - \sum_{m=1}^{n}(m-2)!\left\{\frac{\varphi^{(n-m)}(0)}{\varphi^{(n)}(0)}D_{\chi^m}(p,q)\right.\\ &\left.+(-1)^{m-1}\frac{\varphi^{(n-m)}(1)}{\varphi^{(n)}(0)}D_{\chi^m}(q,p)\right\}\Bigg|\\ &\leq \frac{(n-1)!}{r^n}\left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\right)D_{|\chi|^{n+1}}(p,q).\end{aligned}$$

We also have

$$\begin{aligned}\int_\Omega q(t)f'\left(\frac{q(t)}{p(t)}\right)d\mu(t) &= \int_\Omega \left(q(t)\log\left(\frac{q(t)}{p(t)}\right)+q(t)\right)d\mu(t)\\ &= \int_\Omega q(t)\log\left(\frac{q(t)}{p(t)}\right)d\mu(t)+1 = D_{KL}(q,p)+1,\end{aligned}$$

and

$$\begin{aligned}\int_\Omega p(t)f'\left(\frac{q(t)}{p(t)}\right)d\mu &= \int_\Omega p(t)\left[\log\left[\frac{q(t)}{p(t)}\right]+1\right]d\mu(t)\\ &= -\int_\Omega p(t)\log\left[\frac{p(t)}{q(t)}\right]d\mu(t)+1 = -D_{KL}(p,q)+1.\end{aligned}$$

Therefore, Proposition 4 gives us

$$\begin{aligned}&\Bigg|D_{KL}(q,p) - a\log(a) - \frac{\log(a)+1}{2}(1-a) - \frac{1}{2}D_{KL}(q,p) - \frac{1}{2} - \frac{aD_{KL}(p,q)}{2}\\ &+\frac{a}{2} - \sum_{k=1}^{n}\frac{(-1)^k B_k}{(2k)!}(2k-2)!\left[\int_\Omega \frac{(q(t)-ap(t))^{2k}}{q^{2k-1}(t)}\,d\mu(t) - \frac{D_{\chi^{2k},a}(p,q)}{a^{2k-1}}\right]\Bigg|\\ &= \Bigg|D_{KL}(q,p) - \frac{1}{2}\log(a)(a+1) + (a-1) - \frac{1}{2}(D_{KL}(q,p)+aD_{KL}(p,q))\\ &-\sum_{k=1}^{n}\frac{(-1)^k B_k}{4k^2-2k}\left[a^{2k}D_{\chi^{2k},\frac{1}{a}}(q,p) - \frac{D_{\chi^{2k},a}(p,q)}{a^{2k-1}}\right]\Bigg|\\ &\leq \frac{(2n-1)!}{r^{2n}}\left(\int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\,dt\right)D_{|\chi|^{2n+1},a}(p,q)\\ &= \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{2n}\,dt\right)\frac{D_{|\chi|^{2n+1},a}(p,q)}{r^{2n}}.\end{aligned}$$

In particular, when $a = 1$, we have

$$\begin{aligned}(5.12)\qquad &\left|D_{KL}(q,p) - D_{KL}(p,q) - \sum_{k=1}^{n}\frac{(-1)^k B_k}{2k^2-k}\left[D_{\chi^{2k}}(q,p) - D_{\chi^{2k}}(p,q)\right]\right|\\ &\leq \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{n}\,dt\right)\frac{D_{|\chi|^{2n+1}}(p,q)}{r^{2n}}.\end{aligned}$$

We note that

$$\begin{aligned}\int_\Omega \frac{(q(t)-ap(t))^2}{p(t)} f''\left(\frac{q(t)}{p(t)}\right) d\mu(t) &= \int_\Omega \frac{(q(t)-ap(t))^2}{q(t)}\, d\mu(t) \\ &= 1-2a+a^2\int_\Omega \frac{p(t)^2}{q(t)}\, d\mu(t) \\ &= 1-2a+a^2(D_{\chi^2}(q,p)+1) \\ &= a^2 D_{\chi^2}(q,p)+(1-a)^2.\end{aligned}$$

Note the use of (5.16). Thus, Corollary 3 gives us

$$\Bigg| D_{KL}(q,p) - \frac{1}{2}\log(a)(a+1) + (a-1) - \frac{1}{2}(D_{KL}(q,p) + aD_{KL}(p,q)) + \frac{1}{12}\left[a^2 D_{\chi^2}(q,p) + (1-a)^2 - \frac{1}{a} D_{\chi^2,a}(p,q)\right]\Bigg| \le \frac{D_{|\chi|^3,a}(p,q)}{18\sqrt{3}r^2}.$$

In particular, when $a = 1$, we have

$$\left| D_{KL}(q,p) - D_{KL}(p,q) + \frac{1}{6}\left[D_{\chi^2}(q,p) - D_{\chi^2}(p,q)\right]\right| \le \frac{D_{|\chi|^3}(p,q)}{9\sqrt{3}r^2}. \tag{5.13}$$

Example 3. We consider the convex function $g : (0,\infty) \to \mathbb{R}$, $g(t) = -\log(t)$. We have

$$g^{(k)}(t) = (-1)^k t^{-k}(k-1)!, \quad \text{for } k \ge 1.$$

Thus, $\|g^{(k)}\|_{[r,R]} = r^{-k}$. From Example 1 Part (ii) , we have $I_g(p,q) = D_{KL}(p,q)$. Proposition 3 gives us

$$\begin{aligned}&\Bigg| D_{KL}(p,q) + \log(a) - \frac{1}{\varphi^{(n)}(0)} \sum_{m=1}^{n} (m-1)! \left\{ \frac{\varphi^{(n-m)}(0)}{a^m} D_{\chi^m,a}(p,q) \right. \\ &\left. -\varphi^{(n-m)}(1) \int_\Omega \left(1 - a\frac{p(t)}{q(t)}\right)^m p(t)\, d\mu \right\}\Bigg| \\ &\le \frac{n!}{r^{n+1}} \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\, dt\right) D_{|\chi|^{n+1},a}(p,q).\end{aligned} \tag{5.14}$$

In particular, when $a = 1$, we have

$$\begin{aligned}&\Bigg| D_{KL}(p,q) - \frac{1}{\varphi^{(n)}(0)} \sum_{m=1}^{n} (m-1)! \left\{ \varphi^{(n-m)}(0) D_{\chi^m}(p,q) \right. \\ &\left. -\varphi^{(n-m)}(1) \int_\Omega \left(1 - \frac{p(t)}{q(t)}\right)^m p(t)\, d\mu \right\}\Bigg| \\ &\le \frac{n!}{r^{n+1}} \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\, dt\right) D_{|\chi|^{n+1}}(p,q).\end{aligned} \tag{5.15}$$

We have

$$\int_\Omega q(t) g'\left(\frac{q(t)}{p(t)}\right) d\mu = -\int_\Omega q(t) \left(\frac{p(t)}{q(t)}\right) d\mu = -1$$

and

$$\int_\Omega p(t) g'\left(\frac{q(t)}{p(t)}\right) d\mu = -\int_\Omega \frac{p^2(t)}{q(t)}\, d\mu(t) = -\left[D_{\chi^2}(q,p) + 1\right].$$

Note the use of the following identity:

$$D_{\chi^2}(q,p) = \int_\Omega q(t)\left[\left(\frac{p(t)}{q(t)}\right)^2 - 1\right] d\mu(t) = \int_\Omega \frac{p^2(t)}{q(t)}\, d\mu(t) - 1. \tag{5.16}$$

Proposition 4 gives us

$$\begin{aligned}
&\Bigg| D_{KL}(p,q) + \log(a) + \frac{1}{2a}(1-a) + \frac{1}{2} - \frac{a}{2}(D_{\chi^2}(q,p)+1) \\
&- \sum_{k=1}^{n} \frac{(-1)^k B_k}{2k}\left[\int_\Omega \left(1 - a\frac{p(t)}{q(t)}\right)^{2k} p(t)\, d\mu - \frac{1}{a^{2k}} D_{\chi^{2k},a}(p,q)\right]\Bigg| \\
&\le \left(\int_0^1 |\varphi_{2n}(t)|\, dt\right) \frac{D_{|\chi|^{2n+1},a}(p,q)}{r^{2n+1}}.
\end{aligned} \tag{5.17}$$

In particular, when $a = 1$, we have

$$\begin{aligned}
&\Bigg| D_{KL}(p,q) - \frac{1}{2} D_{\chi^2}(q,p) \\
&- \sum_{k=1}^{n} \frac{(-1)^k B_k}{2k}\left[\int_\Omega \left(1 - \frac{p(t)}{q(t)}\right)^{2k} p(t)\, d\mu - D_{\chi^{2k}}(p,q)\right]\Bigg| \\
&\le \left(\int_0^1 |\varphi_{2n}(t)|\, dt\right) \frac{D_{|\chi|^{2n+1}}(p,q)}{r^{2n+1}}.
\end{aligned} \tag{5.18}$$

Corollary 3 gives us

$$\begin{aligned}
&\Bigg| D_{KL}(p,q) + \log(a) + \frac{1}{2a}(1-a) + \frac{1}{2} - \frac{a}{2}(D_{\chi^2}(q,p)+1) \\
&+ \frac{1}{12}\int_\Omega \left(1 - a\frac{p(t)}{q(t)}\right)^2 p(t)\, d\mu - \frac{1}{12a^2} D_{\chi^2,a}(p,q)\Bigg| \\
&\le \frac{D_{|\chi|^3,a}(p,q)}{9\sqrt{3}r^3}.
\end{aligned} \tag{5.19}$$

In particular, when $a = 1$, we have

$$\left| D_{KL}(p,q) - \frac{2}{3} D_{\chi^2}(q,p) + \frac{1}{12}\left[-1 + \int_\Omega \left(\frac{p(t)}{q(t)}\right)^2 p(t)\, d\mu - D_{\chi^2}(p,q)\right]\right|$$

$$\le \frac{D_{|\chi|^3}(p,q)}{9\sqrt{3}r^3}. \tag{5.20}$$

We note the use of

$$\begin{aligned}
\int_\Omega \left(1 - \frac{p(t)}{q(t)}\right)^2 p(t)\, d\mu &= \int_\Omega \left(p(t) - 2\frac{(p(t))^2}{q(t)} + \left(\frac{p(t)}{q(t)}\right)^2 p(t)\right) d\mu \\
&= 1 - 2(D_{\chi^2}(q,p)+1) + \int_\Omega \left(\frac{p(t)}{q(t)}\right)^2 p(t)\, d\mu \\
&= -1 - 2D_{\chi^2}(q,p) + \int_\Omega \left(\frac{p(t)}{q(t)}\right)^2 p(t)\, d\mu.
\end{aligned}$$

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