Finite-time synchronization of delayed complex dynamic networks via aperiodically intermittent control⋆

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Abstract

In this paper, the finite-time synchronization problem of complex dynamic networks with time delay is studied via aperiodically intermittent control. By compared with the existed results concerning aperiodically intermittent control, some new results are obtained to guarantee the synchronization of networks in a finite time. Especially, a new lemma is proposed to reduce the convergence time. In addition, based on aperiodically intermittent control scheme, the essential condition ensuring finite-time synchronization of dynamic networks is also obtained, and the convergence time is closely related to the topological structure

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of networks and the maximum ratio of the rest width to the aperiodic time span. Finally, a numerical example is provided to verify the validness of the proposed theoretical results.

Keywords: Complex networks; finite-time synchronization; time-delay; aperiodically intermittent control.

1. Introduction

Complex dynamic networks consist of a large number of nodes and edges interconnecting these nodes. Complex networks exist everywhere in real world from science and engineering to society [1–4], such as neural networks, World Wide Web, ecosystems, social networks, etc. Among collective dynamic behaviors, the synchronization problem of complex dynamic networks has drawn widespread attention. This is because it can explain many natural phenomena very well and has potential applications in various fields. In view of this, the synchronization problem of complex networks has been extensively studied. To better solve the synchronization problem of complex networks, lots of effective control schemes have been proposed, for instance, adaptive control [5], sliding control [6], impulsive control [7] and intermittent control [8], etc.

The existed control approaches can be mainly divided into continuous control and discontinuous control. The former one, such as adaptive control and sliding control, has been widely studied and many interesting results have been derived [9–11]. While the latter one, especially impulsive control and intermittent control, has become a hot research topic in different engineering fields, such as manufacturing [12] and secure communication [13], since it is more practical and easier to be implemented.

It is worth noting that intermittent control scheme is studied by decomposing a control time span as a control interval and a rest interval, while impulsive control is only activated at some isolated instants, intermittent control is activated during certain nonzero control intervals and off during other rest intervals. By employing the intermittent control schemes, many researches have been done on the exponential or asymptotical synchronization of complex dynamic networks [14–16], and some excellent results have been obtained. In [14], exponential synchronization of Cohen-Grossberg neural networks via periodically intermittent control was studied. In [15], the stabilization and synchronization of chaotic systems via intermittent control were taken into account. Exponential p-synchronization of non-autonomous Cohen-Grossberg neural networks with reaction-diffusion terms via periodically intermittent control was investigated in [16]. Obviously, the references [14–16] studied that the synchronization problems of complex networks were realized in a infinite time via intermittent control. However, to better show the superiority of convergence time, it is worth studying the finite-time synchronization problems of dynamic complex networks via intermittent control.

Recently, the periodically intermittent control strategy, which is composed of fixed control width and fixed rest width in a period, has drawn much attention [17–19]. However, the requirements of periodically intermittent control are quite restrict as the control strategy, and periodically intermittent control could not explain many aperiodic natural phenomena very well. For example, the generation of wind power is typically aperiodically intermittent. The control operation of aperiodically intermittent control [20, 21] is activated in some uncertain control widths among their own uncertain time spans. Next, we give an elaboration about the aperiodically intermittent control strategy (see Fig. 1): for any time span $[t_m, t_{m+1})$, $t_0 = 0$, $m = 0, 1, 2, \ldots$, $[t_m, s_m)$ and $[s_m, t_{m+1})$

are the mth work time and the mth rest time, respectively, where t_m and s_m denote the start time and the end time of the mth control, t_{m+1} denotes the end time of the mth rest. Note that the intermittent control type becomes a periodic one, when $t_{m+1} - t_m \equiv T$ and $s_m - t_m \equiv \delta$, where $T,\ \delta$ are positive constants, $t_0 = 0,\ m = 0, 1, 2, \ldots$. Hence, compared with periodically intermittent control, the aperiodically intermittent control can be better applied in practice. Thus, it is necessary to study synchronization problems under aperiodically intermittent control.

For all the above proposed intermittent control, the trajectories of the dynamic error systems can reach synchronization over the infinite horizon. However, synchronization needs to be realized in a finite time in the practical engineering process. The finite time technique [22–24] is desirable and widely applied due to its prediction of convergence time. Combining the finite time technique, periodically intermittent control has been widely used to realize finite-time synchronization for complex dynamic networks [25–29]. Yang et al. [25] investigated finite-time synchronization of neural networks with discrete and distributed delays via periodically intermittent memory feedback control. Jing et al. [26] studied the problem of finite-time lag synchronization of delayed neural networks via periodically intermittent control. Mei et al. [27] studied finite-time synchronization of drive-response systems via periodically intermittent adaptive control. Zhang et al. [28] studied global finite-time synchronization of different dimensional chaotic systems and further investigated this issue in [29]. As is known to all, there are many excellent results about using periodically intermittent control to achieve finite-time synchronization of complex networks. Therefore, it is worth to study that using aperiodically intermittent control is to achieve finite-time synchronization of complex networks.

For the aperiodically intermittent control, authors in [30] investigated the aperiodically intermittent control of linearly coupled networks with delays. The synchronization problem of complex networks via aperiodically intermittent pinning control was studied in [31]. It can be seen that both the reference [30] and [31] used aperiodically intermittent control as the control strategy to achieve synchronization of complex networks over the infinite time. Later, based on the finite time technique, the authors in [32] studied finite-time synchronization problem of complex networks via the aperiodically intermittent control. However, we give a more general lemma in this paper than that in [32]. The main difficulty of this article is how to prove the new lemma strictly with mathematical methods. Meanwhile, a new parameter added in the new lemma can be adjusted to make the convergence time shorter. That is to see, the more general lemma provided in this paper has the advantage of shortening the convergence time. Thus, it is valuable and meaningful to further study the finite-time synchronization problem of complex dynamic networks via aperiodically intermittent control.

The contributions of this paper mainly include four parts. Firstly, the main results in this paper about finite-time synchronization problem are obtained based on the famous finite-time stability theory. Meanwhile, compared with [32], we added a new parameter in a new lemma to better reflect superiority in convergence time. Secondly, the controller used in our paper to achieve finite-time synchronization is simpler and easier to be implemented in the field of engineering than that used in [32]. Thirdly, the control method of finite-time synchronization is aperiodically intermittent control, which is more practical and valuable compared with periodic intermittent control schemes. Lastly, a new lemma for obtaining the main results in this paper is rigorous proven with mathematical methods.

This paper is organized as follows. In Section 2, some useful assumptions, lemmas and definitions are given. In Section 3, finite-time synchronization of delayed complex dynamic networks is studied via an aperiodically intermittent feedback controller. In Section 4, a numerical example is given to verify the validness of the proposed theoretical schemes here. Conclusions are finally drawn in Section 5.

Notation: Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrix. For $x\left(t\right) = \left(x_1^T(t),\ x_2^T(t), \ldots, x_N^T(t)\right)^T \in \mathbb{R}^{nN}$, $\|x\| = (\sum\limits_{i=1}^{nN} x_i^2)^{\frac{1}{2}}$. $\lambda_{\max}(A)$ represents the largest eigenvalue of square matrix A. $\otimes$ stands for the notation of Kronecker product. The initial conditions of network (1) and (15) are assumed to be $x(s) = \phi(s),\ y(s) = \varphi(s)$, respectively, where $s \in [-\tau, 0]$, $\phi(s),\ \varphi(s) \in \mathcal{C}([-\tau, 0],\ \mathbb{R}^{nN})$.

2. Preliminaries

In this paper, we consider a class of delayed complex networks each consisting of N nonlinearly coupled identical nodes, with each being an n-dimensional dynamic system, respectively.

The drive networks with time delays are characterized by

$$\dot{x}_i = f_i(t, x_i, x_i(t-\tau)) + \sum_{j=1}^{N} b_{ij} h_j(x_j) + \sum_{j=1}^{N} c_{ij} g_j(x_j(t-\tau)),\ i = 1, 2, \ldots, N, \quad (1)$$

or in a compact form

$$\dot{x} = f(t, x, x(t-\tau)) + (B \otimes I_n) h(x) + (C \otimes I_n) g(x(t-\tau)), \quad (2)$$

where $x_i\left(t\right) = \left(x_{i1}(t),\ x_{i2}(t), \ldots,\ x_{in}(t)\right)^T \in \mathbb{R}^n$ are the state vectors of the ith node, $x\left(t\right) = \left(x_1^T(t),\ x_2^T(t), \ldots, x_N^T(t)\right)^T \in \mathbb{R}^{nN}$ denotes the state vector,

$f(t,x,x(t-\tau)) = (f_1^T(t,x_1,x_1(t-\tau)),\ f_2^T(t,x_2,x_2(t-\tau)),\ldots,f_N^T(t,x_N,x_N(t-\tau)))^T : \mathbb{R}\times\mathbb{R}^{nN}\times\mathbb{R}^{nN}\rightarrow\mathbb{R}^{nN}$ is a smooth nonlinear function, τ is the time delay. $h(x) = (h_1^T, h_2^T,\ \ldots, h_N^T)^T \in \mathbb{R}^{nN}$ and $g(x) = (g_1^T, g_2^T,\ldots,g_N^T)^T \in \mathbb{R}^{nN}$ are the inner connecting functions in each node. While B, $C \in \mathbb{R}^{N\times N}$ are the weight configuration matrices. If there is a connection from node i to node $j\,(j\neq i)$, then the coupling $b_{ij}\neq 0$, $c_{ij}\neq 0$; otherwise, $b_{ij}=c_{ij}=0\,(j=i)$ and $b_{ii} = -\sum_{j=1,j\neq i}^{N} b_{ij}$, $c_{ii} = -\sum_{j=1,j\neq i}^{N} c_{ij}$. Here, the configuration matrices are assumed not to be symmetric or irreducible.

Throughout this paper, we have the following assumptions, lemmas and definitions.

Assumption 1. For the vector valued function $f\left(t,x\left(t\right),x\left(t-\tau\right)\right)$, assume that there exist positive constants $\varepsilon>0$, $\nu>0$ such that f satisfies the semi-Lipschitz condition

$$\begin{aligned}
&\left(y\left(t\right)-x(t)\right)^T\left(f\left(t,y\left(t\right),y\left(t-\tau\right)\right)-f\left(t,x\left(t\right),x\left(t-\tau\right)\right)\right)\\
&\leq\varepsilon\left(y\left(t\right)-x(t)\right)^T\left(y\left(t\right)-x(t)\right)\\
&\quad+\nu\left(y\left(t-\tau\right)-x(t-\tau)\right)^T\left(y\left(t-\tau\right)-x(t-\tau)\right),
\end{aligned}$$

for all $x,y\in R^{nN}$ and $t\geq 0$.

Assumption 2. Functions $h(\cdot)$ and $g(\cdot)$ are Lipshitz, that is, there exist non-negative constants $l_h,\ l_g$ for all $x,\ y\in\mathbb{R}^{nN}$ such that

$$\left\|h(x)-h(y)\right\|\leq l_h\left\|x-y\right\|,\ \ \left\|g(x)-g(y)\right\|\leq l_g\left\|x-y\right\|.$$

Assumption 3. Let $0<\rho<1$ and $\lambda>0$, then there exists a continuous function $w : [0, \infty) \to [0, \infty)$ with $w(0) \geq 0$, for any $0 \leq u \leq t$, such that

$$w(t) - w(u) \leq -\lambda \int_u^t (w(s))^\rho \mathrm{d}s.$$

Assumption 4. [30] For the aperiodically intermittent control strategy, there exist two positive scalars $0 < \theta < \omega < +\infty$, such that, for $m = 0, 1, 2, \ldots$,

$$\begin{aligned} &\inf_m (s_m - t_m) = \theta, \\ &\sup_m (t_{m+1} - t_m) = \omega. \end{aligned} \tag{3}$$

Definition 1. The drive network (1) and the response network (15) are said to be finite-time synchronized if there exists a constant $T^* > 0$ such that

$$\lim_{t \to T^*} \|e(t)\| = \lim_{t \to T^*} \|y(t) - x(t)\| = 0,$$

and

$$\|e(t)\| = 0 \text{ if } t > T^*,$$

where $e(t)$ is the error system of the drive network (1) and the response network (15) with initial conditions ϕ and φ, respectively.

Definition 2. [30] For the aperiodically intermittent control, define

$$\Psi = \limsup_{m \to \infty} \frac{t_{m+1} - s_m}{t_{m+1} - t_m}. \tag{4}$$

Obviously, $0 \leq \Psi < 1$, when $\Psi = 0$, the aperiodically intermittent control becomes continuous control.

Lemma 1. [30] If Assumption 4 holds, then $\Psi \leq 1 - \frac{\theta}{\omega}$.

Lemma 2. [30] For any $m = 0,\ 1,\ 2, \ldots,$ we denote

$$\Psi(t) = \frac{t - s_m}{t - t_m}, \qquad t \in [s_m, t_{m+1}). \tag{5}$$

Then, $\Psi(t)$ is a strictly increasing function and $\Psi(t) \leq \frac{t_{m+1} - s_m}{t_{m+1} - t_m}$.

Lemma 3. [23] For $x_1, x_2, \ldots, x_N \in \mathbb{R}^n$ and $0 < q < 2$, the following inequality holds:

$$\|x_1\|^q + \|x_2\|^q + \cdots + \|x_N\|^q \geq (\|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_N\|^2)^{q/2}. \tag{6}$$

Lemma 4. [32] If $\mathbf{Y}$ and $\mathbf{Z}$ are real matrices with appropriate dimensions, then there exists a positive constant $\varsigma > 0$ such that

$$\mathbf{Y}^T\mathbf{Z} + \mathbf{Z}^T\mathbf{Y} \leq \varsigma \mathbf{Y}^T\mathbf{Y} + \frac{1}{\varsigma}\mathbf{Z}^T\mathbf{Z}.$$

Lemma 5. [24] Assume that a continuous, positive-definite function $V(t)$ on a neighborhood $\tilde{U} \in \mathbb{R}^{nN}$ of the origin, and satisfy the following differential inequality

$$\dot{V}(x(t)) \leq -\beta V^{\eta}(x(t)) - \gamma V(x(t)),\ \forall x(t) \in \tilde{U} \backslash \{0\}, \tag{7}$$

where $\eta \in (0,1)$ and $\beta,\ \gamma > 0$ are constants. Then, for any given $x(t_0)$, $V(t)$ satisfies the following inequality:

$$\begin{aligned} V^{1-\eta}(x(t)) \exp\{\gamma(1-\eta)x(t)\} \leq\ & V^{1-\eta}(x(t_0)) \exp\{\gamma(1-\eta)x(t_0)\} \\ & + \tfrac{\beta}{\gamma}[\exp\{\gamma(1-\eta)x(t_0)\} - \exp\{\gamma(1-\eta)x(t)\}], \\ & t_0 \leq t \leq t_s, \end{aligned} \tag{8}$$

and

$$V(x(t)) \equiv 0,\ \forall\ t \geq t_s,$$

with t_s given by $t_s \leq \frac{\ln(1+\frac{\gamma}{\beta}V^{1-\eta}(0))}{\gamma(1-\eta)}$, for $t_0 = 0$.

Lemma 6. Suppose that function $V(t)$ is continuous and non-negative when $t \in [0, +\infty)$ and satisfies the following conditions:

$$\begin{cases} \dot{V}(t) \leq -\alpha V^{\eta}(t) - p_1 V(t), & t_m \leq t < s_m, \\ \dot{V}(t) \leq p_2 V(t), & s_m \leq t < t_{m+1}, \quad m = 0,\ 1,\ 2, \ldots, \end{cases} \tag{9}$$

where $\alpha,\ p_1,\ p_2 > 0,\ 0 < \eta < 1$. If there exists $\Psi \in (0, 1)$ such that

$$p_1 - (p_1 + p_2)\Psi > 0, \tag{10}$$

where Ψ is defined in Definition 2. Then, we have

$$\begin{aligned} V^{1-\eta}(t)\exp\{(1-\eta)p_1 t\} \leq & \exp\big\{(1-\eta)(p_1+p_2)\Psi t\big\}\Big(V^{1-\eta}(0) + \frac{\alpha}{p_1} \\ & - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\}\exp\big\{-(1-\eta)(p_1+p_2)\Psi t\big\}\Big), \\ & 0 \leq t \leq T^*, \end{aligned}$$

where the settling time T^* is given by

$$T^* \leq \frac{\ln(1+\frac{p_1}{\alpha}V^{1-\eta}(0))}{(1-\eta)\big(p_1 - (p_1+p_2)\Psi\big)}. \tag{11}$$

Proof: Take $M_0 = V^{1-\eta}(0) + \frac{\alpha}{p_1}$ and $W(t) = \exp\{(1-\eta)p_1 t\}V^{1-\eta}(t),$ for $t \geq 0$. Let $Q(t) = W(t) - M_0 + \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\}$. It is easy to see that

$$Q(t) = 0, \quad \text{for} \quad t = 0. \tag{12}$$

In the following, we will prove that

$$Q(t) \leq 0, \quad \text{for} \quad t \in [0, s_0). \tag{13}$$

For $\forall t \in [0, s_0)$, we have

$$\begin{aligned}
\dot{Q}(t) =&(1-\eta)V^{-\eta}(t)\dot{V}(t)\exp\{(1-\eta)p_1 t\} + p_1(1-\eta)\exp\{(1-\eta)p_1 t\} \\
&V^{1-\eta}(t) + \alpha(1-\eta)\exp\{(1-\eta)p_1 t\} \\
\leq&(1-\eta)V^{-\eta}(t)\exp\{(1-\eta)p_1 t\}\big(-\alpha V^{\eta}(t) + p_1 V(t)\big) + p_1(1-\eta) \\
&\exp\{(1-\eta)p_1 t\}V^{1-\eta}(t) + \alpha(1-\eta)\exp\{(1-\eta)p_1 t\} \\
=&0.
\end{aligned}$$

Hence, $Q(t) \leq Q(0) = 0, \quad \text{for} \quad t \in [0, s_0)$.

Let $W_1(t) = \exp\{(1-\eta)p_1 t\}\exp\{-(1-\eta)(p_1+p_2)(t-s_0)\}V^{1-\eta}(t)$, $Q_1(t) = W_1(t) - M_0 + \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\}\exp\{-(1-\eta)(p_1+p_2)(t-s_0)\}$. Next, we prove that for $t \in [s_0, t_1)$,

$$Q_1(t) \leq 0. \tag{14}$$

For $\forall t \in [s_0, t_1)$, we can obtain

$$\begin{aligned}
\dot{Q}_1(t) =&\dot{W}_1(t) + \frac{\alpha}{p_1}(-(1-\eta)p_2)\exp\{(1-\eta)p_1 s_0\}\exp\{-(1-\eta)p_2(t-s_0)\} \\
=&-(1-\eta)p_2\exp\{(1-\eta)p_1 s_0\}\exp\{-(1-\eta)p_2(t-s_0)\}V^{1-\eta}(t) \\
&+\exp\{(1-\eta)p_1 s_0\}\exp\{-(1-\eta)p_2(t-s_0)\}(1-\eta)V^{-\eta}(t)\dot{V}(t) \\
&+\frac{\alpha}{p_1}(-(1-\eta)p_2)\exp\{(1-\eta)p_1 s_0\}\exp\{-(1-\eta)p_2(t-s_0)\} \\
\leq&-\frac{\alpha}{p_1}((1-\eta)p_2)\exp\{(1-\eta)p_1 s_0\}\exp\{-(1-\eta)p_2(t-s_0)\} \\
\leq&0.
\end{aligned}$$

Hence, $Q_1(t) \le Q_1(s_0) = Q(s_0) \le 0$.

Together with $Q_1(t) \le 0$, for $t \in [s_0, t_1)$, we can obtain

$$\begin{aligned} W(t) \le & \exp\{(1-\eta)(p_1+p_2)(t-s_0)\}\big(M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\} \\ & \exp\{-(1-\eta)(p_1+p_2)(t-s_0)\}\big) \\ \le & \exp\{(1-\eta)(p_1+p_2)(t_1-s_0)\}\big(M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\} \\ & \exp\{-(1-\eta)(p_1+p_2)(t_1-s_0)\}\big). \end{aligned}$$

Note that $Q(t) \le 0$, for $t \in [0, s_0)$, we have

$$\begin{aligned} W(t) \le & M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\} \\ \le & \exp\{(1-\eta)(p_1+p_2)(t_1-s_0)\}\big(M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\} \\ & \exp\{-(1-\eta)(p_1+p_2)(t_1-s_0)\}\big). \end{aligned}$$

So,

$$\begin{aligned} W(t) \le & \exp\{(1-\eta)(p_1+p_2)(t_1-s_0)\}\big(M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\} \\ & \exp\{-(1-\eta)(p_1+p_2)(t_1-s_0)\}\big), \end{aligned}$$

for $t \in [0, t_1)$.

Similarly, we can prove that for $t \in [t_1, s_1)$,

$$\begin{aligned} W(t) \le & \exp\{(1-\eta)(p_1+p_2)(t_1-s_0)\}\big(M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\} \\ & \exp\{-(1-\eta)(p_1+p_2)(t_1-s_0)\}\big). \end{aligned}$$

Suppose

$$Q_2 = W(t) - \exp\{(1-\eta)(p_1+p_2)(t_1-s_0)\}\big(M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\}$$
$$\exp\{-(1-\eta)(p_1+p_2)(t_1-s_0)\}\big).$$

It is easy to prove that $Q_2 \leq 0$, $\forall t \in [t_1, s_1)$.

For any $t \in [s_1, t_2)$, by taking $W_2(t) = W_1(t)\exp\{-(1-\eta)(p_1+p_2)(t_1 - s_0)\}\exp\{-(1-\eta)(p_1+p_2)(t-s_1)\}$ and $Q_3(t) = W_2(t) - M_0 + \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\}\exp\{-(1-\eta)(p_1+p_2)(t_1-s_0)\}\exp\{-(1-\eta)(p_1+p_2)(t-s_1)\}$, we can verify $Q_3(t) \leq Q_3(s_1) \leq 0$ similar to the proof of (14).

Therefore,

$$W_2(t) \leq M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\}\exp\{-(1-\eta)(p_1+p_2)(t_1-s_0)\}$$
$$\exp\{-(1-\eta)(p_1+p_2)(t-s_1)\},$$

and,

$$\begin{aligned} W(t) \leq & \exp\{(1-\eta)(p_1+p_2)(t_1-s_0)\}\exp\{(1-\eta)(p_1+p_2)(t-s_1)\}\big(M_0 \\ & - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\}\exp\{-(1-\eta)(p_1+p_2)(t_1-s_0)\} \\ & \exp\{-(1-\eta)(p_1+p_2)(t-s_1)\}\big) \\ \leq & \exp\{(1-\eta)(p_1+p_2)[(t_1-s_0)+(t-s_1)]\}\big(M_0 - \frac{\alpha}{p_1} \\ & \exp\{(1-\eta)p_1 t\}\exp\{-(1-\eta)(p_1+p_2)[(t_1-s_0)+(t-s_1)]\}\big). \end{aligned}$$

By induction, for any integer m, we can deduce the following estimation of $W(t)$ for any t.

For $t_m \leq t < s_m$, we have

$$W(t) \leq \exp\big\{(1-\eta)(p_1+p_2)\sum_{k=1}^{m}(t_k-s_{k-1})\big\}\big(M_0-\frac{\alpha}{p_1}$$
$$\exp\{(1-\eta)p_1t\}\exp\big\{-(1-\eta)(p_1+p_2)\sum_{k=1}^{m}(t_k-s_{k-1})\big\}\big),$$

and for $s_m \leq t < t_{m+1}$, we have

$$W(t) \leq \exp\big\{(1-\eta)(p_1+p_2)\sum_{k=1}^{m}[(t_k-s_{k-1})+(t-s_m)]\big\}\big(M_0-\frac{\alpha}{p_1}$$
$$\exp\{(1-\eta)p_1t\}\exp\big\{-(1-\eta)(p_1+p_2)\sum_{k=1}^{m}[(t_k-s_{k-1})+(t-s_m)]\big\}\big).$$

For $t_m \leq t < s_m$, we have

$$\begin{aligned}W(t) \leq{}& \exp\big\{(1-\eta)(p_1+p_2)\sum_{k=1}^{m}(t_k-s_{k-1})\big\}\big(M_0-\frac{\alpha}{p_1}\exp\{(1-\eta)p_1t\}\\
&\exp\big\{-(1-\eta)(p_1+p_2)\sum_{k=1}^{m}(t_k-s_{k-1})\big\}\big)\\
={}& \exp\big\{(1-\eta)(p_1+p_2)\sum_{k=1}^{m}\frac{t_k-s_{k-1}}{t_k-t_{k-1}}(t_k-t_{k-1})\big\}\big(M_0-\frac{\alpha}{p_1}\\
&\exp\{(1-\eta)p_1t\}\exp\big\{-(1-\eta)(p_1+p_2)\sum_{k=1}^{m}\frac{t_k-s_{k-1}}{t_k-t_{k-1}}(t_k-t_{k-1})\big\}\big)\\
\leq{}& \exp\big\{(1-\eta)(p_1+p_2)\Psi\sum_{k=1}^{m}(t_k-t_{k-1})\big\}\big(M_0-\frac{\alpha}{p_1}\exp\{p_1(1-\eta)t\}\\
&\exp\big\{-(1-\eta)(p_1+p_2)\Psi\sum_{k=1}^{m}(t_k-t_{k-1})\big\}\big)\\
={}& \exp\{(1-\eta)(p_1+p_2)\Psi t_m\}\big(M_0-\frac{\alpha}{p_1}\exp\{(1-\eta)p_1t\}\\
&\exp\{-(1-\eta)(p_1+p_2)\Psi t_m\}\big)\\
\leq{}& \exp\{(1-\eta)(p_1+p_2)\Psi t\}\big(M_0-\frac{\alpha}{p_1}\exp\{(1-\eta)p_1t\}\\
&\exp\{-(1-\eta)(p_1+p_2)\Psi t\}\big).\end{aligned}$$

For $s_m \leq t < t_{m+1}$, we have

$$\begin{aligned}
W(t) \leq & \exp\big\{(1-\eta)(p_1+p_2)\sum_{k=1}^{m}[(t_k - s_{k-1}) + (t - s_m)]\big\}\big(M_0 - \frac{\alpha}{p_1} \\
& \exp\{(1-\eta)p_1 t\}\exp\big\{-(1-\eta)(p_1+p_2)\sum_{k=1}^{m}[(t_k - s_{k-1}) + (t - s_m)]\big\}\big) \\
= & \exp\big\{(1-\eta)(p_1+p_2)\sum_{k=1}^{m}\big(\frac{t_k - s_{k-1}}{t_k - t_{k-1}}(t_k - t_{k-1}) + \frac{t - s_m}{t - t_m}(t - t_m)\big)\big\} \\
& \big(M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\}\exp\big\{-(1-\eta)(p_1+p_2)\sum_{k=1}^{m}\big[\frac{t_k - s_{k-1}}{t_k - t_{k-1}}(t_k \\
& - t_{k-1}) + \frac{t - s_m}{t - t_m}(t - t_m)\big]\big\}\big) \\
\leq & \exp\big\{(1-\eta)(p_1+p_2)\sum_{k=1}^{m}\big[\frac{t_k - s_{k-1}}{t_k - t_{k-1}}(t_k - t_{k-1}) + \frac{t_{m+1} - s_m}{t_{m+1} - t_m}(t - t_m)\big]\big\} \\
& \big(M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\}\exp\big\{-(1-\eta)(p_1+p_2)\Psi\sum_{k=1}^{m}\big((t_k - t_{k-1}) \\
& + (t - t_m)\big)\big\}\big) \\
= & \exp\{(1-\eta)(p_1+p_2)\Psi t\}\big(M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\}\exp\{-(1-\eta)(p_1 \\
& + p_2)\Psi t\}\big).
\end{aligned}$$

From the definition of $W(t)$, we can obtain

$$\begin{aligned}
V^{1-\eta}(t)\exp\{(1-\eta)p_1 t\} \leq & \exp\{(1-\eta)(p_1+p_2)\Psi t\}\big(M_0 - \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 t\} \\
& \exp\{-(1-\eta)(p_1+p_2)\Psi t\}\big).
\end{aligned}$$

With Lemma 5, the settling time T^* can be obtained in the following form $V^{1-\eta}(0) + \frac{\alpha}{p_1} = \frac{\alpha}{p_1}\exp\{(1-\eta)p_1 T^*\}\exp\{-(1-\eta)(p_1+p_2)\Psi T^*\}$.

From (10), one can obtain that

$$T^* = \frac{\ln(1 + \frac{p_1}{\alpha}V^{1-\eta}(0))}{(1-\eta)\big(p_1 - (p_1+p_2)\Psi\big)}.$$

The proof is completed here.

Remark 1. Firstly, it should be noted that Lemma 6 plays an important role in the following theorem, which guarantees the complex networks to achieve finite-time synchronization via the proposed aperiodically intermittent control. Secondly, the difference between the study in this paper and that in [32] is the defined Lyapunov functions in all the control widths, which are $\dot{V}(t) \leq -\alpha V^{\eta}(t),\ t_m \leq t < s_m$ in Lemma 5 of [32] and $\dot{V}(t) \leq -\alpha V^{\eta}(t) - p_1 V(t),\ t_m \leq t < s_m$ in Lemma 6 of this paper. Therefore, Lemma 6 in this paper is more general. Moreover, from mathematical expression of the convergence time (11), it is easy to see that the new added parameter p_1 helps to shorten the convergence time (see Remark 2), the situation of which meets the needs of the actual situation and is of more actual significance.

Next, a proposition will be given as follows.

Proposition 1. Suppose that function $H(\epsilon_1, \epsilon_2) = \frac{\ln\left(1+\frac{\epsilon_1}{\alpha}V^{\frac{1-\mu}{2}}(0)\right)}{\frac{1-\mu}{2}\left(\epsilon_1-(\epsilon_1+\epsilon_2)\Psi\right)}$ is a continuous differential one with positive constants $0 < \mu < 1,\ \alpha > 0,\ 0 < \Psi < 1,\ V(0) > 0$ and $\epsilon_1,\ \epsilon_2 \in (0, +\infty),\ \epsilon_1 - (\epsilon_1 + \epsilon_2)\Psi > 0$. Then $\frac{\partial H}{\partial \epsilon_1} < 0,\ \frac{\partial H}{\partial \epsilon_2} > 0$.

Proof. Calculating the derivative of $H(\epsilon_1, \epsilon_2)$ with ϵ_1, we have

$$\begin{aligned}\frac{\partial H}{\partial \epsilon_1} =& \frac{\frac{1}{1+\frac{\epsilon_1}{\alpha}V^{\frac{1-\mu}{2}}(0)}\frac{1}{\alpha}V^{\frac{1-\mu}{2}}(0)(\frac{1-\mu}{2})\big(\epsilon_1-(\epsilon_1+\epsilon_2)\Psi\big)}{\left(\frac{1-\mu}{2}\big(\epsilon_1-(\epsilon_1+\epsilon_2)\Psi\big)\right)^2} \\ &- \frac{\ln(1+\frac{\epsilon_1}{\alpha}V^{\frac{1-\mu}{2}}(0))(\frac{1-\mu}{2})(1-\Psi)}{\left(\frac{1-\mu}{2}\big(\epsilon_1-(\epsilon_1+\epsilon_2)\Psi\big)\right)^2}.\end{aligned}$$

Denote

$$F(\epsilon_1,\epsilon_2)=\frac{1}{1+\frac{\epsilon_1}{\alpha}V^{\frac{1-\mu}{2}}(0)}\frac{1}{\alpha}V^{\frac{1-\mu}{2}}(0)(\frac{1-\mu}{2})\big(\epsilon_1-(\epsilon_1+\epsilon_2)\Psi\big)$$
$$-\ln(1+\frac{\epsilon_1}{\alpha}V^{\frac{1-\mu}{2}}(0))(\frac{1-\mu}{2})(1-\Psi).$$

Since $F(0,0)=0$, the derivative of $F(\epsilon_1,\epsilon_2)$ with respect to ϵ_1 is given as follows:

$$\frac{\partial F}{\partial \epsilon_1}=\frac{-\frac{1}{\alpha^2}(\frac{1-\mu}{2})V^{1-\mu}(0)\big(\epsilon_1-(\epsilon_1+\epsilon_2)\Psi\big)}{\big(1+\frac{\epsilon_1}{\alpha}V^{\frac{1-\mu}{2}}(0)\big)^2}<0.$$

Therefore, $\frac{\partial H}{\partial \epsilon_1}<0$.

By differentiating the $H(\epsilon_1,\epsilon_2)$ with ϵ_2, we get

$$\frac{\partial H}{\partial \epsilon_2}=\frac{\Psi\ln(1+\frac{\epsilon_1}{\alpha}V^{\frac{1-\mu}{2}}(0))}{\frac{1-\mu}{2}\big(\epsilon_1-(\epsilon_1+\epsilon_2)\Psi\big)^2}>0.$$

This completes the proposition.

3. Finite-time synchronization of delayed complex networks

In this section, we will address finite-time synchronization problem with aperiodically intermittent control technique and finite-time stability theory. Then, an aperiodically intermittent controller is proposed to realize finite-time synchronization for the delayed complex networks.

The response networks can be written in a compact form:

$$\dot{y}=f(t,y,y(t-\tau))+(B\otimes I_n)h(y)+(C\otimes I_n)g(y(t-\tau))+u(t), \tag{15}$$

where $y\in\mathbb{R}^{nN}$ and $u\in\mathbb{R}^{nN}$ is the aperiodically intermittent controller.

Define the synchronization error of (2) and (15) as $e(t) = y(t) - x(t)$, then, we have

$$\begin{aligned}\dot{e}(t) &= \dot{y}(t) - \dot{x}(t) \\ &= f(t, y, y(t-\tau)) - f(t, x, x(t-\tau)) + (B \otimes I_n)h(y) - (B \otimes I_n)h(x) \\ &\quad + (C \otimes I_n)g(y(t-\tau)) - (C \otimes I_n)g(x(t-\tau)) + u(t),\end{aligned} \tag{16}$$

where $e(t) = (e_1^T(t), e_2^T(t), \dots, e_N^T(t))^T$.

The aperiodically intermittent controller is defined as follows:

$$u(t) = \begin{cases} -\xi e - \bar{k}\mathrm{sign}(e)|e|^{\mu}, & t_m \le t < s_m, \\ 0, & s_m \le t < t_{m+1}, \end{cases} \tag{17}$$

where $\xi > 0$ and $\bar{k} > 0$ are tunable parameters, $\mu \in [0, 1)$ is a real number, and $\mathrm{sign}(e) = \mathrm{diag}\{\mathrm{diag}(\mathrm{sign}(e_1)), \mathrm{diag}(\mathrm{sign}(e_2)), \dots, \mathrm{diag}(\mathrm{sign}(e_N))\}$, $|e| = (|e_1|^T, |e_2|^T, \dots, |e_N|^T)^T$.

Our main results are given below.

Theorem 1: Suppose that Assumptions 1-4 hold, if there exist positive constants p_1, p_2, ε, ν, k_1, k_2, s_1, s_2, ξ, λ, l_h and l_g satisfying the following conditions:

(i) $p_1 + 2\varepsilon + s_1\lambda_{\max}(B^TB \otimes I_n) + s_2\lambda_{\max}(C^TC \otimes I_n) + {s_1}^{-1}l_h + k_1\exp\{p_1\tau\} - \frac{k_2}{\lambda} - 2\xi = 0$, and $2\nu + {s_2}^{-1}l_g - k_1 + \frac{k_2}{\lambda} = 0$,

(ii) $2\varepsilon + s_1\lambda_{\max}(B^TB \otimes I_n) + s_2\lambda_{\max}(C^TC \otimes I_n) + {s_1}^{-1}l_h + k_1\exp\{p_1\tau\} - p_2 = 0$, and $2\nu + {s_2}^{-1}l_g - k_1 = 0$,

(iii) $p_1 - (p_1 + p_2)\Psi > 0$,

where $\Psi \in (0, 1)$ is defined in Definition 2. Then, the dynamic networks (2) and (15) with aperiodically intermittent controllers (17) will be synchronized in finite time

$$t \leq \frac{\ln(1 + \frac{p_1}{\alpha} V^{\frac{1-\mu}{2}}(0))}{\frac{1-\mu}{2}\big(p_1 - (p_1 + p_2)\Psi\big)} = T^*, \tag{18}$$

where $V(0) = e^T(0)e(0) + k_1 \exp\{p_1\tau\} \int_{-\tau}^{0} \exp\{p_1(s)\} e^T(s)e(s)\mathrm{ds}$, $e(0)$ is the initial condition of $e(t)$.

Proof. Construct the function

$$V(t) = V_1(t) + V_2(t), \tag{19}$$

where $V_1(t) = e^T(t)e(t)$ and $V_2(t) = k_1 \exp\{p_1\tau\} \int_{t-\tau}^{t} \exp\{p_1(s-t)\} e^T(s)e(s)\mathrm{ds}$.

Next, the derivative of $V(t)$ along the trajectory of (16) can be derived in the following two different cases.

Case 1: When $t_m \leq t < s_m$, for $m = 0, 1, 2, \ldots$,

$$\dot{V}_1(t) = -p_1 V_1(t) + p_1 e^T(t)e(t) + e(t)^T \dot{e}(t) + \dot{e}(t)^T e(t), \tag{20}$$

$$\dot{V}_2(t) = -p_1 V_2(t) + k_1 \exp\{p_1\tau\}\big(e(t)^T e(t) - \exp\{-p_1\tau\} e(t-\tau)^T e(t-\tau)\big). \tag{21}$$

Based on (20), (21) and Assumption 1, it is easy to obtain

$$\begin{aligned}
\dot{V}(t) &= -p_1 V(t) + p_1 e^T(t)e(t) + e(t)^T \dot{e}(t) + \dot{e}(t)^T e(t) + k_1 \exp\{p_1\tau\} e(t)^T e(t) \\
&\quad - k_1 e(t-\tau)^T e(t-\tau) \\
&= -p_1 V_1(t) + p_1 e^T(t)e(t) + e(t)^T \big(f(t, y, y(t-\tau)) - f(t, x, x(t-\tau)) \\
&\quad + (B \otimes I_n)h(y) - (B \otimes I_n)h(x) + (C \otimes I_n)g(y(t-\tau)) \\
&\quad - (C \otimes I_n)g(x(t-\tau)) - \xi e - \bar{k}\mathrm{sign}(e)|e|^{\mu}\big) + \big(f(t, y, y(t-\tau))
\end{aligned}$$

$$
\begin{aligned}
&- f(t,x,x(t-\tau)) + (B\otimes I_n)h(y) - (B\otimes I_n)h(x) + (C\otimes I_n)g(y(t-\tau)) \\
&- (C\otimes I_n)g(x(t-\tau)) - \xi e - \bar{k}\mathrm{sign}(e)|e|^{\mu}\big)^T e(t) \\
&+ k_1 \exp\{p_1\tau\} e(t)^T e(t) - k_1 e(t-\tau)^T e(t-\tau) \\
\le& -p_1 V_1(t) + p_1 e^T(t)e(t) + 2\varepsilon e(t)^T e(t) + 2\nu e(t-\tau)^T e(t-\tau) \\
&+ e(t)^T\big((B\otimes I_n)h(y) - (B\otimes I_n)h(x) + (C\otimes I_n)g(y(t-\tau)) \\
&- (C\otimes I_n)g(x(t-\tau)) - \xi e - \bar{k}\mathrm{sign}(e)|e|^{\mu}\big) + \big((B\otimes I_n)h(y) \\
&- (B\otimes I_n)h(x) + (C\otimes I_n)g(y(t-\tau)) - (C\otimes I_n)g(x(t-\tau)) - \xi e \\
&- \bar{k}\mathrm{sign}(e)|e|^{\mu}\big)^T e(t) + k_1\exp\{p_1\tau\}e(t)^T e(t) - k_1 e(t-\tau)^T e(t-\tau).
\end{aligned} \tag{22}
$$

By using Lemma 4 and Assumption 2, we have

$$
\begin{aligned}
& e(t)^T((B\otimes I_n)h(y) - (B\otimes I_n)h(x)) \\
&\quad + ((B\otimes I_n)h(y) - (B\otimes I_n)h(x))^T e(t) \\
&\le s_1 e(t)^T (B^T B \otimes I_n^T I_n) e(t) \\
&\quad + {s_1}^{-1}(h(y)-h(x))^T(h(y)-h(x)) \\
&\le s_1 e(t)^T(B^T B\otimes I_n)e(t) + {s_1}^{-1} l_h e(t)^T e(t),
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
& e(t)^T((C\otimes I_n)g(y) - (C\otimes I_n)g(x-\tau)) \\
&\quad + ((C\otimes I_n)g(y) - (C\otimes I_n)g(x-\tau))^T e(t) \\
&\le s_2 e(t)^T (C^T C \otimes I_n^T I_n) e(t) \\
&\quad + {s_2}^{-1}(g(y)-g(x-\tau))^T(g(y)-g(x-\tau)) \\
&\le s_2 e(t)^T(C^T C\otimes I_n)e(t) + {s_2}^{-1} l_g e(t-\tau)^T e(t-\tau).
\end{aligned} \tag{24}
$$

In addition, we have

$$-\bar{k}e(t)^T\text{sign}(e)|e|^\mu - \bar{k}(\text{sign}(e)|e|^\mu)^T e(t) = -2\bar{k}|e(t)|^T|e|^\mu \leq -2\bar{k}\Big(e(t)^T e(t)\Big)^{\frac{1+\mu}{2}}. \tag{25}$$

With Assumption 3 and Lemma 3, we have

$$\begin{aligned}
&\frac{k_2}{\lambda}\big(e^T(t)e(t) - e^T(t-\tau)e(t-\tau)\big)\\
&\leq -k_2\int_{t-\tau}^{t}\big(e^T(s)e(s)\big)^{\frac{1+\mu}{2}}\text{ds}\\
&\leq -k_2\Big(\int_{t-\tau}^{t} e^T(s)e(s)\text{ds}\Big)^{\frac{1+\mu}{2}}.
\end{aligned} \tag{26}$$

Therefore, substituting (23)-(26) into (22), we have

$$\begin{aligned}
\dot{V}(t) \leq& -p_1V(t) + p_1e^T(t)e(t) + 2\varepsilon e(t)^T e(t) + 2\nu e(t-\tau)^T e(t-\tau)\\
&+ s_1 e(t)^T(B^TB\otimes I_n)e(t) + {s_1}^{-1}l_h e(t)^T e(t)\\
&+ s_2 e(t)^T(C^TC\otimes I_n)e(t) + {s_2}^{-1}l_g e(t-\tau)^T e(t-\tau) - 2\xi e^T(t)e(t)\\
&- 2\bar{k}\Big(e(t)^T e(t)\Big)^{\frac{1+\mu}{2}} + k_1\exp\{p_1\tau\}e(t)^T e(t) - k_1 e(t-\tau)^T e(t-\tau)\\
&- \frac{k_2}{\lambda}(e^T(t)e(t) - e^T(t-\tau)e(t-\tau)) - k_2\Big(\int_{t-\tau}^{t} e^T(s)e(s)\text{ds}\Big)^{\frac{1+\mu}{2}}\\
=& -p_1V(t) + e^T(t)\Big(p_1 + 2\varepsilon + s_1\lambda_{\max}(B^TB\otimes I_n) + s_2\lambda_{\max}(C^TC\otimes I_n)\\
&+ {s_1}^{-1}l_h + k_1\exp\{p_1\tau\} - \frac{k_2}{\lambda} - 2\xi\Big)e(t) + e^T(t-\tau)\Big(2\nu + {s_2}^{-1}l_g - k_1\\
&+ \frac{k_2}{\lambda}\Big)e(t-\tau) - 2\bar{k}\Big(e(t)^T e(t)\Big)^{\frac{1+\mu}{2}} - k_2\Big(\int_{t-\tau}^{t} e^T(s)e(s)\text{ds}\Big)^{\frac{1+\mu}{2}}.
\end{aligned}$$

According to condition (i), we have

$$\begin{aligned}
\dot{V}(t) \leq & - p_1 V(t) - 2\bar{k}\Big(e(t)^T e(t)\Big)^{\frac{1+\mu}{2}} - k_2\Big(\int_{t-\tau}^{t} e^T(s)e(s)\mathrm{ds}\Big)^{\frac{1+\mu}{2}} \\
= & - p_1 V(t) - 2\bar{k}\Big(e(t)^T e(t)\Big)^{\frac{1+\mu}{2}} \\
& - k_2\Big(k_1 \exp\{p_1\tau\}\Big)^{-\frac{1+\mu}{2}}\Big(k_1 \exp\{p_1\tau\}\int_{t-\tau}^{t} e^T(s)e(s)\mathrm{ds}\Big)^{\frac{1+\mu}{2}} \\
\leq & - p_1 V(t) - \alpha V(t)^{\frac{1+\mu}{2}},
\end{aligned}$$

where $\alpha = \min\Big\{2\bar{k}, k_2\Big(k_1 \exp\{p_1\tau\}\Big)^{-\frac{1+\mu}{2}}\Big\}$.

Case 2: When $s_m \leq t < t_{m+1}$, for $m = 0, 1, 2, \ldots$, it is easy to obtain that

$$\begin{aligned}
\dot{V}(t) \leq & -p_1 V(t) + (p_1 + p_2)V_1(t) - p_2 e^T(t)e(t) + 2\varepsilon e(t)^T e(t) \\
& + s_1 e(t)^T (B^T B \otimes I_n)e(t) + {s_1}^{-1} l_h e(t)^T e(t) + s_2 e(t)^T (C^T C \otimes I_n)e(t) \\
& + 2\nu e(t-\tau)^T e(t-\tau) + {s_2}^{-1} l_g e(t-\tau)^T e(t-\tau) \\
& + k_1 \exp\{p_1\tau\} e(t)^T e(t) - k_1 e(t-\tau)^T e(t-\tau) \\
\leq & -p_1 V(t) + (p_1 + p_2)V(t) - p_2 e^T(t)e(t) + 2\varepsilon e(t)^T e(t) \\
& + s_1 e(t)^T (B^T B \otimes I_n)e(t) + {s_1}^{-1} l_h e(t)^T e(t) + s_2 e(t)^T (C^T C \otimes I_n)e(t) \\
& + 2\nu e(t-\tau)^T e(t-\tau) + {s_2}^{-1} l_g e(t-\tau)^T e(t-\tau) \\
& + k_1 \exp\{p_1\tau\} e(t)^T e(t) - k_1 e(t-\tau)^T e(t-\tau) \\
= & \, p_2 V(t) + e^T(t)\Big(2\varepsilon + s_1 \lambda_{\max}(B^T B \otimes I_n) + s_2 \lambda_{\max}(C^T C \otimes I_n) + {s_1}^{-1} l_h \\
& + k_1 \exp\{p_1\tau\} - p_2\Big)e(t) + e^T(t-\tau)\Big(2\nu + {s_2}^{-1} l_g - k_1\Big)e(t-\tau) \\
\leq & \, p_2 V(t).
\end{aligned}$$

Let $\frac{1+\mu}{2} = \eta$. Therefore, we have

$$\begin{cases} \dot{V}(t) \le -\alpha V^{\eta}(t) - p_1 V(t), & t_m \le t < s_m, \\ \dot{V}(t) \le p_2 V(t), & s_m \le t < t_{m+1}. \end{cases} \tag{27}$$

By Lemma 6, $V(t)$ converges to zero in a finite time T^*, where

$$T^* = \frac{\ln(1 + \frac{p_1}{\alpha} V^{\frac{1-\mu}{2}}(0))}{\frac{1-\mu}{2}\big(p_1 - (p_1 + p_2)\Psi\big)}.$$

Remark 2: From (18), it is obvious that the aperiodically intermittent control constant Ψ, the constant α, the index constant μ and the parameters $p_1,\ p_2$ play an important role in the process of finite-timely synchronizing the error system (16), and the parameters $\Psi,\ \alpha,\ \mu,\ p_1,\ p_2$ are the decision variables of the convergence time. The role of the parameters $\mu,\ \alpha$ has the similarity in aperiodically intermittent control as those in periodically intermittent control discussed in [24], and it is easy to find that a smaller value of Ψ implies a shorter convergence. Moreover, in this paper, we will focus on the effects of the decision variables $p_1,\ p_2$ on the convergence time. Denote $T(p_1, p_2) = \frac{\ln(1+\frac{p_1}{\alpha} V^{\frac{1-\mu}{2}}(0))}{\frac{1-\mu}{2}\big(p_1-(p_1+p_2)\Psi\big)}$, we can prove that $\frac{\partial T}{\partial p_1} < 0,\ \frac{\partial T}{\partial p_2} > 0$ with proposition 1. That is to say, function $T(p_1, p_2)$ is the strictly monotone decreasing function for the variable p_1, while function $T(p_1, p_2)$ is the strictly monotone increasing function for the variable p_2. Hence, a lager value of p_1 will imply a faster convergence speed; a lager value of p_2 will imply a slower convergence.

Remark 3: The authors in [30] studied the synchronization problem for linearly coupled networks by aperiodically intermittent controllers. And later they also make great progress in the aspect of synchronization problem based on the aperiodically intermittent controllers [31, 33–35], and some excellent results have been obtained [36–38]. Based on finite-time stability theorem, we employ the same type of aperiodically intermittent control principle in this paper to ensure the dynamic networks achieving synchronization in a finite time, and the concrete value of convergence time is expressed in (11). As we all know that convergence time is important in practice. Thus, it is very valuable to investigate finite-time synchronization of complex dynamic networks via aperiodically intermittent control.

Remark 4: In practice, the periodically intermittent control is rare, and a general aperiodically intermittent control technique is more economic and has better application value. The framework of the control strategy studied in this paper is shown in Fig. 1. However, a more general type of aperiodically intermittent control is shown in Fig. 2.

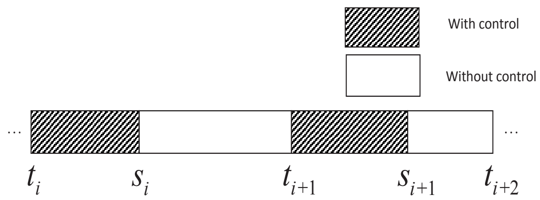

Fig. 1 Sketch map of an aperiodically intermittent control strategy.

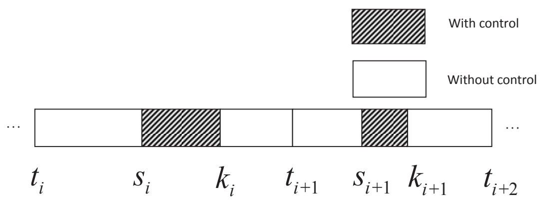

Fig. 2 Sketch map of another type of aperiodically intermittent control strategy.

Obviously, Fig. 2 shows the ith time span $[t_i,\ t_{i+1})$ is composed of control time span $[s_i,\ k_i)$ and rest time $[t_i,\ s_i)$ and $[k_i,\ t_{i+1})$, where $t_0 = 0,\ i = 0, 1, 2, \ldots$. Especially, this type becomes the one discussed in this paper when $s_i - t_i = 0$. The applicable scope of Fig. 2 is more extensive and pragmatic than that of

Fig. 1. When employing the new control strategy, the most difficult thing need to be dealt with is how to strictly prove a lemma as the proof of Lemma 6 in this paper. Hence, this is an open problem and will be further studied in our future work.

4. Numerical simulations

In this section, we give an example to demonstrate the effectiveness of the proposed results in this paper.

The complex networks are described as follows:

$$\begin{aligned}\dot{x}_i(t) =& f_i(t, x_i(t), x_i(t-\tau)) + \sum_{j=1}^{4} b_{ij} h_j(x_j(t)) + \sum_{j=1}^{4} c_{ij} g_j(x_j(t-\tau)) + u_i(t),\\ & i = 1, 2, 3, 4,\end{aligned}$$

where

$$\begin{aligned} f_i(t, x_i(t), x_i(t-\tau)) =& \begin{pmatrix} -a & a & 0 \\ b & -1 & 0 \\ 0 & 0 & -c \end{pmatrix} \begin{pmatrix} x_{i1}(t) \\ x_{i2}(t) \\ x_{i3}(t) \end{pmatrix} \\ & + \begin{pmatrix} 0 \\ x_{i1}(t-\tau) - x_{i1}(t) x_{i3}(t) \\ x_{i1}(t) x_{i2}(t) \end{pmatrix}, \end{aligned}$$

$$h(x(t)) = \sin(x(t)), \;\; g(x(t-\tau)) = 0.5 x(t) \cos(x(t-\tau)),$$

$$B = \begin{pmatrix} -3 & 2 & 1 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -3 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}, \ C = \begin{pmatrix} -3 & 3 & 0 & 0 \\ 3 & -4 & 0 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & 2 & 2 & -4 \end{pmatrix}.$$

The parameters are selected as $a = 10$, $b = 30$, $c = 8/3$, B and C are coupling matrices and the time delay is set to $\tau = 0.05$s. The initial values are given as: $x(0) = (3 + i, 5 + 2i, 7 + 2i)^T$, $y(0) = (-2 + 7i, -5 + 6i, -7 + 8i)^T$ $(i = 1, \dots, 4)$.

The chaotic attractor of the Lorenz system $\dot{x}_i(t) = f_i(t, x_i(t), x_i(t - \tau))$ is shown in Fig. 3. In the following simulation, the design parameters in (17) are selected as $\xi = 4$, $\bar{k} = 15$, $\mu = 1/2$. It is also easy to verify that Assumption 1 holds with $\varepsilon = 42.406$ and $\nu = 0.5$. The control period $t_{m+1} - t_m$ $(m \geq 0)$ is randomly generated between 0.3s and 0.5s, and the ratio of the control width $s_m - t_m$ is randomly generated between 0.3 and 0.7, the trajectories of system errors are illustrated with an aperiodically intermittent controller (17) in Figs. 4-6.

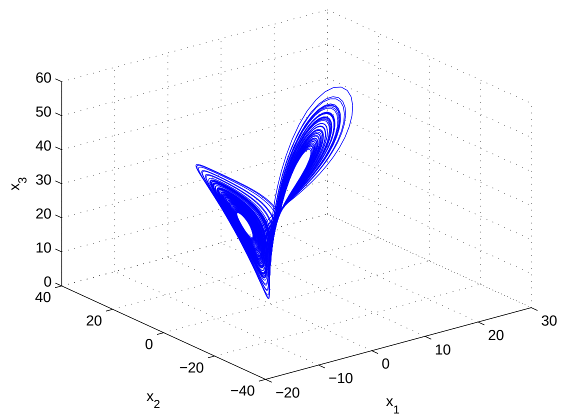

Fig. 3 The chaotic attractor of the delayed Lorenz system.

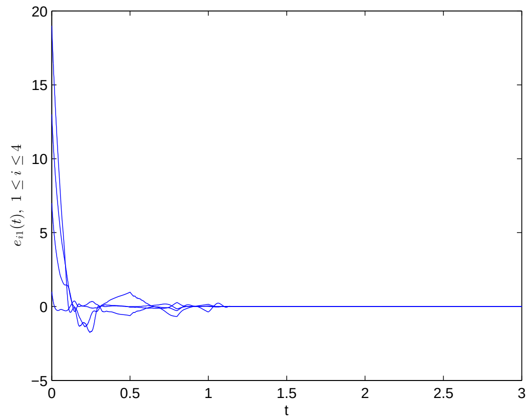

Fig. 4 Trajectories of the synchronization errors e_{i1} $(1 \le i \le 4)$ with control parameters $\xi = 4$, $\bar{k} = 15$, $\mu = 1/2$.

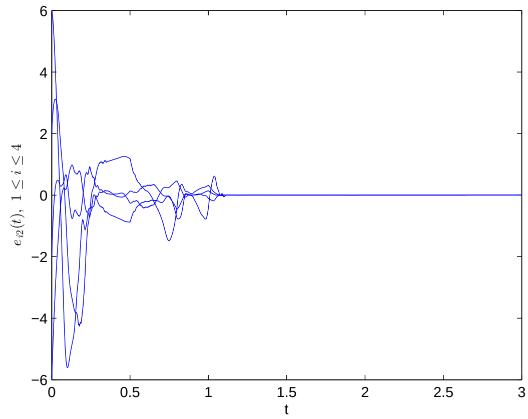

Fig. 5 Trajectories of the synchronization errors e_{i2} $(1 \le i \le 4)$ with control parameters $\xi = 4$, $\bar{k} = 15$, $\mu = 1/2$.

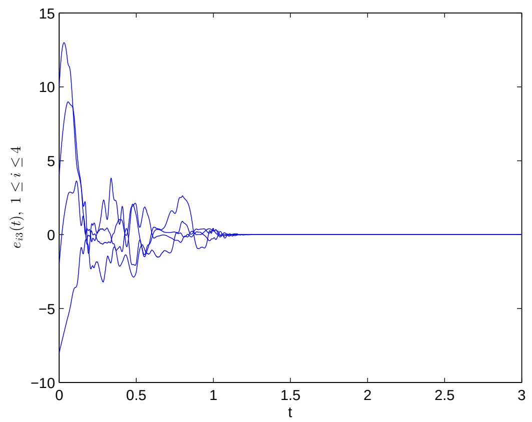

Fig. 6 Trajectories of the synchronization errors e_{i3} $(1 \leq i \leq 4)$ with control parameters $\xi = 4$, $\bar{k} = 15$, $\mu = 1/2$.

In order to highlight the advantage of this article in shorting the convergence time, all parameters are set to fixed values, except that parameter ξ in the controller (17) is set to 15, which is different from the previous one $\xi = 4$. The trajectories of system errors with an aperiodically intermittent controller (17) are shown in Figs. 7-9.

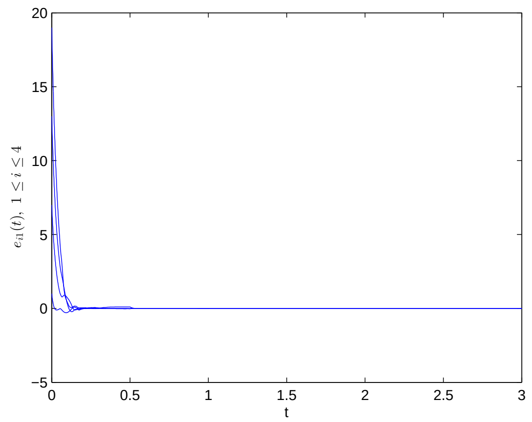

Fig. 7 Trajectories of the synchronization errors e_{i1} $(1 \leq i \leq 4)$ with control parameters $\xi = 15$, $\bar{k} = 15$, $\mu = 1/2$.

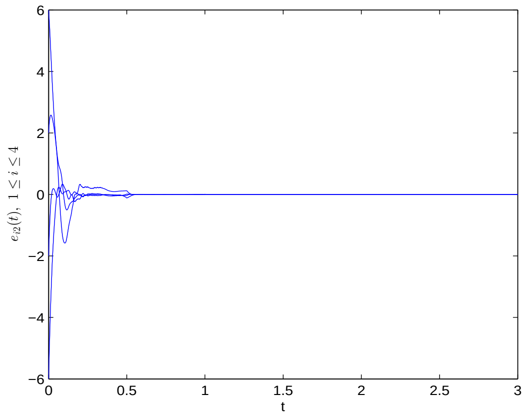

Fig. 8 Trajectories of the synchronization errors e_{i2} $(1 \le i \le 4)$ with control parameters $\xi = 15$, $\bar{k} = 15$,

$\mu = 1/2$.

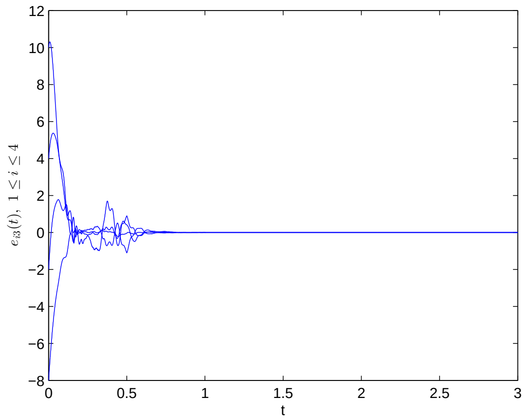

Fig. 9 Trajectories of the synchronization errors e_{i3} $(1 \le i \le 4)$ with control parameters $\xi = 15$, $\bar{k} = 15$,

$\mu = 1/2$.

Remark 5: Comparing Figs.7-9 to Figs. 4-6, it can be seen that the convergence times become shorter with bigger ξ. While from the given convergence time $T(p_1, p_2) = \frac{\ln(1+\frac{p_1}{\alpha}V^{\frac{1-\mu}{2}}(0))}{\frac{1-\mu}{2}\left(p_1-(p_1+p_2)\Psi\right)}$, we know that the convergence time decreases with the increase of parameter p_1. So what is the relationship between the two parameters p_1 and ξ? From the condition (i) of Theorem 1, it is easy to see that the two parameters are positively related. Therefore, in order to shorten the convergence time by the parameter p_1, we only need to adjust the parameter ξ in the controller (17) to a larger value.

5. Conclusions

In this paper, the finite-time synchronization problem of between delayed complex dynamic networks is studied via aperiodically intermittent control. By giving a more general Lemma 6 in this paper, we give rigorous proof process to ensure the synchronization of delayed dynamic networks within a finite time. Moreover, by constructing a piecewise Lyapunov function and applying aperiodically intermittent controller technique, some sufficient conditions are derived, and the convergence time has been expressed in a concrete value, which can be adjusted by some decision parameters. Finally, an example is provided to verify the effectiveness of the proposed results.

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