On exponential and trigonometric functions on nonuniform lattices

M. Kenfack Nangho · M. Foupouagnigni · W. Koepf

Abstract We develop analogs of exponential and trigonometric functions (including the basic exponential function) and derive their fundamental properties: Addition formula, positivity, reciprocal and fundamental relations of trigonometry. We also establish a binomial theorem, characterize symmetric orthogonal polynomials and provide a formula for computing the n^{th}-derivatives for analytic functions on nonuniform lattices (q-quadratic and quadratic variables).

Keywords Basic exponential function · Askey-Wilson polynomials · Symmetric Functions and nonuniform lattices

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1 Introduction

The exponential function can be introduced as solution of the first order differential equation

$$y' = ay, \text{ with } y(0) = 1, \tag{1}$$

where a is a constant. Solution of (1) is

$$y(x) = e^{ax} = \sum_{n=0}^{\infty} \frac{(ax)^n}{n!}, \ x \in \mathbb{R}. \tag{2}$$

Properties of the exponential function (2) such as the positivity, limit and reciprocal, ie.

$$e^x > 0, \ \lim_{x\to+\infty} e^x = +\infty, \ \frac{1}{e^x} = e^{-x}, \tag{3}$$

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M. Kenfack Nangho
kenfnang@gmail.com
Department of Mathematics and Applied Mathematics, University of Pretoria, South Africa
Department of Mathematics and Computer Science, Faculty of Science, University of Dschang, Cameroon

M. Foupouagnigni
foupouagnigni@googlemail.com
Department of Mathematics, Higher Teachers' Training College, University of Yaounde I, Cameroon
African Institute for Mathematical Sciences, Cameroon

W. Koepf
koepf@mathematik.uni-kassel.de
Tel: +49 (0)561/804-4207
Fax: +49 (0)561/804-4646
Institute of Mathematics, University of Kassel, Heinrich-Plett Str. 40, 34132 Kassel, Germany

and the addition formula,

$$e^{ax}e^{ay}=e^{a(x+y)}, \tag{4}$$

play a central role in classical Fourier analysis. The reciprocal of the exponential function appears in the Fourier Transform [9,15]

$$\hat{f}(x)=\int_{-\infty}^{+\infty} f(\xi)e^{-i\xi\, x}d\xi. \tag{5}$$

where f is a piecewise continuous real function over $(-\infty,+\infty)$ satisfying the condition

$$\int_{-\infty}^{+\infty}|f(\xi)|d\xi<\infty.$$

The convergence of the integral in (5) is deduced from the fundamental relation of trigonometry

$$\cos(x)^2+\sin(x)^2=1$$

and Euler's formula

$$e^{ix}=\cos(x)+i\sin(x).$$

Properties of the Fourier Transform (5) are very useful when solving equations of mathematical physics (see [15]). These properties are established by using the properties of the exponential function listed above. A new branch of classical analysis called Fourier analysis on nonuniform lattices has been attracting significant interest (see [18,2,1]). The Fourier analysis is on functions of the variable $x(s)$,

$$x(s)=\begin{cases} c_1\, q^{-s}+c_2\, q^{s}+c_3 & \text{if } q\neq 1\\ c_4\, s^2+c_5\, s+c_6 & \text{if } q=1.\end{cases} \tag{6}$$

The lattice (6) satisfies

$$x(s+n)-x(s)=\gamma_n\,\nabla\, x_{n+1}(s), \tag{7}$$

$$\frac{x(s+n)+x(s)}{2}=\alpha_n\, x_n(s)+\beta_n, \tag{8}$$

for $n=0,\,1,\,\ldots$, with

$$x_\mu(s)=x(s+\frac{\mu}{2}),\ \mu\in\mathbb{C}, \tag{9}$$

where $\mathbb{C}$ is the set of complex numbers and $\nabla\, f(s):=f(s)-f(s-1)$.

The sequences $(\alpha_n),\ (\beta_n),\ (\gamma_n)$ satisfy the following relations

$$\begin{aligned}\alpha_{n+1}-2\,\alpha\,\alpha_n+\alpha_{n-1}&=0,\\ \beta_{n+1}-2\,\beta_n+\beta_{n-1}&=2\,\beta\,\alpha_n,\\ \gamma_{n+1}-\gamma_{n-1}&=2\,\alpha_n,\end{aligned}$$

with the initial values

$$\alpha_0=1,\ \alpha_1=\alpha,\ \beta_0=0,\ \beta_1=\beta, \gamma_0=0,\ \gamma_1=1,$$

and are given explicitly by (see [5,17])

$$\alpha_n=\frac{q^{\frac{n}{2}}+q^{-\frac{n}{2}}}{2},\ \beta_n=\frac{\beta(1-\alpha_n)}{1-\alpha},\ \gamma_n=\frac{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}},\ \text{for } \alpha=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\ q\neq 1. \tag{10}$$

and

$$\alpha_n=1,\ \beta_n=\beta\, n^2,\ \gamma_n=n,\ \text{for } \alpha=1, q=1. \tag{11}$$

Using the approach based on classical orthogonal polynomials (see [19]), Askey, Atakishiyev and Suslov [1] provided an analog of the Fourier Transform for the q-harmonic oscillator. In [1] the Poisson kernel for the continuous Hermite polynomials plays the role of the q-exponential function for the analog of the Fourier integral under consideration (see also [2]). This approach does not allow the establishment of some well know properties of the Fourier transform such as the Fourier inversion formula. The inversion formula, generally used when solving differential equations by using the Fourier transform, is established by using the classical approach of the exponential function (see [15]). To the best of our knowledge, there is no known work investigating the Fourier integral based on the exponential function on nonuniform lattices. This is due to the lack of analogs of the exponential function, the trigonometric functions, and their fundamental properties on nonuniform lattices. In the early 2000's, Suslov [18] in his book "An introduction to basic Fourier Series" introduced the bivariate exponential function on the Askey-Wilson lattice $x(s)=\frac{q^{-s}+q^{s}}{2}=\cos\theta,\, q^s=e^{i\theta}$, $y(z)=\frac{q^{-z}+q^{z}}{2}=\cos\varphi,\, q^z=e^{i\varphi}$

$$\mathcal{E}_q(x,\,y;\,w)=\frac{(w^2;q^2)_\infty}{(w^2q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{4}}w^n}{(q;q)_n}e^{-in\varphi}(-q^{\frac{(1-n)}{2}}e^{i\theta+i\varphi},-q^{\frac{(1-n)}{2}}e^{-i\theta+i\varphi};q)_n,\ |w|<1.$$

By writing

$$\mathcal{E}_q(x,\,y;\,w)=A\mathcal{E}_q(x;\,w)+B\mathcal{E}_q(y;\,-w),\quad (A=A(y),\ B=B(y))$$

with

$$\mathcal{E}_q(x;\,w)=\mathcal{E}_q(x,\,0;\,w), \tag{12}$$

he proved that

$$\mathcal{E}_q(x,\,y;\,w)=\mathcal{E}_q(x;\,w)\mathcal{E}_q(y;\,w).$$

Moreover, he proved that the function $\mathcal{E}_q(x;\,w)$, considered by M.E. Ismail and Zhang[13] with different normalization, satisfies

$$\mathbb{D}_x y=\frac{2q^{\frac{1}{4}}w}{1-q}y,\ \text{with } y(0)=1, \tag{13}$$

where [10]

$$\mathbb{D}_x\, f(x(s))=\frac{f(x_{-1}(s+1))-f(x_{-1}(s))}{x_{-1}(s+1)-x_{-1}(s)}.$$

Despite this important work of Suslov, classical Fourier analysis on nonuniform lattices still needs many fundamental tools such as the reciprocal and the positivity of the exponential function on nonuniform lattice. Let us mention that this last problem has been raised by Suslov in his book (see [18], p. 323). The aim of this work is to:

1. Provide an analog of the power basis on nonuniform lattices;
2. Introduce the analogs of the exponential function and the trigonometric functions on general nonuniform lattices;
3. Establish the analogs of the properties (4) on nonuniform lattices;
4. Establish a binomial theorem on nonuniform lattices;
5. Establish a formula for computing the n^{th}-derivatives of a given function on nonuniform lattices;
6. Characterize symmetric orthogonal polynomials on nonuniform lattices.

2 Preliminary results

Let $\mathbb{S}_x$ be the operator defined by [10]

$$\mathbb{S}_x\, f(x(s))=\frac{f(x(s+\frac{1}{2}))+f(x(s-\frac{1}{2}))}{2}. \tag{14}$$

The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ satisfy the following important relations, known as product and quotient rules

Theorem 1 *[10]*

1. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ satisfy the product rules I*

$$\mathbb{D}_x\left(f(x(s))g(x(s))\right) = \mathbb{S}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{D}_x f(x(s))\,\mathbb{S}_x g(x(s)), \tag{15}$$
$$\mathbb{S}_x\left(f(x(s))g(x(s))\right) = U_2(x(s))\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s)), \tag{16}$$

where U_2 is a polynomial of degree 2

$$U_2(x(s)) = (\alpha^2-1)\,x^2(s) + 2\,\beta\,(\alpha+1)\,x(s) + \delta_x,$$

and δ_x is a constant depending on α, β and the initial values $x(0)$ and $x(1)$ of $x(s)$:

$$\delta_x = \frac{x^2(0)+x^2(1)}{4\alpha^2} - \frac{(2\alpha^2-1)}{2\alpha^2}x(0)\,x(1) - \frac{\beta\,(\alpha+1)}{\alpha^2}(x(0)+x(1)) + \frac{\beta^2\,(\alpha+1)^2}{\alpha^2}. \tag{17}$$

2. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ also satisfy the quotient rules*

$$\mathbb{D}_x\left(\frac{f(x(s))}{g(x(s))}\right) = \frac{\mathbb{S}_x f(x(s))\,\mathbb{D}_x g(x(s)) - \mathbb{D}_x f(x(s))\,\mathbb{S}_x g(x(s))}{U_2(x(s))\,[\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2},$$
$$\mathbb{S}_x\left(\frac{f(x(s))}{g(x(s))}\right) = \frac{U_2(x(s))\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s)) - \mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s))}{U_2(x(s))\,[\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2},$$

provided that $g(x(s)) \neq 0,\ s \in (a,b)$.

3. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ also satisfy the product rules II*

$$\mathbb{D}_x\,\mathbb{S}_x = \alpha\,\mathbb{S}_x\,\mathbb{D}_x + U_1(s)\,\mathbb{D}_x^2, \tag{18}$$

$$\mathbb{S}_x^2 = U_1(s)\,\mathbb{S}_x\,\mathbb{D}_x + \alpha\,U_2(s)\,\mathbb{D}_x^2 + \mathbb{I}, \tag{19}$$

where

$$U_1(s) := U_1(x(s)) = (\alpha^2-1)\,x(s) + \beta\,(\alpha+1),\ \ U_2(s) := U_2(x(s)). \tag{20}$$

Since the action of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ transforms the monomial $x(s)^n$ into a linear combination with complicated coefficients, we used the generalized basis for the lattice $x(s)$ defined by Suslov [17] as

$$[x_m(z)-x_m(s)]^{(n)} = \prod_{j=0}^{n-1}[x_m(z)-x_m(s-j)] = \prod_{j=0}^{n-1}[x_{m-n+1}(z+j)-x_{m-n+1}(s)], \tag{21}$$
$$[x_m(z)-x_m(s)]^{(0)} \equiv 1,$$

where $m \in \mathbb{N} = \{0,\,1,\,2,\,3,...\}$ and $n \in \mathbb{N}^* = \mathbb{N}\setminus\{0\}$, in [7] to define the basis F_n by

$$F_n(x(s)) = \prod_{j=1}^{n}[x(s)-x_j(z_x)],\ n \in \mathbb{N}^*,\ F_0(x(s)) \equiv 1, \tag{22}$$

where z_x, which is a constant term with respect to z but depending on the lattice x, satisfies the relations

$$q^{2z_x} = \frac{c_2}{c_1}q^{-\frac{1}{2}} \tag{23}$$

for the q-quadratic lattice $x(s) = c_1\,q^{-s} + c_2\,q^s + \frac{\beta}{1-\alpha},\ q \neq 1,\ c_2\,c_1 \neq 0$, or the relation

$$z_x = -\frac{1}{4} - \frac{c_5}{2c_4} \tag{24}$$

for the quadratic lattice $x(s) = c_4\,s^2 + c_5\,s + c_6,\ c_4 \neq 0$. This basis enabled us to obtain the following properties:

Theorem 2 *[7]*

$$\mathbb{D}_x F_n(x(s)) = \gamma_n F_{n-1}(x(s)),$$
$$\mathbb{S}_x F_n(x(s)) = \alpha_n F_n(x(s)) + \frac{\gamma_n}{2} \nabla x_{n+1}(z_x) F_{n-1}(x(s)),$$

$$F_{n+1}(x(s)) = (x(s) - x_{n+1}(z_x)) F_n(x(s)) = \prod_{j=1}^{n+1} (x(s) - x_j(z_x)),\ n \geq 0,$$
$$F_n(x_k(z_x)) \neq 0,\ \ \forall n \geq 0,\ \forall k > n \geq 0.$$

Using theorem 2 one can prove a Taylor type theorem:

Theorem 3 *[7] Let $f(x(s))$ be a polynomial of degree n of $x(s)$. f can be expanded in the basis $F_k(x(s))$ as follows*

$$f(x(s)) = \sum_{k=0}^{n} d_k F_k(x(s)),$$

where

$$d_k = \frac{\mathbb{D}_x^k f(x(z_x))}{\gamma_k!},\ \ \gamma_k! = \prod_{j=1}^{k} \gamma_j,\ 0 \leq k \leq n,\ \ \gamma_0! = 1.$$

3 Analog of power basis

We used the basis F_n (see [7]) to develop a method for solving divided-difference equations on nonuniform lattices but, since this basis does not have the symmetry property $B_n(-x) = (-1)^n B_n(x)$ and $B_n(0) = 0$, $n \in \mathbb{N}^*$ it can't play the role of the power basis on nonuniform lattice. So, in this section, we provide an analog of the power basis on nonuniform lattices.

3.1 Analog of the power basis on nonuniform lattices

The generalized basis (21) can be rewritten as

$$[x_p(s) - x_q(z)]^{(n)} = \prod_{j=0}^{n-1} [x_p(s) - x_q(z-j)] = \prod_{j=0}^{n-1} [x_{p-n+1}(s+j) - x_{q-n+1}(z)],$$
$$[x_p(s) - x_q(z)]^{(0)} \equiv 1$$

with $n \in \mathbb{N}^*$, $p \in \mathbb{N}$, and $q \in \mathbb{N}$. So, substituting $p = 0$ and $q = n-1$ into the above equations, we have

$$[x(s) - x_{n-1}(z)]^{(n)} = \prod_{j=0}^{n-1} [x(s) - x_{n-1}(z-j)], \tag{25}$$
$$= \prod_{j=0}^{n-1} [x_{-n+1}(s+j) - x(z)],\ n \geq 1. \tag{26}$$

Taking n for $2n$ and n for $2n-1$ in (25), we obtain respectively

$$[x(s) - x_{2n-1}(z)]^{(2n)} = \prod_{j=0}^{n-1} [x(s) - x(z + \frac{1}{2} + j)][x(s) - x(z - \frac{1}{2} - j)],\ n \geq 1, \tag{27}$$

$$[x(s) - x_{2n-2}(z)]^{(2n-1)} = (x(s) - x(z)) \prod_{j=1}^{n-1} [x(s) - x(z+j)][x(s) - x(z-j)],\ n \geq 2. \tag{28}$$

Now, using the fact that s_0, solution of the equation $x(z)=0$, satisfies

$$x(s_0+t) = -x(s_0-t),\ \forall\, t \in \mathbb{C},$$

(this relation will be proved later) for the lattice $x(s) = c_1 q^{-s} + c_2 q^s$ and for the linear lattice $x(s) = s$, we introduce the analog of the power x^n as follows

$$K_n(x(s)) = x(s)[x(s) - x_{n-2}(s_0)]^{(n-1)}, K_1(x(s)) = x(s),\ K_0(x(s)) \equiv 1, \tag{29}$$

where s_0 is given by $q^{2s_0} = -\frac{c_1}{c_2}$ for the q-quadratic lattice and $s_0 = 0$ for the linear lattice s.

Theorem 4 *1. For the lattice $x(s) = c_1 q^{-s} + c_2 q^s$ and the lattice s, we have*

$$K_n(x(s_0)) = 0, \quad K_n(-x(s)) = (-1)^n K_n(x(s)),\ \ n \in \mathbb{N}^*, \tag{30}$$

$$\mathbb{D}_x\, K_n(x(s)) = \gamma_n K_{n-1}(x(s)). \tag{31}$$

2. For the q-quadratic lattice $x(s) = c_1 q^{-s} + c_2 q^s$ $(c_1 c_2 \neq 0)$ and the linear lattice $x(s) = s$, we have

$$K_{2n}(x(s)) = \prod_{j=0}^{n-1} [x(s)^2 - x_{2j}(s_0)^2] = \left(x(s)^2 - x_{2n-2}(s_0)^2\right) K_{2n-2}(x(s)), \tag{32}$$

$$K_{2n+1}(x(s)) = x(s) \prod_{j=0}^{n-1} [x(s)^2 - x_{2j+1}(s_0)^2] = \left(x(s)^2 - x_{2n-1}(s_0)^2\right) K_{2n-1}(x(s)), \tag{33}$$

$$x(s)\mathbb{S}_x\, K_n = \alpha_n K_{n+1}(x(s)) + (\gamma_{n-1} x_n(s_0)^2 - \alpha\gamma_n x_{n-1}(s_0)^2) K_{n-1}(x(s)). \tag{34}$$

The proof of this theorem uses the following properties:

Proposition 1 *For the lattice $x(s)$ defined by (6), we have*

$$K_n(x(s)) = x(s) \prod_{j=0}^{n-2} x_{-n+2}(s+j), \tag{35}$$

$$K_{2n}(x(s)) = x(s)^2 \prod_{j=1}^{n-1} x(s-j)x(s+j),\ K_2(x(s)) = x(s)^2, \tag{36}$$

$$K_{2n+1}(x(s)) = x(s) \prod_{j=0}^{n-1} x(s - \frac{1}{2} - j)x(s + \frac{1}{2} + j),\ K_1(x(s)) = x(s), \tag{37}$$

$$\mathbb{D}_x K_n(x(s)) = \gamma_n K_{n-1}(x(s)) + (\beta\gamma_{n-1} + \beta_{n-1})[x(s) - x_{n-2}(s_0)]^{(n-1)}. \tag{38}$$

Proof (of Proposition 1)
Relation (35) is obtained by using the definition of K_n, given by (29), the relation (26) and the fact that $x(s_0) = 0$. The equations (36) and (37) are direct consequences of (35). Let us prove (38). Using the definition of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$, and the relations (7) and (8), we obtain by direct computation,

$$\mathbb{D}_x [x(s) - x_{n-1}(s_0)]^{(n)} = \gamma_n [x(s) - x_{n-2}(s_0)]^{(n-1)}, \tag{39}$$

$$\mathbb{S}_x [x(s) - x_{n-1}(s_0)]^{(n)} = \alpha_n x(s)[x(s) - x_{n-2}(s_0)]^{(n-1)} + \beta_n [x(s) - x_{n-1}(s_0)]^{(n)}. \tag{40}$$

Now, observing that $K_n(x(s)) = x(s)[x(s) - x_{n-2}(s_0)]^{(n-1)}$ (see (29)) and applying the product rule (15), we have $\mathbb{D}_x\, K_n(x(s)) = (\alpha\, x(s) + \beta)\mathbb{D}_x[x(s) - x_{n-2}(s_0)]^{(n-1)} + \mathbb{S}_x[x(s) - x_{n-2}(s_0)]^{(n-1)}$. Now, by using (39) and (40) as well as the relation

$$\gamma_n = \alpha\gamma_{n-1} + \alpha_{n-1}, \tag{41}$$

obtained by direct computation, we obtain (38).

Proposition 2

1. *For the* q*-quadratic lattice* $x(s)=c_1q^{-s}+c_2q^s$ $(c_1c_2\neq 0)$ *and the linear lattice* $x(s)=s$*, we have*
$$x(s_0+j)=-x(s_0-j). \tag{42}$$
2. *Let* β_n*,* $n=1,2,...$*, be the constant appearing in (8). We have*
$$\beta_n=0\Leftrightarrow x(s)=c_1q^{-s}+c_2q^s \text{ or } x(s)=c_5s+c_6. \tag{43}$$

Proof (of Proposition 2)

$$x(s)=c_1q^{-s}+c_2q^s\Rightarrow\left(x(s)=0\Leftrightarrow q^{2s_0}=-\frac{c_1}{c_2}\right);\quad x(s)=s\Rightarrow(x(s)=0\Leftrightarrow s_0=0).$$

Therefore, for the q-quadratic lattice $x(s)=c_1q^{-s}+c_2q^s$ $(c_1c_2\neq 0)$, if $x(s)=0$ then

$$q^{s_0}x(s_0+j)=c_1(q^{-j}-q^j)=-q^{s_0}x(s_0-j),\ j\in\mathbb{N}.$$

Since $q^{s_0}\neq 0$, we obtain $x(s_0+j)=-x(s_0-j)$, $j\in\mathbb{N}$. For the linear lattice $x(s)=s$, the result is obvious. Let us prove (43). From (10) and (11), $\beta_n=0,\ n=1,2,...$, if and only if $\beta=0$. Taking $n=1$ in (8), we have $2\beta=x(s+1)+x(s)-2\alpha x_1(s)$. For $x(s)=c_1q^{-s}+c_2q^s+c_3$, we obtain $2\beta=-q^{-\frac{1}{2}}(q^{\frac{1}{2}}-1)^2c_3$. Hence, for the q-quadratic lattices $x(s)=c_1q^{-s}+c_2q^s+c_3$, $\beta=0$ if and only if $c_3=0$, for $q\neq 1$. For the lattices $x(s)=c_4s^2+c_5s+c_6$ (that is $q=1$), we obtain $\beta=0$ if and only if $c_4=0$.

Proof (of Theorem 4) The relation (31) is a direct consequence of (38) and (43). For (32), from the definition of K_n and the use of relation (28), we have

$$K_{2n}(x(s))=x(s)(x(s)-x(s_0))\prod_{j=1}^{n-1}[x(s)-x(s_0-j)][x(s)-x(s_0+j)].$$

Then by using the fact that $x(s_0)=0$ and the fact that $x(s_0+j)=-x(s_0-j)$ (see (42)) for the lattices $x(s)=c_1q^{-s}+c_2q^s$ and $x(s)=s$, we obtain the result. In a similar way, we obtain (33).
Let us prove (34). From the relations (29) and (32), we have

$$x(s)K_{2n}(x(s))=[x(s)-x_{2n}(s_0)]^{(2n+1)}+x_{2n}(s_0)^2[x(s)-x_{2n-2}(s_0)]^{(2n-1)}.$$

Now, by applying $\mathbb{D}_x$ to both sides of the latter equation, the use of the relation $\mathbb{D}_x\,[x(s)-x_{2n-1}(s_0)]^{(2n)}=\gamma_{2n}[x(s)-x_{2n-2}(s_0)]^{(2n-1)}$ (see (39)) and the product rule (15) leads us to

$$\begin{aligned}\mathbb{S}_x\,K_{2n}(x(s))&=(\gamma_{2n+1}-\alpha\gamma_{2n})\,[x(s)-x_{2n-1}(s_0)]^{(2n)}\\&\quad+\left(\gamma_{2n+1}x_{2n}(s_0)^2-\alpha\,\gamma_{2n-1}x_{2n-1}(s_0)^2\right)[x(s)-x_{2n-3}(s_0)]^{(2n-2)}.\end{aligned}$$

The second use of (29) and then the relation (41) leads us to

$$x(s)\mathbb{S}_x\,K_{2n}=\alpha_{2n}K_{2n+1}(x(s))+(\gamma_{2n-1}x_{2n}(s_0)^2-\alpha\gamma_{2n}x_{2n-1}(s_0)^2)K_{2n-1}(x(s)).$$

In a similar way, we obtain from (29) and (32)

$$x(s)\mathbb{S}_x\,K_{2n+1}=\alpha_{2n+1}K_{2n+2}(x(s))+(\gamma_{2n}x_{2n+1}(s_0)^2-\alpha\gamma_{2n+1}x_{2n}(s_0)^2)K_{2n}(x(s)).$$

Having proved Theorem 4, we now give explicitly the basis K_n for specific classes of the lattices $x(s)$ and recover some known results. From relations (35)-(37), we have:

Proposition 3

1. *The basis K_n is explicitly defined for the lattice $x(s) = c_1 q^{-s} + c_2 q^s$ (with $c_1 c_2 \neq 0$) by*

$$K_n(x(s)) = (c_1 q^{-s})^n (1 + \frac{c_2}{c_1} q^{2s})(-\frac{c_2}{c_1} q^{-n+2} q^{2s}; q^2)_{n-1},\ K_0(x(s)) = 1,\ n \geq 1, \tag{44}$$

$$K_{2n}(x(s)) = x(s)^2 (c_1 c_2 q^{-n})^{n-1} (-\frac{c_2}{c_1} q^{2s+2}, -\frac{c_1}{c_2} q^{-2s+2}; q^2)_{n-1},\ n \geq 1, \tag{45}$$

$$K_{2n+1}(x(s)) = x(s)(c_1 c_2)^n q^{-n^2} (-\frac{c_2}{c_1} q^{2s+1}, -\frac{c_1}{c_2} q^{-2s+1}; q^2)_n,\ \ n \geq 0. \tag{46}$$

2. *The basis K_n is explicitly defined for the q-linear lattice $x(s) = q^s$ ($c_1 = 0$, $c_2 = 1$, $c_3 = 0$) by*

$$K_n(x(s)) = x(s)^n.$$

3. *The basis K_n is explicitly defined for the linear lattice $x(s) = s$ ($c_4 = 0$, $c_5 = 1$, $c_6 = 0$) by*

$$\begin{aligned} K_n(x(s)) &= s \left(s + \frac{2-n}{2}\right)_{n-1}, \\ K_{2n}(x(s)) &= (-1)^n \left(-s\right)_n \left(s\right)_n, \\ K_{2n+1}(x(s)) &= (-1)^n s \left(-s + \frac{1}{2}\right)_n \left(s + \frac{1}{2}\right)_n. \end{aligned}$$

From this proposition, we can deduce the following result.

Corollary 1

1. *In the particular case of the q-Racah lattice $x(s) = q^{-s} + \delta\gamma\, q^{s+1}$ ($c_1 = 1, c_2 = \delta\gamma\, q$ and $c_3 = 0$) the equations (44)-(46) read as*

$$K_n(x(s)) = (q^{-s})^n (1 + \delta\gamma q^{2s+1})(-\delta\gamma q^{-n+3} q^{2s}; q^2)_{n-1}, \tag{47}$$

$$K_{2n}(x(s)) = x(s)^2 (\delta\gamma\, q^{-n})^{n-1} (-\delta\gamma q^{2s+3}, -\frac{1}{\delta\gamma} q^{-2s+1}; q^2)_{n-1}, \tag{48}$$

$$K_{2n+1}(x(s)) = x(s)(\delta\gamma\, q)^n q^{-n^2} (-\delta\gamma q^{2s+2}, -\frac{1}{\delta\gamma} q^{-2s}; q^2)_n. \tag{49}$$

2. *In the particular case of the Askey-Wilson lattice $x(s) = \frac{q^{-s}+q^s}{2}$, $q^s = e^{i\theta}$, ($c_1 = c_2 = \frac{1}{2}$, and $c_3 = 0$) the equations (44)-(46) read as*

$$\begin{aligned} K_n(x(s)) &= 2^{-n} e^{-in\theta} (1 + e^{2i\theta})(-q^{2-n} e^{2i\theta}; q^2)_{n-1}, \\ K_{2n}(x(s)) &= 4^{-n+1} x(s)^2 q^{-n(n-1)} (-q^2 e^{2i\theta}, -q^2 e^{-2i\theta}; q^2)_{n-1}, \\ K_{2n+1}(x(s)) &= 4^{-n} x(s) q^{-n^2} (-q e^{2i\theta}, -q e^{-2i\theta}; q^2)_n. \end{aligned}$$

Remark 1 It should be mentioned that for the specific case of the Askey-Wilson lattice our basis K_n coincides (up to a multiplicative factor) with the basis ρ_n used by Ismail [14] (Equation (1.10), page 127)

$$\rho_n(x) = 2^n K_n(x)$$

while the analogs of the power basis on the q-Racah lattice (47)-(49) seem to be new.

In the forthcoming sections, we will use the basis K_n to provide a binomial theorem on nonuniform lattices, introduce the analogs of the exponential, and trigonometric functions on nonuniform lattices, and provide symmetric functions on nonuniform lattices.

3.2 Taylor theorem

In this subsection, we provide a Taylor type theorem by using the basis K_n.

Lemma 1 *(Expansion of Cauchy kernel)*

1. *For the q-quadratic lattice $x(s) = c_1 q^{-s} + c_2 q^s$, the Cauchy kernel can be expanded in the basis K_n as*

$$\frac{1}{x(z)-x(s)} = \sum_{j=0}^{\infty} \frac{x(z)K_j(x(s))}{K_{j+2}(x(z))} + \frac{x(s)}{x(z)^2 - x(s)^2}\left[\frac{x(s)}{x(z)}\prod_{j=1}^{\infty}\left(\frac{1-\frac{x(s)^2}{x_{2j}(s_0)^2}}{1-\frac{x(z)^2}{x_{2j}(s_0)^2}}\right) + \prod_{j=0}^{\infty}\left(\frac{1-\frac{x(s)^2}{x_{2j+1}(s_0)^2}}{1-\frac{x(z)^2}{x_{2j+1}(s_0)^2}}\right)\right],\ s \neq z. \tag{50}$$

2. *For the linear the lattice $x(s) = s$ ($c_4 = 1, c_5 = 1$ and $c_6 = 0$) the Cauchy kernel can be expanded in the basis K_n as*

$$\frac{1}{x(z)-x(s)} = \sum_{j=0}^{\infty} \frac{x(z)K_j(x(s))}{K_{j+2}(x(z))} + \frac{x(s)}{x(z)^2 - x(s)^2}\left[\frac{s}{z}\frac{\sin(s\pi)}{\sin(z\pi)} + \frac{\sin(\frac{2s+1}{2}\pi)}{\sin(\frac{2z+1}{2}\pi)}\right],\ s \neq z.$$

Proof By using the relation $K_n(x(z)) = x(z)[x(z)-x_{n-2}(s_0)]^{(n-1)}$ (see (29)) and the relations (32) and (33) we have

$$\begin{aligned}\frac{x(z)^2 - x(s)^2}{x(z)}\sum_{j=0}^{n}\frac{x(z)K_j(x(s))}{K_{j+2}(x(z))} &= \sum_{j=0}^{[\frac{n}{2}]}\left(\frac{K_{2j}(x(s))}{K_{2j}(x(z))} - \frac{K_{2j+2}(x(s))}{K_{2j+2}(x(z))}\right) + \sum_{j=0}^{[\frac{n}{2}]}\left(\frac{K_{2j+1}(x(s))}{K_{2j+1}(x(z))} - \frac{K_{2j+3}(x(s))}{K_{2j+3}(x(z))}\right),\\ &= 1 - \frac{K_{2m+2}(x(s))}{K_{2m+2}(x(z))} + \frac{x(s)}{x(z)} - \frac{K_{2m+3}(x(s))}{K_{2m+3}(x(z))}\end{aligned}$$

where $m = [\frac{n}{2}]$. From this equation, we obtain

$$\frac{1}{x(z)-x(s)} = \sum_{j=0}^{n}\frac{x(z)K_j(x(s))}{K_{j+2}(x(z))} + \frac{x(z)}{x(z)^2-x(s)^2}\left(\frac{K_{2m+2}(x(s))}{K_{2m+2}(x(z))} + \frac{K_{2m+3}(x(s))}{K_{2m+3}(x(z))}\right).$$

Since

$$\frac{K_{2m+2}(x(s))}{K_{2m+2}(x(z))} = \frac{x(s)^2}{x(z)^2}\prod_{j=1}^{m}\left(\frac{1-\frac{x(s)^2}{x_{2j}(s_0)^2}}{1-\frac{x(z)^2}{x_{2j}(s_0)^2}}\right) \text{ and } \frac{K_{2m+3}(x(s))}{K_{2m+3}(x(z))} = \frac{x(s)}{x(z)}\prod_{j=0}^{m}\left(\frac{1-\frac{x(s)^2}{x_{2j+1}(s_0)^2}}{1-\frac{x(z)^2}{x_{2j+1}(s_0)^2}}\right),$$

the result is obtained when n goes to $+\infty$.

Theorem 5 *Let f be an entire function of the variable $x(s) = c_1 q^{-s} + c_2 q^s$ or $x(s) = s$. The expansion of f in the basis K_n is*

$$f(x(s)) = \sum_{j=0}^{\infty}\frac{\mathbb{D}_x^j f(0)}{\gamma_j!}K_j(x(s)) + \frac{1}{2i\pi}\int_{|x(s)-y|=r} f(y)g(x,y)dy \tag{51}$$

with

$$g(x,y) = \frac{x}{y^2-x^2}\left[\frac{x}{y}\prod_{j=1}^{\infty}\left(\frac{1-\frac{x^2}{x_{2j}(s_0)^2}}{1-\frac{y^2}{x_{2j}(s_0)^2}}\right) + \prod_{j=0}^{\infty}\left(\frac{1-\frac{x^2}{x_{2j+1}(s_0)^2}}{1-\frac{y^2}{x_{2j+1}(s_0)^2}}\right)\right].$$

Proof Let f be an analytic function, and $x(s)$ a complex number. Since $x_n(s-j)$ ($j \in \mathbb{N}^*$ and $n \in \mathbb{N}$) are discrete points of $\mathbb{C}$, there is $r > 0$ such that $x_n(s-j)$ ($j \in \mathbb{N}^*$ and $n \in \mathbb{N}$) do not belong to the circle $C(x(s);r)$. From Cauchy's theorem, we have

$$f(x(s)) = \frac{1}{2i\pi}\int_{|x(s)-y|=r}\frac{f(y)dy}{y-x(s)}.$$

Applying $\mathbb{D}_x^n$ to both sides of this equation, we have

$$\mathbb{D}_x^n f(x(s)) = \frac{1}{2i\pi}\int_{|x(s)-y|=r}\mathbb{D}_x^n\frac{1}{y-x(s)}f(y)dy.$$

Since $\mathbb{D}_x^n\frac{1}{y-x(s)} = \frac{\gamma_n!}{[y-x_n(s)]^{(n+1)}}$ (see [7]), the previous equation becomes

$$\mathbb{D}_x^n f(x(s)) = \frac{\gamma_n!}{2i\pi}\int_{|x(s)-y|=r}\frac{f(y)dy}{[y-x_n(s)]^{(n+1)}}. \tag{52}$$

Taking $s = s_0$ in the latter equation, we obtain

$$\mathbb{D}_x^n f(0) = \frac{\gamma_n!}{2i\pi}\int_{|y|=r}\frac{yf(y)dy}{K_{n+2}(y)}.$$

Now if we integrate both sides of the equation (50) on the circle $C(x(s);r)$ and use Cauchy's formula as well as the latter formula, we obtain (51).

We have the following result for polynomials.

Proposition 4 *Let $f(x(s))$ be a polynomial of degree n of $x(s)$. f can be expanded in the basis $K_j(x(s))$ as follows*

$$f(x(s)) = \sum_{j=0}^{n}\frac{\mathbb{D}_x^j f(0)}{\gamma_j!}K_j(x(s)).$$

Proof Since $K_j(x(s))$, $j \in \mathbb{N}$, is a polynomial of degree j, $\{K_j(x(s)); j \in \mathbb{N}\}$ is a basis of the space of polynomials $\mathbb{C}[x]$. Therefore, there are $a_0, ..., a_n \in \mathbb{C}$ such that $f(x(s)) = \sum_{j=0}^{n} a_j K_j(x(s))$. Applying $\mathbb{D}_x^l$, $l = 0, 1, .., n$, to both sides and using (31), we obtain

$$\mathbb{D}_x^l f(x(s)) = \sum_{j=l}^{n} a_j\gamma_j\gamma_{j-1}...\gamma_{j-l+1}K_{j-l}(x(s)).$$

Taking $x(s) = 0$, we obtain $a_l\gamma_l! = \mathbb{D}_x^l f(0)$, for $K_j(0) = 0, j = 1, 2, ...,$ and $K_0(x(s)) = 1$ (see (29)).

From the latter proposition and the theorems 2 and 3 we can deduce, the following change of basis formulae

Corollary 2 *For the q-quadratic lattice, $x(s) = c_1q^{-s} + c_2q^s$, the bases F_n and K_n are connected as follow*

$$F_n(x(s)) = \sum_{j=0}^{n}\frac{\gamma_n!}{\gamma_{n-j}!\gamma_j!}F_{n-j}(0)K_j(x(s)), \quad K_n(x(s)) = \sum_{j=0}^{n}\frac{\gamma_n!}{\gamma_{n-j}!\gamma_j!}K_{n-j}(x(z_x))F_j(x(s)). \tag{53}$$

4 Binomial theorem on nonuniform lattices

In this section, by using the basis $(F_n)_n$ (see (22)) and the basis $(K_n)_n$ (see (29)) we provide three binomial formulae on nonuniform lattices.
From the equation (25) (respectively the equation (26)), $[x(z)-x_{n-1}(s)]^{(n)}$ is a polynomial of the variable $x(z)$ (respectively $x(s)$). So, $[x(z)-x_{n-1}(s)]^{(n)}$ can be seen as a bivariate polynomial.

Theorem 6 *(Binomial theorem on nonuniform lattices)*

1. *For the q-quadratic lattice $x(s)=c_1q^{-s}+c_2q^s+c_3$ $(c_1c_2\neq 0)$ and the quadratic lattice $x(s)=c_4s^2+c_5s+c_6$, we have*

$$[x(z)-x_{n-1}(s)]^{(n)}=\sum_{k=0}^{n}\frac{\gamma_n!}{\gamma_k!\gamma_{n-k}!}F_k(x(z))(-1)^{n-k}F_{n-k}(x(s)). \tag{54}$$

2. *For the lattice $x(s)=c_1q^{-s}+c_2q^s$ and the linear lattice $x(s)=s$, we have*

$$[x(z)-x_{n-1}(s)]^{(n)}=\sum_{k=0}^{n}\frac{\gamma_n!}{\gamma_k!\gamma_{n-k}!}(-1)^{n-k}x(s)^{-1}K_{n+1-k}(x(s))K_k(x(z)),\ s\neq s_0, \tag{55}$$

$$(x(z)+x(s))\,[x(z)+x_{n-2}(s)]^{(n-1)}=\sum_{k=0}^{n}\frac{\gamma_{n-1}!\,(\gamma_k+\gamma_{n-k})}{\gamma_{n-k}!\gamma_k!}K_{n-k}(x(z))K_k(x(s)). \tag{56}$$

3. *The coefficients $\binom{n}{k}_x:=\frac{\gamma_n!}{\gamma_k!\gamma_{n-k}!}$ in (54) and (55) satisfy the relation*

$$\binom{n}{k}_x=q^{\frac{n-k}{2}}\binom{n-1}{k-1}_x+q^{-\frac{k}{2}}\binom{n-1}{k}_x,\quad 1\leq k<n. \tag{57}$$

Proof By (25), $[x(z)-x_{n-1}(s)]^n=\prod_{j=0}^{n-1}[x(z)-x_{n-1}(s-j)]$ is a polynomial of degree n in $x(z)$. Therefore, by the use of Theorem 3 (with $f(x(z))=[x(z)-x_{n-1}(s)]^{(n)}$), we obtain

$$[x(z)-x_{n-1}(s)]^{(n)}=\sum_{j=0}^{n}d_jF_j(x(z)),\quad d_j=\frac{\mathbb{D}_x^j[x(z)-x_{n-1}(s)]^{(n)}}{\gamma_j!}|_{z=z_x} \tag{58}$$

Next, from relations (25) and (26) we have

$$[x(z+\frac{1}{2})-x_{n-1}(s)]^{(n)}=(x(z+\frac{n}{2})-x(s))[x(z)-x_{n-2}(s)]^{(n-1)},$$

$$[x(z-\frac{1}{2})-x_{n-1}(s)]^{(n)}=(x(z-\frac{n}{2})-x(s))[x(z)-x_{n-2}(s)]^{(n-1)}.$$

Therefore

$$\mathbb{D}_x\,[x(z)-x_{n-1}(s)]^{(n)}=\frac{[x(z+\frac{n}{2})-x(z-\frac{n}{2})][x(z)-x_{n-2}(s)]^{(n-1)}}{\nabla x_1(z)}.$$

The use of (7) transforms $x(z+\frac{n}{2})-x(z-\frac{n}{2})$ into $\gamma_n\nabla x_1(z)$. Thus

$$\mathbb{D}_x\,[x(z)-x_{n-1}(s)]^{(n)}=\gamma_n[x(z)-x_{n-2}(s)]^{(n-1)}.$$

By iterating the above relation, we get

$$\mathbb{D}_x^j\,[x(z)-x_{n-1}(s)]^{(n)}=\frac{\gamma_n!}{\gamma_{n-j}!}[x(z)-x_{n-j-1}(s)]^{(n-j)}. \tag{59}$$

Therefore, from (58) and (22) we obtain (54).
Let us prove (55). Since $[x(z)-x_{n-1}(s)]^{(n)}$ is a polynomial of degree n in the variable $x(z)$ we deduce from the Taylor Proposition 4 and the relation (59) that

$$[x(z)-x_{n-1}(s)]^{(n)}=\sum_{j=0}^{n}\frac{d_{n,j}}{\gamma_j!}K_j(x(z))$$

with $d_{n,j}=\frac{\gamma_n!}{\gamma_{n-j}!}[x(s_0)-x_{n-j-1}(s)]^{(n-j)}$. By using (25) as well as the definition of K_n (see (29)), we transform $[x(s_0)-x_{n-j-1}(s)]^{(n-j)}$ into $(-1)^{n-j}x(s)^{-1}K_{n+1-j}(x(s))$. (56) is a direct consequence of (55). Since $\binom{n}{k}_x=\frac{\gamma_n!}{\gamma_k!\gamma_{n-k}!}=\frac{\gamma_{n-1}!\gamma_n}{\gamma_k!\gamma_{n-k}!}$, the use of the relation $\gamma_n=q^{\frac{n-k}{2}}\gamma_k+q^{-\frac{k}{2}}\gamma_{n-k}$, obtained by direct computation, yields to (57).

Corollary 3

$$[x(z)-x_{n-1}(s)]^{(n)}=(-1)^n[x(s)-x_{n-1}(z)]^{(n)},\quad n\in\mathbb{N}.$$

Proof Let n be a positive integer. Taking $j=n-k$ in (54), we obtain

$$[x(s)-x_{n-1}(z)]^{(n)}=(-1)^n\sum_{j=0}^{n}\frac{\gamma_n!}{\gamma_j!\gamma_{n-j}!}F_j(x(s))(-1)^{n-j}F_{n-j}(x(z))=(-1)^n[x(z)-x_{n-1}(s)]^{(n)}.$$

4.1 Binomial theorem on the lattice $x(s)=c_1q^{-s}+c_2q^s+c_3$

In this subsection, we provide explicit expressions of the binomial formula (54)-(56) on the lattice $x(s)=c_1q^{-s}+c_2q^s+c_3$. From (22), we obtain the following representation of F_n, on the q-quadratic lattice,

$$F_n(x(s))=(-\sqrt{c_1c_2})^nq^{-\frac{n^2}{4}}\left(\sqrt{\frac{c_1}{c_2}}q^{\frac14}q^s,\sqrt{\frac{c_2}{c_1}}q^{\frac14}q^{-s};q^{\frac12}\right)_n, c_1c_2>0,$$

and use it as well as the relation (44) to obtain:

Corollary 4

1. *For the q-quadratic lattice $x(s)=c_1q^{-s}+c_2q^s+c_3$, $c_1c_2>0$, (54) reads as*

$$\begin{aligned}&(c_1q^{-z})^n(q^{\frac{1-n}{2}}q^{z-s},\frac{c_2}{c_1}q^{\frac{1-n}{2}}q^{z+s};q)_n\\&=\sum_{k=0}^{n}\frac{(-1)^kq^{-\frac{n^2}{4}}(\sqrt{c_1c_2})^n(q;q)_n}{(q;q)_{n-k}(q;q)_k}\left(\sqrt{\frac{c_1}{c_2}}q^{\frac14}q^z,\sqrt{\frac{c_2}{c_1}}q^{\frac14}q^{-z};q^{\frac12}\right)_k\left(\sqrt{\frac{c_1}{c_2}}q^{\frac14}q^s,\sqrt{\frac{c_2}{c_1}}q^{\frac14}q^{-s};q^{\frac12}\right)_{n-k}.\end{aligned}$$

2. *For the Askey-Wilson lattice $x=\cos\theta$, ($c_1=c_2=\frac12$, $q^s=e^{i\theta}$ and $q^z=e^{i\varphi}$) (54) reads as*

$$\begin{aligned}&e^{-in\varphi}\left(q^{\frac{1-n}{2}}e^{i(\varphi-\theta)},q^{\frac{1-n}{2}}e^{i(\varphi+\theta)};q\right)_n\\&=\sum_{k=0}^{n}\frac{(-1)^kq^{-\frac{n^2}{4}}(q;q)_n}{(q;q)_{n-k}(q;q)_k}(q^{\frac14}e^{i\varphi},q^{\frac14}e^{-i\varphi};q^{\frac12})_k(q^{\frac14}e^{i\theta},q^{\frac14}e^{-i\theta};q^{\frac12})_{n-k}.\end{aligned}$$

3. *For the q-Racah lattice $x(s)=q^{-s}+\gamma\,\delta\,q^{s+1}$, ($c_1=1$, and $c_2=\delta\gamma\,q$) (54) reads as*

$$\begin{aligned}&(q^{-z})^n(q^{\frac{1-n}{2}}q^{z-s},\delta\gamma\,q^{\frac32-\frac{n}{2}}q^{z+s};q)_n\\&=\sum_{k=0}^{n}\frac{(-1)^kq^{-\frac{n^2}{4}}(\delta\gamma\,q)^{\frac{n}{2}}(q;q)_n}{(q;q)_{n-k}(q;q)_k}\\&\times(\delta^{\frac12}\gamma^{\frac12}q^{\frac34}q^z,\delta^{-\frac12}\gamma^{-\frac12}q^{-\frac14}q^{-z};q^{\frac12})_k(\delta^{\frac12}\gamma^{\frac12}q^{\frac34}q^s,\delta^{-\frac12}\gamma^{-\frac12}q^{-\frac14}q^{-s};q^{\frac12})_{n-k}.\end{aligned}$$

Corollary 5

1. *For the q-quadratic lattice $x(s)=c_1q^{-s}+c_2q^s$, $c_1c_2\neq 0$, (55) reads as*

$$
\begin{aligned}
&(q^{-z})^n(q^{\frac{1-n}{2}}q^{z-s},\frac{c_2}{c_1}q^{\frac{1-n}{2}}q^{z+s};q)_n\\
&=\sum_{j=0}^{n}\frac{(-1)^{n-j}q^{-\frac{(n-j)j}{2}}(q;q)_n}{(q;q)_j(q;q)_{n-j}}(q^{-s})^{n-j}(q^{-z})^j\left(1+\frac{c_2}{c_1}q^{2z}\right)\\
&\times\left(-\frac{c_2}{c_1}q^{-n+1+j}q^{2s};q^2\right)_{n-j}\left(-\frac{c_2}{c_1}q^{-j+2}q^{2z};q^2\right)_{j-1}.
\end{aligned}
$$

2. *For the Askey-Wilson lattice $x=\cos\theta$, ($c_1=c_2=\frac{1}{2}$, $q^s=e^{i\theta}$) (55) reads as*

$$
\begin{aligned}
&e^{-in\varphi}\left(q^{\frac{1-n}{2}}e^{i(\varphi-\theta)},q^{\frac{1-n}{2}}e^{i(\varphi+\theta)};q\right)_n\\
&=\sum_{j=0}^{n}\frac{(-1)^{n-j}q^{-\frac{(n-j)j}{2}}(q;q)_n}{(q;q)_j(q;q)_{n-j}}e^{-i(n-j)\theta}e^{-i\varphi}(1+e^{i2\varphi})\\
&\times(-q^{-n+1+j}e^{i2\theta};q^2)_{n-j}(-q^{-j+2}e^{i2\varphi};q^2)_{j-1}.
\end{aligned}
$$

3. *For the q-Racah lattice $x(s)=q^{-s}+\gamma\,\delta\,q^{s+1}$ ($c_1=1$, and $c_2=\gamma\,\delta\,q$) (55) reads as*

$$
\begin{aligned}
&(q^{-z})^n(q^{\frac{1-n}{2}}q^{z-s},\gamma\,\delta\,q^{\frac{3-n}{2}}q^{z+s};q)_n\\
&=\sum_{j=0}^{n}\frac{(-1)^{n-j}q^{-\frac{(n-j)j}{2}}(q;q)_n}{(q;q)_j(q;q)_{n-j}}(q^{-s})^{n-j}(q^{-z})^j\left(1+\gamma\,\delta\,qq^{2z}\right)\\
&\left(-\gamma\,\delta\,q^{-n+2+j}q^{2s};q^2\right)_{n-j}\left(-\gamma\,\delta\,q^{-j+3}q^{2z};q^2\right)_{j-1}.
\end{aligned}
$$

Corollary 6
For the q-linear lattice $x(z)=q^z=x$ ($c_1=0$, $c_2=1$ and $c_3=0$)(55) and (56) read, respectively, as

$$
[x(z)-x_{n-1}(s)]^{(n)}=\sum_{k=0}^{n}\frac{\gamma_n!}{\gamma_{n-k}!\gamma_k!}x^k(-y)^{n-k},\ y=q^s. \tag{60}
$$

$$
(x(z)+x(s))[x(z)+x_{n-2}(s)]^{(n-1)}=\sum_{k=0}^{n}\frac{\gamma_{n-1}(\gamma_k+\gamma_{n-k})}{\gamma_{n-k}!\gamma_k!}x^k(y)^{n-k},\ y=q^s.
$$

The relation (60) has already been obtained by Suslov (see [18], p.61, Eq. (3.6.2)).

4.2 Binomial theorem on the lattice $x(s)=c_4s^2+c_5s+c_6$

In this subsection, we provide explicit expressions of the binomial formula (54)-(56) on the lattice $x(s)=c_4s^2+c_5s+c_6$.

Corollary 7 *For the quadratic lattice $x(s)=c_4s^2+c_5s+c_6$ the binomial formula (54) reads as*

$$
\begin{aligned}
&(z-s-\frac{n}{2}+\frac{1}{2})_n(z+s-\frac{n}{2}+\frac{1}{2}+\frac{c_5}{c_4})_n\\
&=\sum_{k=0}^{n}\frac{(-1)^{n+k}n!}{4^n(n-k)!k!}(-2z+\frac{1}{2}-\frac{c_5}{c_4},2z+\frac{1}{2}+\frac{c_5}{c_4})_{n-k}(-2s+\frac{1}{2}-\frac{c_5}{c_4},2s+\frac{1}{2}+\frac{c_5}{c_4})_k.
\end{aligned}
$$

Corollary 8

1. *For the linear lattice $x(s) = s$ ($c_4 = 0$, $c_5 = 1$ and $c_6 = 0$) the binomial formula (55) reads as*

$$(z - s + \frac{1-n}{2})_n = (-1)^n(z + \frac{1-n}{2})_n + \sum_{j=1}^{n-1}(-1)^{n-j}\binom{n}{j}(z + \frac{1-n+j}{2})_{n-j}s(s + \frac{2-j}{2})_{j-1}$$
$$+s(s + \frac{2-n}{2})_n.$$

2. *For the linear lattice $x(s) = s$, ($c_4 = 0$, $c_5 = 1$ and $c_6 = 0$) the binomial formula (56) reads as*

$$(z + s)(z + s + \frac{2-n}{2})_{n-1} = z(z + \frac{2-n}{2})_{n-1} + \sum_{j=1}^{n-1}\binom{n}{j}z(z + \frac{2-n+j}{2})_{n-1-j}s(s + \frac{2-j}{2})_{j-1}$$
$$+s(s + \frac{2-n}{2})_{n-1}.$$

5 n^{th}-derivatives of holomorphic functions on nonuniform lattices

In this section, we give a formula for computing the n^{th} derivative of a function of the quadratic, the q-quadratic, the linear and the q-linear variable.

Theorem 7 *Let f be an analytic function. We have*

$$\mathbb{D}_x^n f(x(s)) = \sum_{k=0}^{n}\binom{n}{k}_x(-1)^k\frac{\nabla x_1(s + \frac{n}{2} - k)}{\prod_{j=0}^{n-k}\nabla x_1(s + \frac{j}{2})\prod_{j=1}^{k}\nabla x_1(s - \frac{j}{2})}f(x_{n-2k}(s)), \tag{61}$$

where

$$\binom{n}{k}_x = \begin{cases}\frac{\gamma_n!}{\gamma_{n-k}!\gamma_k!} & \text{if } q \neq 1\\ \frac{n!}{(n-k)!k!} & \text{if } q = 1.\end{cases}$$

Proof Let f be an analytic function. Let $x(s)$ be a complex number, n a positive integer and $r_n > 0$ such that $x_n(s-j)$ $(0 \leq j < n)$ belongs to the interior of the circle $C(x(s); r_n) := C_n$. From Cauchy's Theorem,

$$f(x(s)) = \frac{1}{2i\pi}\int_{C_n}\frac{f(y)dy}{y - x(s)}$$

and by (52)

$$\mathbb{D}_x^n f(x(s)) = \frac{\gamma_n!}{2i\pi}\int_{C_n}\frac{f(y)dy}{[y - x_n(s)]^{(n+1)}}.$$

Using (25), we have

$$\frac{1}{[y - x_n(s)]^{(n+1)}} = \sum_{k=0}^{n}\frac{a_{n,k}}{y - x_n(s-k)},$$

where

$$a_{n,k} = \frac{1}{\prod_{j=0,j\neq k}^{n}(x_n(s-k) - x_n(s-j))}.$$

Therefore,

$$\mathbb{D}_x^n f(x(s)) = \gamma_n!\sum_{k=0}^{n}a_{n,k}\frac{1}{2i\pi}\int_{C_n}\frac{f(y)dy}{y - x_n(s-k)}.$$

Since $x_n(s-j)$ belongs to the interior of C_n it follows from Cauchy's theorem that

$$f(x_n(s-k)) = \frac{1}{2i\pi}\int_{C_n}\frac{f(y)dy}{y-x_n(s-k)}$$

and then

$$\mathbb{D}_x^n f(x(s)) = \gamma_n!\sum_{k=0}^{n} a_{n,k} f(x_n(s-k)).$$

To end the proof, we use relation (7) and the fact that $\gamma_{-j} = -\gamma_j$, $j = 1, 2, 3, ...$, to transform $a_{n,k}$ into

$$\frac{(-1)^k \nabla x_1(s+\frac{n}{2}-k)}{\gamma_k!\gamma_{n-k}!\prod_{j=0}^{n-k}\nabla x_1(s+\frac{j}{2})\prod_{j=1}^{k}\nabla x_1(s-\frac{j}{2})}.$$

Corollary 9

1. *For the q-quadratic lattice $x(s) = c_1q^{-s} + c_2q^s$ (61) reads as*

$$\mathbb{D}_x^n f(x(s)) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}_x a_{n,k} f(x_{n-2k}(s)),$$

where

$$a_{n,k} = \frac{\left(1-\frac{c_2q^{2\,s+n-2\,k}}{c_1}\right)q^{(n-2\,k)s}\,q^{\frac{(n-k)^2+k^2-n+k}{4}}}{(-c_2)^k {c_1}^{n-k}\left(\frac{1}{\sqrt{q}}-\sqrt{q}\right)^n\left(\frac{c_2q^{2\,s}}{c_1},q\right)_{n-k+1}\left(\frac{c_1q^{-2\,s+1}}{c_2},q\right)_k}.$$

2. *For the q-linear lattice $x(s) = q^s = x$, (61) reads as*

$$\mathbb{D}_x^n f(x(s)) = \left(\frac{x^{-1}q^{-\frac{n-1}{4}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\right)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}_x q^{\frac{k(n-1)}{2}} f(xq^{\frac{n-2k}{2}}).$$

3. *For the quadratic lattices $x(s) = c_4s^2 + c_5s + c_6$ (61) reads as*

$$\mathbb{D}_x^n f(x(s)) = \sum_{k=0}^{n}\binom{n}{k}\frac{c_4(2s+n-2k)+c_5}{c_4^{n+1}(-2s+1-\frac{c_5}{c_4})_k(2s+\frac{c_5}{c_4})_{n+1-k}} f(x_{n-2k}(s)).$$

4. *For the linear lattice $x(s) = s$, ($c_4 = 0$, $c_5 = 1$, $c_5 = 0$) (61) reads as*

$$\mathbb{D}_x^n f(s) = \sum_{k=0}^{n}(-1)^k\binom{n}{k} f(s+\frac{n}{2}-k).$$

6 Analogs of the exponential function and the trigonometric functions on nonuniform lattices

6.1 Analog of the exponential function on nonuniform lattices

Replacing $\frac{d}{dx}$, the usual derivative, in (1), by the operator $\mathbb{D}_x$ we have

$$\mathbb{D}_x y(x(s)) = ay(x(s)),\; y(0) = 1, \tag{62}$$

where a is a constant. Introducing the analog of the exponential function $\mathcal{E}(x;a)$ on nonuniform lattices (6) as the solution of Equation (62) we have:

Theorem 8

1. *On the lattice* $x(s)=c_1q^{-s}+c_2q^s$ *or* s, *the exponential function* $\mathcal{E}(x;a)$ *can be formally expanded in the basis* K_n *as*

$$\mathcal{E}(x;a)=\sum_{n=0}^{\infty}\frac{K_n(x(s))}{\gamma_n!}a^n. \tag{63}$$

2. *On q-quadratic lattice* $x(s)=c_1q^{-s}+c_2q^s$, $c_1c_2\neq 0$, *the exponential function* $\mathcal{E}(x;a)$ *can be formally expanded in the basis* F_n *as*

$$\mathcal{E}(x;a)=\zeta_0(a)\sum_{n=0}^{\infty}\frac{F_n(x(s))}{\gamma_n!}a^n, \tag{64}$$

where $\zeta_0(a)$ *is the constant given by* $\zeta_0(a)\sum_{n=0}^{\infty}\frac{F_n(0))}{\gamma_n!}a^n=1$.

Proof From (62), $\forall\, n\in\mathbb{N}$, $\mathbb{D}_x^n\mathcal{E}(0;a)=a^n\in\mathbb{C}$. So, by theorem 5, $\mathcal{E}(x(s);a)$ can be expanded in the basis K_n as

$$\mathcal{E}(x;a)=\sum_{n=0}^{\infty}\frac{K_n(x(s))}{\gamma_n!}a^n+\frac{1}{2i\pi}\int_{|x(s)-y|=r}\mathcal{E}(y;a)g(x,y)dy.$$

$$g(x,y)=\frac{x}{y^2-x^2}\left[\frac{x}{y}\prod_{j=1}^{\infty}\left(\frac{1-\frac{x^2}{x_{2j}(s_0)^2}}{1-\frac{y^2}{x_{2j}(s_0)^2}}\right)+\prod_{j=0}^{\infty}\left(\frac{1-\frac{x^2}{x_{2j+1}(s_0)^2}}{1-\frac{y^2}{x_{2j+1}(s_0)^2}}\right)\right].$$

Noting that the function $x(s)\mapsto\sum_{n=0}^{\infty}\frac{K_n(x(s))}{\gamma_n!}a^n$ is solution to (62), we conclude that it is a formal expansion of $\mathcal{E}(x(s);a)$. Taking into account the expansion of K_n in the basis F_n (see the second equation of (53)) in (63), we obtain

$$\mathcal{E}(x;a)=\sum_{n=0}^{\infty}\frac{K_n(x(s))}{\gamma_n!}a^n=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{K_{n-j}(x(z_x))a^{n-j}}{\gamma_{n-j}!}\frac{F_j(x(s))a^j}{\gamma_j!}.$$

Now, using the product series formula, we obtain

$$\mathcal{E}(x;a)=\zeta_0(a)\sum_{n=0}^{\infty}\frac{F_n(x(s))a^n}{\gamma_n!},\ \zeta_0(a)=\sum_{n=0}^{\infty}\frac{K_n(x(z_x))a^n}{\gamma_n!}.$$

Since $\mathcal{E}(x;a)$ satisfies the initial condition $\mathcal{E}(0;a)=1$, $\zeta_0(a)\sum_{n=0}^{\infty}\frac{F_n(0)a^n}{\gamma_n!}=1$.

Considering (63) and (64) as series expansion of $\mathcal{E}(x;a)$ in the basis K_n and F_n respectively, we provide in the following corollaries their representation in terms of basic hypergeometric or hypergeometric functions and deduced their domains of convergence.

Corollary 10 *The series expansion (63) of the exponential function* $\mathcal{E}(x;a)$ *in the basis* K_n *reads, explicitly, as follows*

1. *For the q-quadratic lattice* $x(s)=c_1q^{-s}+c_2q^s$,

$$\mathcal{E}(x(s),\,a)=1+\sum_{n=1}^{\infty}\frac{(c_1q^{-s})^n(1+\frac{c_2}{c_1}q^{2s})(-\frac{c_2}{c_1}q^{-n+2}q^{2s};q^2)_{n-1}}{\gamma_n!}a^n, \tag{65}$$

$$\begin{aligned}&={}_2\varphi_1\left(\begin{matrix}-\frac{c_2}{c_1}q^{2s},-\frac{c_1}{c_2}q^{-2s}\\ q\end{matrix};q^2,a^2c_1c_2(1-q)^2q^{\frac{1}{2}}\right)\\ &+ax(s)_2\varphi_1\left(\begin{matrix}-\frac{c_2}{c_1}q^{2s+1},-\frac{c_1}{c_2}q^{-2s+1}\\ q^3\end{matrix};q^2,a^2c_1c_2(1-q)^2q^{\frac{1}{2}}\right).\end{aligned} \tag{66}$$

The series (65) is the Taylor expansion of $\mathcal{E}(x(s),a)$ with respect to a in the domain $|a| < \frac{q^{-\frac{1}{4}}}{\sqrt{|c_1c_2||1-q|}}$ where $0 < q \neq 1$.

2. *For the q-linear lattice $x(s) = q^s = x$ ($c_1 = 0$, $c_2 = 1$ and $c_3 = 0$),*

$$\mathcal{E}(x,\, a) = \sum_{n=0}^{\infty} \frac{(ax)^n}{\gamma_n!}.$$

3. *For linear lattice s ($c_4 = 0$, $c_5 = 1$ and $c_6 = 0$),*

$$\begin{aligned}\mathcal{E}(x(s),\, a) &= \sum_{n=0}^{\infty} \frac{s\left(s-\frac{n-2}{2}\right)_{n-1}}{n!} a^n, \qquad (67)\\ &= {}_2F_1\left(\begin{matrix} -s, s \\ \frac{1}{2}\end{matrix} ; -\frac{a^2}{4}\right) + as\,{}_2F_1\left(\begin{matrix} -s+\frac{1}{2}, s+\frac{1}{2} \\ \frac{3}{2}\end{matrix} ; -\frac{a^2}{4}\right).\end{aligned}$$

The series (67) converges for $|a| < 2$.

Corollary 11 *The series expansion (64) of the exponential function $\mathcal{E}(x;a)$ in the basis F_n reads, explicitly, as follows:*

1. *For the q-quadratic lattice $x(s) = c_1q^{-s} + c_2q^s$,*

$$\begin{aligned}\mathcal{E}(x(s);a) &= \frac{(-w;q)_\infty}{(wq^{\frac{1}{2}};q)_\infty} \sum_{n=0}^{\infty} \frac{a^n}{\gamma_n!} F_n(x(s)),\\ &= \frac{(-w;q)_\infty}{(wq^{\frac{1}{2}};q)_\infty} {}_2\phi_1\left(\begin{matrix} \frac{c_2}{c_1}q^{z_x+\frac{1}{2}}q^s, q^{z_x+\frac{1}{2}}q^{-s} \\ -q^{\frac{1}{2}}\end{matrix} ; q^{\frac{1}{2}}, -w\right), \qquad (68)\end{aligned}$$

where $0 < q < 1$ and $w^2 = a^2c_1c_2(1-q)^2q^{-\frac{1}{2}}$. Moreover, the above series is the Taylor expansion of $\mathcal{E}(x(s);a)$ with respect to w or a in the domain $|w| < 1$ that is in the domain $|a| < \frac{q^{\frac{1}{4}}}{\sqrt{|c_1c_2|}(1-q)}$.

2. *For the q-Racah lattice $x(s) = q^{-s} + \gamma\delta\, q^{s+1}$, ($c_1 = 1$, $c_2 = \gamma\delta\, q$ and $q^{z_x} = \gamma^{-\frac{1}{2}}\delta^{-\frac{1}{2}}q^{-\frac{3}{4}}$) (68) reads as*

$$\mathcal{E}(x(s);a) = \frac{(-w;q)_\infty}{(wq^{\frac{1}{2}};q)_\infty} {}_2\phi_1\left(\begin{matrix} \gamma^{\frac{1}{2}}\delta^{\frac{1}{2}}q^{\frac{3}{4}}q^s, \gamma^{-\frac{1}{2}}\delta^{-\frac{1}{2}}q^{-\frac{1}{4}}q^{-s} \\ -q^{\frac{1}{2}}\end{matrix} ; q^{\frac{1}{2}}, -w\right),$$

where $0 < q < 1$ and $w^2 = a^2\gamma\delta\,(1-q)^2q^{\frac{1}{2}}$.

Remark 2 Considering the two Taylor series expansions (63) and (64) of $\mathcal{E}(x(s);\, a)$ (for $0 < q < 1$) , on the q-quadratic lattice $x(s) = c_1q^{-s} + c_2q^s$ with respect to a, we observe that the domain of the first is $|a| < \frac{q^{-\frac{1}{4}}}{\sqrt{|c_1c_2|}(1-q)}$ while the one of the second is $|a| < \frac{q^{\frac{1}{4}}}{\sqrt{|c_1c_2|}(1-q)}$. Therefore, the domain of (63) is larger than the one of (64).

Remark 3 Taking $c_1 = c_2 = \frac{1}{2}$ and $q^s = e^{i\theta}$ in (68) we obtain that the basic exponential function $\mathcal{E}_q(x;\, w)$ is connected to $\mathcal{E}(x;\, a)$ as follows

$$\mathcal{E}_q(x;\, w) = \mathcal{E}(x;\, \frac{2q^{\frac{1}{4}}w}{1-q}).$$

Proposition 5 *For the q-quadratic lattice $x(s) = c_1q^{-s} + c_2q^s$, we have*

$$\begin{aligned}\lim_{n\to\infty} \prod_{j=1}^{n} \left(1 + \frac{x(s)}{x_j(z_x)}\right) &= \sum_{n=0}^{\infty} \frac{K_n(x(s))}{\gamma_n!} \left(\frac{q^{z_x+\frac{1}{2}}}{c_1(1-q)}\right)^n\\ &= \mathcal{E}(x(s);a)\end{aligned}$$

with $a = \frac{q^{z_x+\frac{1}{2}}}{c_1(1-q)}$.

Proof If we substitute $-x$ for x in the relation (53) and then divide the relation by $F_n(0)$, we obtain

$$\prod_{j=1}^{n}\left(1+\frac{x(s)}{x_j(z_x)}\right)=\sum_{j=0}^{n}\frac{\gamma_n!}{\gamma_{n-j}!\gamma_j!}\frac{F_{n-j}(0)}{F_n(0)}K_j(-x(s)). \tag{69}$$

By using the relations $F_n(0)=\left(-c_1q^{-(\frac{n}{4}+z_x+\frac{1}{4})}\right)^n(-q^{\frac{1}{2}};q)_n$ and
$\gamma_n!=q^{-\frac{n(n-1)}{4}}\frac{(q;q)_n}{(1-q)^n}$ which were obtained by direct computation, we transform (69) into

$$\prod_{j=1}^{n}\left(1+\frac{x(s)}{x_j(z_x)}\right)=\sum_{j=0}^{n}\frac{(q\,;q)_n}{(q\,;q)_{n-j}}\frac{(-q^{\frac{1}{2}};q)_{n-j}}{(-q^{\frac{1}{2}};q)_n}\frac{K_j(-x(s))}{\gamma_j!}\left(-\frac{q^{z_x+\frac{1}{2}}}{c_1(1-q)}\right)^j.$$

Now, if n tends to ∞ on both sides of the above relation, and we take into account the fact that $K_n(-x)=(-1)^nK_n(x)$, we obtain the desired result.

6.2 Analogs of the trigonometric functions on nonuniform lattices

In the same way as in [18], for the basic trigonometric functions, the analog of the $cosine$ function on nonuniform lattice $C(x;a)$ and the $sine$ function on nonuniform lattice $S(x;a)$ can be introduced by using the analog of Euler's formula:

$$\mathcal{E}(x;ia)=C(x;a)+iS(x;a).$$

Therefore we deduce from Theorem 8 and the use of the latter relation that:

Proposition 6 *The analog of the* $cosine$ *function on nonuniform lattices* $C(x;a)$ *and of the* $sine$ *function on nonuniform lattices* $S(x;a)$ *can be expanded in the basis* K_n *as*

$$C(x(s);a)=\sum_{n=0}^{\infty}(-1)^n\frac{K_{2n}(x(s))}{\gamma_{2n}!}a^{2n}, \tag{70}$$

$$S(x(s);a)=a\sum_{n=0}^{\infty}(-1)^n\frac{K_{2n+1}}{\gamma_{2n+1}!}a^{2n}. \tag{71}$$

Corollary 12 *The series expansion (70) of* $C(x;a)$ *in the basis* K_n *reads, explicitly, as*

1. *For the q-quadratic lattice* $x(s)=c_1q^{-s}+c_2q^s$*, we have*

$$C(x(s),a)={}_2\varphi_1\left(\begin{matrix}-\frac{c_2}{c_1}q^{2s},-\frac{c_1}{c_2}q^{-2s}\\ q\end{matrix};q^2,-a^2c_1c_2(1-q)^2q^{\frac{1}{2}}\right),$$

2. *For the q-linear lattices* $x(s)=q^s=x$ *(*$c_1=0$*,* $c_2=1$ *and* $c_3=0$*), we have*

$$C(x;a)=\sum_{n=0}^{\infty}(-1)^n\frac{(ax)^{2n}}{\gamma_{2n}!},$$

3. *For the linear lattice* s *(*$c_4=0$*,* $c_5=1$ *and* $c_6=0$*), we have*

$$C(x;a)={}_2F_1\left(\begin{matrix}-s,s\\ \frac{1}{2}\end{matrix};\frac{a^2}{4}\right).$$

Corollary 13 *The series expansion (71) of* $S(x;a)$ *in the basis* K_n *reads, explicitly, as*

1. *For the q-quadratic lattice* $x(s)=c_1q^{-s}+c_2q^s$*, we have*

$$S(x(s),a)=ax(s){}_2\varphi_1\left(\begin{matrix}-\frac{c_2}{c_1}q^{2s+1},-\frac{c_1}{c_2}q^{-2s+1}\\ q^3\end{matrix};q^2,-a^2c_1c_2(1-q)^2q^{\frac{1}{2}}\right),$$

2. *For the q-linear lattices $x(s)=q^s=x$ ($c_1=0$, $c_2=1$ and $c_3=0$), we have*

$$S(x;a)=\sum_{n=0}^{\infty}(-1)^n\frac{(ax)^{2n+1}}{\gamma_{2n+1}!},$$

3. *For the linear lattice s ($c_4=0$, $c_5=1$ and $c_6=0$), we have*

$$S(x;a)=(ax)\,{}_2F_1\left(\begin{matrix}-s+\frac{1}{2},s+\frac{1}{2}\\ \frac{3}{2}\end{matrix};\frac{a^2}{4}\right).$$

7 Analogs of fundamental properties for exponential and trigonometric functions

7.1 Reciprocal of exponential function and fundamental relations of trigonometry

Theorem 9 *(Reciprocal of the exponential function)*

1. *For the q-lattice $x(s)=c_1q^{-s}+c_2q^s+c_3$, we have*

$$\frac{1}{\mathcal{E}(x;a)}=\mathcal{E}(x;-q^{\frac{1}{2}}a). \tag{72}$$

2. *For the lattice $x(s)=c_4s^2+c_5s+c_6$, we have*

$$\frac{1}{\mathcal{E}(x;a)}=\mathcal{E}(x;-a).$$

The proof of this theorem uses the following result.

Proposition 7 *The function $y(x(s))=\mathcal{E}\,(x(s);a)\mathcal{E}\,(x(s);b)$ satisfies the divided-difference equation:*

$$\mathbb{D}_x^2y-2abU_1(x)\mathbb{D}_x\,y-2ab\alpha\,\mathbb{S}_x\,y-(a^2+b^2)y=0,$$

where U_1 is the polynomial $U_1(x)=(\alpha^2-1)x+\beta(\alpha+1)$ (see (20)).

Proof We apply the identity operator as well as the operators $\mathbb{D}_x$, $\mathbb{S}_x$, $\mathbb{S}_x\mathbb{D}_x$ and $\mathbb{D}_x^2$ to the equation

$$y(x(s))=\mathcal{E}\,(x(s);a)\mathcal{E}\,(x(s);b)$$

and use the product rules (15) and (16) to derive the following five equations

$$\begin{cases} X_{0,0} & =y\\ bX_{1,0}+aX_{0,1} & =\mathbb{D}_x\,y\\ abU_2X_{0,0}+X_{1,1} & =\mathbb{S}_x\,y\\ 2a^2b^2U_2U_1X_{0,0}+a(2\alpha\,b^2U_2+1)X_{1,0}+b(2\alpha\,b^2U_2+1)X_{0,1}+2abU_1X_{1,1} & =\mathbb{S}_x\mathbb{D}_x\,y\\ (a^2+2\alpha\,a^2b^2U_2+b^2)X_{0,0}+2ab^2U_1X_{1,0}+2a^2bU_1X_{0,1}+2\alpha\,abX_{1,1} & =\mathbb{D}_x^2y \end{cases}$$

with $X_{0,0}=\mathcal{E}\,(x(s);a)\mathcal{E}\,(x(s);b)$, $X_{1,0}=\mathbb{S}_x\mathcal{E}\,(x(s);a)\mathcal{E}\,(x(s);b)$, $X_{0,1}=\mathcal{E}\,(x(s);a)\mathbb{S}_x\mathcal{E}\,(x(s);b)$ and $X_{1,1}=\mathbb{S}_x\mathcal{E}\,(x(s);a)\mathbb{S}_x\mathcal{E}\,(x(s);b)$. The above system contains 5 linear equations for 4 unknowns, namely $X_{j,k},\ j,k=0,1$. For the solution of this system to exist, it is necessary for $y(x(s))$ to satisfy the determinant condition

$$\begin{vmatrix} 1 & 0 & 0 & 0 & y\\ 0 & b & a & 0 & \mathbb{D}_x\,y\\ abU_2 & 0 & 0 & 1 & \mathbb{S}_x\,y\\ 2a^2b^2U_2U_1 & a(2\alpha\,b^2U_2+1) & b(2\alpha\,a^2U_2+1) & 2abU_1 & \mathbb{S}_x\mathbb{D}_x\,y\\ (a^2+2\alpha\,a^2b^2U_2+b^2) & 2ab^2U_1 & 2a^2bU_1 & 2\alpha\,ab & \mathbb{D}_x^2y \end{vmatrix}=0$$

which is the required second-order divided-difference equation.

Let us mention that computations in the above proof have been made by using the Maple software system (see [16])

Proof (of Theorem 9). According to Proposition 7, for a given a, if there is b such that

$$\mathcal{E}\left(x(s);a\right)\mathcal{E}\left(x(s);b\right)=1,\ \forall\, x(s) \tag{73}$$

then b is solution of

$$a^2+2\alpha\, a\, b+b^2=0. \tag{74}$$

For q-lattices, $\alpha=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2}$ and therefore, (74) gives $b=-q^{\frac{1}{2}}a$ or $b=-q^{-\frac{1}{2}}a$.

If $b=-q^{\frac{1}{2}}a$, (73) becomes (72). If $b=-q^{-\frac{1}{2}}a$, we have from (73)

$$\mathcal{E}\left(x(s);\ a\right)\mathcal{E}\left(x(s);\ -q^{-\frac{1}{2}}a\right)=1.$$

By taking $-q^{\frac{1}{2}}a$ for a, we transform the last relation into (72). For the linear and quadratic lattices, $\alpha=1$ and the solution to (74) is $b=-a$.

Proposition 8

$$\begin{aligned}\mathcal{E}(x;a)&=\mathcal{E}(-x;-a),\quad \mathcal{E}(-x;a)=\mathcal{E}(x;-a),\\ C(-x;a)&=C(x;-a)=C(x;a),\\ S(-x;a)&=S(x;-a)=-S(x;a).\end{aligned}$$

Proof If we substitute x by $-x$ in the relations (63), (70) and (71), and use the symmetry properties $K_n(-x(s))=(-1)^nK_n(x)$ (see (30)), we get the result.

Proposition 9 *(Fundamental Relations of Trigonometry)*

1. *For the q-quadratic lattice $x(s)=c_1q^{-s}+c_2q^s$, we have*

$$C(x;a)C(x;q^{\frac{1}{2}}a)+S(x;a)S(x;q^{\frac{1}{2}}a)=1.$$

2. *For the linear or quadratic lattice $x(s)=c_1s^2+c_2s$, we have*

$$C(x;a)^2+S(x;a)^2=1.$$

7.2 Positivity of the exponential function

Theorem 10 *For the q-quadratic lattice $x(s)=c_1q^{-s}+c_2q^s$, $c_1c_2>0$, the function $\mathcal{E}(x(s);\ a)$, $0<q\neq 1$, has*

1. *The positivity property*

$$\mathcal{E}(x;a)>0,\quad \forall x\in\mathbb{R}, \tag{75}$$

2. *The limit properties*

$$\lim_{x\to\infty}\mathcal{E}(x;a)=+\infty,\quad \lim_{x\to\infty}\mathcal{E}(-x;a)=0,\quad (a>0). \tag{76}$$

Proof We split the proof of the relation (75) into two steps:

For the first step, we prove that $ax\geq\ 0\Rightarrow\ \mathcal{E}(x;a)>0$.
By using the fact that $q^{2s_0}=-\frac{c_1}{c_2}$ into $x_{2j}(s_0)^2$ and $x_{2j+1}(s_0)^2$ we have

$$x_{2j}(s_0)^2=-c_1c_2(q^j-q^{-j})^2 \text{ and } x_{2j+1}(s_0)^2=-c_1c_2(q^{j+\frac{1}{2}}-q^{-j-\frac{1}{2}})^2,$$

thus, from (32) and (33), we obtain

$$K_{2n}(x(s)) = x(s)^2 \prod_{j=1}^{n-1} [x(s)^2 + c_1c_2(q^j - q^{-j})^2],$$

$$K_{2n+1}(x(s)) = x(s) \prod_{j=1}^{n-1} [x(s)^2 + c_1c_2(q^{j+\frac{1}{2}} - q^{-j-\frac{1}{2}})^2].$$

Taking these relations into account in

$$\mathcal{E}(x;a) = 1 + ax(s) + \frac{(ax(s))^2}{\gamma_2!} + \sum_{n=2}^{+\infty} \frac{K_{2n}(x(s))}{\gamma_{2n}!}a^{2n} + \sum_{n=1}^{+\infty} \frac{K_{2n+1}(x(s))}{\gamma_{2n+1}!}a^{2n+1},$$

obtained from (63), yields

$$\begin{aligned}\mathcal{E}(x;a) &= 1 + ax(s) + \frac{(ax(s))^2}{\gamma_2!} + \sum_{n=2}^{+\infty} \frac{x(s)^2 \prod_{j=1}^{n-1}[x(s)^2 + c_1c_2(q^j - q^{-j})^2]}{\gamma_{2n}!}a^{2n} \\ &+ \sum_{n=1}^{+\infty} \frac{x(s) \prod_{j=1}^{n-1}[x(s)^2 + c_1c_2(q^{j+\frac{1}{2}} - q^{-j-\frac{1}{2}})^2]}{\gamma_{2n+1}!}a^{2n+1}. \qquad (77)\end{aligned}$$

Now, using the hypothesis ($ax \geq 0$), the latter equation leads us to $\mathcal{E}(x;a) > 0$.

For the second step, we prove that $ax < 0 \Rightarrow \ \mathcal{E}(x;a) > 0$.

$$ax < 0 \Rightarrow (a < 0 \text{ and } x > 0) \text{ or } (a > 0 \text{ and } x < 0).$$

Let us suppose that $a < 0$ and $x > 0$. From Theorem 9, $\frac{1}{\mathcal{E}(x;a)} = \mathcal{E}(x;b)$, with $b = -q^{\frac{1}{2}}a$ or $b = -q^{-\frac{1}{2}}a$ since $a < 0$, and $q > 0$, $bx > 0$ hence $\mathcal{E}(x;b) > 0$. and therefore, $\mathcal{E}(x;a) > 0$.

In a similar way, we prove that $\mathcal{E}(x;a) > 0$ for ($a > 0$ and $x < 0$). Let us prove (76). Since $a > 0$, if x tends to ∞ in (77), we get $\lim_{x\to\infty} \mathcal{E}(x;a) = +\infty$.

For $\lim_{x\to\infty} \mathcal{E}(-x;a) = 0, \quad a > 0$, the use of Proposition 8 ($\mathcal{E}(-x;a) = \mathcal{E}(x;-a)$) and Theorem 9 allows us to obtain

$$\mathcal{E}(-x;a) = \frac{1}{\mathcal{E}(x;b)} \text{ with } b = q^{\frac{1}{2}}a \text{ or } b = q^{-\frac{1}{2}}a.$$

Therefore, since $q^{-\frac{1}{2}}a > 0$ and $q^{\frac{1}{2}}a > 0$, we obtain the result by using the fact that $\lim_{x\to\infty} \mathcal{E}(x;b) = +\infty,\ b > 0$.

Corollary 14 *The basic exponential function* $\mathcal{E}_q(x,w)$ ($x(s) = \frac{q^{-s}+q^s}{2}$, $q^s = e^{i\theta}$) $(0 < q < 1)$ *has*

1. *The positivity property*

$$\mathcal{E}_q(x;\ w) > 0, \qquad (78)$$

2. *The limit properties*

$$\lim_{x\to\infty} \mathcal{E}_q(x;\ w) = +\infty, \quad \lim_{x\to\infty} \mathcal{E}_q(-x;\ w) = 0, \quad w > 0. \qquad (79)$$

Proof The Askey-Wilson lattice ($x(s) = \frac{q^{-s}+q^s}{2}$, $q^s = e^{i\theta}$) is q-quadratic $x(s) = c_1q^{-s} + c_2q^s$ with $c_1 = c_2 = \frac{1}{2}$ (that is, $c_1c_2 > 0$). Moreover, by Remark 3, $\mathcal{E}_q(x;w) = \mathcal{E}(x;a)$ with $a = \frac{2q^{\frac{1}{4}}w}{1-q}$. Therefore we can deduce (78) from (75) and (79) from (76).

Proposition 10 $y(x) = \mathbb{S}_x\mathcal{E}(x;a)$ *satisfies the second order divided-difference equation*

$$\mathbb{D}_x^2 y - 2a\alpha\, \mathbb{D}_x\, y + a^2 y = 0. \qquad (80)$$

Proof Since $\mathbb{D}_x\,\mathcal{E}(x;a) = a\mathcal{E}(x;a)$, $\mathbb{S}_x\mathbb{D}_x\mathcal{E}(x;a) = a\mathbb{S}_x\mathcal{E}(x;a)$. Taking into account (18) in the latter equation, we have

$$\mathbb{D}_x\mathbb{S}_x\mathcal{E}(x;a) - \alpha\, a\mathbb{S}_x\mathcal{E}(x;a) - a^2U_1\mathcal{E}(x;a) = 0. \tag{81}$$

Applying $\mathbb{D}_x$ to both sides of the above equation, the use of the relation (15) and the fact that $\mathbb{D}_xU_1(x(s)) = (\alpha^2-1)$ and $\mathbb{S}_xU_1(x(s)) = \alpha\, U_1(x(s))$ (obtained by direct computation) we transform (81) into (80) with $y = \mathbb{S}_x\mathcal{E}(x;a)$.

Corollary 15

1. *For the q-quadratic lattice we have*

$$\mathbb{S}_x\mathcal{E}(x;a) = \frac{(w^2;q)_\infty}{(w^2q;q)_\infty}\frac{\mathcal{E}(x;q^{\frac{1}{2}}a)+\mathcal{E}(x;q^{-\frac{1}{2}}a)}{2}.$$

2. *For the linear lattice*

$$\mathbb{S}_x\mathcal{E}(x;a) = \left(1+\frac{a^2}{4}\right)\mathcal{E}(x;a).$$

Proof From Proposition 10, $\mathbb{S}_x\mathcal{E}(x;a)$ is a solution to (80). Looking for a solution of (80) of the form $y(x) = \mathcal{E}(x;r)$, we obtain $y(x) = A\mathcal{E}(x;a)$ if $q=1$ and $y(x) = B\mathcal{E}(x;aq^{\frac{1}{2}}) + C\mathcal{E}(x;aq^{-\frac{1}{2}})$ if $q\neq 1$. From the expansion of $\mathcal{E}(x;a)$ on q-quadratic lattices $x(s) = c_1q^{-s}+c_2q^s$ (resp. on a linear lattice $x(s) = s$) in the basis K_n (see (66), (resp. (67))) and the relations (34), one has $A = \left(1+\frac{a^2}{4}\right)$, $B = C = \frac{(w^2;q)_\infty}{2(w^2q;q)_\infty}$.

7.2.1 Addition formulae for the exponential and trigonometric functions on nonuniform lattices

Here we provide, based on the binomial theorem 6, an addition theorem for the exponential function $\mathcal{E}(x;a)$. Let $K_n(x(z),x(s))$ be the bivariate function on q-quadratic, q-linear and linear lattices

$$K_n(x(z),\,x(s)) = \sum_{j=0}^{n}\frac{\gamma_n!}{\gamma_{n-j}!\gamma_j!}K_{n-j}(x(z))K_j(x(s)). \tag{82}$$

Proposition 11 *$K_n(x(z),x(s))$ has the following properties*

$$K_n(x\,,y) = K_n(y\,,x),\qquad K_n(x\,,0) = K_n(x),\qquad \mathbb{D}_x\,K_n(x\,,y) = \mathbb{D}_y\,K_n(x\,,y).$$

Theorem 11 *(Addition theorem for the exponential function)*

1. *On q-quadratic lattices $x(s) = c_1q^{-s}+c_2q^s$, we have*

$$\mathcal{E}\,(x(z);a)\mathcal{E}(x(s);-a) = \zeta_0(a)\zeta_0(-a)\sum_{n=0}^{\infty}\frac{[x(z)-x_{n-1}(s)]^{(n)}}{\gamma_n!}a^n. \tag{83}$$

2. *On q-quadratic lattices $x(s) = c_1q^{-s}+c_2q^s$ and linear lattice s, we have*

$$\mathcal{E}\,(x(z);a)\mathcal{E}\,(x(s);a) = \sum_{n=0}^{\infty}\frac{K_n(x(z),x(s))}{\gamma_n!}a^n. \tag{84}$$

Proof (83) is a direct consequence of (64) and the binomial theorem (54) while (84) is due to (63) and (82).

By taking $\mathcal{E}\,(x(z),\,x(s);a) = \mathcal{E}\,(x(z);a)\mathcal{E}\,(x(s);a)$ as analog of $e^{x+y} = e^xe^y$, we obtain

Corollary 16 *On q-quadratic lattices, we have*

1.

$$\mathcal{E}\,(x(z),\,x(s);a)=\frac{(w^2;q^2)_\infty}{(w^2q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{[x(z)+x_{n-1}(s)]^{(n)}}{\gamma_n!}a^n,$$
$$=\sum_{n=0}^{\infty}\frac{K_n(x(z),x(s))}{\gamma_n!}a^n.$$

2. $K_n(x(z),x(s))$ *can be expanded in terms of* $[x(z)+x_{n-1}(s)]^{(n)}$ *as*

$$K_n(x(z),x(s))=\sum_{k=0}^{[\frac{n}{2}]}\frac{\gamma_n!(q^{-1};q^2)_k}{(q^2;q^2)_k}\frac{(c_1c_2(1-q)^2q^{\frac{1}{2}})^k}{\gamma_{n-2k}!}[x(z)+x_{n-2k-1}(s)]^{(n-2k)}.$$

8 Characterization of Symmetric orthogonal polynomials on nonuniform lattices

Symmetric orthogonal polynomials play an important role in applications such as Numerical Analysis [12] and in Optics [4]. Orthogonal polynomials of the linear discrete variable have been determined (see [3]) from the three-term recurrence equation

$$P_{-1}(x):=0,\;P_0(x):=1,P_{n+1}(x)=(x-B_n)P_n(x)-C_nP_{n-1}(x).$$

In this section, based on the divided-difference equation [7]

$$\phi(x(s))\,\mathbb{D}_x^2y(x(s))+\psi(x(s))\,\mathbb{S}_x\mathbb{D}_xy(x(s))+\lambda\,y(x(s))=0, \tag{85}$$

where λ is a constant, ϕ and ψ are polynomials of degree at most two and one, respectively, for classical orthogonal polynomials on nonuniform lattices, we characterize symmetric orthogonal polynomials on nonuniform lattices.

Theorem 12 *A sequence of classical orthogonal polynomials on nonuniform lattice is symmetric if and only if their corresponding polynomial coefficients* ϕ *and* ψ *in the equation (85) are of the form*

$$\phi(x)=\phi_2x^2+\phi_0,\;\;\psi(x)=\psi_1x, \tag{86}$$

where $\phi_2,\ \phi_0$ *and* ψ_1 *are constant numbers.*

Proof Let (P_n) be a sequence of classical and symmetric polynomials on nonuniform lattices. Since $(P_n)_n$ is classical (see [8]) there are two polynomials

$$\phi(x))=\phi_2x^2+\phi_1x+\phi_0\;\;\text{and}\quad\psi(x)=\psi_1x+\psi_0$$

of degree at most two and of degree one, respectively, such that

$$\phi(x(s))\,\mathbb{D}_x^2P_n(x(s))+\psi(x(s))\,\mathbb{S}_x\mathbb{D}_xP_n(x(s))+\lambda\,P_n(x(s))=0.$$

Thanking $-x$ for x in this equation, we have

$$\phi(-x(s))\,\mathbb{D}_{-x}^2P_n(-x(s))+\psi(-x(s))\,\mathbb{S}_{-x}\mathbb{D}_{-x}P_n(-x(s))+\lambda\,P_n(-x(s))=0. \tag{87}$$

The following relations

$$\mathbb{D}_{-x}\,P_n(-x)=(-1)^{(n+1)}\mathbb{D}_xP_n(x),\quad\mathbb{S}_{-x}\,P_n(-x)=(-1)^{(n)}\mathbb{S}_xP_n(x),$$

obtained from the definitions of operators $\mathbb{D}_x$ and $\mathbb{S}_x$ as well as the symmetric property $P_n(-x)=(-1)^nP_n(x)$, and the second use of the symmetry, transform the second order equation (87) into

$$\phi(-x(s))\,\mathbb{D}_x^2P_n(x(s))-\psi(-x(s))\,\mathbb{S}_x\mathbb{D}_xP_n(x(s))+\lambda\,P_n(x(s))=0.$$

By identification, we get the result.

Conversely if $(P_n)_n$ is a sequence of classical orthogonal polynomials such that their corresponding polynomial coefficients in (85) are of the form (86), then, by substituting $y(x(s))$ by $P_n(x)$ into (85) and then taking $-x$ for x in the obtained equation, we obtain

$$\phi(x(s))\mathbb{D}_{-x}^2 P_n(-x(s)) - \psi(x(s))\mathbb{S}_{-x}\mathbb{D}_{-x}P_n(-x(s)) + \lambda P_n(-x(s)) = 0$$

which can be easily transformed into

$$\phi(x(s))\mathbb{D}_x^2 P_n(-x(s)) + \psi(x(s))\mathbb{S}_x\mathbb{D}_x P_n(-x(s)) + \lambda P_n(-x(s)) = 0.$$

Therefore, $P_n(-x) = constant P_n(x)$. Since $K_j,\ j \in \mathbb{N}$, is symmetric, by identifying the coefficients of K_n we have $constant = (-1)^n$

From the above theorem and the polynomial coefficients of (86) for well known classical orthogonal polynomials on nonuniform lattices (see [10]), we deduce the following result:

Corollary 17

1. *The Askey-Wilson polynomials are symmetric if and only if* $b = -a$ *and* $d = -c$*;*
2. *The q-Racah polynomials are symmetric if and only if* ($\gamma = -\alpha$ *and* $\beta = \alpha$) *or* ($\gamma = -\frac{\alpha}{\delta}$ *and* $\beta = \frac{\alpha}{\delta^2}$)*;*
3. *The Dual Hahn polynomials are symmetric if and only if* $\gamma = -1$*;*
4. *Continuous q-Jacobi polynomials are symmetric if and only if* $\alpha = \beta$*;*
5. *Continuous q-Hermite polynomials are symmetric.*

The following theorem allows us to expand in the basis K_n a symmetric classical orthogonal polynomials sequence.

Theorem 13 *If*

$$y(x(s)) = \sum_{n=0}^{\infty} d_n\, K_n(x(s)) \tag{88}$$

is a series solution of the equation (85) with polynomial coefficients of the form (86), the coefficients $(d_n)_n$ *satisfy a second-order difference equation*

$$A_n d_{n+2} + B_n d_n = 0, \tag{89}$$

where

$$A_n = \gamma_{n+2}\gamma_{n+1}\phi(x_n(s_0)) + \psi_1\gamma_{n+2}(\gamma_n x_{n+1}(s_0)^2 - \alpha\gamma_{n+1}x_n(s_0)^2)$$
$$B_n = \gamma_n\left(\phi_2\gamma_{n-1} + (\gamma_n - \alpha\,\gamma_{n-1})\psi_1\right) + \lambda.$$

Proof Firstly, we introduce

$$y(x(s)) = \sum_{n=0}^{\infty} d_n\, K_n(x(s))$$

into the equation (85). Secondly, we use the relation

$$\mathbb{D}_x^2 K_n = \gamma_n\gamma_{n-1}K_{n-2}$$

as well as the relations

$$x(s)^2 K_n = K_{n+2} + x_n(s_0)^2 K_n$$
$$x(s)\mathbb{S}_x\, K_n = (\gamma_{n+1} - \alpha\gamma_n)K_{n+1}(x(s)) + (\gamma_{n-1}x_n(s_0)^2 - \alpha\gamma_n x_{n-1}(s_0)^2)K_{n-1}(x(s))$$

(due to (32)-(34)) to transform the obtained equation into

$$\sum_{n=0}^{\infty}\left(A_n d_{n+2} + B_n d_n\right)K_n(x(s)) = 0.$$

The proof is completed by using the fact that K_n is a basis.

The Askey-Wilson polynomials [6]

$$P_n(x;a,b,c,d|q)=\sum_{k=0}^{n}\frac{(q^{-n},q)_k\,\left(a\,b\,c\,d\,q^{n-1},q\right)_k\,(a\,q^{s},q)_k\,(a\,q^{-s},q)_k}{(a\,b,q)_k\,(a\,c,q)_k\,(a\,d,q)_k}\frac{q^k}{(q,q)_k},\;x=\cos\theta, \tag{90}$$

satisfy (85) with [10,7]

$$\begin{aligned}\phi\left(x(s)\right)&=2\left(dcba+1\right)x^2\left(s\right)-\left(a+b+c+d+abc+abd+acd+bcd\right)x\left(s\right)\\&\quad+ab+ac+ad+bc+bd+cd-abcd-1,\\\psi\left(x(s)\right)&=\frac{4\left(abcd-1\right)q^{\frac12}\,x\left(s\right)}{q-1}+\frac{2\left(a+b+c+d-abc-abd-acd-bcd\right)\,q^{\frac12}}{q-1}\end{aligned} \tag{91}$$

and

$$\lambda\,=\lambda_n=-4\,\frac{\left(-1+q^n\right)\left(-q+abcdq^n\right)\sqrt{q}}{\left(q-1\right)^2q^n}. \tag{92}$$

By Corollary 17, these polynomials are symmetric if and only if $b=-a$ and $d=-c$. So, from the above theorem, we have:

Corollary 18

$$\begin{aligned}P_{2n}(x(s)\,;a\,,-a\,,c\,,-c|q)&=a_n\sum_{j=0}^{n}\frac{(q^{-2n}\,;q^2)_j(a^2c^2q^{2n-1}\,;q^2)_j\left(4\,q^2\right)^j q^{2\binom{j}{2}}}{(q\,;q^2)_j(-a^2\,;q^2)_j(-c^2;,q^2)_j(q^2\,;q^2)_j}K_{2j}(x(s)),\\&=q_n(x(s)^2).\end{aligned}$$

$$\begin{aligned}P_{2n+1}(x(s)\,;a\,,-a\,,c\,,-c|q)&=b_n\sum_{j=0}^{n}\frac{(q^{-2n}\,;q^2)_j(a^2c^2q^{2n+1}\,;q^2)_j\left(4\,q^3\right)^j q^{2\binom{j}{2}}}{(-a^2q\,;q^2)_j(-c^2q;,q^2)_j(q^3\,;q^2)_j(q^2\,;q^2)_j}K_{2j+1}(x(s)),\\&=x(s)r_n(x(s)^2).\end{aligned}$$

where $a_n=\frac{(-c^2;q^2)_n(q;q^2)_n(-a^2)^n}{(ac;q)_n(-ac;q)_n(-a^2q;q^2)_n}$ *and* $b_n=\frac{2(q;q^2)_n(-qc^2;q^2)_n(-1)^n(a^n)^2(-1+q^{2n+1})a}{(-1+q)(a^2c^2;q^2)_n(-a^2;q^2)_n(a^2(q^n)^2+1)q^n}$.

9 The divided-difference equation for the function ${}_4\phi_3$

The non-terminating hypergeometric function ${}_2F_1$ and the non-terminating ${}_2\phi_1$ satisfies respectively a second order differential equation and a second order q-difference equation (see [11]). In this section, we prove that the non-terminating Askey-Wilson ${}_4\phi_3$ satisfies the second order divided-difference equation (85).

Theorem 14 *The non-terminating Askey-Wilson function* ${}_4\phi_3$*,* $0<q<1$*,*

$${}_4\phi_3\left(\begin{matrix}t,abcdq^{-1}t^{-1},ae^{i\theta},ae^{-i\theta}\\ab,ac,ad\end{matrix};q,q\right) \tag{93}$$

satisfies the divided-difference equation (85), where coefficients ϕ*,* ψ *are those of the Askey-Wilson polynomials (see (91)) and* λ *is :*

$$\lambda\,=4\,\frac{\left(tcdba-abcd+tq-qt^2\right)\sqrt{q}}{\left(-1+q\right)^2t}. \tag{94}$$

Proof We assume that the non-terminating Askey-Wilson ${}_4\phi_3$, given by (93), satisfies (85). So, writing

$$
{}_4\phi_3\left(\begin{array}{c} t, abcdq^{-1}t^{-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{array}; q, q\right) = \sum_{n=0}^{+\infty} d_n B_n(a,\ x(s))
$$

where

$$
d_n = \frac{(t,q)_n\ \left(a\,b\,c\,d\,q^{n-1},q\right)_n}{(a\,b,q)_n\ (a\,c,q)_n\ (a\,d,q)_n}\frac{q^n}{(q,q)_n},\quad B_n(a,\ x(s)) = (aq^s, aq^{-s};q)_n,
$$

we obtain from [7] (Theorem 11, page 422)

$$
\sum_{k=0}^{8} H_k(\phi_2,\phi_1,\phi_0,\psi_1,\psi_0,\lambda)\, q^{kn} = 0,
$$

where the $H_k(\phi_2,\phi_1,\phi_0,\psi_1,\psi_0,\lambda)$ are linear combinations of λ and the coefficients of ϕ and ψ. Solving the system of linear equations $H_k(\phi_2,\phi_1,\phi_0,\psi_1,\psi_0,\lambda) = 0,\ 0 \le k \le 8$ in terms of λ and the coefficients ϕ_j and ψ_j, we obtain, up to a multiplicative factor, λ (see (94)) and the coefficients of the polynomials given in (91).

Remark 4 Solving the divided-difference equation (85) (by using [7], Theorem 18) with coefficients given by (91) and (94) we recover (93).

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