Zeros of irreducible characters of finite groups

by

Sesuai Yash Madanha

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Declaration

I, Sesuai Yash Madanha, declare that the thesis, which I hereby submit for the degree Doctor of Philosophy at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Dedication

To Mandisa, Nene and my unborn baby

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Title Zeros of irreducible characters of finite groups

Name Sesuai Yash Madanha

Supervisor Prof. Hung Phi Tong-Viet (Binghamton University, New York, USA)

Co-Supervisor Prof. John Eric van den Berg

Co-Supervisor Dr. Tung Thien Le

Department Mathematics & Applied Mathematics

Degree Doctor of Philosophy

Abstract

This work is a contribution to the classification of finite groups with an irreducible character that vanishes on exactly one conjugacy class. Specifically, in this thesis we study finite non-solvable groups G that satisfy the above property when the character is primitive. We show that G has a homomorphic image that is either an almost simple group or a Frobenius group. We then classify all finite non-solvable groups with a faithful primitive irreducible character that vanishes on one conjugacy class. Our results answer two questions of Dixon and Rahnamai Barghi, one partially and the other completely.

A classical theorem of Burnside states that every irreducible character whose character degree is divisible by a prime number vanishes on at least one conjugacy class. Our results imply that if the degree of a primitive irreducible character of a finite group is divisible by two distinct primes, then the character vanishes on at least two conjugacy classes except when the group has a composition factor isomorphic to the Suzuki group ${}^2\mathrm{B}_2(8)$. This is an extension of Burnside's Theorem. Motivated by our result above we show that for M-groups, groups of odd order and groups of derived length at most 3, if the character degree of an irreducible character of a group is divisible by two distinct primes, then the irreducible character vanishes on at least two conjugacy classes.

For nilpotent groups, metabelian groups and groups whose distinct character degrees are pairwise relatively prime, we show that if the character degree of an irreducible character of a group is divisible by n distinct primes, then the irreducible character vanishes on at least n conjugacy classes for any positive integer n. This also holds when the group is solvable and the irreducible character is primitive.

Nomenclature

$\mathrm{Irr}(G)$	The set of irreducible characters of a finite group G
$v(\chi)$	The set of all vanishing elements of χ
$nv(\chi)$	The number of classes on which χ vanishes
$\ker\chi$	The kernel of the character χ
$Z(G)$	The centre of G
$Z(\chi)$	The centre of the character χ
$\mathrm{Aut}(G)$	The automorphism group of G
$\mathrm{Out}(G)$	The outer automorphism group of G
$\mathcal{C}_x$	The conjugacy class containing x
χ_M	The restriction of χ on M
$H \leqslant G$	H is a subgroup of H
$m \leq n$	m is less than or equal to n
$\chi(1)$	The character degree of χ
$\mathbf{N}_G(X)$	The normalizer of subset X in G
$\mathbf{C}_G(x)$	The centralizer of x in G
$\lvert G\rvert$	The order of G
$\langle X\rangle$	The subgroup generated by the subset X

$\gcd(a,b)$	The greatest common divisor of a and b
S_n	The symmetric group of degree n
A_n	The alternating group of degree n
G_α	The point stabilizer of α
α^G	The orbit containing α
D_n	The dihedral group of order n
$G{:}n$	The semidirect product of G with a group of order n
$G \rtimes H$	The semidirect product G with H
$H \lhd G$	H is a normal subgroup of G
G'	The derived subgroup of G
$\tilde{G}$	The Schur cover of G
$M(G)$	The Schur multiplier of G
$\mathcal{M}$	An algebraic group $\mathcal{M}$
$\mathcal{M}^\circ$	The connected component of $\mathcal{M}$
$\mathcal{M}^F$	Finite group of Lie type
χ^G	The induced character of G
$\|G : H\|$	The index of H in G
G^∞	Solvable residual of G
$\mathrm{cd}(G)$	The character degree set of G
$\mathrm{dl}(G)$	The derived length of G
$\mathbb{C}$	The complex number field
$\overline{\mathbb{F}}_p$	The algebraic closure of a finite field of characteristic p

$\mathbb{Q}$	The field of rational numbers
$\vert g\vert$	The order of the element g
$\Phi(G)$	The Frattini subgroup of G
1_G	The identity element of G
$tr(B)$	The trace of the matrix B
$\Phi_n(x)$	The n^{th} cyclotomic polynomial

Contents

Chapter 1

Overview

The study of zeros of irreducible characters has had some applications in representation theory and has also played some role in our understanding of the structure of finite simple groups. In [MSW94], Malle, Saxl and Weigel showed that finite simple classical groups (with one exception) are generated by three involutions using an understanding of zeros of characters of these finite simple groups. In [LM15, LMS16, LM16], the authors classified simple endotrivial modules for quasisimple groups. They used the fact that the corresponding character of the endotrivial module cannot have the value zero for some elements in the quasisimple groups. Recently, using the so-called Steinberg like characters, Malle and Zaleskii [MZ18] classified projective indecomposable modules for finite non-abelian simple groups, again by studying the zeros of characters of finite non-abelian simple groups.

One of the most interesting problems in character theory is determining the structure of a finite group using information given in the character table of that finite group. Many authors have studied the zero entries in a character table of a finite group and their influence on the structure of that finite group and its subgroups (see[Dea90, BCG00, MS04b, MS04a, QZ05, ZS08, ZSS10, ZSW13, TTV18]). We shall list some of the results here. In [Dea90], Deaconescu gave a sufficient condition in terms of zero entries in a row of a character table for the Frattini group of a finite group to be non-trivial. Moretó and Sangroniz [MS04a] bounded the Fitting height of a solvable using the largest number of of zero entries in a row of its character table. The same authors [MS04b] bounded the derived length of a solvable using the largest number of of zero entries in a column of its character table. In [BCG00], Bianchi, Chillag and Gillio classified finite groups in whose character has at most two zeros entries in every row. In 2010, Zhang, Shi and Shen [ZSS10] classified finite groups in whose character has at most three zeros entries in every row. We are particularly interested in a more general problem than ones studied in [BCG00] and [ZSS10]. We study finite groups whose character table has a single zero entry in one of its rows.

Let $\mathrm{Irr}(G)$ be the set of all irreducible characters of a finite group G and let $\chi \in \mathrm{Irr}(G)$. A well-known theorem of Burnside states:

Theorem 1.0.1. *[Isa06, Theorem 3.15] Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$ be non-linear. Then $\chi(g) = 0$ for some $g \in G$.*

Let $\chi \in \mathrm{Irr}(G)$ and define

$$\upsilon(\chi) := \{x \in G \mid \chi(x) = 0\}.$$

Hence $\upsilon(\chi) \neq \emptyset$ for non-linear $\chi \in \mathrm{Irr}(G)$. Let

$$n\upsilon(\chi) = \text{the number of conjugacy classes on which } \chi \text{ vanishes.}$$

Since χ is invariant on conjugacy classes, χ vanishes on at least one conjugacy class, that is, $n\upsilon(\chi) \geq 1$ for non-linear $\chi \in \mathrm{Irr}(G)$. Malle, Navarro and Olsson [MNO00] generalised Burnside's Theorem by showing that we can choose the element to be of prime power order:

Theorem 1.0.2. *[MNO00, Theorem B] Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$ be non-linear. Then there exists $g \in G$ of prime-power order such that $\chi(g) = 0$.*

Many authors have studied finite groups G with a non-linear irreducible character χ with the extremal property that $n\upsilon(\chi) = 1$. Theorem 1.0.2 implies that this conjugacy class contains elements of prime-power order. Zhmud' [Zhm79] was the first to study them. Chillag [Chi99, Corollary 2.4] showed that either $\chi_{G'}$ is irreducible or G is a Frobenius group with a Frobenius complement of order 2 and an abelian Frobenius kernel of odd order. Dixon and Rahnamai Barghi [DRB07, Theorem 9] obtained some partial results when G is solvable and Qian [Qia07] characterised finite solvable groups with this extremal property. Recently, Burness and Tong-Viet [BTV15] studied these groups when χ is imprimitive, being induced from an irreducible character of a maximal subgroup of G.

Dixon and Rahnamai Barghi [DRB07] posed some questions at the end of their paper. Among them were the following:

Question 1. *If G is a finite non-solvable group with an irreducible character χ such that $nv(\chi) = 1$, can G have more than one non-abelian composition factor?*

Question 2. *Let G be a finite non-abelian simple group and let $\chi \in \mathrm{Irr}(G)$. Is it true that if $nv(\chi) = 1$, then one of the following holds:*

(a) $G \cong \mathrm{PSL}_2(5)$, $\chi(1) = 3$;

(b) $G \cong \mathrm{PSL}_2(7)$, $\chi(1) = 3$;

(c) $G \cong \mathrm{PSL}_2(2^a)$, $\chi(1) = 2^a$, *where* $a \geq 2$?

In this thesis we partially answer Question 1 and completely answer Question 2. In order to do so we investigate finite non-solvable groups with a primitive irreducible character that vanishes on a unique conjugacy class. (Refer to Section 2.3 for the definitions of primitive and imprimitive characters.) In particular, we shall establish a reduction theorem:

Theorem 1.0.3. *Let G be a finite non-solvable group. Suppose that $\chi \in \mathrm{Irr}(G)$ is primitive, $nv(\chi) = 1$ and $v(\chi) = \mathcal{C}$. Let $K = \ker\chi$, $Z = Z(\chi)$. Then there exists a normal subgroup M of G such that $Z < M$, $\mathcal{C} \subseteq M \backslash Z$ and M/Z is the unique minimal normal subgroup of the group G/Z. Moreover, one of the following holds:*

(a) G/Z is almost simple and M/K is quasisimple.

(b) G/Z is a Frobenius group with an abelian Frobenius kernel M/Z of order p^{2n}, M/K is an extra-special p-group and Z/K is of order p with K non-solvable.

For case (a) in Theorem 1.0.3 assume that $K = 1$, that is, χ is faithful. Then G/Z is almost simple with socle M/Z where M is quasisimple. Note that χ_M is irreducible and if $\mathcal{C}$ is the unique conjugacy class of zeros of χ in G, then $\mathcal{C}$ is the union of M-conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$ with $r \leq |G : M| = |G/Z : M/Z| \leq |\mathrm{Out}(M/Z)|$. Observe that all zeros of χ_M have the same order which is a power of p for some prime p. Also note that $Z(G) = Z(M)$.

We thus look at this general problem:

Problem 1. *For each quasisimple group M, classify all faithful characters $\chi \in \mathrm{Irr}(M)$ such that:*

(a) χ vanishes on elements of the same p-power order;

(b) the number of conjugacy classes that χ vanishes on is at most the size of the outer automorphism group of $M/Z(M)$;

(c) if $Z(M)$ is non-trivial, then $Z(M)$ is cyclic and of p-power order.

For convenience we shall say a faithful irreducible character χ of a finite group has

$$\text{property } (\star) \text{ if it possesses properties (a)-(c) of Problem 1} \tag{$\star$}$$

.

We completely solve Problem 1.

Theorem 1.0.4. *Let M be a quasisimple group. Suppose that M has a faithful irreducible character χ such that $(\star)$ holds. Then M is one of the following:*

(a) $M \cong \mathrm{PSL}_2(5)$, $\chi(1) = 3$ or $\chi(1) = 4$;

(b) $M \cong \mathrm{SL}_2(5)$, $\chi(1) = 2$ or $\chi(1) = 4$;

(c) $M \cong 3{\cdot}\mathrm{A}_6$, $\chi(1) = 9$;

(d) $M \cong \mathrm{PSL}_2(7)$, $\chi(1) = 3$;

(e) $M \cong \mathrm{PSL}_2(8)$, $\chi(1) = 7$;

(f) $M \cong \mathrm{PSL}_2(11)$, $\chi(1) = 5$ or $\chi(1) = 10$;

(g) $M \cong \mathrm{PSL}_2(q)$, $\chi(1) = q$, where $q \geq 5$;

(h) $M \cong \mathrm{PSU}_3(4)$, $\chi(1) = 13$;

(i) $M \cong {}^2\mathrm{B}_2(8)$, $\chi(1) = 14$.

Using Theorem 1.0.4, we classify finite non-solvable groups with a faithful primitive irreducible character that vanishes on one conjugacy class.

Theorem 1.0.5. *Let G be a finite non-solvable group. Then there exists $\chi \in \mathrm{Irr}(G)$ is faithful, primitive and $nv(\chi) = 1$ if and only if G is one of the following groups:*

(a) $G \cong \mathrm{PSL}_2(5)$, $\chi(1) = 3$ *or* $\chi(1) = 4$*;*

(b) $G \cong \mathrm{SL}_2(5)$, $\chi(1) = 2$ *or* $\chi(1) = 4$*;*

(c) $G \in \{\mathrm{A}_6{:}2_2,\ \mathrm{A}_6{:}2_3,\ 3{\cdot}\mathrm{A}_6{:}2_3\}$, $\chi(1) = 9$ *for all such* $\chi \in \mathrm{Irr}(G)$*;*

(d) $G \cong \mathrm{PSL}_2(7)$, $\chi(1) = 3$*;*

(e) $G \cong \mathrm{PSL}_2(8){:}3$, $\chi(1) = 7$*;*

(f) $G \cong \mathrm{PGL}_2(q)$, $\chi(1) = q$, *where* $q \geq 5$*;*

(g) $G \cong {}^2\mathrm{B}_2(8){:}3$, $\chi(1) = 14$.

Using Theorem 1.0.5 and [BTV15, Theorem 1.5] we partially answer Question 1:

Corollary 1.0.6. *If G is a finite group that has a faithful irreducible character χ such that $nv(\chi) = 1$, then G has at most one non-abelian composition factor.*

The result below follows easily from Theorem 1.0.5:

Corollary 1.0.7. *Let G be a finite non-abelian simple group and let $\chi \in \mathrm{Irr}(G)$. If $nv(\chi) = 1$, then one of the following holds:*

(a) $G \cong \mathrm{PSL}_2(5)$, $\chi(1) = 3$*;*

(b) $G \cong \mathrm{PSL}_2(7)$, $\chi(1) = 3$*;*

(c) $G \cong \mathrm{PSL}_2(2^a)$, $\chi(1) = 2^a$, *where* $a \geq 2$.

Corollary 1.0.7 positively answers Question 2.

We now look at what our results imply as regards to the classical Burnside's Theorem of zeros of characters.

There have been several generalizations of Burnside's Theorem. The first one, which we shall restate here, is that of Malle, Navarro and Olsson [MNO00]:

Theorem 1.0.8. *[MNO00, Theorem B] Let G be a finite group. Then every non-linear irreducible character of G vanishes on an element of prime power order.*

The generalization here is obvious since the result tells us that we can choose the element to be of prime power order. Another generalization is due to Navarro [Nav01].

Theorem 1.0.9. *[Nav01, Theorem A] Let G be a finite group. Let $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$. Then χ_N is not irreducible if and only if χ vanishes on some coset Nx in G.*

Let N be abelian. Since every irreducible character of N is linear, Theorem 1.0.9 implies that χ_N is not irreducible, that is, non-linear if and only if χ vanishes on some coset Nx of N in G. In particular, χ vanishes on some element x in G which is Burnside's Theorem.

[BZ99, Theorem 21.1] is the last generalization we will discuss. Recall that by Burnside's Theorem, $\upsilon(\chi) \neq \emptyset$ for $\chi \in \mathrm{Irr}(G)$ non-linear.

Theorem 1.0.10. *[BZ99, Theorem 21.1] Let $H \leqslant G$ and $\chi \in \mathrm{Irr}(G)$. Then*

$$[\chi_H, \chi_H] \leq 1 + \frac{|\upsilon(\chi) \setminus H|}{|H|}.$$

If $|\upsilon(\chi) \setminus H| < |H|$, then χ_H is irreducible. If $H = \{1\}$, then $[\chi_H, \chi_H] = 1$, that is, $\chi \in \mathrm{Irr}(G)$ vanishes on some element of G.

We propose a new generalization of Burnside's Theorem which gives a connection between the number of prime divisors of character degrees and the number of zeros of characters of a finite group. Burnside's Theorem can be rewritten as follows:

Theorem 1.0.11. ***(Burnside's Theorem)*** *Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$. If $\chi(1)$ is divisible by a prime, then χ vanishes on at least one conjugacy class.*

The above prompts us to ask a more general question:

Question 3. *Let G be a finite group, $\chi \in \mathrm{Irr}(G)$ and n a positive integer. Is it true that if $\chi(1)$ is divisible by n distinct primes, then χ vanishes on at least n conjugacy classes?*

The following is motivated by an implication of one of our main results (Theorem 1.0.5):

Theorem 1.0.12. *Let G be a finite group which has no composition factor isomorphic to ${}^2\mathrm{B}_2(8)$. Let $\chi \in \mathrm{Irr}(G)$ be primitive. If $\chi(1)$ is divisible by two distinct prime numbers, then χ vanishes on at least two conjugacy classes.*

We answer Question 3 in the affirmative for certain finite solvable groups when $n = 2$.

Theorem 1.0.13. *Let G be a finite solvable group and let $\chi \in \mathrm{Irr}(G)$ be non-linear. Suppose that one of the following conditions holds:*

(a) χ is monomial;

(b) G is of odd order;

(c) G has derived length at most 3;

(d) G has a normal Sylow 2-subgroup;

(e) G has a self-normalizing Sylow p-subgroup P and χ vanishes on p-elements for some prime p;

(f) Every maximal subgroup of G is an M-group.

If $\chi(1)$ is divisible by two distinct prime numbers, then χ vanishes on at least two conjugacy classes.

Our strategy is to use results on finite solvable groups with an irreducible character that vanishes on a unique conjugacy class. It is sufficient to show that the character degree of the corresponding character is necessarily of prime power order. Therefore our approach only shows existence of the conjugacy classes and does not tell us if the elements in the conjugacy classes have distinct orders or not. Hence we ask another question with a stronger property:

Question 4. *Let G be a finite solvable group, $\chi \in \mathrm{Irr}(G)$ and n a positive integer. Is it true that if $\chi(1)$ is divisible by n distinct prime numbers, then χ vanishes on at least n elements of pairwise distinct orders?*

This leads to another result.

Theorem 1.0.14. *Let G be a finite solvable group, $\chi \in \mathrm{Irr}(G)$ and n a positive integer. Suppose that one of the following conditions holds:*

(a) χ is primitive;

(b) G is nilpotent;

(c) G is metabelian.

If $\chi(1)$ is divisible by n distinct prime numbers, then χ vanishes on at least n elements of pairwise distinct orders.

However, the answer for Question 4 is negative for finite solvable groups. Our counterexample is [DPSS09, Example 4.2]: Let G be the normalizer of a Sylow 2-subgroup in the Suzuki group ${}^2\mathrm{B}_2(8)$. Then G is a Frobenius group such that the Frobenius complement is of order 7 and the Frobenius kernel is non-abelian. Furthermore, G has an irreducible character of degree is 14, that vanishes only on elements of order 7. Since $\mathrm{cd}(G) = \{1, 7, 14\}$, note that $|\mathrm{cd}(G)| = 3$ and $\gcd(7, 14) \neq 1$ where $\mathrm{cd}(G)$ denotes the character degree set of G. If the character degrees are pairwise relatively prime, then the answer to Question 4 is positive.

Theorem 1.0.15. *Let G be a finite solvable group, $\chi \in \mathrm{Irr}(G)$ and n a positive integer. Suppose that all distinct character degrees of G are pairwise relatively prime. If $\chi(1)$ is divisible by n distinct prime numbers, then χ vanishes on at least n elements of pairwise distinct orders.*

It turns out that for non-solvable groups the answer is also negative for both questions. A counterexample to Question 3 is ${}^2B_2(8){:}3$ which has an irreducible character of degree 14 and this vanishes on exactly one conjugacy class of elements of order 7. This is why we needed to exclude that case in Theorem 1.0.12. A counterexample to Question 4 for finite non-solvable groups is $\mathrm{PSL}_2(11)$ which has an irreducible character of degree 10, that vanishes only on elements of order 5. However, sporadic simple groups and alternating groups satisfy property to Question 3 for arbitrary n. In particular, we prove the following:

Theorem 1.0.16. *Let G be a finite almost simple group such that $S \trianglelefteq G \leqslant \mathrm{Aut}(S)$, where S is either an alternating group or a sporadic simple group. Let $\chi \in \mathrm{Irr}(G)$ and n a positive integer. If $\chi(1)$ is divisible by n distinct prime numbers, then χ vanishes on at least n elements of pairwise distinct orders.*

Theorem 1.0.13(a) and (b) show that Question 3 holds for M-groups and groups of odd order, respectively. At the time of writing, it is not known if Question 3 holds for general finite solvable groups.

The thesis is organized as follows. In Chapter 2 we present some preliminary results that will be needed to prove our main results. We also survey some known results on finite groups with an irreducible character that vanishes on a unique conjugacy class that other authors have proved.

In Chapter 3 we prove Theorems 1.0.3, 1.0.4 and 1.0.5. In Chapter 4 we finish off by proving Theorems 1.0.13, 1.0.14, 1.0.15 and 1.0.16. We describe the properties of a possible counterexample to Question 3 in Chapter 4. In Chapter 5 we conclude the thesis by proposing some possible future work.

NB Part of the work has been published in Communications in Algebra and is found in [Mad19b]. Some of the work has been submitted for consideration for publication and is found in [Mad19c] and [Mad19a].

Chapter 2

Preliminary results

In this chapter we shall present some preliminary results needed to prove our primary results in Chapter 3 and Chapter 4. Most results will be presented without proofs but with references.

2.1 Finite group theory

Let G be a finite group and let $x \in G$. We denote the order of G and x by $|G|$ and $|x|$, respectively. Denote the centralizer of x in G by $\mathbf{C}_G(x)$ and we denote $\mathcal{C}_x = x^G := \{g^{-1}xg \mid g \in G\}$, the conjugacy class containing x. The normalizer of a subset X in G is denoted by $\mathbf{N}_G(X)$. The following result shows the connection between $\mathcal{C}_x$ and $\mathbf{C}_G(x)$.

Lemma 2.1.1. *[Isa08, Corollary 1.5] Let $x \in G$, where G is a finite group, and let $\mathcal{C}_x$ be the conjugacy class containing x. Then $|\mathcal{C}_x| = |G : \mathbf{C}_G(x)|$.*

Let $x, y \in G$. The *commutator* of x and y is denoted by $[x, y] = x^{-1}y^{-1}xy$. If A and B are subsets of G, then

$$[A, B] = \langle [a, b] \mid a \in A, b \in B \rangle,$$

the subgroup generated by the commutators $[a, b]$. The *commutator subgroup* or the *derived subgroup* of G, denoted G', is defined as $G' = [G, G]$.

2.1.1 Permutation groups

In this section we shall state the O'Nan-Scott Theorem. We refer to [DM96] for basic results on permutation groups.

Let Ω be an arbitrary non-empty set. The set of all permutations of Ω (bijections of Ω onto itself) forms a group, under composition of mappings called the *symmetric group* on Ω, denoted by S_n, where $n = |\Omega|$. A *permutation group* is a subgroup of S_n and n is called the *degree of the permutation group*.

Let G be a group and Ω a non-empty set. A *group action* is a map from $\Omega \times G$ to Ω such that $\alpha^1 = \alpha$ and $(\alpha^g)^h = \alpha^{gh}$ for all $g, h \in G$ and α, where the image of (α, g) is denoted by α^g.

If we define a relation on Ω by

$$\alpha \backsim \beta \text{ if and if only there exists } g \in G \text{ such that } \alpha^g = \beta$$

then $\backsim$ is an equivalence relation and the corresponding equivalence classes are called *orbits*. The orbit containing α is denoted by

$$\alpha^G := \{\alpha^g \mid g \in G\}.$$

The *point stabilizer* of α is the subgroup G_α of G, defined by $G_\alpha := \{g \in G \mid \alpha^g = \alpha\}$.

The relation between orbits and stabilizers in given below:

Lemma 2.1.2. *(The Orbit-Stabilizer Property) $|\alpha^G| = |G : G_\alpha|$ for all $\alpha \in \Omega$. In particular, if G is finite, then $|\alpha^G||G_\alpha| = |G|$.*

Lemma 2.1.1 is a special case of the Orbit-Stabilizer Property when G acts on itself by conjugation with the conjugacy class containing an element x as an orbit and its centralizer as the stabilizer.

Let a group A act on another group G via automorphisms. Groups A and G may be viewed as subgroups of the semidirect product $\Gamma = G \rtimes A$. Hence the commutator $[G, A]$ can be calculated as a subgroup of Γ. We have the following result when G is abelian with $|A|$ and $|G|$ relatively prime.

Theorem 2.1.3. *[Isa08, Theorem 4.34] Let a group A act via automorphisms on an abelian group G and assume that A and G are finite and that $\gcd(|G|, |A|) = 1$. Then $G = \mathbf{C}_G(A) \times [G, A]$.*

A group G acting on a set Ω is said to be *transitive* if there is only one orbit, that is, $\alpha^G = \Omega$ for all $\alpha \in \Omega$, or, for all α, β there exists $g \in G$ such that $\alpha^g = \beta$. A transitive group G acting on a set is *regular* if $G_\alpha = \{1\}$ for every $\alpha \in \Omega$. If G is transitive and finite, then G is regular if and only if $|G| = |\Omega|$.

A *block* is a non-empty subset Γ of Ω such that for every $g \in G$, either $\Gamma = \Gamma^g$ or $\Gamma \cap \Gamma^g = \emptyset$. Ω and the singletons $\{\alpha\}$ are called *trivial blocks*. A transitive group G on a set Ω is *primitive* if Ω contains no non-trivial blocks.

Theorem 2.1.4. *[Isa08, Corollary 8.14] Let G be a group that acts transitively on a set Ω with $|\Omega| \geq 2$, and let $H = G_\alpha$, where $\alpha \in \Omega$. Then G is primitive on Ω if and only if H is a maximal subgroup of G.*

A transitive permutation group is called 2-*transitive* if G_α acts transitively on $\Omega\backslash\{\alpha\}$ for every $\alpha \in \Omega$. A 2-transitive permutation group is primitive. G is called *sharply* 2-*transitive* if G is 2-transitive and G acts regularly on the set of pairs of distinct elements of Ω.

The *socle* of a group G, denoted *soc*(G), is defined to be the subgroup generated by the set of all minimal normal subgroups of G. Recall that H is a *characteristic subgroup* of G if for all $\phi \in \mathrm{Aut}(G)$, $\phi(H) = H$. For example, *soc*(G) is a characteristic subgroup G. Note that a minimal normal subgroup of a finite group is a direct product of isomorphic simple groups. If G is a finite solvable group, then a minimal normal subgroup N of G is an elementary abelian p-group for some prime p, that is, N is an abelian group in which every non-trivial element has order p. A primitive permutation group has at most two minimal normal subgroups.

Theorem 2.1.5. *[DM96, Theorem 4.3B] If G is a finite primitive permutation group and K is a minimal normal subgroup of G, then exactly one of the following holds:*

(i) for some prime p and some integer d, K is a regular elementary abelian group of order p^d, and soc$(G) = K = \mathbf{C}_G(K)$;

(ii) K is a regular non-abelian group, $\mathbf{C}_G(K)$ is a minimal normal subgroup of G which is permutation isomorphic to K and soc$(G) = K \times \mathbf{C}_G(K)$;

(iii) K is non-abelian, $\mathbf{C}_G(K) = 1$ and soc$(G) = K$.

The O'Nan-Scott Theorem classifies all finite primitive permutation groups. We shall not reproduce the full statement of the O'Nan-Scott Theorem since we shall only need one of the cases of the statement in Chapter 3.

Theorem 2.1.6. *[DM96, Theorem 4.1A] (O'Nan-Scott Theorem) Let G be a primitive group of degree n and let H be the socle of G. Then one of the following holds:*

(a) H is a regular elementary abelian p-group for some prime p, $n = p^m = |H|$;

(b) H is a non-abelian simple group and $H \lhd G \leqslant \mathrm{Aut}(H)$, that is, G is an almost simple group;

(c) H is isomorphic to a direct product T^m of a non-abelian simple group T and $m \geq 2$.

2.1.1.1 Derangements in transitive permutation groups

Here we refer to a recent book of Burness and Giudici [BG16] for basic notions on derangements. Let G be a transitive permutation group acting on a non-empty set Ω and let $H = G_\alpha$ be the stabilizer of a point α. An element $x \in G$ is called a *derangement* if it fixes no point of Ω or equivalently, if $x^G \cap H$ is empty for all $\alpha \in \Omega$, where x^G is the conjugacy class of x in G. Denote the set of derangements in G by $\Delta(G)$. Then

$$\Delta(G) = G \setminus \textstyle\bigcup_{g \in G} H^g.$$

It turns out the existence of derangements in transitive permutation groups is guaranteed by an old result of Jordan [Jor72]:

Theorem 2.1.7. *(Jordan, 1872) Let G be a transitive permutation group on a finite set Ω with $|\Omega| \geq 2$. Then G contains a derangement.*

A generalization of Jordan's result shows that a finite transitive permutation group always contains a derangement of prime power order:

Theorem 2.1.8. *[FKS81] Let G be a transitive permutation group on a finite set Ω with $|\Omega| \geq 2$. Then G contains a derangement of prime power order.*

In [BTV15], Burness and Tong-Viet considered finite primitive permutation groups which contain one conjugacy class of derangements. By Theorem 2.1.8, the derangements will be of prime power order.

Theorem 2.1.9. *[BTV15, Theorem 1.1] Let G be a finite primitive permutation group with point stabilizer H. Then G contains one conjugacy class of derangements if and only if G is sharply 2-transitive, or $(G, H) = (\mathrm{A}_5, \mathrm{D}_{10})$ or $(\mathrm{PSL}_2(8){:}3, \mathrm{D}_{18}{:}3)$.*

Guralnick [Gur16] showed that primitivity in Theorem 2.1.9 is not necessary, transitivity is sufficient.

Theorem 2.1.10. *[Gur16, Theorem 1.1] Let G be a finite transitive permutation group with point stabilizer H. Then G contains one conjugacy class of derangements if and only if G is sharply 2-transitive, or $(G, H) = (\mathrm{A}_5, \mathrm{D}_{10})$ or $(\mathrm{PSL}_2(8){:}3, \mathrm{D}_{18}{:}3)$. In particular, G is a finite primitive permutation group.*

2.1.2 Frobenius groups

A *Frobenius group* is a transitive permutation group which is not regular but in which only the identity has more than one fixed point. For any two distinct points α, β in Ω we have $G_\alpha \cap G_\beta = 1$.

Let

$$N := \{x \in G \mid x = 1 \text{ or } x \in G \setminus \bigcup_{g \in G} H^g\} = \{x \in G \mid x = 1 \text{ or } x \in \Delta(G)\}.$$

Then N is a normal regular subgroup of G.

A finite 2-transitive Frobenius group has a regular normal abelian subgroup in which each non-trivial element has the same order ([DM96, Theorem 3.4B]). Hence a 2-transitive Frobenius group is a sharply 2-transitive group.

We give below an alternative definition of Frobenius group.

Definition 2.1.11. *G is a Frobenius group if and only if G has a subgroup H with $1 < H < G$ such that $H \cap H^g = 1$ whenever $g \in G \backslash H$. We call such an H a Frobenius complement in G.*

The N defined above is called a *Frobenius kernel*. A Frobenius group is a semidirect product $N{:}H$. Thompson, in his PhD thesis, proved that the Frobenius kernel is nilpotent.

Theorem 2.1.12. *[Isa08, Theorem 6.24] Let N be a Frobenius kernel of a Frobenius group G. Then N is nilpotent.*

Theorem 2.1.13. *Let G be a Frobenius group with complement H and kernel N. Then the following holds:*

(a) $|H| \mid |N| - 1$;

(b) If $|H|$ is even, then N is abelian.

Proof. This follows from [Gro11, Proposition 9.1.8] and [Gro11, Proposition 9.1.10]. □

Below are listed some characterizations of Frobenius groups.

Theorem 2.1.14. *[Isa08, Theorem 6.4] Let N be a normal subgroup of a finite group G and suppose that H is a complement for N in G. The following are equivalent:*

(a) the conjugation action of H on N is Frobenius;

(b) $H \cap H^g = 1$ *for all elements* $g \in G \backslash H$;

(c) $\boldsymbol{C}_G(h) \leqslant H$ *for all non-identity elements* $h \in H$;

(d) $\boldsymbol{C}_G(n) \leqslant N$ *for all non-identity elements* $n \in N$.

Theorem 2.1.15. *[Isa08, Theorem 6.7] Let N be normal subgroup of G, where G is a finite group and suppose that $\boldsymbol{C}_G(n) \leqslant N$ for every non-identity element $n \in N$. Then N is complemented in G, and if $1 < N < G$, then G is a Frobenius group with kernel N.*

Proposition 2.1.16. *[Gro11, Proposition 9.2.3] Let G be a Frobenius group with Frobenius kernel N and Frobenius complement H. Suppose that $1 < N_1 \leqslant N$, $1 < H_1 \leqslant H$ with $H_1 \leqslant \boldsymbol{N}_G(N_1)$. Then $G_1 = N_1 H_1$ is a Frobenius group with Frobenius kernel N_1 and Frobenius complement H_1.*

Proposition 2.1.17. *[Gro11, Theorem 9.2.10] Let G be a Frobenius group with Frobenius complement H. Suppose that $p \mid |H|$ is prime and P is a Sylow p-subgroup of H. If p is odd, then P is cyclic; if $p = 2$, then P is either cyclic or a generalized quaternion group Q_{2^k}, $k \geq 3$.*

The result below exhibits a non-solvable Frobenius complement of a Frobenius group.

Theorem 2.1.18. *[Mei02, Theorem A] Let G be a finite Frobenius group with a Frobenius complement H. If H is perfect, $H \cong \mathrm{SL}_2(5)$.*

We look at some groups related to Frobenius groups.

2.1.2.1 Camina groups

A *Camina group* is a group G such that $|\mathbf{C}_G(g)| = |\mathbf{C}_{G/G'}(gG')|$ for all $g \in G \backslash G'$. An equivalent definition says that G is a Camina group if the conjugacy class of every element $g \in G \backslash G'$ is gG'.

Camina groups were first studied by Camina in [Cam78]. Dark and Scoppola [DS96] classified Camina groups:

Theorem 2.1.19. *[DS96, Corollary] Let G be a group. Then G is a Camina group if and only if one of the following holds:*

(a) G is a Camina p-group of nilpotence class 2 or 3;

(b) G is a Frobenius group with a cyclic Frobenius complement;

(c) G is a Frobenius group with a Frobenius complement isomorphic to Q_8.

2.2 Simple and related groups

We begin this section by stating what is arguably one of the most significant results in mathematics in the twentieth century:

Theorem 2.2.1. ***(Classification of Finite Simple Groups)*** *Let G be a finite simple group. Then G is one of the following:*

(a) G is a cyclic group of prime order;

(b) G is an alternating group of degree at least 5;

(c) G is one of the twenty six sporadic simple groups;

(d) G is a finite group of Lie type.

The largest family of finite simple groups comprises the finite groups of Lie type.

Let G be a finite group. A group G is called a *quasisimple group* if $G = G'$ and $G/Z(G)$ is simple. G is called an *almost simple group* if $S \trianglelefteq G \leqslant \mathrm{Aut}(S)$ for some non-abelian simple group S. Quasisimple and almost simple groups are essential to our arguments in Chapter 3.

2.2.1 Sporadic simple groups

The 26 sporadic simple groups do not fall into any of the infinite families of finite simple groups. The explicit character tables of these groups are found in the Atlas [CCNPW85] and that is sufficient for the arguments in our results.

2.2.2 Symmetric groups, alternating groups and their covers

Recall that S_n denotes the symmetric group of degree n. Note that $|\mathrm{S}_n| = n!$. We call $\pi \in \mathrm{S}_n$ an *r-cycle* if π can be expressed in the form $(i_1, \ldots, i_r)(i_{r+1}) \ldots (i_n)$. A 2-cycle is called a *transposition*. The order of a cycle $(i_1, \ldots, i_r)$ is length r. The inverse of $(i_1, \ldots, i_r)$ is $(i_1, \ldots, i_r)^{-1} = (i_r, i_{r-1}, \ldots, i_1)$. Every $\pi \in \mathrm{S}_n$, $\pi \neq 1$, can be written uniquely as a product of disjoint cycles. The order of π is the lowest common multiple of the lengths of the disjoint cyclic factors of π. Each cycle can be expressed as a product of transpositions $(i_1, \ldots, i_r) = (i_1, i_2)(i_2, i_3) \ldots (i_{r-1}, i_r)$. We call $\pi \in \mathrm{S}_n$, *even* (respectively *odd*) if π is expressible as the product of an even (respectively odd) number of transpositions.

The subgroup A_n of S_n comprising all the even permutations is called the *alternating group of degree n*. It is a normal subgroup of S_n and it is also a simple group for $n \geq 5$ as mentioned in Theorem 2.2.1. A_n is the commutator subgroup of S_n and $|\mathrm{A}_n| = \frac{n!}{2}$.

Also, $\mathrm{Aut}(\mathrm{A}_n) = \mathrm{S}_n$ for all $n \geq 5$ except $n = 6$. The case when $n = 6$ will be dealt with in the subsection of finite groups of Lie type.

A *cycle type* of a permutation is an unordered list of the sizes of the cycles in the cycle decomposition of the permutation. Two permutations in S_n are conjugate if and only if they have the same cycle type. A *partition* α on n, denoted by $\alpha \vdash n$ is a sequence of non-negative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_h)$ which satisfies

(i) $\alpha_1 \geq \alpha_2 \geq \ldots \alpha_h$,

(ii) $\sum_{i=1}^{h} \alpha_i = n$.

The α_i are called the *parts* of α.

A conjugacy class of S_n is either contained in A_n or in $\mathrm{S}_n \backslash \mathrm{A}_n$. Every conjugacy class of S_n contained in A_n is either an A_n-class or splits into two A_n-classes of the same order.

Let $\lambda = (\lambda_1, \ldots, \lambda_h) \vdash n$. Then the corresponding *Young subgroup* of S_n is

$$\mathrm{S}_\lambda = \mathrm{S}_{\{1,\ldots,\lambda_1\}} \times \mathrm{S}_{\{\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2\}} \times \mathrm{S}_{\{n-\lambda_h+1,n-\lambda_h+2,\ldots,n\}}.$$

A *Young diagram* $[\lambda]$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_h) \vdash n$ is an array of n boxes (cells) having h-left justified rows with the ith row the containing λ_i boxes for $1 \leq i \leq h$. The lengths λ'_i of the columns of $[\lambda]$ form another partition λ' of n:

$$\lambda' = (\lambda'_1, \lambda'_2, \ldots), \text{ when } \lambda'_i := \textstyle\sum_j 1, \text{ with } \lambda_j \geq 1.$$

This partition λ' is called the *partition associated with* λ. $[\lambda']$ is called the *Young diagram associated with* $[\lambda]$. $[\lambda']$ arises from $[\lambda]$ by interchanging rows and columns. A partition λ is called *self-associated* if $\lambda = \lambda'$.

If $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $\mu \subseteq \lambda$ as Young diagrams if $\mu_i \leq \lambda_i$ for $i = 1, 2, \ldots, m$. If $\mu \subseteq \lambda$ as *Young diagrams*, then the corresponding skew diagram is the set of cells

$$\lambda \backslash \mu = \{c : c \in \lambda \text{ and } c \notin \mu\}.$$

If $v = (i, j)$ is a node in the diagram of λ, then it has hook

$$H_v = H_{i,j} = \{(i, j') : j' \geq j\} \cup \{(i', j) : i' \geq i\}$$

with corresponding hook length

$$h_v = h_{i,j} = |H_{i,j}|.$$

A *skew hook* or *rim hook*, ξ, is a skew diagram which is edgewise connected and contains no 2×2 subset of cells. The *leg length* of ξ

$$\ell\ell(\xi) := (\text{the number of rows of } \xi){-}1.$$

Note that $\alpha \backslash \alpha_1 = (\alpha_2, \ldots, \alpha_k)$. A *composition* of n is an ordered sequence of non-negative integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$$

such that $\sum_{i=1}^{l} \lambda_i = n$. The integers λ_i are called *parts* of the composition.

Let $M(G)$ denote the Schur multiplier of G (we refer to [Isa06, p. 181] or [HH92, Chapter 1] for a definition). This means that there exist a *Schur cover* $\tilde{G}$ such that $Z(\tilde{G}) \cong M(G)$ and $\tilde{G}/Z(\tilde{G}) \cong G$. The Schur multiplier of S_n has order at most 2 and is trivial if $n \leq 3$ (see [HH92, Theorem 2.8]). If $n \geqslant 4$, then S_n has two non-isomorphic Schur covers except when $n = 6$ ([HH92, Theorem 2.12]). Since the Schur covers have the same character table (see for example [Mor62, Section 2.1]), we shall choose one and say that S_n has a *double cover*, denoted by $\tilde{\mathrm{S}}_n$. The generators and relations of $\tilde{\mathrm{S}}_n$ are given in Section 2.3.1, p. 32.

Theorem 2.2.2. *[HH92, Theorem 2.11] For any positive integer n*

$$M(\mathrm{A}_n) = \begin{cases} 1 & \text{if } n \leq 3 \\ C_6 & \text{if } n = 6 \text{ or } 7 \\ C_2 & \text{for all other } n, \end{cases} \tag{2.2.1}$$

where C_k denotes the cyclic of order k.

Theorem 2.2.3. *[HH92, Theorem 3.8] The conjugacy classes of S_n which split in $\tilde{\mathrm{S}}_n$ are:*

(a) the classes of even permutations which can be written as a product of disjoint cycles with no cycles of even length, and

(b) the classes of odd permutations which can be expressed as a product of disjoint cycles with no two cycles of the same length (including length 1*).*

Expressed in cycle type notation, these conditions are:

(a) $a_{2i} = 0$ *for all* i*;*

(b) $a_i \leq 1$ *for all* i*, and the number of even parts is odd.*

Theorem 2.2.4. *[HH92, Theorem 3.9] The conjugacy classes of* A_n *which split in* $\tilde{\mathrm{A}}_n$ *are:*

(a) the classes of permutations whose decompositions into disjoint cycles have no cycles of even length, and

(b) the classes of permutations which can be expressed as a product of disjoint cycles with at least one cycle of even length and with no two cycles of the same length (including length 1*).*

Expressed in cycle type notation, these conditions are:

(a) $a_{2i} = 0$ *for all* i*;*

(b) $a_i \leq 1$ *for all* i*, and* $a_{2i} = 1$ *for at least one value of* i*.*

2.2.3 Groups of Lie type

In order to define groups of Lie type, we need some results in the theory of algebraic groups. We refer to [Car85] for basic definitions and results. Let $\mathcal{M}$ be an algebraic group. Then connected component of $\mathcal{M}$ containing 1_M is denoted by $\mathcal{M}^\circ$. A simple algebraic group is an algebraic group which has no proper, closed, connected normal subgroups. Let $\mathcal{M}$ be a simple algebraic groups over K, where K is an algebraically closed field of characteristic p, $\overline{\mathbb{F}}_p$. The simple algebraic group over K have been classified (see [Car85, p. 23–26]). In particular, we have groups of these types:

$$\mathrm{A}_n(K), \mathrm{B}_n(K), \mathrm{C}_n(K), \mathrm{D}_n(K), \mathrm{E}_6(K), \mathrm{E}_7(K), \mathrm{E}_8(K), \mathrm{F}_4(K) \text{ and } \mathrm{G}_2(K).$$

A surjective homomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ of algebraic groups with finite kernel is called an *isogeny*. Every simple algebraic group has two isogeny types, the *simply connected type* $\mathcal{M}_{sc}$ with a center whose size is as large as possible and of *adjoint type* $\mathcal{M}_{ad}$ with trivial center. The table below lists the types of algebraic groups and their isogeny types.

Table 2.1: Isogeny Types

$\mathcal{M}$	$\mathcal{M}_{sc}$	$\mathcal{M}_{ad}$	Neither $\mathcal{M}_{sc}$ nor $\mathcal{M}_{ad}$
$\mathrm{A}_n(K)$, $n \geq 1$	$\mathrm{SL}_{n+1}(K)$	$\mathrm{PGL}_{n+1}(K)$	
$\mathrm{B}_n(K)$, $n \geq 2$	$\mathrm{Spin}_{2n+1}(K)$	$\mathrm{SO}_{2n+1}(K)$	
$\mathrm{C}_n(K)$, $n \geq 2$	$\mathrm{S}p_{2n}(K)$	$\mathrm{PCSp}_{2n}(K)$	
$\mathrm{D}_n(K)$, $n \geq 3$ odd	$\mathrm{Spin}_{2n}(K)$	$\mathrm{PCO}^{\circ}_{2n}(K)$	$\mathrm{SO}_{2n}(K)$
$\mathrm{D}_n(K)$, $n \geq 4$ even	$\mathrm{Spin}_{2n}(K)$	$\mathrm{PCO}^{\circ}_{2n}(K)$	$\mathrm{SO}_{2n}(K)$, $\mathrm{HSpin}_{2n}(K)$
$\mathrm{E}_6(K)$	$\mathrm{E}_6(K)_{sc}$	$\mathrm{E}_6(K)_{ad}$	
$\mathrm{E}_7(K)$	$\mathrm{E}_7(K)_{sc}$	$\mathrm{E}_7(K)_{ad}$	
$\mathrm{E}_8(K)$	$\mathrm{E}_8(K)$	$\mathrm{E}_8(K)$	
$\mathrm{F}_4(K)$	$\mathrm{F}_4(K)$	$\mathrm{F}_4(K)$	
$\mathrm{G}_2(K)$	$\mathrm{G}_2(K)$	$\mathrm{G}_2(K)$	

A p-element is an element whose order is a power of p and a p'-element if its order is relatively prime to p, where p is prime. Let $\mathcal{M}$ be an algebraic group. A p'-element is called a *semisimple* element and a p-element is called a *unipotent* element. The Jordan decomposition states that every element $g \in \mathcal{M}$ can be decomposed in this way:

$$g = su = us,$$

where $s \in \mathcal{M}$ is semisimple and $u \in \mathcal{M}$ is unipotent. This decomposition is uniquely determined by g. A unipotent subgroup is a subgroup which consists of unipotent elements.

A maximal closed connected solvable subgroup $\mathcal{B}$ of $\mathcal{M}$ is called a *Borel subgroup*. Borel subgroups are conjugate in $\mathcal{M}$. Every Borel subgroup is self-normalizing, that is, $\mathbf{N}_{\mathcal{M}}(\mathcal{B}) = \mathcal{B}$. A *torus* $\mathcal{T}$ is a subgroup of $\mathcal{M}$ that is isomorphic to a direct product of copies of K^*, the multiplicative group of the field K. Every torus is contained in a maximal torus. All maximal tori are also conjugate in $\mathcal{M}$. Every maximal torus is self-centralizing, that is, $\mathbf{C}_{\mathcal{M}}(\mathcal{T}) = \mathcal{T}$. Every semisimple element is contained in a maximal torus and every unipotent element of $\mathcal{M}$ lies in a closed connected unipotent subgroup.

The *radical* $\mathcal{R}$ of an algebraic group $\mathcal{M}$ is the maximal closed, connected, solvable, normal subgroup of $\mathcal{M}$. $\mathcal{M}$ is called a *semisimple algebraic group* if $\mathcal{M}$ is connected and $\mathcal{R} = 1$. The *unipotent radical* $\mathcal{U}$ of $\mathcal{B}$ is the maximal closed, connected, normal, unipotent subgroup of $\mathcal{B}$. Then $\mathcal{B} = \mathcal{U}{:}\mathcal{T}$, the semi-direct product of $\mathcal{U}$ and $\mathcal{T}$. The *Weyl group* with respect to a torus $\mathcal{T}$, $W := \mathbf{N}_{\mathcal{M}}(\mathcal{T})/\mathbf{C}_{\mathcal{M}}(\mathcal{T})$ is a finite group. $\mathcal{M}$ is called a *reductive algebraic group* if $\mathcal{U} = 1$.

Let $\mathcal{M}$ be a connected algebraic group, $s \in \mathcal{M}$ be semisimple and $\mathcal{T} \leqslant \mathcal{M}$ a maximal torus. Then $s \in \mathbf{C}_{\mathcal{M}}(s)^{\circ}$ ([MT11, Proposition 14.1]). If $\mathcal{M}$ is also reductive, then $\mathbf{C}_{\mathcal{M}}(s)^{\circ}$ is reductive.

Let $\mathcal{M}$ be a linear algebraic group. Then $\mathcal{M}$ is a closed subgroup of $\mathrm{GL}_n(K)$. The map

$$F_q : \mathrm{GL}_n(K) \to \mathrm{GL}_n(K), \qquad (a_{ij}) \mapsto (a_{ij}^q)$$

induces a group homomorphism from $\mathrm{GL}_n(K)$ into itself. F_q is called a *standard Frobenius map*. A *Frobenius map* is a homomorphism $F : \mathrm{GL}_n(K) \to \mathrm{GL}_n(K)$ and some power of F is a standard Frobenius map. Frobenius maps are also called *Steinberg endomorphisms* and we shall use these terms interchangeably.

Definition 2.2.5. *Let $\mathcal{M}$ be a connected reductive algebraic group and let $F : \mathcal{M} \to \mathcal{M}$ be a Steinberg endomorphism. Define $\mathcal{M}^F$ by*

$$\mathcal{M}^F := \{g \in \mathcal{M} : F(g) = g\}.$$

We call $\mathcal{M}^F$ a finite group of Lie type.

Hence $\mathrm{GL}_n^F(K) = \mathrm{GL}_n(q)$, $\mathrm{SL}_n^F(K) = \mathrm{SL}_n(q)$ and $\mathrm{Sp}_{2n}^F(K) = \mathrm{Sp}_{2n}(q)$, where $q = p^n$ with p a prime and n a positive integer. We shall list the properties that $\mathcal{M}^F$ and its dual $(\mathcal{M}^*)^{F^*}$ share. Note that $\mathrm{GL}_n(K)^* = \mathrm{GL}_n(K)$, $\mathrm{SL}_n(K)^* = \mathrm{PGL}_n(K)$ and $\mathrm{Sp}_{2n}(K)^* = \mathrm{SO}_{2n}(K)$. A subgroup $\mathcal{H}$ of $\mathcal{M}$ is F-stable if $F(\mathcal{H}) = \mathcal{H}$ where F is a Steinberg endomorphism.

Proposition 2.2.6. *Let $\mathcal{M}$ be a connected reductive algebraic group and $F : \mathcal{M} \to \mathcal{M}$ a Frobenius map such that $\mathcal{M}^*$ is the dual of $\mathcal{M}$ with a corresponding Frobenius map F^*. Suppose that $\mathcal{T}$ is an F-stable maximal torus and $\mathcal{T}^*$ the corresponding F^*-stable maximal torus. Then the following statements hold:*

(a) $|\mathcal{M}^F| = |(\mathcal{M}^*)^{F^*}|$*;*

(b) $|[\mathcal{M},\mathcal{M}]^F| = |[\mathcal{M}^*,\mathcal{M}^*]^{F^*}|$;

(c) $|\mathcal{T}^F| = |(\mathcal{T}^*)^{F^*}|$.

Proof. This follows from [Car85, Corollary 4.4.2, Propositions 4.4.4 and 4.4.5]. □

The table below gives sizes of centers of finite groups of Lie type of simply connected type. This table is in [MT11, p. 211].

Table 2.2: Sizes of centers of finite groups of Lie type of simply connected type

$\mathcal{M}^F$	$\lvert Z(\mathcal{M}^F)\rvert$	$\mathcal{M}^F$	$\lvert Z(\mathcal{M}^F)\rvert$
$\mathrm{SL}_n(q)$, $n \geq 2$	$\gcd(n, q-1)$	${}^2\mathrm{B}_2(2^{2f+1})$	1
$\mathrm{SU}_n(q)$, $n \geq 3$	$\gcd(n, q+1)$	${}^2\mathrm{G}_2(3^{2f+1})$	1
$\mathrm{Spin}_{2n+1}(q)$, $n \geq 3$	$\gcd(2, q-1)$	$\mathrm{G}_2(q)$	1
$\mathrm{Sp}_{2n}(q)$, $n \geq 2$ odd	$\gcd(2, q-1)$	${}^3\mathrm{D}_4(q)$	1
$\mathrm{Spin}^+_{2n}(q)$, $n \geq 4$ even	$\gcd(2, q-1)^2$	${}^2\mathrm{F}_4(2^{2f+1})$	1
$\mathrm{Spin}^+_{2n}(q)$, $n \geq 5$ odd	$\gcd(4, q-1)$	$\mathrm{F}_4(q)$	1
$\mathrm{Spin}^-_{2n}(q)$, $n \geq 4$ even	$\gcd(2, q-1)$	$\mathrm{E}_6(q)$	$\gcd(3, q-1)$
$\mathrm{Spin}^-_{2n}(q)$, $n \geq 5$ odd	$\gcd(4, q+1)$	${}^2\mathrm{E}_6(q)$	$\gcd(3, q+1)$
		$\mathrm{E}_7(q)$	$\gcd(2, q-1)$
		$\mathrm{E}_8(q)$	1

Tits [MT11, Theorem 24.17] proved that if $\mathcal{M}$ is a simply connected simple linear algebraic group with Steinberg endomorphism $F : \mathcal{M} \to \mathcal{M}$, then $\mathcal{M}^F$ is perfect and $\mathcal{M}^F/Z(\mathcal{M}^F)$ is simple with the following exceptions: $\mathrm{SL}_2(2)$, $\mathrm{SL}_2(3)$, $\mathrm{SU}_3(2)$, $\mathrm{Sp}_4(2)$, $\mathrm{G}_2(2)$, ${}^2\mathrm{B}_2(2)$, ${}^2\mathrm{G}_2(3)$ and ${}^2\mathrm{F}_4(2)$.

In other words, in the case above, $\mathcal{M}^F = M$ is quasisimple. We note that there are some isomorphic groups that arise as both groups of Lie type and as alternating groups. We shall list some of them here:

$$\mathrm{A}_5 \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5),\ \mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$$

$$\mathrm{A}_6 \cong \mathrm{PSL}_2(9) \cong \mathrm{Sp}_4(2)',\ \mathrm{PSU}_3(3) \cong \mathrm{G}_2(2)'$$

$$\mathrm{A}_8 \cong \mathrm{PSL}_4(2),\ \mathrm{PSU}_4(2) \cong \mathrm{Sp}_4(3).$$

Every semisimple element $s \in \mathcal{M}^F$ is contained in an F-stable maximal torus of $\mathcal{M}$.

Let $\mathcal{M}$ be a linear algebraic group and $x \in \mathcal{M}$. We call x *regular* if $\dim \mathbf{C}_{\mathcal{M}}(x)$ is minimal amongst elements in $\mathcal{M}$. Let M be a finite group of Lie type and let $s \in M$ be a semisimple element contained in a maximal torus T. Then s is a regular element of M if $\mathbf{C}_M(s) = T$.

Theorem 2.2.7. *Let M be a finite simple group of Lie type over a field of odd characteristic p that is not isomorphic to* $\mathrm{PSL}_2(q)$. *Then M has an element of order pr where $r \neq p$ is prime.*

Proof. We consider the prime graph of M whose vertices are the primes dividing the order of M and where two vertices r, s are joined by an edge if and only if M contains an element of order rs. By [Wil81, Table Ib-e], we have that the size of the connected component containing p is at least 2, as required. □

Almost simple groups of finite groups of Lie type We shall need the sizes of outer automorphism groups of simple groups of Lie type for some of our arguments. Recorded below is a table of these sizes.

Table 2.3: Sizes of outer automorphism groups of finite groups of Lie type of simply connected type

$\mathcal{M}^F$	$\lvert\mathrm{Out}(M)\rvert$	$\mathcal{M}^F$	$\lvert\mathrm{Out}(M)\rvert$
$\mathrm{SL}_2(q)$, $q = p^f$	$\gcd(2, q-1) \cdot f$	${}^3\mathrm{D}_4(q)$, $q^3 = p^f$	f
$\mathrm{SL}_n(q)$, $n \geq 3$, $q = p^f$	$2 \cdot \gcd(n, q-1) \cdot f$	$\mathrm{G}_2(q)$, $q = p^f$, $p \neq 3$	f
$\mathrm{SU}_n(q)$, $n \geq 3$, $q = p^f$	$\gcd(n, q+1) \cdot f$	$\mathrm{G}_2(q)$, $q = 3^f$	$2 \cdot f$
$\mathrm{Spin}_5(q)$, $q = p^f$	$2 \cdot f$	${}^2\mathrm{G}_2(q)$, $q = 3^f$, f odd	f
$\mathrm{Spin}_n(q)$,$n \geq 3$, $q = p^f$	$\gcd(2, q-1) \cdot f$	$\mathrm{F}_4(q)$, $q = p^f$, $p \neq 2$	f
$\mathrm{Sp}_{2n}(q)$, $n \geq 3$, $q = p^f$	$\gcd(2, q-1) \cdot f$	$\mathrm{F}_4(q)$, $q = 2^f$	$2 \cdot f$
$\mathrm{Spin}_8^+(q)$, $q = p^f$	$3! \cdot \gcd(2, q-1)^2 \cdot f$	${}^2\mathrm{F}_4(q)$, $q = 2^f$, f odd	f
$\mathrm{Spin}_{2n}^+(q)$, $n \geq 6$ even, $q = p^f$	$2 \cdot \gcd(2, q-1)^2 \cdot f$	$\mathrm{E}_6(q)$, $q = p^f$	$2 \cdot \gcd(3, q-1) \cdot f$
$\mathrm{Spin}_{2n}^+(q)$, $n \geq 5$ odd, $q = p^f$	$\gcd(4, q+1) \cdot f$	${}^2\mathrm{E}_6(q)$, $q^2 = p^f$	$\gcd(3, q+1) \cdot f$
$\mathrm{Spin}_{2n}^-(q)$, $n \geq 4$, $q^2 = p^f$	$\gcd(4, q+1) \cdot f$	$\mathrm{E}_7(q)$, $q = p^f$	$\gcd(2, q-1) \cdot f$
${}^2\mathrm{B}_2(q)$, $q = 2^f$, f odd	f	$\mathrm{E}_8(q)$, $q = p^f$	f

2.3 Character theory of finite groups

Let G be a finite group, $\mathbb{F}$ a field and n a positive integer. A *representation* of G over $\mathbb{F}$ of dimension n is a homomorphism from G to the *general linear group* $\mathrm{GL}_n(\mathbb{F})$, the multiplicative group of non-singular $n \times n$ matrices over $\mathbb{F}$.

Given a group G and a representation

$$\mathfrak{X} : G \longrightarrow \mathrm{GL}_n(\mathbb{F}),$$

we have that $\mathfrak{X}$ is uniquely determined by its ordinary character

$$\chi : G \longrightarrow \mathbb{C},\ \chi(g) = tr(\mathfrak{X}(g)).$$

In the scenario above we say $\mathfrak{X}$ affords χ.

Let $\mathfrak{X}$ be a representation of G. Then $\mathfrak{X}$ is *reducible* if for all $g \in G$, $\mathfrak{X}(g)$ can written in the form:

$$\mathfrak{X}(g) = \begin{bmatrix} \mathfrak{Y}(g) & \mathfrak{Z}(g) \\ 0 & \mathfrak{W}(g) \end{bmatrix},$$

where the two diagonal blocks are square. Otherwise $\mathfrak{X}$ is an *irreducible representation*. An *irreducible character* is a character that is afforded by an irreducible representation. If χ is a character such that $\chi = \sum_{i=1}^{k} n_i\chi_i$ and $\chi_i's$ are irreducible characters, then those χ_i with corresponding $n_i > 0$ are called the *irreducible constituents* of χ.

The *character degree* of a character is the value $\chi(1)$. *Linear characters* are characters such that $\chi(1) = 1$. Let χ, ψ be characters of a group G. Then the *inner product* of χ and ψ is defined as:

$$[\chi, \psi] = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

We denote by $\mathrm{Irr}(G)$ the set of all irreducible characters of G. Let $\chi \in \mathrm{Irr}(G)$. The kernel of χ, $\ker\chi := \{g \in G \mid \chi(g) = \chi(1)\}$. Note that $\ker\chi$ is normal in G for $\chi \in \mathrm{Irr}(G)$.

Corollary 2.3.1. *[Isa06, Corollary 2.17] Let χ and ψ be characters of G. Then $[\chi, \psi] = [\psi, \chi]$ is a non-negative integer. Also $[\chi, \chi] = 1$ if and only if χ is irreducible.*

Corollary 2.3.2. *Let G be a group with commutator subgroup G'. Then $G' \leqslant \ker \chi$ for a linear character χ of G.*

Proof. This follows from [Isa06, Corollary 2.23]. □

Let $Z(\chi) := \{g \in G \mid |\chi(g)| = \chi(1)\}$. Hence for $\chi \in \mathrm{Irr}(G)$, $\ker(\chi) \leqslant Z(\chi)$. If H is a subgroup of G and χ is a character of G, then the *restriction of χ on H*, denoted χ_H, is a character on H such that $\chi_H(h) = \chi(h)$ for all $h \in H$.

Lemma 2.3.3. *[Isa06, Lemma 2.27] Let χ be a character of G and let $Z = Z(\chi)$ and $f = \chi(1)$. Let $\mathfrak{X}$ be a representation of G which affords χ. Then*

(a) $Z = \{g \in G \mid \mathfrak{X}(g) = \varepsilon I$ for some $\varepsilon \in \mathbb{C}\}$, is a normal subgroup of G;

(b) $\chi_Z = f\lambda$ for some linear character λ of Z;

(c) $Z/\ker \chi$ is cyclic;

(d) $Z/\ker \chi \leqslant Z(G/\ker \chi)$.

Moreover, if $\chi \in \mathrm{Irr}(G)$, then

(e) $Z/\ker \chi = Z(G/\ker \chi)$.

Definition 2.3.4. *Let H and K be groups and $G = H \times K$ and let φ and ϑ be characters on H and K. Define $\chi = \varphi \times \vartheta$ by $\chi((h, k)) = \varphi(h)\vartheta(k)$ for $h \in H$ and $k \in K$.*

Theorem 2.3.5. *[Isa06, Theorem 4.21] Let H and K be groups and $G = H \times K$. Then those characters of the form $\varphi \times \theta$ where $\varphi \in \mathrm{Irr}(H)$ and $\theta \in \mathrm{Irr}(K)$ are precisely the irreducible characters of G.*

Lemma 2.3.6. *[Isa06, Problem 4.4(a)] Suppose that $G = HK$ with $H \subseteq \boldsymbol{C}_G(K)$. Let $\chi \in \mathrm{Irr}(G)$. Then $\chi_H = \theta(1)\varphi$ and $\chi_K = \varphi(1)\theta$ for some $\theta \in \mathrm{Irr}(H)$ and $\varphi \in \mathrm{Irr}(K)$.*

Let $H \leqslant G$ and let φ be a character of H. Then φ^G, the *induced character* on G, is given by

$$\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^\circ(xgx^{-1}),$$

where φ° is defined by $\varphi^\circ(h) = \varphi(h)$ if $h \in H$ and $\varphi^\circ(y) = 0$ if $y \notin H$. We say that $\chi \in \mathrm{Irr}(G)$ is an induced character or an *imprimitive character* if $\chi = \varphi^G$ for some $\varphi \in \mathrm{Irr}(H)$ where $H < G$. If $\chi \in \mathrm{Irr}(G)$ is not induced from any character of any proper subgroup of G, we say that χ is a *primitive character*.

Clifford theory tells us how characters decompose upon restriction to normal subgroups.

Theorem 2.3.7. *[Isa06, Theorem 6.2] Let H be a normal subgroup of G and $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of χ_H and suppose that $\theta_1, \theta_2, \dots, \theta_t$ are distinct conjugates of θ in G. Then*

$$\chi_H = e \sum_{i=1}^t \theta_i$$

where $e = [\chi_H, \theta]$.

Primitive characters restrict to only one irreducible character constituent upon restriction to a normal subgroup.

Lemma 2.3.8. *[Isa06, Corollary 6.12] Let G be a finite group and $\chi \in \mathrm{Irr}(G)$ be primitive. Then for every normal subgroup N of G, χ_N is a multiple of an irreducible character of N.*

Corollary 2.3.9. *[Isa06, Corollary 6.13] Suppose that G is a finite group that has a faithful primitive character and let A be an abelian normal subgroup of G. Then $A \leqslant Z(G)$.*

Theorem 2.3.10. *[Isa06, Theorem 6.15] Let A be an abelian normal subgroup of G. Then $\chi(1)$ divides $|G{:}A|$ for all $\chi \in \mathrm{Irr}(G)$.*

Let K, L be normal subgroups of G. Then K/L is a *chief factor* of G if there is no normal subgroup M of G such that $L < M < K$.

Theorem 2.3.11. *[Isa06, Theorem 6.18] Let K/L be an abelian chief factor of G. Suppose that $\theta \in \mathrm{Irr}(K)$ is invariant in G. Then one of the following holds:*

(a) $\theta_L \in \mathrm{Irr}(L)$;

(b) $\theta_L = e\varphi$ *for some* $\varphi \in \mathrm{Irr}(L)$ *and* $e^2 = |K{:}L|$*;*

(c) $\theta_L = \sum_{i=1}^{t} \varphi_i$ *where* $\varphi_i \in \mathrm{Irr}(L)$ *are distinct and* $t = |K{:}L|$*.*

Proposition 2.3.12. *[Isa06, Problem 6.3] Let* N *be a normal subgroup of* G *and let* $\chi \in \mathrm{Irr}(G)$ *and* $\theta \in \mathrm{Irr}(N)$ *with* $[\chi_N, \theta] \neq 0$*. Then the following are equivalent;*

(a) $\chi_N = e\theta$ *with* $e^2 = |G{:}N|$*;*

(b) χ *vanishes on* $G\backslash N$ *and* θ *is invariant in* G*;*

(c) χ *is the unique constituent of* θ^G *and* θ *is invariant in* G*.*

We say χ and θ above are fully ramified with respect to G/N.

Let N be a normal subgroup of G. Recall that G is a relative M-group with respect to N if for every $\chi \in \mathrm{Irr}(G)$ there exists H with $N \leqslant H \leqslant G$ and $\sigma \in \mathrm{Irr}(H)$ such that $\sigma^G = \chi$ and $\sigma_N \in \mathrm{Irr}(N)$.

Theorem 2.3.13. *[Isa06, Theorem 6.22] Suppose that* N *is a normal subgroup of* G *and* G/N *is solvable. Suppose, furthermore, that every chief factor of every subgroup of* G/N *has non-square order. Then* G *is a relative* M*-group with respect to* N*.*

Theorem 2.3.14. *[Isa08, Theorem 7.8] Let* G *be a group of order* $p^a q^b$*, where* p *and* q *are primes. Then* G *is solvable.*

Induction of characters is a transitive relation as the result below shows.

Lemma 2.3.15. *Let* H, K *be subgroups of a group* G *and suppose that* φ *is a character of* H*.*

(a) *If* $H \subseteq K \subseteq G$*, then* $(\varphi^K)^G = \varphi^G$*.*

(b) *If* $HK = G$*, then* $(\varphi^G)_K = (\varphi_{H\cap K})^K$*.*

Proof. Statement (a) follows from [Hup98, Theorem 17.3] and (b) follows from [Isa06, Problem 5.2]. □

The character theory of Frobenius groups is well known.

Proposition 2.3.16. *[Gro11, Proposition 9.1.15] Let G be a Frobenius group with complement H and Frobenius kernel N.*

(a) If $1_N \neq \varphi \in \mathrm{Irr}(N)$, then $\varphi^G \in \mathrm{Irr}(G)$;

(b) $\mathrm{Irr}(G) = \mathrm{Irr}(H) \cup \{\varphi^G \mid 1_N \neq \varphi \in \mathrm{Irr}(N)\}$.

Recall that if p is prime, then a p-group G is an extra-special if its center Z is cyclic of order p and the quotient group G/Z is a non-trivial elementary abelian p-group. Seitz [Sei68] classified finite groups with the extremal property that the group has only one non-linear irreducible character:

Theorem 2.3.17. *[Sei68, Theorem] A group G has exactly one non-linear irreducible character if and only if G is isomorphic to one of the following:*

(a) G is an extra-special 2-group;

(b) G is a Frobenius group with an elementary abelian kernel N of order p^n for some prime p and positive integer n, and complement H of order $p^n - 1$.

Theorem 2.3.18. *[GGLMNT14, Corollary] Suppose that G is a finite group with exactly one irreducible character of degree divisible by a prime p. Let P be a Sylow p-subgroup of G. Either P is a normal subgroup of G or $\boldsymbol{N}_G(P)$ is a maximal subgroup of G.*

2.3.1 Symmetric groups, alternating groups and their covers

Let $\mathbb{C}$ be the complex number field. Consider two linear representations of S_λ, $\lambda \vdash n$, over $\mathbb{C}$. The first linear representation is the identity representation IS_λ of S_λ, that is,

$$\mathrm{IS}_\lambda : \mathrm{S}_\lambda \to \mathbb{C}^* \text{ such that } \pi \mapsto 1_{\mathbb{C}^*}.$$

Let the *sgn* π denote the *sign* of a permutation $\pi \in \mathrm{S}_n$ (refer to [JK81, p. 9] for definition). The second linear representation of S_λ, $\lambda \vdash n$, over $\mathbb{C}$ is the alternating representation AS_λ of S_λ, that is,

$$\mathrm{AS}_\lambda : \mathrm{S}_\lambda \to \mathbb{C}^* \text{ such that } \pi \mapsto sgn\ \pi \cdot 1_{\mathbb{C}^*}.$$

If μ is another partition of n, then IS_λ, IS_μ and AS_μ induce representations

$$\mathrm{IS}_\lambda^{\mathrm{S}_n},\ \mathrm{IS}_\mu^{\mathrm{S}_n} \text{ and } \mathrm{AS}_\mu^{\mathrm{S}_n}$$

of S_n.

Theorem 2.3.19. *[JK81, Theorem 2.1.3] If α and λ be partitions of n with S_α and $\mathrm{S}_{\alpha'}$ the Young subgroups corresponding with α and α', then the induced representations $\mathrm{IS}_\lambda^{\mathrm{S}_n}$ and $\mathrm{AS}_{\alpha'}^{\mathrm{S}_n}$ have exactly one ordinary irreducible constituent in common. Furthermore, this irreducible constituent is contained with multiplicity 1 in both $\mathrm{IS}_\alpha^{\mathrm{S}_n}$ and $\mathrm{AS}_{\alpha'}^{\mathrm{S}_n}$.*

The representations $\mathrm{IS}_\alpha^{\mathrm{S}_n}$ and $\mathrm{AS}_{\alpha'}^{\mathrm{S}_n}$ depend only on the partition α of n, since two Young subgroups of S_n corresponding to the same partition $\alpha' \vdash n$ are conjugate subgroups. Hence we denote by $[\alpha]$ this uniquely determined irreducible representation constituent and its equivalence class of representations.

Theorem 2.3.20. *[JK81, Theorem 2.1.11] $\{[\alpha] \mid \alpha \vdash n\}$ is the complete set of equivalence classes of ordinary irreducible representations of S_n.*

We present the Murnaghan-Nakayama Rule which gives character values for any element in S_n and any irreducible character of S_n.

Theorem 2.3.21. ***Murnaghan-Nakayama Rule*** *[JK81, 2.4.7] If λ is a partition of n and $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition of n, then we have*

$$\chi_\alpha^\lambda = \textstyle\sum_\xi (-1)^{\ell\ell(\xi)} \chi_{\alpha\setminus\alpha_1}^{\lambda\setminus\xi},$$

where the sum runs over all rim hooks ξ of λ having α_1 cells.

Theorem 2.3.22. *[JK81, Theorem 2.5.7] Suppose that α is a partition of $n > 1$.*

(a) If $\alpha \neq \alpha'$, then $[\alpha]_{\mathrm{A}_n} = [\alpha']_{\mathrm{A}_n}$ is irreducible.

(b) If $\alpha = \alpha'$, then $[\alpha]_{\mathrm{A}_n} = [\alpha']_{\mathrm{A}_n}$ splits into two irreducible and conjugate characters $[\alpha]^\pm$ of A_n.

A complete system of equivalence classes of ordinary irreducible characters of A_n is therefore

$$\{[\alpha]_{\mathrm{A}_n} \mid \alpha \neq \alpha'\} \cup \{[\alpha]^{\pm} \mid \alpha = \alpha' \vdash n\}.$$

We now want to look at the character theory of covers of alternating groups. We refer to [HH92] for a detailed account.

An irreducible representation $\mathcal{R}$ of $\tilde{\mathrm{S}}_n$ is *negative* if $\mathcal{R}(z) = -I$, where $z \in Z(\mathrm{S}_n)$.

$\tilde{\mathrm{S}}_n$ may be viewed as the group with generators $z, t_1, \ldots, t_{n-1}$ and relations

$$\begin{gathered} z^2 = 1;\ zt_j = t_j z,\ 1 \leq j \leq n-1; \\ t_j^2 = z,\ 1 \leq j \leq n-1; \\ (t_j t_{j+1})^3 = z,\ 1 \leq j \leq n-2; \\ t_j t_k = z t_k t_j, \text{ for } |j-k| > 1 \text{ and } 1 \leq j, k \leq n-1. \end{gathered}$$

(see [HH92, p. 19])

If $n > 2$, the *basic representation* $\mathcal{R}_n$ of $\tilde{\mathrm{S}}_n$ is the complex representation determined by writing $n = 2m+1$ or $2m+2$ for $m \geq 1$, and defining

$$\mathcal{R}_n(t_k) = (2k)^{-\frac{1}{2}}[(k+1)^{\frac{1}{2}} M_k - (k-1)^{\frac{1}{2}} M_{k-1}]$$

for $1 \leq k < n$, where M_k is a matrix of degree 2^m and t_k is as defined above.

The basic representation $\mathcal{R}_n$ is an irreducible character of $\tilde{\mathrm{S}}_n$ ([HH92, Theorem 6.2]). The *basic character* χ_n is the character afforded by the representation $\mathcal{R}_n$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of n. The *weight*, $|\lambda|$, of λ is $\lambda_1 + \lambda_2 + \cdots + \lambda_l$. The *length* of λ is l. The partition λ is *odd* if the number of even parts in λ is odd, and is *even* otherwise. Define

$\mathcal{P} := \{\lambda : \lambda \text{ is a partition of } n\ \}$;

$\mathcal{P}(n) := \{\lambda \in \mathcal{P} : |\lambda| = n\}$;

$\mathcal{P}^{\circ} := \{\lambda \in \mathcal{P} : \text{each } \lambda_i \text{ is odd}\}$.

$\mathcal{P}^{\circ}(n) := \mathcal{P}(n) \cap \mathcal{P}^{\circ}$;

$\mathcal{D} := \{\lambda \in \mathcal{P} : \lambda_i \neq \lambda_j \text{ if } i \neq j\}$;

$\mathcal{D}(n) := \mathcal{P}(n) \cap \mathcal{D}$.

The partition λ is *strict* if its parts are distinct, that is, $\lambda \in \mathcal{D}$. Let $\mathfrak{G}$ be the class of triples (G, z, σ), where G is a finite group, z is an element of order 2 in the centre of G

and σ is a homomorphism from G to $\mathbb{Z}/2\mathbb{Z}$ with $\sigma(z) = 0$. Let $\mathcal{R}$ be a representation of (G, z, σ) ([HH92, Chapter 4]). Then the *associate representation* $\mathcal{R}^a$ is given by

$$\mathcal{R}^a(g) = (-1)^{\sigma(g)}\mathcal{R}(g).$$

A representation $\mathcal{R}$ of (G, z, σ) is *self-associate* if $\mathcal{R}$ is equivalent to $\mathcal{R}^a$. Using [HH92, 8.6], let $<\!\lambda\!>$ denote the unique negative character of $\tilde{\mathrm{S}}_n$ corresponding to λ and let $<\!\lambda\!>^a$ be the associate character of $<\!\lambda\!>$. We shall present part of [HH92, Theorem 8.6] in the following result:

Theorem 2.3.23. *[HH92, Theorem 8.6] Let $n \geq 5$ be an integer. The irreducible negative representations of $\tilde{\mathrm{S}}_n$ and $\tilde{\mathrm{A}}_n$ are given as follows. (All partitions are strict.)*

(a) For each λ in $\mathcal{D}(n)$, there is a negative character $<\!\lambda\!>$ of $\tilde{\mathrm{S}}_n$ which is irreducible in $\tilde{\mathrm{S}}_n$;

(b) The $<\!\lambda\!>$, as λ varies over $\mathcal{D}(n)$, together with $<\!\lambda\!>^a$ when λ is odd, are a complete non-redundant list of the irreducible negative characters of $\tilde{\mathrm{S}}_n$;

(c) When λ is odd ($<\lambda> \neq <\lambda>^a$), the character $<\lambda>$ restricts to an irreducible character for $\tilde{\mathrm{A}}_n$ (which is also the restriction of $<\lambda>^a$). If λ is even, then $<\!\lambda\!>^a = <\!\lambda\!>$ and the restriction of $<\!\lambda\!>$ to $\tilde{\mathrm{A}}_n$ is a sum of two distinct conjugate irreducible characters;

(d) The restrictions in (c) give a non-redundant list of the irreducible negative representations for $\tilde{\mathrm{A}}_n$.

We end this section by presenting prime power degree representations of alternating and symmetric groups. Let f_λ be the character degree of an irreducible character identified by λ in S_n.

Theorem 2.3.24. *[BBOO01, Theorem 2.4] Let λ be a partition of n. Then $f_\lambda = p^r$ for some prime p, $r \geq 1$, if and only if one of the following occurs:*

$$n = p^r + 1,\ \lambda = (p^r, 1) \text{ or } (2, 1^{p^r-1}),\ f_\lambda = p^r,$$

or we are in one of the following exceptional cases:

$$\begin{array}{llll} n=4 & : & \lambda=(2^2) & f_\lambda=2 \\ n=5 & : & \lambda=(2^2,1) \ \mathit{or} \ (3,2) & f_\lambda=5 \\ n=6 & : & \lambda=(4,2) \ \mathit{or} \ (2^2,1^2) & f_\lambda=3^2 \\ & : & \lambda=(3^2) \ \mathit{or} \ (2^3) & f_\lambda=5 \\ & : & \lambda=(3,2,1) & f_\lambda=2^4 \\ n=8 & : & \lambda=(5,2,1) \ \mathit{or} \ (3,2,1^3) & f_\lambda=2^6 \\ n=9 & : & \lambda=(7,2) \ \mathit{or} \ (2^5,1^5) & f_\lambda=3^3. \end{array}$$

Let $\overline{f}_\lambda$ be the character degree of an irreducible character identified by λ in A_n.

Theorem 2.3.25. *[BBOO01, Theorem 5.1] Let λ be a partition of n. Then $\overline{f}_\lambda = p^r$ for some prime p, $r \geq 1$, if and only if one of the occurs:*

$$n = p^r + 1 > 3,\ \lambda = (p^r, 1) \ \mathit{or} \ (2, 1^{p^r-1}),\ \overline{f}_\lambda = p^r,$$

or we are in one of the following exceptional cases:

$$\begin{array}{llll} n=5 & : & \lambda=(2^2,1) \ \mathit{or} \ (3,2) & \overline{f}_\lambda=5 \\ & : & \lambda=(3,1^2) & \overline{f}_\lambda=3 \\ n=6 & : & \lambda=(4,2) \ \mathit{or} \ (2^2,1^2) & \overline{f}_\lambda=3^2 \\ & : & \lambda=(3^2) \ \mathit{or} \ (2^3) & \overline{f}_\lambda=5 \\ & : & \lambda=(3,2,1) & \overline{f}_\lambda=2^3 \\ n=8 & : & \lambda=(5,2,1) \ \mathit{or} \ (3,2,1^3) & \overline{f}_\lambda=2^6 \\ n=9 & : & (7,2) \ \mathit{or} \ (2^5,1^5) & \overline{f}_\lambda=3^3. \end{array}$$

2.3.2 Deligne-Lusztig Theory for finite groups of Lie type

We refer to [Car85] and [DM91] for basic results on the Deligne-Lusztig Theory for irreducible characters of finite groups of Lie type. Let $M = \mathcal{M}^F$ where $\mathcal{M}$ is a connected reductive algebraic group over an algebraically closed field K of characteristic p with Steinberg endomorphism F. Let the pair $(\mathcal{M}^*, F^*)$ be the dual of $(\mathcal{M}, F)$ with $M^* = (\mathcal{M}^*)^{F^*}$. We have that the set of all irreducible characters of M, $\mathrm{Irr}(M)$, can be written as a disjoint union $\bigsqcup \mathcal{E}(M, (s^*))$ of Lusztig series corresponding to M^*-conjugacy classes of semisimple elements $s^* \in M^*$. If $\mathbf{C}_{\mathcal{M}^*}(s^*)$ is connected, then the

Lusztig series $\mathcal{E}(M,(s^*))$ contains a unique irreducible *semisimple character*, χ_{s^*}, of degree $|M^*:\mathbf{C}_{M^*}(s^*)|_{p'}$ (If $n = p^a m$ such that $\gcd(p,m)=1$. Then the $n_{p'} = \dfrac{n}{p^a} = m$ and $n_p = p^a$). The characters in the Lusztig series corresponds to M^*-conjugacy classes of semisimple elements, so χ_{s^*} and χ_{r^*} are equal if and only if s^* and r^* are conjugate elements of M^*. The irreducible characters contained in the Lusztig series $\mathcal{E}(M,(1^*))$ are called *unipotent characters*.

We may view the Deligne-Lusztig theory as an analogue to the Jordan decomposition for irreducible characters into semisimple characters and unipotent characters.

Let M be a finite simple group of Lie type in characteristic p distinct from the Tits group ${}^2\mathrm{F}_4(2)'$. Then M has an irreducible character of degree $|M|_p$, called the Steinberg character of M and denoted $\mathrm{St_G}$. The *Steinberg character* has the property that $\mathrm{St(g)} = 0$ for all p-singular elements g of M by Theorem 2.4.1 which we state in the next section.

2.4 Zeros of characters

We begin this section with a result of Brauer which gives us a sufficient condition for a character to vanish on p-singular elements.

Theorem 2.4.1. *[Isa06, Theorem 8.17] Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. If $p \nmid |G|/\chi(1)$ for some prime p, then $\chi(g) = 0$ for all p-singular elements g of G.*

We say G is of *p-defect zero* if it has a irreducible character χ satisfying the hypothesis of Theorem 2.4.1.

Theorem 2.4.2. *[MNO00, Theorems 3.4 and Theorem 5.1] [BO04, Theorem 1.2] Let G be a finite simple group or a symmetric group and let $\chi \in \mathrm{Irr}(G)$ be non-linear. Then there exists $g \in G$ of prime order such that $\chi(g) = 0$.*

For finite simple groups of Lie type, it turns out that we can choose four conjugacy classes with prime order elements as the result below states:

Theorem 2.4.3. *[MNO00, Theorem 5.1] Let G be a finite simple group of Lie type. Then there exist four conjugacy classes of elements of prime order in G such that every non-linear $\chi \in \mathrm{Irr}(G)$ vanishes on at least one of them.*

Lemma 2.4.4. *[Qia07, Lemma 2.2] Let G be a finite group. For any non-linear $\chi \in \mathrm{Irr}(G)$, if $\upsilon(\chi) \subseteq N$ for some normal subgroup N of G, then gcd$(\chi(1), |G : N|) = 1$ and $Z(\chi) \leqslant N$.*

Lemma 2.4.5. *[Chi99, Proposition 2.7] Let G be a finite non-abelian group. Assume that every $\chi \in \mathrm{Irr}(G)$ vanishes on at most one conjugacy class. Then G is Frobenius with a complement of order 2 and an abelian odd-order kernel.*

2.4.1 Symmetric groups, alternating groups and their covers

We present a more precise result on zeros of characters of symmetric and alternating groups.

Theorem 2.4.6. *[BO04, Theorem 1.2] Let χ be any non-linear irreducible character of the symmetric group S_n or the alternating group A_n. If $\chi(1)$ is not a power of 2, then χ vanishes on some element of odd prime order.*

Theorem 2.4.7. *[HH92, Theorem 8.7] Let $\lambda \in \mathcal{D}(n)$ have length l, and let $g \in \tilde{\mathrm{S}}_m$.*

(a) Let λ be odd. If g projects to cycle type λ which is neither in $\mathcal{P}^{\circ}(m)$ nor equal to λ, then $<\lambda> (g) = 0$.

(b) Let λ be even. If g does not project to a cycle type in $\mathcal{P}^{\circ}(m)$, then $<\lambda> (g) = 0$.

2.4.2 The Special Linear Groups $\mathrm{SL}_2(q)$, $q \geq 4$

The explicit character tables of $\mathrm{SL}_2(q)$ and $\mathrm{PSL}_2(q)$ are found in [Dor71, Geh02, Ada02]. We use the notation in [Dor71, Chapter 38]. Let $\mathbb{F}_q$ be the finite field of q elements. By theory, $q = p^n$ for some prime p and positive integer n. Let ν be a generator of the cyclic group $\mathbb{F}_q^* = \mathbb{F}_q \backslash \{0\}$ and let τ be a generator of $\mathbb{F}_{q^2}^*$, and $\gamma = \tau^{q-1}$. Put

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; z = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \; c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$d = \begin{bmatrix} 1 & 0 \\ \nu & 1 \end{bmatrix}, \ a = \begin{bmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{bmatrix}, \ b = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{bmatrix}.$$

If q is odd, then every element of $\mathrm{SL}_2(q)$ is conjugate to one of $1, z, c, cz, d, dz, a^l$ for $1 \leq l \leq (q-3)/2$, or b^m for $1 \leq m \leq (q-1)/2$. If q is even, then every element of $\mathrm{SL}_2(q) = \mathrm{PSL}_2(q)$ is conjugate to one of $1, c, a^l$ for $1 \leq l \leq [(q-2)/2]$, or b^m for $1 \leq m \leq [q/2]$, where $[x]$ denotes the greatest integer less than or equal to x.

The outer automorphism group of $\mathrm{PSL}_2(q)$, $q = p^f$, is of order df, $d = \gcd(2, q-1)$. It is generated by a *diagonal automorphism* δ and a *field automorphism* φ. The diagonal automorphism of $\mathrm{PSL}_2(q)$ is an automorphism induced by conjugation on $\mathrm{SL}_2(q)$ by the matrix

$$M = \begin{bmatrix} \nu & 0 \\ 0 & 1 \end{bmatrix}$$

and these automorphisms act on elements of $\mathrm{SL}_n(q)$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\delta} = \begin{bmatrix} a & \nu^{-1}b \\ \nu c & d \end{bmatrix} \text{and} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\varphi} = \begin{bmatrix} a^p & b^p \\ c^p & d^p \end{bmatrix}.$$

Lemma 2.4.8. *[Whi13, Lemma 3.1] Let q be odd. In $\mathrm{SL}_2(q)$, the diagonal automorphism δ interchanges the conjugacy classes of c and d, interchanges the conjugacy classes of cz and dz, and fixes all other conjugacy classes.*

Lemma 2.4.9. *[Whi13, Lemma 3.2] Assume notation as above and let $1 \leq k < f$. In $\mathrm{SL}_2(q)$, the automorphism φ^k sends:*

(a) the conjugacy class of a^l to the conjugacy class of a^r, where $1 \leq r \leq [(q-2)/2]$ and

$r \equiv \pm\, lp^k \pmod{q-1}$;

(b) the conjugacy class of b^m to the conjugacy class of b^s, where $1 \leq s \leq [q/2]$ and

$$s \equiv \pm\ mp^k \pmod{q+1};$$

and fixes all other conjugacy classes.

Theorem 2.4.10. *[Dor71, Theorem 38.1] Let $G = \mathrm{SL}_2(q)$, with $q \geq 5$ an odd prime. Put $\varepsilon = (-1)^{(q-1)/2}$. Let $\rho \in \mathbb{C}$ be a primitive $(q-1)$th root of unity and $\sigma \in \mathbb{C}$ a primitive $(q+1)$th root of unity. Then the complex character table of G is*

	1	z	c	d	a^l	b^l
1_G	1	1	1	1	1	1
ϕ	q	q	0	0	1	-1
χ_i	$q+1$	$(-1)^i(q+1)$	1	1	$p^{il}+p^{-il}$	0
θ_j	$q-1$	$(-1)^j(q-1)$	-1	-1	0	$-(\sigma^{jm}+\sigma^{-jm})$
ξ_1	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}(1+\sqrt{\varepsilon q})$	$\frac{1}{2}(1-\sqrt{\varepsilon q})$	$(-1)^l$	0
ξ_2	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}(1-\sqrt{\varepsilon q})$	$\frac{1}{2}(1+\sqrt{\varepsilon q})$	$(-1)^l$	0
η_1	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}(-1+\sqrt{\varepsilon q})$	$\frac{1}{2}(-1-\sqrt{\varepsilon q})$	0	$(-1)^{m+1}$
η_2	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}(-1-\sqrt{\varepsilon q})$	$\frac{1}{2}(-1+\sqrt{\varepsilon q})$	0	$(-1)^{m+1}$

for $1 \leq i \leq (q-3)/2$, $1 \leq j \leq (q-1)/2$, $1 \leq l \leq (q-3)/2$, $1 \leq m \leq (q-1)/2$.

(The columns for the conjugacy classes (zc) and (zd) are missing in this table. These values are obtained from the relations

$$\chi(zc) = \frac{\chi(z)}{\chi(1)}\chi(c),\ \chi(zd) = \frac{\chi(z)}{\chi(1)}\chi(d),$$

for all irreducible characters χ of G.)

Theorem 2.4.11. *[Dor71, Theorem 38.2] Let $G = \mathrm{SL}_2(q)$, with $q = 2^n$. Let $\rho \in \mathbb{C}$ be a primitive $(q-1)$th root of unity and $\sigma \in \mathbb{C}$ a primitive $(q+1)$th root of unity. Then the complex character table of G is*

	1	z	a^l	b^m
1_G	1	1	1	1
ϕ	q	0	1	-1
χ_i	$q+1$	1	$\rho^{il}+\rho^{-il}$	0
θ_j	$q-1$	-1	0	$-(\sigma^{jm}+\sigma^{-jm})$

for $1 \leq i \leq (q-2)/2$, $1 \leq j \leq q/2$, $1 \leq l \leq (q-2)/2$, $1 \leq m \leq q/2$.

(The columns for the conjugacy classes (zc) and (zd) are missing in this table. These values are obtained from the same pair of relations shown above.)

Since $\mathrm{PSL}_2(q) = \mathrm{SL}_n(q)/Z(\mathrm{SL}_2(q)) = \mathrm{SL}_2(q)/\langle z\rangle$, we may obtain the explicit character tables of $\mathrm{PSL}_2(q)$ for odd q as follows:

Theorem 2.4.12. *[Geh02, Theorem 4.7] Let $G = \mathrm{PSL}_2(q)$, with $q \geq 5$ an odd prime. Let $\rho \in \mathbb{C}$ be a primitive $(q-1)$th root of unity and $\sigma \in \mathbb{C}$ a primitive $(q+1)$th root of unity.*

(a) If $q \equiv 1 \pmod 4$, then the complex character table of G is:

	$\langle z\rangle$	$\langle z\rangle c$	$\langle z\rangle d$	$\langle z\rangle a^l$	$\langle z\rangle a^{\frac{q-1}{4}}$	$\langle z\rangle b^m$
1_G	1	1	1	1	1	1
ϕ	q	0	0	1	1	-1
χ_i	$q+1$	1	1	$\rho^{il}+\rho^{-il}$	$\rho^{i\frac{q-1}{4}}+\rho^{-i\frac{q-1}{4}}$	0
θ_j	$q-1$	-1	-1	0	0	$-(\sigma^{jm}+\sigma^{-jm})$
ξ_1	$\frac{1}{2}(q+1)$	$\frac{1}{2}(\sqrt{q}+1)$	$\frac{1}{2}(1-\sqrt{q})$	$(-1)^l$	$(-1)^{\frac{q-1}{4}}$	0
ξ_2	$\frac{1}{2}(q+1)$	$\frac{1}{2}(1-\sqrt{q}))$	$\frac{1}{2}(\sqrt{q}+1)$	$(-1)^l$	$(-1)^{\frac{q-1}{4}}$	0

where $i = 2, 4, 6, \ldots, (q-5)/2$, $j = 2, 4, 6, \ldots, (q-1)/2$, $1 \leq l \leq (q-5)/4$, $1 \leq m \leq (q-1)/4$.

(b) If $q \equiv -1 \pmod 4$, then the complex character table of G is:

	$\langle z\rangle$	$\langle z\rangle c$	$\langle z\rangle d$	$\langle z\rangle a^l$	$\langle z\rangle b^m$	$\langle z\rangle b^{\frac{q+1}{4}}$
1_G	1	1	1	1	1	1
ϕ	q	0	0	1	-1	-1
χ_i	$q+1$	1	1	$\rho^{il}+\rho^{-il}$	0	0
θ_j	$q-1$	-1	-1	0	$-(\sigma^{jm}+\sigma^{-jm})$	$-(\sigma^{j\frac{q+1}{4}}+\rho^{-j\frac{q+1}{4}})$
η_1	$\frac{q-1}{2}$	$\frac{\sqrt{-q}-1}{2}$	$-\frac{1}{2}(1+\sqrt{-q})$	0	$(-1)^{m+1}$	$(-1)^{\frac{q+1}{4}+1}$
η_2	$\frac{q-1}{2}$	$\frac{1-\sqrt{q}}{2}$	$\frac{1}{2}(\sqrt{q}+1)$	0	$(-1)^{m+1}$	$(-1)^{\frac{q+1}{4}+1}$

where $i = 2, 4, 6, \ldots, (q-3)/2$, $j = 2, 4, 6, \ldots, (q-3)/2$, $1 \leq l \leq (q-3)/4$, $1 \leq m \leq (q-3)/4$.

2.4.3 Special Linear Groups

A famous result of Zsigmondy is given below (see for example [Zsi92] or [HB82, Theorem 8.3]). A new proof of this result has been presented in [Roi97].

Theorem 2.4.13. *(**Zsigmondy's Theorem**) Let q, n be integers greater than 1. Then except in the cases $n = 2$, $q = 2^a - 1$ and $n = 6$, $q = 2$, there is a prime l with the following properties:*

(a) l divides $q^n - 1$;

(b) l does not divide $q^i - 1$ whenever $0 < i < n$;

(c) l does not divide n.

Let $q, n \geq 2$ be integers. Suppose that $(q, n) \neq (2, 6)$ and if $n = 2$ assume that $q + 1$ is not a power of 2. Then by Zsigmondy's Theorem 2.4.13, a Zsigmondy prime divisor $l(n)$ always exists. A Zsigmondy prime divisor is defined as a prime $l(n)$ such that $l(n) \mid q^n - 1$ but $l(n) \nmid \prod_{i=1}^{n-1}(q^i - 1)$.

The table below shows Zsigmondy primes l_i for the orders of corresponding tori T_i. Note that elements of order l_i in the torus T_i are regular elements. It was shown in [MNO00] that almost all characters of simple groups vanish on elements of order l_1 or l_2 whenever l_1 and l_2 exist.

Table 2.4: Tori and Zsigmondy primes for classical groups of Lie type

M	$\lvert T_1\rvert$	$\lvert T_2\rvert$	l_1	l_2
A_n	$(q^{n+1}-1)/(q-1)$	$q^n - 1$	$l(n+1)$	$l(n)$
${}^2\mathrm{A}_n$ ($n \geq 3$ odd)	$(q^{n+1}-1)/(q+1)$	$q^n + 1$	$l(n+1)$	$l(2n)$
${}^2\mathrm{A}_n$ ($n \geq 2$ even)	$(q^{n+1}+1)/(q+1)$	$q^n - 1$	$l(2n+2)$	$l(n)$
$\mathrm{B}_n, \mathrm{C}_n$ ($n \geq 3$ odd)	$q^n + 1$	$q^n - 1$	$l(2n)$	$l(n)$
$\mathrm{B}_n, \mathrm{C}_n$ ($n \geq 2$ even)	$q^n + 1$	$(q^{n-1}+1)(q+1)$	$l(2n)$	$l(2n-2)$
D_n ($n \geq 5$ odd)	$q^n - 1$	$q^{n-1}+1)(q+1)$	$l(n)$	$l(2n-2)$
D_n ($n \geq 4$ even)	$(q^{n-1}-1)(q-1)$	$(q^{n-1}+1)(q+1)$	$l(n-1)$	$l(2n-2)$
${}^2\mathrm{D}_n$ ($n \geq 4$)	$q^n + 1$	$(q^{n-1}+1)(q-1)$	$l(2n)$	$l(2n-2)$

Proposition 2.4.14. *Let $G = \mathrm{PSL}_3(q)$, $q \geq 2$. Then every non-linear irreducible character of G vanishes either on an element of Zsigmondy prime order l_1 or l_2, on an involution, or on a regular unipotent element of prime order.*

Proof. This follows from [MNO00, Lemmas 5.3 and 5.4]. □

Theorem 2.4.15. *Let $G = \mathrm{PSL}_n(q)$, with $n \geq 2$, $q \geq 2$ and $(n, q) \notin \{(2, 2), (2, 3)\}$. Let $\mathcal{C}_i$, $i = 1, 2$ be conjugacy classes of regular elements of G in T_i with the following properties:*

(a) If n is even, then elements in $\mathcal{C}_1$ have order $(q^{n/2} + 1)/\gcd(2, q - 1)$ and those in $\mathcal{C}_2$ have order $(q^{n-1} - 1)/\gcd(n, q - 1)$;

(b) If n is odd, then elements in $\mathcal{C}_1$ have order $(q^n - 1)/((q - 1)\gcd(n, q - 1))$ and those in $\mathcal{C}_2$ have order $q^{(n-1)/2} + 1$.

Then every non-linear character $\chi \in \mathrm{Irr}(G)$ that is not the Steinberg character vanishes either on $\mathcal{C}_1$ or on $\mathcal{C}_2$.

Let $\chi \in \mathrm{Irr}(G)$ be unipotent and not the Steinberg character. If Zsigmondy primes l_1 and l_2 exist, then χ is of l_1-defect zero or l_2-defect zero.

Proof. This follows from the proof of [MSW94, Theorem 2.1]. □

2.4.4 Special Unitary Groups

Proposition 2.4.16. *Let $G = \mathrm{PSU}_3(q)$, $q \geq 2$. Then every non-linear $\chi \in \mathrm{Irr}(G)$ vanishes either on an element of Zsigmondy prime order l_1 or l_2, an involution, or on a regular unipotent element.*

Proof. This follows from [MNO00, Lemmas 5.3 and 5.4]. □

Theorem 2.4.17. *Let $G = \mathrm{PSU}_n(q)$, with $n \geq 2$, $q \geq 2$ and $(n, q) \notin \{(2, 2), (2, 3)\}$. Let $\mathcal{C}_i$, $i = 1, 2$ be conjugacy classes of regular elements of G in T_i with the following properties:*

(a) If n is even, then elements in $\mathcal{C}_1$ have order $(q^{n/2} + (-1)^{n/2})/\gcd(2, q - 1)$ and those in $\mathcal{C}_2$ have order $(q^{n-1} + 1)/\gcd(n, q + 1)$;

(b) If n is odd, then elements in $\mathcal{C}_1$ have order $(q^n+1)/((q+1)\gcd(n,q+1))$ and those in $\mathcal{C}_2$ have order $q^{(n-1)/2}+(-1)^{(n-1)/2}$.

Then every non-linear character $\chi \in \mathrm{Irr}(G)$ that is not the Steinberg character vanishes either on $\mathcal{C}_1$ or on $\mathcal{C}_2$.

Let $\chi \in \mathrm{Irr}(G)$ be unipotent and not the Steinberg character. If Zsigmondy primes l_1 and l_2 exist, then χ is of l_1-defect zero or l_2-defect zero.

Proof. This follows from the proof of [MSW94, Theorem 2.2]. □

2.4.5 Symplectic Groups and Orthogonal Groups

Theorem 2.4.18. *Let $G \in \{\mathrm{Sp}_{2n}(q) \mid n \geq 2\} \cup \{\mathrm{PSO}_{2n+1}(q) \mid n \geq 3$ and q odd$\} \cup \{\mathrm{PSO}^-_{2n}(q) \mid n \geq 4\} \cup \{\mathrm{PSO}^+_{2n}(q) \mid n \geq 5,\ n$ odd$\}$. Suppose that $\mathcal{C}_i$ is a conjugacy class of regular elements in T_i, $i = 1, 2$. Then every non-linear character $\chi \in \mathrm{Irr}(G)$ that is not the Steinberg character vanishes either on regular elements of $\mathcal{C}_1$ or regular elements of $\mathcal{C}_2$.*

Proof. This follows from [MSW94, Theorems 2.3-2.6] □

2.5 Imprimitive characters that vanish on one conjugacy class

We now look at finite groups with an imprimitive character that vanish on one conjugacy class. Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$ be imprimitive such that $\chi = \varphi^G$ for some $\varphi \in \mathrm{Irr}(H)$ where H is a proper subgroup of G. Let $N := H_G$ denote the largest normal subgroup G contained in H. Then G/N is a transitive permutation group on the set Ω of right cosets of H/N in G/N with point stabilizer H/N. If $x \in G$ is a derangement, then $\varphi^G(x) = 0$ by the definition of an imprimitive character. If χ is an imprimitive character that vanishes on one conjugacy class, then G/N is a transitive permutation group that has one conjugacy class of derangements. By Theorem 2.1.10, G/N is a primitive permutation group with one conjugacy class of derangements. This means that H is a maximal subgroup of G. The result below now follows by [BTV15, Theorem 1.6]:

Theorem 2.5.1. *[BTV15, Theorem 1.6] Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$. Suppose that $\chi = \varphi^G$ for some $\varphi \in \mathrm{Irr}(H)$, where H is a subgroup of G with $nv(\chi) = 1$. Then H is a maximal subgroup of G. Let $N = H_G$. Then one of the following holds:*

(a) *G is a Frobenius group with an abelian odd-order kernel $H = G'$ of index 2;*

(b) *G/N is a 2-transitive Frobenius group with an elementary abelian kernel M/N of order p^n for some prime p and integer $n \geqslant 1$, and a complement H/N of order $p^n - 1$. Moreover, $M' = N$ and one of the following holds:*

  (i) *M is a Frobenius group with kernel M' and $p^n = p > 2$;*

  (ii) *M is a Frobenius group with kernel $K \lhd G$ such that $G/K \cong \mathrm{SL}_2(3)$ and $M/K \cong \mathrm{Q}_8$;*

  (iii) *M is a Camina p-group;*

(c) *$G/N \cong \mathrm{PSL}_2(8){:}3$, $H/N \cong \mathrm{D}_{18}{:}3$ and N is a nilpotent $7'$-group;*

(d) *$G/N \cong \mathrm{A}_5$, $H/N \cong \mathrm{D}_{10}$ and N is a 2-group.*

Chapter 3

Primitive characters that vanish on one conjugacy class

In this chapter we shall present Theorems 1.0.3, 1.0.4 and 1.0.5 which are restated bearing the numbering of Chapter 3.

Theorem 3.0.1. *Let G be a finite non-solvable group. Suppose that $\chi \in \mathrm{Irr}(G)$ is primitive, $n\upsilon(\chi) = 1$ and $\upsilon(\chi) = \mathcal{C}$. Let $K = \ker\chi$, $Z = Z(\chi)$. Then there exists a normal subgroup M of G such that $Z < M$, $\mathcal{C} \subseteq M \setminus Z$ and M/Z is the unique minimal normal subgroup of the group G/Z. Moreover, one of the following holds:*

(a) G/Z is almost simple and M/K is quasisimple;

(b) G/Z is a Frobenius group with an abelian kernel M/Z of order p^{2n}, M/K is an extra-special p-group and Z/K is of order p.

Recall that a faithful irreducible character χ of a finite group M has property $(\star)$ if

(a) χ vanishes on elements of the same p-power order;

(b) the number of conjugacy classes that χ vanishes on is at most the size of the outer automorphism group of $M/Z(M)$;

(c) if $Z(M)$ is non-trivial, then $Z(M)$ is cyclic and of p-power order.

Theorem 3.0.2. *Let M be a quasisimple group. Suppose that M has a faithful irreducible character χ such that $(\star)$ holds. Then M is one of the following:*

(a) $M \cong \mathrm{PSL}_2(5)$, $\chi(1) = 3$ *or* $\chi(1) = 4$;

(b) $M \cong \mathrm{SL}_2(5)$, $\chi(1) = 2$ *or* $\chi(1) = 4$;

(c) $M \cong 3{\cdot}\mathrm{A}_6$, $\chi(1) = 9$;

(d) $M \cong \mathrm{PSL}_2(7)$, $\chi(1) = 3$;

(e) $M \cong \mathrm{PSL}_2(8)$, $\chi(1) = 7$;

(f) $M \cong \mathrm{PSL}_2(11)$, $\chi(1) = 5$ *or* $\chi(1) = 10$;

(g) $M \cong \mathrm{PSL}_2(q)$, $\chi(1) = q$, *where* $q \geq 5$;

(h) $M \cong \mathrm{PSU}_3(4)$, $\chi(1) = 13$;

(i) $M \cong {}^2\mathrm{B}_2(8)$, $\chi(1) = 14$.

Theorem 3.0.3. *Let G be a finite non-solvable group. Then $\chi \in \mathrm{Irr}(G)$ is faithful, primitive and $nv(\chi) = 1$ if and only if G is one of the following groups:*

(a) $G \cong \mathrm{PSL}_2(5)$, $\chi(1) = 3$ *or* $\chi(1) = 4$;

(b) $G \cong \mathrm{SL}_2(5)$, $\chi(1) = 2$ *or* $\chi(1) = 4$;

(c) $G \in \{\mathrm{A}_6{:}2_2,\ \mathrm{A}_6{:}2_3,\ 3{\cdot}\mathrm{A}_6{:}2_3\}$, $\chi(1) = 9$ *for all such* $\chi \in \mathrm{Irr}(G)$;

(d) $G \cong \mathrm{PSL}_2(7)$, $\chi(1) = 3$;

(e) $G \cong \mathrm{PSL}_2(8){:}3$, $\chi(1) = 7$;

(f) $G \cong \mathrm{PGL}_2(q)$, $\chi(1) = q$, *where* $q \geq 5$;

(g) $G \cong {}^2\mathrm{B}_2(8){:}3$, $\chi(1) = 14$.

3.1 Preliminaries

In this section we shall present some number theory results needed in subsequent sections.

Lemma 3.1.1. *If $2^a - 1 = q$, where q is a power of a prime, then q is a prime.*

Proof. This follows from [HB82, IX, Lemma 2.7]. □

Lemma 3.1.2. *Let p be a prime and f a positive integer. Then the following statements hold:*

(a) If $q = 2^{2f+1} > 8$, then $f + 1 < (q - 2)/2$.

(b) If $q = p^f > 32$, then $2f + 1 < (q - 7)/4$.

(c) If $q = p^f > 11$, then $6f + 1 < (q^2 + q - 2)/9$.

(d) If $q = p^f > 13$, then $6f + 1 < (q^2 - q - 2)/9$.

Proof. We shall prove these results by induction on $f > 1$. For (a) assume that the statement is true for $f = k$, that is, if $q_k = 2^{2k+1} > 8$, then $k + 1 < (q_k - 2)/2$. If $q_{k+1} = 2^{2(k+1)+1} = 4q_k > 8$, then $(k + 1) + 1 < (q_k - 2)/2 + 1 = q_k/2 < (4q_k - 2)/2 = (q_{k+1} - 2)/2$.

For (b) since the largest f arises in the case when $p = 2$, it is sufficient to prove the statement when $p = 2$. Assume that if $q_k = 2^k > 32$, then $2k + 1 < (q_k - 7)/4$. Then $q_{k+1} = 2^{k+1} = 2 \cdot 2^k = 2q_k$ implies $2(k + 1) + 1 < (q_k - 7)/4 + 2 = q_k/4 - 1/4 < q_k/2 - 7/4 = (2q_k - 7)/4 = (q_{k+1} - 7)/4$ as required.

For (c) let $q = 2^f$. Assume that if $q_k = p^k > 11$, then $6k+1 < (q_k^2 + q - k - 2)/9$. Then $q_{k+1} = 2^{k+1} = 2q_k > 11$, $6(k + 1) + 1 < (q_k^2 + q_k - 2)/9 + 6 = q_k^2/9 + q_k/9 + 52/9 < 4q_k^2/9 + 2q_k/9 + 52/9 = (q_{k+1}^2 + q_{k+1} - 2)/9$ as required.

For (d) let $p = 2$ and assume that if $q_k = 2^k > 13$, then $6k + 1 < (q_k^2 - q_k - 2)/9$. Then $6(k + 1) + 1 < (q_k^2 - q_k - 2)/9 + 6 = (q_k^2 - q_k + 52)/9 < 2(q_k^2 - q_k - 1)/9 < (4q_k^2 - 2q_k - 2)/9 < q_{k+1}^2 - q_{k+1} - 2)/9$ and the result follows. □

Lemma 3.1.3. *[Wak08, Lemma 3.1.1] Let $q = p^f$ for some prime p and positive integer f. Then the number $q^2 + q + 1$ cannot be written in the form l^m with l prime and $m > 1$. The number $q^2 - q + 1$ is of the form l^m with l prime and $m > 1$, only for $q = 19$.*

Lemma 3.1.4. *Let $q = p^f$ for some prime p and positive integer f. Suppose that a and b are positive integers and c is a non-negative integer.*

(a) If $q - 1 = 2^a 3^b$ and $q + 1 = 2^c$, then $q = 3$ or 7;

(b) *If* $q-1=2^a$ *and* $q+1=2^a3^b$, *then* $q=3,5$ *or* 17*;*

(c) *If* $q-1=2^a$ *and* $q+1=2^b5^c$, *then* $q=3$ *or* 9*;*

(d) *If* $q-1=2^a5^b$ *and* $q+1=2^c$, *then* $q=3$.

Proof. (a) If $b=0$, then $q-1=2^a$ and since $q+1=2^c$, we have that $q=3$. If $2=(q+1)-(q-1)=2^c-2^a3^b=2^a(2^{c-a}-3^b)$, then $a=1$ and $2^{c-1}-3^b=1$, that is, $3^b+1=2^{c-1}$. By Lemma 3.1.1, $b=1$ and so $q=7$.

(b) If $b=0$, then $q=3$. Otherwise $2=2^a(3^b-2^{c-a})$ and so $a=1$ and $3^b-1=2^{c-a}$. By Zsigmondy's Theorem 2.4.13, there is a Zsigmondy prime $l \mid (3^b-1)$ except when $b\leq 2$. If $b=1$, then $q=5$ and if $b=2$, then $q=17$.

(c) If $c=0$, then $q=3$. If $c\geq 1$, then $a>b$. Now $2=2^b5^c-2^a=2^b(5^c-2^{a-b})$. Since $b\geq 1$, we have that $b=1$ and $5^c-2^{a-b}=1$, that is, $5^c-1=2^{a-1}$. By Zsigmondy's Theorem 2.4.13, there exists a Zsigmondy prime $l \mid (5^c-1)$ unless $c=1$. Hence $q+1=10$ and so $q=9$.

(d) If $b=0$, then $q=3$. If $b\geq 1$, then $2=2^c-2^a5^b=2^a(2^{c-a}-5^b)$. Hence $a=1$ and $2^{c-1}-5^b=1$ which implies that $5^b+1=2^{c-1}$. By Lemma 3.1.1, $b=1$ which yields $q=11$ as the only possibility. This is a contradiction since $q+1=12$ is not a power of 2. □

Theorem 3.1.5. *Let M be a simple group. Then M has an element of order $2r$ for some odd prime r except when:*

(a) $M\cong {}^2\mathrm{B}_2(q)$, $q=2^{2f+1}$*;*

(b) $M\cong \mathrm{PSL}_2(q)$, $q\geq 4$*;*

(c) $M\cong \mathrm{PSL}_3(4)$.

Proof. This follows from [Suz61, III, Theorem 5]. □

Lemma 3.1.6. *[Bla94, Theorem 1] Assume that G is a quasisimple group and let $z\in Z(G)$. Then one of the following holds:*

(a) $|z|=6$ *and* $G/Z(G)\cong \mathrm{A}_6,\mathrm{A}_7,\mathrm{Fi}_{22},\mathrm{PSU}_6(2)$ *or* ${}^2\mathrm{E}_6(2)$;

(b) $|z|=6$ *or* 12 *and* $G/Z(G)\cong \mathrm{PSL}_3(4),\mathrm{PSU}_4(3)$ *or* M_{22};

(c) $|z| = 2$ *or* 4, $G/Z(G) \cong \mathrm{PSL}_3(4)$, *and* $Z(G)$ *is non-cyclic;*

(d) z *is a commutator.*

Lemma 3.1.7. *Suppose that a finite group* G *has a faithful irreducible character* χ *such that* $nv(\chi) = 1$ *and* $v(\chi) = \mathcal{C}$, *with* $\mathcal{C} \subseteq K \setminus L$, K/L *abelian chief factor of* G. *Then* $L = Z(G)$ *has order* p, K *is an extra-special* p*-group and* χ *is primitive.*

Proof. This follows from [DRB07, Propositions 1 and 4]. □

Lemma 3.1.8. *Let* G *be a finite group with* $[x, y] \in Z(G)$ *for some* $x, y \in G$. *If* $\chi \in \mathrm{Irr}(G)$ *is faithful with* $\chi(x) \neq 0$, *then* $[x, y] = 1$.

Proof. We use the argument in [MNO00, Lemma 2.1]. Suppose that $z = [x, y] \neq 1$. Then $xz = x^y$ and $\chi(x) = \chi(x^y) = \chi(xz) = \chi(x)\lambda(z)$, where $\lambda \in \mathrm{Irr}(Z(G))$ such that $\chi_{Z(G)} = \chi(1)\lambda$. Dividing by $\chi(x)$, we have $1 = \lambda(z)$. On the other hand, $z \neq 1$ implies that $\lambda(z) \neq 1$ since λ is faithful. The result follows from this contradiction. □

Lemma 3.1.9. *Let* G *be a finite non-solvable group. Let* χ *be a faithful primitive irreducible character of* G *such that* $nv(\chi) = 1$. *Put* $v(\chi) = \mathcal{C}$ *and* $Z = Z(G)$. *Then:*

(a) there exists a normal subgroup M *of* G *such that* $Z < M$, $\mathcal{C} \subseteq M \backslash Z$ *and* M/Z *is the unique minimal normal subgroup of the group* G/Z.

Let N *be a normal subgroup of* G.

(b) If $N \cap \mathcal{C} = \emptyset$, *then* $N \leqslant Z$;

(c) If χ_N *is reducible, then* $N \leqslant Z$. *If* χ_N *is irreducible, then* $\mathcal{C} \subseteq N$ *and* $M \leqslant N$.

Proof. We first show (b) and the first part of (c). Note that since N is a normal subgroup of G either $N \cap \mathcal{C} = \emptyset$ or $\mathcal{C} \subseteq N$. For (b), if $N \cap \mathcal{C} = \emptyset$, then since χ vanishes on one conjugacy class, namely $\mathcal{C}$, we have that χ does not vanish on N. Since χ is primitive, $\chi_N = e\psi$, for some $\psi \in \mathrm{Irr}(N)$ and positive integer e by Lemma 2.3.8. By Burnside's Theorem, the only characters which do not vanish on any conjugacy class are the linear characters, so $\psi(1) = 1$. Using Corollary 2.3.2, $N' \leqslant \ker\psi \leqslant \ker\chi = 1$. Hence N is an abelian normal subgroup and since χ is faithful and primitive, $N \leqslant Z$ by Corollary 2.3.9 and (b) follows.

If χ_N is reducible, then $[\chi_N, \chi_N] \geq 2$ using Corollary 2.3.1. By Theorem 1.0.10, we have $2 \leq [\chi_N, \chi_N] \leq 1 + \frac{|\mathcal{C}\backslash N|}{|N|}$ which implies that $\mathcal{C}\backslash N$ is not empty. Since N is normal in G we deduce that $\mathcal{C} \cap N = \emptyset$. This means that $N \leqslant Z$ by (b). Hence the first part of (c) holds.

We now prove (a). Choose $M \lhd G$ such that M/Z is a minimal normal subgroup of G/Z. If $\mathcal{C} \nsubseteq M$, then $\mathcal{C} \cap M = \emptyset$ and $M \leqslant Z$ by (b), a contradiction. Thus $\mathcal{C} \subseteq M \setminus Z$. Suppose that M/Z is not unique and let M_1/Z be another minimal normal subgroup of G/Z. Then $\mathcal{C} \subseteq M_1$ by using a similar argument as above, and so $\mathcal{C} \subseteq M \cap M_1 = Z$. But this is a contradiction since $\mathcal{C}$ cannot be contained in Z. Hence M/Z is unique and (a) follows.

To establish the last part of (c), suppose that χ_N is irreducible and $N \cap \mathcal{C} = \emptyset$. Then $N \leqslant Z$ by (b). Thus N is abelian and χ is linear contradicting the fact that χ vanishes on $\mathcal{C}$. It follows that $\mathcal{C} \subseteq N$. We claim that $cz \in \mathcal{C}$ for all $z \in Z$, $c \in \mathcal{C}$. Suppose that $\mathfrak{X}$ is a representation affording χ. Then $\mathfrak{X}$ is a scalar representation on Z and $\mathfrak{X}(z)$ is a scalar of the form λI by Lemma 2.3.3(a). Evaluating, we get $\chi(cz) = tr(\mathfrak{X}(cz)) = tr(\lambda\mathfrak{X}(c)) = \lambda\chi(c) = 0$, that is, $cz \in \mathcal{C}$. We have that $Z < N$ and N/Z is a normal subgroup of G/Z. By (a), M/Z is the only minimal normal subgroup of G/Z, implying that $M/Z \leqslant N/Z$, that is, $M \leqslant N$ and the result follows. □

Lemma 3.1.10. *Let G be a finite group. Let χ be a faithful irreducible character of G such that $nv(\chi) = 1$. Put $v(\chi) = \mathcal{C}$ and $Z = Z(G)$. If Z is non-trivial, then every non-trivial $z \in Z$ is a commutator. Moreover, $z = [x, y]$ for some $x, y \in \mathcal{C}$ and Z is cyclic of prime power order.*

Proof. Let Z be non-trivial. We show that every non-trivial element z of Z is a commutator. Now $cz \in \mathcal{C}$ for $c \in \mathcal{C}$ by Lemma 3.1.9. This means there exists $g \in G$ such that $cz = g^{-1}cg$ and therefore $z = c^{-1}g^{-1}cg$ as required. To prove the lemma's last assertion, suppose that z is non-trivial and $z = [x, y] = x^{-1}y^{-1}xy$, where $x, y \in G$ and $x \notin \mathcal{C}$. By Lemma 3.1.8, $z = [x, y] = 1$, a contradiction. Hence the result follows.

We know that Z is cyclic by Lemma 2.3.3(d). We may assume that $c \in \mathcal{C}$ is of order p^r for some positive integer r using Theorem 2.4.2. Then $z^{p^r} = c^{p^r}z^{p^r} = (cz)^{p^r} = (g^{-1}cg)^{p^r} = g^{-1}c^{p^r}g = 1$ and so Z is of prime power order. □

3.2 A reduction theorem

In this section we reduce our main problem to almost simple and quasisimple cases. In the following proposition we follow the method of proof employed in Lemma 2.3 and Theorem 1.1 of [Qia07] with χ primitive.

Proposition 3.2.1. *Under the hypothesis and notation of Lemma 3.1.9, suppose further that M/Z is abelian. Then G/Z is a Frobenius group with an abelian kernel M/Z of order p^{2n} for some prime p and $n \in \mathbb{N}$, M is an extra-special p-group and Z is of order p.*

Proof. There exists a normal subgroup M of G such that $\mathcal{C} \subseteq M\backslash Z$ by Lemma 3.1.9(a). If χ_M is reducible, then $M \leqslant Z$ by Lemma 3.1.9(b). This contradicts the choice of M. Hence χ_M is irreducible. By Lemma 2.3.3(c), χ_Z is reducible since χ is non-linear. Thus M is an extra-special p-group with Z of order p by Lemma 3.1.7.

Using Theorem 2.3.11 we deduce that $\chi_Z = f\varphi$, where $f^2 = |M/Z| = p^{2n}$ for some positive integers f and n and linear character φ of Z. It follows that $\chi(1) = f\varphi(1) = f = p^n$ and hence $\chi(1)$ is a prime power. It follows from Lemma 2.4.4 that $p \nmid |G{:}M|$ and hence M is the unique Sylow p-subgroup of G.

Now we show that $G/Z = \overline{G}$ is a Frobenius group with kernel $M/Z = \overline{M}$. Suppose that $|\mathbf{C}_{\overline{M}}(\overline{x})| > 1$ for some non-trivial p'-element $\overline{x}$ of $\overline{G}$. Let $\overline{Y} = \langle\overline{x}\rangle\overline{M}$, $\overline{T} = [\overline{M}, \langle\overline{x}\rangle]$ with $Y = \langle x\rangle M$. Then $\overline{Y}' = [\overline{M}\langle\overline{x}\rangle, \overline{M}\langle\overline{x}\rangle] = [\overline{M}, \overline{M}\langle\overline{x}\rangle][\langle\overline{x}\rangle, \overline{M}\langle\overline{x}\rangle] = [\overline{M}, \overline{M}][\langle\overline{x}\rangle, \overline{M}][\langle\overline{x}\rangle, \overline{M}][\langle\overline{x}\rangle, \langle\overline{x}\rangle] = [\overline{M}, \langle\overline{x}\rangle] = \overline{T}$ since $[\overline{M}, \overline{M}] = 1$, $[\langle\overline{x}\rangle, \langle\overline{x}\rangle] = 1$.

We claim that $\overline{T} = \overline{Y}' < \overline{M}$. Since $\langle\overline{x}\rangle\overline{M}$ is a semidirect product of $\langle\overline{x}\rangle$ and $\overline{M}$, $\langle\overline{x}\rangle$ acts via automorphisms on $\overline{M}$. By Theorem 2.1.3 we have $\overline{M} = \mathbf{C}_{\overline{M}}(\langle\overline{x}\rangle) \times [\overline{M}, \langle\overline{x}\rangle]$ because $(|\overline{M}|, |\langle\overline{x}\rangle|) = 1$. Since $\mathbf{C}_{\overline{M}}(\overline{x})$ is non-trivial, it follows that $\overline{Y}' < \overline{M}$ as required.

Let $\chi_M = \rho$ and $\psi = \chi_Y$ so that $\rho = \chi_M = \psi_M$, and let δ be an irreducible constituent of χ_T. Observe that M/Z is an abelian chief factor of G and $\rho = \chi_M$ is irreducible, so it is G-invariant. Moreover, $\rho_Z = \chi_Z = \chi(1)\mu$ for some G-invariant $\mu \in \mathrm{Irr}(Z)$. Using Theorem 2.3.11, we can see that ρ is fully ramified in M/Z and by Proposition 2.3.12, ρ vanishes on $M\backslash Z$. Note that $Z \leqslant T$. Thus ρ vanishes on $M\backslash T$. It then follows that $\psi(1) = \rho(1) > \delta(1)$. Note that Y/T is abelian, hence every chief factor of every subgroup of Y/T has non-square order. Also $T \leq M$ is solvable.

By Theorem 2.3.13, Y is a relative M-group with respect to T. We have that $\psi = \lambda^Y$, where $\lambda \in \mathrm{Irr}(B)$, $T < B \leqslant Y$, and $\lambda_T = \delta$. We now show that $B < Y$. Suppose that $Y = B$. Whence ρ_T is irreducible. Then $\rho_T = \chi_T = \delta$, which is a contradiction since $\rho(1) > \delta(1)$ by the argument in the paragraph above. Hence $B < Y$.

Let $\mathcal{C}_1$ be the on which ψ vanishes on. It follows that $Y \backslash B \subseteq \mathcal{C}_1 \subseteq \mathcal{C} \subset M$. Thus $Y \backslash B \subseteq M$, that is, $M \cup B = Y$, a contradiction since M and B are proper subgroups of Y. Thus $\mathbf{C}_{\overline{M}}(\overline{x})$ is trivial for any non-trivial p'-element $\overline{x}$ of $\overline{G}$. Thus $\mathbf{C}_{\overline{M}}(\overline{m}) \leqslant \overline{M}$ for all non-trivial $\overline{m} \in \overline{M}$. By Theorem 2.1.15, G/Z is a Frobenius group with kernel M/Z.

Finally, we show that $M < G$. If $\overline{M} = \overline{G}$, that is, if Z is a maximal normal subgroup of G, then G/Z is simple and abelian. Hence G/Z is cyclic, whence G is abelian, a contradiction since χ is non-linear. Hence the result follows. □

Recall that G^{∞} denotes the *solvable residual* of a group G, the smallest normal subgroup N of G such that G/N is solvable.

Proposition 3.2.2. *Let G be a finite non-solvable group. Let χ be a faithful primitive irreducible character of G such that $nv(\chi) = 1$. Put $v(\chi) = \mathcal{C}$ and $Z = Z(G)$. Suppose that M is a normal subgroup of G such that $Z < M$ and M/Z is a non-abelian minimal normal subgroup of G/Z. Then G/Z is almost simple and M is quasisimple.*

Proof. By Lemma 3.1.10, we may assume that Z is a p-group and all elements in $\mathcal{C}$ are p-elements. We claim that M is perfect. If $\chi_{M'}$ is reducible, then $M' \leqslant Z$ by Lemma 3.1.9(c). This implies that M is solvable, contradicting the hypothesis that M/Z is non-abelian. Hence $\chi_{M'}$ is irreducible and it follows that $M \leqslant M'$ using Lemma 3.1.9(c). Thus M is perfect as claimed. Since M/Z is a non-abelian chief factor of G, we may write $M/Z = T_1/Z \times T_2/Z \times \cdots \times T_k/Z$, where the T_i/Z are isomorphic non-abelian simple groups.

Suppose that $k = 1$. Note that M is quasisimple because $M/Z = T_1/Z$ is simple and M is perfect. If $M = G$, then the result follows. Suppose that $M < G$. We first claim that there exists a maximal subgroup H of G such that $\mathcal{C} \nsubseteq H$. Otherwise $\mathcal{C}$ is contained in every maximal subgroup of G, that is, $\mathcal{C} \subseteq \Phi(G)$, the Frattini subgroup of G. We infer from Lemma 3.1.9, that $M \leq \Phi(G)$, contradicting the hypothesis that M is non-abelian since $\Phi(G)$ is nilpotent. Hence our claim is true. We conclude that there exists a maximal subgroup H of G such that $\mathcal{C} \nsubseteq H$. Since H_G is a normal subgroup of G, $\mathcal{C} \cap H_G = \emptyset$ or $\mathcal{C} \subseteq H_G$. Thus $\mathcal{C} \cap H_G = \emptyset$ because $\mathcal{C} \nsubseteq H$. It follows that H/H_G is a maximal subgroup of G/H_G, G/H_G is a primitive permutation group on the right cosets of H/H_G in G/H_G; and $H_G \leqslant Z$ by Lemma 3.1.9(b). Suppose that $H_G \neq Z$. Since Z/H_G is an abelian normal subgroup of G/H_G, we have that G/H_G has an abelian minimal normal subgroup, S/H_G, which is central in G/H_G. By Theorem 2.1.5, $\mathbf{C}_{G/H_G}(S/H_G) = G/H_G = S/H_G$ and so $G = Z$, which implies that χ is linear contradicting the fact that χ vanishes on $\mathcal{C}$ by Burnside's Theorem. Hence $H_G = Z$. If $H = Z$, then G/Z is simple. Let $xH \in G/H$ be a non-trivial element of G/H. Then $\langle x \rangle H$ is a subgroup of G/H and since $H < \langle x \rangle H$, $G = \langle x \rangle H$, that is, G/H is cyclic. Since $H = Z$, we have that G is abelian, a contradiction. Hence $Z \neq H$.

Since M/Z is the unique minimal normal subgroup of G/Z, it follows that G/Z is an almost simple group with socle M/Z by Theorem 2.1.6.

Now we assume that $k \geq 2$ and seek for a contradiction.

Assume that $Z = \{1_G\}$. Then $M = T_1 \times T_2 \times \cdots \times T_k$. Since χ_M is irreducible, we have that $\chi_M = \theta_1 \times \theta_2 \times \cdots \times \theta_k$ where $\theta_i \in \mathrm{Irr}(T_i)$. Clearly θ_1 is non-linear. Let $a_i \in T_1$ be such that $\theta_1(a_1) = 0$, and let $a_2 \in T_2$ be a p'-element. We have $\chi(a_1) = \chi(a_1 a_2) = 0$. This implies that $a_1, a_1 a_2 \in \mathcal{C}$ are p-elements, a contradiction.

Assume that $|Z| > 1$ and each T_i^{∞} is simple. Note that T_i/Z is simple, whence

$$T_i = T_i^{\infty} Z = T_i^{\infty} \times Z.$$

Then $M = (T_1^{\infty} T_2^{\infty} \cdots T_k^{\infty}) \times Z$ is not perfect, a contradiction.

Assume that $|Z| > 1$ and T_i^{∞} is not simple for some i. We may assume that T_1^{∞} is not simple. Now T_1^{∞} is quasisimple since T_1^{∞} is perfect and

$$T_1^{\infty}/Z(T_1^{\infty}) \cong T_1^{\infty}/(Z \cap T_1^{\infty}) \cong T_1^{\infty} Z/Z = T_1/Z$$

is simple. Now $Z(T_1^{\infty}) = T_1^{\infty} \cap Z = \langle z_1 \rangle > \{1_G\}$, for some p-element $z_1 \in Z$, noting that Z is a cyclic p-group. We claim that z_1 is a commutator in T_1^{∞}. If not, then by Lemma 3.1.6, T_1^{∞} must be one of the groups in cases (a), (b), (c). Since z_1 is of prime power order, this rules out cases (a) and (b). But $Z(T_1^{\infty})$ is cyclic so case (c) does not hold, a contradiction. Thus by Lemma 3.1.10, $z_1 = [x, y]$ for some $x, y \in T_1^{\infty}$. Lemma 3.1.8 implies that $x, y \in \mathcal{C} \cap T_1^{\infty}$. Now let $s_2 \in T \setminus Z$ be a nontrivial p'-element.

Applying Lemma 3.1.8 again, we see that $[x, s_2] = [y, s_2] = 1_G$. In particular, ys_2 is not a p-element and so $ys_2 \notin \mathcal{C}$. This also implies, by Lemma 3.1.8, that $[x, ys_2] = 1_G$. However, since $[x, s_2] = [y, s_2] = 1_G$, we have $[x, ys_2] = [x, y] = z_1$, and this leads to the contradiction $z_1 = 1_G$. Now the proof is complete. □

Theorem 3.2.3. *Let G be a finite non-solvable group. Suppose that $\chi \in \mathrm{Irr}(G)$ is primitive, $n\upsilon(\chi) = 1$ and $\upsilon(\chi) = \mathcal{C}$. Let $K = \ker\chi$, $Z = Z(\chi)$. Then there exists a normal subgroup M of G such that $Z < M$, $\mathcal{C} \subseteq M \setminus Z$ and M/Z is the unique minimal normal subgroup of the group G/Z. Moreover, one of the following holds:*

(a) G/Z is almost simple and M/K is quasisimple;

(b) G/Z is a Frobenius group with an abelian kernel M/Z of order p^{2n}, M/K is an extra-special p-group and Z/K is of order p with K non-solvable.

Proof. Note that $\mathcal{C}$ is a conjugacy class of G/K, χ is faithful on G/K and $Z/K = Z(G/K)$ by Lemma 2.3.3(f). Moreover, $\chi \in \mathrm{Irr}(G/K)$ is primitive, faithful and vanishes on one conjugacy class $\mathcal{C}$. If G/K is solvable, that is, M/Z is abelian, then by Proposition 3.2.1, (b) holds. Otherwise G/K is non-solvable. Therefore the result follows from Proposition 3.2.2 in the case where M/Z is non-abelian. □

3.3 Quasisimple groups with a character vanishing on elements of the same order

In this section we prove Theorem 3.0.2.

Let m, n pe positive integers. Then by $m||n$, we mean that $m|n$ but $m^2 \nmid n$. If $l > 2$ not dividing q, the multiplicative order of q modulo l is denoted by $d_l(q)$. Below is a recent result of Lübeck and Malle [LM16]:

Theorem 3.3.1. *[LM16, Theorem 1] Let $l > 2$ be a prime and M a finite quasisimple group of l-rank at least 3. Then for any non-linear character $\chi \in \mathrm{Irr}(M)$ there exists an l-singular element $g \in M$ with $\chi(g) = 0$, unless either M is of Lie type in characteristic l, or $l = 5$ and one of the following hold:*

(a) $M \cong \mathrm{PSL}_5(q)$, $5||(q-1)$ and χ is unipotent;

(b) $M \cong \mathrm{PSU}_5(q)$, $5||(q+1)$ *and* χ *is unipotent;*

(c) $M \cong Ly$ *and* $\chi(1) \in \{48174, 11834746\}$*;*

(d) $M \cong \mathrm{E}_8(q)$ *with* q *odd,* $d_l(q) = 4$ *and* χ *is one of the characters in the Lusztig series of type* D_8*.*

3.3.1 Sporadic Groups

Using the Atlas [CCNPW85] we have the following result:

Theorem 3.3.2. *Let* M *be a quasisimple group such that* $M/Z(M)$ *is a sporadic simple group. Then every irreducible non-trivial character of* M *fails to satisfy* $(\star)$*.*

3.3.2 Alternating groups

Firstly we consider our problem when the center $Z(M)$ is trivial, i.e., when M is an alternating group. Recall that λ is a partition of n and χ_λ is an irreducible character of S_n or A_n corresponding to λ. We require the following results:

Proposition 3.3.3. *Let* $M = \mathrm{A}_n$ *or* S_n*,* $n > 8$*, and let* $\chi \in \mathrm{Irr}(M)$*. Then* $\chi(g) = 0$ *for some* $g \in M$ *of even order. Moreover, if the degree of* χ *is a power of* 2*, we can choose* $g \in M$ *whose order is* 4 *such that* $\chi(g) = 0$*.*

Proof. The first assertion follows from the proof of [LMS16, Proposition 4.3]. Now suppose that the degree of χ is a power of 2. By Theorems 2.3.24 and 2.3.25, $\chi_\lambda(1) = 2^r$, where $\lambda = (2^r, 1)$ or $\lambda = (2, 1^{2^r-1})$, and $n = 2^r + 1$. Now using the Murnaghan-Nakayama Rule 2.3.21 we shall give the appropriate choices for g.

Either $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$. If $n \equiv 3 \pmod 4$, then using the proof of [DPSS09, Proposition 2.4] we have $g = (4, 2^{(n-5)/2}, 1)$. If $n \equiv 1 \pmod 4$, then take $g = (4^2, 2^{(n-9)/2}, 1)$. □

Theorem 3.3.4. *Let* $M = \mathrm{A}_n$*,* $n \geq 5$*. If* $(\star)$ *holds, then* $M \cong \mathrm{A}_5$ *or* A_6*.*

Proof. Using the Atlas [CCNPW85] we infer that the only alternating groups with the desired property are A_5 and A_6 when $n \leq 13$. Suppose that $n > 13$.

We consider first the case when χ vanishes on a 2-element. Suppose that $\chi(1)$ is not a power of 2. Then by Theorem 2.4.6, χ vanishes on some element of odd prime order, implying that χ vanishes on at least two conjugacy classes of elements of distinct orders, a contradiction. Hence $\chi(1)$ is a power of 2. The result then follows by Proposition 3.3.3 and Theorem 2.4.2.

Now suppose that χ vanishes on a $2'$-element. By Proposition 3.3.3 we have that χ vanishes on an element of even order. Hence χ vanishes on at least two elements of distinct orders and the result follows. $\square$

We now consider our problem when $Z(M)$ is non-trivial.

Lemma 3.3.5. *Let $M = \tilde{\mathrm{A}}_n$ and suppose that $n \geq p+4$, where p is an odd prime. Suppose that $\chi \in \mathrm{Irr}(M)$ is faithful. Then χ vanishes on some p-singular element g of M.*

Proof. If $\lambda \in \mathcal{D}(n)$ is even, then the result follows using the proof of [LMS16, Theorem 4.5]. Now suppose that $\lambda \in \mathcal{D}(n)$ is odd. Then the characters $\chi_\lambda^\pm \in \mathrm{Irr}(\tilde{\mathrm{S}}_n)$ are the same and irreducible when restricted to $\tilde{\mathrm{A}}_n$. Let $g \in \tilde{\mathrm{A}}_n$ be an element which projects to a cycle type $\mu = (p, 2^2, 1^{n-p-4})$. Then $\lambda \neq \mu$ and so by Theorem 2.4.7, $\chi_\lambda^\pm(g) = 0$. $\square$

Theorem 3.3.6. *Let M be a quasisimple group such that $M/Z(M) \cong \mathrm{A}_n$, $n \geq 5$ and $Z(M) \neq \{1_M\}$. If $(\star)$ holds, then $M \cong 2{\cdot}\mathrm{A}_5$ with $p=2$ or $M \cong 3{\cdot}\mathrm{A}_6$ with $p=3$.*

Proof. Checking in the Atlas [CCNPW85] we see that the result is true when $n \leq 13$. Let $n \geq 14$ and $\chi \in 2{\cdot}\mathrm{A}_5$ be faithful. Then χ vanishes on an element whose order is not a power of p since χ vanishes on a 3-singular element and a 5-singular element by Lemma 3.3.5. The result follows by Theorem 2.4.2. $\square$

3.3.3 Groups of Lie type

We recall some definitions. Let $\mathcal{M}$ be a simple, simply connected algebraic group over $\overline{\mathbb{F}}_p$, the algebraic closure of a finite field of characteristic p and let $F : \mathcal{M} \to \mathcal{M}$ be a Frobenius map such that $M := \mathcal{M}^F$, the finite group of fixed points. Let $\mathcal{M}^*$ denote the dual group of $\mathcal{M}$ with corresponding Frobenius map $F^* : \mathcal{M}^* \to \mathcal{M}^*$. Then

$M^* := (\mathcal{M}^*)^{F^*}$ is the dual group of M. If $\mathcal{T}$ is an F-stable maximal torus of $\mathcal{M}$, then $T = \mathcal{T}^F$ and $T^* = (\mathcal{T}^*)^{F^*}$.

The following result will be essential:

Lemma 3.3.7. *[GM12, Lemma 3.2] Suppose that M is a finite group of fixed points as defined above. Let $x \in M$ be semisimple and $\chi \in \mathcal{E}(M, s^*)$ be an irreducible character of M with $\chi(x) \neq 0$. Then there is a maximal torus $T \leqslant M$ with $x \in T$ such that $T^* \leqslant \boldsymbol{C}_{M^*}(s^*)$ for a torus $T^* \leqslant M^*$ which is a dual group of T.*

Lemma 3.3.8. *[LM16, Remark 2.2] Let $\mathcal{M}$ be a connected reductive group with Steinberg morphism F and $\mathcal{T}$ an F-stable maximal torus of $\mathcal{M}$. Let l be a prime dividing $|\mathcal{T}^F|$. If $\mathcal{T}^F$ contains a regular element, then $\mathcal{T}^F$ also contains an l-singular regular element.*

Proof. If there exists a regular element whose order is divisible by l, the result follows. Let $t \in \mathcal{T}^F$ be a regular element whose order is prime to l and let $u \in \mathcal{T}^F$ be an element of order l. Then tu has order $l|t|$ and some power of tu equals t, hence tu is also regular. □

3.3.4 Classical groups

3.3.4.1 Special Linear Groups $\mathrm{SL}_2(q)$

We consider $\mathrm{SL}_2(q)$, $q = p^n$, where p is prime and n a positive integer. The character tables of $\mathrm{SL}_2(q)$ and $\mathrm{PSL}_2(q)$ are found in Section 2.4.2. Since $\mathrm{SL}_2(4) \cong \mathrm{PSL}_2(5) \cong \mathrm{A}_5$, $\mathrm{SL}_2(5) \cong 2{\cdot}\mathrm{A}_5$ and $\mathrm{A}_6 \cong \mathrm{PSL}_2(9)$, we will not consider these cases here. The sizes of the outer automorphism groups of finite groups of Lie type are displayed in Table 2.3. In particular, $|\mathrm{Out}(\mathrm{PSL}_2(q))| = \gcd(2, q-1){\cdot}f$ where $q = p^f$, p a prime and f a positive integer.

Proposition 3.3.9. *Let M be a quasisimple group such that $M/Z(M) = \mathrm{PSL}_2(q)$, $q = p^n$, where p is prime, n is a positive integer, $q \geq 7$ and $q \neq 9$. If $(\star)$ holds, then M is one of the following:*

(a) $M \cong \mathrm{PSL}_2(7)$, $\chi(1) = 3$;

(b) $M \cong \mathrm{PSL}_2(8)$, $\chi(1) = 7$;

(c) $M \cong \mathrm{PSL}_2(11)$, $\chi(1) = 5$ *or* $\chi(1) = 10$;

(d) $M \cong \mathrm{PSL}_2(q)$, $\chi(1) = q$.

Proof. If we consider $\mathrm{PSL}_2(q)$ or $\mathrm{SL}_2(q)$, $7 \leq q \leq 32$, then inspection of the character tables in the Atlas [CCNPW85] shows that M must coincide with one of the types (a)-(d). We may thus consider $q > 32$. We first suppose that q is odd. With reference to the character table of $\mathrm{SL}_2(q)$ displayed in Theorem 2.4.10, the faithful characters of M are the ones labelled by χ_i when i is odd, θ_j when j is odd, ξ_1 and ξ_2 when $q \equiv 3 \pmod 4$ ($\varepsilon = -1$), and η_1 and η_2 when $q \equiv 1 \pmod 4$ ($\varepsilon = 1$). This is because $\chi_i(z) = (-1)^i(q+1) = -(q+1)$ and $\chi_i(1) = q+1$, $\theta_j(z) = (-1)^j(q-1) = -1(q-1)$ and $\theta_j(1) = q-1$, $\xi_1(z) = \xi_2(z) = \frac{1}{2}\varepsilon(q+1) = -\frac{1}{2}(q+1)$ and $\xi_1(1) = \xi_2(1) = \frac{1}{2}(q+1)$, $\eta_1(z) = \eta_2(z) = -\varepsilon\frac{1}{2}(q-1) = -\frac{1}{2}(q-1)$ and $\eta_1(1) = \eta_2(1) = \frac{1}{2}(q-1)$.

Let $\chi \in \{\chi_i \mid i \text{ is odd}\}$. Then χ vanishes on $(q-1)/2$ conjugacy classes of elements represented by b^m, $1 \leq m \leq (q-1)/2$. Hence χ vanishes on more than $2f$ conjugacy classes by Lemma 3.1.2. Since the size of the outer automorphism group of $M/Z(M) = \mathrm{PSL}_2(q)$ is $2f$, χ does not satisfy hypothesis (b) Property ($\star$).

Let $\chi \in \{\theta_j \mid j \text{ is odd}\}$. Then χ vanishes on $(q-3)/2$ conjugacy classes of elements represented by a^l, $1 \leq l \leq (q-3)/2$. By the argument in the paragraph above, χ fails to satisfy ($\star$).

Now suppose that $\chi \in \{\xi_i \mid i = 1, 2 \text{ and } q \equiv 3 \pmod 4\}$. Then $\varepsilon = -1$ and χ vanishes on $(q-1)/2$ conjugacy classes and the result follows.

Lastly, let $\chi \in \{\eta_i \mid i = 1, 2 \text{ and } q \equiv 1 \pmod 4\}$. Then $\varepsilon = 1$ and χ vanishes on $(q-3)/2$ conjugacy classes. Again the result follows.

Now let $M = \mathrm{PSL}_2(q)$ with q odd. The character tables of $\mathrm{PSL}_2(q)$ are exhibited in Theorem 2.4.12. The faithful characters of M are those labelled ϕ, θ_j when j is even, χ_i when i is even, ξ_1 and ξ_2 when $q \equiv 1 \pmod 4$ ($\varepsilon = 1$), and η_1 and η_2 when $q \equiv 3 \pmod 4$ ($\varepsilon = -1$). This is because $\phi(z) = \phi(1) = p$, $\theta_j(z) = \theta(j)(1) = q-1$, $\chi_i(z) = \chi_i(1) = q+1$, $\xi_1(z) = \xi_2(z) = \xi_1(1) = \xi_2(1) = \frac{1}{2}(q+1)$ and $\eta_1(z) = \eta_2(z) = \eta_1(1) = \eta_2(1) = \frac{1}{2}(q-1)$.

Let us consider ϕ, the Steinberg character of M. Note that ϕ vanishes on two conjugacy classes represented by c and d, both of order p. Now M must coincide with type (d) of the statement of the proposition because the size of the outer automorphism group of M is $2f$.

Consider $\chi \in \{\chi_i \mid i \text{ is even}\}$. Then χ vanishes on $(q-1)/4$ conjugacy classes if $q \equiv 1 \pmod 4$ and χ vanishes on $(q-3)/4$ conjugacy classes if $q \equiv 3 \pmod 4$ by Theorem 2.4.12. Since the size of the outer automorphism group of M is $2f$, χ does not satisfy hypothesis (b) of Property $(\star)$ by Lemma 3.1.2.

Now consider $\chi \in \{\theta_j \mid j \text{ is even}\}$. Then χ vanishes on $(q-1)/4$ conjugacy classes when $q \equiv 1 \pmod 4$ and χ vanishes on $(q-3)/4$ conjugacy classes when $q \equiv 3 \pmod 4$, again by Theorem 2.4.12. In both cases χ vanishes on more than $2f$ conjugacy classes and we are done.

Let $\chi \in \{\xi_i \mid i = 1, 2 \text{ and } q \equiv 1 \pmod 4\}$. Then $\varepsilon = 1$ and χ vanishes on $(q-1)/4$ conjugacy classes so χ vanishes on more than $2f$ conjugacy classes by Lemma 3.1.2 and the result follows.

Now take $\chi \in \{\eta_i \mid i = 1, 2 \text{ and } q \equiv 3 \pmod 4\}$. Such a χ vanishes on $(q-3)/4$ conjugacy classes represented by a^l and so it fails to satisfy Property $(\star)$ by the argument in the paragraph above.

Finally, we consider $\mathrm{SL}_2(q)$ where q is even. Its character table is exhibited in Theorem 2.4.11. Note that since $\gcd(2, q-1) = 1$, $M = \mathrm{SL}_2(q) = \mathrm{PSL}_2(q)$. We may assume that $q \geq 32$. We consider first the Steinberg character ϕ of $\mathrm{PSL}_2(q)$. Then ϕ vanishes on one conjugacy class c. Hence M is a group of type (d) of our proposition. Consider χ_i, $1 \leq i \leq (q-2)/2$. Then χ_i vanishes on elements of the form b^m, $1 \leq m \leq q/2$. Hence χ vanishes on $q/2$ conjugacy classes. Also θ_j vanishes on at least $(q-2)/2$ conjugacy classes. Clearly the number of conjugacy classes is more than the size of the outer automorphism group of M in all these cases which contradicts hypothesis (b) of Property $(\star)$. Hence the result follows. $\square$

3.3.4.2 Special Linear Groups distinct from $\mathrm{SL}_2(q)$

Theorem 3.3.10. *Let M be a quasisimple group such that $M/Z(M)$ is a finite simple group of Lie type over a field of characteristic p, distinct from $\mathrm{PSL}_2(q)$. If $(\star)$ holds, then M is one of the following:*

(a) $M \cong \mathrm{PSU}_3(4)$, $\chi(1) = 13$;

(b) $M \cong {}^2\mathrm{B}_2(8)$, $\chi(1) = 14$.

We shall show that Theorem 3.3.10 holds by means of a series of propositions and also the whole of Section 3.3.5.

We first show that the Steinberg character of a classical group of Lie type does not satisfy $(\star)$:

Lemma 3.3.11. *Let M be a finite classical simple group of Lie type over a field of characteristic p, distinct from $\mathrm{PSL}_2(q)$. Then the Steinberg character χ does not satisfy $(\star)$.*

Proof. Suppose that $p = 2$. Then χ is of 2-defect zero and so χ vanishes on every 2-singular element of M. In particular, χ vanishes on an involution. By Theorem 3.1.5, M has an element of order $2r$ for some odd prime r except when $M \cong \mathrm{PSL}_3(4)$. The character table of $\mathrm{PSL}_3(4)$ exhibited in the Atlas [CCNPW85] confirms our conclusion for this special case. We may thus assume that M has an element g of order $2r$ with r as above. Then χ vanishes on g and so vanishes on 2 elements of distinct orders.

Now suppose that p is odd. Then χ is of p-defect zero and so χ vanishes on every p-singular element of M. In particular, χ vanishes on a unipotent element of order p. Now M has an element g of order pr, r prime, since the size of the connected component containing p is at least 2 by Theorem 2.2.7. Hence $\chi(g) = 0$ and the result follows. □

Let $\mathcal{M} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ and F be the standard Frobenius map. The conjugacy classes of F-stable maximal tori of $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ and $\mathrm{SL}_n(\overline{\mathbb{F}}_p)$ are characterised by conjugacy classes of S_n. Recall that conjugacy classes of S_n are parametrised by cycle types. Thus if $\mathcal{T} \leqslant \mathrm{GL}_n((\overline{\mathbb{F}}_p)$ corresponds to $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathrm{S}_n$ with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$, then $|T| = |\mathcal{T}^F| = \prod_i^m (q^{\lambda_i} - 1)$ and if $\mathcal{T} \leqslant \mathrm{SL}_n(\overline{\mathbb{F}}_p)$, then $(q-1)|T| = (q-1)|\mathcal{T}^F| = \prod_i^m (q^{\lambda_i} - 1)$.

Lemma 3.3.12. *[LM15, Lemma 4.1] Let $\lambda \vdash n$ be a partition, and $\mathcal{T}$ a corresponding F-stable maximal torus of $\mathrm{SL}_n(\overline{\mathbb{F}}_p)$. Assume that either all parts of λ are distinct, or $q \geq 3$ and at most two parts of λ are equal. Then $T = \mathcal{T}^F$ contains regular elements.*

Lemma 3.3.13. *[LM15, Lemma 3.2] Let $\mathcal{H} \leqslant \mathrm{PGL}_n(\overline{\mathbb{F}}_p)$ be a reductive subgroup containing F-stable maximal tori corresponding to cycle types $\lambda_1, \lambda_2, ..., \lambda_r$. If no intransitive or imprimitive subgroup of S_n contains elements of all these cycle types then $\mathcal{H} = \mathrm{PGL}_n(\overline{\mathbb{F}}_p)$.*

To use this result we may assume that $\mathcal{M}$ is connected reductive with a Steinberg endomorphism $F : \mathcal{M} \to \mathcal{M}$ and $M := \mathcal{M}^F$. If $\mathcal{T}^* \leqslant \mathbf{C}_{\mathcal{M}^*}(s^*)$, then since $\mathcal{T}^*$ is connected we have that $\mathcal{T}^* \leqslant \mathbf{C}^\circ_{\mathcal{M}^*}(s^*)$ is a reductive subgroup of $\mathcal{M}^*$ ([MT11, Theorem 14.2]).

Proposition 3.3.14. *Let $G = \mathrm{SL}_n(q)$, $4 \leq n \leq 6$, $q \geq 2$. Suppose that G is of l-rank at least 2, where l is an odd prime. Then every non-linear irreducible unipotent character vanishes on an l-singular element.*

Proof. This follows from the proof [LM15, Proposition 3.8]. □

Since $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$ and $\mathrm{PSL}_2(7)$ is considered in Theorem 1.0.4, we may assume that $n \geq 3$ and $q \geq 3$ going forward.

Proposition 3.3.15. *Let M be a quasisimple group such that $M/Z(M) = \mathrm{PSL}_3(q)$, and $q \geq 3$. Then every non-trivial faithful irreducible character of M fails to satisfy ($\star$).*

Proof. Using explicit character tables in the Atlas [CCNPW85], we may assume that $q \geq 13$. First consider $Z(M) \neq \{1_M\}$. Now, $|Z(M)| = 3$, $3 \mid (q-1)$ and by ($\star$), χ vanishes on a 3-element. Note that unipotent characters are not faithful when $Z(M) \neq \{1_M\}$. We may thus assume that χ is not unipotent. Then χ lies in the Lusztig series $\mathcal{E}(M, s^*)$ of a semisimple element s^* in the dual group $M^* = \mathrm{PGL}_3(q)$. Let T_1 and T_2 be tori of M corresponding to the partitions (3) and (2)(1), respectively. By Lemma 3.3.12, the tori T_1 and T_2 contain regular elements. We claim that χ vanishes on regular elements in T_1 or in T_2. Otherwise, by Lemma 3.3.7, $\mathbf{C}_{M^*}(s^*)$ contains conjugates of the duals T_1^* and T_2^*. This means that the corresponding reductive subgroup $\mathbf{C}^\circ_{\mathcal{M}^*}(s^*)$ contains $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$. Invoking Lemma 3.3.13, we have that $\mathbf{C}^\circ_{\mathcal{M}^*}(s^*) = \mathrm{PGL}_3(\overline{\mathbb{F}}_p)$, that is, $\mathbf{C}_{M^*}(s^*) = \mathrm{PGL}_3(q)$ and so χ is unipotent, contradicting our assumption that χ is not unipotent. Hence χ vanishes on regular elements in T_1 or in T_2. Suppose that χ

vanishes on regular elements in T_1. Note that $|T_1|$ is divisible by a Zsigmondy prime l_1 and T_1 contains regular elements of order l_1. Since $\gcd(l_1, 3) = 1$, χ vanishes on at least two elements of distinct orders, contradicting Property ($\star$). We may thus assume χ vanishes on regular elements in T_2. If $q+1$ is not a power of 2, then $|T_2|$ is divisible by a Zsigmondy prime l_2. By the same argument as above, we may infer that χ vanishes on at least 2 elements of distinct orders. Suppose that $q+1$ is a power of 2. This means that $|T_2|$ is even and hence T_2 contains elements of even order by Lemma 3.3.8. Hence χ vanishes on an element of even order and the result follows.

Suppose that $M = \mathrm{PSL}_3(q)$. Then χ is not the Steinberg character by Lemma 3.3.11. By Theorem 2.4.15, χ vanishes on regular elements in T_1 or in T_2. Suppose that χ vanishes on regular elements of T_1. Note that $|T_1|$ is divisible by a Zsigmondy prime l_1. If $|T_1|$ is divisible by two distinct primes, then the result follows by Lemma 3.3.8. Suppose that $|T_1|$ is a prime power. Then $|T_1| = (q^2+q+1)/\gcd(3, q-1)$ must be prime by Lemma 3.1.3. Put $|T_1| = \frac{q^3-1}{(q-1)\gcd(3,q-1)} = \frac{q^2+q+1}{\gcd(3,q-1)} = l_1$. Then G has $\frac{l_1-1}{3} = \frac{q^2+q-2}{3\gcd(3,q-1)}$ conjugacy classes whose elements are of order l_1. Now $|\mathrm{Out}(M)| \leq 2 \cdot \gcd(3, q-1) \cdot f$. By Lemma 3.1.2, $6f + 1 \leq \frac{q^2+q-2}{9}$ and so (ii) of ($\star$) fails to hold. Suppose that χ vanishes on regular elements in T_2. By Theorem 2.4.15, χ vanishes on elements of order $q+1$. If q is odd, then $q+1$ is even. In particular, $q+1$ is not prime. By Theorem 2.4.3, χ vanishes on an element of prime order which means that χ vanishes on two elements of distinct orders, contradicting Property ($\star$). Hence we may assume that q is even so that $q+1$ is odd. We may assume that $q+1$ is prime by the above argument. Since $|T_2| = (q^2-1)/\gcd(3, q-1)$ and $(q-1)/\gcd(3, q-1) \neq 1$, we have that $|T_2|$ is divisible by at least two distinct primes. Hence there exists a prime l such that $l \mid (q-1)$ which entails the existence of an l-singular regular element in $|T_2|$ by Lemma 3.3.8. By Theorem 2.4.15, χ vanishes on this l-singular element. Hence χ vanishes on two elements of distinct orders, a contradiction. Hence the result follows. $\square$

We illustrate part of the proof above with an example.

Example 3.3.16. *Let $M = \mathrm{PSL}_3(8)$. Consider characters of degree 73. These vanish on elements of order 73. Then $|T_1| = 73$ is prime and M has $\frac{73-1}{3} = 24$ conjugacy classes with elements of order 73. Note that $|\mathrm{Out}(M)| = 6 < 24$ as expected.*

Proposition 3.3.17. *Suppose that M is quasisimple such that $M/Z(M) \cong \mathrm{PSL}_n(q)$, $n \geq 4$ and $q \geq 2$. Then every non-trivial faithful irreducible character of M fails to satisfy ($\star$).*

Proof. Firstly, suppose that $n \geq 4$ and $q = 2$. For M isomorphic to $\mathrm{PSL}_4(2)$ or $\mathrm{PSL}_5(2)$ we have explicit character tables in the Atlas [CCNPW85] and for $M/Z(M)$ isomorphic to $\mathrm{PSL}_6(2)$ or $\mathrm{PSL}_7(2)$, we obtain explicit character tables in Magma [BCP97]. Hence we may assume that $n \geq 8$. Then $3 = q + 1$. Now $(q + 1)^4 \mid |T|$ for a torus T corresponding to the partition $(n - 8)(2)(2)(2)(2)$. It follows that M is of 3-rank at least 4. Hence by Theorem 3.3.1, χ vanishes on a 3-singular element. On the other hand, by Theorem 2.4.15, if n is even, χ vanishes on an element of order $q^{n/2} + 1$ or one of order $q^{n-1} - 1$, and if n is odd, then χ vanishes on an element of order $q^n - 1$ or one of order $q^{(n-1)/2} + 1$. Note that in all of the aforementioned cases, the orders of elements on which χ vanishes, exceeds 3. Each such order is thus either relatively prime to 3 or the element is 3-singular. In the former case, χ vanishes on at least two elements of distinct orders as required. In the latter, χ vanishes on an element that is not of prime order. Using Theorem 2.4.2, χ also vanishes on an element of prime order. Hence χ vanishes on at least two elements of distinct orders.

Suppose that $M = \mathrm{SL}_n(q)$, $n \geq 4$, $q \geq 3$ with $|Z(M)| \neq 1$. By ($\star$), $|Z(M)|$ is a power of a prime l that divides $q - 1$ and χ necessarily vanishes on an l-element. We claim that χ also vanishes on an l_1-element or an l_2-element. Suppose the contrary. First note that χ is not unipotent since χ is faithful in M. Hence χ lies in the Lusztig series $\mathcal{E}(M, s^*)$ of a semisimple element s^* in the dual group $M^* = \mathrm{PGL}_n(q)$. Let T_1 and T_2 denote maximal tori corresponding to the partitions (n) and $(n - 1)(1)$. Note that T_1 and T_2 contain regular elements by Lemma 3.3.12. By Lemma 3.3.7, $\mathbf{C}_{M^*}(s^*)$ contains conjugates of the dual tori T_1^* and T_2^*. The corresponding reductive subgroup $\mathbf{C}^{\circ}_{\mathcal{M}^*}(s^*)$ contains $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$. Using Lemma 3.3.13, we infer that $\mathbf{C}^{\circ}_{\mathcal{M}^*}(s^*) = \mathrm{PGL}_n(\overline{\mathbb{F}}_p)$ and so s^* is central. Hence $s^* = 1$ and χ is unipotent thus contradicting the assumption that χ is not unipotent. Hence our claim is true and the result follows.

Suppose that $M = \mathrm{PSL}_n(q)$, $n \geq 4$, $q \geq 3$. Assume that χ is not unipotent. Let T_1, T_2 and T_3 be tori of M corresponding to (n), $(n - 1)(1)$ and $(n - 2)(2)$, respectively. These tori contain regular elements by Lemma 3.3.12. We claim that χ vanishes on regular elements in at least two of these tori. Otherwise, $\mathbf{C}_{M^*}(s^*)$ contains conjugates of the dual tori T_i^* and T_j^* of T_i and T_j, respectively, $i \neq j$, $1 \leq i, j \leq 3$, where χ lies in the Lusztig series $\mathcal{E}(M, s^*)$. The corresponding reductive subgroup $\mathbf{C}^{\circ}_{\mathcal{M}^*}(s^*)$ contains $\mathcal{T}_i^*$ and $\mathcal{T}_j^*$. It follows from Lemma 3.3.13 that $\mathbf{C}^{\circ}_{\mathcal{M}^*}(s^*) = \mathrm{PGL}_n(\overline{\mathbb{F}}_p)$, that is, χ is unipotent, a contradiction. The claim is thus true. Now for $|T_1|$ and $|T_2|$ note that corresponding Zsigmondy primes l_1 and l_2 exist. Hence χ vanishes on at least two elements of distinct orders l_1, l_2 or some positive integer that divides $|T_3|$.

In light of the above, we may assume that χ is unipotent. Note that for $|T_1|$ and $|T_2|$, corresponding Zsigmondy primes l_1 and l_2 exist. By Theorem 2.4.15, χ is of l_1-defect zero or l_2-defect zero. Suppose that $n \geq 9$. Consider a torus T of M corresponding to the partition $(n-9)(3)(3)(3)$. There exists a Zsigmondy prime l dividing $q^3 - 1$ such that M is of l-rank at least 3. By Theorem 3.3.1, χ vanishes on an l-singular element. Since $\gcd(l_1, l) = \gcd(l_2, l) = 1$, the result follows. Hence we may assume that $n \leq 8$. For $n \geq 6$, note that if there is a prime l such that $l \mid (q-1)$ or $l \mid (q+1)$, then M is of l-rank at least 3. It is sufficient to find an odd prime l such that $l \mid (q-1)$ or $l \mid (q+1)$ but l does not divide $|Z(M)|$.

Suppose that $n = 8$. If $\gcd(8, q-1) = 1$, then there exists an odd prime l such that $l \mid (q-1)$ and the result follows. Assume that $\gcd(8, q-1) \neq 1$. If there exists an odd prime l such that $l \mid (q-1)$, then we are done. We may assume that $q - 1 = 2^a$, $a \geq 1$. Then q is odd. If there exists an odd prime l such that $l \mid (q+1)$, then we are done. Otherwise, $q + 1 = 2^b$, $b \geq 2$. Then $q = 3$. For $M = \mathrm{PSL}_8(3)$, $|T_1| = \frac{3^8-1}{3-1}$ and $|T_2| = 3^8 - 1$ are both divisible by two distinct primes. Since χ is of l_1-defect zero or of l_2-defect zero, we have that χ vanishes on two elements of distinct orders.

Suppose that $n = 7$. We first consider q even. If $\gcd(7, q-1) = 1$, then there exists an odd prime l such that $l \mid (q-1)$. If $\gcd(7, q-1) \neq 1$, then since q is even, there exists an odd prime $l \neq 7$ such that $l \mid (q+1)$. Assume that q is odd. Suppose that $\gcd(7, q-1) = 1$. Then there exists an odd prime l such that either $l \mid (q-1)$ or $l \mid (q+1)$ unless $q = 3$. If $q \neq 3$, then we have an odd prime l and M is of l-rank at least 3. If $q = 3$, then using Magma [BCP97] to calculate the character table of $\mathrm{PSL}_7(3)$, we conclude that χ does not satisfy $(\star)$. Suppose that $\gcd(7, q-1) = 7$. If $q + 1$ is not a power of 2, then there exists an odd prime l such that $l \mid (q+1)$ and we are done. We may thus assume that $q + 1 = 2^a$, $a \geq 3$. Note that 3 divides one of $q - 1$, q or $q + 1$. We know that $3 \nmid q + 1$. Suppose that $3 \mid (q-1)$. Then 3 is the desired odd prime. Thus $3 \mid q$, that is, $q = 3^f$, $f \geq 1$. This implies that $q = 2^a - 1 = 3^f$. By Lemma 3.1.1, $f = 1$, that is, $q = 3$, a contradiction since $\gcd(7, q-1) = 7$.

Suppose that $n = 6$. Let $\gcd(6, q-1) = 2$. Then q is odd. If $q \neq 3$, then there exists an odd prime l such that $l \mid (q-1)$ or $l \mid (q+1)$. In this case, M is of l-rank at least 3 and we are done. If $q = 3$, then we consider the orders of the tori T_1 and T_2. Now $|T_1|$ and $|T_2|$ are divisible by two distinct primes. Since χ is of l_1-defect zero or of l_2-defect zero, χ vanishes on at least two elements of distinct orders. Let $\gcd(6, q-1) = 3$. Then q is even. For such q there exists an odd prime $l \neq 3$ such that $l \mid (q+1)$. The result follows since the l-rank of M is 3. Let $\gcd(6, q-1) = 6$. Then q is odd. If there exists an odd prime $l \neq 3$ such that $l \mid (q-1)$, then we are done. We may assume that $q - 1 = 2^a 3^b$. If there exists an odd prime l such that $l \mid (q+1)$, then again, we are done. We may thus assume that $q + 1 = 2^c$. Then by Lemma 3.1.4, $q = 7$ since $\gcd(6, q-1) = 6$. Since χ is of l_1-defect zero or of l_2-defect zero and since $|T_1|$ and $|T_2|$ are both divisible by two distinct primes, the result follows.

Suppose that $n = 5$. By Theorem 3.3.1, it is sufficient to show that χ vanishes on an l-singular element with $l \neq 5$, an odd prime. Let q be even and note that $q \geq 3$. If $\gcd(5, q-1) = 1$, then there exists an odd prime $l \neq 5$ such that $l \mid (q-1)$ and M is of l-rank at least 3. If $\gcd(5, q-1) \neq 1$, then there exists an odd prime $l \neq 5$ such that $l \mid (q+1)$. Note that M is of l-rank 2. By the proof of Proposition 3.3.14, χ vanishes on an l-singular element. Now assume that q is odd. Suppose that $\gcd(5, q-1) = 1$. Then there exists an odd prime $l \neq 5$ such that either $l \mid (q-1)$ or $l \mid (q+1)$ with the following exception: $q - 1 = 2^a$, $a \geqslant 1$ and $q + 1 = 2^b 5^c$, $b \geqslant 1$, $c \geqslant 0$. By Lemma 3.1.4, $q = 3$ or 9. If $q = 3$, then using Magma [BCP97] to calculate the character table of $\mathrm{PSL}_5(3)$, we conclude that χ does not satisfy ($\star$). Let $q = 9$. In this case we look at the orders of T_1 and T_2. Now $|T_1| = \frac{9^5-1}{9-1} = 11^2 \cdot 61$ and $|T_2| = 9^4 - 1 = 2^5 \cdot 5 \cdot 41$. Since χ is either of l_1-defect zero or of l_2-defect zero, χ vanishes on at least two elements of distinct orders. Assume that $\gcd(5, q-1) = 5$. If there exists an odd prime $l \neq 5$ such that $l \mid (q-1)$ or $l \mid (q+1)$, then M is of l-rank at least 2 and we are done. Hence the only exception we have is when $q - 1 = 2^a 5^b$ and $q + 1 = 2^c$. Lemma 3.1.4 entails $q = 3$ which contradicts the assumption $\gcd(5, q-1) = 5$.

Finally, suppose that $n = 4$. If $\gcd(4, q-1) = 1$, then there exists an odd prime l such that $l \mid (q-1)$ and the result follows. Assume that $\gcd(4, q-1) \neq 1$. If there exists an odd prime l such that $l \mid (q-1)$, then M is of l-rank at least 2 and we are done. Suppose that $q-1$ is a power of 2. If that $q+1$ is not a power of 2, then there exists an odd prime l such that $l \mid (q+1)$ and the result follows. Assume that $q+1$ is a power of 2. Then $q=3$. For $M = \mathrm{PSL}_4(3)$, we have that $|T_1|$ and $|T_2|$ are both divisible by two distinct primes. Since χ is of l_1-defect zero or of l_2-defect zero, χ vanishes on two elements of distinct orders. This concludes our argument. □

3.3.4.3 Special Unitary Groups

Let $\mathcal{M} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ and F be the twisted Frobenius morphism. The conjugacy classes of F-stable maximal tori of $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ and $\mathrm{SU}_n(\overline{\mathbb{F}}_p)$ are also characterised by conjugacy classes of S_n. If $\mathcal{T} \leqslant \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ corresponds to $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathrm{S}_n$ with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$, then $|T| = |\mathcal{T}^F| = \prod_i^m (q^{\lambda_i} - (-1)^{\lambda_i})$ whilst if $\mathcal{T} \leqslant \mathrm{SU}_n(\overline{\mathbb{F}}_p)$, then $(q+1)|T| = (q+1)|\mathcal{T}^F| = \prod_i^m (q^{\lambda_i} - (-1)^{\lambda_i})$.

Lemma 3.3.18. *[LM15, Lemma 4.1] Let $\lambda \vdash n$ be a partition, and $\mathcal{T}$ a corresponding F-stable maximal torus of $\mathrm{SU}_n(\overline{\mathbb{F}}_p)$. Assume that either all parts of λ are distinct, or $q \geq 3$ and at most two parts of λ are equal. Then $T = \mathcal{T}^F$ contains regular elements.*

Lemma 3.3.19. *[LM15, Lemma 3.2] Let $\mathcal{H} \leqslant \mathrm{PGU}_n(\overline{\mathbb{F}}_p)$ be a reductive subgroup containing F-stable maximal tori corresponding to cycle types $\lambda_1, \lambda_2, ..., \lambda_r$. If no intransitive or imprimitive subgroup of S_n contains elements of all these cycle types then $\mathcal{H} = \mathrm{PGU}_n(\overline{\mathbb{F}}_p)$.*

Proposition 3.3.20. *Let $G = \mathrm{SU}_n(q)$, $n \geqslant 4$. Suppose that G is of l-rank at least 2, where l is an odd prime. Then every non-linear irreducible unipotent character vanishes on an l-singular element.*

Proof. Follows from the proof of [LM15, Proposition 4.2]. □

Proposition 3.3.21. *Let M be a quasisimple group such that $M/Z(M) = \mathrm{PSU}_3(q)$ and $q \geq 3$. If ($\star$) holds, then $M = \mathrm{PSU}_3(4)$ with $\chi(1) = 13$.*

Proof. We may conclude from the character tables displayed in Atlas [CCNPW85] that $M = \mathrm{PSU}_3(4)$ when $3 \leq q \leq 11$. Assume that $q \geq 13$. Note that χ is not the Steinberg character. We first consider the case $M = \mathrm{SU}_3(q)$ and $|Z(M)| \neq 1$. Since we are only considering faithful characters, χ is not unipotent. Then $|Z(M)| = 3$ and $3 \mid (q+1)$. By (iii) of ($\star$), χ vanishes on a 3-element. We have that T_1 and T_2 correspond to the cycle types (3) and (2)(1) and so T_1 and T_2 contain regular elements by Lemma 3.3.18. Using the same argument as in Proposition 3.3.15 we have that χ vanishes on regular elements in T_1 or in T_2. If χ vanishes on regular elements in T_1, then χ vanishes on an element of Zsigmondy prime order l_1. Since $\gcd(l_1, 3) = 1$, the result follows. If χ vanishes on regular elements in T_2, then χ vanishes either on an element of Zsigmondy prime order l_2 if $q-1$ is not a power of 2, or on an element of even order if $q-1$ is a power of 2. Since such orders are relatively prime to 3, χ vanishes on at least two elements of distinct orders.

Let $M = \mathrm{PSU}_3(q)$. By Proposition 2.4.16, χ vanishes on regular elements in T_1 or in T_2. Assume the former. Note $|T_1|$ is divisible by a Zsigmondy prime l_1. If $|T_1|$ is divisible by two distinct primes, then by Lemma 3.3.8, χ vanishes on at least two elements of distinct orders. Note that T_1 is cyclic by [Gag73, Section 3.3]. If $|T_1| = l_1^a$, $a > 1$, then χ vanishes on two elements of distinct orders l_1^a and l_1, which contradicts Property ($\star$). We may thus assume that $|T_1| = \frac{q^3+1}{(q+1)\gcd(3,q+1)} = \frac{q^2-q+1}{\gcd(3,q+1)} = l_1$. If $q = 13$, then M has 52 conjugacy classes of order 53, $|\mathrm{Out}(M)| = 2$ and we are done. We thus assume that $q \geq 16$. Then M has $\frac{l_1-1}{3} = \frac{q^2-q-2}{3\cdot\gcd(3,q+1)}$ conjugacy classes whose elements are of order l_1. Now $|\mathrm{Out}(M)| \leq 2 \cdot \gcd(3, q+1) \cdot f$. By Lemma 3.1.2, $6f + 1 \leq \frac{q^2-q-2}{9}$ and (ii) of ($\star$) fails to hold.

We now consider the case where χ vanishes on regular elements in T_2. By Theorem 2.4.17, χ vanishes on an element of order $q-1$. On the other hand, χ vanishes on: an element of Zsigmondy prime order l_2, an involution, or a regular unipotent element by the proof of Proposition 2.4.16. Therefore χ vanishes on at least two elements of distinct orders. $\square$

Proposition 3.3.22. *Let M be a quasisimple group such that $M/Z(M) \cong \mathrm{PSU}_n(q)$, $n \geq 4$ and $q \geq 2$. Then every non-trivial faithful irreducible character of M fails to satisfy ($\star$).*

Proof. We consider the case $M/Z(M) \cong \mathrm{PSU}_n(2)$ first. Consulting of the character tables for $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_5(2)$ and $\mathrm{PSU}_6(2)$ displayed in Atlas [CCNPW85], and for $\mathrm{PSU}_7(2)$ and $\mathrm{PSU}_8(2)$ derived from Magma [BCP97], disposes of the case $n < 9$, so we may assume that $n \geq 9$. Suppose that $|Z(M)| \neq 1$. This means that $|Z(M)| = 3$ and $3 \mid (q+1)$. Note that χ is not unipotent. By $(\star)$, χ vanishes on a 3-element. We claim that χ vanishes on regular elements in T_1 or in T_2. Assume that this claim is not true. Then $\mathbf{C}_{M^*}(s^*)$ contains conjugates of the dual tori T_1^* and T_2^* of T_1 and T_2, where χ lies in the Lusztig series $\mathcal{E}(M, s^*)$. The corresponding reductive subgroup $\mathbf{C}^\circ_{\mathcal{M}^*}(s^*)$ contains the tori $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$. By Lemma 3.3.19, $\mathbf{C}^\circ_{\mathcal{M}^*}(s^*) = \mathrm{PGU}_n(\overline{\mathbb{F}}_2)$ and s^* is central. Hence $\mathbf{C}_{M^*}(s^*) = \mathrm{PGU}_n(2)$, and so χ is unipotent, a contradiction. The claim is true and χ vanishes either on an l_1-element or on an l_2-element. Hence χ vanishes on at least two elements of distinct orders.

Assume that $Z(M) = \{1_M\}$. Consider the first case where χ is non-unipotent. Then χ lies in the Lusztig series $\mathcal{E}(M, s^*)$ of s^* in the dual $M^* = \mathrm{PGU}_n(2)$. Let T_1, T_2 and T_3 be tori of M corresponding to (n), $(n-1)(1)$ and $(n-2)(2)$, respectively. We shall use the same argument as employed in the proof of Proposition 3.3.17 (third paragraph) to show that χ vanishes on regular elements in at least two of these tori. Assume that χ does not vanish on any of T_1, T_2 or T_3. $\mathbf{C}_{M^*}(s^*)$ contains conjugates of the dual tori T_i^* and T_j^* of T_i and T_j, respectively, $i \neq j$, $1 \leq i, j \leq 3$, where χ lies in the Lusztig series $\mathcal{E}(M, s^*)$. The corresponding reductive subgroup $\mathbf{C}^\circ_{\mathcal{M}^*}(s^*)$ contains $\mathcal{T}_i^*$ and $\mathcal{T}_j^*$. It follows from Lemma 3.3.19 that $\mathbf{C}^\circ_{\mathcal{M}^*}(s^*) = \mathrm{PGU}_n(\overline{F}_p)$. Hence $\mathbf{C}^\circ_{\mathcal{M}^*}(s^*) = \mathrm{PGU}_n(q)$, that is, s^* is central and χ is unipotent, a contradiction. The claim is thus true. Note that all these tori contain elements of order l_1, l_2 and a Zsigmondy prime dividing $|T_3|$. Hence the result follows.

Suppose that χ is unipotent. By Theorem 2.4.17, χ is of l_1-defect zero or l_2-defect zero. Consider a torus T of M corresponding to the partition $(n-9)(3)(3)(3)$. Then there exists a Zsigmondy prime $l = l(6)$ such that $l \mid (q^3+1)$. Hence M is of l-rank at least 3. Since $n \geq 9$, χ vanishes on an l-singular element by Theorem 3.3.1. Hence the result follows.

Suppose that $M/Z(M) \cong \mathrm{PSU}_n(q)$, $n \geq 4$, $q \geq 3$. Assume that $|Z(M)| \neq 1$. By $(\star)$, $|Z(M)|$ is a power of a prime $l = q+1$ and χ vanishes on an l-element. The same argument as employed in the proof of Proposition 3.3.17 (second paragraph), shows that χ also vanishes on an l_1-element or an l_2-element.

If $M \cong \mathrm{PSU}_n(q)$ with χ non-unipotent, then the result follows using in the proof of Proposition 3.3.17 (third paragraph).

Assume that χ is unipotent. By the proof of Theorem 2.4.17, χ is of l_1-defect zero or l_2-defect zero. Suppose that $n \geq 9$ and consider a torus T of M corresponding to partition $(n-9)(3)(3)(3)$. Then there exists a Zsigmondy prime $l = l(6)$ dividing q^3+1 and M is of l-rank at least 3. By Theorem 3.3.1, χ vanishes on an l-singular element and the result follows. Hence we may assume that $n \leq 8$.

Suppose that $n = 8$. Recall that $q \geq 3$. If $\gcd(8, q+1) = 1$, then there exist an odd prime $l \mid (q+1)$ such that M is of l-rank at least 3. By Theorem 3.3.1, χ vanishes on an l-singular element and the result follows since $\gcd(l_1, l) = \gcd(l_2, l) = 1$. If $\gcd(8, q+1) \neq 1$, then q is odd and there exists an odd prime l such that $l \mid (q-1)$ unless $q = 3$. If $q \neq 3$, then M is of l-rank at least 3 and hence χ vanishes on an l-singular element for an odd prime l dividing $q-1$. For $q = 3$, we have that $|T_1|$ and $|T_2|$ are both divisible by two distinct primes since χ is of l_1-defect zero or of l_2-defect zero. Hence the result follows.

Suppose that $n = 7$. We first consider the case q is even. If $\gcd(7, q+1) = 1$, then there exists an odd prime $l \neq 7$ such that $l \mid (q+1)$. If $\gcd(7, q+1) \neq 1$, then since q is even, there exists an odd prime $l \neq 7$ such that $l \mid (q-1)$. In both cases, M is of l-rank and so χ vanishes on an l-singular element. Since $\gcd(l_1, l) = \gcd(l_2, l) = 1$, χ vanishes on at least two elements of distinct orders. Assume that q is odd. Suppose that $\gcd(7, q+1) = 1$. Then there exists an odd prime $l \neq 7$ such that $l \mid (q-1)$ or $l \mid (q+1)$ unless $q = 3$. If $q \neq 3$, we have that M is of l-rank at least 3 with l an odd prime, and implies that χ vanishes on at least two elements of distinct orders. If $q = 3$, we calculate the character table of $\mathrm{PSL}_7(3)$ using Magma [BCP97]. Examining the character table, we conclude that χ does not satisfy $(\star)$. Suppose that $\gcd(7, q+1) = 7$. If $q-1$ is not a power of 2, then there exists an odd prime $l \neq 7$ such that $l \mid (q-1)$. Hence M is of l-rank more than 3 and $\gcd(l_1, l) = \gcd(l_2, l) = 1$. Thus χ vanishes on two elements of distinct orders, a contradiction to $(\star)$. We may thus assume that $q - 1 = 2^a$, $a > 1$. Since $3 \nmid (q-1)$ we must have that $3 \mid (q+1)$. Then 3 is the desired odd prime since M is of 3-rank at least 3. Thus $3 \mid q$, whence, $q = 3^f$, $f \geq 1$. This implies that $q - 1 = 3^f - 1 = 2^a$. By Zsigmondy's Theorem, there is a Zsigmondy prime l that divides $3^f - 1$ unless $f \leq 2$. If $f = 1$, then $q = 3$, contradicting the hypothesis that $\gcd(7, q+1) = 7$. If $f = 2$, then $q = 9$, again contradicting the hypothesis that $\gcd(7, q+1) = 7$.

Suppose that $n = 6$. Let $\gcd(6, q + 1) = 2$. Then q is odd. If $q \neq 3$, then there exists an odd prime l such that $l \mid (q - 1)$ or $l \mid (q + 1)$ and the result follows noting that M must be of l-rank at least 3. Let $\gcd(6, q + 1) = 3$. Then q is even. Noting that $q \geq 4$, there must exist an odd prime l that divides $q - 1$. Hence χ vanishes on an l-singular element since the l-rank of M is 6. Let $\gcd(6, q+1) = 6$. Then q is odd. If there exists an odd prime $l \neq 3$ such that $l \mid (q + 1)$, then the result follows. We may assume that $q + 1 = 2^a 3^b$. Then there exists an odd prime $l \neq 3$ that divides $q - 1$ and the result follows unless $q - 1 = 2^c$. For such q it follows from Lemma 3.1.4 that $q = 5$ or 17 since $\gcd(6, q + 1) = 6$. In both cases, the result follows since $|T_1|$ and $|T_2|$ are both divisible by two distinct primes.

Suppose that $n = 5$. By Theorem 3.3.1, it is sufficient to show that χ vanishes on an l-singular element with $l \neq 5$, an odd prime. Let q be even and note that $q \geq 3$. If $\gcd(5, q + 1) = 1$, then there exists an odd prime $l \neq 5$ such that $l \mid (q + 1)$ and M is of l-rank at least 3. If $\gcd(5, q + 1) \neq 1$, then there exists an odd prime $l \neq 5$ such that $l \mid (q - 1)$. Note that M is of l-rank 2. By the proof of Proposition 3.3.20, χ vanishes on an l-singular element. Now assume that q is odd. Suppose that $\gcd(5, q + 1) = 1$. Then there exists an odd prime $l \neq 5$ such that either $l \mid (q - 1)$ or $l \mid (q + 1)$ with the following exception: $q - 1 = 2^a$, $a \geqslant 1$ and $q + 1 = 2^b 5^c$, $b \geqslant 1$, $c \geqslant 0$. By Lemma 3.1.4, $q = 3$ or 9. If $q = 3$, then using Magma [BCP97] to calculate the character table of $\mathrm{PSU}_5(3)$, we conclude that χ does not satisfy ($\star$). Let $q = 9$. In this case we look at the orders of T_1 and T_2. Now $|T_1| = \frac{9^5+1}{9+1} = 5 \cdot 11811$ and $|T_2| = 9^4 - 1 = 2^5 \cdot 5 \cdot 41$. Since χ is either of l_1-defect zero or of l_2-defect zero, χ vanishes on at least two elements of distinct orders. Assume that $\gcd(5, q + 1) = 5$. If there exists an odd prime $l \neq 5$ such that $l \mid (q-1)$ or $l \mid (q+1)$, then M is of l-rank at least 2 and we are done by Proposition 3.3.20. Hence the only exception we have is when $q - 1 = 2^a 5^b$ and $q + 1 = 2^c$. Lemma 3.1.4 entails $q = 3$ which contradicts the assumption $\gcd(5, q + 1) = 5$.

Suppose that $n = 4$. If $\gcd(4, q + 1) = 1$, then there exists an odd prime l such that $l \mid (q + 1)$ and the result follows noting that M is of l-rank at least 3. Assume that $\gcd(4, q + 1) \neq 1$. Then there exists an odd prime l such that $l \mid (q - 1)$ or $l \mid (q + 1)$. If $q \neq 3$, then M is of l-rank at least 2 and we are done. If $q = 3$, we have that $|T_1|$ and $|T_2|$ are both divisible by two distinct primes and the result follows. $\square$

3.3.4.4 Symplectic Groups and Special Orthogonal Groups

Let $\mathcal{M}$ be a simple, simply connected algebraic group of type B_n, C_n or D_n over $\overline{\mathbb{F}}_p$ and $F : \mathcal{M} \to \mathcal{M}$ be a Frobenius morphism such that $M := \mathcal{M}^F$. Then the $\mathcal{M}^F$-conjugacy classes of F-stable maximal tori of $\mathcal{M}$ are characterised by the conjugacy classes of W, the Weyl group. Now W is isomorphic to the wreath product $C_2 \wr \mathrm{S}_n$. If $\mathcal{M}$ is of type B_n or C_n, then the conjugacy classes of W are parametrised by pairs of partitions $(\lambda, \mu) \vdash n$. In particular, if a maximal torus $T = \mathcal{T}^F$ corresponds to a partition $(\lambda, \mu) = ((\lambda_1, \lambda_2, ..., \lambda_r), (\mu_1, \mu_2, ..., \mu_s)) \vdash n$, then

$$|T| = \textstyle\prod_{i=1}^{r}(q^{\lambda_i} - 1)\prod_{j=1}^{s}(q^{\mu_j} + 1)$$

and $\mathcal{T}^F$ contains cyclic subgroups of orders $q^{\lambda_i} - 1$ and $q^{\mu_j} + 1$ for all i and for all j. If $\mathcal{M}$ is of type D_n, then the $\mathcal{M}^F$-conjugacy classes of F-stable maximal tori of $\mathcal{M}$ are characterised by pairs of partitions $(\lambda, \mu) \vdash n$ such that μ has an even number of parts if $\mathcal{M}^F$ is the split orthogonal group $\mathrm{SO}^+_{2n}(q)$ and μ has an odd number of parts if $\mathcal{M}^F$ is the non-split orthogonal group $\mathrm{SO}^-_{2n}(q)$. Now $|T|$ is the same as in the case when $\mathcal{M}$ is of type B_n or C_n, that is,

$$|T| = \textstyle\prod_{i=1}^{r}(q^{\lambda_i} - 1)\prod_{j=1}^{s}(q^{\mu_j} + 1).$$

Lemma 3.3.23. *[LM16, Lemma 2.1] Let $\mathcal{M}$ be a simple, simply connected classical group of type B_n, C_n or D_n defined over $\mathbb{F}_p$ with corresponding Steinberg morphism F. Let $(\lambda, \mu) = ((\lambda_1, \lambda_2, ..., \lambda_r), (\mu_1, \mu_2, ..., \mu_s))$ be a pair of partitions of n, and $\mathcal{T}$ a corresponding F-stable maximal torus of $\mathcal{M}$. Then $T = \mathcal{T}^F$ contains regular elements if one of the following is fulfilled:*

(a) $q > 3$, $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ and $\mu_1 < \mu_2 < \cdots < \mu_s$;

(b) $q \in \{2, 3\}$, $\lambda_1 < \lambda_2 < \cdots < \lambda_r$, $\mu_1 < \mu_2 < \cdots < \mu_s$, $\lambda_i \neq 2$ for $1 \leq i \leq r$, and if $\mathcal{M}$ is of type B_n or C_n, then also $\lambda_i \neq 1$ for $1 \leq i \leq r$; or

(c) $\mathcal{M}$ is of type D_n, $2 < \lambda_1 < \lambda_2 < \cdots < \lambda_r$ and $1 = \mu_1 = \mu_2 < \mu_3 < \cdots < \mu_s$.

Lemma 3.3.24. *[LM16, Lemma 2.3] Let $\mathcal{M}$ be a simple algebraic group of type B_n, C_n (with $n \geq 2$) or D_n (with $n \geq 4$) with Frobenius endomorphism F such that $\mathcal{M}^F$ is a classical group. Let Λ be a set of pairs of partitions (λ, μ) of n. Assume the following:*

(a) *there is no k satisfying $1 \leq k \leq n-1$ such that all $(\lambda, \mu) \in \Lambda$ are of the form $(\lambda_1, \mu_1) \sqcup (\lambda_2, \mu_2)$ with $(\lambda_1, \mu_1) \vdash k$;*

(b) *the greatest common divisor of all parts of all $(\lambda, \mu) \in \Lambda$ is 1; and*

(c) *if $\mathcal{G}$ is of type B_n, then there exist pairs $(\lambda, \mu) \in \Lambda$ for which μ has an odd number of parts, and for which μ has an even number of parts.*

If $s \in \mathcal{M}^F$ is semisimple such that $\boldsymbol{C}_{\mathcal{M}}(s)$ contains maximal tori of $\mathcal{M}$ corresponding to each $(\lambda, \mu) \in \Lambda$, then s is central.

Theorem 3.3.25. *[LM16, Theorem 4.1] Let G be one of the groups $\mathrm{Spin}_{2n+1}(q)$ for odd q and $n \geq 3$, $\mathrm{Sp}_{2n}(q)$ for any q and $n \geq 2$, or $\mathrm{Spin}^{\pm}_{2n}(q)$ for any q and $n \geq 4$. Let l be an odd prime that does not divide q such that the Sylow l-subgroups of G are non-cyclic. Then any non-unipotent irreducible character of G vanishes on some l-singular regular semisimple element, except for the two cases $G = \mathrm{Sp}_4(2)$ and $G = \mathrm{Sp}_8(2)$.*

We first consider a quasisimple group M such that $M/Z(M) \cong \mathrm{PSp}_4(q)$. Since $\mathrm{Sp}_4(2)' \cong \mathrm{PSL}_2(9)$ and $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$, and the groups $\mathrm{PSL}_2(9)$ and $\mathrm{PSU}_4(2)$ were dealt with in Propositions 3.3.9 and 3.3.22, respectively, we shall restrict our attention to the case $q \geq 4$ in the result below.

Proposition 3.3.26. *Let M be a quasisimple group such that $M/Z(M) \cong \mathrm{PSp}_4(q)$, $q \geq 4$. Then every non-trivial faithful irreducible character of M fails to satisfy ($\star$).*

Proof. Since the character tables $\mathrm{Sp}_4(4)$ and $\mathrm{Sp}_4(5)$ are in Atlas [CCNPW85] we may assume that $q \geq 7$. Suppose that q is even. Then the result follows from the generic character tables given in Chevie [GHLMP96] or in [Sri68]. We may thus assume that q is odd. For this case we first suppose $|Z(M)| \neq 1$. Note that χ is not unipotent. Then $|Z(M)| = 2$ and by ($\star$), χ vanishes on a 2-element. For each prime l such that $l \mid (q-1)$ or $l \mid (q+1)$, the Sylow l-subgroups of G are non-cyclic. Since $q \neq 3$, there exists an odd prime l such that $l \mid q-1$ or $l \mid q+1$. By Theorem 3.3.25, χ vanishes on an l-singular element. But $\gcd(2, l) = 1$, so χ vanishes on at least two elements of distinct orders, as required.

Suppose that $M \cong \mathrm{PSp}_4(q)$. We may assume that χ is not the Steinberg character. The same methods used in the proof of Theorem 2.4.18 show that χ vanishes on regular elements in T_1 or in T_2. In particular, we may choose two conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_2$ in T_1 and T_2, respectively, such that χ vanishes on $\mathcal{C}_1$ or $\mathcal{C}_2$. Now $\mathcal{C}_2$ may contain elements which are not of Zsigmondy prime order. In that case, the result follows since we know that χ vanishes on elements of prime order by Theorem 2.4.3. Hence we may assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ contain elements of Zsigmondy prime orders. Suppose that χ vanishes on elements in T_2. Note that $|T_2| = (q^2-1)/2$ is even. Hence T_2 contains a regular element of even order by Lemma 3.3.8 and χ vanishes on this element. This means that χ vanishes on two elements of distinct orders, contradicting ($\star$). Suppose now that χ vanishes on elements of T_1. Note that T_1 is cyclic by [Gag73, Section 4.5]. If $|T_1|$ is not prime, then there exist at least 2 elements of distinct orders on which χ vanishes. We may assume that $|T_1| = \frac{q^2+1}{2}$ is prime. Then there are $\frac{\frac{(q^2+1)}{2}-1}{4} = \frac{q^2-1}{8}$ conjugacy classes with elements of order $\frac{q^2+1}{2}$. On other the hand, $|\mathrm{Out}(M/Z(M))| \leq 4f$, where $q = p^f$ with p a prime and $f \geq 1$. By Lemma 3.1.2, $4f+1 < \frac{q^2-1}{8}$ and the result follows by ($\star$)(ii). □

Let $\mathcal{S} = \{\mathrm{PSp}_{2n}(q) \mid n \geq 3\} \cup \{\mathrm{PSO}_{2n+1}(q) \mid n \geq 3\} \cup \{\mathrm{PSO}^{\pm}_{2n}(q) \mid n \geq 4\}$.

Lemma 3.3.27. *(a) Let $G \cong \mathrm{PSO}^+_n(q)$ with $n \geq 6$ even. Then every non-linear character $\chi \in \mathrm{Irr}(G)$ that is not the Steinberg character vanishes either on elements of Zsigmondy prime order l_1 , l_2 (defined in Table 2.4), or l_3, where l_3 is the Zsigmondy prime $l_3 = l(2n-4)$.*

(b) Let $G \cong \mathrm{PSO}^+_8(q)$ with $q > 2$. Then every non-linear character $\chi \in \mathrm{Irr}(G)$ that is not the Steinberg character vanishes either on elements of Zsigmondy prime order l_1 , l_2 (defined in Table 2.4) or l_3 where l_3 is the Zsigmondy prime $l_3 = l(2n-4)$.

(c) Let $G \in \mathcal{S}\backslash\{\mathrm{PSO}^+_{2n}(q)\}$. Then every non-linear character $\chi \in \mathrm{Irr}(G)$ that is not the Steinberg character vanishes either on elements of Zsigmondy prime order l_1 , l_2 (defined in Table 2.4), or is of l_3-defect zero, where $l_3 = l(n-1)$.

(d) Let $G \cong \mathrm{PSO}^+_{2n}(q)$ with $n \geq 5$ odd. Then every non-linear character $\chi \in \mathrm{Irr}(G)$ that is not the Steinberg character vanishes on elements of order l_1 or l_2.

Proof. This follows from the proof of [MNO00, Lemmas 5.3-5.6] □

Proposition 3.3.28. *Let M be a quasisimple group such that $M/Z(M) \in \mathcal{S}$. Then every non-trivial faithful irreducible character of M fails to satisfy ($\star$).*

Proof. Note that χ is not the Steinberg character by Lemma 3.3.11. We first consider the case where $M \in \mathcal{S}$ with $q = 2$. Consulting of the character tables for $\mathrm{Sp}_{2n}(2) \cong \mathrm{SO}_{2n+1}(2)$, $3 \leq n \leq 4$ and $\mathrm{PSO}^{\pm}_{2n}(2)$, $4 \leq n \leq 5$ displayed in Atlas [CCNPW85] allows us to dispose of the cases $n < 5$ (for $\mathrm{Sp}_{2n}(2)$) and $n < 6$ (for $\mathrm{PSO}^{\pm}_{2n}(2)$), so we may assume that $n \geq 5$ and $n \geq 6$, respectively. Since $q + 1 = 3$, M is of 3-rank at least 5. By Theorem 3.3.1, χ vanishes on a 3-singular element. For $M \cong \mathrm{Sp}_{2n}(2)$, either χ vanishes on elements of order l_1 or elements of order l_2 as can be seen in Table 2.4, or χ is of l_3-defect zero, where $l_3 = l(n-1)$ (the last case only arising when n is even) by Lemma 3.3.27. Note that Zsigmondy primes l_1, l_2, l_3 exist and $\gcd(l_1, 3) = \gcd(l_2, 3) = \gcd(l_3, 3) = 1$. Hence χ vanishes on at least two elements of distinct orders. For $M \cong \mathrm{PSO}^{\pm}_{2n}(2)$, $n \geq 6$, χ vanishes on elements of order l_1, or l_2, or χ is of l_3-defect zero, where $l_3 = l(2n-4)$ (the last case only arising when n is even) by Lemma 3.3.27. Since the Zsigmondy primes l_1, l_2 and l_3 exist, the result follows.

Henceforth we may assume that $q \geq 3$ and $n \geq 3$. Suppose that $|Z(M)| \neq 1$. Then $\gcd(2, q-1) = 2$ and by ($\star$), χ vanishes on a 2-element. We want to show that χ also vanishes on an element of Zsigmondy prime order. Note that χ is not unipotent and so χ lies in the Lusztig series $\mathcal{E}(M, s^*)$ of s^* in the dual M^*. Let T_1, T_2 and T_3 be tori of M corresponding to $((n), -)$, $(-, (n))$ and $(-, (n-1, 1))$, respectively. These tori contain regular elements by Lemma 3.3.23. We claim that χ vanishes on regular elements in at least one of these tori. Otherwise, by Lemma 3.3.7, $\mathbf{C}_{M^*}(s^*)$ contains conjugates of the dual tori T_i^*, $1 \leq i \leq 3$. The corresponding reductive subgroup $\mathbf{C}_{\mathcal{M}^*}(s^*)$ contains conjugates of the dual tori $\mathcal{T}_i^*$, $1 \leq i \leq 3$. It follows from Lemma 3.3.24 that $\mathbf{C}_{\mathcal{M}^*}(s^*)$ is central. Hence $\mathbf{C}_{M^*}(s^*) = M^*$, that is, χ is unipotent, a contradiction. The claim is thus true. Now for T_1, T_2 and T_3 note that there exist Zsigmondy primes l_1, l_2 and l_3 in respect of $|T_1|$, $|T_2|$ and $|T_3|$. Hence χ vanishes on at least two elements of distinct orders and we are done.

Suppose that $Z(M) = \{1_M\}$. Let us consider $M \cong \mathrm{PSp}_{2n}(q)$ ($n \geq 3$) or $\mathrm{PSO}_{2n+1}(q)$ ($n \geq 3$) or $\mathrm{PSO}^{-}_{2n}(q)$ ($n \geq 4$) with $q \geq 3$. By Lemma 3.3.27, χ vanishes on elements of order l_1, or l_2, or χ is of l_3-defect zero, where $l_3 = l(n-1)$ (the last case only arising when n is even). In all cases Zsigmondy primes exist. There exists an odd prime l

such that $l \mid (q-1)$ or $l \mid (q+1)$ except when $q = 3$. Note that M is of l-rank at least 3. If $q \neq 3$, then by Theorem 3.3.1, χ vanishes on an l-singular element. Since $\gcd(l_1, l) = \gcd(l_2, l) = \gcd(l_3, l) = 1$, the result follows. We are left with case when $q = 3$. If $n \geq 6$, then M has a torus T corresponding to $(-, (n-6, 2, 2, 2))$, i.e., M is of l-rank at least 3, where $l \mid (q^2+1)$. The result follows again. Hence we may assume that $n \leq 5$, that is, $M \in \{\mathrm{PSp}_6(3), \mathrm{PSp}_8(3), \mathrm{PSp}_{10}(3), \mathrm{PSO}_7(3), \mathrm{PSO}_9(3), \mathrm{PSO}_{11}(3), \mathrm{PSO}^-_{10}(3)\}$. Explicit character tables for $\mathrm{PSp}_6(3)$ and $\mathrm{PSO}_7(3)$ are given in Atlas [CCNPW85]. The rest may be constructed using Magma [BCP97]. The conclusion of the proposition can be verified in respect of each of these cases.

Suppose that $Z(M) = \{1_M\}$ and $M \cong \mathrm{PSO}^+_{2n}(q)$ with $n \geq 4$ and $q \geq 3$. Assume that n is odd. Then Zsigmondy primes l_1 and l_2 exist, and χ vanishes on regular elements in T_1 or in T_2 by Lemma 3.3.27. If $q \neq 3$, then there exists an odd prime l such that $l \mid (q-1)$ or $l \mid (q+1)$ and M is of l-rank at least 3. Hence χ vanishes on an l-singular element by Theorem 3.3.1, thus χ vanishes on at least 2 elements of distinct orders. Let $q = 3$. If $n \geq 7$, then consider a torus corresponding to the cycle type $(-, (n-6, 2, 2, 2))$. It follows that M is of l-rank at least 3 with $l \mid (q^2+1)$ and by Theorem 3.3.1, χ vanishes on an l-singular element. Hence we may assume $n \leq 5$. Hence $n = 5$ and so $M \cong \mathrm{PSO}^+_{10}(3)$. Consulting of this group's character table constructed using Magma [BCP97] shows that the proposition's conclusion is met in this instance.

Suppose that $n \geq 4$ is even. By Lemma 3.3.27, χ vanishes on regular elements of order l_1 or l_2, or χ is of l_3-defect where $l_3 = l(2n-4)$. An argument, similar to that used in the paragraph above, allows us to dispose of the case $q \neq 3$. Suppose then $q = 3$. Now if $n \geq 6$, then M is of l-rank at least 3 where l is an odd prime dividing q^2+1. This entails $l = 5$. By Theorem 3.3.1, χ vanishes on a 5-singular element. Since $\gcd(l_1, 5) = \gcd(l_2, 5) = \gcd(l_3, 5) = 1$, the result follows. If $n = 4$, then $M \cong \mathrm{PSO}^+_8(3)$. Consulting of this group's character table displayed in the Atlas [CCNPW85] shows that the proposition's conclusion is met in this instance. □

3.3.5 Exceptional groups

3.3.5.1 Exceptional groups of small Lie rank

Let $\mathcal{L} = \{{}^2\mathrm{B}_2(q) \mid q = 2^{2f+1}, f \geq 1\} \cup \{{}^2\mathrm{G}_2(q^2) \mid q^2 = 3^{2f+1}, f \geq 1\} \cup \{{}^2\mathrm{F}_4(q^2) \mid q^2 > 2\} \cup \{\mathrm{G}_2(q) \mid q \geq 3\} \cup \{{}^3\mathrm{D}_4(q), q \geq 2\}$.

Proposition 3.3.29. *Let M be a quasisimple group such that $M/Z(M) \in \mathcal{L}$. If $(\star)$ holds, then $M \cong {}^2\mathrm{B}_2(8)$ with $\chi(1) = 14$.*

Proof. The simple group $M = {}^2\mathrm{B}_2(8)$ satisfies the conclusion of our proposition from its character table in the Atlas [CCNPW85]. Let $M = {}^2\mathrm{B}_2(q)$, $q > 8$. For the remaining groups, inspection of the explicit character tables displayed in Atlas [CCNPW85] and the generic ordinary character tables shown in Chevie [GHLMP96], reveals that every non-trivial character of the given group, fails to satisfy either condition (i) or (ii) of $(\star)$. Hence the result follows. □

3.3.5.2 Exceptional finite groups of large Lie rank

The table below shows the Zsigmondy primes l_i for the corresponding tori T_i. It was shown in [MNO00] that every non-trivial irreducible character which is not the Steinberg character, vanishes on an element of order l_i for some $i \in \{1, 2, 3\}$. Recall that the n^{th} *cyclotomic polynomial* over $\mathbb{Q}$, denoted Φ_n, is equal to

$$\Phi_n(x) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} (x - e^{\frac{2\pi i k}{n}})$$

Table 3.1: Tori and Zsigmondy primes for exceptional of groups of Lie type

M	$\lvert T_1\rvert$	$\lvert T_2\rvert$	$\lvert T_3\rvert$	l_1	l_2	l_3
$\mathrm{F}_4(q)$	Φ_{12}	Φ_8		$l(12)$	$l(8)$	
$\mathrm{E}_6(q)$	$\Phi_{12}\Phi_3$	Φ_9	$\Phi_8\Phi_2\Phi_1$	$l(12)$	$l(9)$	$l(8)$
${}^2\mathrm{E}_6(q)$	Φ_{18}	$\Phi_{12}\Phi_6$	$\Phi_8\Phi_2\Phi_1$	$l(18)$	$l(12)$	$l(8)$
$\mathrm{E}_7(q)$	$\Phi_{18}\Phi_2$	$\Phi_{14}\Phi_2$	$\Phi_{12}\Phi_3\Phi_1$	$l(18)$	$l(14)$	$l(12)$
$\mathrm{E}_8(q)$	Φ_{30}	Φ_{24}	Φ_{20}	$l(30)$	$l(24)$	$l(20)$

Lemma 3.3.30. *[MNO00, Lemma 5.9] Let $M \in \{\mathrm{F}_4(q), \mathrm{E}_6(q), {}^2\mathrm{E}_6(q), \mathrm{E}_7(q), \mathrm{E}_8(q)\}$. Then every non-trivial irreducible character of M which is not the Steinberg character, vanishes on elements of order l_1, l_2 or l_3 as can be seen in Table 3.1.*

Proposition 3.3.31. *Let M be a quasisimple exceptional group of Lie type over a field of characteristic p, and of rank at least 4. Then every non-trivial faithful irreducible character of M fails to satisfy ($\star$).*

Proof. Note that the group M must be one of the types : F_4, E_6, ${}^2\mathrm{E}_6$, E_7 or E_8.

Let $M = \mathrm{F}_4(q)$, $q \geq 2$. Then M is simple. Inspection of the character table for $\mathrm{F}_4(2)$ displayed in the Atlas [CCNPW85], disposes of the case $q = 2$, so that we may assume $q \geq 3$. From Lemma 3.3.30 we have that χ vanishes either on regular elements of order l_1, l_2 or $l_3 = l(3)$, or χ is of p-defect zero if it coincides with the Steinberg character. On the other hand, there exist an odd prime l such that $l \mid (q-1)$ or $l \mid (q+1)$ and M is of l-rank at least 3 unless $q = 3$. If $q \neq 3$, then by Theorem 3.3.1, we have that χ vanishes on an l-singular element and since $\gcd(l_1, l) = \gcd(l_2, l) = \gcd(l_3, l) = \gcd(p, l) = 1$, the result follows. The construction of a character table for $M = \mathrm{F}_4(3)$ using Magma [BCP97], disposes of the case $q = 3$.

Now suppose that $M = \mathrm{E}_6(q)$, $q \geq 2$ with $Z(M) = \{1_M\}$. From Lemma 3.3.30 we have that χ vanishes on regular elements of order l_1, l_2 or l_3, or χ is of p-defect zero if it coincides with is the Steinberg character. On the other hand, M is of l-rank at least 3 for an odd prime l such that $l \mid (q^3 - 1)$. By Theorem 3.3.1, χ vanishes on an l-singular element. Since $\gcd(l_1, 1) = \gcd(l_2, l) = \gcd(l_3, l) = 1$, we are done. Now suppose that $|Z(M)| \neq 1$, that is, $|Z(M)| = 3$. By ($\star$), χ vanishes on a 3-element. Using the above argument, χ vanishes on an l-singular element, where l is an odd prime such that $l \mid (q^3 - 1)$. Since $\gcd(3, l) = 1$, the result follows.

Suppose that $M = {}^2\mathrm{E}_6(q)$, $q \geq 2$ with $Z(M) = \{1_M\}$. By Lemma 3.3.30 we have that χ vanishes on regular elements of order l_1, l_2 or l_3, or χ is of p-defect zero if it coincides with the Steinberg character. On the other hand, M is of l-rank at least 3 for an odd prime l such that $l \mid (q^6 - 1)$. By Theorem 3.3.1, χ vanishes on an l-singular element. Since $\gcd(l_1, 1) = \gcd(l_2, l) = \gcd(l_3, l) = 1$, the result follows. Now suppose that $|Z(M)| \neq 1$, so $|Z(M)| = 3$ and $q = 3b - 1$ for some positive integer $b \geq 2$. Inspection of the character table for ${}^2\mathrm{E}_6(2)$ given in the Atlas [CCNPW85] disposes of the case $q = 2$, so that we may assume $q \geq 5$. It is sufficient to show that M is of l-rank at least 3 for some prime $l \neq 3$. Such a candidate for l is an odd prime l such that $l \mid (q^6 - 1)$.

By Lemma 3.3.30, we have that χ vanishes on regular elements of order l_1, l_2 or l_3, or

χ is of p-defect zero if it coincides with the Steinberg character. By Theorem 3.3.1, χ vanishes on an l-singular element, where l is an odd prime such that $l \mid (q^6 - 1)$. Hence the result follows. The same arguments used above, may be applied to $M = \mathrm{E}_7(q)$ and $M = \mathrm{E}_8(q)$ in the case when M is simple.

Now suppose that $M = \mathrm{E}_7(q)$ and $|Z(M)| \neq 1$. Using ($\star$), we see that χ vanishes on a 2-element. By Theorem 3.3.1, χ vanishes on an l-singular element, where l is an odd prime such that $l \mid (q^6 - 1)$. Hence the result follows. □

3.4 Non-solvable groups with a character vanishing on one class

We begin this section by showing the primitivity of the characters in Propositions 3.4.2, 3.4.3 and 3.4.4. Imprimitive characters for quasisimple groups were described by Hiss, Husen and Magaard in [HHM15], and Hiss and Magaard [HM19]. This description can be used to determine which characters are primitive, for at least the cases where $G = M$ in Theorem 3.2.3. However, here we shall adopt a different approach.

In light of Theorem 2.5.1 and Propositions 3.4.2 and 3.4.3, we need only check the primitivity of characters in $\mathrm{PSL}_2(8){:}3$ and A_5. For $(G, H) = (\mathrm{A}_5, \mathrm{D}_{10})$, $|G : H| = 6$ and in Proposition 3.4.2(a) we do not have characters of degree greater than or equal to 6. Also for $(G, H) = (\mathrm{PSL}_2(8){:}3, \mathrm{D}_{18}{:}3)$, $|(\mathrm{PSL}_2(8){:}3) : (\mathrm{D}_{18}{:}3)| = 14$ but the character in Proposition 3.4.3(b) has degree not greater than or equal to 14. We have thus proved:

Theorem 3.4.1. *The irreducible characters of finite groups in Propositions 3.4.2(a)-(d), 3.4.3(a)-(c) and 3.4.4 are primitive.*

3.4.1 Symmetric and alternating groups

Proposition 3.4.2. *Let G be a finite group with a composition factor isomorphic to A_n, $n \geq 5$. Then G has a faithful irreducible character χ such that $nv(\chi) = 1$ with $v(\chi) = \mathcal{C}$ if and only if G is one of the following:*

(a) $G \cong \mathrm{A}_5$, $\chi_2(1) = \chi_3(1) = 3$ or $\chi_4(1) = 4$;

(b) $G \cong 2{\cdot}\mathrm{A}_5$, $\chi_6(1) = \chi_7(1) = 2$ *or* $\chi_8(1) = 4$;

(c) $G \cong \mathrm{S}_5$, $\chi_6(1) = \chi_7(1) = 5$;

(d) $G \in \{\mathrm{A}_6{:}2_2,\ \mathrm{A}_6{:}2_3,\ 3{\cdot}\mathrm{A}_6{:}2_3\}$, $\chi(1) = 9$ *for all such* $\chi \in \mathrm{Irr}(G)$.

Proof. Suppose that G has a faithful irreducible character χ with $v(\chi) = \mathcal{C}$. By Theorem 3.2.3, we have that there exist normal subgroups M and Z such that G/Z is almost simple and M is quasisimple with χ_M irreducible and χ_M vanishing on $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ with $m \geq 1$ such that $\mathcal{C} = \bigcup_{i=1}^{m} \mathcal{C}_i$. By the argument preceding Problem 1 in Chapter 1, it is sufficient to only consider groups G such that M is of the type listed in the statements of Theorems 3.3.4 and 3.3.6. This means that M is isomorphic to A_5, A_6, $2{\cdot}\mathrm{A}_5$ or $3{\cdot}\mathrm{A}_6$. Using GAP [GAP16] or Atlas [CCNPW85] the result follows. Lastly, by Theorem 3.4.1, all the characters appearing in the statement of the proposition are primitive. □

3.4.2 Almost simple groups of Lie type

We note that $\mathrm{PGL}_2(q) = \mathrm{PSL}_2(q) \rtimes \langle\delta\rangle$ where δ is a diagonal automorphism and $|\langle\delta\rangle| = 2$. Also, $\mathrm{Aut}(\mathrm{PSL}_2(q)) = \mathrm{PGL}_2(q) \rtimes \langle\varphi\rangle$, where φ is a field automorphism and $|\langle\varphi\rangle| = f$, where f is a positive integer.

Proposition 3.4.3. *Let G be a finite group with a composition factor isomorphic to $\mathrm{PSL}_2(q)$, where $q \geq 4$ is a prime power. Then G has a faithful irreducible character χ such that $nv(\chi) = 1$ if and only if G is one of the following:*

(a) $G \cong \mathrm{PSL}_2(7)$, $\chi(1) = 3$;

(b) $G \cong \mathrm{PSL}_2(8){:}3$, $\chi(1) = 7$;

(c) $G \cong \mathrm{PGL}_2(q)$, $\chi(1) = q$.

Proof. Suppose that G has a faithful irreducible character χ. By Theorem 3.2.3, we have that there exist normal subgroups M and Z such that G/Z is almost simple and M is quasisimple. By the argument used in the proof of Proposition 3.4.2, M is isomorphic to one of the groups listed in Proposition 3.3.9. Suppose that $M \cong \mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(8)$

or $\mathrm{PSL}_2(11)$. Consulting of the relevant character tables in the Atlas [CCNPW85], shows that (a) or (b) hold.

We now consider the case of Proposition 3.3.9(d). When q is even, $G = M = \mathrm{PSL}_2(q) = \mathrm{PGL}_2(q)$ and (c) follows from Theorem 3.3.9(d). Now suppose that q is odd and let $M = \mathrm{PSL}_2(q)$ and $G = \mathrm{PGL}_2(q)$. Note that for $\mathrm{PGL}_2(q)$ the Steinberg character ϕ has values ± 1 for all elements outside $\mathrm{PSL}_2(q)$ by [Ste51, Section 2]. Moreover, $\mathrm{PGL}_2(q)$ has one conjugacy class of order p (recall that $q = p^m$ with m a positive integer). That the Steinberg character extends to $\mathrm{PGL}_2(q)$ follows from [Fei93]. Hence $G = \mathrm{PGL}_2(q)$ satisfies the conclusion as required.

Now suppose that $M = \mathrm{PSL}_2(q) < G \leqslant \mathrm{Aut}(\mathrm{PSL}_2(q))$ and $G \ncong \mathrm{PGL}_2(q)$. We want to show that every χ of G vanishes on at least two conjugacy classes of G. In light of Proposition 3.3.9(d), we need only consider the Steinberg character ϕ of $\mathrm{PSL}_2(q)$. By Lemma 2.4.4, we have $\gcd(|G : \mathrm{PSL}_2(q)|, q) = 1$. Hence ϕ is of p-defect zero in G. We show that $\mathrm{PGL}_2(q) \leqslant H$. By Lemma 2.4.8, the action of δ makes the conjugacy classes represented by c and d of $\mathrm{PSL}_2(q)$ into one conjugacy class. On the other hand, by Lemma 2.4.9, for $1 \leq k < f$, φ^k fixes these conjugacy classes represented by c and d of $\mathrm{PSL}_2(q)$, so G has two conjugacy classes of elements of order p. Therefore G necessarily contains δ and hence $\mathrm{PGL}_2(q)$, and has only one conjugacy class of order p, $\mathcal{C}$ say. Thus $G = G \cap \mathrm{PGL}_2(q) \rtimes \langle \varphi \rangle$. Now $|\mathbf{C}_{\mathrm{PGL}_2(q)}(c)| = \dfrac{|\mathrm{PGL}_2(q)|}{|\mathcal{C}|}$. This means that $|\mathbf{C}_G(c)| = \dfrac{|G|}{|\mathcal{C}|} = \dfrac{|G : \mathrm{PGL}_2(q)||\mathrm{PGL}_2(q)|}{|\mathcal{C}|}$. Since $\gcd(|G : \mathrm{PGL}_2(q)|, q) = \gcd(|\langle \varphi \rangle|, q) = 1$, there must exist $x \in G \backslash \mathrm{PGL}_2(q)$ of order $r \nmid q$, for some prime r. Note that $x \in \mathbf{C}_G(c)$. It follows that cx has order pr. Since ϕ is of p-defect zero in G, ϕ vanishes on cx. Since ϕ vanishes on c, we have that ϕ vanishes on two distinct classes of G as required.

By Theorem 3.4.1, all the characters appearing in the statement of the proposition are primitive. □

Proposition 3.4.4. *Let G be a finite group with a composition factor isomorphic to a finite simple group of Lie type distinct from $\mathrm{PSL}_2(q)$. Then G has a faithful irreducible character χ such that $nv(\chi) = 1$ if and only if $G = {}^2\mathrm{B}_2(8){:}3$ with $\chi(1) = 14$.*

Proof. Suppose that $\chi \in \mathrm{Irr}(G)$ is faithful, primitive and vanishes on one conjugacy class. By Theorem 3.2.3, there exist normal subgroups M and Z such that G/Z is almost simple and M is quasisimple. By the argument used in the proof of Proposition 3.4.2, M is isomorphic to ${}^2\mathrm{B}_2(8)$ or $\mathrm{PSU}_3(4)$. Consulting of the chacracter tables these two groups in Atlas [CCNPW85] eliminates $\mathrm{PSU}_3(4)$ and the result follows.

Conversely, if $G = {}^2\mathrm{B}_2(8){:}3$ with $\chi(1) = 14$, then by Theorem 3.4.1, χ is primitive. $\square$

3.5 Questions of Dixon and Rahnamai Barghi

We restate and prove a renumbered version of Corollary 1.0.6.

Corollary 3.5.1. *If G is a finite group that has a faithful irreducible character χ such that $nv(\chi) = 1$, then G has at most one non-abelian composition factor.*

Proof. Suppose that G is non-solvable. If χ is primitive, then G satisfies condition (a) or (b) of Theorem 3.2.3. By the proof of Theorem 3.2.3, G is solvable when G satisfies condition (b). If G satisfies (b), the result follows since $\mathrm{Out}(M/Z)$ is solvable.

Suppose that χ is imprimitive. By Theorem 2.5.1, the non-solvable cases correspond with conditions 2.5.1(b)(iii), (c) and (d). For (b)(iii) it is well known that if H/N is a non-solvable complement, then it has only one non-abelian composition factor. For (c) and (d) it is clear that G has only one non-abelian composition factor. Hence the result follows. $\square$

Note that for the imprimitive case, χ need not be faithful. We also restate and renumber Corollary 1.0.7 which answers Question 2.

Corollary 3.5.2. *Let G be a finite non-abelian simple group and let $\chi \in \mathrm{Irr}(G)$. If $nv(\chi) = 1$, then one of the following holds:*

(a) $G \cong \mathrm{PSL}_2(5)$, $\chi(1) = 3$;

(b) $G \cong \mathrm{PSL}_2(7)$, $\chi(1) = 3$;

(c) $G \cong \mathrm{PSL}_2(2^a)$, $\chi(1) = 2^a$, *where* $a \geqslant 2$.

Combining Theorems 3.0.1, 3.0.3 and 2.5.1, we have the following:

Theorem 3.5.3. *Let G be a finite group that has a non-linear irreducible character χ such that $nv(\chi) = 1$. Then there exists a maximal subgroup H and normal subgroups M, N, K and Z of G (where appropriate), such that one of the following holds:*

(a) *G is a Frobenius group with an abelian odd-order kernel $H = G'$ of index 2;*

(b) *G/N is a 2-transitive Frobenius group with an elementary abelian kernel M/N of order p^n for some prime p and integer $n \geq 1$, and a complement H/N of order $p^n - 1$. Moreover, $M' = N$ and one of the following holds:*

  (i) *M is a Frobenius group with kernel M' and $p^n = p > 2$;*

  (ii) *M is a Frobenius group with kernel $K \lhd G$ such that $G/K \cong \mathrm{SL}_2(3)$ and $M/K \cong \mathrm{Q}_8$;*

  (ii) *M is a Camina p-group;*

(c) *$G/N \cong \mathrm{PSL}_2(8){:}3$, $H/N \cong \mathrm{D}_{18}{:}3$ and N is a nilpotent $7'$-group;*

(d) *$G/N \cong \mathrm{A}_5$, $H/N \cong \mathrm{D}_{10}$ and N is a 2-group;*

(e) *$G/K \cong \mathrm{PSL}_2(5)$;*

(f) *$G/K \cong \mathrm{SL}_2(5)$;*

(g) *$G/K \in \{\mathrm{A}_6{:}2_2,\ \mathrm{A}_6{:}2_3,\ 3{\cdot}\mathrm{A}_6{:}2_3\}$;*

(h) *$G/K \cong \mathrm{PSL}_2(7)$;*

(i) *$G/K \cong \mathrm{PSL}_2(8){:}3$;*

(j) *$G/K \cong \mathrm{PGL}_2(q)$;*

(k) *$G/K \cong {}^2\mathrm{B}_2(8){:}3$;*

(l) *G/Z is a Frobenius group with an abelian kernel M/Z of order p^{2n}, M/K is an extra-special p-group and Z/K is of order p for some prime p.*

Chapter 4

Character degrees and zeros of irreducible characters

4.1 Introduction

We begin the chapter by recalling some definitions. A character χ of a finite group G is called *monomial* if $\chi = \lambda^G$ for some linear character λ of H, where $H \leqslant G$. A group G is called an *M-group* if every irreducible character of G is monomial. Supersolvable groups are M-groups and M-groups are solvable groups (see [Isa06, Theorem 6.22 and Corollary 5.13]). Let $\mathrm{dl}(G)$ denote the derived length of G. A famous result of Taketa [Isa06, Theorem 6.12] states that $\mathrm{dl}(G) \leq \mathrm{cd}(G)$ when G is an M-group.

Let G be a finite group G and $g \in G$. Then g is *non-vanishing* in G if for every $\chi \in \mathrm{Irr}(G)$, $\chi(g) \neq 0$. A conjecture of Isaacs, Navarro and Wolf [INW99] claims that every non-vanishing element of a solvable group is contained in the Fitting subgroup of that group. They settled the conjecture for elements of odd order [Isa06, Theorem D]. A *vanishing class* $\mathcal{C}$ of G is a conjugacy class on which some irreducible character of G vanishes and *vanishing class size* is the number of elements in a vanishing class. We shall restate and renumber Theorems 1.0.13, 1.0.14, 1.0.15 and 1.0.16 in this chapter.

Theorem 4.1.1. *Let G be a finite solvable group and let $\chi \in \mathrm{Irr}(G)$ be non-linear. Suppose that one of the following conditions holds:*

(a) χ is monomial;

(b) G is of odd order;

(c) G has derived length at most 3;

(d) G has a normal Sylow 2-subgroup;

(e) G has a self-normalizing Sylow p-subgroup P and χ vanishes on p-elements for some prime p;

(f) Every maximal subgroup of G is an M-group.

If $\chi(1)$ is divisible by two distinct primes, then χ vanishes on at least two conjugacy classes.

Theorem 4.1.2. *Let G be a finite solvable group, $\chi \in \mathrm{Irr}(G)$ and let n be a positive integer. Suppose that one of the following conditions holds:*

(a) χ is primitive;

(b) G is nilpotent;

(c) G is metabelian.

If $\chi(1)$ is divisible by n distinct prime numbers, then χ vanishes on at least n elements of pairwise distinct orders.

Theorem 4.1.3. *Let G be a finite solvable group, $\chi \in \mathrm{Irr}(G)$ and let n be a positive integer. Suppose that all distinct character degrees of G are relatively prime. If $\chi(1)$ is divisible by n distinct prime numbers, then χ vanishes on at least n elements of pairwise distinct orders.*

Theorem 4.1.4. *Let G be a finite almost simple group such that $S \trianglelefteq G \leqslant \mathrm{Aut}(S)$, where S is either an alternating group or a sporadic simple group. Let $\chi \in \mathrm{Irr}(G)$ and n be a positive integer. If $\chi(1)$ is divisible by n distinct prime numbers, then χ vanishes on at least n elements of pairwise distinct orders.*

4.2 Preliminaries

Lemma 4.2.1. *[BBERA10, Theorem 3.1.2] Let $G = G_1 G_2 \cdots G_n$ be a product of cyclic groups $G_1, G_2, \ldots, G_n$. Then G is supersolvable.*

Theorem 4.2.2. *Let G be a finite supersolvable group. If p is the largest prime dividing the order of G, then the corresponding Sylow p-subgroup is a normal subgroup of G.*

Proof. The result follows from [Bec71, Theorems 6.2.5 and 6.2.6]. □

Theorem 4.2.3. *[INW99, Theorem A] Suppose that a group G has a normal Sylow p-subgroup P. Then all elements of $Z(P)$ are non-vanishing in G.*

Theorem 4.2.4. *[Bro16, Theorem B] Let G be a finite group and suppose that every vanishing class size of G is square free. Then G is supersolvable.*

Lemma 4.2.5. *Let G be a finite solvable group and let $\chi \in \mathrm{Irr}(G)$ be non-linear. Suppose that $\chi(1)$ is divisible by two distinct primes, but χ vanishes on one conjugacy class. Then there exist normal subgroups M and N and a maximal subgroup H of G such that G/N is a Frobenius group with a cyclic Frobenius kernel M/N of order p and a Frobenius complement of order $p-1$, and M is a Frobenius group with a Frobenius complement of order p and a Frobenius kernel N with $M' = N$.*

Proof. If χ is primitive, then the result follows by Theorem 1.0.14(c). Suppose that χ is imprimitive. By Theorem 2.5.1, we need only consider the solvable cases, that is, cases (a) and (b)(i)-(iii) of Theorem 2.5.1.

For case (a) we have that since H is abelian, φ is linear and $\chi = \varphi^G(1) = |G : H|\varphi(1) = 2$, contradicting our hypothesis.

For case (b)(ii) we have that M is a Frobenius group with a Frobenius kernel K such that $G/K \cong \mathrm{SL}_2(3)$ and $M/K \cong \mathrm{Q}_8$. Note that $|M/K|$ is even. Invoking Proposition 2.1.13, we see that K is abelian. By Theorem 2.3.10, $\chi(1)$ divides $|M/K|$ since $\gcd(\chi(1), |G : M|) = 1$ by Lemma 2.4.4, and so $\chi(1) = 2^s$, $s \leq 3$.

For case (b)(iii) since $\gcd(\chi(1), |G : M|) = 1$ by Lemma 2.4.4 and so χ divides $|M| = p^m$, $m \geq 1$. Hence we are left with case (b)(i) and the result then follows. □

Lemma 4.2.6. *[Isa06, Problem 12.3] Let G be a finite solvable group. If all distinct character degrees of G are relatively prime, then $|\mathrm{cd}(G)| \leq 3$.*

Theorem 4.2.7. *[Isa06, Theorem 12.5] Let G be a finite solvable group and let m be a positive integer. If $\mathrm{cd}(G) = \{1, m\}$, then at least one of the following holds:*

(a) G has an abelian normal subgroup of index m.

(b) $m = p^e$ for some prime p and G is a direct product of a p-group and an abelian group.

Theorem 4.2.8. *[Isa06, Corollary 12.6] Let G be a finite group. If $|\mathrm{cd}(G)| = 2$, then G is metabelian.*

Theorem 4.2.9. *[Isa06, Theorem 12.15] Let G be a finite group. If $|\mathrm{cd}(G)| = 3$, then* $\mathrm{dl}(G) \leq 3$.

Theorem 4.2.10. *Let G be a finite solvable group. Let $\chi \in \mathrm{Irr}(G)$ be primitive. Suppose that $\chi(1) = p_1^{a_1} p_2^{a_2} ... p_n^{a_n}$, where the $p_i's$ are distinct prime numbers and $a_i's$ positive integers for $i \in \{1, 2, ..., n\}$. Then $\chi = \alpha_{p_1}\alpha_{p_2}...\alpha_{p_n}$, where $\alpha_{p_i} \in \mathrm{Irr}(G)$ is primitive and is of p_i-power degree for $i \in \{1, 2, ..., n\}$.*

Proof. This follows from [Isa18, Theorem 2.17]. □

We recall the definition of an element of S_n, denoted by π_{np}.

If p is a prime number let

$$n = a_0 + a_1 p + \cdots + a_k p^k,\ 0 \leq a_i \leq p-1,\ a_k \neq 0,$$

be the p-adic expansion of n. Let $\pi_{np} \in \mathrm{S}_n$ be an element which is a product of a_1 p-cycles, a_2 p^2-cycles $\ldots$ and a_k p^k-cycles, i.e., $\pi_{np} = ((p^k)^{a_k}, \ldots, (p^2)^{a_2}, p^{a_1})$.

Lemma 4.2.11. *Let $\chi_\lambda \in \mathrm{Irr}(\mathrm{S}_n)$ and let $\rho \in \mathrm{Irr}(\mathrm{A}_n)$. If $p \mid \chi_\lambda(1)$, then $\chi_\lambda(\pi_{np}) = 0$. Furthermore, if $p \mid \rho(1)$ is odd, then $\pi_{np} \in \mathrm{A}_n$ and $\rho(\pi_{np}) = 0$.*

Proof. The first assertion is [MNO00, Theorem 4.1]. The second assertion follows from [MNO00, Theorem 4.2] and the first remark after [MNO00, Theorem 4.2]. □

4.3 Proof of main results

Proof of Theorem 4.1.1. In order to establish the theorem's conclusion in respect of each of the conditions listed, it is sufficient to show that if χ vanishes on one conjugacy class, then $\chi(1)$ is a prime power.

For (a), we have $\chi = \phi^G$, for some linear character $\phi \in \mathrm{Irr}(H)$, where H is a proper subgroup of G. Then G/H_G is a transitive permutation group on the set Ω of right cosets of H/H_G in G/H_G, with point stabilizer H/H_G. Note that G/H_G has one class of derangements. By Theorem 2.1.10, G/H_G is a primitive permutation group. This implies that H is a maximal subgroup of G. Since $|G : H| = p$, by Lemma 4.2.5, we have $\chi(1) = p$, as required.

For (b), suppose the contrary. Then G is the group in Lemma 4.2.5. The result follows noting that $|G|$ is even, contradicting our hypothesis.

For (c), first note that G/M is cyclic which implies that $G' \leqslant M$. If $G' < M$, then M/G' is abelian and so $N \leqslant G'$ since $M' = N$. Since M/N is cyclic of order p, we have that $G' = N$, a contradiction since G/N is not abelian. Hence $G' = M$. Note that $M' = N$. Since G has derived length at most 3, we must have that N is abelian. Now $\gcd(\chi(1), |G/M|) = 1$ by Lemma 2.4.4, and by Theorem 2.3.10, $\chi(1)$ divides $|G/N|$. Hence $\chi(1)$ divides $|M/N| = p$, which means that $\chi(1) = p$, thus concluding our argument.

For (d), suppose the contrary. By Theorem 1.0.2, χ vanishes on p-elements for some prime p. Let P be a Sylow p-subgroup of G. From Lemma 4.2.5, note that $M/N = PN/N$. If T is the normal Sylow 2-subgroup of G, then TPN/N is a direct product of TN/N and $PN/N = M/N$. This is a contradiction since G/N is a Frobenius group with a Frobenius kernel $M/N = PN/N$.

For (e), again suppose the contrary. Using Theorem [Isa08, Theorem 5.13], we infer that G has a normal p-complement K, that is, $|G/K| = p$. Hence G/N is a direct product of K/N and M/N, a contradiction.

For (f), suppose the contrary. We have that χ is imprimitive. Choose $H \leqslant G$ minimal, such that there exists $\phi \in \mathrm{Irr}(H)$ with $\chi = \phi^G$. Using the transitivity of the induced character in Lemma 2.3.15, we have that ϕ is primitive. Then G/H_G is a transitive permutation group on the set Ω of right cosets of H/H_G in G/H_G, with point stabilizer

H/H_G. Note that G/H_G has one class of derangements. By Theorem 2.1.10, G/H_G is a primitive permutation group. This implies that H is a maximal subgroup of G. By hypothesis, χ is monomial. This means that ϕ is both monomial and primitive and hence ϕ is linear. Since $|G : H| = p^m$ for some positive integer m by Lemma 4.2.5, we have $\chi(1) = p$, as required. □

Observe that the proof of (b) above shows that if G is a finite group of odd order, then G has no imprimitive irreducible character that vanishes on one conjugacy class.

Proof of Theorem 4.1.2. Suppose that χ is primitive. Suppose that $\chi(1) = p_1^{a_1}p_2^{a_2}...p_n^{a_n}$, where $p_i's$ are distinct primes and each a_i a positive integer for $i \in \{1, 2, \ldots, n\}$. Then by Theorem 4.2.10, $\chi = \alpha_{p_1}\alpha_{p_2}\ldots\alpha_{p_n}$, where each $\alpha_{p_i} \in \mathrm{Irr}(G)$ is primitive and of p_i-power degree. Since $\alpha_{p_i}(1) \neq 1$, there is a p_i-element g_i such that $\alpha_{p_i}(g_i) = 0$ for each $i \in \{1, 2, \ldots, n\}$. It follows that $\chi(g_i) = \alpha_{p_1}(g_i)\ldots\alpha_{p_{i-1}}(g_i)\alpha_{p_i}(g_i)\alpha_{p_{i+1}}(g_i)\ldots\alpha_{p_n}(g_i) = 0$. Therefore $g_1, g_2, ..., g_n$ are elements of distinct orders on which χ vanishes.

For (b), suppose that G is nilpotent and suppose that $\chi(1) = p_1^{k_1}p_2^{k_2}...p_n^{k_n}$, where the p_i's are distinct primes and each k_i a positive integer for $i \in \{1, 2, \ldots, n\}$. Then $\chi = \psi_{P_1} \times \psi_{P_2} \times \cdots \times \psi_{P_n}$ for some $\psi_{P_i} \in \mathrm{Irr}(P_i)$ and Sylow p_i-subgroup P_i of G, $i \in \{1, 2, ..., n\}$. We may consider $g_i \in P_i$ such that $\psi_{P_i}(g_i) = 0$. Then

$$\chi(g_i) = \psi_{P_1}(g_i)\psi_{P_2}(g_i)\ldots\psi_{P_{i-1}}(g_i)\psi_{P_i}(g_i)\psi_{P_{i+1}}(g_i)\ldots\psi_{P_n}(g_i) = 0.$$

Therefore $g_1, g_2, ..., g_n$ are p_i-elements of distinct orders on which χ vanishes.

For (c), we consider the case where G is metabelian. Since G/G' is abelian, we have by Theorem 2.3.13, that G is a relative M-group with respect to G'. This entails the existence of a subgroup K of H with $G' \leqslant K \leqslant G$, and $\psi \in \mathrm{Irr}(K)$ such that $\psi^G = \chi$ and $\chi_{G'} \in \mathrm{Irr}(G')$. Note that K is normal in G, ψ^G vanishes on $G\backslash K$ and ψ is linear. If $\chi(1) = p_1^{a_1}p_2^{a_2}...p_n^{a_n}$, where the p_i's are distinct primes and each a_i a positive integer for $i \in \{1, 2, \ldots, n\}$, then $p_1^{a_1}p_2^{a_2}...p_n^{a_n} = |G : H|\psi(1) = |G : H|$, whence G/H has a p_i-element g_i and χ vanishes on g_i for each $i \in \{1, 2, \ldots, n\}$. Hence the result follows. □

Proof of Theorem 4.1.3. By Lemma 4.2.6, we have that $|\mathrm{cd(G)}| \leq 3$. If $|\mathrm{cd(G)}| = 2$, then G is metabelian by Theorem 4.2.8 and the result follows by Theorem 4.1.2(c).

Hence we may assume that $\operatorname{cd}(G) = \{1, m, n\}$, with $m, n \geq 2$ positive integers such that $\gcd(m, n) = 1$. Note that $\mathrm{dl(G)} \leq 3$ by Theorem 4.2.9. Again, we may assume that $\mathrm{dl(G)} = 3$ in light of Theorem 4.1.2(c). If $|\mathrm{cd(G/G'')}| = 3$, then the result again follows from Theorem 4.1.2(c). Suppose then that $|\mathrm{cd(G/G'')}| = 2$, that is, $\operatorname{cd}(G/G'') = \{1, m\}$. By Theorem 4.2.7, G has an normal subgroup A such that A/G'' is abelian and either $|G : A| = m$ with G/A abelian since G/A has no non-linear irreducible characters, or $|G : A| = p^f$, where $m = p^e$ and e, f are positive integers, with G/A nilpotent. Note that since G'' is abelian, n divides $|G : G''|$ by Theorem 2.3.10. On the other hand, $\gcd(m, n) = 1$, so n divides $|A : G''|$. Now G/A is nilpotent and hence by Theorem 2.3.13, G is a relative M-group with respect to A. Hence for all $\chi \in \operatorname{Irr}(G)$ such that $\chi(1) = m$, we have that $\chi = \varphi^B$, where $\varphi \in \operatorname{Irr}(G)$, $A \leqslant B \leqslant G$ and $\varphi_A \in \operatorname{Irr}(A)$. If $B < G$, then $\chi(1) = |G : B|\varphi(1)$ and so $\gcd(m, n) \neq 1$, a contradiction. Hence $B = G$ and χ_A is irreducible. Note that A/G'' and G'' are abelian, and so A is metabelian. The result then follows by Theorem 4.1.2(c). □

Inspection of the Atlas [CCNPW85] shows that the following holds for sporadic simple groups:

Lemma 4.3.1. *Let G be a finite almost simple group such that $S \trianglelefteq G \leqslant \operatorname{Aut}(S)$, where S is a sporadic simple group and let $\chi \in \operatorname{Irr}(G)$. If $\chi(1)$ is divisible by n distinct prime numbers, then χ vanishes on at least n elements of pairwise distinct orders.*

Proof of Theorem 4.1.4. Consulting of the Atlas [CCNPW85] establishes the result for G such that $\mathrm{A}_n \trianglelefteq G \leqslant \operatorname{Aut}(\mathrm{A}_n)$, where $5 \leq n \leq 7$. For S_n, $n \geq 8$, the result is true by Lemma 4.2.11. We thus consider the case $G = \mathrm{A}_n$ for $n \geq 8$. If $\chi(1)$ is odd, the result is true by Lemma 4.2.11. If $\chi(1)$ is even, the result is true by Lemmas 4.2.11 and 3.3.3. □

4.4 Properties of a counterexample

We conclude the chapter by describing a counterexample to Question 3 when $n = 2$.

Theorem 4.4.1. *Let G be a finite solvable group and $\chi \in \operatorname{Irr}(G)$. Suppose that $\chi(1)$ is divisible by two distinct primes, but χ vanishes on a unique conjugacy class. Then the following hold:*

(a) *G has normal subgroups M, N and a maximal subgroup H such that G/N is a Frobenius group with a cyclic kernel M/N of order p for some prime p, and a cyclic complement H/N of order $p-1$. In particular, G/N is supersoluble and $M = PN$ is a Frobenius group with kernel N and Frobenius complement P;*

(b) *G has derived length at least 4, N contains at least one non-abelian normal Sylow r-subgroup R of G, r an odd prime. In particular, $r \nmid |H/N|$ and $\chi(1)$ is odd;*

(c) *There exists a primitive character $\phi \in \mathrm{Irr}(H)$ such that $\phi_N \in \mathrm{Irr}(N)$ and $(\phi_N)^M = \chi_M$. Moreover, M is an M-group, but H is not an M-group;*

(d) *P is not self-normalizing and $x \in \mathbf{N}_G(P)$ for all $x \in G \backslash N$.*

If $|M|$ is even and T is the the Sylow 2-subgroup of M, then the following also hold:

(e) *M is not supersolvable and $|Z(T)| = 2^{2s}$ for some positive integer s;*

(f) *$\phi \in \mathrm{Irr}(H)$ is not faithful.*

Proof. The first part of (a) follows from Lemma 4.2.5. Note that G/N is supersolvable by Lemma 4.2.1, since it is a product of cyclic groups.

The fact that G has derived length at least 4 is obvious in light of Theorem 4.1.1(c). Hence N is non-abelian and so must have a non-abelian Sylow r-subgroup R of G for some r dividing $\chi(1)$. By Lemma 2.4.4, $r \nmid |G : M| = |H/N|$ since $\mathcal{C} \subseteq M$ by Proposition 3.1.7. Since r does not divide $|H : N|$ and N is nilpotent, the Sylow r-subgroup R is characteristic in N and so normal in G. Inasmuch as $|H/N|$ we must have that $\chi(1)$ is odd by Lemma 2.4.4. Hence (b) holds.

For (c), note that χ is not primitive by Theorem 1.0.12. Choose a subgroup H of G minimal such that there exists $\phi \in \mathrm{Irr}(H)$ with $\chi = \phi^G$. By the transitivity of character induction, we have that ϕ is primitive. Using the same argument as in Section 2.5, it follows that H is maximal in G. Since H/N is cyclic, by Theorem 2.3.13 there exist a subgroup K of H with $N \leqslant K \leqslant H$ and $\psi \in \mathrm{Irr}(K)$ such that $\psi^H = \phi$ and $\phi_N \in \mathrm{Irr}(N)$. The the primitivity of ϕ implies that $K = H$, whence, ϕ_N is irreducible. Note that $(\phi_N)^M = \chi_M$ by Lemma 2.3.15 since $G = HM$, $H \cap M = N$ and $(\phi^G)_M = \chi_M$. To establish the second assertion in (c), observe that since M is a Frobenius group, we have that every irreducible character of M is either an irreducible character of P with kernel N, or is induced from an irreducible character of N by Proposition 2.3.16. Irreducible characters of P are all linear, however. Hence the non-linear characters of M are induced from irreducible characters of N. But since N is nilpotent, all its characters are monomial. By transitivity of characters, all characters of M are monomial, as required. Now if H is an M-group, since ϕ is primitive, it follows that ϕ is linear, a contradiction.

The first assertion made in (d) follows from Theorem 4.1.1(e), whilst the second holds inasmuch as M is a normal subgroup and by the Frattini argument, $G = \mathbf{N}_G(P)M$.

To establish (e), note that if M is supersolvable, then PT is a supersolvable subgroup of M. By Theorem 4.2.2, $P \lhd PT$ and $PT = P \times T$ which contradicts the fact that $\mathbf{C}_M(P) = P$. Hence M is not supersolvable.

Suppose that $|Z(T)| \neq 2^{2s}$, for any positive integer s. Note that $Z(T)P$ is a Frobenius group with kernel $Z(T)$ and Frobenius complement P by Proposition 2.1.16. Also note that $Z(T)$ is a set of non-vanishing elements by Theorem 4.2.3, and non-trivial elements of P are vanishing elements. Let $x \in P$. Then $|G/\mathbf{C}_{Z(T)P}(x)| = |Z(T)|$ is the size of the conjugacy class containing x. Since $|Z(T)|$ is square free, $Z(T)P$ is supersolvable by Theorem 4.2.4. Now since $p > 2$, $P \lhd Z(T)P$ by Theorem 4.2.2. Hence $Z(T)P = Z(T) \times P$, a contradiction because $\mathbf{C}_G(x) = P$. Thus (e) follows.

For (f), if $\phi \in \mathrm{Irr}(H)$ is faithful, then $Z(T)$ is cyclic and $Z(T)P$ is supersolvable by Lemma 4.2.1. By the argument in (e) above, $Z(T)P = Z(T) \times P$ and the result follows. $\square$

Chapter 5

Future Work

5.1 Classification of groups with a character vanishing on one conjugacy class

In Chapter 3, we classified finite non-solvable groups with an irreducible that vanishes on one conjugacy class. Even though our result is a major step towards the classification of finite groups with an irreducible that vanishes on exactly one conjugacy class, the problem is still open. We restate Theorem 3.5.3, what is currently known about this classification problem:

Theorem 5.1.1. *Let G be a finite group that has a non-linear irreducible character χ such that $nv(\chi) = 1$. Then there exists a maximal subgroup H and normal subgroups M, N, K and Z of G (where appropriate), such that one of the following holds:*

(a) *G is a Frobenius group with an abelian odd-order kernel $H = G'$ of index 2;*

(b) *G/N is a 2-transitive Frobenius group with an elementary abelian kernel M/N of order p^n for some prime p and integer $n \geq 1$, and a complement H/N of order $p^n - 1$. Moreover, $M' = N$ and one of the following holds:*

  (i) *M is a Frobenius group with kernel M' and $p^n = p > 2$;*

  (ii) *M is a Frobenius group with kernel $K \lhd G$ such that $G/K \cong \mathrm{SL}_2(3)$ and $M/K \cong \mathrm{Q}_8$;*

(ii) *M is a Camina p-group;*

(c) *$G/N \cong \mathrm{PSL}_2(8){:}3$, $H/N \cong \mathrm{D}_{18}{:}3$ and N is a nilpotent $7'$-group;*

(d) *$G/N \cong \mathrm{A}_5$, $H/N \cong \mathrm{D}_{10}$ and N is a 2-group;*

(e) *$G/K \cong \mathrm{PSL}_2(5)$;*

(f) *$G/K \cong \mathrm{SL}_2(5)$;*

(g) *$G/K \in \{\mathrm{A}_6{:}2_2,\ \mathrm{A}_6{:}2_3,\ 3{\cdot}\mathrm{A}_6{:}2_3\}$;*

(h) *$G/K \cong \mathrm{PSL}_2(7)$;*

(i) *$G/K \cong \mathrm{PSL}_2(8){:}3$;*

(j) *$G/K \cong \mathrm{PGL}_2(q)$;*

(k) *$G/K \cong {}^2\mathrm{B}_2(8){:}3$;*

(l) *G/Z is a Frobenius group with an abelian kernel M/Z of order p^{2n}, M/K is an extra-special p-group and Z/K is of order p for some prime p.*

Further work could be done by investigating:

(a) The structure of the normal subgroup K in cases (e)-(l) of Theorem 3.5.3;

(b) The converse of cases (c) and (d) in Theorem 3.5.3.

5.2 Character degrees and zeros of characters

Question 3 which we proposed in Chapter 1 is still open. It is logical to first study this question for finite solvable groups and then for finite non-solvable groups which have no composition factors isomorphic to ${}^2\mathrm{B}_2(8)$ (see Theorem 1.0.12).

A counterexample to Question 4 is provided in the paragraph following Theorem 1.0.14. In that counterexample, note that the character degree of the irreducible character of the given group G is even. This prompts the following refinement of Question 4:

Question 5. *Let G be a finite solvable group, $\chi \in \mathrm{Irr}(G)$ and n a positive integer. Is it true that if $\chi(1)$ is divisible by n distinct prime numbers, then χ vanishes on at least $n-1$ elements of pairwise distinct orders?*

5.3 One zero in a column of a character table

Dual to the classification of finite groups with an irreducible character that vanishes on exactly one conjugacy class is the classification of finite groups whose character table has a column with exactly one zero entry, that is, finite groups with an element on which exactly one irreducible character vanishes. Some work has been done on zeros in columns of a character table of a finite group (see [MS04a] [ZSW13]). In [QZ05] and [TTV18], the authors classified finite groups G with an element g such that $\chi(g) \neq \varphi(g)$ for every $\chi \neq \varphi \in \mathrm{Irr}(G)$. If $\chi(g) = 0$ for some prime $\chi \in \mathrm{Irr}(G)$, then it means that the column of the character table of G labelled by the conjugacy class $\mathcal{C}_g$ has exactly one zero entry. Hence some of the arguments in [QZ05] and [TTV18] will come in handy in the classification of finite groups whose character table has a column with exactly one zero entry.

Bibliography

[Ada02] J. Adams. *Character Tables of* PGL(2), SL(2), PGL(2) *and P*SL(2) *over a finite field.* unpublished notes, 2002, available at www.math.umd.edu/%7Ejda/characters/characters.pdf

[BBERA10] A. Ballester-Bolinches, R. Esteban-Romero, and M. Asaad. *Products of finite groups*, Volume 53. (Walter de Gruyter, 2010).

[BBOO01] A. Balog, C. Bessenrodt, J. Olsson, K. Ono. Prime power degree representations of the symmetric and alternating groups. *Journal of the London Mathematical Society*, **64**(2):344–356, 2001.

[Bec71] H. Bechtell. *The theory of groups.* (Addison-Wesley, 1971).

[BZ99] Ya. G. Berkovich and E. M. Zhmud. *Characters of Finite Groups, Part (2)*, Mathematical Monographs. Volume 181. (American Mathematical Society, 1999).

[BO04] C. Bessenrodt and J. Olsson. Weights of partitions and character zeros. *The Electronic Journal of Combinatorics*, **11**(2):R5, 2004.

[BCG00] M. Bianchi, D. Chillag, and A. Gillio. Finite groups in which every irreducible character vanishes on at most two conjugacy classes. *Houston Journal of Mathematics*, **26**(3):451–461, 2000.

[Bla94] H. Blau. A fixed point theorem for central elements in quasisimple groups. *Proceedings of the American Mathematical Society*, **122**(1):79–84, 1994.

[BCP97] W. Bosma, J. Cannon and C. Playoust. The MAGMA algebra system I: the user language. *Journal of Symbolic Computation*, **24**:235–265, 1997.

[Bro16] J. Brough. On vanishing criteria that control finite group structure. *Journal of Algebra*, **458**:207–215, 2016.

[BG16] T. C. Burness and M. Giudici. *Classical groups, derangements and primes*, Volume 25. (Cambridge University Press, 2016).

[BTV15] T. C. Burness and H. P. Tong-Viet. Derangements in primitive permutation groups, with an application to character theory. *The Quarterly Journal of Mathematics*, **66**(1):63–96, 2015.

[Cam78] A. R. Camina. Some conditions which almost characterize Frobenius groups. *Israel Journal of Mathematics*, **31**(2):153–160, 1978.

[Car85] R. W. Carter. *Finite Groups of Lie Type : Conjugacy Classes and Complex Characters*. (A Wiley-Interscience Publication, Pure and Applied Mathematics, John Wiley and Sons Inc., New York, 1985).

[Chi99] D. Chillag. On zeros of characters of finite groups. *Proceedings of the American Mathematical Society*, **127**(4):977–983, 1999.

[CCNPW85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson. *Atlas of Finite Groups* (Oxford University Press, Clarendon, 1985).

[DS96] R. Dark and C. M. Scoppola. On Camina groups of prime power order. *Journal of Algebra*, **181**(3):787–802, 1996.

[Dea90] M. Deaconescu. Zeros of characters and the Frattini subgroup. *Elemente der Mathematik*, **45**:80–81, 1990.

[DM91] F. Digne and J. Michel. *Representations of Finite Groups of Lie Type*. London Mathematical Society Student Texts, Volume 21. (Cambridge University Press, 1991).

[DM96] J. D. Dixon and B. Mortimer. *Permutation Groups*. Graduate Texts in Mathematics, Volume 163. (Springer-Verlag, New York, 1996).

[DRB07] J. D. Dixon and A. Rahnamai Barghi. Irreducible characters which are zero on only one conjugacy class. *Proceedings of the American Mathematical Society*, **135**(1):41–45, 2007.

[DPSS09] S. Dolfi, E. Pacifici, L. Sanus, P. Spiga. On the orders of zeros of irreducible characters. *Journal of Algebra* **321**(1):345–352, 2009.

[Dor71] L. Dornhoff. *Group Representation Theory. Part A: Ordinary Representation Theory*, Pure and Applied Mathematics, Volume 7. (Marcel Dekker, Inc., New York, 1971).

[FKS81] B. Fein, W. Kantor and W. Schache., Relative Brauer groups, II. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, **328**:39–57, 1981.

[Fei93] W. Feit. Extending Steinberg characters, In *Linear algebraic groups and their representations (Los Angeles, CA, 1992).* Contemporary Mathematics. Volume 153 (American Mathematical Society, 1993).

[Gag73] P. C. Gager. Maximal tori in finite groups of Lie type. PhD Thesis, University of Warwick, 1973.

[GHLMP96] M. Geck, G. Hiss, F. Lübeck, G. Malle and G. Pfeiffer. CHEVIE–A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras. *Applicable Algebra in Engineering, Communication and Computing* **7**(3):175–210, 1996.

[Geh02] K. E. Gehles. Ordinary characters of finite special linear groups. MSc thesis, University of St Andrews, 2002.

[GGLMNT14] D. Goldstein, R. Guralnick, M. Lewis, A. Moretó, G. Navarro, and P. H. Tiep. Groups with exactly one irreducible character of degree divisible by p. *Algebra & Number Theory*, **8**(2):397–428, 2014.

[Gro11] L. C. Grove. *Groups and characters*, Volume 30 (John Wiley & Sons, 2011).

[Gur16] R. Guralnick. Conjugacy classes of derangements in finite transitive groups. *Proceedings of the Steklov Institute of Mathematics*, **292**(1):112–117, 2016.

[GM12] R. M. Guralnick and G. Malle. Products of conjugacy classes and fixed point spaces. *Journal of the American Mathematical Society*, **25**(1):77–121, 2012.

[HHM15] G. Hiss, W. Husen, and K. Magaard. *Imprimitive irreducible modules for finite quasisimple groups*, Volume 234. (Memoirs of the American Mathematical Society, 2015).

[HM19] G. Hiss and K. Magaard. Imprimitive irreducible modules for finite quasisimple groups, II. *Transactions of the American Mathematical Society*, **371**:833–882, 2019.

[HH92] P. N. Hoffman, J. F. Humphreys. *Projective representations of the symmetric groups. Q-functions and shifted tableaux.* (Oxford University Press, Clarendon, 1992).

[Hup98] B. Huppert. *Character theory of finite groups*, Volume 25. (Walter de Gruyter, 1998).

[HB82] B. Huppert and N. Blackburn. *Finite Groups II.* Volume 242 (Berlin-Heidelberg-New York, 1982).

[Isa06] I. M. Isaacs. *Character Theory of Finite Groups*, Volume 69. Graduate Studies in Mathematics, (American Mathematical Society, Providence, Rhode Island, 2006).

[Isa08] I. M. Isaacs. *Finite Group Theory*, Volume 92. Graduate Studies in Mathematics, (American Mathematical Society, Providence, Rhode Island, 2008).

[Isa18] I. M. Isaacs. *Characters of solvable groups*, Volume 189. Graduate Studies in Mathematics, (American Mathematical Society, Providence, Rhode Island, 2018).

[INW99] I. M. Isaacs, G. Navarro and T. R. Wolf. Finite group elements where no irreducible character vanishes. *Journal of Algebra*, **222**(2):413–423, 1999.

[JK81] G. James and A. Kerber. *The Representation Theory of the Symmetric Group*, Volume 16. Encyclopedia of Mathematics and its Applications. (Addison- Wesley Publishing Co., 1981).

[Jor72] C. Jordan. Recherches sur les substitutions. *Journal de Mathématiques Pures et Appliquées*, **17**:351–367, 1872.

[LM15] C. Lassueur and G. Malle. Simple endotrivial modules for linear, unitary and exceptional groups. *Mathematische Zeitschrift*, **280**(3–4):1047–1074, 2015.

[LMS16] C. Lassueur, G. Malle, E. Schulte. Simple endotrivial modules for quasi-simple groups. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, **712**:141–174, 2016.

[LM16] F. Lübeck, G. Malle. A Murnaghan-Nakayama rule for values of unipotent characters in classical groups. *Representation Theory*, **20**:139–161, 2016.

[Mad19a] S. Y. Madanha. Character degrees and zeros of characters. *preprint*, 2019.

[Mad19b] S. Y. Madanha. On a question of Dixon and Rahnamai Barghi. *Communications in Algebra*, **47**(8):3064–3075, 2019.

[Mad19c] S. Y. Madanha. Zeros of primitive characters of finite groups. *preprint*, 2019.

[MNO00] G. Malle, G. Navarro, J. B. Olsson. Zeros of characters of finite groups. *Journal of Group Theory*, **3**(4):353–368, 2000.

[MSW94] G. Malle, J. Saxl, T. Weigel. Generation of classical groups. *Geometriae Dedicata*, **49**(1):85–116, 1994.

[MT11] G. Malle and D. Testerman. *Linear Algebraic Groups and Finite Groups of Lie Type*. Volume 133. Cambridge Studies in Advanced Mathematics. (Cambridge University Press, 2011).

[MZ18] G. Malle and A. E. Zalesskii. Steinberg-like characters for finite simple groups. *preprint, available at arxiv.org/abs/1712.08401v2*, 2018.

[Mei02] U. Meierfrankenfeld. Perfect Frobenius complements. *Archiv der Mathematik*, **79**(1):19–26, 2002.

[MS04a] A. Moretó and J. Sangroniz. On the number of conjugacy classes of zeros of characters. *Israel Journal of Mathematics*, **142**(1):163–187, 2004.

[MS04b] A. Moretó and J. Sangroniz. On the number of zeros in the columns of the character table of a group. *Journal of Algebra*, **279**(2):726–736, 2004.

[Mor62] A. O. Morris. The spin representation of the symmetric group. *Proceedings of the London Mathematical Society*, **12**:55–76, 1962.

[Nav01] G. Navarro. Irreducible restriction and zeros of characters. *Proceedings of the American Mathematical Society*, **129**(6):1643–1645, 2001.

[Qia07] G. Qian. Finite solvable groups with an irreducible character vanishing on just one class of elements. *Communications in Algebra*, **35**(7):2235–2240, 2007.

[QZ05] G. Qian and Y. Zhu. Solvable groups admitting an element on which distinct irreducible characters take distinct values. *Chinese Journal of Contemporary Mathematics*, **26**(1):1–8, 2005.

[Roi97] M. Roitman. On Zsigmondy primes. *Proceedings of the American Mathematical Society*, **125**:1913–1919, 1997.

[Sei68] G. Seitz. Finite groups having only one irreducible representation of degree greater than one. *Proceedings of the American Mathematical Society*, **19**(2):459–461, 1968.

[Sri68] B Srinivasan. The characters of the finite symplectic group $\mathrm{Sp}(4;q)$. *Transactions of the American Mathematical Society*, **131**(2):488–525, 1968.

[Ste51] R. Steinberg, The representations of $\mathrm{GL}(3;q)$, $\mathrm{GL}(4;q)$, $\mathrm{PGL}(3;q)$ and $\mathrm{PGL}(4;q)$. *Canadian Journal of Mathematics*, **3**:225–235, 1951.

[Suz61] M. Suzuki. Finite groups with nilpotent centralizers. *Transactions of the American Mathematical Society*, **99**(3):425–470, 1961.

[TTV18] P. H. Tiep and H. P. Tong-Viet. Finite groups with many values in a column of the character table. *Journal of Algebra and Its Applications*, **17**(10):1850196, 20 pages, 2018.

[GAP16] The GAP Group. GAP-Groups, Algorithms and Programming. Version 4.8.4. (2016) http://www.gap-system.org.

[Wak08] T. P. Wakefield. *Verifying Huppert's conjecture for the simple groups of Lie type of rank two*. PhD thesis, Kent State University, 2008.

[Whi13] D. White, Character degree extensions of $\mathrm{PSL}_2(q)$ and $\mathrm{SL}_2(q)$, *Journal of Group Theory* **16**:1–33, 2013.

[Wil81] J. S. Williams. Prime graph components of finite groups. *Journal of Algebra*, **69**:487–513, 1981.

[ZSW13] J. Zhang, Z. Shen, and S. Wu. A note on the number of zeros in the columns of the character table of a group. *Journal of Algebra and Its Applications*, **12**(02):1250150, 10 pages, 2013.

[ZS08] J. Zhang and W. Shi. Two dual questions on zeros of characters of finite groups. *Journal of Group Theory*, **11**(5):697–708, 2008.

[ZSS10] J. S. Zhang, W. J. Shi and Z. C. Shen. Finite groups in which every irreducible character vanishes on at most three conjugacy classes. *Journal of Group Theory*, **13**(6):799–819, 2010.

[Zhm79] E. M. Zhmud'. On finite groups having an irreducible complex character with one class of zeros. *Doklady Akademii Nauk SSSR*, **20**:795–797, 1979.

[Zsi92] K. Zsigmondy. Zur Theorie der Potenzreste. *Monatshefte für Mathematik und Physik*, **3**(1):265–284, 1892.

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Index